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Asymptotic Stability of Bresse System with One Infinite Memory in the Longitudinal Displacements

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Abstract. The asymptotic stability of one-dimensional linear Bresse systems under infinite memories was proved by Guesmia and Kafini (Math Methods Appl Sci 38:2389–2402, 2015) under three infinite memories, Guesmia and Kirane (Z Angew Math Phys 67:1–39, 2016) under two infinite memories, and De Lima Santos et al. (Q Appl Math 73:23–54, 2015) under one infinite memory acting on the shear angle displacements. The subject of this paper is to complete these results by proving that the asymptotic stability of Bresse systems holds also under one infinite memory acting on the longitudinal displacements.

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Keywords. Bresse system, infinite memory, asymptotic behavior, energy method, integral inequalities.

1. Introduction

In this paper, we consider a Bresse system in one-dimensional open bounded domain under the homogeneous Dirichlet–Neumann–Neumann boundary conditions and with one infinite memory acting on the third equation (longitudinal displacements)

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x + \psi + lw)_x - lk_3(w_x - l\varphi) = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x + \psi + lw) = 0, \\ \rho_1 w_{tt} - k_3(w_x - l\varphi)_x + lk_1(\varphi_x + \psi + lw) + \displaystyle\int_0^{+\infty} g(s)w_{xx}(x, t-s)\,\mathrm{d}s = 0, \\ \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = \varphi(L,t) = \psi_x(L,t) = w_x(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \\ w(x,-t) = w_0(x,t),\ w_t(x,0) = w_1(x), \end{cases} \tag{1.1}$$

where $(x,t) \in]0, L[\times\mathbb{R}_+$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function, and L, l, ρ_i, $k_i \in \mathbb{R}_+^*$.

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The Bresse system [3] is known as the circular arch problem, where φ, w and ψ represent, respectively, the vertical, longitudinal and shear angle displacements. For more details, we refer to [15] and [16] (see also [12] and [13]).

The stability of Bresse systems with (local or global) frictional dampings was obtained by several researchers in the last few years; see [1], [7], [19] and [21] for the case of one frictional damping acting on the shear angle displacements, [2], [23] and [24] for the case of two frictional dampings, and [4], [20], [22] and [24] for the case of three frictional dampings. When each equation is controlled by a frictional damping, the exponential stability of Bresse systems was proved regardless to the speeds of wave propagations given by

$$s_1 = \sqrt{\frac{k_1}{\rho_1}}, \quad s_2 = \sqrt{\frac{k_2}{\rho_2}} \quad \text{and} \quad s_3 = \sqrt{\frac{k_3}{\rho_1}}. \tag{1.2}$$

When at least one equation is free, the obtained stability estimate is of exponential or polynomial type depending on some relations between s_i. When only one frictional damping is considered, it was proved that the exponential stability is equivalent to

$$s_1 = s_2 = s_3. \tag{1.3}$$

Similar stability results were proved in [8], [17] and [18] in case where the Bresse system is coupled with one or two heat equations in a certain manner.

The stability of Bresse systems with memories was also recently studied. When the three equations are controlled via infinite memories of the form

$$\int_0^{+\infty} g_1(s)\varphi_{xx}(x,t-s)\,\mathrm{d}s, \ \int_0^{+\infty} g_2(s)\psi_{xx}(x,t-s)\,\mathrm{d}s \text{ and}$$
$$\int_0^{+\infty} g_3(s)w_{xx}(x,t-s)\,\mathrm{d}s,$$

where $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ are differentiable, non-increasing and integrable functions on $\mathbb{R}_+$, the stability was proved in [12] regardless to s_i. The obtained decay estimate in [12] depends only on the arbitrary growth at infinity of $s \mapsto g_i(s)$. When only two memories are considered, the stability of Bresse systems was proved in [13], where the decay rate depends also on s_i and the smoothness of initial data.

As far as we know, the first stability result for Bresse systems with only one infinite memory is the one obtained in [6] under

$$\int_0^{+\infty} g(s)\psi_{xx}(x,t-s)\,\mathrm{d}s \tag{1.4}$$

acting on the shear angle displacements (the second equation in (1.1)), where $g : \mathbb{R}_+ \to \mathbb{R}_+$ converges esponentially to zero at infinity.

Our objective in this paper is to prove that the asymptotic stability of Bresse systems holds also under one infinite memory acting on the longitudinal displacements; that is (1.1) is stable, where the decay rate of solutions depends on the arbitrary growth at infinity of the kernel g, the speeds of wave propagations (1.2) and the smoothness of initial data.

The paper is organized as follows. In Sect. 2, we present our hypotheses and state our main results. The proof of our main results will be given in Sect. 3.

2. Hypotheses and Main Results

Following the method of [5], we consider the functional

$$\eta(x,t,s) = w(x,t) - w(x,t-s) \quad \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+. \tag{2.1}$$

This functional satisfies

$$\begin{cases} \eta_t + \eta_s - w_t = 0 & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \eta_x(0,t,s) = \eta_x(L,t,s) = 0 & \text{in } \mathbb{R}_+\times\mathbb{R}_+, \\ \eta(x,t,0) = 0 & \text{in }]0,L[\times\mathbb{R}_+. \end{cases} \tag{2.2}$$

Let $\eta^0(x,s) = \eta(x,0,s)$, $U^0 = \left(\varphi_0, \psi_0, w_0, \varphi_1, \psi_1, w_1, \eta^0\right)^T$, $U = (\varphi, \psi, w, \varphi_t, \psi_t, w_t, \eta)^T$ and

$$g^0 = \int_0^{+\infty} g(s)\,\mathrm{d}s. \tag{2.3}$$

Then the system (1.1) takes the following abstract form:

$$\begin{cases} U_t = \mathcal{A}U, \\ U(t=0) = U^0, \end{cases} \tag{2.4}$$

where $\mathcal{A}$ is the linear operator defined by

$$\mathcal{A}U = \begin{pmatrix} \varphi_t \\ \psi_t \\ w_t \\ \frac{k_1}{\rho_1}\varphi_{xx} - \frac{l^2k_3}{\rho_1}\varphi + \frac{k_1}{\rho_1}\psi_x + \frac{l}{\rho_1}(k_1+k_3)w_x \\ -\frac{k_1}{\rho_2}\varphi_x + \frac{k_2}{\rho_2}\psi_{xx} - \frac{k_1}{\rho_2}\psi - \frac{lk_1}{\rho_2}w \\ -\frac{l}{\rho_1}(k_1+k_3)\varphi_x - \frac{lk_1}{\rho_1}\psi + \frac{1}{\rho_1}\left(k_3 - g^0\right)w_{xx} - \frac{l^2k_1}{\rho_1}w + \frac{1}{\rho_1}\int_0^{+\infty} g\eta_{xx}\,\mathrm{d}s \\ w_t - \eta_s \end{pmatrix}.$$

Let

$$L_0 = \left\{ v:\, \mathbb{R}_+ \to H_*^1(]0,L[),\ \int_0^L\int_0^{+\infty} g v_x^2\,\mathrm{d}s\,\mathrm{d}x < +\infty \right\} \tag{2.5}$$

and

$$\mathcal{H} = H_0^1(]0,L[) \times \left(H_*^1(]0,L[)\right)^2 \times L^2(]0,L[) \times \left(L_*^2(]0,L[)\right)^2 \times L_0, \tag{2.6}$$

where

$$L_*^2(]0,L[) = \left\{ v \in L^2(]0,L[),\ \int_0^L v\,\mathrm{d}x = 0 \right\} \tag{2.7}$$

and

$$H_*^1(]0,L[) = \left\{ v \in H^1(]0,L[),\ \int_0^L v\,\mathrm{d}x = 0 \right\}. \tag{2.8}$$

The domain $D(\mathcal{A})$ of $\mathcal{A}$ is defined by

$$D(\mathcal{A})=\Big\{V=(v_1,\cdots,v_7)^T\in\mathcal{H},\ \mathcal{A}V\in\mathcal{H},\ v_7(0)=0,\ \partial_x v_2(0)=\partial_x v_3(0)=0,\\ \partial_x v_2(L)=\partial_x v_3(L)=0,\ \partial_x v_7(\cdot,0)=\partial_x v_7(\cdot,L)=0\Big\}. \tag{2.9}$$

Remark 2.1. As in [13], integrating on $]0,L[$ the second and third equations in (1.1), and using the boundary conditions, we verify that

$$\partial_{tt}\left(\int_0^L \psi\,\mathrm{d}x\right)+\frac{k_1}{\rho_2}\int_0^L \psi\,\mathrm{d}x+\frac{lk_1}{\rho_2}\int_0^L w\,\mathrm{d}x=0 \tag{2.10}$$

and

$$\partial_{tt}\left(\int_0^L w\,\mathrm{d}x\right)+\frac{l^2k_1}{\rho_1}\int_0^L w\,\mathrm{d}x+\frac{lk_1}{\rho_1}\int_0^L \psi\,\mathrm{d}x=0. \tag{2.11}$$

Therefore, (2.10) implies that

$$\int_0^L w\,\mathrm{d}x=-\frac{\rho_2}{lk_1}\partial_{tt}\left(\int_0^L \psi\,\mathrm{d}x\right)-\frac{1}{l}\int_0^L \psi\,\mathrm{d}x. \tag{2.12}$$

Substituting (2.12) into (2.11), we get

$$\partial_{tttt}\left(\int_0^L \psi\,\mathrm{d}x\right)+\left(\frac{k_1}{\rho_2}+\frac{l^2k_1}{\rho_1}\right)\partial_{tt}\left(\int_0^L \psi\,\mathrm{d}x\right)=0. \tag{2.13}$$

Let $l_0=\sqrt{\frac{k_1}{\rho_2}+\frac{l^2k_1}{\rho_1}}$. Then, solving (2.13), we find

$$\int_0^L \psi\,\mathrm{d}x=\tilde{c}_1\cos(l_0t)+\tilde{c}_2\sin(l_0t)+\tilde{c}_3t+\tilde{c}_4, \tag{2.14}$$

where $\tilde{c}_1,\cdots,\tilde{c}_4$ are real constants. By combining (2.12) and (2.14), we get

$$\int_0^L w\,\mathrm{d}x=\tilde{c}_1\left(\frac{\rho_2l_0^2}{lk_1}-\frac{1}{l}\right)\cos(l_0t)+\tilde{c}_2\left(\frac{\rho_2l_0^2}{lk_1}-\frac{1}{l}\right)\sin(l_0t)-\frac{\tilde{c}_3}{l}t-\frac{\tilde{c}_4}{l}. \tag{2.15}$$

Let $(\tilde{\psi}_0(x),\tilde{w}_0(x))=(\psi_0(x),w_0(x,0))$. Using the initial data of ψ and w in (1.1), we see that

$$\begin{cases}\tilde{c}_1=\frac{k_1}{\rho_2l_0^2}\int_0^L\tilde{\psi}_0\,\mathrm{d}x+\frac{lk_1}{\rho_2l_0^2}\int_0^L\tilde{w}_0\,\mathrm{d}x,\\ \tilde{c}_2=\frac{k_1}{\rho_2l_0^3}\int_0^L\psi_1\,\mathrm{d}x+\frac{lk_1}{\rho_2l_0^3}\int_0^L w_1\,\mathrm{d}x,\\ \tilde{c}_3=\left(1-\frac{k_1}{\rho_2l_0^2}\right)\int_0^L\psi_1\,\mathrm{d}x-\frac{lk_1}{\rho_2l_0^2}\int_0^L w_1\,\mathrm{d}x,\\ \tilde{c}_4=\left(1-\frac{k_1}{\rho_2l_0^2}\right)\int_0^L\tilde{\psi}_0\,\mathrm{d}x-\frac{lk_1}{\rho_2l_0^2}\int_0^L\tilde{w}_0\,\mathrm{d}x.\end{cases}$$

Let

$$\tilde{\psi}=\psi-\frac{1}{L}\left(\tilde{c}_1\cos(l_0t)+\tilde{c}_2\sin(l_0t)+\tilde{c}_3t+\tilde{c}_4\right) \tag{2.16}$$

and

$$\tilde{w} = w - \frac{1}{L}\left(\tilde{c}_1\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right)\cos(l_0 t) + \tilde{c}_2\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right)\sin(l_0 t) - \frac{\tilde{c}_3}{l}t - \frac{\tilde{c}_4}{l}\right). \tag{2.17}$$

Then, from (2.14) and (2.15), one can check that

$$\int_0^L \tilde{\psi}\,\mathrm{d}x = \int_0^L \tilde{w}\,\mathrm{d}x = \int_0^L \tilde{\eta}\,\mathrm{d}x = 0, \tag{2.18}$$

where

$$\tilde{\eta}(x,t,s) = \tilde{w}(x,t) - \tilde{w}(x,t-s) \quad \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+.$$

Therefore, Poincaré's inequality

$$\exists\, c_0 > 0:\ \int_0^L v^2\,\mathrm{d}x \le c_0\int_0^L v_x^2\,\mathrm{d}x, \quad \forall v \in H_*^1(]0,L[) \cup H_0^1(]0,L[) \tag{2.19}$$

is applicable for $\tilde{\psi}$, $\tilde{w}$ and $\tilde{\eta}$. In addition, $(\varphi,\tilde{\psi},\tilde{w})$ satisfies the boundary conditions and the first three equations in (1.1) with initial data

$$\psi_0 - \frac{1}{L}(\tilde{c}_1+\tilde{c}_4),\ \psi_1 - \frac{1}{L}(l_0\tilde{c}_2+\tilde{c}_3),\ w_0 - \frac{1}{L}\left(\tilde{c}_1\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right) - \frac{\tilde{c}_4}{l}\right) \text{ and}$$
$$w_1 - \frac{1}{L}\left(\tilde{c}_2 l_0\left(\frac{\rho_2 l_0^2}{lk_1} - \frac{1}{l}\right) - \frac{\tilde{c}_3}{l}\right)$$

instead of ψ_0, ψ_1, w_0 and w_1, respectively. In the sequel, we work with $\tilde{\psi}$, $\tilde{w}$ and $\tilde{\eta}$ instead of ψ, w and η, respectively, but, for simplicity of notation, we use ψ, w and η.

Now, the following hypothesis guarantees the well-posedness of (2.4):

(H1) Assume that the function $g:\mathbb{R}_+\to\mathbb{R}_+$ is differentiable, non-increasing and integrable on $\mathbb{R}_+$, and there exists a positive constant k_0 such that, for any

$$(\varphi,\psi,w)^T \in H_0^1(]0,L[)\times\left(H_*^1(]0,L[)\right)^2,$$

we have

$$k_0\int_0^L\left(\varphi_x^2+\psi_x^2+w_x^2\right)\mathrm{d}x$$
$$\le \int_0^L\left(k_2\psi_x^2 + k_1(\varphi_x+\psi+lw)^2 + k_3(w_x-l\varphi)^2 - g^0 w_x^2\right)\mathrm{d}x. \tag{2.20}$$

Moreover, assume that there exists a positive constant β such that

$$-\beta g(s) \le g'(s), \quad \forall s\in\mathbb{R}_+. \tag{2.21}$$

We notice that, under the hypothesis **(H1)**, the sets L_0 and $\mathcal{H}$ are Hilbert spaces equipped with the inner products that generate the norms, for $v\in L_0$ and $V=(v_1,\cdots,v_7)^T\in\mathcal{H}$,

$$\|v\|_{L_0}^2 = \int_0^L\int_0^{+\infty} g v_x^2\,\mathrm{d}s\,\mathrm{d}x \tag{2.22}$$

and

$$\|V\|_{\mathcal{H}}^2 = \int_0^L \left(k_2(\partial_x v_2)^2 + k_1(\partial_x v_1 + v_2 + l v_3)^2 + k_3(\partial_x v_3 - l v_1)^2 - g^0(\partial_x v_3)^2\right) \mathrm{d}x$$
$$+ \int_0^L \left(\rho_1 v_4^2 + \rho_2 v_5^2 + \rho_1 v_6^2\right) \mathrm{d}x + \|v_7\|_{L_0}^2. \tag{2.23}$$

Exactly as in [13] one can prove that $\mathcal{A}$ generates a C_0-semigroup of contractions in $\mathcal{H}$ by proving that $-\mathcal{A}$ is maximal monotone (it is enough to neglect the second memory in the first system considered in [13], and the proof is the same as in [13]), and deduce the following well-posedness results of (2.4).

Theorem 2.2. *Assume that* **(H1)** *holds. Let $n \in \mathbb{N}$ and $U^0 \in D(\mathcal{A}^n)$. Then (2.4) has a unique solution*

$$U \in \cap_{k=0}^n C^{n-k}\left(\mathbb{R}_+; D\left(\mathcal{A}^k\right)\right). \tag{2.24}$$

To get the stability of (2.4), we consider the following additional hypothesis:

(H2) Assume that $g(0) > 0$, and there exist a positive constant α and an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} G'(t) = +\infty$$

such that

$$g'(s) \leq -\alpha g(s), \quad \forall s \in \mathbb{R}_+ \tag{2.25}$$

or

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \mathrm{d}s + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty. \tag{2.26}$$

Let us consider the energy functional E associated to (2.4) defined by

$$E(t) = \frac{1}{2}\|U(t)\|_{\mathcal{H}}^2. \tag{2.27}$$

First, we consider the case (1.3).

Theorem 2.3. *Assume that* **(H1)**, **(H2)** *and (1.3) are satisfied. Let $U^0 \in \mathcal{H}$ be such that*

$$(2.25) \text{ holds} \quad \text{or} \quad \sup_{t \in \mathbb{R}_+} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L \left(\eta_x^0(x, s-t)\right)^2 \mathrm{d}x \, \mathrm{d}s < +\infty. \tag{2.28}$$

Then there exist positive constants $\tilde{l}$ and $\tilde{g}$ (not depending neither on l nor on g) so that, if

$$l < \tilde{l} \quad \text{and} \quad g^0 < \tilde{g}, \tag{2.29}$$

then there exist positive constants c' and c'' satisfying

$$E'(t) \leq c'' \tilde{G}^{-1}(c' t), \quad \forall t \in \mathbb{R}_+, \tag{2.30}$$

where

$$\tilde{G}(s) = \int_s^1 \frac{1}{G_0(\tau)}\,d\tau \quad \textit{and} \quad G_0(s) = \begin{cases} s & \textit{if (2.25) } \text{holds,} \\ sG'(s) & \textit{if (2.26) } \text{holds.} \end{cases} \tag{2.31}$$

When (1.3) does not hold and $s_1 = s_2$; that is

$$s_1 = s_2 \quad \text{and} \quad s_1 \neq s_3, \tag{2.32}$$

we prove the following weaker stability result.

Theorem 2.4. *Assume that* **(H1)**, **(H2)** *and (2.32) are satisfied. Let* $n \in \mathbb{N}^*$ *and* $U^0 \in D(\mathcal{A}^n)$ *be such that*

$$\textit{(2.25) holds} \quad \textit{or} \quad \sup_{t\in\mathbb{R}_+} \max_{k=0,\cdots,n} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L \left(\partial_s^k \eta_x^0(x, s-t)\right)^2 \mathrm{d}x\,\mathrm{d}s < +\infty. \tag{2.33}$$

Then there exist positive constants $\tilde{l}$ *and* $\tilde{g}$ *(not depending neither on* l *nor on* g*) such that, if (2.29) holds, then there exists a positive constant* c_n *satisfying*

$$E(t) \le c_n G_n\left(\frac{c_n}{t}\right), \quad \forall t > 0, \tag{2.34}$$

where $G_m(s) = G_1(sG_{m-1}(s))$, *for* $m = 2, \cdots, n$, $G_1 = G_0^{-1}$ *and* G_0 *is defined in (2.31).*

Remark 2.5. 1. If (2.25) holds, then (2.30) and (2.34) give, respectively, for some positive constants d_1 and d_2,

$$E(t) \le d_1 e^{-d_2 t}, \quad \forall t \in \mathbb{R}_+ \tag{2.35}$$

and

$$E(t) \le \frac{d_1}{t^n}, \quad \forall t > 0. \tag{2.36}$$

The particular estimates (2.35) and (2.36) (for $n = 1$) coincide with the ones of [6] obtained in case (1.4) with g converging exponentially to zero at infinity.

2. Condition (2.26) (introduced in [10]) allows $s \mapsto g(s)$ to have a decay rate at infinity arbitrarily close to $\frac{1}{s}$. For specific examples of g and U^0 satisfying (2.26) and (2.33), respectively, and the corresponding decay rates (2.30) and (2.34), see [10] and [11].

3. Proof of (2.30) and (2.34)

3.1. Preliminaries Lemmas

We will use c, c_{y_1}, c_{y_1,y_2}, $\cdots$, throughout the rest of this paper, to denote generic positive constants which depend continuously on the initial data U^0 and some constants y_1, y_2, $\cdots$, introduced in the proof.

First, simple computations (see [13]), we see that

$$E'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'\eta_x^2\,\mathrm{d}s\,\mathrm{d}x. \tag{3.1}$$

Recalling that g is non-increasing, (3.1) implies that E is non-increasing.

We start our lemmas by considering the following functional:

$$I(t) = -\rho_1 \int_0^L w_t \int_0^{+\infty} g(s)\eta \,\mathrm{d}s\,\mathrm{d}x. \tag{3.2}$$

Lemma 3.1. *For any $\delta_0 > 0$, there exists $c_{\delta_0,l,g^0,g(0)} > 0$ such that*

$$I'(t) \leq -\rho_1 g^0 \int_0^L w_t^2 \,\mathrm{d}x + \delta_0 E(t) + c_{\delta_0,l,g^0,g(0)} \int_0^L \int_0^{+\infty} \left(g(s) - g'(s)\right) \eta_x^2 \,\mathrm{d}s\,\mathrm{d}x. \tag{3.3}$$

Proof. First, noticing that

$$\begin{aligned} \partial_t \int_0^{+\infty} g(s)\eta \,\mathrm{d}s &= \partial_t \int_{-\infty}^t g(t-s)(w(t)-w(s))\,\mathrm{d}s \\ &= \int_{-\infty}^t g'(t-s)(w(t)-w(s))\,\mathrm{d}s + \left(\int_{-\infty}^t g(t-s)\,\mathrm{d}s\right) w_t; \end{aligned}$$

that is

$$\partial_t \int_0^{+\infty} g(s)\eta \,\mathrm{d}s = \int_0^{+\infty} g'(s)\eta \,\mathrm{d}s + g^0 w_t. \tag{3.4}$$

Second, using Young's and Hölder's inequalities, we get the following inequality: for any $\lambda > 0$, there exists $c_\lambda > 0$ such that, for any $v \in L^2(]0, L[)$ and $f \in \{\eta, \eta_x\}$,

$$\left|\int_0^L v \int_0^{+\infty} g(s) f \,\mathrm{d}s\,\mathrm{d}x\right| \leq \lambda \int_0^L v^2 \,\mathrm{d}x + c_{\lambda,g^0} \int_0^L \int_0^{+\infty} g(s) f^2 \,\mathrm{d}s\,\mathrm{d}x. \tag{3.5}$$

Similarly,

$$\left|\int_0^L v \int_0^{+\infty} g'(s) f \,\mathrm{d}s\,\mathrm{d}x\right| \leq \lambda \int_0^L v^2 \,\mathrm{d}x - c_{\lambda,g(0)} \int_0^L \int_0^{+\infty} g'(s) f^2 \,\mathrm{d}s\,\mathrm{d}x. \tag{3.6}$$

Now, direct computations, using the third equation in (1.1), integrating by parts and using the boundary conditions and (3.4), yield

$$\begin{aligned} I'(t) = &-\rho_1 g^0 \int_0^L w_t^2 \,\mathrm{d}x + \int_0^L \left(\int_0^{+\infty} g(s)\eta_x \,\mathrm{d}s\right)^2 \mathrm{d}x \\ &+ k_3 \int_0^L (w_x - l\varphi) \int_0^{+\infty} g(s)\eta_x \,\mathrm{d}s\,\mathrm{d}x \\ &+ lk_1 \int_0^L (\varphi_x + \psi + lw) \int_0^{+\infty} g(s)\eta \,\mathrm{d}s\,\mathrm{d}x \\ &- \rho_1 \int_0^L w_t \int_0^{+\infty} g'(s)\eta \,\mathrm{d}s\,\mathrm{d}x - g^0 \int_0^L w_x \int_0^{+\infty} g(s)\eta_x \,\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

Using (3.5) and (3.6) for the last four terms in the above equality, Poincaré's inequality (2.19) for η, and (2.20) and Hölder's inequality to estimate

$$\int_0^L w_x^2 \,\mathrm{d}x \quad \text{and} \quad \left(\int_0^{+\infty} g(s)\eta_x \,\mathrm{d}s\right)^2,$$

respectively, we get (3.3). □

Lemma 3.2. *Let*

$$J(t) = -\rho_2 \int_0^L (\varphi_x + \psi + lw)\psi_t \,\mathrm{d}x - \frac{k_2\rho_1}{k_1}\int_0^L \psi_x\varphi_t \,\mathrm{d}x. \tag{3.7}$$

Then, for any ϵ_0, $\epsilon_1 > 0$, there exists $c_{\epsilon_0,l} > 0$ such that

$$J'(t) \le k_1 \int_0^L (\varphi_x + \psi + lw)^2 \,\mathrm{d}x + \frac{lk_2k_3\epsilon_1}{2k_1}\int_0^L (w_x - l\varphi)^2 \,\mathrm{d}x + \frac{lk_2k_3}{2k_1\epsilon_1}\int_0^L \psi_x^2 \,\mathrm{d}x$$
$$+ (-\rho_2 + \epsilon_0)\int_0^L \psi_t^2 \,\mathrm{d}x + c_{\epsilon_0,l}\int_0^L w_t^2 \,\mathrm{d}x + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_t\varphi_{xt} \,\mathrm{d}x. \tag{3.8}$$

Proof. By exploiting the first two equations in (1.1), integrating by parts and using the boundary conditions, we get

$$J'(t) = k_1 \int_0^L (\varphi_x + \psi + lw)^2 \,\mathrm{d}x + \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_t\varphi_{xt} \,\mathrm{d}x - \rho_2\int_0^L \psi_t^2 \,\mathrm{d}x$$
$$- \rho_2 l\int_0^L \psi_t w_t \,\mathrm{d}x - \frac{lk_2k_3}{k_1}\int_0^L (w_x - l\varphi)\psi_x \,\mathrm{d}x.$$

Applying Young's inequality for the last two terms in the above equality, we find (3.8). □

Lemma 3.3. *Let*

$$K(t) = \rho_1 \int_0^L (\varphi_x + \psi + lw)w_t \,\mathrm{d}x + \frac{k_3\rho_1}{k_1}\int_0^L (w_x - l\varphi)\varphi_t \,\mathrm{d}x$$
$$-\frac{\rho_1}{k_1}\int_0^L \varphi_t \int_0^{+\infty} g(s)w_x(t-s)\mathrm{d}s \;\mathrm{d}x. \tag{3.9}$$

Then, for any δ_0, $\epsilon_0 > 0$, there exist $c_{\delta_0,l,g^0,g(0)}$, $c_{\epsilon_0,l} > 0$ such that

$$K'(t) \le -lk_1 \int_0^L (\varphi_x + \psi + lw)^2 \,\mathrm{d}x + \frac{lk_3^2}{k_1}\int_0^L (w_x - l\varphi)^2 \,\mathrm{d}x + \delta_0 E(t)$$
$$- \tfrac{lk_3g^0}{k_1}\int_0^L (w_x - l\varphi)w_x \,\mathrm{d}x + \int_0^L \left(c_{\epsilon_0,l}w_t^2 + \epsilon_0\psi_t^2\right) \mathrm{d}x - \frac{lk_3\rho_1}{k_1}\int_0^L \varphi_t^2 \,\mathrm{d}x$$
$$+ \rho_1\left(\tfrac{k_3}{k_1} - 1\right)\int_0^L w_{xt}\varphi_t \,\mathrm{d}x + c_{\delta_0,l,g^0,g(0)}\int_0^L\int_0^{+\infty} (g(s) - g'(s))\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x. \tag{3.10}$$

Proof. First, we notice that

$$\partial_t \int_0^{+\infty} g(s)w_x(t-s)\,\mathrm{d}s = \partial_t \int_{-\infty}^t g(t-s)w_x(s)\,\mathrm{d}s$$
$$= g(0)w_x(t) + \int_{-\infty}^t g'(t-s)w_x(s)\,\mathrm{d}s$$
$$= -\int_0^{+\infty} g'(s)w_x(t)\,\mathrm{d}s + \int_0^{+\infty} g'(s)w_x(t-s)\,\mathrm{d}s;$$

that is

$$\partial_t \int_0^{+\infty} g(s)w_x(t-s)\,\mathrm{d}s = -\int_0^{+\infty} g'(s)\eta_x\,\mathrm{d}s. \tag{3.11}$$

Now, using the first and third equations in (1.1), integrating by parts, recalling (3.11) and using the boundary conditions, we find

$$\begin{aligned}K'(t) &= -lk_1\int_0^L(\varphi_x+\psi+lw)^2\,\mathrm{d}x+\frac{lk_3^2}{k_1}\int_0^L(w_x-l\varphi)^2\,\mathrm{d}x+\rho_1\left(\frac{k_3}{k_1}-1\right)\int_0^L\varphi_t w_{xt}\,\mathrm{d}x\\&+l\rho_1\int_0^L w_t^2\,\mathrm{d}x-\frac{lk_3\rho_1}{k_1}\int_0^L\varphi_t^2\,\mathrm{d}x-\frac{lk_3g^0}{k_1}\int_0^L(w_x-l\varphi)w_x\,\mathrm{d}x+\rho_1\int_0^L\psi_t w_t\,\mathrm{d}x\\&+\tfrac{\rho_1}{k_1}\int_0^L\varphi_t\int_0^{+\infty}g'(s)\eta_x\,\mathrm{d}s\,\mathrm{d}x+\frac{lk_3}{k_1}\int_0^L(w_x-l\varphi)\int_0^{+\infty}g(s)\eta_x\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}$$

By applying (3.5), (3.6) and Young's inequality for the last three terms in the above equality and exploiting (2.20), we deduce (3.10). □

Lemma 3.4. *Let*

$$\begin{aligned}P(t)=-\rho_1k_3\int_0^L(w_x-l\varphi)\int_0^x w_t(y,t)\,dy\,\mathrm{d}x-\rho_1k_1\int_0^L\varphi_t\\ \times\int_0^x(\varphi_x+\psi+lw)(y,t)\,dy\,\mathrm{d}x.\end{aligned}\tag{3.12}$$

Then, for any ϵ_0, δ_0, $\epsilon_2>0$, there exist $c_{\epsilon_0,l}$, $c_{\delta_0,g^0}>0$ such that

$$\begin{aligned}P'(t)&\le k_1^2\int_0^L(\varphi_x+\psi+lw)^2\,\mathrm{d}x-k_3^2\int_0^L(w_x-l\varphi)^2\,\mathrm{d}x+\delta_0E(t)\\&+\left(-\rho_1k_1+\tfrac{\rho_1k_1\epsilon_2}{2}+\epsilon_0\right)\int_0^L\varphi_t^2\,\mathrm{d}x+c_{\epsilon_0,l}\int_0^L w_t^2\,\mathrm{d}x+\frac{c_0\rho_1k_1}{2\epsilon_2}\int_0^L\psi_t^2\,\mathrm{d}x\\&+k_3g^0\int_0^L(w_x-l\varphi)w_x\,\mathrm{d}x+c_{\delta_0,g^0}\int_0^L\int_0^{+\infty}g(s)\eta_x^2\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}\tag{3.13}$$

Proof. By exploiting the first and third equations in (1.1), integrating by parts and using (2.18) and the boundary conditions, we get

$$\begin{aligned}P'(t)=\rho_1k_3&\int_0^L w_t^2\,\mathrm{d}x-\rho_1k_1\int_0^L\varphi_t^2\,\mathrm{d}x-k_3^2\int_0^L(w_x-l\varphi)^2\,\mathrm{d}x\\&+k_1^2\int_0^L(\varphi_x+\psi+lw)^2\,\mathrm{d}x\\&+k_3g^0\int_0^L(w_x-l\varphi)w_x\,\mathrm{d}x-\rho_1\int_0^L\varphi_t\int_0^x(k_1\psi_t(y,t)\\&+l(k_1-k_3)w_t(y,t))\,dy\,\mathrm{d}x\\&-k_3\int_0^L(w_x-l\varphi)\int_0^{+\infty}g(s)\eta_x\,\mathrm{d}s\,\mathrm{d}x.\end{aligned}\tag{3.14}$$

Noticing that the functions

$$x\mapsto\int_0^x\psi_t(y,t)\,dy\quad\text{and}\quad x\mapsto\int_0^x w_t(y,t)\,dy$$

vanish at 0 and L (because of (2.18)), then, applying (2.19), we have

$$\int_0^L\left(\int_0^x\psi_t(y,t)\,dy\right)^2\mathrm{d}x\le c_0\int_0^L\psi_t^2\,\mathrm{d}x\tag{3.15}$$

and

$$\int_0^L \left(\int_0^x w_t(y,t)\,dy\right)^2 \mathrm{d}x \le c_0 \int_0^L w_t^2\,\mathrm{d}x. \tag{3.16}$$

By applying Young's inequality and (3.5) for the last two terms in (3.14), and recalling (2.20), (3.15) and (3.16), we conclude (3.13). □

Lemma 3.5. *Let*

$$R(t) = \int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t + \rho_1 w w_t)\,\mathrm{d}x. \tag{3.17}$$

Then, for any $\delta_0 > 0$, there exists $c_{\delta_0,g^0} > 0$ such that

$$R'(t) \le \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 - k_2\psi_x^2 - k_1(\varphi_x+\psi+lw)^2 - k_3(w_x - l\varphi)^2\right)\mathrm{d}x$$
$$+ g^0 \int_0^L w_x^2\,\mathrm{d}x + \delta_0 E(t) + c_{\delta_0,g^0}\int_0^L\int_0^{+\infty} g(s)\eta_x^2\,\mathrm{d}s\,\mathrm{d}x. \tag{3.18}$$

Proof. By exploiting the first three equations in (1.1), integrating by parts and using the boundary conditions, we get

$$R'(t) = \int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + \rho_1 w_t^2 - k_2\psi_x^2 - k_1(\varphi_x+\psi+lw)^2 - k_3(w_x - l\varphi)^2\right)\mathrm{d}x$$
$$+ g^0\int_0^L w_x^2\,\mathrm{d}x - \int_0^L w_x \int_0^{+\infty} g(s)\eta_x\,\mathrm{d}s\,\mathrm{d}x.$$

Applying (2.20) and (3.5) for the last term in the above equality, we arrive at (3.18). □

Let N, N_1, N_2, N_3, $N_4 > 0$ and

$$F := NE + N_1 I + N_2 P + \frac{N_3}{l}K + N_4 R + J. \tag{3.19}$$

Then, by combining (3.3), (3.8), (3.10), (3.13) and (3.18), and exploiting (3.1) to estimate the integral of $-g'\eta_x^2$, we obtain

$$F'(t) \le \int_0^L \left(l_1\varphi_t^2 + l_2\psi_t^2 + l_3 w_t^2 + \left(l_4 + \frac{lk_2k_3}{2k_1\epsilon_1}\right)\psi_x^2 + \left(l_5 + \frac{lk_2k_3\epsilon_1}{2k_1}\right)(w_x - l\varphi)^2\right)\mathrm{d}x$$
$$+ \int_0^L \left(l_6(\varphi_x+\psi+lw)^2 + g^0 N_4 w_x^2 + g^0 l_7(w_x - l\varphi)w_x\right)\mathrm{d}x + \delta_0 c_{l,N_1,\cdots,N_4}E(t)$$
$$+ (N - c_{\delta_0,l,g^0,g(0),N_1,N_3})E'(t) + c_{\delta_0,l,g^0,g(0),N_1,\cdots,N_4}\int_0^L\int_0^{+\infty} g(s)\eta_x^2\,\mathrm{d}s\,\mathrm{d}x$$
$$+ \left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_t\varphi_{xt}\,\mathrm{d}x$$
$$+ \frac{N_3\rho_1}{l}\left(\frac{k_3}{k_1} - 1\right)\int_0^L w_{xt}\varphi_t\,\mathrm{d}x + \epsilon_0 c_{l,N_2,N_3}\int_0^L (\varphi_t^2 + \psi_t^2)\,\mathrm{d}x$$
$$+ c_{l,\epsilon_0,N_2,N_3}\int_0^L w_t^2\,\mathrm{d}x, \tag{3.20}$$

where

$$l_1 = \rho_1 k_1 \left(\frac{\epsilon_2}{2} - 1\right) N_2 - \frac{\rho_1 k_3 N_3}{k_1} + \rho_1 N_4, \quad l_2 = \frac{c_0 \rho_1 k_1 N_2}{2\epsilon_2} + \rho_2 N_4 - \rho_2,$$
$$l_3 = -\rho_1 g_0 N_1 + \rho_1 N_4, \quad l_4 = -k_2 N_4, \quad l_5 = -k_3^2 N_2 + \frac{k_3^2 N_3}{k_1} - k_3 N_4,$$
$$l_6 = k_1 + k_1^2 N_2 - k_1 N_3 - k_1 N_4 \quad \text{and} \quad l_7 = k_3 \left(N_2 - \frac{N_3}{k_1}\right).$$

At this point, we choose carefully the constants N, N_i, ϵ_i and δ_0 to get suitable values of l_i. We choose $\epsilon_2 \in \left]0, \frac{2(k_1+k_3)}{k_1}\right[$. After we pick $N_2 > 0$ small enough so that

$$\frac{c_0 \rho_1 k_1}{2\rho_2 \epsilon_2} N_2 < 1, \quad \left(k_1 \left(\frac{\epsilon_2}{2} - 1\right) + \frac{c_0 \rho_1 k_1}{2\rho_2 \epsilon_2} - k_3\right) N_2 < 1,$$
$$\left(\frac{k_3 \epsilon_2}{2} + \frac{k_1 + k_3}{k_1} \left(\frac{c_0 \rho_1 k_1}{2\rho_2 \epsilon_2} - k_3\right)\right) N_2 < 1 \text{ and } \frac{c_0 \rho_1}{2\rho_2 \epsilon_2}(k_1 + k_3) N_2 < 1.$$

Next we fix $N_3 > 0$ such that

$$N_3 < \frac{k_1}{k_3} \left(1 + \left(k_3 - \frac{c_0 \rho_1 k_1}{2\rho_2 \epsilon_2}\right) N_2\right)$$

and

$$N_3 > \max\left\{k_1 \left(1 + \frac{c_0 \rho_1}{2\rho_2 \epsilon_2}\right) N_2, \frac{k_1^2}{k_3} \left(\frac{\epsilon_2}{2} - 1\right) N_2, \frac{k_1}{k_1 + k_3} \left(1 + \frac{k_1 \epsilon_2}{2} N_2\right)\right\}.$$

Noticing that N_3 exists according to the choice of N_2. Then we choose $N_4 > 0$ so that

$$N_4 < \min\left\{1 - \frac{c_0 \rho_1 k_1}{2\rho_2 \epsilon_2} N_2, \frac{k_3}{k_1} N_3 + k_1 \left(1 - \frac{\epsilon_2}{2}\right) N_2\right\}$$

and

$$N_4 > \max\left\{-k_3 N_2 + \frac{k_3}{k_1} N_3, 1 + k_1 N_2 - N_3\right\}.$$

The constant N_4 exists thanks to the choice of N_3, N_2 and ϵ_2. By virtue of the choice of ϵ_2, N_2, N_3 and N_4, we see that

$$\max\{l_1, l_2, l_4, l_5, l_6\} < 0. \tag{3.21}$$

At this step, we choose $\epsilon_1 = \sqrt{\frac{l_5}{l_4}}$ and we put $\tilde{l} = \frac{2k_1}{k_2 k_3} \sqrt{l_4 l_5}$ (ϵ_1 and $\tilde{l}$ are well defined from (3.21)). Then, if l satisfies the first inequality in (2.29), we get

$$l_4 + \frac{l k_2 k_3}{2k_1 \epsilon_1} < 0 \quad \text{and} \quad l_5 + \frac{l k_2 k_3 \epsilon_1}{2k_1} < 0. \tag{3.22}$$

On the other hand, from (2.20) and Young's inequality, we find that

$$\begin{aligned}&\int_0^L \left(g^0 N_4 w_x^2 + g^0 l_7 (w_x - l\varphi) w_x\right) \mathrm{d}x \\ &\quad\le g^0 \int_0^L \left(\left(N_4 + \frac{1}{2k_3}\right) w_x^2 + \frac{k_3 l_7^2}{2}(w_x - l\varphi)^2\right) \mathrm{d}x \\ &\quad\le g^0 \left(\frac{1}{k_0}\left(N_4 + \frac{1}{2k_3}\right) + \frac{l_7^2}{2}\right) \int_0^L \left(k_2\psi_x^2 + k_1(\varphi_x + \psi + lw)^2 + k_3(w_x - l\varphi)^2\right) \mathrm{d}x.\end{aligned} \tag{3.23}$$

Let

$$\tilde{g} := \frac{2k_0}{2\left(N_4 + \frac{1}{2k_3}\right) + k_0 l_7^2} \min\left\{-\frac{1}{k_2}\left(l_4 + \frac{lk_2k_3}{2k_1\epsilon_1}\right), -\frac{1}{k_3}\left(l_5 + \frac{lk_2k_3\epsilon_1}{2k_1}\right), -\frac{1}{k_1} l_6\right\}$$

($\tilde{g} > 0$ according to (3.21) and (3.22)). Because ϵ_1, l_4, l_5, l_6 and l_7 do not depend neither on l nor on g, then $\tilde{l}$ and $\tilde{g}$ do not depend neither on l nor on g. So, if g^0 satisfies the second inequality in (2.29), then

$$\begin{aligned}\lambda_0 := \max &\begin{cases} l_4 + \frac{lk_2k_3}{2k_1\epsilon_1} + g^0 k_2\left(\frac{1}{k_0}\left(N_4 + \frac{1}{2k_3}\right) + \frac{l_7^2}{2}\right), l_5 + \frac{lk_2k_3\epsilon_1}{2k_1} \\ +g^0 k_3\left(\frac{1}{k_0}\left(N_4 + \frac{1}{2k_3}\right) + \frac{l_7^2}{2}\right), \end{cases} \\ &\qquad l_6 + g^0 k_1\left(\frac{1}{k_0}\left(N_4 + \frac{1}{2k_3}\right) + \frac{l_7^2}{2}\right)\Bigg\} < 0,\end{aligned} \tag{3.24}$$

and therefore, using (3.23),

$$\begin{aligned}&\int_0^L \left(\left(l_4 + \frac{lk_2k_3}{2k_1\epsilon_1}\right)\psi_x^2 + \left(l_5 + \frac{lk_2k_3\epsilon_1}{2k_1}\right)(w_x - l\varphi)^2\right. \\ &\qquad \left. + l_6(\varphi_x + \psi + lw)^2 + g^0 N_4 w_x^2 + g^0 l_7 (w_x - l\varphi) w_x\right) \mathrm{d}x \\ &\quad\le \lambda_0 \int_0^L \left(\psi_x^2 + (w_x - l\varphi)^2 + (\varphi_x + \psi + lw)^2\right) \mathrm{d}x.\end{aligned} \tag{3.25}$$

After, because l, l_1, l_2, N_2 and N_3 do not depend on ϵ_0, we can choose $\epsilon_0 > 0$ small enough such that

$$l_1 + \epsilon_0 c_{l,N_2,N_3} < 0 \quad \text{and} \quad l_2 + \epsilon_0 c_{l,N_2,N_3} < 0, \tag{3.26}$$

and then we fix N_1 large enough so that

$$l_3 + c_{l,\epsilon_0,N_2,N_3} < 0. \tag{3.27}$$

Therefore, from (3.24), (3.26) and (3.27) we see that

$$\max\left\{\frac{1}{\rho_1}(l_1 + \epsilon_0 c_{l,N_2N_3}), \frac{1}{\rho_2}(l_2 + \epsilon_0 c_{l,N_2N_3}), \frac{1}{\rho_1}(l_3 + c_{l,N_2N_3,\epsilon_0}), \frac{\lambda_0}{k_2}, \frac{\lambda_0}{k_3}, \frac{\lambda_0}{k_1}\right\} < 0.$$

Finally, we choose $\delta_0 > 0$ small enough such that

$$\begin{aligned}\tilde{c}_1 := -2\max&\left\{\frac{1}{\rho_1}(l_1 + \epsilon_0 c_{l,N_2N_3}), \frac{1}{\rho_2}(l_2 + \epsilon_0 c_{l,N_2N_3}),\right. \\ &\left.\times \frac{1}{\rho_1}(l_3 + c_{l,N_2N_3,\epsilon_0}), \frac{\lambda_0}{k_2}, \frac{\lambda_0}{k_3}, \frac{\lambda_0}{k_1}\right\} - \delta_0 c_{l,N_1,\cdots,N_4} > 0.\end{aligned}$$

Then, using (2.23) and (2.27), and recalling that $\frac{k_2\rho_1}{k_1} - \rho_2 = 0$ (because $s_1 = s_2$ thanks to (1.3) or (2.32)), we deduce from (3.20) and (3.25) that

$$F'(t) \leq -\tilde{c}_1 E(t) + (N-c)E'(t) + c\int_0^L \int_0^{+\infty} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x + \frac{N_3\rho_1}{l}\left(\frac{k_3}{k_1} - 1\right)\int_0^L w_{xt}\varphi_t \,\mathrm{d}x. \tag{3.28}$$

Now, we estimate the integral of $g\eta_x^2$ in (3.28). When (2.25) holds, we see that, by virtue of (3.1),

$$\int_0^L \int_0^{+\infty} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \leq -\frac{2}{\alpha}E'(t). \tag{3.29}$$

When (2.26) holds, we apply Lemma 3.6 [11] (in the particular case $B = -\partial_{xx}$ and $\|\cdot\| = \|\cdot\|_{L^2(]0,L[)}$) to get the following inequality.

Lemma 3.6. *There exists a positive constant c such that, for any $\tau_0 > 0$, we have*

$$G'(\tau_0 E(t))\int_0^L \int_0^{+\infty} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \leq -cE'(t) + c\tau_0 E(t) G'(\tau_0 E(t)). \tag{3.30}$$

Proof. See Lemma 3.6 [11]. □

Using (3.29) and (3.30), we get, for the two cases (2.25) and (2.26) and for any $\tau_0 > 0$,

$$\frac{G_0(\tau_0 E(t))}{\tau_0 E(t)}\int_0^L \int_0^{+\infty} g(s)\eta_x^2 \,\mathrm{d}s\,\mathrm{d}x \leq cG_0(\tau_0 E(t)) - cE'(t) - c\frac{G_0(\tau_0 E(t))}{\tau_0 E(t)}E'(t), \tag{3.31}$$

where G_0 is defined in (2.31). By multiplying (3.28) by $\frac{G_0(\tau_0 E(t))}{E(t)}$ and combining with (3.31), we obtain, for any $\tau_0 > 0$,

$$\frac{G_0(\tau_0 E(t))}{E(t)}F'(t) \leq -(\tilde{c}_1 - c\tau_0)G_0(\tau_0 E(t)) + \left((N-c)\frac{G_0(\tau_0 E(t))}{E(t)} - c\tau_0\right)E'(t) + \frac{G_0(\tau_0 E(t))}{E(t)}\frac{N_3\rho_1}{l}\left(\frac{k_3}{k_1} - 1\right)\int_0^L w_{xt}\varphi_t \,\mathrm{d}x. \tag{3.32}$$

On the other hand, from (2.20), (2.23) and (2.27), we deduce that there exists a positive constant γ (independent of N) satisfying

$$\left|N_1 I + N_2 P + \frac{N_3}{l}K + N_4 R + J\right| \leq \gamma E,$$

which, combined with (3.19), implies that

$$(N-\gamma)E \leq F \leq (N+\gamma)E. \tag{3.33}$$

Choosing N so that $N > \max\{\gamma, c\}$ (c is the constant in (3.32)) and using (3.32), (3.33) and $E' \leq 0$, we deduce that $F \sim E$ and

$$\frac{G_0(\tau_0 E(t))}{E(t)}F'(t) \leq -(\tilde{c}_1 - c\tau_0)G_0(\tau_0 E(t)) - c\tau_0 E'(t) + \frac{G_0(\tau_0 E(t))}{E(t)}\frac{N_3\rho_1}{l}\left(\frac{k_3}{k_1} - 1\right)\int_0^L w_{xt}\varphi_t \,\mathrm{d}x. \tag{3.34}$$

Let $\tilde{\tau} > 0$ and

$$\tilde{F} = \tilde{\tau}\left(\frac{G_0(\tau_0 E(t))}{E(t)}F + c\tau_0 E(t)\right). \tag{3.35}$$

Because $\frac{G_0(\tau_0 E(t))}{E(t)}$ is non-increasing, then, thanks to (3.33),

$$c\tilde{\tau}\tau_0 E \le \tilde{F} \le \tilde{\tau}\left((N+\gamma)\frac{G_0(\tau_0 E(0))}{E(0)} + c\tau_0\right)E. \tag{3.36}$$

We have, using (3.34), (3.35) and the fact that $\frac{G_0(\tau_0 E)}{E}$ is non-increasing,

$$\tilde{F}'(t) \le -\tilde{\tau}(\tilde{c}_1 - c\tau_0)G_0(\tau_0 E(t)) + \tilde{\tau}\frac{G_0(\tau_0 E(t))}{E(t)}\frac{N_3\rho_1}{l}\left(\frac{k_3}{k_1} - 1\right)\int_0^L w_{xt}\varphi_t \,\mathrm{d}x. \tag{3.37}$$

3.2. Proof of (2.30)

Let us choose $\tilde{\tau} > 0$ such that

$$\tilde{F} \le \tau_0 E \quad \text{and} \quad \tilde{F}(0) \le 1. \tag{3.38}$$

According to (1.3), the coefficient of the integral in (3.37) vanishes, and hence, by choosing $\tau_0 > 0$ small enough such that $\tilde{c}_1 - c\tau_0 > 0$ and using the first inequality in (3.38), we get, for $c' = \tilde{\tau}(\tilde{c} - c\tau_0)$,

$$\tilde{F}' \le -c' G_0(\tilde{F}), \tag{3.39}$$

whereupon

$$(\tilde{G}(\tilde{F}))' \ge c', \tag{3.40}$$

where $\tilde{G}$ is defined in (2.31). Integrating (3.40) over $[0, t]$ yields

$$\tilde{G}(\tilde{F}(t)) \ge c't + \tilde{G}(\tilde{F}(0)). \tag{3.41}$$

Because $\tilde{F}(0) \le 1$ (from (3.38)), $\tilde{G}(1) = 0$ and $\tilde{G}$ is decreasing, we obtain from (3.41) that $\tilde{G}(\tilde{F}(t)) \ge c't$, which implies that $\tilde{F}(t) \le \tilde{G}^{-1}(c't)$. Then (3.36) gives (2.30).

3.3. Proof of (2.34)

In this section, we treat the case when (2.32) holds. We need to estimate the last integral in (3.37) using the following systems resulting from differentiating (1.1) with respect to time t:

$$\begin{cases} \rho_1\varphi_{ttt} - k_1(\varphi_{xt} + \psi_t + lw_t)_x - lk_3(w_{xt} - l\varphi_t) = 0, \\ \rho_2\psi_{ttt} - k_2\psi_{xxt} + k_1(\varphi_{xt} + \psi_t + lw_t) = 0, \\ \rho_1 w_{ttt} - k_3(w_{xt} - l\varphi_t)_x + lk_1(\varphi_{xt} + \psi_t + lw_t) \\ \quad + \displaystyle\int_0^{+\infty} g(s) w_{xxt}(x, t-s)\,\mathrm{d}s = 0, \\ \varphi_t(0,t) = \psi_{xt}(0,t) = w_{xt}(0,t) = \varphi_t(L,t) = \psi_{xt}(L,t) = w_{xt}(L,t) = 0. \end{cases} \tag{3.42}$$

Thanks to Theorem 2.2, we have, for any initial data $U^0 \in D(\mathcal{A})$, the system (3.42) has a unique solution U satisfying

$$\partial_t U \in C(\mathbb{R}_+; \mathcal{H}).$$

Let $U^0 \in D(\mathcal{A})$ and $\tilde{E}$ be the energy of (3.42) defined by

$$\tilde{E}(t) = \frac{1}{2}\|\partial_t U(t)\|^2_{\mathcal{H}}. \tag{3.43}$$

Similarly to (3.1), we have

$$\tilde{E}'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\eta_{xt}^2 \,\mathrm{d}s\,\mathrm{d}x \le 0; \tag{3.44}$$

so $\tilde{E}$ is non-increasing. We use an idea introduced in [9] to get this lemma.

Lemma 3.7. *For any $\epsilon > 0$, there exists $c_\epsilon > 0$ such that*

$$\left|\frac{N_3\rho_1}{l}\left(\frac{k_3}{k_1}-1\right)\int_0^L w_{xt}\varphi_t \,\mathrm{d}x\right| \le c_\epsilon \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2 \,\mathrm{d}s\,\mathrm{d}x + \epsilon E(t) - c_\epsilon E'(t). \tag{3.45}$$

Proof. We have, by the definition of η,

$$\begin{aligned} &\tfrac{N_3\rho_1}{l}\left(\tfrac{k_3}{k_1}-1\right)\int_0^L \varphi_t w_{xt}\,\mathrm{d}x \\ &\quad= \tfrac{N_3\rho_1}{g^0 l}\left(\tfrac{k_3}{k_1}-1\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\eta_{xt}\,\mathrm{d}s\,\mathrm{d}x \\ &\qquad+ \tfrac{N_3\rho_1}{g^0 l}\left(\tfrac{k_3}{k_1}-1\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)w_{xt}(t-s)\,\mathrm{d}s\,\mathrm{d}x. \end{aligned} \tag{3.46}$$

Using (3.5) (for $f = \eta_{xt}$ and $v = \varphi_t$) and (2.27), we get, for all $\epsilon > 0$,

$$\begin{aligned} &\left|\frac{N_3\rho_1}{g^0 l}\left(\frac{k_3}{k_1}-1\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)w_{xt}\eta\,\mathrm{d}s\,\mathrm{d}x\right| \\ &\quad\le \frac{\epsilon}{2}E(t) + c_\epsilon \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2\,\mathrm{d}s\,\mathrm{d}x. \end{aligned} \tag{3.47}$$

On the other hand, by integrating with respect to s and using the definition of η, we obtain

$$\begin{aligned} \int_0^L \varphi_t \int_0^{+\infty} g(s)w_{xt}(t-s)\,\mathrm{d}s\,\mathrm{d}x &= -\int_0^L \varphi_t \int_0^{+\infty} g(s)\partial_s(w_x(t-s))\,\mathrm{d}s\,\mathrm{d}x \\ &= \int_0^L \varphi_t\left(g(0)w_x(t) + \int_0^{+\infty} g'(s)w_x(t-s)\,\mathrm{d}s\right)\mathrm{d}x \\ &= -\int_0^L \varphi_t \int_0^{+\infty} g'(s)\eta_x\,\mathrm{d}s\,\mathrm{d}x. \end{aligned}$$

Therefore, using (3.6) (for $f = \eta_x$ and $v = \varphi_t$) and (3.1),

$$\left|\frac{N_3\rho_1}{l}\left(\frac{k_3}{k_1}-1\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)w_{xt}(t-s)\,\mathrm{d}s\,\mathrm{d}x\right| \le \frac{\epsilon}{2}E(t) - c_\epsilon E'(t). \tag{3.48}$$

Inserting (3.47) and (3.48) into (3.46), we obtain (3.45). □

Now, by combining (3.45), (3.37) with $\tilde{\tau} = 1$ and choosing $\epsilon = \frac{\tilde{c}_1}{2}$, we get

$$\tilde{F}'(t) \leq -\left(\frac{\tilde{c}_1}{2} - c\tau_0\right) G_0(\tau_0 E(t)) - c\frac{G_0(\tau_0 E(0))}{E(0)} E' + c\frac{G_0(\tau_0 E(t))}{E(t)} \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2 \,\mathrm{d}s\,\mathrm{d}x. \tag{3.49}$$

Similarly to (3.29) and (3.30), using (3.44), we find

$$\int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2 \,\mathrm{d}s\,\mathrm{d}x \leq -\frac{2}{\alpha}\tilde{E}'(t) \tag{3.50}$$

when (2.25) holds. When (2.26) holds, there exists a positive constant c such that, for any $\tau_0 > 0$, we have as for (3.30) (see the proof of Lemma 3.6 [11])

$$G'(\tau_0 E(t)) \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2 \,\mathrm{d}s\,\mathrm{d}x \leq -c\tilde{E}'(t) + c\tau_0 E(t) G'(\tau_0 E(t)). \tag{3.51}$$

From (3.50) and (3.51), we find that, in both cases (2.25) and (2.26),

$$\frac{G_0(\tau_0 E(t))}{\tau_0 E(t)} \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2 \,\mathrm{d}s\,\mathrm{d}x \leq cG_0(\tau_0 E(t)) - c\frac{G_0(\tau_0 E(t))}{\tau_0 E(t)}\tilde{E}'(t) - c\tilde{E}'(t). \tag{3.52}$$

Inserting (3.52) in (3.49), choosing $\tau_0 > 0$ small enough such that $\frac{\tilde{c}_1}{2} - c\tau_0 > 0$, and using the fact that $\frac{G_0(\tau_0 E)}{E}$ is non-increasing, we find, for some $\tilde{c}_2 > 0$,

$$G_0(\tau_0 E(t)) \leq -\tilde{c}_2 \tilde{F}'(t) - c\left(1 + \frac{G_0(\tau_0 E(0))}{E(0)}\right)\left(E'(t) + \tilde{E}'(t)\right). \tag{3.53}$$

By integration with respect to t and using (3.36), we get, for some $\tilde{c}_3 > 0$,

$$\int_S^T G_0(\tau_0 E(t))\,dt \leq \tilde{c}_3\left(1 + \frac{G_0(\tau_0 E(0))}{E(0)}\right)\left(E(S) + \tilde{E}(S)\right), \quad \forall T \geq S \geq 0. \tag{3.54}$$

Choosing $S = 0$ in (3.54) and using the fact that $G_0(\tau_0 E)$ is non-increasing, we obtain

$$G_0(\tau_0 E(T))T \leq \int_0^T G_0(E(t))\,dt \leq \tilde{c}_3\left(1 + \frac{G_0(\tau_0 E(0))}{E(0)}\right)\left(E(0) + \tilde{E}(0)\right). \tag{3.55}$$

Because G_0^{-1} is increasing, (2.34) for $n = 1$ is deduced from (3.55) with

$$c_1 = \max\left\{\frac{1}{\tau_0}, \tilde{c}_3\left(1 + \frac{G_0(\tau_0 E(0))}{E(0)}\right)\left(E(0) + \tilde{E}(0)\right)\right\}.$$

By induction on n, one can prove that (2.34) holds, for $n = 2, 3, \cdots$; see [11] and [13].

Remark 3.8. We give in this remark some general comments and open problems.

1. Our stability results (2.30) and (2.34) hold under the smallness conditions (2.29) on l and g^0, and the boundedness conditions (2.28) and (2.33) on the initial data η^0. It is interesting to drop these conditions or determine the biggest values of $\tilde{l}$ and $\tilde{g}$ in (2.29) for which (2.30) and (2.34) hold.
2. Another interesting question concerns the stability of (1.1) when $s_1 \neq s_2$.
3. The case where only one infinity memory is considered in the vertical displacements; that is the integral in (1.1) is replaced by

$$\int_0^{+\infty} g(s)\varphi_{xx}(x,t-s)\,\mathrm{d}s$$

and considered on the first equation in (1.1), seems very delicate. The particular case of Timoshenko systems ((1.1) with $l = 0$) under infinity memory and/or frictional damping in the vertical displacements was studied in [14] and some stability estimates were proved.

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