On Scattering theory for Lindblad operators

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Lindblad operators, quantum dynamical semigroups

Physical system

Quantum system

- Quantum particle
- Quantum "target" localized in a suitable sense
- Environment

Hilbert space

- Hilbert space for the particle $\mathcal{H}_p$
- Hilbert space for the target $\mathcal{H}_t$
- Hilbert space for the environment $\mathcal{H}_E$
- Total Hilbert space

$$\mathcal{H}_{\mathrm{tot}} = \mathcal{H}_p \otimes \mathcal{H}_t \otimes \mathcal{H}_E.$$

Hamiltonian

$$H_{\mathrm{tot}} = H_p \otimes \mathrm{Id} \otimes \mathrm{Id} \; + \mathrm{Id} \otimes H_t \otimes \mathrm{Id} + \mathrm{Id} \otimes \mathrm{Id} \otimes H_E + H_I.$$

Reduced dynamics

Schrödinger picture

- (Mixed) states of the full system = density matrices $\rho \in \mathcal{J}_1(\mathcal{H}_{\mathrm{tot}})$, $\rho \geq 0$, $\mathrm{tr}(\rho) = 1$
- Evolution in the Schrödinger picture : $\rho(t) = e^{-itH_{\mathrm{tot}}} \rho e^{itH_{\mathrm{tot}}}$

Reduced dynamics

- Initial state of the target and the environment : fixed reference state $\rho^R_{t,E}$
- Reduced dynamics : for any initial state $\rho_p \in \mathcal{J}_1(\mathcal{H}_p)$ of the particle,

$$\rho_p(t) = \mathrm{tr}_{t,E}\big(e^{-itH_{\mathrm{tot}}}(\rho_p \otimes \rho^R_{t,E})e^{itH_{\mathrm{tot}}}\big)$$

Dynamical map

(Irreversible) dynamics for the particle = map $\Lambda : [0, \infty) \ni t \mapsto \Lambda_t \in \mathcal{L}(\mathcal{J}_1(\mathcal{H}_p))$ s.t.

- $t \mapsto \Lambda_t$ is a strongly continuous one-parameter semigroup on $\mathcal{J}_1(\mathcal{H}_p)$
- $\forall t \geq 0$, Λ_t is trace preserving
- $\forall t \geq 0$, Λ_t is positive

Quantum dynamical semigroups and their generators (I)

Theorem ([Kossakowski '72], [Ingarden, Kossakowski '75])

Let $\mathcal{H}$ be a complex, separable Hilbert space. Necessary and sufficient conditions for an operator L on $\mathcal{J}_1^{\mathrm{sa}}(\mathcal{H})$ to be the generator of a strongly continuous, trace preserving, positive one-parameter semigroup on $\mathcal{J}_1^{\mathrm{sa}}(\mathcal{H})$ are that

- $\mathcal{D}(L)$ is dense in $\mathcal{J}_1^{\mathrm{sa}}(\mathcal{H})$
- $\mathrm{Ran}(\mathrm{Id} - L) = \mathcal{J}_1^{\mathrm{sa}}(\mathcal{H})$
- L is dissipative (i.e. $\mathrm{tr}(\mathrm{sgn}(\rho) L\rho) \leq 0$ for all $\rho \in \mathcal{D}(L)$)
- $\mathrm{tr}(L\rho) = 0$ for all $\rho \in \mathcal{D}(L)$

Definition ([Lindblad '76])

Let $\mathcal{H}$ be a complex, separable Hilbert space. A quantum dynamical semigroup on $\mathcal{J}_1(\mathcal{H})$ is a map $\Lambda : [0, \infty) \ni t \mapsto \Lambda_t \in \mathcal{L}(\mathcal{J}_1(\mathcal{H}))$ s.t.

- $t \mapsto \Lambda_t$ is a strongly continuous one-parameter semigroup on $\mathcal{J}_1(\mathcal{H})$
- $\forall t \geq 0$, Λ_t is trace preserving
- $\forall t \geq 0$, Λ_t is completely positive (i.e. $\forall n \in \mathbb{N}$, $\Lambda_t \otimes \mathrm{Id} \in \mathcal{L}(\mathcal{J}_1(\mathcal{H} \otimes \mathbb{C}^n))$ is positive)

Quantum dynamical semigroups and their generators (II)

Theorem ([Lindblad '76])

Let $\mathcal{H}$ be a complex, separable Hilbert space. The generator

$$\mathcal{L} := \mathrm{s}\text{-}\lim_{t\to 0}(-it)^{-1}(\Lambda_t - \mathrm{Id})$$

of a norm continuous quantum dynamical semigroup $t \mapsto \Lambda_t \in \mathcal{L}(\mathcal{J}_1(\mathcal{H}))$ is of the form

$$\mathcal{L} = \underbrace{[H_0, \cdot]}_{\mathrm{ad}(H_0)} - \frac{i}{2}\sum_{j\in\mathbb{N}}\{C_j^* C_j, \cdot\} + i\sum_{j\in\mathbb{N}} C_j \cdot C_j^*, \tag{1}$$

with H_0 self-adjoint and bounded and $C_j \in \mathcal{L}(\mathcal{H})$. Write $\Lambda_t = \{e^{-it\mathcal{L}}\}$.

Definition

An operator of the form (1) with H_0, C_j unbounded is called a Lindblad operator.

Quantum dynamical semigroups and their generators (III)

Proposition ([Davies '76])

Let H_0 be a self-adjoint operator on $\mathcal{H}$ and let $C_j \in \mathcal{L}(\mathcal{H})$ for $j \in \{1, 2, \dots, n\}$. Then the operator $\mathcal{L}$ in (1) with domain

$$\mathcal{D}(\mathcal{L}) = \mathcal{D}(\mathrm{ad}(H_0)) = \{\rho \in \mathcal{J}_1(\mathcal{H}), \rho(\mathcal{D}(H_0)) \subset \mathcal{D}(H_0) \text{ and } H_0\rho - \rho H_0 \text{ defined on } \mathcal{D}(H_0) \text{ extends to an element of } \mathcal{J}_1(\mathcal{H})\}$$

generates a quantum dynamical semigroup $\{e^{-it\mathcal{L}}\}$ on $\mathcal{J}_1(\mathcal{H})$

Assumption

We suppose that the reduced dynamics for the particle is given by a quantum dynamical semigroup associated with a Lindblad operator

$$\mathcal{L} = [H_0, \cdot] - \frac{i}{2}\{C^*C, \cdot\} + iC \cdot C^*$$

with H_0 a self-adjoint operator and $C \in \mathcal{L}(\mathcal{H}_p)$. If the particle is initially in the state $\rho \in \mathcal{J}_1(\mathcal{H}_p)$, $\rho \geq 0$, $\mathrm{tr}(\rho) = 1$, the state of the particle at time $t \geq 0$ is given by $\rho(t) = e^{-it\mathcal{L}}\rho$. It solves the quantum master equation (quantum mechanical Fokker-Planck equation)

$$i\rho'(t) = \mathcal{L}\rho(t)$$

Scattering theory for Lindblad operators

References

[Davies '80], [Alicki '81], [Alicki, Frigerio '83]

Free dynamics

Generated by $\mathcal{L}_0 := [H_0, \cdot]$

Aim

Suppose that the interaction is "not too strong". Prove that for any initial state ρ which is not a "bound state", there exists a scattering state ρ_+ such that

$$\lim_{t \to +\infty} \left\| e^{-it\mathcal{L}} \rho - e^{-it\mathcal{L}_0} \rho_+ \right\|_{\mathcal{J}_1(\mathcal{H})} = 0$$

Wave operators

$$\Omega^+(\mathcal{L}, \mathcal{L}_0) := \underset{t \to +\infty}{\text{s-lim}}\ e^{-it\mathcal{L}} e^{it\mathcal{L}_0}, \quad \Omega^-(\mathcal{L}_0, \mathcal{L}) := \underset{t \to +\infty}{\text{s-lim}}\ e^{it\mathcal{L}_0} e^{-it\mathcal{L}}$$

Scattering operator

$$\sigma := \Omega^-(\mathcal{L}_0, \mathcal{L}) \Omega^+(\mathcal{L}, \mathcal{L}_0)$$

Main results

Results (I)

Lindblad operator

Let H_0 be self-adjoint on $\mathcal{H}$ and $C \in \mathcal{L}(\mathcal{H})$. For all $\rho \in \mathcal{D}(\mathrm{ad}(H_0))$,

$$\begin{aligned}\mathcal{L}(\rho) &= H_0\rho - \rho H_0 - \frac{i}{2}C^*C\rho - \frac{i}{2}\rho C^*C + iC\rho C^* \\ &= H\rho - \rho H^* + iC\rho C^*,\end{aligned}$$

with

$$H := H_0 - \frac{i}{2}C^*C.$$

Theorem ([Davies '80], [Falconi, F., Fröhlich, Schubnel])

Suppose that there exists a dense subset $\mathcal{D} \subset \mathcal{H}$ such that, for all $u \in \mathcal{D}$,

$$\int_{\mathbb{R}} \big\| C^*Ce^{-itH_0}u \big\|_{\mathcal{H}} dt < \infty.$$

Let $\mathcal{L}_0(\rho) = H_0\rho - \rho H_0$. Then

$$\Omega^+(\mathcal{L}, \mathcal{L}_0) = \underset{t\to+\infty}{\text{s-lim}}\ e^{-it\mathcal{L}}e^{it\mathcal{L}_0} \text{ exists on } \mathcal{J}_1(\mathcal{H}).$$

Results (II)

Theorem ([Falconi, F., Fröhlich, Schubnel])

Suppose that there exists a positive constant $\mathrm{c}_0 < 2$ such that

$$\int_{\mathbb{R}} \| C e^{-itH_0} u \|_{\mathcal{H}}^2 dt \le \mathrm{c}_0^2 \| u \|_{\mathcal{H}}^2,$$

for all $u \in \mathcal{H}$. Let $\mathcal{L}_0(\rho) = H_0 \rho - \rho H_0$. Then

$$\Omega^+(\mathcal{L}, \mathcal{L}_0) = \underset{t \to +\infty}{\text{s-lim}}\ e^{-it\mathcal{L}} e^{it\mathcal{L}_0} \text{ exists on } \mathcal{J}_1(\mathcal{H}),$$

$$\Omega^-(\mathcal{L}_0, \mathcal{L}) = \underset{t \to +\infty}{\text{s-lim}}\ e^{it\mathcal{L}_0} e^{-it\mathcal{L}} \text{ exists on } \mathcal{J}_1(\mathcal{H}).$$

If in addition $\mathrm{c}_0 < 2 - \sqrt{2}$, then $\Omega^+(\mathcal{L}, \mathcal{L}_0)$ and $\Omega^-(\mathcal{L}_0, \mathcal{L})$ are invertible in $\mathcal{L}(\mathcal{J}_1(\mathcal{H}))$ and the operators $\mathcal{L}$ and $\mathcal{L}_0$ are similar.

Kato smoothness

Main assumption

- If there exists $c_0 > 0$ such that

$$\int_{\mathbb{R}} \|Ce^{-itH_0}u\|^2_{\mathcal{H}} dt \leq c_0^2 \|u\|^2_{\mathcal{H}},$$

  for all $u \in \mathcal{H}$, C is said to be H_0-smooth ([Kato '66])

- It is equivalent to

$$\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \left\| C\left((H_0 - z)^{-1} - (H_0 - \bar{z})^{-1} \right) C^* \right\|_{\mathcal{H}} \leq 2\pi c_0.$$

- Useful observation : the following estimate is *always* satisfied :

$$\int_0^\infty \|Ce^{-itH}u\|^2_{\mathcal{H}} dt \leq \|u\|^2_{\mathcal{H}}.$$

Weaker assumptions

Theorem ([Falconi, F., Fröhlich, Schubnel])

Suppose that C is H_0-smooth and that there exists a positive constant $\tilde{c}_0 < 1$ such that

$$\int_0^\infty \|Ce^{-itH}u\|_{\mathcal{H}}^2 dt \leq \tilde{c}_0{}^2 \|u\|_{\mathcal{H}}^2, \tag{2}$$

for all $u \in \mathcal{H}$. Let $\mathcal{L}_0(\rho) = H_0\rho - \rho H_0$. Then

$$\Omega^+(\mathcal{L}, \mathcal{L}_0) = \operatorname*{s-lim}_{t\to+\infty} e^{-it\mathcal{L}} e^{it\mathcal{L}_0} \text{ exists on } \mathcal{J}_1(\mathcal{H}),$$

$$\Omega^-(\mathcal{L}_0, \mathcal{L}) = \operatorname*{s-lim}_{t\to+\infty} e^{it\mathcal{L}_0} e^{-it\mathcal{L}} \text{ exists on } \mathcal{J}_1(\mathcal{H}).$$

If in addition $\tilde{c}_0 < 1/2$, then $\Omega^+(\mathcal{L}, \mathcal{L}_0)$ and $\Omega^-(\mathcal{L}_0, \mathcal{L})$ are invertible in $\mathcal{L}(\mathcal{J}_1(\mathcal{H}))$ and the operators $\mathcal{L}$ and $\mathcal{L}_0$ are similar.

Remark

Assumption that (2) holds with $\tilde{c}_0 < 1$ is equivalent to assuming that the inverse semigroup $\{e^{itH}\}_{t\geq 0}$ is uniformly bounded in $\mathcal{L}(\mathcal{H})$.

Examples (I)

Non-relativistic quantum particle in $\mathbb{R}^3$

- Hilbert space $L^2(\mathbb{R}^3) \otimes \mathfrak{h}$ (where $\mathfrak{h}$ is a complex, finite dimensional Hilbert space)
- Effective dynamics of the particle generated by

$$\mathcal{L} := \mathrm{ad}(-\Delta + H_{\mathrm{int}}) - \frac{i}{2}\{C^*C, \cdot\} + iC \cdot C^* = \mathcal{L}_0 - \frac{i}{2}\{C^*C, \cdot\} + iC \cdot C^*,$$

with $H_{\mathrm{int}} \geq 0$ acting on $\mathfrak{h}$.

Assumptions

- Explicit form may be very complicated
- Rigorous derivation = open problem in general ([Davies '74] Weak coupling limit for finite dimensional system coupled to a free heat bath)
- Heuristic argument : assuming that the interaction with the environment induces decoherence in position space, it is "reasonable" to assume that

$$C = g(x) \otimes S$$

with $S \in \mathcal{L}(\mathfrak{h})$ and $g : \mathbb{R}^3 \to \mathbb{R}$ with sufficiently fast decay.

Examples (II)

Corollary ([Falconi, F., Fröhlich, Schubnel])

Suppose that $\| C|x| \|_{\mathcal{L}(\mathcal{H})} < 2\pi^{-1/2}$, then $\Omega^+(\mathcal{L},\mathcal{L}_0)$ and $\Omega^-(\mathcal{L}_0,\mathcal{L})$ exist on $\mathcal{J}_1(L^2(\mathbb{R}^3 \otimes \mathfrak{h}))$. If $\| C|x| \|_{\mathcal{L}(\mathcal{H})} < (2-\sqrt{2})\pi^{-1/2}$, then $\Omega^+(\mathcal{L},\mathcal{L}_0)$ and $\Omega^-(\mathcal{L}_0,\mathcal{L})$ exist and are invertible.

Optimal smoothness estimate

Suffices to use ([Simon '91])

$$\int_{\mathbb{R}} \| |x|^{-1} e^{it\Delta} \varphi \|_2^2 dt \le \pi \|\varphi\|_2^2,$$

for all $\varphi \in L^2(\mathbb{R}^3)$.

Other smoothness inequalities

$$\int_{\mathbb{R}} \| \langle x \rangle^{-1} (1-\Delta)^{\frac{1}{4}} e^{it\Delta} \varphi \|_2^2 dt \le \frac{\pi}{2} \|\varphi\|_2^2,$$

$$\int_{\mathbb{R}} \| V(x) e^{it\Delta} \varphi \|_2^2 dt \le \frac{\|V^2\|_{\mathrm{R}}}{2\pi} \|\varphi\|_2^2, \quad \|W\|_{\mathrm{R}}^2 := \int_{\mathbb{R}^3} \frac{|W(x)||W(y)|}{|x-y|^2} dxdy.$$

Capture (I)

Lindblad operator

Let V be relatively compact w.r.t. H_0. For all $\rho \in \mathcal{D}(\mathrm{ad}(H_0))$,

$$\mathcal{L}(\rho) = (H_0 + V)\rho - \rho(H_0 + V) - \frac{i}{2} C^* C \rho - \frac{i}{2} \rho C^* C + i C \rho C^*$$
$$= H\rho - \rho H^* + i C \rho C^*,$$

with

$$H = H_0 + V - \frac{i}{2} C^* C = H_V - \frac{i}{2} C^* C$$

Assumption

The spectrum of H_0 is purely absolutely continuous, the singular continuous spectrum of H_V is empty and H_V has at most finitely many eigenvalues with finite multiplicities. The wave operators

$$W_\pm(H_V, H_0) := \operatorname*{s-lim}_{t \to \mp\infty} e^{itH_V} e^{-itH_0}, \quad W_\pm(H_0, H_V) := \operatorname*{s-lim}_{t \to \mp\infty} e^{itH_0} e^{-itH_V} \Pi_{\mathrm{ac}}(H_V),$$

exist on $\mathcal{H}$ and are asymptotically complete in the sense that

$$\mathrm{Ran}(W_\pm(H_V, H_0)) = \mathrm{Ran}(\Pi_{\mathrm{ac}}(H_V)) = \mathrm{Ran}(\Pi_{\mathrm{pp}}(H_V))^\perp,$$
$$\mathrm{Ran}(W_\pm(H_0, H_V)) = \mathcal{H}.$$

Capture (II)

Subspaces associated to H, H^*

With $H = H_0 + V - \frac{i}{2} C^* C$, consider the subspaces

- $\mathcal{H}_{\mathrm{b}}(H)$ = closure of the vector space generated by the set of eigenvectors with *real* eigenvalues of H.
- $\mathcal{H}_{\mathrm{d}}(H) := \{u \in \mathcal{H}, \lim_{t\to+\infty} \|e^{-itH}u\| = 0\}$.
- $\mathcal{H}_{\mathrm{d}}(H^*) := \{u \in \mathcal{H}, \lim_{t\to+\infty} \|e^{itH^*}u\| = 0\}$.

Definition : Modified wave operator ([Davies '80])

Let Π be the orthogonal projection with kernel $\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{d}}(H)$. Modified wave operator $\tilde{\Omega}^-(\mathcal{L}_0, \mathcal{L})$ defined by

$$\tilde{\Omega}^-(\mathcal{L}_0, \mathcal{L}) := \underset{t\to+\infty}{\text{s-lim}}\, e^{it\mathcal{L}_0}\big(\Pi e^{-it\mathcal{L}}(\cdot)\Pi\big).$$

Capture (III)

Theorem ([Falconi, F., Fröhlich, Schubnel])

Suppose that there exists a positive constant $\mathrm{c}_V < 2$ such that

$$\int_{\mathbb{R}} \| C e^{-itH_V} \Pi_{\mathrm{ac}}(H_V) u \|_{\mathcal{H}}^2 dt \le \mathrm{c}_V^2 \| \Pi_{\mathrm{ac}}(H_V) u \|_{\mathcal{H}}^2,$$

for all $u \in \mathcal{H}$. Let $\mathcal{L}_0(\rho) = H_0 \rho - \rho H_0$. Then the modified wave operator $\tilde{\Omega}^-(\mathcal{L}_0, \mathcal{L})$ exists on $\mathcal{J}_1(\mathcal{H})$. For all $\rho \in \mathcal{J}_1(\mathcal{H})$, $\rho \ge 0$, $\mathrm{tr}(\rho) = 1$, we have that

$$0 \le \mathrm{tr}(\tilde{\Omega}^-(\mathcal{L}_0, \mathcal{L})\rho) \le 1,$$

and $\mathrm{tr}(\tilde{\Omega}^-(\mathcal{L}_0, \mathcal{L})\rho)$ is interpreted as the probability that the particle initially in the state ρ eventually escapes from the target.

Example (I)

Non-relativistic quantum particle in $\mathbb{R}^3$

Effective dynamics of the particle

$$\begin{aligned}\mathcal{L} &= \mathrm{ad}(-\Delta + V(x) + H_{\mathrm{int}}) - \frac{i}{2}\{C^*C, \cdot\} + iC \cdot C^* \\ &= \mathcal{L}_0 + \mathrm{ad}(V(x)) - \frac{i}{2}\{C^*C, \cdot\} + iC \cdot C^*,\end{aligned}$$

with $H_{\mathrm{int}} \geq 0$ acting on $\mathfrak{h}$, V real-valued.

Conditions on V

Suppose that

- There exists $\mathrm{C} > 0$ s.t. for all $x \in \mathbb{R}$,

$$|V(x)| \leq \mathrm{C}\langle x\rangle^{-2-\varepsilon}, \quad \varepsilon > 0.$$

- 0 is neither an eigenvalue nor a resonance of H_V.

[Ben-Artzi, Klainerman '91] : there exists $\mathrm{c}_1 > 0$ s.t.

$$\int_{\mathbb{R}} \| \langle x\rangle^{-1-\varepsilon} e^{-itH_V} \Pi_{\mathrm{ac}}(H_V) u\|_2^2 dt \leq \mathrm{c}_1^2 \|u\|_2^2,$$

for all $u \in L^2(\mathbb{R}^3)$.

Example (II)

Corollary ([Falconi, F., Fröhlich, Schubnel])

Suppose that the previous conditions on V are satisfied and that

$$\left\| C \langle x \rangle^{1+\varepsilon} \right\|_{\mathcal{L}(\mathcal{H})} < 2\mathrm{c}_1^{-1},$$

for some $\varepsilon > 0$. Let $\mathcal{L}_0 = \mathrm{ad}(-\Delta + H_{\mathrm{int}})$. Then the modified wave operator $\tilde{\Omega}^-(\mathcal{L}_0, \mathcal{L})$ exists on $\mathcal{J}_1(\mathcal{H})$.

Ideas of the proofs

Existence of $\Omega^+(\mathcal{L}, \mathcal{L}_0)$

Theorem ([Falconi, F., Fröhlich, Schubnel])

Recall

$$\begin{aligned}\mathcal{L}(\rho) &= H_0\rho - \rho H_0 - \frac{i}{2}C^*C\rho - \frac{i}{2}\rho C^*C + iC\rho C^* \\ &= \mathcal{L}_0(\rho) - \frac{i}{2}C^*C\rho - \frac{i}{2}\rho C^*C + iC\rho C^*\end{aligned}$$

Suppose that there exists a dense subset $\mathcal{D} \subset \mathcal{H}$ such that, for all $u \in \mathcal{D}$,

$$\int_{\mathbb{R}} \left\| C^*Ce^{-itH_0}u \right\|_{\mathcal{H}} dt < \infty.$$

Then

$$\Omega^+(\mathcal{L}, \mathcal{L}_0) = \underset{t\to+\infty}{\text{s-lim}}\; e^{-it\mathcal{L}}e^{it\mathcal{L}_0} \text{ exists on } \mathcal{J}_1(\mathcal{H}).$$

Idea of the proof ([Davies '80])

- Cook's argument
- Cyclicity of the trace

Existence of $\Omega^-(\mathcal{L}_0, \mathcal{L})$ (I)

Theorem ([Falconi, F., Fröhlich, Schubnel])

Suppose that there exists a positive constant $c_0 < 2$ such that

$$\int_{\mathbb{R}} \| C e^{-itH_0} u \|_{\mathcal{H}}^2 dt \leq c_0^2 \| u \|_{\mathcal{H}}^2,$$

for all $u \in \mathcal{H}$. Then

$$\Omega^-(\mathcal{L}_0, \mathcal{L}) = \underset{t \to +\infty}{\text{s-lim}}\, e^{it\mathcal{L}_0} e^{-it\mathcal{L}} \text{ exists on } \mathcal{J}_1(\mathcal{H}).$$

Idea of the proof

- Let $\mathcal{L}_H(\rho) = H\rho - \rho H^*$, with $H = H_0 - iC^*C/2$, and write

$$e^{it\mathcal{L}_0} e^{-it\mathcal{L}} = e^{it\mathcal{L}_0} e^{-it\mathcal{L}_H} + e^{it\mathcal{L}_0} \big(e^{-it\mathcal{L}} - e^{-it\mathcal{L}_H} \big).$$

- First term :

$$e^{it\mathcal{L}_0} e^{-it\mathcal{L}_H} \rho = e^{itH_0} e^{-itH} \rho e^{itH^*} e^{-itH_0} \underset{t \to \infty}{\to} W_- \rho W_-^*,$$

with $W_- = W_-(H_0, H)$ (assuming we can prove it exists on $\mathcal{H}$)

Existence of $\Omega^-(\mathcal{L}_0, \mathcal{L})$ (II)

Idea of the proof

- Second term

$$e^{it\mathcal{L}_0}\left(e^{-it\mathcal{L}} - e^{-it\mathcal{L}_H}\right)\rho = \int_0^t e^{is\mathcal{L}_0} e^{i(t-s)\mathcal{L}_0} e^{-i(t-s)\mathcal{L}_H} C(e^{-is\mathcal{L}}\rho)C^* ds$$

$$\underset{t\to\infty}{\to} \int_0^\infty e^{is\mathcal{L}_0} W_- C(e^{-is\mathcal{L}}\rho) C^* W_-^* ds,$$

provided we can justify taking the limit

- Scattering theory for dissipative operators in Hilbert space,

$$W_+(H, H_0) = \underset{t\to+\infty}{\text{s-lim}}\, e^{-itH} e^{itH_0},$$

$$W_-(H_0, H) = \underset{t\to+\infty}{\text{s-lim}}\, e^{itH_0} e^{-itH}$$

[Martin '75], [Mochizuki '76], [Davies '78, '80], [Simon '79], [Kadowaki '02]

Invertibility of the wave operators

Theorem ([Falconi, F., Fröhlich, Schubnel])

Suppose that there exists a positive constant $\mathrm{c}_0 < 2 - \sqrt{2}$ such that

$$\int_{\mathbb{R}} \| C e^{-itH_0} u \|_{\mathcal{H}}^2 dt \leq \mathrm{c}_0^2 \| u \|_{\mathcal{H}}^2,$$

for all $u \in \mathcal{H}$. Then $\Omega^+(\mathcal{L}, \mathcal{L}_0)$ and $\Omega^-(\mathcal{L}_0, \mathcal{L})$ are invertible in $\mathcal{L}(\mathcal{J}_1(\mathcal{H}))$ and the operators $\mathcal{L}$ and $\mathcal{L}_0$ are similar.

Idea of the proof

- Introduce $e^{-it\mathcal{L}_H} = e^{-itH} \cdot e^{itH^*}$
- Estimate the Dyson series

Scattering theory for dissipative operators in Hilbert space

Proposition ([Falconi, F., Fröhlich, Schubnel])

Suppose that there exists $\mathrm{c}_0 > 0$ such that

$$\int_{\mathbb{R}} \| C e^{-itH_0} u \|^2_{\mathcal{H}} dt \leq \mathrm{c}_0^2 \| u \|^2_{\mathcal{H}},$$

for all $u \in \mathcal{H}$. Let $H = H_0 - iC^*C/2$. Then

$$W_+(H, H_0) = \underset{t \to +\infty}{\text{s-lim}}\, e^{-itH} e^{itH_0}, \quad W_-(H_0, H) = \underset{t \to +\infty}{\text{s-lim}}\, e^{itH_0} e^{-itH}$$

exist on $\mathcal{H}$ and

- $W_+(H, H_0)$ is injective,
- $\mathrm{Ran}(W_-(H_0, H))$ is dense in $\mathcal{H}$.

If $\mathrm{c}_0 < 2$, then $W_+(H, H_0)$ and $W_-(H_0, H)$ are bijective.

Remark

- Bijectivity in the case where $\mathrm{c}_0 < 2$: result close to [Kato '66]
- Do not need the assumption that $\sup_{z \in \mathbb{C} \setminus \mathbb{R}} \| C(H_0 - z)^{-1} C^* \| < \infty$

The case of capture

Theorem ([Falconi, F., Fröhlich, Schubnel])

Let $H = H_0 + V - iC^*C/2$. Suppose that there exists a positive constant $\mathrm{c}_V < 2$ s.t.

$$\int_{\mathbb{R}} \| C e^{-itH_V} \Pi_{\mathrm{ac}}(H_V) u \|_{\mathcal{H}}^2 dt \leq \mathrm{c}_V^2 \| \Pi_{\mathrm{ac}}(H_V) u \|_{\mathcal{H}}^2,$$

for all $u \in \mathcal{H}$. Then

$$W_+(H, H_0) = \underset{t \to +\infty}{\text{s-lim}}\, e^{-itH} e^{itH_0},$$

exists on $\mathcal{H}$, is injective, and its range is equal to

$$\mathrm{Ran}(W_+(H, H_0)) = \big(\mathcal{H}_{\mathrm{b}}(H) \oplus \mathcal{H}_{\mathrm{d}}(H^*)\big)^{\perp}.$$

Remark

- Dissipative Schrödinger operators with small imaginary part [Wang, Zhu '14]
- Possible to relax the smallness condition in examples by remarking that if $V = 0$,

  $$\mathrm{Ran}(W_+(H, H_0)) = \big\{ u \in \mathcal{H},\ \exists M_u > 0,\ \forall t \geq 0,\ \| e^{itH} u \| \leq M_u \big\}$$

  [Goldberg '08] (Schrödinger operators without real resonances)

Thank you !