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Bresse system with infinite memories

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In this paper, we consider a one-dimensional linear Bresse system with infinite memories acting in the three equations of the system. We establish well-posedness and asymptotic stability results for the system under some conditions imposed into the relaxation functions regardless to the speeds of wave propagations. Copyright © 2014 John Wiley & Sons, Ltd.

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1. Introduction

The Bresse system is known as the circular arch problem and is given by the following equations:

$$\begin{cases} \rho_1\varphi_{tt} = Q_x + lN + F_1, \\ \rho_2\psi_{tt} = M_x - Q + F_2, \\ \rho_3 w_{tt} = N_x - lQ + F_3, \end{cases} \tag{1.1}$$

where

$$N = k_0\left(w_x - l\varphi\right), Q = k(\varphi_x + lw + \psi), M = b\psi_x$$

and $\rho_1, \rho_2, \rho_3, l, k, k_0, b$ are positive constants.

As in [1], we use N, Q and M to denote, respectively, the axial force, the shear force and the bending moment. By w, φ and ψ we are denoting, respectively, the longitudinal, vertical and shear angle displacements. Here

$$\rho_1 = \rho A, \rho_2 = \rho I, k_0 = EA,$$
$$k = k'GA, b = EI, l = R^{-1}.$$

To the material properties, we use ρ for density, E for modulus of elasticity, G for the shear modulus, k' for the shear factor, A for the cross-sectional area, I for the second moment of area of the cross section and R for the radius of curvature, and we assume that all these quantities are positive. Finally, by F_i we are denoting external forces in $]0, L[\times]0, +\infty[$ together with initial conditions and Dirichlet boundary conditions or Dirichlet–Neumann boundary conditions. For more details, we refer to [2].

If we consider $F_1 = F_3 = 0$ and $F_2 = -\gamma\psi_t$ with $\gamma > 0$, we obtain the system obtained by Bresse [3] in 1856, which consists of three coupled wave equations and is more general than the well-known Timoshenko system, where the longitudinal displacement is not considered: $l = 0$ [4, 5].

The third equation in (1.1) can be negligible [6], and the lack of exponential decay to the first and second equations was assured by Muñoz Rivera and Racke [7] using boundary conditions of type Dirichlet–Neumann.

Concerning the asymptotic behavior of the Bresse system (or circular arch problem), we have only a few results. The most important is due to Liu and Rao [8], where the authors considered a thermoelastic Bresse system (with two dissipative mechanisms) and proved that the solutions decay exponentially to zero if and only if the velocities of wave propagations are the same. Otherwise, the solutions decay polynomially to zero with rates $t^{-4+\epsilon}$ or $t^{-6+\epsilon}$ provided that the boundary conditions is of Dirichlet–Neumann–Neumann or

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Dirichlet–Dirichlet–Dirichlet type, respectively, where ϵ is an arbitrary positive constant. Alabau-Boussouira *et al.* [1] considered only one dissipative mechanism and get a polynomial decay $t^{-4+\epsilon}$ for any boundary condition.

In [8], Liu and Rao considered a thermoelastic Bresse system that consists of three wave equations and two heat equations coupled in a certain pattern. The two wave equations about the longitudinal displacement and the shear angle displacement are effectively globally damped by the dissipation from the two heat equations. The wave equation about the vertical displacement is subject to a weak thermal damping and indirectly damped through the coupling. They established exponential energy decay rate when the vertical and longitudinal waves have the same speeds of wave propagations. Otherwise, a polynomial-type decay is established.

In their paper, Wehbe and Yousef [9] studied the stabilization of the elastic Bresse systems damped by two locally distributed feedbacks with initial and boundary conditions. They established the exponential stability for this system in the case of the same speeds of wave propagations of the equation of the vertical displacement and the equation of the rotation angle of the system. When the speeds of wave propagations are different, the nonexponential decay rate is proved and a polynomial-type decay rate is obtained. The frequency domain method and the multiplier technique are applied in their proof.

For the Timoshenko system, along with the new theory of Green and Naghdi [10], Messaoudi and Said-Houari [11] considered a Timoshenko system of thermoelasticity of type III of the form

$$\begin{cases} \rho_1\varphi_{tt} - K\left(\varphi_x + \psi\right)_x = 0 & \text{in }]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + K\left(\varphi_x + \psi\right) + \beta\theta_x = 0 & \text{in }]0,L[\times\mathbb{R}_+, \\ \rho_3\theta_{tt} - \delta\theta_{xx} + \gamma\psi_{ttx} - k\theta_{txx} = 0 & \text{in }]0,L[\times\mathbb{R}_+, \end{cases} \tag{1.2}$$

where φ, ψ and θ are functions of (x,t), which model the transverse displacement of the beam, the rotation angle of the filament and the difference temperature, respectively. They proved an exponential decay in the case of equal wave speeds $\left(\frac{K}{\rho_1} = \frac{b}{\rho_2}\right)$. This result was later established by Messaoudi and Said-Houari [12] for system (1.2) in the presence of a viscoelastic damping of the form

$$\int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds$$

acting in the second equation. Moreover, the case of nonequal speeds $\left(\frac{K}{\rho_1} \neq \frac{b}{\rho_2}\right)$ was studied, and a polynomial decay result was proved for solutions with smooth initial data. A more general decay result, from which the exponential and polynomial rates of decay are only special cases, was also established by Kafini [13]. In this paper, the viscoelastic damping of the form

$$\int_0^{t} g(t-s)\theta_{xx}(x,s)ds$$

is acting in the third equation only.

The problem of stability of abstract hyperbolic systems with infinite memory was investigated by Guesmia [14]. The approach used in [14] allowed the kernel function to have decay at infinity arbitrary close to $\frac{1}{t}$. In [15], Guesmia *et al.* applied this approach for various types of Timoshenko systems. For more results concerning materials with 'finite' or 'infinite' memory, we refer to [16–19]. Concerning the stability of Bresse systems with local and global dampings, we refer to [20–23]. Decay rates for Bresse system with arbitrary nonlinear localized damping were also obtained by Charles *et al.*[24].

In this work, we will study the Bresse system with infinite memories acting in the three equations. So, our system with the initial–boundary conditions takes the form

$$\begin{cases} \rho_1\varphi_{tt} - k_1\left(\varphi_x + \psi + lw\right)_x - lk_3\left(w_x - l\varphi\right) + \int_0^{+\infty} g_1(s)\varphi_{xx}\left(x,t-s\right)ds = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1\left(\varphi_x + \psi + lw\right) + \int_0^{+\infty} g_2(s)\psi_{xx}\left(x,t-s\right)ds = 0, \\ \rho_1 w_{tt} - k_3\left(w_x - l\varphi\right)_x + lk_1\left(\varphi_x + \psi + lw\right) + \int_0^{+\infty} g_3(s)w_{xx}\left(x,t-s\right)ds = 0, \\ \varphi\left(0,t\right) = \psi\left(0,t\right) = w\left(0,t\right) = \varphi\left(L,t\right) = \psi\left(L,t\right) = w\left(L,t\right) = 0, \\ \varphi\left(x,-t\right) = \varphi_0(x,t), \varphi_t\left(x,0\right) = \varphi_1(x), \\ \psi\left(x,-t\right) = \psi_0(x,t), \psi_t\left(x,0\right) = \psi_1(x), \\ w\left(x,-t\right) = w_0(x,t), w_t\left(x,0\right) = w_1(x), \end{cases} \tag{P}$$

where $(x,t) \in]0,L[\times\mathbb{R}_+$, $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ are given functions and L, l, ρ_i, k_i are positive constants. The infinite integrals in (P) represent the infinite memories. The derivative of a generic function f with respect to a variable y is noted f_y or $\partial_y f$. If f has only one variable, its derivative is noted f'.

Our goal is to study the well-posedness and asymptotic stability of this system in terms of the growth at infinity of the kernels g_i and without paying any attention to the speeds of wave propagations defined by

$$\frac{k_1}{\rho_1}, \quad \frac{k_2}{\rho_2} \quad \text{and} \quad \frac{k_3}{\rho_1}. \tag{1.3}$$

We prove, under suitable conditions on the initial data and the memories g_i, that the system is well-posed and its energy converges to zero when time goes to infinity, and we provide a connection between the decay rate of energy and the growth of g_i at infinity. The proof is based on the semigroup's theory for the well-posedness, and the energy method and the approach introduced by Guesmia [14], for the stability.

The paper is organized as follows. In Section 2, we present our assumptions on g_i and state and prove the well-posedness of (P). Section 3 is devoted to the statement and proof of the asymptotic stability.

2. Well-posedness of (P)

We introduce, as in [25], the new variables

$$\begin{cases} \eta_1(x,t,s) = \varphi(x,t) - \varphi(x,t-s) & \text{in} \quad]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \eta_2(x,t,s) = \psi(x,t) - \psi(x,t-s) & \text{in} \quad]0,L[\times\mathrm{R}_+\times\mathbb{R}_+, \\ \eta_3(x,t,s) = w(x,t) - w(x,t-s) & \text{in} \quad]0,L[\times\mathrm{R}_+\times\mathbb{R}_+. \end{cases} \tag{2.1}$$

These functionals satisfy

$$\begin{cases} \partial_t\eta_1 + \partial_s\eta_1 - \varphi_t = 0, & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \partial_t\eta_2 + \partial_s\eta_2 - \psi_t = 0, & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \partial_t\eta_3 + \partial_s\eta_3 - w_t = 0, & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \eta_i(0,t,s) = \eta_i(L,t,s) = 0, & \text{in } \mathbb{R}_+\times\mathbb{R}_+, i=1,2,3, \\ \eta_i(x,t,0) = 0, & \text{in }]0,L[\times\mathbb{R}_+, i=1,2,3. \end{cases} \tag{2.2}$$

In order to convert our problem to a system of first-order ordinary differential equations, we note the following:

$$\eta_i^0(x,s) = \eta_i(x,0,s), i=1,2,3, \tag{2.3}$$

$$U = (\varphi,\psi,w,\varphi_t,\psi_t,w_t,\eta_1,\eta_2,\eta_3)^T \tag{2.4}$$

and

$$U^0(x) = \left(\varphi_0(x,0),\psi_0(x,0),w_0(x,0),\varphi_1(x),\psi_1(x),w_1(x),\eta_1^0(x,.),\eta_2^0(x,.),\eta_3^0(x,.)\right)^T.$$

Then (P) is equivalent to the following abstract system:

$$\begin{cases} \partial_t U = \mathcal{A}U, \\ U(x,0) = U^0(x), \end{cases} \tag{2.5}$$

where $\mathcal{A}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} \varphi_t \\ \psi_t \\ w_t \\ \frac{1}{\rho_1}\left(k_1 - \int_0^{+\infty} g_1(s)ds\right)\varphi_{xx} - \frac{l^2k_3}{\rho_1}\varphi + \frac{k_1}{\rho_1}\psi_x + \frac{l}{\rho_1}(k_1+k_3)w_x + \frac{1}{\rho_1}\int_0^{+\infty} g_1(s)\partial_{xx}\eta_1 ds \\ -\frac{k_1}{\rho_2}\varphi_x + \frac{1}{\rho_2}\left(k_2 - \int_0^{+\infty} g_2(s)ds\right)\psi_{xx} - \frac{k_1}{\rho_2}\psi - \frac{lk_1}{\rho_2}w + \frac{1}{\rho_2}\int_0^{+\infty} g_2(s)\partial_{xx}\eta_2 ds \\ -\frac{l}{\rho_1}(k_1+k_3)\varphi_x - \frac{lk_1}{\rho_1}\psi + \frac{1}{\rho_1}\left(k_3 - \int_0^{+\infty} g_3(s)ds\right)w_{xx} - \frac{l^2k_1}{\rho_1}w + \frac{1}{\rho_1}\int_0^{+\infty} g_3(s)\partial_{xx}\eta_3 ds \\ \varphi_t - \partial_s\eta_1 \\ \psi_t - \partial_s\eta_2 \\ w_t - \partial_s\eta_3 \end{pmatrix}. \tag{2.6}$$

We define the functional space of U as follows.

$$\mathcal{H} = \left(H_0^1\left(]0,L[\right)\right)^3 \times \left(L^2\left(]0,L[\right)\right)^3 \times H_1^* \times H_2^* \times H_3^*, \tag{2.7}$$

where

$$H_i^* = \left\{ v : \mathbb{R}_+ \to H_0^1(]0,L[), \int_0^L \int_0^{+\infty} g_i(s) v_x^2(s)\,ds\,dx < +\infty \right\}. \tag{2.8}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$D(\mathcal{A}) = \{U \in \mathcal{H};\ \mathcal{A}U \in \mathcal{H},\ \eta_i(x,t,0) = 0, i = 1,2,3\}. \tag{2.9}$$

Now, to get the well-posedness of (P), we assume that the functions g_i satisfy the following hypothesis:

(H1) $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ are differentiable non-increasing functions and integrable on $\mathbb{R}_+$ such that there exists a positive constant k_0 satisfying, for any $(\varphi, \psi, w) \in \left(H_0^1(]0,L[)\right)^3$,

$$\begin{aligned} k_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) dx \le& \int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) dx \\ &- \int_0^L \left(\left(\int_0^{+\infty} g_1(s)ds\right)\varphi_x^2 + \left(\int_0^{+\infty} g_2(s)ds\right)\psi_x^2 + \left(\int_0^{+\infty} g_3(s)ds\right) w_x^2\right) dx. \end{aligned} \tag{2.10}$$

Remark 2.1

By contradiction arguments, it is easy to see that there exists a positive constant $\bar{k}_0$ such that, for any $(\varphi, \psi, w) \in \left(H_0^1(]0,L[)\right)^3$,

$$\bar{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) dx \le \int_0^L \left(k_2\psi_x^2 + k_1\left(\varphi_x + \psi + lw\right)^2 + k_3(w_x - l\varphi)^2\right) dx. \tag{2.11}$$

Therefore, if

$$g_i^0 := \int_0^{+\infty} g_i(s)ds < \bar{k}_0, \quad i = 1,2,3, \tag{2.12}$$

then (2.10) is satisfied with

$$k_0 = \bar{k}_0 - \max\left\{g_1^0, g_2^0, g_3^0\right\}.$$

On the other hand, thanks to Poincaré inequality, there exists a positive constant $\tilde{k}_0$ such that, for any $(\varphi, \psi, w) \in \left(H_0^1(]0,L[)\right)^3$,

$$\int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) dx \le \tilde{k}_0 \int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) dx. \tag{2.13}$$

Thus, the right-hand side of the inequality (2.10) defines a norm on $\left(H_0^1(]0,L[)\right)^3$ for (φ, ψ, w) equivalent to the usual norm of $\left(H^1(]0,L[)\right)^3$.

Under hypothesis (H1), the sets H_i^* and $\mathcal{H}$ are Hilbert spaces equipped, respectively, with the inner products that generate the norms

$$\|\eta_i\|_{H_i^*}^2 = \int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2\,ds\,dx$$

and

$$\begin{aligned} \|U\|_{\mathcal{H}}^2 =& \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 + k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) dx \\ &- \int_0^L \left(g_1^0\varphi_x^2 + g_2^0\psi_x^2 + g_3^0 w_x^2\right) dx + \|\eta_1\|_{H_1^*}^2 + \|\eta_2\|_{H_2^*}^2 + \|\eta_3\|_{H_3^*}^2. \end{aligned}$$

Now, the domain of $D(\mathcal{A})$ is dense in $\mathcal{H}$, and a simple computation implies that, for $U \in D(\mathcal{A})$,

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = -\frac{1}{2}\int_0^L g_1(s)\int_0^{+\infty} \partial_s (\partial_x \eta_1)^2 \, dsdx - \frac{1}{2}\int_0^L g_2(s)\int_0^{+\infty} \partial_s (\partial_x \eta_2)^2 \, dsdx - \frac{1}{2}\int_0^L g_3(s)\int_0^{+\infty} \partial_s (\partial_x \eta_3)^2 \, dsdx.$$

Integration by parts, using (H1) and the boundary conditions in (2.2), yields

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} = \frac{1}{2}\int_0^L \int_0^{+\infty} \left(g_1'(s)(\partial_x \eta_1)^2 + g_2'(s)(\partial_x \eta_2)^2 + g_3'(s)(\partial_x \eta_3)^2\right) dsdx \tag{2.14}$$

and then, because, for any $i = 1, 2, 3$, the kernel g_i is non-increasing,

$$\langle \mathcal{A}U, U\rangle_{\mathcal{H}} \le 0. \tag{2.15}$$

This implies that $\mathcal{A}$ is a dissipative operator. Next, we prove that $Id - \mathcal{A}$ is surjective. Let $F = (f_1, \cdots, f_9)^T \in \mathcal{H}$. We prove the existence of $V = (v_1, \cdots, v_9) \in D(\mathcal{A})$ solution of the equation

$$(Id - \mathcal{A})V = F. \tag{2.16}$$

The first three equations of (2.16) give

$$v_4 = v_1 - f_1, \quad v_5 = v_2 - f_2 \text{ and } \quad v_6 = v_3 - f_3. \tag{2.17}$$

Using (2.17), the last three equations of (2.16) imply

$$\partial_s v_7 + v_7 = v_1 + f_7 - f_1, \quad \partial_s v_8 + v_8 = v_2 + f_8 - f_2, \quad \partial_s v_9 + v_9 = v_3 + f_9 - f_3.$$

By integrating these three differential equations and using the fact that $v_7(0) = v_8(0) = v_9(0) = 0$, we get

$$\begin{aligned} v_7 &= (1 - e^{-s})(v_1 - f_1) + \int_0^s e^{\tau - s} f_7(\tau)d\tau, \\ v_8 &= (1 - e^{-s})(v_2 - f_2) + \int_0^s e^{\tau - s} f_8(\tau)d\tau \end{aligned} \tag{2.18}$$

and

$$v_9 = (1 - e^{-s})(v_3 - f_3) + \int_0^s e^{\tau - s} f_9(\tau)d\tau.$$

Inserting (2.18) into the fourth, fifth and sixth equations of (2.16), multiplying them by $\rho_1\tilde{v}_1$, $\rho_2\tilde{v}_2$ and $\rho_1\tilde{v}_3$, respectively, and integrating their sum over $]0, L[$, we get

$$a\left((v_1, v_2, v_3)^T, (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right) = \tilde{a}\left((\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right), \ \forall (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T \in \left(H_0^1(]0, L[)\right)^3, \tag{2.19}$$

where

$$\begin{aligned} a\left((v_1, v_2, v_3)^T, (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3)^T\right) = &\int_0^L (k_1(\partial_x v_1 + v_2 + lv_3)(\partial_x \tilde{v}_1 + \tilde{v}_2 + l\tilde{v}_3) + k_3(\partial_x v_3 - lv_1)(\partial_x \tilde{v}_3 - l\tilde{v}_1))\, dx \\ &+ \int_0^L (\rho_1 v_1 \tilde{v}_1 + \rho_2 v_2 \tilde{v}_2 + \rho_1 v_3 \tilde{v}_3)dx \\ &+ \int_0^L \left(-\tilde{g}_1^0 \partial_x v_1 \partial_x \tilde{v}_1 + (k_2 - \tilde{g}_2^0)\partial_x v_2 \partial_x \tilde{v}_2 - \tilde{g}_3^0 \partial_x v_3 \partial_x \tilde{v}_3\right) dx, \end{aligned} \tag{2.20}$$

$$\tilde{g}_i^0 = \int_0^{+\infty} e^{-s} g_i(s)ds$$

and

$$\tilde{a}\left((\tilde{v}_1,\tilde{v}_2,\tilde{v}_3)^T\right) = \int_0^L (\rho_1(f_1+f_4)\tilde{v}_1 + \rho_2(f_2+f_5)\tilde{v}_2 + \rho_1(f_3+f_6)\tilde{v}_3)\,dx$$
$$+ \int_0^L ((g_1^0-\tilde{g}_1^0)\partial_x f_1 \partial_x \tilde{v}_1 + (g_2^0-\tilde{g}_2^0)\partial_x f_2 \partial_x \tilde{v}_2 + (g_3^0-\tilde{g}_3^0)\,\partial_x f_3 \partial_x \tilde{v}_3)\,dx$$
$$- \int_0^L \left(\int_0^{+\infty} g_1(s)\int_0^s e^{\tau-s}\partial_x f_7(\tau)d\tau ds\right)\partial_x\tilde{v}_1 dx$$
$$- \int_0^L \left(\int_0^{+\infty} g_2(s)\int_0^s e^{\tau-s}\partial_x f_8(\tau)d\tau ds\right)\partial_x\tilde{v}_2 dx$$
$$- \int_0^L \left(\int_0^{+\infty} g_3(s)\int_0^s e^{\tau-s}\partial_x f_9(\tau)d\tau ds\right)\partial_x\tilde{v}_3 dx.$$

Thanks to (2.10) and (2.13), we have that a is a bilinear continuous coercive form on $\left(H_0^1(]0,L[)\right)^3 \times \left(H_0^1(]0,L[)\right)^3$, and $\tilde{a}$ is a linear continuous form on $\left(H_0^1(]0,L[)\right)^3$. Then, using Lax–Milgram theorem [26], we deduce that (2.19) has a unique solution $(v_1,v_2,v_3)^T \in \left(H_0^1(]0,L[)\right)^3$. Thus, using (2.17), (2.18) and classical regularity arguments, we conclude that (2.16) admits a unique solution $V \in D(\mathcal{A})$. Therefore, $Id - \mathcal{A}$ is surjective.

Finally, thanks to the Lumer–Phillips theorem [26,27], we deduce that $\mathcal{A}$ generates a C_0-semigroup of contraction in $\mathcal{H}$, which gives the following well-posedness results of (P) [27,28]:

Theorem 2.1
Assume that (H1) holds. For any $U^0 \in \mathcal{H}$, (2.5) has a unique weak solution

$$U \in C(\mathbb{R}_+;\mathcal{H}).$$

Moreover, if $U^0 \in D(A)$, then

$$U \in C(\mathbb{R}_+;D(\mathcal{A})) \cap C^1(\mathbb{R}_+;\mathcal{H}).$$

3. Stability of (P)

In this section, we prove the stability of (P), where the decay rate of solution is explicitly specified in function of g_i and where no restriction is considered on the speeds of wave propagations (1.3). We consider the following additional hypothesis:

(H2) There exist positive constants δ_i and an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0,+\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} G'(t) = +\infty \tag{3.1}$$

such that $g_i(0) > 0$, and, for any $i = 1,2,3$, one of the following two conditions is satisfied:

$$g_i'(s) \le -\delta_i g_i(s), \quad \forall s \in \mathbb{R}_+ \tag{3.2}$$

or

$$\int_0^{+\infty} \frac{g_i(s)}{G^{-1}(-g_i'(s))}ds + \sup_{s\in\mathbb{R}_+} \frac{g_i(s)}{G^{-1}(-g_i'(s))} < +\infty. \tag{3.3}$$

Theorem 3.1
Assume that (H1) and (H2) are satisfied, and let $U^0 \in \mathcal{H}$ such that, for any $i = 1,2,3$,

$$(3.2)\ \text{holds or}\ \ \exists M_i \ge 0 : \int_0^L (\partial_x \eta_i^0)^2 dx \le M_i, \quad \forall s > 0. \tag{3.4}$$

Then there exist positive constants c', c'' and ϵ_0 such that

$$\|U(t)\|_{\mathcal{H}}^2 \le c'' e^{-c't} \quad \text{if (3.2) holds, for any } i = 1,2,3, \tag{3.5}$$

and

$$\|U(t)\|_{\mathcal{H}}^2 \le c'' H^{-1}(c't) \quad \text{otherwise,} \tag{3.6}$$

where

$$H(s) = \int_s^1 \frac{1}{\tau G'(\epsilon_0 \tau)} d\tau, \quad \forall s \in]0,1]. \tag{3.7}$$

Remark 3.1
Condition (3.2) implies that g_i converges at least exponentially to zero and then the exponential stability (3.5) of (P) is obtained only when all the functions g_i converge at least exponentially to zero without restrictions on η_i^0.

Remark 3.2
Condition (3.3), introduced in [14], allows g_i to have a decay rate arbitrarily close to $\frac{1}{t}$, and the decay rate in (3.6) depends on g_i, which has the weakest decay.

Remark 3.3
Let us consider this simple example (for other examples, see [14, 15]). Let $g_i(t) = \frac{d_i}{(1+t)^{q_i}}$ for $q_i > 1$, and $d_i > 0$ be small enough so that (2.12) is satisfied. Condition (3.2) does not hold, but condition (3.3) holds with $G(t) = t^{1+\frac{1}{p}}$, for any $p \in]0, \frac{q-1}{2}[$, where $q = \min\{q_i\}$. Then (3.6) gives, for all $p \in]0, \frac{q-1}{2}[$,

$$\|U(t)\|_{\mathcal{H}}^2 \le \frac{c'}{(1+t)^p}. \tag{3.8}$$

Proof of Theorem 3.1
We have only to prove (3.5) and (3.6) for $U^0 \in D(\mathcal{A})$, so the calculations are justified, and therefore, (3.5) and (3.6) remain valid for $U^0 \in \mathcal{H}$ by density arguments. The proof is based on the multipliers method and an approach of [14] to estimate the memory terms in case (3.3). First, we consider the following functionals:

$$I_1(t) = -\rho_1 \int_0^L \varphi_t \int_0^{+\infty} g_1(s)\eta_1 ds dx, \tag{3.9}$$

$$I_2(t) = -\rho_2 \int_0^L \psi_t \int_0^{+\infty} g_2(s)\eta_2 ds dx \tag{3.10}$$

and

$$I_3(t) = -\rho_1 \int_0^L w_t \int_0^{+\infty} g_3(s)\eta_3 ds dx. \tag{3.11}$$

□

Lemma 3.1
The functionals I_i satisfy, for any $\delta > 0$,

$$\begin{aligned} I_1'(t) \le& -\rho_1\left(g_1^0 - \delta\right)\int_0^L \varphi_t^2 dx + \delta \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2\right) dx \\ &+ c_\delta \int_0^L \int_0^{+\infty} g_1(s)(\partial_x \eta_1)^2 ds dx - c_\delta \int_0^L \int_0^{+\infty} g_1'(s)(\partial_x \eta_1)^2 ds dx, \end{aligned} \tag{3.12}$$

$$\begin{aligned} I_2'(t) \le& -\rho_2(g_2^0 - \delta)\int_0^L \psi_t^2 dx + \delta \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2\right) dx \\ &+ c_\delta \int_0^L \int_0^{+\infty} g_2(s)(\partial_x \eta_2)^2 ds dx - c_\delta \int_0^L \int_0^{+\infty} g_2'(s)(\partial_x \eta_2)^2 ds dx \end{aligned} \tag{3.13}$$

and

$$I_3'(t) \le -\rho_1 \left(g_3^0 - \delta\right) \int_0^L w_t^2 dx + \delta \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2\right) dx$$
$$+ c_\delta \int_0^L \int_0^{+\infty} g_3(s)(\partial_x \eta_3)^3 dsdx - c_\delta \int_0^L \int_0^{+\infty} g_3'(s)(\partial_x \eta_3)^2 dsdx, \tag{3.14}$$

where g_i^0 is defined by (2.12) and c_δ is a positive constant depending on δ.

Proof
Direct computations, using the first equation of (P), integrating by parts and using the fact that

$$\partial_t \int_0^{+\infty} g_1(s)\eta_1 ds = \partial_t \int_0^{+\infty} g_1(t-s)(\varphi(t) - \varphi(s))ds$$
$$= \int_0^{+\infty} g_1'(t-s)(\varphi(t) - \varphi(s))ds + \left(\int_0^{+\infty} g_1(t-s)ds\right)\varphi_t$$
$$= \int_0^{+\infty} g_1'(s)\eta_1 ds + g_1^0 \varphi_t,$$

yield

$$I_1'(t) = -\rho_1 g_1^0 \int_0^L \varphi_t^2 dx - \rho_1 \int_0^L \varphi_t \int_0^{+\infty} g_1'(s)\eta_1 dsdx$$
$$+ k_1 \int_0^L (\varphi_x + \psi + lw) \int_0^{+\infty} g_1(s)\partial_x \eta_1 dsdx - lk_3 \int_0^L (w_x - l\varphi) \int_0^{+\infty} g_1(s)\eta_1 dsdx$$
$$- \int_0^L \varphi_x \left(\int_0^{+\infty} g_1(s)\partial_x \eta_1 ds\right) dx + \int_0^L \left(\int_0^{+\infty} g_1(s)\partial_x \eta_1 ds\right)^2 dx.$$

Using Young's, Poincaré (for η_1) and Hölder's inequalities for the last five terms of this equality, and (2.11) to estimate $\int_0^L \varphi_x^2 dx$, we get (3.12).

Similarly, using the second and third equations of (P), we find (3.13) and (3.14). □

Lemma 3.2
There exist positive constants c_1 and c_2 such that the functional

$$I_4(t) = \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t + \rho_1 w w_t) dx \tag{3.15}$$

satisfies

$$I_4'(t) \le \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2\right) dx$$
$$- c_1 \int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2\right) dx \tag{3.16}$$
$$+ c_2 \int_0^L \int_0^{+\infty} \left(g_1(s)(\partial_x \eta_1)^2 + g_2(s)(\partial_x \eta_2)^2 + g_3(s)(\partial_x \eta_3)^2\right) dsdx.$$

Proof
By exploiting equations of (P) and integrating by parts, we get

$$I_4'(t) = \int_0^L \left(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + \rho_1 w_t^2\right) dx - k_1 \int_0^L (\varphi_x + \psi + lw)^2 dx$$
$$- k_3 \int_0^L (w_x - l\varphi)^2 dx + g_1^0 \int_0^L \varphi_x^2 dx - \left(k_2 - g_2^0\right) \int_0^L \psi_x^2 dx + g_3^0 \int_0^L w_x^2 dx$$
$$- \int_0^L \varphi_x \int_0^{+\infty} g_1(s)\partial_x \eta_1 dsdx - \int_0^L \psi_x \int_0^{+\infty} g_2(s)\partial_x \eta_2 dsdx \tag{3.17}$$
$$- \int_0^L w_x \int_0^{+\infty} g_3(s)\partial_x \eta_3 dsdx.$$

Using Young's and Hölder's inequalities for the last three terms of this equality, we get, for all $\epsilon > 0$, a positive constant c_ϵ such that

$$-\int_0^L \varphi_x \int_0^{+\infty} g_1(s)\partial_x\eta_1 dsdx - \int_0^L \psi_x \int_0^{+\infty} g_2(s)\partial_x\eta_2 dsdx - \int_0^L w_x \int_0^{+\infty} g_3(s)\partial_x\eta_3 dsdx$$

$$\leq \epsilon\int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) dx + c_\epsilon \int_0^L\int_0^{+\infty} \left(g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right) dsdx.$$

Inserting this inequality into (3.17) and using (2.10), we find

$$\begin{aligned} I_4'(t) \leq & \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2\right) dx - (k_0 - \epsilon)\int_0^L \left(\varphi_x^2 + \psi_x^2 + w_x^2\right) dx \\ & + c_\epsilon \int_0^L\int_0^{+\infty} \left(g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right) dsdx. \end{aligned} \tag{3.18}$$

Then, choosing $0 < \epsilon < k_0$ and inserting (2.13) in (3.18), we get (3.16) with $c_1 = \frac{k_0-\epsilon}{\tilde{k}_0}$ and $c_2 = c_\epsilon$.

Now, let $N_1, N_2 > 0$ and

$$I_5 = N_1 E + N_2(I_1 + I_2 + I_3) + I_4, \tag{3.19}$$

where E is the energy functional associated to (P) and defined by

$$E(t) = \frac{1}{2}\|U(t)\|_{\mathcal{H}}^2. \tag{3.20}$$

First, note that E is non-increasing according to (2.5), (2.14) and (2.15),

$$E'(t) = \frac{1}{2}\int_0^L\int_0^{+\infty} \left(g_1'(s)(\partial_x\eta_1)^2 + g_2'(s)(\partial_x\eta_2)^2 + g_3'(s)(\partial_x\eta_3)^2\right) dsdx \leq 0. \tag{3.21}$$

Now, using (3.12)–(3.14) with $\delta = \frac{1}{N_2^2}$, (3.16) and (3.21), we get

$$\begin{aligned} I_5'(t) \leq & -\left(c_1 - \frac{3}{N_2}\right)\int_0^L \left(\psi_x^2 + (\varphi_x + \psi + lw)^2 + (w_x - l\varphi)^2\right) dx \\ & - \rho_1\left(N_2 g_1^0 - \frac{1}{N_2} - 1\right)\int_0^L \varphi_t^2 dx - \rho_2\left(N_2 g_2^0 - \frac{1}{N_2} - 1\right)\int_0^L \psi_t^2 dx \\ & - \rho_1\left(N_2 g_3^0 - \frac{1}{N_2} - 1\right)\int_0^L w_t^2 dx \\ & + \left(\frac{N_1}{2} - c_{N_2}\right)\int_0^L\int_0^{+\infty} \left(g_1'(s)(\partial_x\eta_1)^2 + g_2'(s)(\partial_x\eta_2)^2 + g_3'(s)(\partial_x\eta_3)^2\right) dsdx \\ & + c_{N_2}\int_0^L\int_0^{+\infty} \left(g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right) dsdx, \end{aligned}$$

where $c_{N_2} = N_2 c_\delta + c_2$. We choose N_2 large enough so that

$$\min\left\{c_1 - \frac{3}{N_2}, N_2 \min\left\{g_i^0\right\} - \frac{1}{N_2} - 1\right\} > 0$$

(note that $g_i^0 > 0$ because g_i is continuous non-negative and $g_i(0) > 0$) and we find, for some positive constants c_3 and c_4,

$$\begin{aligned} I_5'(t) \leq & -c_3 E(t) + \left(\frac{N_1}{2} - c_4\right)\int_0^L\int_0^{+\infty} \left(g_1'(s)(\partial_x\eta_1)^2 + g_2'(s)(\partial_x\eta_2)^2 + g_3'(s)(\partial_x\eta_3)^2\right) dsdx \\ & + c_4\int_0^L\int_0^{+\infty} \left(g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right) dsdx. \end{aligned} \tag{3.22}$$

On the other hand, by (2.10) and definition of E and I_i, there exists a positive constant N_3 (not depending on N_1) such that

$$(N_1 - N_3)E \le I_5 \le (N_1 + N_3)E. \tag{3.23}$$

Thus, choosing $N_1 > \max\{2c_4, N_3\}$ and using the fact that $g_i' \le 0$,

$$I_5'(t) \le -c_3 E(t) + c_4 \int_0^L \int_0^{+\infty} \left(g_1(s)(\partial_x\eta_1)^2 + g_2(s)(\partial_x\eta_2)^2 + g_3(s)(\partial_x\eta_3)^2\right) dsdx. \tag{3.24}$$

Now, we estimate the terms $\int_0^L \int_0^{+\infty} g_i(s)(\partial_x\eta_i(s))^2 dsdx$. □

Lemma 3.3
For any $i = 1, 2, 3$, there exist positive constants d_i and $\tilde{d}_i$ such that, for any $\epsilon_0 > 0$, the following inequalities hold:

$$\int_0^L \int_0^{+\infty} g_i(s)(\partial_x\eta_i)^2 dsdx \le -d_i E'(t) \quad \text{if } \ (3.2) \text{ holds} \tag{3.25}$$

and

$$G'(\epsilon_0 E(t)) \int_0^L \int_0^{+\infty} g_i(s)(\partial_x\eta_i)^2 dsdx \tag{3.26}$$
$$\le -\tilde{d}_i E'(t) + \tilde{d}_i \epsilon_0 E(t) G'(\epsilon_0 E(t)) \quad \text{if } \ (3.3) \text{ holds and (3.2) does not hold.}$$

Proof
When (3.2) holds, using (3.21), we get (3.25) with $d_i = \frac{2}{\delta_i}$. □

When (3.3) holds and (3.2) does not hold, we follow an approach of Guesmia [14]. Let us consider the case where (3.2) does not hold for $i = 1$. Therefore, (3.3) holds for $i = 1$. Then, using (2.10), (3.4) and (3.21), we have

$$\begin{aligned}
\int_0^L (\partial_x\eta_1)^2 dx &\le 2\int_0^L \left(\varphi_x^2(x,t) + \varphi_x^2(x,t-s)\right) dx \\
&\le 4 \sup_{t\ge 0} \int_0^L \varphi_x^2(x,t)dx + 2\sup_{s>0} \int_0^L (\partial_x\varphi_0)^2(x,s)dx \\
&\le \frac{8}{k_0}E(0) + 2\sup_{s>0}\int_0^L \left(2(\partial_x\eta_1^0)^2(x,s) + 2(\partial_x\varphi_0)^2(x,0)\right) dx \\
&\le \frac{16}{k_0}E(0) + 4M_1.
\end{aligned}$$

Similar estimates hold for η_2 and η_3; that is, for $b_i = \frac{16}{k_0}E(0) + 4M_i$,

$$\int_0^L (\partial_x\eta_i)^2 dx \le b_i, \quad \forall t, s \in \mathbb{R}^+. \tag{3.27}$$

Recall that, if $E(t_0) = 0$ for some $t_0 \ge 0$, then $E(t_0) = 0$ for any $t \ge t_0$ as E is non-increasing and non-negative. Therefore, by continuity of E, (3.6) holds. Hence, without loss of generality, we assume that $E(t) > 0$ for any $t \in \mathbb{R}_+$. Similarly, if $g_i'(s_0) = 0$ for some $s_0 \ge 0$, then $g_i(s_0) = 0$ because of (3.3). So, $g_i(s) = 0$ for any $s \ge s_0$ as g_i is non-increasing and non-negative. Therefore,

$$\int_0^{+\infty} g_i(s)(\partial_x\eta_i)^2 ds = \int_0^{s_0} g_i(s)(\partial_x\eta_i)^2 ds.$$

Hence, without loss of generality, we assume that $g_i'(s) < 0$ for any $s \in \mathbb{R}_+$.

Now, let ϵ_0, τ_i, $s_i > 0$ and $K(s) = \frac{s}{G^{-1}(s)}$, for $s \in \mathbb{R}_+$. Thanks to the properties of G, $K(0) = G'(0) = 0$ and K is non-decreasing. Therefore, using (3.27),

$$K\left(-s_i g_i'(s)\int_0^L (\partial_x \eta_i)^2 dx\right) \le K(-b_i s_i g_i'(s)).$$

Then

$$\begin{aligned}
\int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 ds dx &= \frac{1}{\tau_i G'(\epsilon_0 E(t))}\int_0^{+\infty} G^{-1}\left(-s_i g_i'(s)\int_0^L (\partial_x \eta_i)^2 dx\right) \\
&\quad \times \frac{\tau_i G'(\epsilon_0 E(t)) g_i(s)}{-s_i g_i'(s)} K\left(-s_i g_i'(s)\int_0^L (\partial_x \eta_i)^2 dx\right) ds \\
&\le \frac{1}{\tau_i G'(\epsilon_0 E(t))}\int_0^{+\infty} G^{-1}\left(-s_i g_i'(s)\int_0^L (\partial_x \eta_i)^2 dx\right) \frac{\tau_i G'(\epsilon_0 E(t)) g_i(s)}{-s_i g_i'(s)} K(-b_i s_i g_i'(s)) ds \\
&\le \frac{1}{\tau_i G'(\epsilon_0 E(t))}\int_0^{+\infty} G^{-1}\left(-s_i g_i'(s)\int_0^L (\partial_x \eta_i)^2 dx\right) \frac{b_i \tau_i G'(\epsilon_0 E(t)) g_i(s)}{G^{-1}(-b_i s_i g_i'(s))} ds.
\end{aligned}$$

We denote by G^* the dual function of G defined by

$$G^*(t) = \sup_{s \in \mathbb{R}_+} \{ts - G(s)\}, \quad \forall t \in \mathbb{R}_+.$$

Thanks to (H2), G' is increasing and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, and then, for any $t \in \mathbb{R}_+$, the function $s \mapsto ts - G(s)$ reaches its maximum on $\mathbb{R}_+$ at the unique point $(G')^{-1}(t)$. Therefore,

$$G^*(t) = t(G')^{-1}(t) - G((G')^{-1}(t)), \quad \forall t \in \mathbb{R}_+.$$

Using Young's inequality

$$t_1 t_2 \le G(t_1) + G^*(t_2),$$

for

$$t_1 = G^{-1}\left(-s_i g_i'(s)\int_0^L (\partial_x \eta_i)^2 dx\right) \quad \text{and} \quad t_2 = \frac{b_i \tau_i G'(\epsilon_0 E(t)) g_i(s)}{G^{-1}(-b_i s_i g_i'(s))},$$

we get

$$\begin{aligned}
\int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 ds dx &\le \frac{-s_i}{\tau_i G'(\epsilon_0 E(t))}\int_0^L \int_0^{+\infty} g_i'(s)(\partial_x \eta_i)^2 ds dx \\
&\quad + \frac{1}{\tau_i G'(\epsilon_0 E(t))}\int_0^{+\infty} G^*\left(\frac{b_i \tau_i G'(\epsilon_0 E(t)) g_i(s)}{G^{-1}(-b_i s_i g_i'(s))}\right) ds.
\end{aligned}$$

Using (3.21) and the fact that

$$G^*(t) \le t(G')^{-1}(t),$$

we obtain

$$\int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 ds dx \le \frac{-2s_i}{\tau_i G'(\epsilon_0 E(t))} E'(t) + b_i \int_0^{+\infty} \frac{g_i(s)}{G^{-1}(-b_i s_i g_i'(s))} (G')^{-1}\left(\frac{b_i \tau_i G'(\epsilon_0 E(t)) g_i(s)}{G^{-1}(-b_i s_i g_i'(s))}\right) ds.$$

Thanks to (3.3),

$$\sup_{s \in \mathbb{R}_+} \frac{g_i(s)}{G^{-1}(-g_i'(s))} = a_i < +\infty.$$

Then, using the fact that $(G')^{-1}$ is non-decreasing (thanks to (H2)) and choosing $s_i = \frac{1}{b_i}$, we get

$$\int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 dsdx \le \frac{-2}{b_i \tau_i G'(\epsilon_0 E(t))} E'(t) + b_i (G')^{-1} \left(b_i a_i \tau_i G'(\epsilon_0 E(t))\right) \int_0^{+\infty} \frac{g_i(s)}{G^{-1}(-g_i'(s))} ds.$$

Choosing $\tau_i = \frac{1}{b_i a_i}$ and using the fact that

$$\int_0^{+\infty} \frac{g_i(s)}{G^{-1}(-g_i'(s))} ds = l_i < +\infty$$

thanks to (3.3), we obtain

$$\int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 dsdx \le \frac{-2a_i}{G'(\epsilon_0 E(t))} E'(t) + b_i l_i \epsilon_0 E(t),$$

which implies (3.26) with $\tilde{d}_i = \max\{2a_i, b_i l_i\}$.

Now, if (3.2) holds, for all $i \in \{1,2,3\}$, then (3.24) and (3.25) imply that

$$I_5'(t) \le -c_3 E(t) - c_4(d_1 + d_2 + d_3)E'(t). \tag{3.28}$$

Let

$$F = I_5 + c_4(d_1 + d_2 + d_3)E.$$

Thanks to (3.23) and (3.28), we have $F' \le -c'F$, where

$$c' = \frac{c_3}{N_1 + N_3 + c_4(d_1 + d_2 + d_3)}.$$

Integrating over $[CR0, t]$, we arrive at

$$F(t) \le F(0)e^{-c't},$$

which, thanks to (3.20) and (3.23), gives (3.5) with

$$c'' = \frac{2F(0)}{N_1 - N_3 + c_4(d_1 + d_2 + d_3)}.$$

If (3.2) does not hold at least for one $i \in \{1,2,3\}$, then, according to (3.25) and (3.26), we see that

$$G'(\epsilon_0 E(t)) \int_0^L \int_0^{+\infty} g_i(s)(\partial_x \eta_i)^2 dsdx \le -\alpha_i G'(\epsilon_0 E(t))E'(t) - \beta_i E'(t) + \epsilon_0 \beta_i G'(\epsilon_0 E(t))E(t), \tag{3.29}$$

where

$$\alpha_i = \begin{cases} d_i & \text{if (3.2) holds,} \\ 0 & \text{otherwise} \end{cases}$$

and

$$\beta_i = \begin{cases} 0 & \text{if (3.2) holds,} \\ \widetilde{d_i} & \text{otherwise.} \end{cases}$$

Thus, multiplying (3.24) by $G'(\epsilon_0 E(t))$ and using (3.29), we get

$$G'(\epsilon_0 E(t))I_5'(t) \le -(c_3 - c_4\epsilon_0(\beta_1 + \beta_2 + \beta_3))E(t)G'(\epsilon_0 E(t)) - c_4(\beta_1 + \beta_2 + \beta_3)E'(t) - c_4(\alpha_1 + \alpha_2 + \alpha_3)G'(\epsilon_0 E(t))E'(t).$$

Choosing

$$0 < \epsilon_0 < \frac{c_3}{c_4(\beta_1 + \beta_2 + \beta_3)}$$

(note that ϵ_0 is well defined as $\beta_1 + \beta_2 + \beta_3 > 0$ because (3.2) does not hold at least for one of the kernels), we get

$$G'(\epsilon_0 E(t))I_5'(t) + c_4 \left(\beta_1 + \beta_2 + \beta_3 + (\alpha_1 + \alpha_2 + \alpha_3)G'(\epsilon_0 E(t))\right) E'(t) \leq -c_5 E(t)G'(\epsilon_0 E(t)), \tag{3.30}$$

where

$$c_5 = c_3 - c_4\epsilon_0(\beta_1 + \beta_2 + \beta_3).$$

Let

$$F = \tau \left(G'(\epsilon_0 E)I_5 + c_4 \left(\beta_1 + \beta_2 + \beta_3 + (\alpha_1 + \alpha_2 + \alpha_3)G'(\epsilon_0 E)\right)E\right), \tag{3.31}$$

where $\tau > 0$. The fact that $G'(\epsilon_0 E)$ is non-increasing (due to **(**H2**)** and (3.21)) and $I_5 \geq 0$ (thanks to (3.23)) imply that

$$\left(G'(\epsilon_0 E)\right)' I_5 \leq 0 \quad \text{and} \quad \left(G'(\epsilon_0 E)\right)' E \leq 0.$$

Therefore, using (3.30), we get

$$F' \leq -c_5 \tau E G'(\epsilon_0 E). \tag{3.32}$$

Thanks to (3.23) and the fact that

$$G'(\epsilon_0 E(t)) \leq G'(\epsilon_0 E(0)),$$

we can choose $\tau > 0$ small enough such that

$$F \leq E \quad \text{and} \quad F(0) \leq 1, \tag{3.33}$$

and we find, for $c' = c_5\tau$ (note that $s \mapsto sG'(\epsilon_0 s)$ is non-decreasing),

$$F' \leq -c' F G'(\epsilon_0 F), \tag{3.34}$$

which implies that $(H(F))' \geq c'$, where H is defined in (3.7). Then, by integrating over $[0, t]$, we obtain

$$H(F(t)) \geq c't + H(F(0)).$$

Because $F(0) \leq 1$, $H(1) = 0$ and H is decreasing, we arrive at

$$H(F(t)) \geq c't.$$

Because H^{-1} is decreasing, we deduce that $F(t) \leq H^{-1}(c't)$. Then (3.20), (3.31) and dropping the positive terms $G'(\epsilon_0 E)I_5$ and $(\alpha_1 + \alpha_2 + \alpha_3)G'(\epsilon_0 E)E$ give (3.6) with

$$c'' = \frac{2}{\tau c_4(\beta_1 + \beta_2 + \beta_3)}.$$

This completes the proof of Theorem 3.1.

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