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ON LINEAR ELASTICITY SYSTEMS WITH VARIABLE COEFFICIENTS

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1. Introduction and statement of the results

Let Ω be a non-empty bounded open set in $\mathbb{R}^n (n = 1, 2, \ldots)$ having a boundary Γ of class C^2 and let $a_{ijkl}, i, j, k, l = 1, \ldots, n$, be functions in $W^{2,\infty}(\Omega \times \mathbb{R})$ such that

$$a_{ijkl} = a_{klij} = a_{jikl} \quad \text{and} \quad a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \alpha\varepsilon_{ij}\varepsilon_{ij} \quad \text{in } \Omega \times \mathbb{R} \tag{1.1}$$

for some $\alpha > 0$ and for every symmetric tensor ε_{ij}. (Here and in the sequel we shall use the summation convention for repeated indices.)

For a given function $u = (u_1, \ldots, u_n) : \Omega \times \mathbb{R} \to \mathbb{R}^n$, we shall use the notation

$$\varepsilon_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i}), \quad \sigma_{ij} = a_{ijkl}\varepsilon_{kl}, \quad \text{in } \Omega \times \mathbb{R},$$

where $u_{i,j} = \partial u_i/\partial x_j$ and $u_{j,i} = \partial u_j/\partial x_i$. If it is necessary to be more precise, we shall write $\varepsilon_{ij}(u)$ and $\sigma_{ij}(u)$ instead of $\varepsilon_{ij}, \sigma_{ij}$.

Consider the problem

$$\begin{cases} u_i'' - \sigma_{ij,j} = 0 \quad \text{in } \Omega \times \mathbb{R}, \\ u_i = 0 \quad \text{on } \Gamma \times \mathbb{R}, \\ u_i(0) = u_i^0 \quad \text{and} \quad u_i'(0) = u_i^1 \quad \text{in } \Omega, \\ i = 1, \ldots, n, \end{cases} \tag{1.2}$$

where $'$ denotes $\partial/\partial t$, $\sigma_{ij,j} = \partial\sigma_{ij}/\partial x_j$ and $u_i(0)$, $u_i'(0)$ denote, respectively, the functions $x \longmapsto u_i(x, 0)$, $x \longmapsto u_i'(x, 0)$.

In order to formulate the definition of a solution to (1.2) we introduce three real Hilbert spaces H, V and W by setting

$$H = (L^2(\Omega))^n, \quad \|v\|_H^2 = \int_\Omega v_i v_i \, dx,$$

$$V = (H_0^1(\Omega))^n, \quad \|v\|_V^2 = \int_\Omega \varepsilon_{ij}(v)\varepsilon_{ij}(v) \, dx,$$

where $H_0^1(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma\}$ (by the Korn inequality, it is clear that this expression defines a norm on V), and

$$W = (H^2(\Omega) \cap H_0^1(\Omega))^n, \quad \|v\|_W^2 = \int_\Omega (\Delta v_i \Delta v_i + \varepsilon_{ij}(v)\varepsilon_{ij}(v))\, dx.$$

Identifying H with its dual H' we have

$$W \subset V \subset H = H' \subset V' \subset W'$$

with dense and compact imbedings.

Setting

$$z := u', \quad U := (u, z), \quad \mathcal{A}(t)U := (z, Au),$$
$$A = [A_i]_{i=1,\dots,n} \quad \text{and} \quad A_i u_i = \sigma_{ij,j}(u),$$

we may write problem (1.2) in the following form:

$$U'(t) - \mathcal{A}(t)U(t) = 0 \quad \text{in } \mathbb{R},$$
$$U(0) = (u^0, u^1).$$

It is natural to introduce the Hilbert space $\mathcal{H} := V \times H$ and to consider $\mathcal{A}(t)$ as an operator acting in $\mathcal{H}$:

$$D(\mathcal{A}(t)) := W \times V$$

is independent of t.

For all $t \in \mathbb{R}$, the bounded linear operator $\mathcal{A}(t)$ is the infinitesimal generator of a strongly continuous semigroup of contractions and its domain is clearly densely and continuously imbeding in $\mathcal{H}$ (cf. Lagnese [**9**]). On the other hand, from the assumption $a_{ijkl} \in W^{2,\infty}(\Omega \times \mathbb{R})$ we have for any $U \in W \times V$ that the mapping $t \to \mathcal{A}(t)U$ is continuously differentiable in $\mathcal{H}$. Then we recall (cf. Pazy [**12**; ch. 5] and note that (1.2) is a time-reversible problem) that problem (1.2) is well-posed in the following sense.

1. For every $(u^0, u^1) \in V \times H$, system (1.2) has a unique solution (defined in a suitable weak sense) satisfying

$$u \in C(\mathbb{R}; V) \cap C^1(\mathbb{R}; H).$$

2. If $(u^0, u^1) \in W \times V$ then the solution (called a strong solution) is more regular:

$$u \in C(\mathbb{R}; W) \cap C^1(\mathbb{R}; V) \cap C^2(\mathbb{R}; H).$$

3. The energy of the (weak) solution, defined by the formula

$$E(t) = \frac{1}{2}\int_\Omega (u_i' u_i' + \sigma_{ij}\varepsilon_{ij})\,dx, \quad t \in \mathbb{R}, \tag{1.3}$$

is a non-negative function and satisfies the identity

$$E'(t) = \frac{1}{2}\int_\Omega a_{ijkl}'\varepsilon_{kl}\varepsilon_{ij}\,dx, \quad \forall t \in \mathbb{R}. \tag{1.4}$$

Fix a point $x^0 = (x_1^0, \ldots, x_n^0) \in \mathbb{R}^n$, let $m(x) = x - x^0$, $R = \|m\|_{L^\infty(\Omega)}$ and fix a measurable partition Γ_0, Γ_1 of Γ such that

$$\Gamma_0 = \{x \in \Gamma : (x - x^0)\cdot \nu(x) \le 0\} \quad \text{and} \quad \Gamma_1 = \Gamma \setminus \Gamma_0, \tag{1.5}$$

where ν denotes the outward unit normal vector to Γ. (For example, we may always choose $\Gamma_0 = \emptyset$ and $\Gamma_1 = \Gamma$.) Let $\gamma \in]-\infty, 2[$ and $\lambda \ge 0$ be two constants satisfying

$$(x_p - x_p^0)(\partial_p a_{ijkl})\varepsilon_{ij}\varepsilon_{kl} \le \gamma a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \quad \text{in } \Omega \times \mathbb{R}, \tag{1.6}$$

and

$$|a_{ijkl}'\varepsilon_{ij}\varepsilon_{kl}| \le \lambda a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \quad \text{in } \Omega \times \mathbb{R} \tag{1.7}$$

for every symmetric tensor ε_{ij}, where $\partial_p a_{ijkl} = \partial a_{ijkl}/\partial x_p$. Assume that

$$\gamma \ge 2(1-n) \quad \text{and} \quad \frac{4\lambda\sqrt{(2/\alpha)}R}{2-\gamma} < 1. \tag{1.8}$$

Let T be a real number such that

$$T > -\frac{1}{\lambda}\log\left(1 - \frac{4\lambda\sqrt{(2/\alpha)}R}{2-\gamma}\right). \tag{1.9}$$

If $\lambda = 0$, then we take $T > 4\sqrt{(2/\alpha)}R/(2-\gamma)$.

Then we have the following results.

THEOREM 1.1. *Assume (1.1), (1.5)–(1.9). Then there exist two positive constants c_1 and c_2 such that every strong solution of (1.2) satisfies the inequalities*

$$c_1 E(0) \le \int_0^T \int_{\Gamma_1} \sigma_{ij}\varepsilon_{ij}\,d\Gamma\,dt \le c_2 E(0). \tag{1.10}$$

Remark 1.1. The first inequality in (1.10) cannot hold for arbitrarily small T. The condition $T > 2\sqrt{(2/\alpha)}R$ is the best possible if the system is isotropic; i.e. when

$$a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}),$$

where λ and μ are the positive Lamé constants (see Komornik [**7**] and note that, in this case, $\gamma = \lambda = 0$).

Remark 1.2. By a simple density argument, the second estimate in (1.10) allows us to define the trace of $\sigma_{ij}\varepsilon_{ij}$ on $\Gamma_1 \times \mathbb{R}$ as an element of $L^2_{\text{loc}}(\Gamma_1 \times \mathbb{R})$, for every weak solution of (1.2).

Remark 1.3. Theorem 1.1 means that in some sense the observation of the solution in a neighbourhood of the boundary during a sufficiently large time allows one to determine the initial data. Indeed, for sufficiently large time T, if two solutions of (1.2) coincide in Γ_1, then the boundary integral in (1.10), for their difference, vanishes and therefore the energy initial of their difference is equal to zero by the first inequality in (1.10). From the unicity of solution, this implies that the two solutions are identical.

Applying the Hilbert uniqueness method (HUM) introduced by Lions [**11**] we shall deduce from Theorem 1.1 an exact controllability result for the non-homogeneous system

$$\begin{cases} y_i'' - \sigma_{ij,j}(y) = 0 \quad \text{in } \Omega \times \mathbb{R}, \\ y_i = \vartheta_i \quad \text{on } \Gamma \times \mathbb{R}, \\ y_i(0) = y_i^0 \quad \text{and} \quad y_i'(0) = y_i^1 \quad \text{in } \Omega, \\ i = 1, \ldots, n. \end{cases} \tag{1.11}$$

THEOREM 1.2. *Assume (1.1), (1.5)–(1.9). Then for any given $y^0, \tilde{y}^0 \in (L^2(\Omega))^n$ and $y^1, \tilde{y}^1 \in (H^{-1}(\Omega))^n$ there exists $\vartheta \in L^2_{\text{loc}}(\mathbb{R}; (L^2(\Gamma))^n)$ such that the solution of (1.11) satisfies*

$$y(T) = \tilde{y}^0 \quad \text{and} \quad y'(T) = \tilde{y}^1 \quad \text{in } \Omega.$$

Moreover, we may assume that ϑ vanishes outside of $\Gamma_1 \times (0, T)$.

Concerning the observability and the controllability for system (1.2), we note that the case a_{ijkl} = constant was studied by Alabau and Komornik [**1**]. Theorems 1.1 and 1.2 extend the results of [**1**] to the case where a_{ijkl} are functions of class $W^{2,\infty}(\Omega \times \mathbb{R})$.

In the second half of this paper we shall study the uniform stabilizability of elasticity systems by applying suitable dissipative boundary feedbacks. Consider the problem

$$\begin{cases} u_i'' - \sigma_{ij,j} = 0 \quad \text{in } \Omega \times \mathbb{R}^+, \\ u_i = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}^+, \\ \sigma_{ij}\nu_j + au_i + bu_i' = 0 \quad \text{on } \Gamma_1 \times \mathbb{R}^+, \\ u_i(0) = u_i^0 \quad \text{and} \quad u_i'(0) = u_i^1 \quad \text{in } \Omega, \\ i = 1, \dots, n, \end{cases} \tag{1.12}$$

where a and b are given non-negative numbers. (It is easy to generalize our results to the case where a and b are non-negative functions of class $C^1(\bar{\Gamma}_1)$.) Indeed, define the energy of the solutions of (1.12) by

$$E(t) = \frac{1}{2}\int_\Omega (u_i'u_i' + \sigma_{ij}\varepsilon_{ij})\,dx + \frac{1}{2}\int_{\Gamma_1} au_iu_i\,d\Gamma, \tag{1.13}$$

for all $t \in \mathbb{R}^+$. The energy E is non-negative and we have

$$E'(t) = \frac{1}{2}\int_\Omega a'_{ijkl}\varepsilon_{kl}\varepsilon_{ij}\,dx - \int_{\Gamma_1} bu_i'u_i'\,d\Gamma \le 0, \quad \forall t \ge 0.$$

We assume that

$$a'_{ijkl}\varepsilon_{kl}\varepsilon_{ij} \le 0 \quad \text{in } \Omega \times \mathbb{R}^+ \tag{1.14}$$

for every symmetric tensor ε_{ij}. Then the energy E is non-increasing for $t \in \mathbb{R}^+$.

We shall consider system (1.12) under conditions (1.1), (1.5) and

$$|x - x_0| = R \quad \text{for all } x \in \Gamma_1. \tag{1.15}$$

Condition (1.15) is satisfied for all domain Ω having a part Γ_1 of its boundary which is a sphere; for example

$$\Omega = \{x \in \mathbb{R}^n : r < |x - x_0| < R\},$$

where $0 < r < R$ and $\Gamma_0 = \{x \in \Gamma : |x - x_0| = r\}$ or $r = 0$ and Γ_0 is empty.

THEOREM 1.3. *Assume (1.1), (1.5), (1.6), (1.14), (1.15) and $a < (2-\gamma)\alpha/4R$. Then there exists a positive number ω such that all (weak) solutions of (1.12) satisfy the energy estimate*

$$E(t) \le E(0)e^{1-\omega t}, \quad \textit{for all } t \ge 0. \tag{1.16}$$

If Γ_0 has a positive measure, then the result also holds for $a = 0$.

Remark 1.4. The proof of Theorem 1.3 will be obtained by applying a Liapunov-type method based on an integral inequality applied earlier in Komornik [**7**, **8**].

Remark 1.5. If a_{ijkl} = constant (i.e. $\lambda = \gamma = 0$) then the condition on a reduces to $a < \alpha/2R$. In this case we obtain a better condition on a than the one given by Alabau and Komornik [**2**].

2. Observability: proof of Theorem 1.1

First we prove the following lemma.

LEMMA 2.1. *Assume (1.7). Then we have*

$$-\lambda E(t) \leq E'(t) \leq \lambda E(t) \quad \forall t \in \mathbb{R}^+, \tag{2.1}$$

$$e^{-\lambda t} E(0) \leq E(t) \leq e^{\lambda t} E(0) \quad \forall t \in \mathbb{R}^+. \tag{2.2}$$

Proof. From (1.7) we have

$$-\lambda a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \leq -|a'_{ijkl}\varepsilon_{ij}\varepsilon_{kl}| \leq a'_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \leq \lambda a_{ijkl}\varepsilon_{ij}\varepsilon_{kl}.$$

Then by (1.3) and (1.4) we obtain (2.1). By Gronwall's inequality we deduce (2.2) from (2.1). The proof of Lemma 2.1 follows. □

Now, fix a number T which satisfies (1.9) and an arbitrary function $h \in (W^{1,\infty}(\Omega))^n$. We deduce from (1.2) that

$$\begin{aligned} 0 &= \int_0^T \int_\Omega (h_m u_{i,m})(u_i'' - \sigma_{ij,j})\,dx\,dt \\ &= \left[\int_\Omega h_m u_{i,m} u_i'\,dx\right]_0^T - \int_0^T \int_\Gamma h_m u_{i,m} \sigma_{ij} \nu_j\,d\Gamma\,dt \\ &\quad + \int_0^T \int_\Omega (h_{m,j}\sigma_{ij}u_{i,m} + h_m \sigma_{ij} u_{i,jm} - \tfrac{1}{2} h_m (u_i' u_i')_m)\,dx\,dt. \end{aligned}$$

Since

$$\sigma_{ij} u_{i,jm} = \sigma_{ij}\varepsilon_{ij,m} = \tfrac{1}{2}(\sigma_{ij}\varepsilon_{ij})_m - \tfrac{1}{2}(\partial_m a_{ijkl})\varepsilon_{kl}\varepsilon_{ij}, \tag{2.3}$$

integrating by parts the last two terms in the last integral and then multiplying by 2, we obtain the following identity:

$$\int_0^T \int_\Gamma (2h_m u_{i,m}\sigma_{ij}\nu_j + (h \cdot \nu)(u_i' u_i' - \sigma_{ij}\varepsilon_{ij}))\,d\Gamma\,dt$$

$$= \left[\int_\Omega 2h_m u_{i,m} u_i' \, dx\right]_0^T - \int_0^T \int_\Omega h_m (\partial_m a_{ijkl}) \varepsilon_{kl} \varepsilon_{ij} \, dx \, dt$$
$$+ \int_0^T \int_\Omega (2h_{m,j} \sigma_{ij} u_{i,m} + (\operatorname{div} h)(u_i' u_i' - \sigma_{ij} \varepsilon_{ij})) \, dx \, dt. \qquad (2.4)$$

Note that in the proof of identity (2.4) we did not use the boundary conditions in (1.2).

Using the assumption $h_m \in W^{1,\infty}(\Omega)$, $a_{ijkl} \in W^{2,\infty}(\Omega \times \mathbb{R})$, estimate (2.2) and the Korn inequality, the right-hand side of (2.4) can be easily majorized by $cE(0)$, where c is a positive constant. Furthermore, we deduce from the homogeneous Dirichlet boundary condition in (1.2) that

$$u_i' = 0 \quad \text{and} \quad u_{i,m} \nu_j = u_{i,\nu} \nu_m \nu_j = u_{i,j} \nu_m \quad \text{on } \Gamma,$$

and hence

$$h_m u_{i,m} \sigma_{ij} \nu_j = (h \cdot \nu) \sigma_{ij} u_{i,j} = (h \cdot \nu) \sigma_{ij} \varepsilon_{ij}.$$

Therefore, the left-hand side of (2.4) reduces to

$$\int_0^T \int_\Gamma (h \cdot \nu) \sigma_{ij} \varepsilon_{ij} \, d\Gamma \, dt.$$

Choosing h such that $h = \nu$ on Γ (cf. [**6**, Lemma 2.1, p. 18]) the second inequality in (1.10) follows with $c_2 = c$.

Now choosing $h(x) = x - x_0$, the identity (2.4) reduces to

$$\int_0^T \int_\Gamma (h \cdot \nu) \sigma_{ij} \varepsilon_{ij} \, d\Gamma \, dt$$
$$= \left[\int_\Omega 2h_m u_{i,m} u_i' \, dx\right]_0^T + \int_0^T \int_\Omega ((2-n) \sigma_{ij} \varepsilon_{ij} + n u_i' u_i') \, dx \, dt$$
$$- \int_0^T \int_\Omega h_m (\partial_m a_{ijkl}) \varepsilon_{kl} \varepsilon_{ij} \, dx \, dt.$$

Furthermore, we also deduce from (1.2) that

$$0 = \int_0^T \int_\Omega u_i (u_i'' - \sigma_{ij,j}) \, dx \, dt$$
$$= \left[\int_\Omega u_i u_i' \, dx\right]_0^T - \int_0^T \int_\Gamma u_i \sigma_{ij} \nu_j \, d\Gamma \, dt + \int_0^T \int_\Omega (\sigma_{ij} \varepsilon_{ij} - u_i' u_i') \, dx \, dt$$
$$= \left[\int_\Omega u_i u_i' \, dx\right]_0^T + \int_0^T \int_\Omega (\sigma_{ij} \varepsilon_{ij} - u_i' u_i') \, dx \, dt.$$

Multiplying this equality by $n-1+\gamma/2$ and combining with the preceding identity we obtain that

$$\begin{aligned}\int_0^T \int_\Gamma (h\cdot\nu)\sigma_{ij}\varepsilon_{ij}\,d\Gamma\,dt &= \left[\int_\Omega \left(2h_m u_{i,m} + \left(n-1+\frac{\gamma}{2}\right)u_i\right)u_i'\,dx\right]_0^T - \int_0^T\int_\Omega h_m(\partial_m a_{ijkl})\varepsilon_{kl}\varepsilon_{ij}\,dx\,dt \\ &\quad + \int_0^T\int_\Omega \left(\left(1+\frac{\gamma}{2}\right)\sigma_{ij}\varepsilon_{ij} + \left(1-\frac{\gamma}{2}\right)u_i'u_i'\right)dx\,dt\end{aligned}$$

since, by (1.6), we have

$$-\int_0^T\int_\Omega h_m(\partial_m a_{ijkl})\varepsilon_{kl}\varepsilon_{ij}\,dx\,dt \geq -\gamma\int_0^T\int_\Omega \sigma_{ij}\varepsilon_{ij}\,dx\,dt,$$

whence

$$\begin{aligned}R\int_0^T\int_{\Gamma_1}\sigma_{ij}\varepsilon_{ij}\,d\Gamma\,dt & \qquad\qquad (2.5)\\ \geq (2-\gamma)\int_0^T E(t)\,dt - \left|\left[\int_\Omega\left(2h_m u_{i,m} + \left(n-1+\frac{\gamma}{2}\right)u_i\right)u_i'\,dx\,dt\right]_0^T\right|.\end{aligned}$$

Let us majorize the last integral. For each fixed i we have

$$\begin{aligned}&\left\|2h_m u_{i,m} + \left(n-1+\frac{\gamma}{2}\right)u_i\right\|_{L^2(\Omega)}^2 - \|2h_m u_{i,m}\|_{L^2(\Omega)}^2 \\ &= \int_\Omega\left(\left(n-1+\frac{\gamma}{2}\right)^2 u_i^2 + 4\left(n-1+\frac{\gamma}{2}\right)h_m u_{i,m}u_i\right)dx \\ &= \int_\Omega\left(\left(n-1+\frac{\gamma}{2}\right)^2 u_i^2 - 2\left(n-1+\frac{\gamma}{2}\right)nu_i^2\right)dx \\ &\quad + \int_\Gamma 2\left(n-1+\frac{\gamma}{2}\right)h_m\nu_m u_i^2\,d\Gamma \\ &= (1-n^2+\tfrac{1}{4}\gamma(\gamma-4))\int_\Omega u_i^2\,dx \leq 0\end{aligned}$$

(cf. (1.8)). Therefore, for any fixed $\delta>0$ we have

$$\left|\int_\Omega\left(2h_m u_{i,m} + \left(n-1+\frac{\gamma}{2}\right)u_i\right)u_i'\,dx\right|$$

$$\leq 2R\sum_{i=1}^{n}\left(\int_\Omega |\nabla u_i|^2\,dx\right)^{1/2}\left(\int_\Omega (u_i')^2\,dx\right)^{1/2}$$

$$\leq R\delta\int_\Omega u_{i,m}u_{i,m}\,dx + R\delta^{-1}\int_\Omega u_i'u_i'\,dx. \tag{2.6}$$

Furthermore, applying the Green formula and using the boundary condition in (1.2) we have

$$\int_\Omega u_{i,m}u_{i,m}\,dx = \int_\Omega 2\varepsilon_{im}u_{i,m}\,dx - \int_\Omega u_{m,i}u_{i,m}\,dx = \int_\Omega (2\varepsilon_{im}\varepsilon_{im} - u_{m,m}u_{i,i})\,dx,$$

i.e.

$$\int_\Omega 2\varepsilon_{im}\varepsilon_{im}\,dx = \int_\Omega u_{i,m}u_{i,m}\,dx + \int_\Omega |\operatorname{div} u|^2\,dx. \tag{2.7}$$

It follows from (1.1) and (2.7) that

$$\int_\Omega u_{i,m}u_{i,m}\,dx \leq \frac{2}{\alpha}\int_\Omega \sigma_{im}\varepsilon_{im}\,dx.$$

Substituting into (2.6), choosing $\delta = \sqrt{\alpha/2}$ and using definition (1.3) of the energy we obtain

$$\left|\int_\Omega \left(2h_m u_{i,m} + \left(n-1+\frac{\gamma}{2}\right)u_i\right)u_i'\,dx\right| \leq 2\sqrt{\frac{2}{\alpha}}RE(t).$$

Therefore, we deduce from (2.5) the inequality

$$R\int_0^T\int_{\Gamma_1}\sigma_{ij}\varepsilon_{ij}\,d\Gamma\,dt \geq (2-\gamma)\int_0^T E(t)\,dt - 2\sqrt{\frac{2}{\alpha}}R(E(0)+E(T)).$$

Suppose $E(T) \geq E(0)$. Now from (2.1) we have

$$E(t) \geq \frac{1}{\lambda}((1-e^{-\lambda t})E(t))'.$$

Then we obtain

$$R\int_0^T\int_{\Gamma_1}\sigma_{ij}\varepsilon_{ij}\,d\Gamma\,dt \geq \left(\frac{1}{\lambda}(2-\gamma)(1-e^{-\lambda T}) - 4\sqrt{\frac{2}{\alpha}}R\right)E(T)$$

$$\geq \left(\frac{1}{\lambda}(2-\gamma)(1-e^{-\lambda T}) - 4\sqrt{\frac{2}{\alpha}}R\right)E(0)$$

and the first estimate of (1.10) follows with

$$c_1 = \frac{1}{R\lambda}(2-\gamma)(1-e^{-\lambda T}) - 4\sqrt{\frac{2}{\alpha}}.$$

(From assumption (1.9) we have $c_1 > 0$.) With the same reasoning we can argue the case $E(T) \leq E(0)$, and the proof is thus complete.

In the next section we give an equivalent form of the integral in (1.10).

LEMMA 2.2. *Assume (1.1) and put*

$$\beta = \sum_{i,j,k,l} \|a_{ijkl}\|^2_{L^\infty(\Gamma\times\mathbb{R})}.$$

Then every strong solution of (1.2) satisfies on $\Gamma \times \mathbb{R}$ *the inequalities*

$$\frac{\alpha}{2}\sigma_{ij}\varepsilon_{ij} \leq \sum_{i=1}^{n} |\sigma_{ij}\nu_j|^2 \leq \frac{\beta}{\alpha}\sigma_{ij}\varepsilon_{ij}. \tag{2.8}$$

Applying the density argument, estimate (2.8) also remains valid for weak solution.

Proof. The proof of the second inequality of (2.8) does not use the boundary conditions:

$$\begin{aligned}\sum_i |\sigma_{ij}\nu_j|^2 &\leq \sum_{i,j} |\sigma_{ij}|^2 = \sum_{i,j} |a_{ijkl}\varepsilon_{kl}|^2 \\ &\leq \sum_{i,j}\left(\sum_{k,l} \|a_{ijkl}\|^2_{L^\infty(\Gamma\times\mathbb{R})} \sum_{k,l} |\varepsilon_{ij}|^2\right) = \beta\varepsilon_{kl}\varepsilon_{kl} \leq \frac{\beta}{\alpha}\sigma_{ij}\varepsilon_{ij}.\end{aligned}$$

For the proof of the first inequality of (2.8), without the boundary conditions in (1.2) we have

$$\begin{aligned}u_{i,j}u_{i,j} &= \frac{1}{2}\sum_{i,j}(u_{i,j}+u_{j,i})^2 - u_{i,j}u_{j,i} \\ &= 2\varepsilon_{ij}\varepsilon_{ij} - u_{i,\nu}\nu_j u_{j,\nu}\nu_i = 2\varepsilon_{ij}\varepsilon_{ij} - |\operatorname{div} u|^2,\end{aligned}$$

i.e.

$$2\varepsilon_{ij}\varepsilon_{ij} = u_{i,j}u_{i,j} + |\operatorname{div} u|^2 \quad \text{on } \Gamma. \tag{2.9}$$

(Compare with (2.7).) Therefore,

$$\sigma_{ij}\varepsilon_{ij} = \sigma_{ij}u_{i,j} = \sigma_{ij}\nu_j u_{i,\nu} \le \Big(\sum_i |\sigma_{ij}\nu_j|^2\Big)^{1/2} (u_{i,\nu}u_{i,\nu})^{1/2}$$

$$\le \Big(\sum_i |\sigma_{ij}\nu_j|^2\Big)^{1/2} (2\varepsilon_{ij}\varepsilon_{ij})^{1/2} \le \Big(\sum_i |\sigma_{ij}\nu_j|^2\Big)^{1/2} (2\alpha^{-1}\sigma_{ij}\varepsilon_{ij})^{1/2}$$

and the first inequality in (2.8) follows. □

3. Controllability: proof of Theorem 1.2

Let us first study the well-posedness of the non-homogeneous problem (1.11). Let us multiply the equation in (1.11) by an arbitrary solution of problem (1.2) and integrate by parts formally. Then we obtain that

$$\begin{aligned} 0 &= \int_0^T \int_\Omega u_i(y_i'' - \sigma_{ij,j}(y))\,dx\,dt \\ &= \int_0^T \int_\Omega (u_i'' - \sigma_{ij,j}(u))y_i\,dx\,dt + \Big[\int_\Omega u_i y_i' - u_i' y_i\,dx\Big]_0^T \\ &\quad + \int_0^T \int_\Gamma (-u_i\sigma_{ij}(y)\nu_j + \sigma_{ij}(u)\nu_j y_i)\,d\Gamma\,dt \\ &= \Big[\int_\Omega (u_i y_i' - u_i' y_i)\,dx\Big]_0^T + \int_0^T \int_\Gamma \sigma_{ij}(u)\nu_j \vartheta_i\,d\Gamma\,dt. \end{aligned}$$

Hence, putting

$$H = (H^{-1}(\Omega))^n \times (L^2(\Omega))^n \quad \text{and} \quad H' = (H_0^1(\Omega))^n \times (L^2(\Omega))^n$$

for simplicity, we have

$$\begin{aligned} &\langle (y'(T), -y(T)), (u(T), u'(T))\rangle_{H,H'} \\ &= \langle (y_{i1}, -y_{i0}), (u_{i0}, u_{i1})\rangle_{H,H'} - \int_0^T \int_\Gamma \sigma_{ij}(u)\nu_j \vartheta_i\,d\Gamma\,dt. \end{aligned} \tag{3.1}$$

This leads to the following.

Definition. A solution y of (1.11) is a continuous function such that $(y', -y) : \mathbb{R} \to H$ satisfies identity (3.1) for all $T \in \mathbb{R}$ and for all (weak) solutions of problem (1.2).

This definition is justified by the following.

THEOREM 3.1. *Assume (1.1). Then for any given*

$$y^0 \in (L^2(\Omega))^n, \quad y^1 \in (H^{-1}(\Omega))^n \quad \textit{and} \quad \vartheta \in L^2_{\text{loc}}(\mathbb{R}; (L^2(\Gamma))^n)$$

the problem (1.11) has a unique solution satisfying

$$y \in C(\mathbb{R}; (L^2(\Omega))^n) \cap C^1(\mathbb{R}; (H^{-1}(\Omega))^n).$$

Furthermore, the linear map $(y^0, y^1, \vartheta) \mapsto y$ is continuous with respect to these topologies.

Proof. We apply Theorem 1.1. Thanks to the second estimate in (1.10) the right-hand side of equality (3.1) defines a bounded linear form of $(y^0, y^1) \in H'$ for each T. Since the linear map $(u^0, u^1) \mapsto (u(T), u'(T))$ is an automorphism of H' (because problem (1.2) is reversible), the right-hand side of equality (3.1) can also be considered as a bounded linear form of $(y(T), y'(T)) \in H'(= H'')$, and we conclude the existence of a unique element $(y'(T), -y(T)) \in H$ satisfying (3.1).

Since the bounded linear form used in this proof depends continuously on $T \in \mathbb{R}$, the solution y has the regularity required in the theorem. Finally, the bounded linear form clearly depends continuously on (y^0, y^1, ϑ), hence y also has this property. □

We now turn to the proof of Theorem 1.2.

The main idea is to seek a control in the form $\vartheta_i = \sigma_{ij}(u)\nu_j$, where u is the solution of (1.2) for some suitable initial data. (Thanks to Theorem 1.1 and to Lemma 2.2 these controls have the required regularity for the well-posedness of (1.11).)

Case 1: $\tilde{y}^0 = \tilde{y}^1 = 0$. Let $(u^0, u^1) \in H'$ arbitrarily, first solve problem (1.2), then solve the problem

$$\begin{cases} y_i'' - \sigma_{ij,j}(y) = 0 \quad \text{in } \Omega \times \mathbb{R}, \\ y_i = 0 \quad \text{on } \Gamma_0 \times \mathbb{R}, \\ y_i = \sigma_{ij}(u)\nu_j \quad \text{on } \Gamma_1 \times \mathbb{R}, \\ y_i(T) = y_i'(T) = 0 \quad \text{in } \Omega, \\ i = 1, \dots, n, \end{cases} \tag{3.2}$$

and set

$$\Lambda(u^0, u^1) = (y'(0), -y(0)).$$

(Problem (3.2) is well-posed in an analogous sense as (1.11) in Theorem 3.1: $\exists! y \in C(\mathbb{R}; (L^2(\Omega))^n) \cap C^1(\mathbb{R}; (H^{-1}(\Omega))^n)$ satisfies (3.2); because it is a time-reversible problem.) Obviously, $\Lambda : H' \to H$ is a bounded linear map. Applying the HUM, it is sufficient to show that Λ is onto. Indeed, then for any given $(y^0, y^1) \in (L^2(\Omega))^n \times (H^{-1}(\Omega))^n$ it will suffice to choose the control ϑ defined by $\vartheta_i = 0$ on $\Gamma_0 \times \mathbb{R}^+$ and $\vartheta_i = \sigma_{ij}(u)\nu_j$ on $\Gamma_1 \times \mathbb{R}^+$, where u is the solution of (1.2) corresponding to $(u^0, u^1) = \Lambda^{-1}(y^1, -y^0)$.

We shall prove that Λ is in fact an isomorphism. We have

$$\begin{aligned}
&\int_0^T \int_\Omega u_i(y_i'' - \sigma_{ij,j}(y))\,dx\,dt \\
&\quad = \left[\int_\Omega (u_i y_i' - u_i' y_i)\,dx\right]_0^T + \int_0^T \int_\Omega (u_i'' - \sigma_{ij,j}(u))y_i\,dx\,dt \\
&\qquad + \int_0^T \int_\Gamma (-u_i\sigma_{ij}(y)\nu_j + \sigma_{ij}(u)\nu_j y_i)\,d\Gamma\,dt,
\end{aligned}$$

using the definition of Λ, the system (1.2) and (3.2). We obtain

$$\langle \Lambda(u^0, u^1), (u^0, u^1)\rangle_{H,H'} = \int_0^T \int_{\Gamma_1} \sum_i |\sigma_{ij}(u)\nu_j|^2\,d\Gamma\,dt. \tag{3.3}$$

Applying the first estimate of (1.10) in Theorem 1.1 and Lemma 2.2, we conclude from the identity (3.3) that Λ is coercive. Applying the Lax–Milgram theorem we conclude that Λ is an isomorphism.

Case 2: $\tilde{y}^0$ or $\tilde{y}^1 \neq 0$. We take $y = z + w$, where z is the solution of the problem

$$\begin{cases}
z_i'' - \sigma_{ij,j}(z) = 0 & \text{in } \Omega \times \mathbb{R}, \\
z_i = 0 & \text{on } \Gamma \times \mathbb{R}, \\
z_i(T) = \tilde{y}_i^{\,0} \quad \text{and} \quad z'(T) = \tilde{y}_i^{\,1} & \text{in } \Omega, \\
i = 1, \dots, n
\end{cases}$$

and w is the solution of the problem

$$\begin{cases}
w_i'' - \sigma_{ij,j}(w) = 0 & \text{on } \Omega \times \mathbb{R}, \\
w_i = \vartheta_i & \text{in } \Gamma \times \mathbb{R}, \\
w_i(0) = y_i^0 - z_i(0) \quad \text{and} \quad w_i'(0) = y_i^1 - z_i'(0) & \text{on } \Omega, \\
w_i(T) = w_i'(T) = 0 & \text{on } \Omega, \\
i = 1, \dots, n.
\end{cases}$$

The proof of Theorem 1.2 is now complete.

4. Stabilizability: proof of Theorem 1.3

The well-posedness of problem (1.12) can be established by standard methods of evolution systems (cf. Pazy [**12**, ch. 5]) as in the study of system (1.2). We have the following.

THEOREM 4.1. *Assume (1.1). Then for every given* $(u^0, u^1) \in (H^1_{\Gamma_0}(\Omega))^n \times H$, *problem (1.12) has a unique (weak) solution satisfying*

$$u \in C(\mathbb{R}^+; (H^1_{\Gamma_0}(\Omega))^n) \cap C^1(\mathbb{R}^+; H),$$

where $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$.

If $(u^0, u^1) \in W \times V$, *then the corresponding strong solution is more regular:*

$$u \in C(\mathbb{R}^+; W) \cap C^1(\mathbb{R}^+; V) \cap C^2(\mathbb{R}^+; H).$$

Let us turn to the proof of Theorem 1.3. All the computations which follow will be justified for a strong solution. Since the constant ω in (1.16) will not depend on $E(0)$, once estimates (1.16) are established for regular solutions, they will also be satisfied for all weak solutions by an easy density argument. For this, we shall prove that $\int_0^\infty E(t)\,dt \le (1/\omega)E(0)$, with ω the positive constant not depending on $E(0)$, and by [**5**, Theorem 8.1] we deduce estimate (1.16).

First we show the dissipativity of problem (1.12).

LEMMA 4.1. *The function* $E : \mathbb{R}^+ \to \mathbb{R}^+$ *is non-increasing and*

$$\begin{aligned} &E(0) - E(T) \\ &\quad = -\frac{1}{2}\int_0^T \int_\Omega a'_{ijkl}\varepsilon_{kl}\varepsilon_{ij}\,dx\,dt + \int_0^T \int_{\Gamma_1} b u'_i u'_i\,d\Gamma\,dt, \quad 0 \le T < \infty. \end{aligned} \tag{4.1}$$

Proof. We have

$$\begin{aligned} E' &= \int_\Omega (u'_i u''_i + \sigma_{ij}\varepsilon'_{ij} + \tfrac{1}{2}a'_{ijkl}\varepsilon_{kl}\varepsilon_{ij})\,dx + \int_{\Gamma_1} a u_i u'_i\,d\Gamma \\ &= \int_\Omega (u'_i\sigma_{ij,j} + \sigma_{ij}u'_{i,j})\,dx + \frac{1}{2}\int_\Omega a'_{ijkl}\varepsilon_{kl}\varepsilon_{ij}\,dx + \int_{\Gamma_1} a u_i u'_i\,d\Gamma \\ &= \int_{\Gamma_1} u'_i\sigma_{ij}\nu_j\,d\Gamma + \frac{1}{2}\int_\Omega a'_{ijkl}\varepsilon_{kl}\varepsilon_{ij}\,dx + \int_{\Gamma_1} a u_i u'_i\,d\Gamma \\ &= \frac{1}{2}\int_\Omega a'_{ijkl}\varepsilon_{kl}\varepsilon_{ij}\,dx - \int_{\Gamma_1} b u'_i u'_i\,d\Gamma \le 0; \end{aligned}$$

integrating between 0 and T we obtain (4.1).

Let $0 \leq T \leq \infty$ arbitrarily. We have

$$\begin{aligned} 0 &= \int_0^T \int_\Omega u_i(u_i'' - \sigma_{ij,j})\,dx\,dt \\ &= \left[\int_\Omega u_i u_i'\,dx\right]_0^T - \int_0^T \int_\Gamma u_i \sigma_{ij}\nu_j\,d\Gamma\,dt + \int_0^T \int_\Omega (\sigma_{ij}\varepsilon_{ij} - u_i'u_i')\,dx\,dt, \end{aligned}$$

whence

$$\int_0^T \int_\Gamma u_i\sigma_{ij}\nu_j\,d\Gamma\,dt = \left[\int_\Omega u_i u_i'\,dx\right]_0^T + \int_0^T \int_\Omega (\sigma_{ij}\varepsilon_{ij} - u_i'u_i')\,dx\,dt. \qquad (4.2)$$

Multiply (4.2) by $n-1+\gamma/2+2aR/\alpha$ and combine with (2.4) such that $h(x) = x - x_0$. Writing

$$M_i = 2(x_m - x_m^0)u_{i,m} + \left(n - 1 + \frac{\gamma}{2} + \frac{2aR}{\alpha}\right)u_i$$

for simplicity, we have

$$\begin{aligned} &\left(1 - \frac{\gamma}{2} - \frac{2aR}{\alpha}\right)\int_0^T \int_\Omega (\sigma_{ij}\varepsilon_{ij} + u_i'u_i')\,dx\,dt + \left[\int_\Omega M_i u_i'\,dx\right]_0^T \\ &= \int_0^T \int_\Gamma (M_i\sigma_{ij}\nu_j + (h\cdot\nu)(u_i'u_i' - \sigma_{ij}\varepsilon_{ij}))\,d\Gamma\,dt - \frac{4aR}{\alpha}\int_0^T \int_\Omega \sigma_{ij}\varepsilon_{ij}\,dx\,dt \\ &\quad + \int_0^T \int_\Omega (x_m - x_m^0)(\partial_m a_{ijkl})\varepsilon_{kl}\varepsilon_{ij}\,dx\,dt - \gamma\int_0^T \int_\Omega \sigma_{ij}\varepsilon_{ij}\,dx\,dt, \end{aligned}$$

by (1.6). The last part of this equality is negative; taking into account definition (1.13) of the energy, we can rewrite it in the following form:

$$\begin{aligned} &\left(2 - \gamma - \frac{4aR}{\alpha}\right)\int_0^T E(t)\,dt + \left[\int_\Omega M_i u_i'\,dx\right]_0^T \\ &\leq \left(1 - \frac{\gamma}{2} - \frac{2aR}{\alpha}\right)\int_0 \int_{\Gamma_1} a u_i u_i\,d\Gamma\,dx - \frac{4aR}{\alpha}\int_0^T \int_\Omega \sigma_{ij}\varepsilon_{ij}\,dx\,dt \\ &\quad + \int_0^T \int_\Gamma (M_i\sigma_{ij}\nu_j + (h\cdot\nu)(u_i'u_i' - \sigma_{ij}\varepsilon_{ij}))\,d\Gamma\,dt. \end{aligned}$$

Now using the boundary conditions in (1.12) we obtain

$$\left(2 - \gamma - \frac{4aR}{\alpha}\right)\int_0^T E(t)\,dt + \left[\int_\Omega M_i u_i'\,dx\right]_0^T$$

$$
\begin{aligned}
&\leq \int_0^T \int_{\Gamma_0} (h \cdot \nu)\sigma_{ij}\varepsilon_{ij}\, d\Gamma\, dt - \frac{4aR}{\alpha} \int_0^T \int_\Omega \sigma_{ij}\varepsilon_{ij}\, dx\, dt \\
&+ \int_0^T \int_{\Gamma_1} \left(\left(1 - \frac{\gamma}{2} - \frac{2aR}{\alpha} \right) au_i u_i - M_i(au_i + bu_i') \right. \\
&\left. + (h \cdot \nu)(u_i' u_i' - \sigma_{ij}\varepsilon_{ij}) \right) d\Gamma\, dt. \qquad (4.3)
\end{aligned}
$$

(The term on Γ_0 is obtained in the same way as in the proof of Theorem 1.1.)

Next we transform the integral over Γ_1. Applying the Green formula twice and using the boundary condition on Γ_0 we have

$$
\begin{aligned}
\int_\Omega u_{m,i} u_{i,m}\, dx &= \int_\Gamma (u_{m,i} u_i \nu_m - u_{m,m} u_i \nu_i)\, d\Gamma + \int_\Omega u_{m,m} u_{i,i}\, dx \\
&= \int_{\Gamma_1} (2\varepsilon_{mi} u_i \nu_m - u_{i,m} u_i \nu_m - \varepsilon_{mm} u_i \nu_i)\, d\Gamma + \int_\Omega \varepsilon_{mm}\varepsilon_{ii}\, dx.
\end{aligned}
$$

On the other hand

$$
\int_\Omega u_{m,i} u_{i,m}\, dx = \int_\Omega (2\varepsilon_{mi} u_{i,m} - u_{i,m} u_{i,m})\, dx = \int_\Omega (2\varepsilon_{mi}\varepsilon_{mi} - u_{i,m} u_{i,m})\, dx
$$

and therefore

$$
\int_\Omega (2\varepsilon_{mi}\varepsilon_{mi} - u_{i,m} u_{i,m} - \varepsilon_{mm}\varepsilon_{ii})\, dx = \int_{\Gamma_1} (2\varepsilon_{mi} u_i \nu_m - u_{i,m} u_i \nu_m - \varepsilon_{mm} \nu_i u_i)\, d\Gamma. \qquad (4.4)
$$

Multiplying (4.4) by $2aR$ and using the relation $h = R\nu$ on Γ_1 we deduce the following equality:

$$
\begin{aligned}
&\int_0^T \int_{\Gamma_1} (-2ah_m u_{i,m} u_i)\, d\Gamma\, dt \\
&= \int_0^T \int_\Omega (4aR\varepsilon_{mi}\varepsilon_{mi} - 2aRu_{i,m}u_{i,m} - 2aR|\operatorname{div} u|^2)\, dx\, dt \\
&+ \int_0^T \int_{\Gamma_1} (2aR(\operatorname{div} u)(\nu \cdot u) - 4aR\varepsilon_{mi} u_i \nu_m)\, d\Gamma\, dt. \qquad (4.5)
\end{aligned}
$$

Next we obtain by a similar computation that

$$
\int_0^T \int_\Omega u_{m,i} u_{i,m}'\, dx\, dt
$$

$$= \int_0^T \int_\Gamma (u_{m,i} u_i' \nu_m - u_{m,m} \nu_i u_i')\, d\Gamma\, dt + \int_0^T \int_\Omega u_{m,m} u_{i,i}'\, dx\, dt$$

$$= \int_0^T \int_{\Gamma_1} (2\varepsilon_{mi} u_i' \nu_m - u_{i,m} u_i' \nu_m - \varepsilon_{mm} \nu_i u_i')\, d\Gamma\, dt + \left[\frac{1}{2} \int_\Omega |\operatorname{div} u|^2\, dx \right]_0^T.$$

Furthermore,

$$\int_0^T \int_\Omega u_{m,i} u_{i,m}'\, dx\, dt = \int_0^T \int_\Omega (2\varepsilon_{mi} \varepsilon_{mi}' - u_{i,m} u_{i,m}')\, dx\, dt = \left[\int_\Omega (\varepsilon_{mi}\varepsilon_{mi} - \tfrac{1}{2} u_{i,m} u_{i,m})\, dx \right]_0^T.$$

Again using the relation $h = R\nu$ on Γ_1, it follows that

$$\begin{aligned} &\int_0^T \int_{\Gamma_1} (-2bh_m u_{i,m} u_i')\, d\Gamma\, dt \\ &\quad = \left[\int_\Omega (2bR\varepsilon_{mi}\varepsilon_{mi} - bRu_{i,m}u_{i,m} - bR|\operatorname{div} u|^2)\, dx \right]_0^T \\ &\qquad + \int_0^T \int_{\Gamma_1} (2bR(\operatorname{div} u)(\nu \cdot u') - 4bR\varepsilon_{mi} u_i' \nu_m)\, d\Gamma\, dt. \end{aligned} \tag{4.6}$$

Substituting equalities (4.5) and (4.6) into (4.3) and using the equality $h \cdot \nu = R$ on Γ_1 and $h \cdot \nu \leq 0$ on Γ_0, we obtain

$$\begin{aligned} &\left(2 - \gamma - \frac{4aR}{\alpha}\right) \int_0^T E(t)\, dt \\ &\leq \left[\int_\Omega (-M_i u_i' + 2bR\varepsilon_{mi}\varepsilon_{mi} - bRu_{i,m}u_{i,m} - bR|\operatorname{div} u|^2)\, dx \right]_0^T \\ &\quad + \int_0^T \int_\Omega \left(4aR\varepsilon_{mi}\varepsilon_{mi} - 2aRu_{i,m}u_{i,m} - 2aR|\operatorname{div} u|^2 - \frac{4aR}{\alpha}\sigma_{ij}\varepsilon_{ij} \right) dx\, dt \\ &\quad + \int_0^T \int_{\Gamma_1} \left(\left(1 - \frac{\gamma}{2} - \frac{2aR}{\alpha}\right) au_i u_i - \left(n - 1 + \frac{\gamma}{2} + \frac{2aR}{\alpha}\right) u_i(au_i + bu_i') \right. \\ &\quad + R(u_i' u_i' - \sigma_{ij}\varepsilon_{ij}) + 2aR(\operatorname{div} u)(\nu \cdot u) - 4aR\varepsilon_{mi} u_i \nu_m \\ &\quad \left. + 2bR(\operatorname{div} u)(\nu \cdot u') - 4bR\varepsilon_{mi} u_i' \nu_m \right) d\Gamma\, dt. \end{aligned} \tag{4.7}$$

Let us majorize the right-hand side of this identity. Using the definition of the energy and the Korn inequality,

$$\left| \int_\Omega (-M_i u_i' + 2bR\varepsilon_{mi}\varepsilon_{mi} - bRu_{i,m}u_{i,m} - bR|\operatorname{div} u|^2)\, dx \right| \le c_1 E(t)$$

and

$$\begin{aligned} &\left| -b\left(n - 1 + \frac{\gamma}{2} + \frac{2aR}{\alpha}\right) \int_0^T \int_{\Gamma_1} u_i u_i' \, d\Gamma\, dt \right| \\ = &\left| \frac{b}{4}\left(2n - 2 + \gamma + \frac{4aR}{\alpha}\right) \left[\int_{\Gamma_1} u_i u_i \right]_0^T \right| \le c_2 E(t) \end{aligned}$$

with some constants c_1 and c_2 independent of $E(0)$ and of T.

By condition (1.1) we have

$$\begin{aligned} &\int_\Omega \left(4aR\varepsilon_{ij}\varepsilon_{ij} - 2aRu_{i,m}u_{i,m} - 2aR|\operatorname{div} u|^2 - \frac{4aR}{\alpha}\sigma_{ij}\varepsilon_{ij}\right) dx \\ \le &\int_\Omega \left(\frac{4aR}{\alpha}\sigma_{ij}\varepsilon_{ij} - \frac{4aR}{\alpha}\sigma_{ij}\varepsilon_{ij}\right) dx = 0. \end{aligned}$$

Applying Lemma 4.1 and using (1.14) we deduce that

$$R\int_0^T \int_{\Gamma_1} u_i' u_i' \, d\Gamma\, dt \le \frac{R}{b}(E(0) - E(T)) \le \frac{R}{b}E(0).$$

Then we deduce from identity (4.7) (also using (1.1) on $-R\sigma_{ij}\varepsilon_{ij}$) the following inequality:

$$\begin{aligned} &\left(2 - \gamma - \frac{4aR}{\alpha}\right) \int_0^T E(t)\, dt \\ \le\ & c_3 E(0) + \left(2 - n - \gamma - \frac{4aR}{\alpha}\right) \int_0^T \int_{\Gamma_1} a u_i u_i \, dx \\ &+ \int_0^T \int_{\Gamma_1} (-R\alpha\varepsilon_{ij}\varepsilon_{ij} + 2aR(\operatorname{div} u)(u \cdot \nu) - 4aR\varepsilon_{mi}u_i\nu_m \\ &+ 2bR(\operatorname{div} u)(u' \cdot \nu) - 4bR\varepsilon_{mi}u_i'\nu_m)\, d\Gamma\, dt. \end{aligned} \tag{4.8}$$

Here, $c_3 = 2c_1 + 2c_2 + R/b$. For any fixed $\delta > 0$ we have

$$2aR(\operatorname{div} u)u \cdot \nu \quad \le \quad \delta|\operatorname{div} u|^2 + a^2R^2\delta^{-1}|u|^2,$$

$$\begin{aligned} 2bR(\operatorname{div} u)u' \cdot \nu &\leq \delta|\operatorname{div} u|^2 + b^2R^2\delta^{-1}|u'|^2, \\ -4aR\varepsilon_{mi}u_i\nu_m &\leq \delta\varepsilon_{mi}\varepsilon_{mi} + 4a^2R^2\delta^{-1}|u|^2, \\ -4bR\varepsilon_{mi}u_i'\nu_m &\leq \delta\varepsilon_{mi}\varepsilon_{mi} + 4b^2R^2\delta^{-1}|u'|^2. \end{aligned}$$

Substituting them into (4.8) and using the inequality $|\operatorname{div} u|^2 \leq \varepsilon_{mi}\varepsilon_{mi}$, we obtain

$$\begin{aligned} &\left(2-\gamma-\frac{4aR}{\alpha}\right)\int_0^T E(t)\,dt \leq c_3E(0) \\ &\quad + \int_0^T\int_{\Gamma_1}\left(\left(2-n-\gamma-\frac{4aR}{\alpha}+5aR^2\delta^{-1}\right)a|u|^2\right. \\ &\quad \left. + (4\delta-\alpha R)\varepsilon_{mi}\varepsilon_{mi} + 5bR^2\delta^{-1}b|u'|^2\right)d\Gamma\,dt. \end{aligned}$$

Using (1.14) and (4.1) we have

$$5bR^2\delta^{-1}\int_0^T\int_{\Gamma_1} b|u'|^2\,d\Gamma\,dt \leq 5bR^2\delta^{-1}(E(0)-E(T)) \leq 5bR^2\delta^{-1}E(0).$$

Substituting into the preceding inequality and choosing $\delta = \alpha R/4$, we conclude that

$$\left(2-\gamma-\frac{4aR}{\alpha}\right)\int_0^T E(t)\,dt \leq c_4E(0) + \left(2-n-\gamma+16\frac{aR}{\alpha}\right)\int_0^T\int_{\Gamma_1} a|u|^2\,d\Gamma\,dt \tag{4.9}$$

with $c_4 = c_3 + 20bR/\alpha$.

Applying a method of Conrad and Rao [**3**] we shall prove the following lemma.

LEMMA 4.2. *For any given $\epsilon > 0$, there exists a constant $c_5 > 0$ such that*

$$\int_0^T\int_{\Gamma_1} |u|^2\,d\Gamma\,dt \leq c_5E(0) + \epsilon\int_0^T E(t)\,dt$$

for all $T \geq 0$.

Assuming this lemma, choosing $\epsilon > 0$ such that $(2-n-\gamma+16aR/\alpha)a\epsilon < 2-\gamma-4aR/\alpha$ if $n < 2-\gamma+16aR/\alpha$, we deduce from (4.9) the inequality

$$\int_0^T E(t)\,dt \leq cE(0), \quad \text{for all } T \geq 0,$$

where c is a constant independent of $E(0)$ and of T. Then we conclude that

$$\int_0^\infty E(t)\,dt \leq cE(0)$$

and obtain (1.16) with $\omega = 1/c$.

It remains to prove Lemma 4.2. For every $t \geq 0$ let us denote by $\eta(t)$ the solution of the problem

$$\begin{cases} -\sigma_{ij,j} = 0 & \text{in } \Omega, \\ \eta = u & \text{on } \Gamma. \end{cases}$$

Then we have

$$\|\eta\|_{(L^2(\Omega))^n} \leq c\|u\|_{(L^2(\Gamma))^n} \leq c\sqrt{E(t)}, \tag{4.10}$$

where c is a positive constant not depending on u. Applying (4.10) with u' instead of u we also obtain

$$\|\eta'\|_{(L^2(\Omega))^n} \leq c\|u'\|_{(L^2(\Gamma))^n} \leq c\sqrt{|E'|}. \tag{4.11}$$

Let us also observe that

$$\int_\Omega \sigma_{ij}(\eta)\varepsilon_{ij}(u-\eta)\,dx = -\int_\Omega \sigma_{ij,j}(\eta)(u_i-\eta_i)\,dx + \int_\Gamma \sigma_{ij}(\eta)\nu_j(u_i-\eta_i)\,d\Gamma = 0; \tag{4.12}$$

hence

$$\int_\Omega \sigma_{ij}(\eta)\varepsilon_{ij}(u)\,dx = \int_\Omega \sigma_{ij}(\eta)\varepsilon_{ij}(\eta)\,dx \geq 0. \tag{4.13}$$

Integrating by parts on $\Omega \times [0,T]$ and using the boundary conditions on u and on η we deduce the following inequality:

$$\begin{aligned} 0 &= \int_0^T \int_\Omega \eta_i(u_i'' - \sigma_{ij,j}(u))\,dx\,dt \\ &= \left[\int_\Omega \eta_i u_i'\,dx\right]_0^T + \int_0^T \int_\Omega (-\eta_i' u_i' + \sigma_{ij}(u)\varepsilon_{ij}(\eta))\,dx\,dt \\ &\quad + \int_0^T \int_\Gamma u_i(au_i + bu_i')\,d\Gamma\,dt. \end{aligned}$$

Using (4.13) we obtain

$$a\int_0^T \int_\Gamma |u|^2\,d\Gamma\,dt \leq -\left[\int_\Omega \eta_i u_i'\,dx + \frac{b}{2}\int_\Gamma |u|^2\,d\Gamma\right]_0^T + \int_0^T \int_\Omega \eta_i' u_i'\,dx\,dt. \tag{4.14}$$

By the definition of the energy and inequality (4.10) we have

$$\left|\left[\int_\Omega \eta_i u_i'\,dx + \frac{b}{2}\int_\Gamma |u|^2\,d\Gamma\right]_0^T\right| \leq cE(0).$$

Furthermore, using (4.11) and applying the Poincaré inequality we obtain

$$\begin{aligned}\int_0^T \int_\Omega \eta_i' u_i' \, dx \, dt &\leq \int_0^T \|\eta'\|_{(L^2(\Omega))^n} \|u'\|_{(L^2(\Omega))^n} \, dt \\ &\leq c \int_0^T \sqrt{|E'|}\sqrt{E} \, dt \leq \int_0^T \left(a\epsilon E + \frac{c^2}{4a\epsilon}(-E') \right) dt \\ &\leq a\epsilon \int_0^T E(t) \, dt + c' E(0).\end{aligned}$$

Substituting these inequalities into (4.14), then the lemma follows.

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