MATHEMATICAL METHODS IN THE APPLIED SCIENCES
Math. Meth. Appl. Sci. (2009)
Published online in Wiley InterScience
(www.interscience.wiley.com) DOI: 10.1002/mma.1125
MOS subject classification: 35 B 35; 35 L 20; 35 L 70

General energy decay estimates of Timoshenko systems with frictional versus viscoelastic damping

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Communicated by R. Racke

SUMMARY

In this paper we consider the following Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt}-k_1(\varphi_x+\psi)_x=0 & \text{in } (0,L)\times\mathbb{R}_+ \\ \rho_2\psi_{tt}-k_2\psi_{xx}+\displaystyle\int_0^t g(t-\tau)(a(x)\psi_x(\tau))_x\,\mathrm{d}\tau+k_1(\varphi_x+\psi)+b(x)h(\psi_t)=0 & \text{in } (0,L)\times\mathbb{R}_+ \end{cases}$$

with Dirichlet boundary conditions and initial data where a, b, g and h are specific functions and ρ_1, ρ_2, k_1, k_2 and L are given positive constants. We establish a general stability estimate using the multiplier method and some properties of convex functions. Without imposing any growth condition on h at the origin, we show that the energy of the system is bounded above by a quantity, depending on g and h, which tends to zero as time goes to infinity. Our estimate allows us to consider a large class of functions h with general growth at the origin. We use some examples (known in the case of wave equations and Maxwell system) to show how to derive from our general estimate the polynomial, exponential or logarithmic decay. The results of this paper improve and generalize some existing results in the literature and generate some interesting open problems. Copyright © 2009 John Wiley & Sons, Ltd.

KEY WORDS: general decay estimate; nonlinear damping; relaxation function; Timoshenko; viscoelastic; convexity; asymptotic behavior

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Contract/grant sponsor: KFUPM; contract/grant number: IN080408

 Received 12 May 2008

1. INTRODUCTION

A simple model describing the transverse vibration of a beam, which was developed in [1], is given by the following system of coupled hyperbolic equations:

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x & \text{in } (0, L) \times \mathbb{R}_+ \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) & \text{in } (0, L) \times \mathbb{R}_+ \end{cases} \tag{1}$$

where t denotes the time variable and x is the space variable along the beam of length L in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section and the shear modulus.

Kim and Renardy [2] considered (1) together with two boundary controls of the form

$$K\varphi(L, t) - Ku_x(L, t) = \alpha u_t(L, t) \quad \text{on } \mathbb{R}_+$$

$$EI\varphi_x(L, t) = -\beta\varphi_t(L, t) \quad \text{on } \mathbb{R}_+$$

and used the multiplier techniques to establish an exponential decay result for the natural energy of (1). They also provided numerical estimates to the eigenvalues of the operator associated with system (1). Raposo *et al.* [3] studied (1) with homogeneous Dirichlet boundary conditions and two linear frictional dampings. Precisely, they looked into the following system:

$$\begin{cases} \rho_1 u_{tt} - K(u_x - \varphi)_x + u_t = 0 & \text{in } (0, L) \times \mathbb{R}_+ \\ \rho_2 \varphi_{tt} - b\varphi_{xx} + K(u_x - \varphi) + \varphi_t = 0 & \text{in } (0, L) \times \mathbb{R}_+ \\ u(0, L) = u(L, t) = \varphi(0, t) = \varphi(L, t) = 0 & \text{on } \mathbb{R}_+ \end{cases} \tag{2}$$

and proved that the energy associated with (2) decays exponentially. Soufyane and Wehbe [4] showed that it is possible to stabilize (1) uniformly by using a unique locally distributed feedback. Therefore, they considered

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x & \text{in } (0, L) \times \mathbb{R}_+ \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi) - b\varphi_t & \text{in } (0, L) \times \mathbb{R}_+ \\ u(0, t) = u(L, t) = \varphi(0, t) = \varphi(L, t) = 0 & \text{on } \mathbb{R}_+ \end{cases} \tag{3}$$

where b is a positive and continuous function that satisfies

$$b(x) \geqslant b_0 > 0 \quad \forall x \in [a_0, a_1] \subset [0, L]$$

In fact, they proved that the uniform stability of (3) holds if and only if the wave speeds are equal ($K/\rho = EI/I_\rho$); otherwise, only the asymptotic stability has been proved. This result has been recently improved by Muñoz Rivera and Racke [5], where an exponential decay of the solution

DOI: 10.1002/mma

energy of (3) has been established, allowing b to be with an indefinite sign. In addition, Muñoz Rivera and Racke [6] treated a system of the form

$$\begin{cases} \rho_1\varphi_{tt}-\sigma(\varphi_x,\psi)_x=0 & \text{in } (0,L)\times\mathbb{R}_+ \\ \rho_2\psi_{tt}-b\psi_{xx}+K(\varphi_x+\psi)+\gamma\theta_x=0 & \text{in } (0,L)\times\mathbb{R}_+ \\ \rho_3\theta_t-K\theta_{xx}+\gamma\psi_{xt}=0 & \text{in } (0,L)\times\mathbb{R}_+ \end{cases}$$

where φ, ψ and θ are functions of (x,t) model transverse displacement of the beam, the rotation angle of the filament and the difference temperature, respectively. Under appropriate conditions of σ, ρ_i, b, K and γ, they proved several exponential decay results for the linearized system and nonexponential stability result for the case of different wave speeds of propagation. In addition, Muñoz Rivera and Racke [7] considered the following nonlinear Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt}-\sigma(\varphi_x,\psi)_x=0 & \text{in } (0,L)\times\mathbb{R}_+ \\ \rho_2\psi_{tt}-b\psi_{xx}+K(\varphi_x+\psi)+d\psi_t=0 & \text{in } (0,L)\times\mathbb{R}_+ \end{cases}$$

with homogeneous boundary conditions and proved that the system is exponentially stable if and only if $K/\rho_1=b/\rho_2$ and a polynomial stability otherwise. Alabau-Boussouira [8] extended the results of Muñoz Rivera and Racke [7] to the case of nonlinear feedback $\alpha(\psi_t)$, instead of $d\psi_t$, where α is a globally Lipchitz function satisfying some growth conditions at the origin.

Ammar-Khodja *et al.* [9] considered a linear Timoshenko-type system with memory of the form

$$\begin{cases} \rho_1\varphi_{tt}-K(\varphi_x+\psi)_x=0 & \text{in } (0,L)\times\mathbb{R}_+ \\ \rho_2\psi_{tt}-b\psi_{xx}+\displaystyle\int_0^t g(t-s)\psi_{xx}(s)\,\mathrm{d}s+K(\varphi_x+\psi)=0 & \text{in } (0,L)\times\mathbb{R}_+ \end{cases} \tag{4}$$

together with homogeneous boundary conditions. They used the multiplier techniques and proved that the system is uniformly stable if and only if the wave speeds are equal $(K/\rho_1=b/\rho_2)$ and g decays uniformly. Precisely, they proved an exponential decay if g decays at an exponential rate and polynomially if g decays at a polynomial rate. They also required some extra technical conditions on both g' and g'' to obtain their result. These extra technical conditions were eliminated by Messaoudi and Mustafa [10], where weaker conditions on g were imposed and a more general decay estimate was obtained.

The feedback of memory type has also been used by Santos [11]. He considered a Timoshenko system and showed that the presence of two feedbacks of memory type at a portion of the boundary stabilizes the system uniformly. He also obtained the rate of decay of the energy, which is exactly the rate of decay of the relaxation functions. Shi and Feng [12] investigated a nonuniform Timoshenko beam and showed that, under some locally distributed controls, the vibration of the beam decays exponentially. A similar result has also been obtained by Ammar-Khodja *et al.* [13] for a nonuniform Timoshenko system.

DOI: 10.1002/mma

In the present paper we are concerned with

$$\begin{cases} \rho_1\varphi_{tt}-k_1(\varphi_x+\psi)_x=0 & \text{in } (0,L)\times\mathbb{R}_+ \\ \rho_2\psi_{tt}-k_2\psi_{xx}+k_1(\varphi_x+\psi)+\displaystyle\int_0^t g(t-\tau)(a(x)\psi_x(\tau))_x\,\mathrm{d}\tau+b(x)h(\psi_t)=0 & \text{in } (0,L)\times\mathbb{R}_+ \\ \varphi(0,t)=\varphi(L,t)=\psi(0,t)=\psi(L,t)=0 & \text{on } \mathbb{R}_+ \\ \varphi(x,0)=\varphi_0(x),\ \varphi_t(x,0)=\varphi_1(x) & \text{on } (0,L) \\ \psi(x,0)=\psi_0(x),\ \psi_t(x,0)=\psi_1(x) & \text{on } (0,L) \end{cases} \tag{5}$$

in the case of equal speeds of propagation ($k_1/\rho_1=k_2/\rho_2$). Therefore, without loss of generality, we take $\rho_1=\rho_2=k_1=k_2=1$ and $L=1$. Our aim in this paper is to investigate the effect of both frictional and viscoelastic dampings, where each one of them can vanish on the whole domain or in a part of it. In addition, we would like to see the influence of these dissipations on the rate of decay of solutions. Of course, the most interesting case occurs when we have simultaneous and complementary damping mechanisms. Precisely, we obtain an explicit and general decay rate, depending on g and h, for the energy of solutions without any growth assumption on h at the origin and under weaker conditions on the relaxation function g. More precisely, we intend to obtain a general relation between the decay rate for the energy (when t goes to infinity) and the functions g and h. In particular, we can consider the cases where h degenerates near the origin polynomially, between polynomially and exponentially, exponentially or faster than exponentially. This kind of growth was considered (in less general form) by Komornik [14], Martinez [15], Lasiecka and co-workers [16–19], Liu and Zuazua [20] and Alabau-Boussouira [21] for the wave equation and Eller *et al.* [22] for the Maxwell system.

Our proof combines arguments from [9, 10, 23–27] and some properties of convex functions, in particular, the dual function of convex function to use the general Young's inequality and Jensen's inequality (instead of Hölder inequality widely used in the classical case of linear or polynomial growth of h at the origin) in objective to prove our general decay estimate (estimate (10) below) under a general growth of h at the origin (hypothesis (H2) below). These arguments of convexity were introduced and developed by Lasiecka and co-workers [16–19, 28, 29] and used by Liu and Zuazua [20], Alabau-Boussouira [21] and Eller *et al.* [22].

Our results generalize the ones cited above and improve some of them where only exponential or polynomial estimates were obtained. In particular, the one of Ammar-Khodja *et al.* [9] where the hypotheses imposed on g are stronger than (H3) and (H4). Additionally, our proof is simpler.

This paper is organized as follows: in Section 2, we list our hypotheses and state the main results of this paper. In Section 3, we prove our main theorem. Finally, we conclude and give some comments and open questions in Section 4.

2. PRELIMINARIES

In order to state our main result we make the following hypotheses:

(H1) $a,\, b:[0,1]\to\mathbb{R}_+$ are such that

$$a\in C^1([0,1]),\quad b\in L^\infty([0,1])$$
$$a=0 \quad\text{or}\quad a(0)+a(1)>0,\qquad \inf_{x\in[0,1]}\{a(x)+b(x)\}>0$$

DOI: 10.1002/mma

(H2) $h:\mathbb{R}\to\mathbb{R}$ is a differentiable nondecreasing function such that there exist constants ε', c_1, $c_2>0$ and a convex and increasing function $H:\mathbb{R}_+\to\mathbb{R}_+$ of class $C^1(\mathbb{R}_+)\cap C^2((0,\infty))$ satisfying $H(0)=0$ and H is linear on $[0,\varepsilon']$ or ($H'(0)=0$ and $H''>0$ on $(0,\varepsilon']$) such that

$$c_1|s|\leqslant|h(s)|\leqslant c_2|s| \quad \text{if } |s|\geqslant\varepsilon'$$

$$s^2+h^2(s)\leqslant H^{-1}(sh(s)) \quad \text{if } |s|\leqslant\varepsilon'$$

(H3) $g:\mathbb{R}_+\to\mathbb{R}_+$ is a differentiable function such that

$$g(0)>0, \quad 1-\|a\|_\infty\int_0^{+\infty} g(s)\,\mathrm{d}s=l>0$$

(H4) There exists a nonincreasing differentiable function $\xi:\mathbb{R}_+\to\mathbb{R}_+$ satisfying

$$g'(s)\leqslant-\xi(s)g(s) \quad \forall s\geqslant 0$$

Remarks

1. We note that in the case where $a\neq 0$, by hypothesis (H1), which was considered by Cavalcanti and Oquendo [27] for the wave equation, we have either $a(0)>0$ or $a(1)>0$. Therefore, without loss of generality, we take in this case $a(0)>0$ in the whole paper.
2. If h satisfies

$$h_0(|s|)\leqslant|h(s)|\leqslant h_0^{-1}(|s|) \quad \text{if } |s|\leqslant\varepsilon'$$

for a function $h_0:\mathbb{R}_+\to\mathbb{R}_+$ satisfying $h_0'(0)=0$ and $h_0'>0$ on $(0,\sqrt{\varepsilon'/2}]$ or h_0 is linear on $[0,\sqrt{\varepsilon'/2}]$, and such that the function $s\mapsto\sqrt{s}h_0(\sqrt{s})$, $s\geqslant 0$, is convex, increasing and of class $C^1(\mathbb{R}_+)\cap C^2((0,\infty))$, then condition (H2) is satisfied for $H(s)=\sqrt{s/2}h_0(\sqrt{s/2})$. On the other hand, h satisfies (H2) for any $\varepsilon''\in(0,\varepsilon']$ instead of ε', with some c_1' and $c_2'>0$ instead of c_1 and c_2, respectively. This kind of hypotheses, with $\varepsilon'=1$, was considered by Komornik [14], Martinez [15], Liu and Zuazua [20] and Alabau-Boussouira [21].
3. The condition (H2), with $\varepsilon'=1$, was introduced and employed by Lasiecka and co-workers [16–19, 28, 29] in their study of the asymptotic behavior of solutions of nonlinear wave equations with nonlinear boundary damping where they obtained decay estimates that depend on the solution of an explicit nonlinear ordinary differential equation. On the other hand, using the method developed by Lasiecka and co-workers [16–19], we can also consider general growth conditions on h at infinity and prove similar general decay estimates for uniformly bounded solutions.
4. The condition (H4) was considered by Messaoudi and Mustafa [10] to study the stability of (10) in the (particular) case $b=0$ and $a=1$.

For completeness we state, without proof, an existence and regularity result.

Proposition 2.1
Let (φ_0,φ_1), $(\psi_0,\psi_1)\in H_0^1(0,1)\times L^2(0,1)$ be given. Assume that (H1)–(H3) are satisfied, then problem (5) has a unique global (weak) solution

$$\varphi,\psi\in C(\mathbb{R}_+;H_0^1(0,1))\cap C^1(\mathbb{R}_+;L^2(0,1)) \tag{6}$$

DOI: 10.1002/mma

Moreover,

1. If

$$(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$$

then the (strong) solution satisfies

$$\varphi, \psi \in L^\infty(\mathbb{R}_+; H^2(0,1) \cap H_0^1(0,1)) \cap W^{1,\infty}(\mathbb{R}_+; H_0^1(0,1)) \cap W^{2,\infty}(\mathbb{R}_+; L^2(0,1)) \tag{7}$$

2. If h is linear and

$$(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in (H^2(0,1) \cap H_0^1(0,1)) \times H_0^1(0,1)$$

then the (classical) solution satisfies

$$\varphi, \psi \in C(\mathbb{R}_+; H^2(0,1) \cap H_0^1(0,1)) \cap C^1(\mathbb{R}_+; H_0^1(0,1)) \cap C^2(\mathbb{R}_+; L^2(0,1))$$

Remark
This result can be proved using standard arguments such as the Galerkin method.

Now, we introduce the energy functional

$$E(t) := \frac{1}{2} \int_0^1 \left(\varphi_t^2 + \psi_t^2 + \left(1 - a(x) \int_0^t g(s)\, \mathrm{d}s\right) \psi_x^2 + (\varphi_x + \psi)^2 \right) \mathrm{d}x + \frac{1}{2}(g \circ \psi_x) \tag{8}$$

where, for all $v \in L^2(0,1)$,

$$(g \circ v)(t) = \int_0^1 a(x) \int_0^t g(t-s)(v(t) - v(s))^2\, \mathrm{d}s\, \mathrm{d}x \tag{9}$$

We are now ready to state our main stability result.

Theorem 2.2
Let $(\varphi_0, \varphi_1), (\psi_0, \psi_1) \in H_0^1(0,1) \times L^2(0,1)$ be given. Assume that (H1)–(H4) are satisfied, then there exist positive constants c', c'' and ε_0 for which the (weak) solution of problem (5) satisfies

$$E(t) \leqslant c'' H_1^{-1}\left(c' \int_0^t \xi(s)\, \mathrm{d}s\right) \quad \forall t \geqslant 0 \tag{10}$$

where $H_1(t) = \int_t^1 (1/H_2(s))\, \mathrm{d}s$,

$$H_2(t) = \begin{cases} t & \text{if } H \text{ is linear on } [0, \varepsilon'] \\ tH'(\varepsilon_0 t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } (0, \varepsilon'] \end{cases}$$

and $\xi = 1$ if $a = 0$.

Remarks

1. Because H_2 is convex (thanks to the fact that H is convex), then $\lim_{t \to 0} H_1(t) = +\infty$ and, then, if $\int_0^{+\infty} \xi(t)\, \mathrm{d}t = +\infty$, we have the strong stability of (5); that is,

$$\lim_{t \to +\infty} E(t) = 0$$

DOI: 10.1002/mma

2. If $-g'/g$ is differentiable and nonincreasing, we can take $\xi=-g'/g$, and then our estimate (10) becomes

$$E(t)\leqslant c'' H_1^{-1}\left(c'\ln\frac{g(0)}{g(t)}\right)\quad \forall t\geqslant 0$$

Examples

We now present several illustrating examples of growth on h at the origin and the corresponding decay estimates. This kind of growth conditions on h was considered, in less general form, by Komornik [14], Matinez [15], Lasiecka and co-workers [16–19], Liu and Zuazua [20] and Alabau-Boussouira [21] for the wave equation and Eller *et al.* [22] for the Maxwell system.

1. *Polynomial growth of* h: If

$$c_1'|s|^q\leqslant |h(s)|\leqslant c_2'|s|^{1/q}\quad \text{on } [-\varepsilon',\varepsilon']$$

for some $c_1', c_2'>0$ and $q\geqslant 1$ (then (H2) is satisfied for $H(s)=cs^{(q+1)/2}$ with $c>0$), then there exist $c', c''>0$ such that for all $t\geqslant 0$

$$E(t)\leqslant c'' \mathrm{e}^{-c'\int_0^t \xi(s)\,\mathrm{d}s}\quad \text{if } q=1$$

$$E(t)\leqslant \left(c'\int_0^t \xi(s)\,\mathrm{d}s+c''\right)^{-(2/(q-1))}\quad \text{if } q>1$$

2. *Exponential growth of* h: If

$$h_0(|s|)\leqslant |h(s)|\leqslant h_0^{-1}(|s|)\quad \text{on } [-\varepsilon',\varepsilon']$$

where $h_0(s)=(1/s)\mathrm{e}^{-s^{-\gamma}}$ and $\gamma>0$ (then (H2) is satisfied for $H(s)=\mathrm{e}^{-c_1's^{-\gamma/2}}$ when s is near 0, and hence $H_2(s)=cs^{-\gamma/2}\mathrm{e}^{-c_1's^{-\gamma/2}}$ when s is near 0 and $H_1(s)\leqslant c_2''\mathrm{e}^{c_1''s^{-\gamma/2}}$ on $(0,1]$ for some c, c_1', c_1'', $c_2''>0$), then there exist $c', c'', c'''>0$ such that

$$E(t)\leqslant c'''\left(\ln\left(c'\int_0^t \xi(s)\,\mathrm{d}s+c''\right)\right)^{-2/\gamma}\quad \forall t\geqslant 0$$

3. *Faster than exponential growth of* h: If

$$h_0(|s|)\leqslant |h(s)|\leqslant h_0^{-1}(|s|)\quad \text{on } [-\varepsilon',\varepsilon']$$

where $h_0(s)=(1/s)h_n(s)$, $\gamma>0$ and

$$h_1(s)=\mathrm{e}^{-s^{-\gamma}}\quad \text{and}\quad h_n(s)=\mathrm{e}^{-1/h_{n-1}(s)},\quad n=2,3,\ldots$$

then (as in example 2) there exist $c', c'', c'''>0$ such that

$$E(t)\leqslant c'''\left(\bar{h}_n\left(\int_0^t \xi(s)\,\mathrm{d}s\right)\right)^{-2/\gamma}\quad \forall t\geqslant 0$$

where

$$\bar{h}_1(t)=\ln(c't+c'')\quad \text{and}\quad \bar{h}_n(t)=\ln(\bar{h}_{n-1}(t)),\quad n=2,3,\ldots$$

DOI: 10.1002/mma

4. *Between polynomial and exponential growth of* h: If

$$h_0(|s|)\leqslant|h(s)|\leqslant h_0^{-1}(|s|) \quad \text{on } [-\varepsilon',\varepsilon']$$

where $h_0(s)=(1/s)\mathrm{e}^{-(h_n(s))^\gamma}$, $\gamma>1$ and

$$h_1(s)=-\ln s \quad \text{and} \quad h_n(s)=\ln(h_{n-1}(s)), \quad n=2,3,\ldots$$

(then (H2) is satisfied for $H(s)=\mathrm{e}^{-(h_n(\sqrt{s/2}))^\gamma}$ when s is near 0, and hence

$$H_2(s)=-cs^{1/2}(h_n(c_1's^{1/2}))^{\gamma-1}h_n'(c_1's^{1/2})\mathrm{e}^{-(h_n(c_1's^{1/2}))^\gamma}$$

when s is near 0 and

$$H_1(s)\leqslant c_1''\mathrm{e}^{(h_n(c_1's^{1/2}))^\gamma}$$

on $(0,1]$ for some c, c_1', $c_1''>0$), then there exist c', c'', $c'''>0$ such that

$$E(t)\leqslant c'''\mathrm{e}^{-2\bar{h}_n(\int_0^t\xi(s)\,\mathrm{d}s)} \quad \forall t\geqslant 0$$

where

$$\bar{h}_1(t)=(\ln(c't+c''))^{1/\gamma} \quad \text{and} \quad \bar{h}_n(t)=\mathrm{e}^{\bar{h}_{n-1}(t)}, \quad n=2,3,\ldots$$

3. PROOF OF THE MAIN RESULT

In this section we prove our main stability result. The key point to show the general decay estimate (10) is to construct a Lyapunov functional F, equivalent to E, which satisfies, for positive constants c' and t_0,

$$F'(t)\leqslant -c'\xi(t)H_2(F(t)) \quad \forall t\geqslant t_0$$

where H_2 is defined in Theorem 2.2. For this purpose, we define several functionals that allow us to obtain the desired estimates and establish several lemmas.

Lemma 3.1
Let (φ,ψ) be the (weak) solution of (5). Then the energy functional satisfies

$$\begin{aligned} E'(t) &= -\frac{1}{2}g(t)\int_0^1 a(x)\psi_x^2\,\mathrm{d}x-\int_0^1 b(x)\psi_t h(\psi_t)\,\mathrm{d}x+\frac{1}{2}(g'\circ\psi_x)\\ &\leqslant -\int_0^1 b(x)\psi_t h(\psi_t)\,\mathrm{d}x+\frac{1}{2}(g'\circ\psi_x)\leqslant 0 \end{aligned} \tag{11}$$

Proof
By multiplying equations in (5) by φ_t and ψ_t, respectively, and integrating over $(0,1)$, using integration by parts, hypotheses (H1)–(H4) and some manipulations as in [9], we obtain (11) for any strong solution. This equality remains valid for weak solutions by a simple density argument. □

DOI: 10.1002/mma

Next, we use a function α introduced by Cavalcanti and Oquendo [27], which helps in establishing some desired estimates in the case where $a\neq 0$. By using the fact that $a(0)>0$ and a is continuous, then there exists $\varepsilon>0$ such that $\inf_{x\in[0,\varepsilon]} a(x)\geqslant\varepsilon$. Set

$$d=\min\left\{\varepsilon, \inf_{x\in[0,1]}\{a(x)+b(x)\}\right\}>0$$

and let $\alpha\in C^1([0,1])$ be such that $0\leqslant\alpha\leqslant a$ and

$$\alpha(x)=0 \quad \text{if } a(x)\leqslant\frac{d}{4}$$

$$\alpha(x)=a(x) \quad \text{if } a(x)\geqslant\frac{d}{2}$$

To simplify the computations we set

$$g\odot v=\int_0^1 \alpha(x)\int_0^t g(t-s)(v(t)-v(s))\,\mathrm{d}s\,\mathrm{d}x$$

for all $v\in L^2(0,1)$ and use c, throughout this paper, to denote a generic positive constant.

Lemma 3.2 (*Cavalcanti and Oquendo* [27])
The function α is not identically zero and satisfies

$$\inf_{x\in[0,1]}\{\alpha(x)+b(x)\}\geqslant\frac{d}{2}$$

Proof
For all $x\in[0,\varepsilon]$, we have $a(x)\geqslant\varepsilon\geqslant d>d/2$; therefore, by definition, $\alpha(x)=a(x)\geqslant\varepsilon$; hence, α is not identically zero over $[0,1]$.

On the other hand, if $a(x)\geqslant d/2$, then $\alpha(x)\geqslant d/2$, which implies that $\alpha(x)+b(x)\geqslant d/2$. If $a(x)<d/2$, then, by (H1), $b(x)>d/2$. Consequently, $\alpha(x)+b(x)\geqslant d/2$. Therefore, $\inf_{x\in[0,1]}\{\alpha(x)+b(x)\}\geqslant d/2$. □

Lemma 3.3 (*Cavalcanti and Oquendo* [27])
There exists a positive constant c such that

$$(g\odot v)^2\leqslant c g\circ v_x$$

for all $v\in H_0^1(0,1)$.

Proof
Let $S_a=\{x\in[0,1]: a(x)>d/4\}$. We should note that, by definition of d, $0\in S_a$; hence, $\partial S_a\cap\partial(0,1)\neq\emptyset$ and $\operatorname{supp}\alpha\subset S_a$:

$$(g\odot v)^2=\left(\int_{\operatorname{supp}\alpha}\alpha(x)\int_0^t g^{1/2}(t-s)g^{1/2}(t-s)(v(t)-v(s))\,\mathrm{d}s\,\mathrm{d}x\right)^2$$

DOI: 10.1002/mma

By using Hölder's inequality, a variant of Poincaré's inequality (see [27]), we get

$$(g\odot v)^2 \leqslant c\left(\int_0^t g(s)\,\mathrm{d}s\right)\left(\int_{\operatorname{supp}\alpha}\int_0^t g(t-s)(v(t)-v(s))^2\,\mathrm{d}s\,\mathrm{d}x\right)$$
$$\leqslant c\int_{S_a}\int_0^t g(t-s)(v_x(t)-v_x(s))^2\,\mathrm{d}s\,\mathrm{d}x$$

Recalling the definition of S_a, we arrive at

$$(g\odot v)^2\leqslant c\int_{S_a} a(x)\int_0^t g(t-s)(v_x(t)-v_x(s))^2\,\mathrm{d}s\,\mathrm{d}x\leqslant c g\circ v_x \qquad \square$$

Lemma 3.4
Under the assumptions (H1)–(H4), the functional I_1 defined by

$$I_1(t):=-\int_0^1 \alpha(x)\psi_t\int_0^t g(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\,\mathrm{d}x$$

satisfies, along the (weak) solution, the estimate

$$I_1'(t)\leqslant -\left(\int_0^t g(s)\,\mathrm{d}s-\delta\right)\int_0^1 \alpha(x)\psi_t^2\,\mathrm{d}x+\delta\int_0^1(\varphi_x+\psi)^2\,\mathrm{d}x+c\delta\int_0^1\psi_x^2\,\mathrm{d}x$$
$$-\frac{c}{\delta}g'\circ\psi_x+c\left(\delta+\frac{1}{\delta}\right)g\circ\psi_x+\frac{c}{\delta}\int_0^1 b(x)h^2(\psi(t))\,\mathrm{d}x \tag{12}$$

for all $\delta>0$.

Proof
By using equations in (5), we get

$$I_1'(t)=-\int_0^1\alpha\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\,\mathrm{d}x-\int_0^1\alpha\psi_t^2\left(\int_0^t g(s)\,\mathrm{d}s\right)\mathrm{d}x$$
$$-\int_0^1\alpha\left[\psi_{xx}-\int_0^t g(t-s)(a(x)\psi_x(s))_x\,\mathrm{d}s-\varphi_x-\psi-b(x)h(\psi_t)\right]$$
$$\times\int_0^t g(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\,\mathrm{d}x$$
$$=-\int_0^1\alpha\psi_t\int_0^t g'(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\,\mathrm{d}x-\int_0^1\alpha\psi_t^2\left(\int_0^t g(s)\,\mathrm{d}s\right)\mathrm{d}x$$
$$+\int_0^1\alpha\psi_x\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,\mathrm{d}s\,\mathrm{d}x+\int_0^1\alpha(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\,\mathrm{d}x$$
$$-\int_0^1\alpha a\left(\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,\mathrm{d}s\right)\mathrm{d}x$$

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$$+\int_0^1 \alpha'\left(\psi_x - a\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\right)\mathrm{d}x$$

$$+\int_0^1 b(x)h(\psi_t)\int_0^t g(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\,\mathrm{d}x$$

We now estimate the terms in the right side of the above equality as follows:

By using Young's inequality and Lemma 3.3 (for g') we obtain, for all $\delta>0$,

$$-\int_0^1 \alpha\psi_t \int_0^t g'(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\,\mathrm{d}x \leqslant \delta\int_0^1 \alpha(x)\psi_t^2\,\mathrm{d}x - \frac{c}{\delta} g'\circ\psi_x$$

Similarly, we have

$$-\int_0^1 \alpha\psi_x \int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,\mathrm{d}s\,\mathrm{d}x \leqslant \delta\int_0^1 \psi_x^2\,\mathrm{d}x + \frac{c}{\delta} g\circ\psi_x$$

$$-\int_0^1 \alpha(\varphi_x+\psi)\int_0^t g(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\,\mathrm{d}x \leqslant \delta\int_0^1 (\varphi_x+\psi)^2\,\mathrm{d}x + \frac{c}{\delta} g\circ\psi_x$$

$$\begin{aligned}
&-\int_0^1 \alpha a\left(\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,\mathrm{d}s\right)\mathrm{d}x\\
&\leqslant \delta'\int_0^1 a\left(\int_0^t g(t-s)(\psi_x(s)-\psi_x(t)+\psi_x(t))\,\mathrm{d}s\right)^2\mathrm{d}x\\
&\quad+\frac{c}{\delta'}\int_0^1 a\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,\mathrm{d}s\right)^2\mathrm{d}x\\
&\leqslant 2\delta'\int_0^1 a\psi_x^2\left(\int_0^t g(s)\,\mathrm{d}s\right)^2\mathrm{d}x + \left(2\delta'+\frac{c}{\delta'}\right)\int_0^1 a\left(\int_0^t g(t-s)(\psi_x(t)-\psi_x(s))\,\mathrm{d}s\right)^2\mathrm{d}x\\
&\leqslant c\delta'\int_0^1 \psi_x^2\,\mathrm{d}x + c\left(\delta'+\frac{1}{\delta'}\right)g\circ\psi_x \leqslant \delta\int_0^1 \psi_x^2\,\mathrm{d}x + c\left(\delta+\frac{1}{\delta}\right)g\circ\psi_x
\end{aligned}$$

$$\begin{aligned}
&\int_0^1 \alpha'\left(\psi_x - a\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)\left(\int_0^t g(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\right)\mathrm{d}x\\
&\leqslant \delta\int_0^1 \psi_x^2\,\mathrm{d}x + c\left(\delta+\frac{1}{\delta}\right)g\circ\psi_x
\end{aligned}$$

and

$$\int_0^1 b(x)h(\psi_t)\int_0^t g(t-s)(\psi(t)-\psi(s))\,\mathrm{d}s\,\mathrm{d}x \leqslant \delta\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x + c\left(\delta+\frac{1}{\delta}\right)g\circ\psi_x$$

By combining all the above estimates, the assertion of Lemma 3.4 is proved. □

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Lemma 3.5
Under the assumptions (H1)–(H4), the functional I_2 defined by

$$I_2(t):=-\int_0^1(\psi\psi_t+\varphi\varphi_t)\,\mathrm{d}x$$

satisfies, along the (weak) solution, the estimate

$$I_2'(t)\leqslant-\int_0^1(\psi_t^2+\varphi_t^2)\,\mathrm{d}x+\int_0^1(\varphi_x+\psi)^2\,\mathrm{d}x+c\int_0^1\psi_x^2\,\mathrm{d}x+cg\circ\psi_x+c\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x \tag{13}$$

Proof
By exploiting Equations (5) and repeating the same procedure as in the above, we have

$$\begin{aligned}
I_2'(t)=&-\int_0^1(\psi_t^2+\varphi_t^2)\,\mathrm{d}x-\int_0^1\varphi(\psi_x+\varphi_{xx})\,\mathrm{d}x\\
&-\int_0^1\psi\left[\psi_{xx}-\int_0^t g(t-s)(a(x)\psi_x(s))_x\,\mathrm{d}s-\varphi_x-\psi-b(x)h(\psi_t)\right]\mathrm{d}x\\
=&-\int_0^1(\psi_t^2+\varphi_t^2)\,\mathrm{d}x+\int_0^1\psi_x^2\,\mathrm{d}x-\int_0^1 a(x)\psi_x\left(\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)\mathrm{d}x\\
&+\int_0^1(\varphi_x+\psi)^2\,\mathrm{d}x+\int_0^1 b(x)\psi h(\psi_t)\,\mathrm{d}x\\
\leqslant&-\int_0^1(\psi_t^2+\varphi_t^2)\,\mathrm{d}x+\int_0^1(\varphi_x+\psi)^2+c\int_0^1\psi_x^2\,\mathrm{d}x+cg\circ\psi_x+c\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x
\end{aligned}$$

This completes the proof of Lemma 3.5. □

Lemma 3.6
Assume that (H1)–(H4) hold. Then, the functional I_3 defined by

$$I_3(t):=-\int_0^1\psi_t(\varphi_x+\psi)\,\mathrm{d}x+\int_0^1\psi_x\varphi_t\,\mathrm{d}x-\int_0^1 a(x)\varphi_t\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\,\mathrm{d}x$$

satisfies, along the (weak) solution, the estimate

$$\begin{aligned}
I_3'(t)\leqslant&\left[\left(\psi_x-a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)\varphi_x\right]_{x=0}^{x=1}-(1-\varepsilon)\int_0^1(\varphi_x+\psi)^2\,\mathrm{d}x\\
&+\varepsilon\int_0^1\varphi_t^2\,\mathrm{d}x-\frac{c}{\varepsilon}g'\circ\psi_x+\frac{c}{\varepsilon}\int_0^1\psi_x^2\,\mathrm{d}x+\int_0^1\psi_t^2\,\mathrm{d}x+\frac{c}{\varepsilon}\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x
\end{aligned}\tag{14}$$

for any $0<\varepsilon<1$.

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Proof
By exploiting Equations (5) and repeating the same procedure as in the above, we have

$$
\begin{aligned}
I_3'(t) = & \int_0^1 (\varphi_x+\psi)\left[\psi_{xx}-\int_0^t g(t-s)(a(x)\psi_x(s))_x\,\mathrm{d}s-\varphi_x-\psi-b(x)h(\psi_t)\right]\mathrm{d}x \\
& +\int_0^1 (\varphi_{xt}+\psi_t)\psi_t\,\mathrm{d}x+\int_0^1 \psi_{xt}\varphi_t\,\mathrm{d}x+\int_0^1 \psi_x(\varphi_x+\psi)_x\,\mathrm{d}x \\
& -\int_0^1 a(x)(\varphi_x+\psi)_x\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\,\mathrm{d}x-\int_0^1 a(x)\varphi_t\left(g(0)\psi_x+\int_0^t g'(t-s)\psi_x(s)\,\mathrm{d}s\right)\mathrm{d}x \\
= & \left[\psi_x-a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\,\varphi_x\right]_{x=0}^{x=1} \\
& -\int_0^1 (\varphi_x+\psi)^2\,\mathrm{d}x-\int_0^1 b(x)(\varphi_x+\psi)h(\psi_t)\,\mathrm{d}x+\int_0^1 \psi_t^2\,\mathrm{d}x \\
& +g(t)\int_0^1 a(x)\psi_x\varphi_t\,\mathrm{d}x-\int_0^1 a(x)\varphi_t\int_0^t g'(t-s)(\psi_x(s)-\psi_x(t))\,\mathrm{d}s\,\mathrm{d}x
\end{aligned}
$$

By using Young's inequality, (14) is established. □

Lemma 3.7
Assume that (H1)–(H4) hold. Let $m\in C^1([0,1])$ be a function satisfying $m(0)=-m(1)=2$. Then there exists $c>0$ such that for any $\varepsilon>0$ the functionals I_4 and I_5 defined by

$$
I_4=\int_0^1 m(x)\psi_t\left(\psi_x-a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)\mathrm{d}x, \quad I_5=\int_0^1 m(x)\varphi_t\varphi_x\,\mathrm{d}x
$$

satisfy, along the (weak) solution,

$$
\begin{aligned}
I_4'(t)\leqslant & -\left(\left(\psi_x(1,t)-a(1)\int_0^t g(t-s)\psi_x(1,s)\,\mathrm{d}s\right)^2+\left(\psi_x(0,t)-a(0)\int_0^t g(t-s)\psi_x(0,s)\,\mathrm{d}s\right)^2\right) \\
& +\varepsilon\int_0^1 (\varphi_x+\psi)^2\,\mathrm{d}x+\frac{c}{\varepsilon}\left(\int_0^1 \psi_x^2\,\mathrm{d}x+g\circ\psi_x\right)+c\left(\int_0^1 (\psi_t^2+b(x)h^2(\psi_t))\,\mathrm{d}x-g'\circ\psi_x\right)
\end{aligned}
$$

and

$$
I_5'(t)\leqslant-(\varphi_x^2(1,t)+\varphi_x^2(0,t))+c\int_0^1 (\varphi_t^2+\varphi_x^2+\psi_x^2)\,\mathrm{d}x
$$

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Proof

By exploiting Equations (5) and repeating the same procedure as in the above, we have

$$
\begin{aligned}
I_4'(t) &= \int_0^1 m\left(\psi_x - a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)_x \left(\psi_x - a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)\mathrm{d}x \\
&\quad - \int_0^1 m\left(\psi_x - a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)(\varphi_x+\psi+b(x)h(\psi_t))\,\mathrm{d}x \\
&\quad + \int_0^1 m(x)\psi_t\left(\psi_{xt} - a(x)g(0)\psi_x - a(x)\int_0^t g'(t-s)\psi_x(s)\,\mathrm{d}s\right)\mathrm{d}x \\
&= -\left(\left(\psi_x(1,t)-a(1)\int_0^t g(t-s)\psi_x(1,s)\,\mathrm{d}s\right)^2 + \left(\psi_x(0,t)-a(0)\int_0^t g(t-s)\psi_x(0,s)\,\mathrm{d}s\right)^2\right) \\
&\quad - \frac{1}{2}\int_0^1 m'(x)\left(\psi_x - a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)^2\mathrm{d}x \\
&\quad - \int_0^1 m(x)\left(\psi_x - a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)(\varphi_x+\psi+b(x)h(\psi_t))\,\mathrm{d}x - \frac{1}{2}\int_0^1 m'(x)\psi_t^2\,\mathrm{d}x \\
&\quad + \int_0^1 m(x)a(x)\psi_t\left(\int_0^t g'(t-s)(\psi_x(t)-\psi_x(s))\,\mathrm{d}s\right)\mathrm{d}x + g(t)\int_0^1 m(x)a(x)\psi_x\psi_t\,\mathrm{d}x
\end{aligned}
$$

By using Young's inequality and Lemma 3.3, the first estimate of Lemma 3.7 is established.
Similarly, we can prove the second estimate of Lemma 3.7. □

Lemma 3.8

Assume that (H1)–(H4) hold. Then, the functional I_6 defined by

$$I_6(t) := I_3(t) + \frac{1}{4\varepsilon} I_4(t) + \varepsilon I_5(t)$$

satisfies, along the (weak) solution, the estimate

$$
\begin{aligned}
I_6'(t) \leqslant & -\left(\frac{3}{4} - c\varepsilon\right)\int_0^1 (\varphi_x+\psi)^2\,\mathrm{d}x + c\varepsilon\int_0^1 \varphi_t^2\,\mathrm{d}x + \frac{c}{\varepsilon}\int_0^1 \psi_t^2\,\mathrm{d}x \\
& + \frac{c}{\varepsilon}\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x + \frac{c}{\varepsilon}\int_0^1 \psi_x^2\,\mathrm{d}x - \frac{c}{\varepsilon} g'\circ\psi_x + \frac{c}{\varepsilon} g\circ\psi_x
\end{aligned} \tag{15}
$$

for any $0<\varepsilon<1$.

Proof

By using Lemmas 3.6 and 3.7, Young's and Poincaré's inequalities and the fact that

$$\varphi_x^2 \leqslant 2(\psi+\varphi_x)^2 + 2\psi^2$$

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and

$$\left(\psi_x - a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)\varphi_x \leqslant \varepsilon\varphi_x^2 + \frac{1}{4\varepsilon}\left(\psi_x - a(x)\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\right)^2$$

we obtain (15). □

Let $I_7(t) := I_6(t) + 2c\varepsilon I_2(t)$. By using Lemmas 3.5 and 3.8, and fixing ε small enough, we obtain

$$\begin{aligned} I_7'(t) \leqslant & -\frac{1}{2}\int_0^1 (\varphi_x+\psi)^2\,\mathrm{d}x - \tau\int_0^1 \varphi_t^2\,\mathrm{d}x + c\int_0^1 \psi_t^2\,\mathrm{d}x + c\int_0^1 \psi_x^2\,\mathrm{d}x \\ & + c\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x + cg\circ\psi_x - cg'\circ\psi_x \end{aligned} \tag{16}$$

where $\tau = c\varepsilon$.

As in [9], we use the multiplier w given by the solution of

$$-w_{xx} = \psi_x, \quad w(0) = w(1) = 0 \tag{17}$$

Lemma 3.9
The (weak) solution of (17) satisfies

$$\int_0^1 w_x^2\,\mathrm{d}x \leqslant \int_0^1 \psi^2\,\mathrm{d}x$$

and

$$\int_0^1 w_t^2\,\mathrm{d}x \leqslant \int_0^1 \psi_t^2\,\mathrm{d}x$$

Proof
We multiply Equation (17) by w, integrate by parts and use the Cauchy–Schwarz inequality to get

$$\int_0^1 w_x^2\,\mathrm{d}x \leqslant \int_0^1 \psi^2\,\mathrm{d}x$$

Next, we differentiate (17) with respect to t to obtain, by similar calculations,

$$\int_0^1 w_{xt}^2\,\mathrm{d}x \leqslant \int_0^1 \psi_t^2\,\mathrm{d}x$$

Poincaré's inequality then yields

$$\int_0^1 w_t^2\,\mathrm{d}x \leqslant \int_0^1 \psi_t^2\,\mathrm{d}x$$

This completes the proof of Lemma 3.9. □

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Lemma 3.10
Under the assumptions (H1)–(H4), the functional I_8 defined by

$$I_8(t):=\int_0^1 (\psi\psi_t + w\varphi_t)\,\mathrm{d}x$$

satisfies, along the (weak) solution, the estimate

$$I_8'(t)\leqslant -\frac{l}{2}\int_0^1 \psi_x^2\,\mathrm{d}x+\frac{c}{\varepsilon}\int_0^1 \psi_t^2\,\mathrm{d}x+\varepsilon\int_0^1 \varphi_t^2\,\mathrm{d}x+c\left(\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x+g\circ\psi_x\right) \tag{18}$$

for any $0<\varepsilon<l$ (l is defined in (H3)).

Proof
By exploiting Equations (5) and integrating by parts, we have

$$\begin{aligned}
I_8'(t) &= \int_0^1 (\psi_t^2-\psi_x^2)\,\mathrm{d}x+\int_0^1 a(x)\psi_x\int_0^t g(t-s)\psi_x(s)\,\mathrm{d}s\,\mathrm{d}x\\
&\quad -\int_0^1 \psi(\varphi_x+\psi+b(x)h(\psi_t))\,\mathrm{d}x-\int_0^1 w_x(\varphi_x+\psi)\,\mathrm{d}x+\int_0^1 w_t\varphi_t\,\mathrm{d}x\\
&\leqslant \int_0^1 \psi_t^2\,\mathrm{d}x-\frac{l}{2}\int_0^1 \psi_x^2\,\mathrm{d}x+c\left(\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x+g\circ\psi_x\right)\\
&\quad +\int_0^1 (w_x^2-\psi^2)\,\mathrm{d}x+\frac{c}{\varepsilon}\int_0^1 \varphi_t^2\,\mathrm{d}x+\varepsilon\int_0^1 w_t^2
\end{aligned}$$

Lemma 3.9 gives the desired result. □

For $N_1, N_2, N_3>1$, let

$$I_9(t):=N_1E(t)+N_2I_1(t)+N_3I_8(t)+I_7(t)$$

and let $t_0>0$ and $g_0=\int_0^{t_0} g(s)\,\mathrm{d}s>0$. By combining (11), (12), (16), (18) and taking $\delta=1/4N_2$, we arrive at

$$\begin{aligned}
I_9'(t) &\leqslant -\left(N_2g_0-\frac{1}{4}\right)\int_0^1 (\alpha(x)+b(x))\psi_t^2\,\mathrm{d}x+c\frac{N_3}{\varepsilon}\int_0^1 \psi_t^2\,\mathrm{d}x\\
&\quad -N_1\int_0^1 b(x)\psi_t h(\psi_t)\,\mathrm{d}x+\left(N_2g_0-\frac{1}{4}\right)\int_0^1 b(x)\psi_t^2\,\mathrm{d}x+c(N_2^2+N_3)\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x\\
&\quad -\left(\frac{lN_3}{2}-c-\frac{c}{N_2}\right)\int_0^1 \psi_x^2\,\mathrm{d}x-(c-N_3\varepsilon)\int_0^1 \varphi_t^2\,\mathrm{d}x-\frac{1}{4}\int_0^1 (\varphi_x+\psi)^2\,\mathrm{d}x\\
&\quad +\left(\frac{N_1}{2}-cN_2^2-c\right)g'\circ\psi_x+c(N_2^2+N_3)g\circ\psi_x
\end{aligned} \tag{19}$$

for all $t\geqslant t_0$ and $0<\varepsilon<l$.

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Now, if $a \neq 0$, we choose N_3 large enough so that

$$\frac{lN_3}{2} > c$$

and then ε small enough so that

$$\varepsilon < \frac{c}{N_3}$$

Next, we choose N_2 large enough so that

$$N_2 g_0 - \frac{1}{4} > \frac{2cN_3}{d\varepsilon}, \quad \frac{lN_3}{2} - c - \frac{c}{N_2} > 0$$

Finally, we choose N_1 large enough so that

$$\frac{N_1}{2} - cN_2^2 - c \geqslant 0$$

By using (H3), we arrive at

$$I_9'(t) \leqslant -c\int_0^1 (\alpha(x)+b(x))\psi_t^2\,\mathrm{d}x + c\int_0^1 b(x)(\psi_t^2 + h^2(\psi_t))\,\mathrm{d}x$$
$$-c\int_0^1 (\psi_x^2+\varphi_t^2)\,\mathrm{d}x - \frac{1}{4}\int_0^1 (\varphi_x+\psi)^2\,\mathrm{d}x + cg\circ\psi_x \tag{20}$$

Lemma 3.2 and estimate (8) then lead to

$$I_9'(t) \leqslant -cE(t) + cg\circ\psi_x + c\int_0^1 b(x)(\psi_t^2 + h^2(\psi_t))\,\mathrm{d}x \tag{21}$$

for all $t \geqslant t_0$.

If $a = 0$, then $I_1 = 0$ and $I_9(t) := N_1 E(t) + N_3 I_8(t) + I_7(t)$. Then (19) takes the form

$$I_9'(t) \leqslant c\frac{N_3}{\varepsilon}\int_0^1 \psi_t^2\,\mathrm{d}x - N_1\int_0^1 b(x)\psi_t h(\psi_t)\,\mathrm{d}x + cN_3\int_0^1 b(x)h^2(\psi_t)\,\mathrm{d}x$$
$$-\left(\frac{N_3}{2} - c\right)\int_0^1 \psi_x^2\,\mathrm{d}x - (c - N_3\varepsilon)\int_0^1 \varphi_t^2\,\mathrm{d}x - \frac{1}{4}\int_0^1 (\varphi_x+\psi)^2\,\mathrm{d}x$$

This implies (21) by repeating the above procedure of choosing the constants and taking into account the fact that $\inf_{x\in[0,1]}\{b(x)\} \geqslant 0$.

On the other hand, we can choose N_1 even larger (if needed) so that

$$I_9(t) \sim E(t) \tag{22}$$

Now we estimate the last integral of (21). For this reason, we consider the following partition of $(0,1)$ (where ε' is defined in (H2)):

$$\Omega^+ = \{x \in (0,1) : |\psi_t| > \varepsilon'\} \quad \text{and} \quad \Omega^- = \{x \in (0,1) : |\psi_t| \leqslant \varepsilon'\} \tag{23}$$

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From (H2) and (11), we easily show that

$$\begin{aligned}\int_{\Omega^+} b(x)(\psi_t^2+h^2(\psi_t))\,\mathrm{d}x &\leqslant c\int_{\Omega^+} b(x)\psi_t h(\psi_t)\,\mathrm{d}x\\ &\leqslant c\int_0^1 b(x)\psi_t h(\psi_t)\,\mathrm{d}x \leqslant -cE'(t)\end{aligned} \tag{24}$$

Case 1: H is linear on $[0,\varepsilon']$: Then there exist c_1', $c_2'>0$ such that $c_1'|s|\leqslant|h(s)|\leqslant c_2'|s|$ for all $s\in\mathbb{R}_+$, and then (24) is satisfied on all (0,1). Using (21) and (24), we deduce that

$$(I_9(t)+cE(t))'\leqslant -cH_2(E(t))+cg\circ\psi_x \tag{25}$$

Case 2: $H'(0)=0$ and $H''>0$ on $(0,\varepsilon']$: In this case, let H^* denote the dual function of the convex function H in the sense of Young (for the definition, see [30, p. 64]). Because $H''>0$ on $(0,\varepsilon_1]$ and $H'(0)=0$ and (H2) is still satisfied for any $\varepsilon''\in(0,\varepsilon']$, we can assume, without loss of generality, that H' defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$. Then H^* is the Legendre transform of H, which is given by

$$H^*(s)=s(H')(s)-H[(H')^{-1}(s)]\quad \forall s\in\mathbb{R}_+$$

see Arnold [30, pp. 61–62] and Liu and Zuazua [20] for this matter.

By using Jensen's inequality (see [31]) and (11), we deduce that

$$\begin{aligned}\int_{\Omega^-} b(x)(\psi_t^2+h^2(\psi_t))\,\mathrm{d}x &\leqslant c\int_{\Omega^-} b(x)H^{-1}(\psi_t h(\psi_t))\,\mathrm{d}x\\ &\leqslant c\int_{\Omega^-} H^{-1}(b(x)\psi_t h(\psi_t))\,\mathrm{d}x\leqslant cH^{-1}\left(\int_0^1 b(x)\psi_t h(\psi_t)\,\mathrm{d}x\right)\\ &\leqslant cH^{-1}(-cE'(t))\end{aligned} \tag{26}$$

Therefore, we deduce from (21), (24) and (26) that

$$I_9'(t)\leqslant -cE(t)+cH^{-1}(-cE'(t))-cE'(t)+cg\circ\psi_x\quad \forall t\geqslant t_0 \tag{27}$$

Using Young's inequality (see [30, p. 64]) and the fact that

$$H^*\leqslant s(H')(s),\quad E'\leqslant 0,\quad H''\geqslant 0$$

and choosing $\varepsilon_0>0$ small enough, we obtain, for $c_0>0$ large enough,

$$\begin{aligned}(H'(\varepsilon_0 E(t))(I_9(t)+cE(t))+c_0E(t))' &= \varepsilon_0 E'(t)H''(\varepsilon_0E(t))(I_9(t)+cE(t))\\ &\quad +H'(\varepsilon_0E(t))(I_9'(t)+cE'(t))+c_0E'(t)\\ &\leqslant -cH'(\varepsilon_0E(t))E(t)+cH'(\varepsilon_0E(t))H^{-1}(-cE'(t))\\ &\quad +c_0E'(t)+cH'(\varepsilon_0E(t))g\circ\psi_x\\ &\leqslant -cH'(\varepsilon_0E(t))E(t)+cH^*(H'(\varepsilon_0E(t)))-cE'(t)\\ &\quad +c_0E'(t)+cg\circ\psi_x\end{aligned}$$

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$$\leqslant -cH'(\varepsilon_0 E(t))E(t) + c\varepsilon_0 H'(\varepsilon_0 E(t))E(t) + cg\circ\psi_x$$

$$\leqslant -cH'(\varepsilon_0 E(t))E(t) + cg\circ\psi_x = -cH_2(E(t)) + cg\circ\psi_x$$

Let

$$I_{10}(t) = \begin{cases} I_9(t) + cE(t) & \text{if } H \text{ is linear on } [0,\varepsilon'] \\ H'(\varepsilon_0 E(t))(I_9(t) + cE(t)) + c_0 E(t) & \text{if } H'(0) = 0 \text{ and } H'' > 0 \text{ on } (0,\varepsilon'] \end{cases}$$

Using (22), we have

$$I_{10}(t) \sim E(t)$$

and exploiting (25), we easily deduce that

$$I'_{10}(t) \leqslant -cH_2(E(t)) + cg\circ\psi_x \quad \forall t \geqslant t_0$$

As in [10], taking in account (11) and (H4), we obtain

$$\begin{aligned}(\xi(t)I_{10}(t))' &= \xi'(t)I_{10}(t) + \xi(t)I'_{10}(t) \\ &\leqslant -c\xi(t)H_2(E(t)) + c\xi(t)g\circ\psi_x \\ &\leqslant -c\xi(t)H_2(E(t)) + c(\xi g)\circ\psi_x \\ &\leqslant -c\xi(t)H_2(E(t)) - cg'\circ\psi_x \\ &\leqslant -c\xi(t)H_2(E(t)) - cE'\end{aligned}$$

Let $F(t) = \varepsilon(\xi(t)I_{10}(t) + cE(t))$, where $0 < \varepsilon < \bar{\varepsilon}$ and $\bar{\varepsilon}$ is a positive constant satisfying

$$\xi(t)I_{10}(t) + cE(t) \leqslant \frac{1}{\bar{\varepsilon}}E(t) \quad \forall t \geqslant 0$$

We also have $F \sim E$ and

$$\begin{aligned}F'(t) &\leqslant -c\varepsilon\xi(t)H_2(E(t)) \leqslant -c\varepsilon\xi(t)H_2(\bar{\varepsilon}(\xi(t)I_{10}(t) + cE(t))) \\ &\leqslant -c\varepsilon\xi(t)H_2(\varepsilon(\xi(t)I_{10}(t) + cE(t))) = -c\varepsilon\xi(t)H_2(F(t))\end{aligned}$$

A simple integration over (t_0, t) then yields

$$F(t) \leqslant H_1^{-1}\left(c\varepsilon\int_0^t \xi(s)\,\mathrm{d}s + H_1(F(t_0)) - c\varepsilon\int_0^{t_0}\xi(s)\,\mathrm{d}s\right)$$

where $H_1(t) = \int_t^1 (1/H_2(s))\,\mathrm{d}s$.

Since $\lim_{t\to 0} H_1(t) = +\infty$ and

$$0 \leqslant F(t_0) \leqslant \frac{\varepsilon}{\bar{\varepsilon}}E(t_0) \leqslant \frac{\varepsilon}{\bar{\varepsilon}}E(0)$$

we may choose ε small enough so that

$$H_1(F(t_0)) - c\varepsilon\int_0^{t_0}\xi(s)\,\mathrm{d}s \geqslant 0$$

DOI: 10.1002/mma

consequently, $F(t) \leqslant H_1^{-1}(c\varepsilon \int_0^t \xi(s)\,\mathrm{d}s)$. Therefore, there exist constants c', $c''>0$ for which

$$E(t) \leqslant c'' H_1^{-1}\left(c' \int_0^t \xi(s)\,\mathrm{d}s\right) \quad \forall t \geqslant 0$$

which gives (10). This completes the proof of the theorem.

Remarks

1. By taking $a \equiv 1$ and $b \equiv 0$, (5) reduces to the system studied in [9]. In this case our result is established under weaker conditions on g. Precisely, we do not require anything on g'' as in (1.6) and (1.7) of [9]. We only need g to be differentiable satisfying (H3) and (H4) (see also [32]).
2. Our results are still true if we consider variable coefficients (depending only on space variable): $\rho_1(x)$, $\rho_2(x)$, $k_1(x)$ and $k_2(x)$ such that ρ_1, ρ_2, k_1, $k_2 \in C^1(0, L)$, $k_1/\rho_1 = k_2/\rho_2$, $\inf k_1/\rho_1 > 0$ and $\sup k_1/\rho_1 < +\infty$.

4. COMMENTS AND OPEN QUESTIONS

In this paper, we considered a model of Timoshenko beams with frictional and viscoelastic dampings in the case of same speeds of propagation and proved a general decay estimate, which led to precise decay rates of the energy. Many interesting questions are still open and have to be studied. Some of them are the following:

1. The optimality of our estimate (10).
2. The case of different speeds of propagation. The only known results in this case were proved in the case of a damping h satisfying (H2) with H linear. See [6–8].
3. The semilinear case; that is, we add $f_1(\varphi)$ and $f_2(\psi)$ to the first two equations of (5), respectively, where f_1 and f_2 are given functions.
4. The case of function g satisfying (H3) and (H4) with arbitrary function ξ (not necessarily nonincreasing).

ACKNOWLEDGEMENTS

This work has been partially funded by KFUPM under Project #IN080408. Special thanks go to I. Lasiecka for fruitful discussions.

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DOI: 10.1002/mma