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WELL-POSEDNESS AND ASYMPTOTIC STABILITY FOR THE LAMÉ SYSTEM WITH INFINITE MEMORIES IN A BOUNDED DOMAIN

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Abstract. In this work, we consider the Lamé system in 3-dimension bounded domain with infinite memories. We prove, under some appropriate assumptions, that this system is well-posed and stable, and we get a general and precise estimate on the convergence of solutions to zero at infinity in terms of the growth of the infinite memories.

1. Introduction and position of the problem.

Let Ω be a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$. Let us consider the following Lamé system with infinite memories:

$$\begin{cases} u'' - \Delta_e u + \displaystyle\int_0^{+\infty} g(s)\Delta u(t-s)ds = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ u = 0, & \text{on } \partial\Omega \times \mathbb{R}_+ \end{cases} \tag{1}$$

with initial conditions

$$\begin{cases} u(x,-t) = u_0(x,t), & \text{in } \Omega \times \mathbb{R}_+, \\ u'(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{2}$$

where $' = \frac{\partial}{\partial t}$ and u_0 and u_1 are given history and initial data. Here Δ denotes the Laplacian operator and Δ_e denotes the elasticity operator, which is the 3×3 matrix-valued differential operator defined by

$$\Delta_e u = \mu\Delta u + (\lambda+\mu)\nabla \operatorname{div} u, \quad u = (u_1, u_2, u_3)^T$$

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and λ and μ are the Lamé constants which satisfy the conditions

$$\mu > 0, \quad \lambda + \mu \geq 0. \tag{3}$$

Moreover,

$$g(s) = \begin{pmatrix} g_1(s) & 0 & 0 \\ 0 & g_2(s) & 0 \\ 0 & 0 & g_3(s) \end{pmatrix},$$

where $g_i : \mathbb{R}_+ \to \mathbb{R}_+$ are given functions which represent the dissipative terms. In the particular case $\lambda + \mu = 0$, $\Delta_e = \mu\Delta$ gives a vector Laplacian; that is (1) describes the vector wave equation.

The problem of well-posedness and stability and/or the obtention of bounded estimates for elasticity systems in general, and the Lamé system in particular, has attracted considerable attention in recent years, where diverse types of dissipative mechanisms have been introduced and several stability and boundedness results have been obtained. The main problem concerning the stability and/or boundedness of estimates of solutions in the presence of finite or infinite memory is to determine the largest class of memory functions which guarantees the stability and/or boundedness of estimates for the system, and the best estimate on the decay rate and/or the bound for solutions in terms of the memory function. Let us recall here some known results in this direction related to our goals, addressing problems of existence, uniqueness and asymptotic behavior of solutions.

1. Damping controls. Real progress has been realized during the last three decades, in particular, in the works of Lagnese [19, 20], Komornik [18], Martinez [21], Aassila [1], Alabau and Komornik [2], Horn [14, 15], Guesmia [8, 9], and Bchatnia and Daoulatli [4]. In [19], Lagnese proved some uniform stability results of elasticity systems with linear feedback and under some technical assumptions on the elasticity tensor. In particular, these results do not hold in the linear homogeneous isotropic case for which the elasticity tensor depends on two parameters called Lamé constants. In [20], Lagnese obtained uniform stability estimates for linear homogeneous isotropic and bidimensional elasticity systems under a linear boundary feedback. Komornik [18] proved the same estimates for the homogeneous isotropic system in 1-dimension and 2-dimension and under a linear boundary feedback. The estimates of Komornik [18] are even optimal when the domain is a ball from $\mathbb{R}^3$. Martinez [21] generalized the results of Komornik [18] to the case of elasticity systems of cubic crystals under a nonlinear boundary feedback. For these systems, the elasticity tensor depends on three parameters.

Aassila [1] proved the strong stability of a homogeneous isotropic elasticity system with an internal nonlinear feedback in domains of finite Lebesgue measure, but no stability estimate on the decay rate of solutions was given. Alabau and Komornik [2] studied an anisotropic elasticity system with constant coefficients and linear boundary feedback. Under certain geometric conditions, they obtained some exact controllability and uniform stability results, where the decay rate of solutions is given explicitly in terms of the parameters of the system. The proof of [2] is based on the multipliers method and some new identities. Horn [14, 15] obtained some stability results for homogeneous isotropic elasticity systems under weaker geometric conditions. The key of the proof in [14, 15] is a combination of the multipliers method and the microlocal analysis. Guesmia [8, 9] considered the problem of observability, exact controllability and stability of general elasticity systems with variable coefficients depending on both time and space variables in bounded domains or of a finite Lebesgue measure.

The results of [8, 9] hold under linear or nonlinear, global or local feedbacks, and they generalize and improve, in some cases, the decay rate obtained by Alabau and Komornik [2]. Recently, Bchatnia and Daoulatli [4] considered the case of the Lamé system in a three-dimensional bounded domain with local nonlinear damping and external force, and obtained several boundedness and stability estimates depending on the growth of the damping and the external forces. The control region considered in [4] satisfies the famous geometric optical condition (GOC).

For the stability of other kind of coupled hyperbolic systems, let us mention the following results. Guesmia [7] considered a coupled wave-Petrovsky system with two nonlinear internal dampings, and showed some polynomial and exponential stability estimates. Alabau, Cannarsa and Komornik [3] considered a coupled system of two abstract hyperbolic equations with linear weak coupling of order zero and only one damping acting on the first equation, and proved that this system is not exponentially stable and the asymptotic behavior of solutions is at least of polynomial type with decay rates depending on the smoothness of initial data. The method introduced and developed in [3] is based on a general estimate on the asymptotic behavior of solutions in terms of higher order initial energies. Some extensions of the results of [3] to the nonlinear and nondissipative cases are given by Guesmia [10] in the particular case of coupled wave equations. Recently, the stability of a coupled Euler-Bernoulli and wave equations with linear weak coupling and clamped boundary conditions for the Euler-Bernoulli equation was considered in Tebou [23]. The decay estimates obtained in [23] are of polynomial type with decay rates smaller than the ones obtained in [3], but the abstract framework introduced in [3] does not include the case considered in [23]. See also the references of [3, 10, 16, 23] for further results related to the stability of coupled hyperbolic equations.

2. Memory controls. The asymptotic stability with finite or infinite memories of hyperbolic partial differential equations has been the subject of many works in the last few years. Let us mention here some works in this direction.

In the case where the memory function converges exponentially to zero, it was proved that the system is exponentially stable; that is, the solution converges exponentially to zero (see [6] and the references therein for abstract dissipative systems). When g does not converge exponentially to zero at infinity, the stability of such systems has been proved in [11], where general decay estimates depending on the growth of the memory function at infinity were obtained. The approach of [11] was applied in [12, 13] to, respectively, the wave equation and different kinds of Timoshenko systems. See [11, 12, 13] for more known results in the literature concerning the stability with finite or infinite memory.

In all stability results with memory cited above, the coupling terms are not a part of the principal operator but additional terms in the system. Concerning the Lamé system with infinite memories (1)-(2) considered in this work, the unique coupling is given in the principal operator, and as far as we know, there is no stability and/or boundedness results in the literature. Our aim in this work is to prove that the stability and/or boundedness of our system holds with infinite memories and to obtain a general decay connection (exponential, polynomial, or others) between the decay rates of the solutions and the growth of the memory functions.

The paper is organized as follows: in Section 2, we prove the global existence and uniqueness of solutions of (1)-(2). Section 3 is devoted to state the main results of this work, that is, the stability of the system (1)-(2). Finally, in Section 4, we prove the stability results.

2. **Well-posedness.** In this section, we prove the existence and uniqueness of solutions of (1)-(2) using semigroup theory. We consider the following hypothesis:

(H1) The functions g_i are nonnegative, differentiable and nonincreasing such that

$$\mu - \int_0^{+\infty} g_i(s)ds > 0, \qquad i = 1, 2, 3. \tag{4}$$

Following the idea of [5], we consider

$$\eta(x,t,s) = u(x,t) - u(x,t-s), \quad \text{in } \Omega \times \mathbb{R}_+ \times \mathbb{R}_+. \tag{5}$$

Consequently, we obtain

$$\begin{cases} \eta(x,t,0) = 0, & \text{in } \Omega \times \mathbb{R}_+, \\ \eta(x,t,s) = 0, & \text{on } \partial\Omega \times \mathbb{R}_+ \times \mathbb{R}_+, \\ \eta_0(x,s) := \eta(x,0,s) = u_0(x,0) - u_0(x,s), & \text{in } \Omega \times \mathbb{R}_+. \end{cases} \tag{6}$$

Clearly, (5) gives

$$\eta_t(x,t,s) + \eta_s(x,t,s) = u'(x,t), \quad \text{in } \Omega \times \mathbb{R}_+ \times \mathbb{R}_+, \tag{7}$$

where $\eta_t = \dfrac{\partial \eta}{\partial t}$ and $\eta_s = \dfrac{\partial \eta}{\partial s}$. By combining (1) and (5), we obtain the following equation:

$$\begin{aligned} u'' - \left(\mu Id - \int_0^{+\infty} g(s)ds\right) \triangle u - (\lambda+\mu)\nabla \operatorname{div} u \\ - \int_0^{+\infty} g(s)\Delta\eta(x,t,s)ds = 0, \qquad \text{in } \Omega \times \mathbb{R}_+, \end{aligned} \tag{8}$$

where $Id = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$.

Let $\mathcal{H} = (H_0^1(\Omega))^3 \times (L^2(\Omega))^3 \times L_g$, where

$$\begin{aligned} L_g = \quad & \left\{ v = (v_1, v_2, v_3)^T : \mathbb{R}_+ \longrightarrow \left(H_0^1(\Omega)\right)^3, \right. \\ & \left. \int_0^{+\infty} g_i(s) \int_\Omega |\nabla v_i(s)|^2 dxds < +\infty, \ i = 1, 2, 3. \right\} \end{aligned}$$

The set L_g is a Hilbert space endowed with the inner product, for $v = (v_1, v_2, v_3)^T$ and $w = (w_1, w_2, w_3)^T$ in L_g,

$$\langle v, w \rangle_{L_g} = \sum_{i=1}^3 \int_0^{+\infty} g_i(s) \int_\Omega \nabla v_i(s) \cdot \nabla w_i(s) dxds.$$

Thanks to (3) and (4), the set $\mathcal{H}$ is also a Hilbert space endowed with the inner product defined, for $v = (v_1, v_2, v_3)^T \in \mathcal{H}$ and $w = (w_1, w_2, w_3)^T \in \mathcal{H}$, by

$$\langle v, w \rangle_{\mathcal{H}} = \int_\Omega \left[\sum_{i=1}^3 \left((\mu - \alpha_i) \nabla v_1^i \cdot \nabla w_1^i + v_2^i \cdot w_2^i \right) + (\lambda+\mu) \operatorname{div} v_1 \operatorname{div} w_1 \right] dx + \langle v_3, w_3 \rangle_{L_g},$$

where we denote $v_i = (v_i^1, v_i^2, v_i^3)$, $w_i = (w_i^1, w_i^2, w_i^3)$ and $\alpha_i = \displaystyle\int_0^{+\infty} g_i(s)ds$.

Now, let $U = (u, u', \eta)^T$ and $U_0 = (u_0(\cdot,0), u_1, \eta_0)^T$. Thanks to (7) and (8), (1)-(2) is equivalent to the abstract linear first-order Cauchy problem

$$\begin{cases} U'(t) = AU(t), & \text{on } \mathbb{R}_+, \\ U(0) = U_0, \end{cases} \tag{9}$$

where A is the linear operator defined by

$$A = \begin{pmatrix} 0 & 1 & 0 \\ \Delta_e - \left(\int_0^{+\infty} g(s)ds\right)\Delta & 0 & \int_0^{+\infty} g(s)\Delta ds \\ 0 & 1 & -\partial_s \end{pmatrix},$$

where $\partial_s = \frac{\partial}{\partial s}$. The domain $D(A)$ of A is given by

$$D(A) = \left\{ V = (v_1, v_2, v_3)^T \in \mathcal{H},\ AV \in \mathcal{H} \text{ and } v_3(0) = 0 \right\}$$

and endowed with the graph norm

$$\|V\|_{D(A)} = \|V\|_{\mathcal{H}} + \|AV\|_{\mathcal{H}}.$$

Now, we prove that $A : D(A) \longrightarrow \mathcal{H}$ is a maximal monotone operator; that is, A is dissipative and $Id - A$ is surjective. Indeed, a simple calculation implies that, for any $V = (v_1, v_2, v_3)^T \in D(A)$,

$$\begin{aligned} &\langle AV, V\rangle_{\mathcal{H}} \qquad (10) \\ = &\left\langle \begin{pmatrix} v_2 \\ \Delta_e v_1 - \left(\int_0^{+\infty} g(s)ds\right)\Delta v_1 + \int_0^{+\infty} g(s)\Delta v_3 ds \\ v_2 - \partial_s v_3 \end{pmatrix}, \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \right\rangle_{\mathcal{H}} \\ = &\int_\Omega [\sum_{i=1}^3 (\mu - \alpha_i)\nabla v_2^i \cdot \nabla v_1^i + \left(\Delta_e v_1^i - \left(\int_0^{+\infty} g_i(s)ds\right)\Delta v_1^i\right. \\ &\left. + \sum_{i=1}^3 \int_0^{+\infty} g_i(s)\Delta v_3^i ds\right) \cdot v_2^i + (\lambda + \mu)\operatorname{div} v_2 \operatorname{div} v_1]dx \\ &+ \sum_{i=1}^3 \int_0^{+\infty} g_i(s)\int_\Omega \nabla(v_2^i - \partial_s v_3^i)\nabla v_3^i(s)dxds \\ = &-\sum_{i=1}^3 \int_0^{+\infty} g_i(s)\int_\Omega \partial_s\left(\nabla v_3^i\right)\nabla v_3^i(s)dxds \\ = &\frac{1}{2}\sum_{i=1}^3 \int_0^{+\infty} g_i'(s)\int_\Omega |\nabla v_3^i|^2 dxds \le 0, \end{aligned}$$

since g_i is nonincreasing. This implies that A is dissipative. On the other hand, we prove that $Id - A$ is surjective; that is, for any $W = (w_1, w_2, w_3) \in \mathcal{H}$, there exists $V = (v_1, v_2, v_3) \in D(A)$ satisfying

$$(Id - A)V = W. \tag{11}$$

The first and last equations of (11) are equivalent to

$$v_2 = v_1 - w_1 \tag{12}$$

and

$$v_3 + \partial_s v_3 = v_1 - w_1 + w_3. \tag{13}$$

By integrating the equation (13) with respect to s and noting that $v_3(0)=0$, we obtain

$$v_3(s)=\left(\int_0^s (v_1-w_1+w_3(\tau))\, e^{\tau} d\tau\right) e^{-s}. \tag{14}$$

Using (12) and (14), the second equation of (11) becomes

$$\begin{aligned} & v_1-\left[\Delta_e-\left(\int_0^{+\infty} e^{-s} g(s) ds\right) \Delta\right] v_1 \qquad (15)\\ = \;& w_1+w_2+\int_0^{+\infty} e^{-s} g(s) \Delta\left(\int_0^s e^{\tau}(w_3(\tau)-w_1) d\tau\right) ds. \end{aligned}$$

It is sufficient to prove that (15) has a solution v_1 in $\left(H^2(\Omega) \cap H_0^1(\Omega)\right)^3$, and then we replace in (12) and (14) to conclude that (11) has a solution $V \in D(A)$. So we multiply (15) by a test function $\varphi_1 \in \left(H_0^1(\Omega)\right)^3$ and we integrate by parts, obtaining the following variational formulation of (15):

$$a(v_1, \varphi_1)=l(\varphi_1), \qquad \forall \varphi_1 \in \left(H_0^1(\Omega)\right)^3, \tag{16}$$

where

$$a(v_1, \varphi_1)=\int_{\Omega}\left[v_1 \varphi_1+\sum_{i=1}^3\left((\mu-\alpha_i) \nabla v_1^i \cdot \nabla \varphi_1^i\right)+(\lambda+\mu) \operatorname{div} v_1 \operatorname{div} \varphi_1\right] dx$$

and

$$l(\varphi_1)=\int_{\Omega}(w_1+w_2) \cdot \varphi_1-\left(\int_0^{+\infty} e^{-s} g(s)\left(\int_0^s e^{\tau} \nabla(w_3(\tau)-w_1) d\tau\right) ds\right) \cdot \nabla \varphi_1 dx.$$

It is clear that a is a bilinear and continuous form on $\left(H_0^1(\Omega)\right)^3 \times\left(H_0^1(\Omega)\right)^3$, and l is a linear and continuous form on $\left(H_0^1(\Omega)\right)^3$. On the other hand, (3) and (4) imply that there exists a positive constant a_0 such that

$$\begin{aligned} a(v_1, v_1) &= \int_{\Omega}\left[|v_1|^2+\sum_{i=1}^3(\mu-\alpha_i)|\nabla v_i|^2 dx+(\lambda+\mu)|\operatorname{div} v_1|^2\right] dx \\ &\geq a_0\|v_1\|_{\left(H_0^1(\Omega)\right)^3}^2, \quad \forall v_1 \in \left(H_0^1(\Omega)\right)^3, \end{aligned}$$

which implies that a is coercive. Therefore, using the Lax-Milgram Theorem, we conclude that (16) has a unique solution v_1 in $\left(H_0^1(\Omega)\right)^3$. By classical regularity arguments, we conclude that the solution v_1 of (16) belongs into $\left(H^2(\Omega) \cap H_0^1(\Omega)\right)^3$ and satisfies (15). Consequently, using (12) and (14), we deduce that (11) has a unique solution $V \in D(A)$. This proves that $Id-A$ is surjective.

Finally, using the Lummer-Phillips Theorem (see [22]), we find that A is an infinitesimal generator of a linear C_0-semigroup on $\mathcal{H}$. Consequently, applying semigroup theory to (9) (see [17, 22]), we get the following well-posedness results of (1)-(2):

Theorem 2.1. *Assume that (3) and (H1) are satisfied. Then, for any*

$$(u_0(\cdot, 0), u_1) \in\left(H_0^1(\Omega)\right)^3 \times\left(L^2(\Omega)\right)^3,$$

the system (1)-(2) has a unique weak solution

$$u \in \mathcal{C}\left(\mathbb{R}_{+},\left(H_0^1(\Omega)\right)^3\right) \cap \mathcal{C}^1\left(\mathbb{R}_{+},\left(L^2(\Omega)\right)^3\right).$$

Moreover, if $(u_0(\cdot,0),u_1) \in \left(H^2(\Omega)\cap H_0^1(\Omega)\right)^3 \times \left(H_0^1(\Omega)\right)^3$, *the solution of (1)-(2) is classical; that is*

$$u \in \mathcal{C}\left(\mathbb{R}_+, \left(H^2(\Omega)\cap H_0^1(\Omega)\right)^3\right) \cap \mathcal{C}^1\left(\mathbb{R}_+, \left(H_0^1(\Omega)\right)^3\right) \cap \ \mathcal{C}^2\left(\mathbb{R}_+, \left(L^2(\Omega)\right)^3\right).$$

3. **Stability.** In this section, we state our stability results for problem (1)-(2). For this purpose, we start with the following hypotheses:

(H2)

$$\alpha_i := \int_0^{+\infty} g_i(s)ds > 0, \qquad i = 1,2,3. \tag{17}$$

(H3) For any $i = 1,2,3$,

$$\exists \gamma_i > 0, \quad g_i'(s) \le -\gamma_i g_i(s), \quad \forall s \in \mathbb{R}_+ \tag{18}$$

or there exists an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+)\cap C^2(]0,+\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t\to+\infty} G'(t) = +\infty,$$

and

$$\int_0^{+\infty} \frac{g_i(s)}{G^{-1}(-g_i'(s))} ds + \sup_{s\in\mathbb{R}_+} \frac{g_i(s)}{G^{-1}(-g_i'(s))} < +\infty. \tag{19}$$

Remark 1. The condition (19) introduced in [11] is satisfied by any positive function g_i of class $C^1(\mathbb{R}_+)$ with $g_i' < 0$ and g_i is integrable on $\mathbb{R}_+$ (see [11, 12, 13] for explicit examples).

The classical energy of any weak solution u of (1)-(2) at time t is defined by

$$E_u(t) = \frac{1}{2}\int_\Omega \left(\sum_{k=1}^{3}(\mu-\alpha_k)\left|\nabla u_k\right|^2 + (\lambda+\mu)\left|\operatorname{div} u\right|^2 + \left|u'\right|^2\right)dx. \tag{20}$$

We define the "modified" energy functional of the weak solution u by

$$E(t) = E_u(t) + \frac{1}{2} g\circ\nabla\eta,$$

where

$$\begin{aligned} g\circ\nabla\eta &= \int_\Omega\int_0^{+\infty} Tr\left(g(s)\nabla\eta\cdot\nabla\eta\right)dsdx \\ &= \sum_{i=1}^{3}\int_0^{+\infty} g_i(s)\int_\Omega |\nabla u_i(x,t)-\nabla u_i(x,t-s)|^2 dxds. \end{aligned} \tag{21}$$

Now, we give our main stability results.

Theorem 3.1. *Assume that (3) and (H1)-(H3) are satisfied such that (18) holds or there exists a positive constant* m_i *satisfying*

$$\int_\Omega \left|\nabla\eta_0^i\right|^2 dx \le m_i, \ \forall s\in\mathbb{R}_+. \tag{22}$$

Then there exist positive constants c', c'' *and* ϵ_0 *for which* E *satisfies*

$$E(t) \le c'' e^{-c't}, \quad \forall t\in\mathbb{R}_+ \tag{23}$$

if (18) is satisfied, for any $i=1,2,3$, *and*

$$E(t) \le c'' G_1^{-1}(c't), \quad \forall t\in\mathbb{R}_+ \tag{24}$$

otherwise, where

$$G_1(s) = \int_s^1 \frac{1}{\tau G'(\epsilon_0 \tau)} d\tau \ (s \in]0,1]). \tag{25}$$

Remark 2. Theorem 3.1 shows that the exponential stability (23) of (1)-(2) holds when each function g_i satisfies (18) (which implies that g_i converges exponentially to zero at infinity). Otherwise, the weak stability (24) holds. For precise examples illustrating (24), see [11, 12, 13].

4. **Proof of Theorem 3.1.** First, we prove Lemmas 4.1-4.3 for classical solutions and we note that these results remain valid for any weak solution by simple density arguments. These Lemmas are well known in the case of the wave equation or Timoshenko systems, see, for example, [11, 12, 13] and the references therein. On the other hand, we can assume that $E(t) > 0$, for any $t \in \mathbb{R}_+$, without loss of generality. Otherwise, if $E(t_0) = 0$, for some $t_0 \in \mathbb{R}_+$, then $E(t) = 0$, for all $t \geq t_0$, because E is positive and nonincreasing, and then (23) and (24) are satisfied.

Lemma 4.1. *The "modified" energy functional satisfies, along the solution u of (1)-(2),*

$$E'(t) = \frac{1}{2} g' \circ \nabla \eta \leq 0, \tag{26}$$

where

$$g' \circ \nabla \eta = \sum_{i=1}^{3} \int_0^{+\infty} g_i'(s) \int_\Omega |\nabla u_i(x,t) - \nabla u_i(x,t-s)|^2 dx ds.$$

Proof. By multiplying (1) by u', integrating over Ω and using integration by parts, we get easily (26) (as in (10)). □

Lemma 4.2. *The functional*

$$\Phi(t) = \int_\Omega u \cdot u' dx \tag{27}$$

satisfies, along the solution u of (1)-(2) and for any $\varepsilon > 0$ and for some positive constant c_1,

$$\Phi'(t) \leq \int_\Omega |u'|^2 dx - \sum_{i=1}^{3} (\mu - \varepsilon - \alpha_i) \int_\Omega |\nabla u_i|^2 dx - \int_\Omega (\lambda + \mu) |\operatorname{div} u|^2 dx + \frac{c_1}{\varepsilon} g \circ \nabla \eta, \qquad \forall t \geq 0. \tag{28}$$

Proof. By differentiating (27) and using (8), Young's inequality and (21), we obtain

$$\begin{aligned} \Phi'(t) &= \int_\Omega |u'|^2 dx - \mu \int_\Omega |\nabla u_i|^2 dx - \int_\Omega (\lambda + \mu) |\operatorname{div} u|^2 dx \\ &\quad + \int_0^{+\infty} g(s) \int_\Omega \nabla u \cdot (\nabla u - \nabla \eta) dx ds \\ &= \int_\Omega |u'|^2 dx - \sum_{i=1}^{3} (\mu - \alpha_i) \int_\Omega |\nabla u_i|^2 dx - \int_\Omega (\lambda + \mu) |\operatorname{div} u|^2 dx \end{aligned}$$

$$
\begin{aligned}
&-\sum_{i=1}^{3}\int_0^{+\infty} g_i(s)\int_\Omega \nabla u_i\cdot\nabla\eta_i dxds\\
\leq\ & \int_\Omega |u'|^2dx-\sum_{i=1}^{3}(\mu-\alpha_i)\int_\Omega|\nabla u_i|^2dx-\int_\Omega(\lambda+\mu)|\operatorname{div}u|^2dx\\
&+\varepsilon\int_\Omega|\nabla u|^2dx+\frac{c_1}{\varepsilon}g\circ\nabla\eta,
\end{aligned}
$$

which gives (28). □

Lemma 4.3. *The functional*

$$
\Psi(t)=-\sum_{i=1}^{3}\int_0^{+\infty} g_i(s)\int_\Omega u_i'\eta_i dxds \tag{29}
$$

satisfies, along the solution u of (1)-(2), for any $\varepsilon_1,\varepsilon_2>0$ and for some positive constant $c_2>0$,

$$
\begin{aligned}
\Psi'(t)\ \leq\ & -\sum_{i=1}^{3}(\alpha_i-\varepsilon_1)\int_\Omega|u_i'|^2dx \\
&+\varepsilon_2\left(\int_\Omega|\nabla u|^2dx+\int_\Omega|\operatorname{div}u|^2dx\right)-\frac{c_2}{\varepsilon_1}g'\circ\nabla\eta+\frac{c_2}{\varepsilon_2}g\circ\nabla\eta,\ \ \forall t\geq 0.
\end{aligned} \tag{30}
$$

Proof. Multiplying (1) by $\int_0^{+\infty} g(s)\eta(t,s)ds$ and integrating over Ω, we get

$$
\begin{aligned}
0\ =\ & \sum_{i=1}^{3}\int_0^{+\infty} g_i(s)\int_\Omega u_i''\eta_i dxds \\
&-\sum_{i=1}^{3}\int_\Omega(\mu\Delta u_i+(\lambda+\mu)\nabla\operatorname{div}u_i)\left(\int_0^{+\infty}g_i(s)\eta_i ds\right)dx\\
&+\sum_{i=1}^{3}\int_\Omega\left(\int_0^{+\infty}g_i(s)\Delta u_i(t-s)ds\right)\left(\int_0^{+\infty}g_i(s)\eta_i ds\right)dx.
\end{aligned} \tag{31}
$$

Using (7), we get

$$
\begin{aligned}
\Psi'(t)\ =\ & -\sum_{i=1}^{3}\int_0^{+\infty} g_i(s)\int_\Omega u_i''\eta_i dxds-\sum_{i=1}^{3}\int_\Omega u_i'\int_0^{+\infty}g_i(s)\partial_t\eta_i dsdx\\
=\ & -\sum_{i=1}^{3}\int_0^{+\infty} g_i(s)\int_\Omega u_i''\eta_i dxds-\sum_{i=1}^{3}\alpha_i\int_\Omega|u_i'|^2dx\\
&+\sum_{i=1}^{3}\int_\Omega u_i'\int_0^{+\infty}g_i(s)\partial_s\eta_i dsdx.
\end{aligned}
$$

By integrating by parts with respect to s in the last term of this equality, we obtain (note that $\eta_i(x,t,0)=0$ and $\lim_{s\to+\infty}g_i(s)\eta_i(x,t,s)=0$ thanks to (6) and because η_s

$\in L_g$)

$$\Psi'(t) = -\sum_{i=1}^{3}\int_\Omega u_i''\int_0^{+\infty} g_i(s)\eta_i ds dx - \sum_{i=1}^{3}\alpha_i\int_\Omega |u_i'|^2 dx - \sum_{i=1}^{3}\int_\Omega u_i'\int_0^{+\infty} g_i'(s)\eta_i ds dx. \tag{32}$$

Combining (31) and (32), we obtain

$$\begin{aligned}\Psi'(t) = & -\sum_{i=1}^{3}\alpha_i\int_\Omega |u_i'|^2 dx - \sum_{i=1}^{3}\int_\Omega u_i'\int_0^{+\infty} g_i'(s)\eta_i ds dx \\ & +\sum_{i=1}^{3}\int_\Omega (\mu\nabla u_i + (\lambda+\mu)\operatorname{div} u_i)\left(\int_0^{+\infty} g_i(s)\nabla\eta_i ds\right)dx \\ & -\sum_{i=1}^{3}\int_\Omega\left(\alpha_i\nabla u_i - \int_0^{+\infty} g_i(s)\nabla\eta_i ds\right)\left(\int_0^{+\infty} g_i(s)\nabla\eta_i ds\right)dx.\end{aligned}$$

By applying Cauchy-Schwarz inequality, Young's inequality and Poincaré inequality (allowed by the boundary condition in (6)) for the last three terms of this equality and using (4), we obtain (30). □

Now, we prove our main stability results (23) and (24).

Proof of Theorem 3.1. Let $L = NE + M\Phi + \Psi$, for M, $N > 0$. By definition of Φ, Ψ and E, there exist two constants d_1 and d_2 such that $|\Phi| \le d_1 E$ and $|\Psi| \le d_2 E$. Therefore,

$$(N - Md_1 - d_2)E \le L \le (N + Md_1 + d_2)E.$$

Then, for

$$N > Md_1 + d_2, \tag{33}$$

we get $L \sim E$. On the other hand, (26), (28) and (30) imply that

$$\begin{aligned} & L'(t) \\ \le & \left(\frac{N}{2} - \frac{c_2}{\varepsilon_1}\right) g'\circ\nabla\eta - \sum_{i=1}^{3}(\alpha_i - \varepsilon_1 - M)\int_\Omega |u_i'|^2 dx + \left(\frac{c_2}{\varepsilon_2} + \frac{Mc_1}{\varepsilon}\right) g\circ\nabla\eta \\ & -\sum_{i=1}^{3}(M(\mu - \alpha_i - \varepsilon) - (1+\hat{c})\varepsilon_2)\int_\Omega |\nabla u_i|^2 dx - (\lambda+\mu)M\int_\Omega |\operatorname{div} u|^2 dx,\end{aligned}$$

where $\hat{c} > 0$ satisfies

$$\int_\Omega |\operatorname{div} u|^2 dx \le \hat{c}\int_\Omega |\nabla u|^2 dx.$$

We choose $0 < \varepsilon < \mu - \max_{1\le i\le 3}\{\alpha_i\}$ and $0 < \varepsilon_1 < \min_{1\le i\le 3}\{\alpha_i\}$ (this is possible thanks to (4) and (17)). Next, we choose M and ε_2 such that $0 < M < \min_{1\le i\le 3}\{\alpha_i\} - \varepsilon_1$ and $0 < \varepsilon_2 < \frac{M}{1+\hat{c}}(\mu - \max_{1\le i\le 3}\{\alpha_i\} - \varepsilon)$. These choices imply that $\alpha_i - \varepsilon_1 - M$ and $M(\mu - \alpha_i - \varepsilon) - (1+\hat{c})\varepsilon_2$ are positive constants. Therefore, we obtain, for some β, c_3, $c_4 > 0$,

$$L'(t) \le -\beta E(t) + \left(\frac{N}{2} - c_3\right) g'\circ\nabla\eta + c_4 g\circ\nabla\eta,\ \forall t \ge 0.$$

Finally, we choose N large enough so that $N > \max\{2c_3, Md_1 + d_2\}$, which implies that $\frac{N}{2} - c_3 \geq 0$ and (33) holds; and we get

$$L'(t) \leq -\beta E(t) + c_4 g \circ \nabla\eta, \ \forall t \in \mathbb{R}_+. \tag{34}$$

To estimate $g \circ \nabla\eta$ we follow the approach introduced by the second author in [11].

If (18) is satisfied, then

$$g_i \circ \nabla\eta_i \leq \frac{-1}{\gamma_i} g_i' \circ \nabla\eta_i. \tag{35}$$

If (18) does not hold, we follow the approach of [11]. Let us consider $G^*(t) = \sup_{s\in\mathbb{R}_+} \{ts - G(s)\}$, for $t \in \mathbb{R}_+$, the dual function of G. Thanks to (H3), G' is increasing and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, and then, for any $t \in \mathbb{R}_+$, the function $s \longmapsto ts - G(s)$ reaches its maximum on $\mathbb{R}_+$ at the unique point $s = (G')^{-1}(t)$. Therefore

$$G^*(t) = t(G')^{-1}(t) - G\left((G')^{-1}(t)\right), \forall t \in \mathbb{R}_+.$$

Let δ, τ_1, $\tau_2 > 0$ (will be fixed later on). Using the generalized Young's inequality: $t_1 t_2 \leq G(t_1) + G^*(t_2)$, we get

$$\begin{aligned} &g_i \circ \nabla\eta_i \\ &= \frac{1}{\tau_1 G'(\delta E(t))} \int_0^{+\infty} G^{-1}(-\tau_2 g_i'(s) \int_\Omega |\nabla\eta_i|^2 dx) \frac{\tau_1 G'(\delta E(t))\, g_i(s) \int_\Omega |\nabla\eta_i|^2 dx}{G^{-1}\left(-\tau_2 g_i'(s) \int_\Omega |\nabla\eta_i|^2 dx\right)} ds \\ &\leq \frac{1}{\tau_1 G'(\delta E(t))} \int_0^{+\infty} [-\tau_2 g_i'(s) \int_\Omega |\nabla\eta_i|^2 dx + G^*(\frac{\tau_1 G'(\delta E(t))\, g_i(s) \int_\Omega |\nabla\eta_i|^2 dx}{G^{-1}\left(-\tau_2 g_i'(s) \int_\Omega |\nabla\eta_i|^2 dx\right)})] ds. \end{aligned}$$

Because $s \longmapsto \frac{s}{G^{-1}(s)}$ and G^* are increasing functions and thanks to the fact that $\int_\Omega |\nabla\eta_i|^2 \leq c_5$ for some positive constant c_5 and all $s \geq 0$ (since E is nonincreasing, see (22) and (26)), we get

$$g_i \circ \nabla\eta_i \leq -\frac{\tau_2}{\tau_1 G'(\delta E(t))} g_i' \circ \nabla\eta_i + \frac{1}{\tau_1 G'(\delta E(t))} \int_0^{+\infty} G^*\left(\frac{c_5 \tau_1 G'(\delta E(t))\, g_i(s)}{G^{-1}\left(-c_5 \tau_2 g_i'(s)\right)}\right) ds.$$

By exploiting (26) and $G^*(t) \leq t(G')^{-1}(t)$, we get

$$\begin{aligned} g_i \circ \nabla\eta_i \quad &\leq -\frac{2\tau_2}{\tau_1 G'(\delta E(t))} E'(t) \\ &+ c_5 \int_0^{+\infty} \frac{g_i(s)}{G^{-1}\left(-c_5 \tau_2 g_i'(s)\right)} (G')^{-1}\left(\frac{c_5 \tau_1 G'(\delta E(t))\, g_i(s)}{G^{-1}\left(-c_5 \tau_2 g_i'(s)\right)}\right) ds. \end{aligned}$$

Choosing $\tau_2 = \frac{1}{c_5}$ and using (H3), we obtain

$$g_i \circ \nabla\eta_i \leq -\frac{2}{c_5 \tau_1 G'(\delta E(t))} E'(t) + c_5 (G')^{-1}\left(c_6 \tau_1 G'(\delta E(t))\right) \int_0^{+\infty} \frac{g_i(s)}{G^{-1}\left(-g_i'(s)\right)} ds,$$

for $c_6 = c_5 \sup_{s\in\mathbb{R}_+} \frac{g_i(s)}{G^{-1}\left(-g_i'(s)\right)}$. Choosing $\tau_1 = \frac{1}{c_6}$ and using again (H3), we get, for any $\delta > 0$,

$$G'(\delta E(t))\, g_i \circ \nabla\eta_i \leq -c_7 E'(t) + c_8 \delta E(t) G'(\delta E(t)), \quad \forall t \in \mathbb{R}_+, \tag{36}$$

where $c_7 = \dfrac{2c_6}{c_5}$ and $c_8 = c_5 \displaystyle\int_0^{+\infty} \frac{g_i(s)}{G^{-1}\left(-g_i'(s)\right)} ds$.

Case 1. If (18) is satisfied, for any $i = 1, 2, 3$, then (34) and (35) imply that

$$L'(t) \leq -\beta E(t) - c_9 g' \circ \nabla\eta, \ \forall t \in \mathbb{R}_+,$$

where $c_9 = c_4 \max\limits_{i=1,2,3} \left\{ \frac{1}{\gamma_i} \right\}$.

Let $F = L + 2c_9 E$. Using (26), we get

$$F'(t) \leq -\beta E(t), \quad \forall t \in \mathbb{R}_+. \tag{37}$$

Because $L \sim E$, then $F \sim E$. Therefore, (37) implies that

$$F'(t) \leq -c' F(t), \quad \forall t \in \mathbb{R}_+,$$

for some $c' > 0$. By integrating this differential inequality, we get

$$F(t) \leq F(0) e^{-c't}, \quad \forall t \in \mathbb{R}_+.$$

Thus, thanks to $F \sim E$, we get (23).

Case 2. If (18) is not satisfied at least for one $i \in \{1, 2, 3\}$, then, multiplying (34) by $G'(\delta E(t))$ and using (35) and (36), we obtain

$$\begin{aligned} & G'(\delta E(t)) L'(t) + c_7 c_4 (\beta_1 + \beta_2 + \beta_3) E'(t) \\ \leq \ & -(\beta - c_4 c_8 \delta (\beta_1 + \beta_2 + \beta_3)) E(t) G'(\delta E(t)) \\ & -c_{10} \left((1 - \beta_1) + (1 - \beta_2) + (1 - \beta_3) \right) E'(t) G'(\delta E(t)), \end{aligned}$$

where $c_{10} = 2c_4 \max\limits_{i=1,2,3} \left\{ \frac{1}{\gamma_i} \right\}$ and

$$\beta_i = \begin{cases} 0 \text{ if (18) holds,} \\ 1 \text{ if (18) does not hold.} \end{cases}$$

Choosing δ small enough so that $0 < \delta < \dfrac{\beta}{c_4 c_8 (\beta_1 + \beta_2 + \beta_3)}$, and letting

$$F = \tau \left(G'(\delta E(t)) L + c_7 c_4 (\beta_1 + \beta_2 + \beta_3) E + c_{10} (3 - \beta_1 - \beta_2 - \beta_3) E G'(\delta E) \right)$$

with $\tau > 0$, we deduce, for $\beta_0 = \beta - c_4 c_8 (\beta_1 + \beta_2 + \beta_3) \delta$,

$$F'(t) \leq -\tau \beta_0 E(t) G'(\delta E(t)), \quad \forall t \in \mathbb{R}_+, \tag{38}$$

since $E' G''(\delta E) \leq 0$. As $L \sim E$ and $G'(\delta E)$ is nonnegative and nonincreasing, we easily deduce that $F \sim E$. On the other hand, we choose $\tau > 0$ small enough so that $F \leq E$ and $F(0) \leq 1$. Thus, we deduce from (38) that

$$F'(t) \leq -\tau \beta_0 F(t) G'(\delta F(t)), \quad \forall t \in \mathbb{R}_+. \tag{39}$$

Inequality (39) implies that $(G_1(F))' \geq \tau \beta_0$, where $G_1(t) = \displaystyle\int_t^1 \frac{1}{s G'(\delta s)} ds$, for $t \in]0, 1]$. Consequently, we get, for $c' = \tau \beta_0$,

$$F(t) \leq G_1^{-1}(c't), \quad \forall t \in \mathbb{R}_+.$$

Finally, recalling that $F \sim E$, we obtain the desired result (24). □

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