DYNAMICS OF KLEIN-GORDON ON A COMPACT SURFACE NEAR A HOMOCLINIC ORBIT

by

Benoît Grébert, Tiphaine Jézéquel and Laurent Thomann

Abstract. — We consider the Klein-Gordon equation (KG) on a Riemannian surface M
$$\partial_t^2 u - \Delta u - m^2 u + u^{2p+1} = 0, \quad p \in \mathbb{N}^*, \quad (t,x) \in \mathbb{R} \times M,$$
which is globally well-posed in the energy space. This equation has a homoclinic orbit to the origin, and in this paper we study the dynamics close to it. Using a strategy from Groves-Schneider, we get the existence of a large family of heteroclinic connections to the center manifold that are close to the homoclinic orbit during all times. We point out that the solutions we construct are not small.

Contents

1. Introduction, statement of the main results

1.1. General introduction. — Denote by M a compact Riemannian manifold without boundary of dimension 1,2 or 3 and denote by $\Delta = \Delta_M$ the Laplace-Beltrami operator on M. In this paper we

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are concerned with the following nonlinear Klein-Gordon (KG) equation

$$\text{(KG)} \qquad \begin{cases} \partial_t^2 u - \Delta u - m^2 u + u^{2p+1} = 0, \quad (t,x) \in \mathbb{R} \times M, \\ u(0,x) = u_0(x), \quad \partial_t u(0,x) = u_1(x), \end{cases}$$

where $p \geq 1$ is an integer, and $(u_0, u_1) \in H^1(M) \times L^2(M)$ are real-valued.

It is well-known that there exists a Hilbert basis of $L^2(M)$ composed with eigenfunctions $(e_n)_{n \geq 0}$ of Δ. Moreover (see *e.g.* [**15**]), there exists a sequence $0 = \lambda_0 < \lambda_1 \leq \cdots \leq \lambda_n \leq \ldots$ so that

$$-\Delta e_n = \lambda_n^2 e_n, \quad n \geq 0.$$

In the sequel, we define the scalar product on $L^2(M)$ by $\langle f, g \rangle = \dfrac{1}{\operatorname{Vol} M} \displaystyle\int_M fg$, where $\operatorname{Vol} M$ denotes the volume of M, we assume that $\|e_n\|_{L^2} = 1$ and we set $e_0 = 1$.

We make following assumptions

Assumption 1. — *The parameter m satisfies $0 < m < \lambda_1$.*

Assumption 2. — *The manifold M and the integer p satisfy either:*

- *M is any compact manifold without boundary of dimension 1 or 2 and $p \geq 1$*

- *M is any compact manifold without boundary of dimension 3 and $p = 1$.*

Moreover, up to a rescaling, we can assume that $\operatorname{Vol} M = 1$.

The stationary solutions of (KG) (solutions which only depend on the space variable) are exactly the constants $u = 0$, $u = m^{1/p}$ and $u = -m^{1/p}$ (see Lemma 2.1). The origin is an equilibrium with an unstable direction. In fact, the eigenvalues of $-\Delta - m^2$ are the $(\lambda_k^2 - m^2)_{k \in \mathbb{N}}$. Since $0 < m < \lambda_1$, the case $k = 0$ only, gives the hyperbolic directions, corresponding to the solution $\exp(mt)$ for $t > 0$ (resp. $\exp(-mt)$ for $t < 0$). It turns out that (KG) admits a homoclinic orbit to the origin. Indeed, the following x−independent function is a solution to (KG)

$$\alpha(t) = \frac{m^{1/p}(p+1)^{1/(2p)}}{\big(\cosh(pmt)\big)^{1/p}},$$

and in the sequel we will refer to

$$h(t) = (\alpha(t), \beta(t)), \quad \text{with} \quad \dot{\beta}(t) = \alpha(t),$$

as the time homoclinic solution (to the origin) of (KG), see Section 1.5. In this work we aim to study the dynamics of (KG) near this particular trajectory, and we will show the existence of solutions which remain close for all times to this exact temporal solution.

In the case $M = \mathbb{S}^1$ we can precise the dynamics around the equilibrium $u = m^{1/p}$ (the study near $u = -m^{1/p}$ is similar since the non-linearity is odd). We linearize the equation near this point $(u = m^{1/p} + v)$ and we are lead to study the spectrum of $(-\partial_x^2 + 2pm^2)v$. This operator on $\mathbb{S}^1$ is self adjoint and has pure point spectrum $j^2 + 2pm$, $j \in \mathbb{N}$. Using the Birkhoff normal form theory,

Bambusi [1] (see also [3]) has shown that, for a generic choice of m (in order to avoid resonances between the frequencies $\sqrt{j^2 + 2pm}$, $j \in \mathbb{N}$), the solutions of (KG) with initial datum of the form $m^{1/p} + v_0$ with v_0 small enough, remain close to the equilibrium point $m^{1/p}$ for very long time. See also the work [8] of J.-M. Delort for quasi-linear equations.

Moreover, using KAM theory, C. E. Wayne [28] and J. Pöschel [26] have proved that, for a generic choice of m, there exist many quasi-periodic solutions near $w = m^{1/p}$. See also the recent work of Berti-Biasco-Procesi [4] for derivative wave equations. Observe that quasi-periodic solutions can be constructed even if the equilibrium has a finite number of hyperbolic directions (see [5]).

In higher dimension and for a general manifold, few is known. In the case M is a Zoll manifold, Bambusi-Delort-Grébert-Sjeftel [2] have developed a Birkhoff normal form theory for (KG) near an elliptic equilibrium. Up to now, there is no KAM-type result for (KG) in dimension greater than two.

In this work, we describe some possible behaviours of (KG) near the homoclinic orbit. We state the existence of solutions that travel from a neighbourhood of the origin to turn around the equilibrium $m^{1/p}$, close to the homoclinic connection to 0 (see Theorems 1.1 and 1.2). We stress out that these results do not require non resonance conditions.

The existence of homoclinic or heteroclinic connections to periodic or quasi-periodic solutions is a question of interest: for example, in the case of parabolic PDE's on a one dimensional bounded domain, homoclinic orbits to equilibrium points can exist but there cannot exist any homoclinic connections to periodic orbits (see [19]). In some other cases some homoclinic connections to small periodic orbits exist while homoclinic orbits to 0 do not exist or remain an open question (see for instance [20, 21, 18]). More generally, such homoclinic connections to the center manifold often appear when a homoclinic orbit does not persist after a perturbation of the system (see for instance [27]). We refer also to [6], where the existence of homoclinic solutions to a beam equation is studied.

In this paper, we get the existence of a large family of homoclinic connections to the center manifold (more precisely heteroclinic connections to some solutions lying in the center manifold). The proof is based on a perturbative method which is classical (see for instance [17, 20]) for finite dimensional reversible systems (systems which anticommute with a symmetry S): the key idea is that if a solution hits the reversibility plane $\{u \mid u = Su\}$, then the latter solution is reversible. This method requires more computations in the infinite dimensional cases. It was already performed in a situation close to ours by M. Groves and G. Schneider [12, 13, 14]: but in their case, they work in the neighbourhood of a bifurcation and get small homoclinic solutions, while in our case the size of the solutions is of order 1 and this requires some additional work on the linearised system, which is one of our main contributions in this paper.

Finally we also mention the recent book of Nakanishi & Schlag [24] on invariant manifolds in the context of dispersive Hamiltonian PDEs.

1.2. A first motivation for studying (KG). — Recall that the usual non linear Klein-Gordon equation reads

$$(1.1) \qquad \partial_t^2 w - \Delta w + w - f(w) = 0, \quad (t, x) \in \mathbb{R} \times M,$$

where f is a non linear function. Let us show that if there exists a nonzero equilibrium, the equation near the smallest equilibrium is of the form (KG) (but with a general non linearity). The equilibrium of (1.1) are the real constants w_0 satisfying $f(w_0) - w_0 = 0$. Observe that if one performs the change

of coordinates $u = w - w_0$, then there exists a non linear function g such that u satisfies

$$\partial_t^2 u - \Delta u - (f'(w_0) - 1)u - g(u) = 0.$$

Given that f is non linear, $f'(0) = 0$ and $f'(0) - 1 < 0$. Thus, if f is $\mathcal{C}^1$ and if there exists an equilibrium $w_0 \neq 0$, then by the intermediate value theorem, we get that the smallest equilibrium w_0 satisfies $f'(w_0) - 1 \geq 0$. This means that near the smallest non zero equilibrium of (1.1), the equation is of the form

$$\partial_t^2 u - \Delta u - m^2 u - g(u) = 0.$$

For instance in this paper we consider a nonlinearity $g(u) = u^{2p+1}$, $u = m^{1/p}$ is a (stable) equilibrium and the change of variable $u = m^{1/p} + w$ leads to

$$\partial_t^2 w - \Delta w + 2m^2 w - f(w) = 0, \quad (t, x) \in \mathbb{R} \times M$$

with a nonlinearity $f(w) = (w + m^{\frac{1}{p}})^{2p+1} - m^{\frac{2p+1}{p}} - (2p + 1)m^2 w$.

1.3. Another motivation of the problem. — Here we are inspired from the works [12, 13, 14] of M. Groves and G. Schneider. Consider the non linear wave equation on the real line

$$(1.2) \qquad \partial_t^2 w - \partial_x^2 w + w - w^{2p+1} = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{R}.$$

One can be interested in the construction of "modulated pulse solutions " for (1.2) which are solutions of the form

$$(1.3) \qquad w(t, x) = u\Big(\frac{x - \beta t}{c_\beta}, \frac{t - \beta x}{c_\beta}\Big),$$

where $u(s, y)$ is a 2π−periodic function in y, and where $\beta \in (0, 1)$ and $c_\beta > 0$. In the particular case $c_\beta = (1 - \beta^2)^{1/2}$, (1.3) is the Lorentz transform which preserves (1.2) and in general we get

$$(1.4) \qquad \partial_s^2 u - \partial_y^2 u - \frac{c_\beta^2}{1 - \beta^2}\big(u - u^{2p+1}\big) = 0, \quad (s, y) \in \mathbb{R} \times \mathbb{S}^1.$$

Now let $m \in (0, 1)$ and $\beta \in (0, 1)$, then we can choose $c_\beta > 0$ so that $\frac{c_\beta^2}{1 - \beta^2} = m^2 \in (0, 1)$, which is (KG) up to the change of unknown $u \longmapsto u^{1/p}w$. As a consequence, for each $m \in (0, 1)$, any solution to (1.4) provides a one-parameter family of solutions to (1.2).

1.4. Hamiltonian structure of (KG). — Denote by $v = \partial_t u$ and introduce

$$(1.5) \qquad H = \frac{1}{2} \int_M \Big(|\nabla_x u|^2 + v^2 - m^2 u^2\Big) + \frac{1}{2p + 2} \int_M u^{2p+2}.$$

Then, the system (KG) is equivalent to

$$(1.6) \qquad \dot{u} = \frac{\delta H}{\delta v}, \quad \dot{v} = -\frac{\delta H}{\delta u}.$$

We write

$$u(t, x) = \sum_{n=0}^{\infty} a_n(t)e_n(x), \quad v(t, x) = \sum_{n=0}^{\infty} b_n(t)e_n(x),$$

where

$$(a_n)_{n\in\mathbb{N}} \in h^1(\mathbb{N}, \mathbb{R}) := \big\{ x = (x_n)_{n\in\mathbb{N}} \mid \|x\|_{h^1}^2 = \sum_{n\in\mathbb{N}} (1+\lambda_n^2)|x_n|^2 < \infty \big\},$$

$$(b_n)_{n\in\mathbb{N}} \in \ell^2(\mathbb{N}, \mathbb{R}) := \big\{ x = (x_n)_{n\in\mathbb{N}} \mid \|x\|_{\ell^2}^2 = \sum_{n\in\mathbb{N}} |x_n|^2 < \infty \big\},$$

in such a way that to the continuous phase space $\mathcal{X} := H^1 \times L^2$ corresponds the discrete one $h^1 \times \ell^2$. We endow this space with the natural norm

$$\|X\| = \|u\|_{H^1} + \|v\|_{L^2}, \quad \text{for} \quad X = (u, v).$$

We define the two dimensional subspace $\mathcal{X}_h \subset \mathcal{X}$ spanned by the stable/unstable modes (hyperbolic modes) of X

$$(1.7) \qquad \mathcal{X}_h = \big\{ X \in \mathcal{X} \mid \forall\, n \geq 1, \quad a_n = b_n = 0 \big\},$$

and we observe that $\mathcal{X}_h$ is an invariant subspace for (1.6). We also define the infinite dimensional subspace $\mathcal{X}_c \subset \mathcal{X}$ spanned by the center modes (elliptic modes) of X

$$(1.8) \qquad \mathcal{X}_c = \big\{ X \in \mathcal{X} \mid a_0 = b_0 = 0 \big\}.$$

We clearly have $\mathcal{X} = \mathcal{X}_h \oplus \mathcal{X}_c$. Denote by $P : \mathcal{X} \longrightarrow \mathcal{X}_h$ the projection onto $\mathcal{X}_h$ and define $Q = I - P : \mathcal{X} \longrightarrow \mathcal{X}_c$ the projection onto $\mathcal{X}_c$. In the sequel, for $X \in \mathcal{X}$, we use the notation $X_h = PX$ and $X_c = QX$.

In the coordinates $(a_n, b_n)_{n\geq 0}$, the Hamiltonian in (1.5) reads

$$(1.9) \qquad H = \frac{1}{2} \sum_{n=0}^{\infty} \Big[(\lambda_n^2 - m^2)a_n^2 + b_n^2 \Big] + \frac{1}{2p+2} \int_M \Big(\sum_{k=0}^{\infty} a_k e_k(x) \Big)^{2p+2} \mathrm{d}x,$$

and the system (1.6) becomes

$$(1.10) \qquad \begin{cases} \dot{a}_n = b_n, \quad n \geq 0 \\[2mm] \dot{b}_n = -(\lambda_n^2 - m^2)a_n - \displaystyle\int_M \Big(\sum_{k=0}^{\infty} a_k e_k(x) \Big)^{2p+1} e_n(x)\mathrm{d}x, \quad n \geq 0. \end{cases}$$

1.5. The homoclinic orbit. — The space-stationary solutions of (KG) exactly correspond to the solutions of (1.10) satisfying $a_n = b_n = 0$ for $n \geq 1$. In this case, the equation on (a_0, b_0) reads

$$(1.11) \qquad \begin{cases} \dot{a}_0 = b_0 \\[2mm] \dot{b}_0 = m^2 a_0 - a_0^{2p+1}, \end{cases}$$

and this system possesses a homoclinic solution to 0 which we will denote in the sequel by

$$h : t \mapsto (\alpha(t), \beta(t)),$$

and we will denote by $\mathcal{K}_0$ the curve which is described (see Figure 1). Indeed we can explicitly compute

$$(1.12) \qquad \alpha(t) = \frac{m^{1/p}(p+1)^{1/(2p)}}{\big(\cosh(pmt) \big)^{1/p}}, \quad \beta(t) = -m^{1/p+1}(p+1)^{1/(2p)} \frac{\sinh(pmt)}{\big(\cosh(pmt) \big)^{1/p+1}},$$

and we have the bounds

$$(1.13) \qquad |\alpha(t)| \leq Ce^{-m|t|}, \quad |\beta(t)| \leq Ce^{-m|t|}, \quad \text{for all} \quad t \in \mathbb{R}.$$

For $\eta > 0$ denote by $\mathcal{K}_\eta$ the trajectory of (1.11) given by the initial conditions $a_0(0) = \eta$, $b_0(0) = 0$ (see Figure 1). In our forthcoming paper [10], we study the long time stability of the trajectory $\mathcal{K}_\eta$ for $0 < \eta \ll 1$, by the flow of the system (1.10).
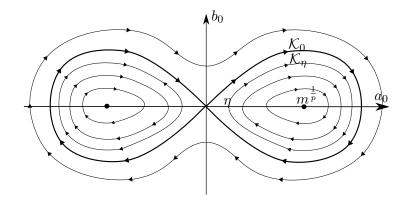

FIGURE 1. Phase portrait for the space-stationary set $a_n = b_n = 0$ for $n \geq 1$.

1.6. The main results on the Klein-Gordon equation. — Under Assumptions 1 and 2, the equation (KG) is globally well-posed in $\mathcal{X} = H^1(M) \times L^2(M)$ (see Theorem 2.2 in Section 2).

Our main result describes possible dynamics near the homoclinic orbit: We show that there are many solutions which stay close to h. The proof of this result is inspired from the work [12] of M. Groves and G. Schneider. In our case, the novelty is that we have not to assume m to be small, *i.e.* we are not dealing with small solutions of (KG).

For any function $f \in L^2(M)$, denote by $f_c = f - \int_M f$. Then we can state

Theorem 1.1. — *Let M, p and m satisfy Assumptions 1 and 2. There exist $C > 0$ and $\varepsilon_0 > 0$ so that the following holds: Let $(f,g) \in H^1(M) \times L^2(M)$ be such that $\|f_c\|_{H^1} + \|g_c\|_{L^2} \leq \varepsilon_0$, then there exists a solution u to (KG) so that*

$$(1.14) \qquad \partial_t u(0,\cdot) = g_c, \quad u_c(0,\cdot) = f_c,$$

and so that for all $t \geq 0$

$$(1.15) \qquad \|u(t,\cdot) - \alpha(t)\|_{H^1} + \|\partial_t u(t,\cdot) - \beta(t)\|_{L^2} \leq C\big(\|f_c\|_{H^1} + \|g_c\|_{L^2}\big).$$

Furthermore, as $t \longrightarrow +\infty$, $u(t,\cdot)$ tends to a solution of the local center manifold W^c (see Definition 4.6).

Notice that Theorem 1.1 is not a stability result since we only fix the elliptic part of the initial datum, and the statement of the theorem is that there exists a way to define the hyperbolic part such

that the solution satsifies (1.15). For example if $f_c = g_c = 0$, the corresponding u is the homoclinic orbit h.

The equation (KG) is reversible. More precisely, the equation (KG) is invariant under the transformation $u(t, x) \longmapsto u(-t, x)$. This corresponds to the symmetry S defined by

$$S(a_n) = a_n \quad S(b_n) = -b_n, \quad \text{for} \quad n \geq 0.$$

This is a symmetry of reversibility: Namely, denote by V_H the vector field (1.10), then $V_H \circ S = -S \circ V_H$.

Using this symmetry we are able to precise the result of Theorem 1.1 in the case $g = 0$.

Theorem 1.2. — *Let M, p and m satisfy Assumptions 1 and 2. There exist $C > 0$ and $\varepsilon_0 > 0$ so that the following holds: Let $f \in H^1(M)$ be such that $\|f_c\|_{H^1} \leq \varepsilon_0$, then there exists a solution u to (KG) so that*

$$(1.16) \qquad\qquad \partial_t u(0, \cdot) = 0, \quad u_c(0, \cdot) = f_c,$$

for all $t \in \mathbb{R}$

$$u(-t) = Su(t),$$

and so that

$$\|u(t, \cdot) - \alpha(t)\|_{H^1} + \|\partial_t u(t, \cdot) - \beta(t)\|_{L^2} \leq C\|f_c\|_{H^1}.$$

Furthermore, as $t \longrightarrow +\infty$, $u(t, \cdot)$ tends to a solution of the local center manifold W^c.

Since u is reversible, observe that if u tends to w when $t \longrightarrow +\infty$, then u tends to Sw when $t \longrightarrow -\infty$.

Remark 1.3. — Theorem 1.2 proves the existence of heteroclinic orbits from some orbit on the center manifold W^c to another one. It would be interesting to precise the dynamics on the center manifold. Actually we can expect the existence of invariant tori on W^c which are perturbations of the tori obtained for the linearized equation around the homoclinic orbit (see section 3). Nevertheless, to prove this, we would need a KAM theory for infinite dimensional tori which is an open problem up to now (see however [**7**]). Notice that even if such theory would exist, it would prove that most of the tori, but not all of them, of the linearized system are preserved. Thus it would be impossible to conclude that our heteroclinic orbits tends to KAM tori of W^c when $t \to \pm\infty$ as we can expect.

1.7. Plan of the paper. — In Section 2 we prove Theorem 2.2. Section 3 is devoted to the study of the linearisation of (KG) around the homoclinic orbit (Theorem 3.2). In Section 4 we construct the center manifold for the Klein-Gordon equation and prove Theorem 1.1. In the appendix, we recall a result which is useful in the proof of Theorem 3.2.

Notations. — *In this paper $c, C > 0$ denote constants the value of which may change from line to line. These constants will always be universal, or depend on the fixed quantities m and p.*
We denote by $\mathbb{N}$ the set of the non negative integers, and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$. We set $\mathcal{X} = H^1(M) \times L^2(M)$.

2. Preliminaries: Proof of the first results on (KG)

2.1. Stationary solutions of (KG). — We look for the time-independent solutions to (KG), which correspond to equilibrium points for the system (1.10).

Lemma 2.1. — *The stationary solutions to* (KG) *are* $u = 0$, $u = m^{1/p}$ *and* $u = -m^{1/p}$.

Proof. — Assume that $u(t, x) = g(x) \in H^1(M)$ is solution to (KG). We apply ∇ to the equation and get
$$-\Delta \nabla g - m^2 \nabla g + (2p+1)g^{2p}\nabla g = 0.$$
We multiply this equation with ∇g and integrate (by parts) on M

$$(2.1) \qquad \int_M \left(\Delta g\right)^2 + (2p+1)\int_M g^{2p}|\nabla g|^2 = m^2 \int_M |\nabla g|^2.$$

Now, write $g = \sum_{k=0}^{\infty} g_k e_k$. Since $(\lambda_k)_{k \geq 0}$ is non decreasing and $0 < m < \lambda_1$ we have

$$m^2 \int_M |\nabla g|^2 = m^2 \sum_{k=1}^{\infty} \lambda_k^2 g_k^2 < \sum_{k=1}^{\infty} \lambda_k^4 g_k^2 = \int_M \left(\Delta g\right)^2.$$

This inequality together with (2.1) implies that $\nabla g = 0$, and we conclude. $\qquad\square$

2.2. Global well-posedness of (KG). — The aim of this section is to prove the following result.

Theorem 2.2. — *Let M, p and m satisfy Assumptions 1 and 2. Then for all $(u_0, u_1) \in \mathcal{X}$ there exists a unique solution to* (KG)
$$u \in \mathcal{C}^0\big(\mathbb{R}; H^1(M)\big) \cap \mathcal{C}^1\big(\mathbb{R}; L^2(M)\big).$$
Moreover, u is bounded in the energy space: $\sup_{t \in \mathbb{R}} \|(u, \partial_t u)\|_{\mathcal{X}} \leq C.$

The proof is classical: it relies on a fixed point argument in $H^1(M)$ combined with the conservation of the energy.

To begin with, for $u = \sum_{k=0}^{\infty} a_k e_k$ and $\partial_t u = \sum_{k=1}^{\infty} b_k e_k$ we set

$$(2.2) \qquad J := \frac{1}{2}\sum_{k=1}^{\infty} \left[(\lambda_k^2 - m^2)a_k^2 + b_k^2\right].$$

We also define

$$(2.3) \qquad U := u_c = \sum_{k=1}^{\infty} a_k e_k$$

the spectral projection away the mode 0. Observe that under Assumption 1 there exist $C_1, C_2 > 0$ so that for all $(u, \partial_t u) \in \mathcal{X}$

$$(2.4) \qquad C_1 \|u_c, \partial_t u_c\|_{\mathcal{X}} \leq J^{1/2} \leq C_2 \|u_c, \partial_t u_c\|_{\mathcal{X}},$$

because $\|u_c, \partial_t u_c\|_{\mathcal{X}} = \|u_c\|_{H^1} + \|\partial_t u_c\|_{L^2}$ and

$$\|u_c\|_{H^1}^2 + \|\partial_t u_c\|_{L^2}^2 = \sum_{k=1}^{\infty} \left(\lambda_k^2 a_k^2 + b_k^2\right).$$

In the sequel we will need

Lemma 2.3. — *Let $u \in H^1(M)$. Then*

- *When M has dimension 1 or 2, for all $2 \leq q < +\infty$*

$$\|U\|_{L^q(M)} \leq C_q J^{1/2}.$$

- *When M has dimension 3, for all $2 \leq q \leq 6$*

$$\|U\|_{L^q(M)} \leq C_q J^{1/2}.$$

Proof. — By Sobolev, in each of the previous cases, there exists $C_q > 0$ so that for all $U \in H^1(M)$ we have $\|U\|_{L^q(M)} \leq C_q \|U\|_{H^1(M)}$ and the result follows. $\qquad\square$

We then can prove the following a priori estimate

Lemma 2.4. — *Let $u(t, \cdot)$ be a solution of* (KG) *and denote by H^0 the constant value of $H(u(t, \cdot))$ along this trajectory. For all t for which the solution is defined, we have*

$$(2.5) \qquad \int_M \left((\partial_t u)^2 + |\nabla u|^2\right) = b_0^2 + \sum_{k=1}^{\infty} \left(b_k^2 + \lambda_k^2 a_k^2\right) \leq 2H^0 + \frac{p}{p+1} m^{2+2/p}.$$

Furthermore there exists $C \equiv C(H^0) > 0$ such that $|a_0(t)| \leq C$ for all t.

Proof. — We assume that M has dimension 1 or 2. By the conservation of the energy for (KG), we have

$$\int_M \left((\partial_t u)^2 + |\nabla u|^2\right) + \frac{1}{p+1} \int_M u^{2p+2} = 2H^0 + m^2 \int_M u^2.$$

We apply the Young inequality

$$c_1 c_2 = (\varepsilon c_1)(\varepsilon^{-1} c_2) \leq \frac{\varepsilon^q}{q} c_1^q + \frac{1}{r\varepsilon^r} c_2^r, \quad c_1, c_2 \geq 0, \quad \varepsilon > 0, \quad \frac{1}{q} + \frac{1}{r} = 1,$$

with $c_1 = u^2$, $c_2 = 1$, $q = p + 1$ and $\varepsilon = m^{-2/(p+1)}$ and deduce (2.5).

Recall the notations (2.2) and (2.3). Then we have

$$(2.6) \qquad H^0 = \frac{1}{2}(b_0^2 - m^2 a_0^2) + J + \frac{1}{2(p+1)} \int_M \left(a_0 + U\right)^{2p+2} dx.$$

By (2.5), b_0 and J are bounded and moreover by Lemma 2.3 and (2.5) we deduce that $\int_M |U|^q dx \leq C J^{q/2}$ for all $q \geq 2$. Thus (2.6) gives H^0 as a polynomial in a_0 with bounded coefficients and thus a_0 has to be bounded by a constant depending only on H^0.

The argument is similar when M has dimension 3 and $p = 1$. $\qquad\square$

Remark 2.5. — It is easy to check that the points $u = \pm m^{1/p}$ correspond to the minimum of H, and that for this choice $H^0 = -\dfrac{p}{2(p+1)} m^{2+2/p}$. In particular, the r.h.s of (2.5) is nonnegative.

Proof of Theorem 2.2. — First we show that (KG) is locally well-posed. Let $(u_0, u_1) \in \mathcal{X}$ and for $T > 0$ define the space

$$E_T = \big\{ u \in \mathcal{C}^0\big([-T,T]; H^1(M)\big) \cap \mathcal{C}^1\big([-T,T]; L^2(M)\big), \ \sup_{|t| \leq T} \|(u, \partial_t u)\|_{\mathcal{X}} \leq 2\|(u_0, u_1)\|_{\mathcal{X}} \big\}.$$

For $f : \mathbb{R} \longrightarrow \mathbb{R}$, a continuous and bounded function, we can define a bounded operator

$$f(-\Delta) : L^2(M) \longrightarrow L^2(M)$$

by

$$f(-\Delta)u = \sum_{k=0}^{+\infty} f(\lambda_k^2) a_k e_k, \quad u = \sum_{k=0}^{+\infty} a_k e_k.$$

With this definition, denote by $K(t)$ the free wave propagator

$$K(t)(u_0, u_1) = \cos(t\sqrt{-\Delta})u_0 + \frac{\sin(t\sqrt{-\Delta})}{\sqrt{-\Delta}}u_1.$$

Then we show that the mapping Φ defined by

$$\Phi(t)u = K(t)(u_0, u_1) + \int_0^t \frac{\sin(t-s)\sqrt{-\Delta}}{\sqrt{-\Delta}}\big(m^2 u - u^{2p+1}\big)(s)\mathrm{d}s,$$

is a contraction of E_T for T small enough. Namely, by the Sobolev embeddings and the fact that $\|K\|_{L(\mathcal{X},\mathcal{X})} \leq C$, we obtain that we can take $T = c_0 \min\big(m^{-2}, \|(u_0, u_1)\|_{\mathcal{X}}^{-2p}\big)$ where $c_0 > 0$ is a small absolute constant.

In order to prove the global well-posedness, we iterate the previous argument. We then obtain a sequence of times T_n with $T_{n+1} = c_0 \min\big(m^{-2}, \|(u(T_n), \partial_t u(T_n)\|_{\mathcal{X}}^{-2p}\big)$ such that the solution is defined on $(-\sum_{n=0}^{\infty} T_n, \sum_{n=0}^{\infty} T_n)$. By Lemma 2.4, for any $t > 0$, $\|(u(t), \partial_t u(t))\|_{\mathcal{X}}^{-2p} \geq C(H^0)$, hence $\sum_{n \geq 0} T_n = +\infty$ which gives the result. $\qquad\square$

Remark 2.6. — The result of Theorem 2.2 indeed holds for any $m \in \mathbb{R}$.

3. The linearized equation around the homoclinic orbit

In this section, we study the system

$$(3.1) \qquad \begin{cases} \dot{a}_0 = b_0 \\ \dot{b}_0 = m^2 a_0 - a_0^{2p+1}, \end{cases}$$

and for $n \geq 1$

$$(3.2) \qquad \begin{cases} \dot{a}_n = b_n \\ \dot{b}_n = -(\lambda_n^2 - m^2)a_n - (2p+1)a_0^{2p} a_n. \end{cases}$$

Indeed, if one considers that a_0 is given by (3.1), the system (3.2) is the linearisation of (KG) around the solution $u = a_0$. In particular, denoting $V(t) = (2p+1)a_0^{2p}(t)$ and $w = \sum_{k \geq 1} a_k e_k$, (3.2) is equivalent to the following linear wave equation with time-dependent potential

$$(3.3) \qquad \partial_t^2 w - \Delta w - m^2 w + V(t)w = 0.$$

Notice that the system (3.1), (3.2) is not Hamiltonian for the canonical structure, but it is reversible. On the other hand, for a_0 given, the system (3.2) is Hamiltonian for the canonical structure.

We now study the linear evolution of (3.2), which will be useful in the proof of Theorem 1.1. Introduce the notations

$$Z = \begin{pmatrix} z \\ \partial_t z \end{pmatrix}, \quad \mathcal{L}^t = \begin{pmatrix} 0 & 1 \\ \Delta + m^2 - (2p+1)\alpha^{2p}(t) & 0 \end{pmatrix},$$

then (3.1), (3.2) can also be written

$$\partial_t Z = \mathcal{L}^t Z.$$

Proposition 3.1. — *The restriction $\mathcal{L}^t_c : \mathcal{X}_c \longrightarrow \mathcal{X}_c$ generates a two parameter group $K(t,\tau)$ so that*

$$\sup_{t,\tau \in \mathbb{R}} \|K(t,\tau)\|_{L(\mathcal{X}_c, \mathcal{X}_c)} \leq C.$$

In other words, every solution to $\partial_t Z_c = \mathcal{L}^t_c Z_c$ is bounded.

Proposition 3.1 is a direct consequence of the next result. Define the tori by

$$\mathcal{T}_c := \left\{ (a_k, b_k)_{k \geq 1} \mid (\lambda_n^2 - m^2)a_n^2 + b_n^2 = c_n^2 \right\}, \quad c \in \ell^2(\mathbb{N}, \mathbb{R}),$$

then we have

Theorem 3.2. — *Let $(a_0(0), b_0(0)) \in \mathcal{K}_0$ and $(a_n(0), b_n(0))_{n \geq 1} \in h^1 \times \ell^2$. Then the corresponding solution of (3.1), (3.2) is homoclinic to $\mathcal{T}_c$ for some $c \in \ell^2(\mathbb{N}, \mathbb{R})$.*

This result can also be interpreted as a linear scattering result for the system (3.2). The infinite dimensional system (3.1), (3.2) has homoclinic orbits to invariant tori of arbitrary large dimension (finite or infinite). These tori correspond to the case $a_0 = b_0 = 0$ which is stable by (3.1). In that case (3.2) becomes the standard harmonic oscillator in infinite dimension with frequency vector $\omega(m) = (\omega_n)_{n \geq 1}$ and $\omega_n = \sqrt{\lambda_n^2 - m^2}$.

Proof. — Take the homoclinic (α, β) in the plane (a_0, b_0), and observe that $\int_{\mathbb{R}} \alpha^{2p}(t)\mathrm{d}t < \infty$. Now, for $n \geq 1$ we introduce the norms on $\mathbb{R}^2$, $|x, y|_n = \left((\lambda_n^2 - m^2)x^2 + y^2\right)^{1/2}$ and the norm on $h^1 \times \ell^2$ $\|(a_n, b_n)_{n \geq 1}\| = \left(\sum_{n \geq 1} |a_n, b_n|_n^2\right)^{1/2}$. Then for all $n \geq 1$, we deduce from Lemma A.1 that the trajectory (a_n, b_n) satisfies $|a_n, b_n|_n \leq C|a_n(0), b_n(0)|_n$. Next, we denote by S the flow of the linear part of (3.2), *i.e.* with $a_0 = 0$. Then the solution of (3.2) reads

$$(3.4) \qquad \begin{pmatrix} a_n(t) \\ b_n(t) \end{pmatrix} = S(t) \begin{pmatrix} a_n(0) \\ b_n(0) \end{pmatrix} - (2p+1) \int_0^t S(t-s) \begin{pmatrix} 0 \\ \alpha^{2p}(s)a_n(s) \end{pmatrix} \mathrm{d}s.$$

By construction S preserves the norm $\big|\,.\,\big|_n$ and thus

$$\int_0^{+\infty} \left| S(-s) \begin{pmatrix} 0 \\ \alpha^{2p}(s)a_n(s) \end{pmatrix} \right|_n \mathrm{d}s = \int_0^{+\infty} \alpha^{2p}(s)|a_n(s)|\mathrm{d}s$$

$$(3.5) \qquad\qquad\qquad\qquad\qquad \leq C|a_n(0), b_n(0)|_n,$$

where we used the boundedness of a_n and the integrability of α^{2p}. Denote by

$$\begin{pmatrix} a_n^+ \\ b_n^+ \end{pmatrix} = \begin{pmatrix} a_n(0) \\ b_n(0) \end{pmatrix} - (2p+1) \int_0^{+\infty} S(-s) \begin{pmatrix} 0 \\ \alpha^{2p}(s)a_n(s) \end{pmatrix} \mathrm{d}s,$$

therefore by (3.4) we get

$$\left| \begin{pmatrix} a_n(t) \\ b_n(t) \end{pmatrix} - S(t) \begin{pmatrix} a_n^+ \\ b_n^+ \end{pmatrix} \right|_n \leq (2p+1) \int_t^{+\infty} \alpha^{2p}(s)|a_n(s)|\mathrm{d}s$$

$$\leq C|a_n(0), b_n(0)|_n \int_t^{+\infty} \alpha^{2p}(s)\mathrm{d}s,$$

which in turn implies

$$\left\| \begin{pmatrix} a_n(t) \\ b_n(t) \end{pmatrix} - S(t) \begin{pmatrix} a_n^+ \\ b_n^+ \end{pmatrix} \right\| \longrightarrow 0, \quad \text{when } t \longrightarrow +\infty.$$

Finally we remark that the trajectory $t \mapsto S(t) \begin{pmatrix} a_n^+ \\ b_n^+ \end{pmatrix}$ lives on the torus $\mathcal{T}_c$ with $c_n^2 = (\lambda_n^2 - m^2)(a_n^+)^2 + (b_n^+)^2$ for $n \in \mathbb{N}$. $\square$

Remark 3.3. — If $\omega(m)$ is non resonant (for example when $M = \mathbb{S}^1$, this happens for a generic choice of the mass m, see [**3**]), $S(t)\big(a_n^+, b_n^+\big)_{t \geq 0}$ densely covers the torus $\mathcal{T}_c$.

Remark 3.4. — Notice that in the proof of Theorem 3.2 we crucially use that $a_0 = \alpha$, *i.e.* that we linearize around the homoclinic orbit. A natural question is to ask whether this result still holds if (a_0, b_0) is any periodic solution of (3.1). In that case $\int_{\mathbb{R}} a_0^{2p}(t)\mathrm{d}t = +\infty$ and Lemma A.1 does not apply. In particular it is not clear at all that the trajectories remain bounded. This seems to be a difficult and interesting problem related to the reducibility of (3.3) when V is not a priori small (the case V small and quasi-periodic in time can be solved by a KAM approach, see [**9, 11**] for the last results in the Schrödinger case).

Denote by $\mathcal{L}_h^t := \mathcal{L}_{|\mathcal{X}_h}^t = \begin{pmatrix} 0 & 1 \\ m^2 - (2p+1)\alpha^{2p}(t) & 0 \end{pmatrix}$ the restriction of $\mathcal{L}^t$ to $\mathcal{X}_h$. We can identify the space $\mathcal{X}_h = \mathbb{R}^2$ and we denote by $\langle \, , \, \rangle$ the Euclidian scalar product. Then

Lemma 3.5. — *The equation*

$$\partial_t Z_h = \mathcal{L}_h^t Z_h,$$

admits two solutions $\sigma(t)$ and $\rho(t)$ defined on $\mathbb{R}_+$ so that

(3.6) $$\sigma(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \rho(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

(see Figure 2) and

(3.7) $$|\sigma(t)| \leq Ce^{-mt}, \quad |\rho(t)| \leq Ce^{mt}, \quad \text{for all } t \geq 0.$$

The dual basis $\{\sigma^\star, \rho^\star\}$ to $\{\sigma, \rho\}$ in $\mathcal{X}_h$ satisfies

(3.8) $$|\sigma^\star(t)| \leq Ce^{mt}, \quad |\rho^\star(t)| \leq Ce^{-mt}, \quad \text{for all } t \geq 0.$$
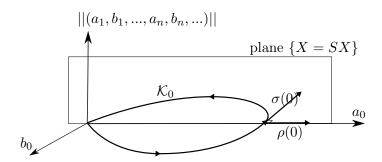
FIGURE 2. Linearized system at $t = 0$ and reversibility plane.

Proof. — Let $\mu \in \mathbb{R}$, then $t \mapsto \mu \begin{pmatrix} \dot{\alpha}(t) \\ \ddot{\alpha}(t) \end{pmatrix} = \mu \begin{pmatrix} \beta(t) \\ \dot{\beta}(t) \end{pmatrix}$ is a solution of $\partial_t Z_h = \mathcal{L}_h^t Z_h$. Then in view of (1.12) we deduce that this solution corresponds to σ and we obtain the first bound in (3.7). With a suitable choice of μ we get (3.6).

To find ρ, we write $\rho = \begin{pmatrix} \gamma \\ \dot{\gamma} \end{pmatrix}$, and look for γ of the form $\gamma = \dot{\alpha}z$, when $t \geq 1$. Then z satisfies the equation $\dot{\alpha}\ddot{z} + 2\ddot{\alpha}\dot{z} = 0$ and therefore

$$z(t) = z(1) + \dot{z}(1)(\dot{\alpha}(1))^2 \int_1^t (\dot{\alpha}(\tau))^{-2} \mathrm{d}\tau, \quad \forall t \geq 1.$$

It is then straightforward to check that $\gamma = \dot{\alpha}z$ can be extended to a $\mathcal{C}^\infty$ function on $\mathbb{R}$. Then the bound on σ gives $|z(t)| \leq Ce^{2mt}$ and $|\dot{z}(t)| \leq Ce^{2mt}$ for all $t \geq 0$, which in turn implies that $|\gamma(t)| \leq Ce^{mt}$ and $|\dot{\gamma}(t)| \leq Ce^{mt}$ for all $t \geq 1$. Hence the second bound in (3.7). We can moreover choose the initial value ρ of the form claimed in the statement of the lemma.

Then we can explicitly compute

$$\sigma^\star = \frac{1}{\beta\dot{\gamma} - \gamma\dot{\beta}} \begin{pmatrix} \dot{\gamma} \\ -\gamma \end{pmatrix}, \quad \rho^\star = \frac{1}{\beta\dot{\gamma} - \gamma\dot{\beta}} \begin{pmatrix} -\dot{\beta} \\ \beta \end{pmatrix},$$

where $\beta\dot{\gamma} - \gamma\dot{\beta}$ is a non vanishing constant, since it is the Wronskian of the equation

$$\ddot{y} - m^2 y + (2p+1)\alpha^{2p} y = 0.$$

The bound (3.8) then follows from (3.7). $\qquad\square$

4. Homoclinic orbits to the center manifold

In this section we prove Theorem 1.1. As we mentioned in the introduction, this part closely follows the argument of Groves-Schneider [14]. Nevertheless in contrast to their work, we are not dealing with m small and thus with small solutions. Even if some results already appear in [14], we reproduce here all the proofs for the convenience of the reader. First we recall the general strategy, illustrated by Figure 3.

1. To begin with, we consider a spatially truncated system for which the non-linearity sees only the elliptic (or central) modes of small amplitude. We construct global solutions to this system that are close to the homoclinic orbit, namely the hyperbolic part is close to $h = (\alpha, \beta)$ for all times $t \geq 0$, and the elliptic part is small only for $t \leq \varepsilon^{-1}$ where ε is a small parameter. (Subsection 4.2) In the next steps we will prove that the elliptic part of these solutions actually stay small for all t, *i.e.* that the set of all these solutions is the global center-stable manifold W^{cs} of the truncated system, and thus a local center-stable manifold for (KG) at time $t = 0$.

2. We construct a global center manifold W^c for the truncated system by constructing the solutions that are close to the origin for all times. (Subsection 4.3)

3. The crucial point consists in linking the two previous steps: we prove that the solutions of the truncated system constructed in step 1 actually tend to some solutions contained in the local center manifold as $t \to +\infty$ and thus remain small for all times $t \geq 0$. As the central part remains unconditionally small, we conclude that these solutions are actually solutions of the original system (without truncation). This gives us a global center stable manifold $\tilde{W}^{cs}$ which is parametrized by the initial value V_s of the stable part that we add to h and by V_c the initial value of the elliptic part, both being small. (Subsection 4.4)

4. For the proof of Theorem 1.2, it remains to prove that we can choose V_s and V_c in such a way that the solution hits the reversibility plane $b_n = 0$ for all n: namely the solution at $t = 0$ belongs to the reversibility plane as soon as $V_s = 0$, and V_c is symmetric. The corresponding solutions are then automatically symmetric and thus describe heteroclinic connections between two symmetric solutions of the center manifold. (Subsection 4.5)
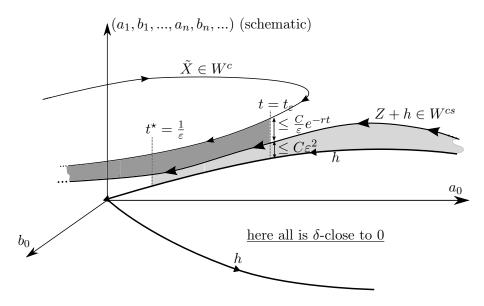

FIGURE 3. Schematic overview of the strategy and the notations. We will see later that we take $\delta = \varepsilon^{3/2}$ with $\varepsilon \ll 1$.

4.1. Preliminary results. — To begin with, we write an equivalent formulation of (KG). Denote by

$$(4.1) \qquad X = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \Lambda = \begin{pmatrix} 0 & 1 \\ \Delta + m^2 & 0 \end{pmatrix}, \quad F(X) = \begin{pmatrix} 0 \\ -u^{2p+1} \end{pmatrix}.$$

The equation (KG) is equivalent to

$$(4.2) \qquad \partial_t X = \Lambda X + F(X).$$

By the result of Theorem 2.2, the equation (4.2) admits, for a given initial condition $X(0) \in \mathcal{X}$ a unique global solution $X \in \mathcal{X}$ so that $\sup_{t \in \mathbb{R}} \|X(t)\| \leq C(\|X(0)\|_{\mathcal{X}})$. We fix $C_0 > 0$ large enough, so that all the initial conditions $X(0)$ we consider in the sequel satisfy $\|X(0)\| \leq C_0$. Therefore all the solutions X we are going to consider satisfy

$$\sup_{t \in \mathbb{R}} \|X(t)\| \leq C(C_0).$$

We denote $\mathcal{D}_c$ the projection on $\mathcal{X}_c$ of such initial conditions:

$$\mathcal{D}_c := \big\{ QX \mid \|X\| \leq C_0 \big\}.$$

Let $\theta \in \mathcal{C}^\infty([0,\infty),\mathbb{R})$ be a cut-off function so that

$$\theta(s) = \begin{cases} 1, & s \leq \delta \\ 0, & s \geq 2\delta, \end{cases}$$

where δ is a parameter much smaller than 1.
For any function $G : \mathcal{X} \longrightarrow \mathbb{R}$, denote by

$$\underline{G}(X) = G\big(X\theta(\|X_c\|)\big).$$

Lemma 4.1. — *Consider the function F given by (4.1) and let $X, X' \in \mathcal{X}$. Assume moreover that $|X_h|, |X'_h| \leq \delta < 1$. Then there exists a constant $C > 0$ such that*

$$(4.3) \qquad \|\underline{F}(X)\| \leq C\delta^3,$$

$$(4.4) \qquad \|\underline{F}(X) - \underline{F}(X')\| \leq C\delta^2 \|X - X'\|.$$

Proof. — Writing $X = \begin{pmatrix} u \\ v \end{pmatrix}$ and $u = a + U$ with $a \in \mathcal{X}_h$ and $U \in \mathcal{X}_c$ we have by the fact that $|a| \leq \delta$ and the Sobolev embedding

$$\begin{aligned} \|\underline{F}(X)\| &\leq \big\| \big(|a| + |U|\theta(\|U\|_{H^1})\big)^{2p+1} \big\|_{L^2(M)} \\ &\leq C\delta^{2p+1} + C\|U\|_{L^{2(2p+1)}}^{2p+1}\theta(\|U\|_{H^1}) \\ &\leq C\delta^{2p+1} + C\|U\|_{H^1}^{2p+1}\theta(\|U\|_{H^1}). \end{aligned}$$

This gives (4.3), since $\|U\|_{H^1}\theta(\|U\|_{H^1}) \leq C\delta$.
To get (4.4), just remark that for $|u|, |u'| \leq C\delta$

$$\left\| F\begin{pmatrix} u \\ v \end{pmatrix} - F\begin{pmatrix} u' \\ v' \end{pmatrix} \right\|^2 \leq \int_M |u^{2p+1} - u'^{2p+1}|^2 \mathrm{d}x \leq C\delta^4 \int_M |u - u'|^2 \mathrm{d}x,$$

and then apply this inequality to $\begin{pmatrix} u \\ v \end{pmatrix} = X\theta(\|X_c\|)$ and $\begin{pmatrix} u' \\ v' \end{pmatrix} = X'\theta(\|X'_c\|)$.   □

We write the linearisation of (4.2) around the homoclinic orbit $h = \begin{pmatrix} \alpha \\ \beta \end{pmatrix}$. Set $X = Z + h$ and define

(4.5) $$\mathcal{N}(Z) := F(Z + h) - F(h) - \mathrm{d}F[h].Z,$$

with

$$\mathrm{d}F[h] = \begin{pmatrix} 0 & 0 \\ -(2p+1)\alpha^{2p}(t) & 0 \end{pmatrix}.$$

As a result, Z satisfies the equation

(4.6) $$\partial_t Z = \mathcal{L}^t(Z) + \mathcal{N}(Z).$$

Similarly as in Lemma 4.1 we have

Lemma 4.2. — *Consider the function $\mathcal{N}$ defined in (4.5) and let $Z, Z' \in \mathcal{X}$ with $|Z_h|, |Z'_h| \leq \delta$ and with $\|Z + h\|, \|Z' + h\| \leq C_1$*

(4.7) $$\|\underline{\mathcal{N}}(Z)\| \leq C\delta^2,$$

$$\|\underline{\mathcal{N}}(Z)\| \leq C\|Z\|^2,$$

(4.8) $$\|\underline{\mathcal{N}}(Z) - \underline{\mathcal{N}}(Z')\| \leq C\delta\|Z - Z'\|.$$

4.2. The local center-stable manifold. — The equation (4.6) is equivalent to the system

(4.9) $$\begin{cases} \partial_t Z_h = \mathcal{L}_h^t Z_h + P\mathcal{N}(Z), \\ \partial_t Z_c = \mathcal{L}_c^t Z_c + Q\mathcal{N}(Z). \end{cases}$$

We will be interested in solutions to (4.9) for which the center part is small, that's why we introduce the following truncated system

(4.10) $$\begin{cases} \partial_t Z_h = \mathcal{L}_h^t Z_h + P\underline{\mathcal{N}}(Z), \\ \partial_t Z_c = \mathcal{L}_c^t Z_c + Q\underline{\mathcal{N}}(Z). \end{cases}$$

We now look for solutions to (4.10) with small hyperbolic part. Fix $0 < r < m$ and denote by

$$\mathcal{E}_r^+ = \big\{ Z \in \mathcal{C}(\mathbb{R}_+, \mathcal{X}), \ \|Z\|_r := \sup_{t \geq 0} \mathrm{e}^{-rt}\|Z(t)\| < \infty \big\}.$$

We introduce a new small parameter ε which will quantify the size of Z, namely $\|Z\| \leq C\varepsilon^2$. We will choose $\varepsilon = \delta^{2/3}$ in such way the truncation θ can be removed.

Proposition 4.3. — *Let $V_c \in \mathcal{X}_c$ with $\|V_c\| \leq \varepsilon^2$ and let $V_s \in \mathbb{R}$ with $|V_s| \leq \varepsilon^2$. Then there exists a unique solution $Z \equiv Z_{V_c, V_s}$ to (4.10) in*

$$\mathcal{B}_\delta^+ = \big\{ Z \in \mathcal{E}_r^+, \ \sup_{t \geq 0} |Z_h(t)| \leq \delta \big\},$$

and so that

(4.11) $$\langle Z_h(0), \sigma^\star(0) \rangle = V_s, \quad Z_c(0) = V_c.$$

Moreover, this solution satisfies

$$\sup_{t \geq 0} |Z_h(t)| \leq C\varepsilon^2, \tag{4.12}$$

and

$$\|Z_c\| \leq C\varepsilon^2, \quad \text{for all} \quad 0 \leq t \leq \varepsilon^{-1}. \tag{4.13}$$

This result means, that for any fixed $\|V_c\| \leq \varepsilon^2$, $|V_s| \leq \varepsilon^2$, there exists a unique choice of $V_u = \langle Z_h(0), \rho^\star(0) \rangle \in \mathbb{R}$ so that the corresponding solution has a small hyperbolic part for all times and a central part small for relatively long time (comparing with the time necessary in order that the homoclinic orbit h reaches a neighbourhood of radius ε^2 of the origin). Let us call *local center-stable manifold* the set of all the solutions verifying this property:

Definition 4.4. — For $\|V_c\| \leq \varepsilon^2$ and $|V_s| \leq \varepsilon^2$, denote by Z_{V_c,V_s} the function given by Proposition 4.3 and

$$X_{V_c,V_s} := Z_{V_c,V_s} + h.$$

We define W^{cs} by

$$W^{cs} = \bigcup_{\|V_c\| \leq \varepsilon^2, |V_s| \leq \varepsilon^2} \big\{ X_{V_c,V_s}(0) \big\}.$$

Theorem 4.10 below will ensure that for ε sufficiently small, this set is actually the standard center-stable manifold of the equilibrium 0 for the initial equation (4.2).

Proof. — We define the map $\mathcal{F} : \mathcal{E}_r^+ \longrightarrow \mathcal{E}_r^+$

$$
\begin{aligned}
\mathcal{F}Z(t) \;=\; & V_s\sigma(t) + K(t,0)V_c + \int_0^t \langle\, P\underline{\mathcal{N}}(Z)(\tau), \sigma^\star(\tau)\,\rangle \mathrm{d}\tau\, \sigma(t) \\
& - \int_t^{+\infty} \langle\, P\underline{\mathcal{N}}(Z)(\tau), \rho^\star(\tau)\,\rangle \mathrm{d}\tau\, \rho(t) + \int_0^t K(t,\tau)Q\underline{\mathcal{N}}(Z)(\tau)\mathrm{d}\tau,
\end{aligned}
$$

and show that for $\delta > 0$ small enough, $\mathcal{F}$ is a contraction in $\mathcal{B}_\delta^+$.
• To begin with, we show that $\mathcal{F} : \mathcal{B}_\delta^+ \longrightarrow \mathcal{B}_\delta^+$. Since $P = \int_M$, we have

$$|\langle\, P\underline{\mathcal{N}}(Z), \sigma^\star\,\rangle| \leq |P\underline{\mathcal{N}}(Z)||\sigma^\star| \leq \|\underline{\mathcal{N}}(Z)\|_{L^2(M)}\mathrm{e}^{m\tau} \leq \|\underline{\mathcal{N}}(Z)\|\mathrm{e}^{m\tau}$$

Let $|Z_h| \leq \delta$, then by (4.7), for all $t \geq 0$

$$
\begin{aligned}
\big|\big(\mathcal{F}(Z)\big)_h(t)\big| \;\leq\; & C|V_s|\mathrm{e}^{-mt} + C\int_0^t \|\underline{\mathcal{N}}(Z)\|\mathrm{e}^{m\tau}\mathrm{d}\tau\,\mathrm{e}^{-mt} + C\int_t^{+\infty} \|\underline{\mathcal{N}}(Z)\|\mathrm{e}^{-m\tau}\mathrm{d}\tau\,\mathrm{e}^{mt} \\
\leq\; & C(\varepsilon^2 + \delta^2) \leq C\varepsilon^2.
\end{aligned}
\tag{4.14}
$$

On the other hand by Proposition 3.1 and (4.7)

$$\big\|\big(\mathcal{F}(Z)\big)_c(t)\big\| \leq C\|V_c\| + C\int_0^t \|\underline{\mathcal{N}}(Z)\|\mathrm{d}\tau \leq C(\|V_c\| + \delta^2 t), \tag{4.15}$$

which in turn implies that $\|(\mathcal{F}(Z))_c\|_r < \infty$.

$\bullet$ We now show that $\mathcal{F}$ is a contraction in $\mathcal{B}_\delta^+$. By Lemma 4.2

$$\left|\big(\mathcal{F}(Z_1)\big)_h(t) - \big(\mathcal{F}(Z_2)\big)_h(t)\right| \le$$

$$\le C\int_0^t \|\underline{\mathcal{N}}(Z_1) - \underline{\mathcal{N}}(Z_2)\| \mathrm{e}^{m\tau} \mathrm{d}\tau \mathrm{e}^{-mt} + C\int_t^{+\infty} \|\underline{\mathcal{N}}(Z_1) - \underline{\mathcal{N}}(Z_2)\| \mathrm{e}^{-m\tau} \mathrm{d}\tau \mathrm{e}^{mt}$$

$$\le C\delta \mathrm{e}^{-mt}\|Z_1 - Z_2\|_r \int_0^t \mathrm{e}^{(m+r)\tau} \mathrm{d}\tau + C\delta \mathrm{e}^{mt}\|Z_1 - Z_2\|_r \int_t^{+\infty} \mathrm{e}^{(-m+r)\tau} \mathrm{d}\tau,$$

and therefore

(4.16) $$\|\big(\mathcal{F}(Z_1)\big)_h(t) - \big(\mathcal{F}(Z_2)\big)_h(t)\|_r \le C\delta\|Z_1 - Z_2\|_r.$$

Similarly

$$\|\big(\mathcal{F}(Z_1)\big)_c(t) - \big(\mathcal{F}(Z_2)\big)_c(t)\| \quad \le \quad C\int_0^t \|\underline{\mathcal{N}}(Z_1) - \underline{\mathcal{N}}(Z_2)\| \mathrm{d}\tau$$

$$\le \quad C\delta\|Z_1 - Z_2\|_r \int_0^t \mathrm{e}^{r\tau} \mathrm{d}\tau,$$

and then

(4.17) $$\|\big(\mathcal{F}(Z_1)\big)_c(t) - \big(\mathcal{F}(Z_2)\big)_c(t)\|_r \le C\delta\|Z_1 - Z_2\|_r.$$

Thus, (4.16) and (4.17) show that $\mathcal{F}$ is a contraction whenever $0 < \delta < 1$ is small enough, and we can deduce that there exists a unique fixed point $Z \in \mathcal{B}_\delta^+$. By definition of $\mathcal{F}$, it is clear that this solution satisfies (4.11).

$\bullet$ Finally, in view of the choice $\varepsilon = \delta^{2/3}$, the bound (4.12) comes from (4.14) and (4.13) is a direct consequence of (4.15). $\qquad\square$

4.3. The local center manifold. — In this subsection we want to construct solutions that remain close to the origin for all time. Actually this will be achieved by constructing a local center manifold. Again we consider the approximation of the initial problem (4.2), in which we truncate the non-linearity

(4.18) $$\begin{cases} \partial_t X_h = \Lambda_h X_h + P\underline{F}(X), \\ \partial_t X_c = \Lambda_c X_c + Q\underline{F}(X). \end{cases}$$

The next result shows that there exist solutions to (4.18) with small hyperbolic components. For $0 < r < m$ we define

$$\mathcal{E}_r = \big\{ X \in \mathcal{C}\big(\mathbb{R}_+, \mathcal{X}\big),\ \|X\|_r := \sup_{t\in\mathbb{R}} \mathrm{e}^{-r|t|}\|X(t)\| < \infty \big\}.$$

Proposition 4.5. — *Let $V_c \in \mathcal{D}_c$. Then there exists a unique solution to (4.18) in*

$$\mathcal{B}_\delta = \big\{ X \in \mathcal{E}_r,\ \sup_{t\in\mathbb{R}} |X_h(t)| \le \delta \big\},$$

and so that

(4.19) $$QX(0) = V_c.$$

Moreover, this solution satisfies

(4.20)
$$\sup_{t\in\mathbb{R}} |X_h(t)| \le C\delta^3.$$

The set of these solutions is the center manifold of 0 for equation (4.18) :

Definition 4.6. — For $V_c \in \mathcal{D}_c$, denote by X_{V_c} the function given by Proposition 4.5. We define W^c, a global center manifold for (4.18) (and a local center manifold for (KG)), by

$$W^c = \bigcup_{V_c\in\mathcal{D}_c, t\in\mathbb{R}} \{X_{V_c}(t)\}.$$

Proof. — We proceed similarly as in the proof of Proposition 4.3. Denote by $\sigma_0 = \begin{pmatrix} 1 \\ -m \end{pmatrix}$ and $\rho_0 = \begin{pmatrix} 1 \\ m \end{pmatrix}$. The dual basis of $\{\sigma_0, \rho_0\}$ reads $\{\sigma_0^\star, \rho_0^\star\}$ with $\sigma_0^\star = -\frac{1}{2m} \begin{pmatrix} -m \\ 1 \end{pmatrix}$ and $\rho_0^\star = \frac{1}{2m} \begin{pmatrix} m \\ 1 \end{pmatrix}$. It is clear that $(\sigma_0 \mathrm{e}^{-mt}, \rho_0 \mathrm{e}^{mt})$ form a basis of the solutions of $\partial_t X_h = \Lambda_h X_h$. Denote by K the propagator of the equation $\partial_t X_c = \Lambda_c X_c$. Then for $X \in \mathcal{E}_r$ we define

(4.21)
$$\mathcal{G}X(t) = K(t)V_c + \int_{-\infty}^t \langle P\underline{F}(X)(\tau), \sigma_0^\star \mathrm{e}^{m\tau} \rangle \mathrm{d}\tau\, \sigma_0 \mathrm{e}^{-mt}$$
$$- \int_t^{+\infty} \langle P\underline{F}(X)(\tau), \rho_0^\star \mathrm{e}^{-m\tau} \rangle \mathrm{d}\tau\, \rho_0 \mathrm{e}^{mt} + \int_0^t K(t-\tau)Q\underline{F}(X)(\tau)\mathrm{d}\tau.$$

It is straightforward to check that any fixed point of $\mathcal{G}$ in $\mathcal{B}_\delta$ satisfies (4.18) and $QX(0) = V_c$.
• Firstly, we show that $\mathcal{G} : \mathcal{B}_\delta \longrightarrow \mathcal{B}_\delta$. Let $X \in \mathcal{B}_\delta$, then by (4.3)

$$|(\mathcal{G}(X))_h(t)| \le C \int_{-\infty}^t \|\underline{F}(X)\|\mathrm{e}^{m\tau}\mathrm{d}\tau \mathrm{e}^{-mt} + C\int_t^{+\infty} \|\underline{F}(X)\|\mathrm{e}^{-m\tau}\mathrm{d}\tau \mathrm{e}^{mt}$$

(4.22)
$$\le C\delta^3.$$

Next, by (4.3) again

$$\|(\mathcal{G}(X))_c(t)\| \le C\|V_c\| + C\int_0^t \|\underline{F}(X)\|\mathrm{d}\tau \le C + C\delta^3 t,$$

which in turn implies that $\|(\mathcal{G}(X))_c\|_r < \infty$. Hence, with (4.22) we get $\mathcal{G}(X) \in \mathcal{B}_\delta$.
• In a second time, by (4.4) we can write

$$|(\mathcal{G}(X_1))_h(t) - (\mathcal{G}(X_2))_h(t)| \le$$
$$\le C\int_{-\infty}^t \|\underline{F}(X_1) - \underline{F}(X_2)\|\mathrm{e}^{m\tau}\mathrm{d}\tau \mathrm{e}^{-mt} + C\int_t^{+\infty} \|\underline{F}(X_1) - \underline{F}(X_2)\|\mathrm{e}^{-m\tau}\mathrm{d}\tau \mathrm{e}^{mt}$$
$$\le C\delta^2 \mathrm{e}^{-mt}\|X_1 - X_2\|_r \int_{-\infty}^t \mathrm{e}^{m\tau+r|\tau|}\mathrm{d}\tau + C\delta^2 \mathrm{e}^{mt}\|X_1 - X_2\|_r \int_t^{+\infty} \mathrm{e}^{-m\tau+r|\tau|}\mathrm{d}\tau,$$

and therefore

(4.23)
$$\|(\mathcal{G}(X_1))_h(t) - (\mathcal{G}(X_2))_h(t)\|_r \le C\delta^2\|X_1 - X_2\|_r.$$

Similarly

$$
\begin{aligned}
\left\|\big(\mathcal{G}(X_1)\big)_c(t) - \big(\mathcal{G}(X_2)\big)_c(t)\right\| &\le C\int_0^t \|\underline{F}(X_1) - \underline{F}(X_2)\|\mathrm{d}\tau \\
&\le C\delta^2\|X_1 - X_2\|_r \int_0^t \mathrm{e}^{r|\tau|}\mathrm{d}\tau,
\end{aligned}
$$

and then

$$
(4.24) \qquad \left\|\big(\mathcal{G}(X_1)\big)_c(t) - \big(\mathcal{G}(X_2)\big)_c(t)\right\|_r \le C\delta^2\|X_1 - X_2\|_r.
$$

Thus, (4.23) and (4.24) show that $\mathcal{G}$ is a contraction in $\mathcal{B}_\delta$ whenever $0 < \delta < 1$ is small enough. The bound (4.20) is given by (4.22). $\qquad\square$

We are now able to give a parametrisation of the local center manifold given in Definition 4.6. Let $\Psi : \mathcal{D}_c \longrightarrow \mathcal{X}_h$ the map defined by

$$
\Psi(V_c) = (X_{V_c})_h(0),
$$

then

$$
(4.25) \qquad W^c = \big\{(\Psi(V_c), V_c),\ V_c \in \mathcal{D}_c\big\},
$$

and, in view of (4.21), Ψ is quadratic at the origin, $\Psi(V_c) = \mathcal{O}(\|V_c\|^2)$. This is a particular case of a result of Mielke [23].

As a consequence we can prove that the origin is Lyapunov stable within the center manifold:

Lemma 4.7. — *Let X be a solution of* (4.18) *which lies in W^c. Assume that for some time $t^\star > 0$, $\|X_c(t^\star)\| \le C\varepsilon^2$. Then for all time $t \in \mathbb{R}$*

$$
(4.26) \qquad \|X_c(t)\| \le C\varepsilon^2.
$$

Proof. — It is here convenient to work in the coordinates $(a_n, b_n)_{n \ge 1}$. Recall that from (2.6) the Hamiltonian of (4.2) reads

$$
H^0 = \frac{1}{2}(b_0^2 - m^2 a_0^2) + J + \frac{1}{2(p+1)}\int_M \big(a_0 + U\big)^{2p+2}\mathrm{d}x,
$$

and observe that $J/C \le \|X_c\|^2 \le CJ$. We first show that $|H^0| \le C\varepsilon^4$. By assumption, $J(t^\star) \le C\varepsilon^4$, while by (4.20) we obtain

$$
(4.27) \qquad |a_0(t)|, |b_0(t)| \le C\delta^6 = C\varepsilon^9 \le C\varepsilon^2 \quad \text{for all} \quad t \in \mathbb{R}.
$$

Next, using Lemma 2.3 we obtain, for $t = t^\star$

$$
\frac{1}{2(p+1)}\int_M \big(a_0 + U\big)^{2p+2}\mathrm{d}x \le C\varepsilon^4,
$$

which proves the claim. Next, with (4.27) we can write for all $t \in \mathbb{R}$

$$
J(t) \le H^0 - \frac{1}{2}(b_0^2 - m^2 a_0^2) \le C\varepsilon^4,
$$

which implies (4.26). $\qquad\square$

4.4. The local center-stable manifold is global. — Let $\|V_c\| \leq \varepsilon^2$, $|V_s| \leq \varepsilon^2$ and consider the solution $Z = Z_{V_c,V_s}$ to (4.10) given by Proposition 4.3. Set

$$(4.28) \qquad t_\varepsilon := \frac{4}{m} \ln \frac{1}{\varepsilon}.$$

By (1.13), with this choice we have $|h(t)| \leq C\varepsilon^2$ for all $t \geq t_\varepsilon/2$. Next, we define $\Theta \in \mathcal{C}^\infty(\mathbb{R}, \mathbb{R})$ so that

$$\Theta(t) = \begin{cases} 0, & t \leq t_\varepsilon/2 \\ 1, & t \geq t_\varepsilon, \end{cases}$$

and with $|\Theta'(t)| \leq C/t_\varepsilon$ for all $t \in \mathbb{R}$. We set

$$(4.29) \qquad Y(t) = \Theta(t)\big(Z(t) + h(t)\big).$$

Proposition 4.8. — *Consider the function Y defined in (4.29). Then there exists $\widetilde{X}$ solution of (4.18) within W^c so that for all $t \in \mathbb{R}$*

$$(4.30) \qquad \|(Y - \widetilde{X})(t)\| \leq C\varepsilon^{-2} e^{-rt} \quad \text{and} \quad |(Y - \widetilde{X})_h(t)| \leq C\varepsilon^2.$$

Proof. — The function Y satisfies the equation

$$\partial_t Y = \Lambda Y + \underline{F}(Y) + \underline{S},$$

with

$$(4.31) \qquad \underline{S}(t) := \Theta'(t)(Z+h)(t) + \Theta(t)\underline{F}\big((Z+h)(t)\big) - \underline{F}\big(\Theta(t)(Z+h)(t)\big).$$

Consider now a solution $\widetilde{X}$ of the equation $\partial_t \widetilde{X} = \Lambda \widetilde{X} + \underline{F}(\widetilde{X})$ and set $\Delta = Y - \widetilde{X}$. Then Δ satisfies

$$(4.32) \qquad \partial_t \Delta = \Lambda \Delta + \underline{F}(Y) - \underline{F}(Y - \Delta) + \underline{S}.$$

The proof of the proposition consists in finding a solution of (4.32) so that (4.30) holds and so that $\widetilde{X}(0) := Y(0) - \Delta(0) \in W^c$.

Set

$$\mathcal{E}_r^- = \Big\{ \Delta \in \mathcal{C}(\mathbb{R}, \mathcal{X}), \ \|\Delta\|_r := \sup_{t \in \mathbb{R}} e^{rt}\|\Delta(t)\| < \infty \Big\}.$$

and

$$\mathcal{B}_\delta^- = \Big\{ \Delta \in \mathcal{E}_r^-, \ \sup_{t \in \mathbb{R}} |\Delta_h(t)| \leq \delta \Big\},$$

where again we choose $\delta^2 = \varepsilon^3$.

We claim that we can define a contraction $\mathcal{J} : \mathcal{B}_\delta^- \longrightarrow \mathcal{B}_\delta^-$ by

$$\begin{aligned}
\mathcal{J}\Delta(t) \ = \ & \int_{-\infty}^t \langle P\big(\underline{F}(Y) - \underline{F}(Y-\Delta) + \underline{S}\big)(\tau), \sigma_0^\star e^{m\tau} \rangle \mathrm{d}\tau \, \sigma_0 e^{-mt} \\
& - \int_t^{+\infty} \langle P\big(\underline{F}(Y) - \underline{F}(Y-\Delta) + \underline{S}\big)(\tau), \rho_0^\star e^{-m\tau} \rangle \mathrm{d}\tau \, \rho_0 e^{mt} \\
& - \int_t^{+\infty} K(t-\tau)Q\big(\underline{F}(Y) - \underline{F}(Y-\Delta) + \underline{S}\big)(\tau)\mathrm{d}\tau.
\end{aligned}$$

It is clear that the fixed point will satisfy (4.32). In a second time we will check that (4.30) holds.

• We begin by estimating $\underline{\mathcal{S}}$. First observe that by Proposition 4.3

$$(4.33) \qquad \|Z(t)\| \leq C\varepsilon^2 \quad \text{for all} \quad 0 \leq t \leq \varepsilon^{-1}.$$

Then, for $\varepsilon > 0$ small enough we have $t_\varepsilon < \varepsilon^{-1}$ and we can write for all $t \in [t_\varepsilon/2, t_\varepsilon]$

$$(4.34) \qquad \|Z(t)\| \leq \sup_{t \in [t_\varepsilon/2, t_\varepsilon]} \|Z(t)\| \mathrm{e}^{-r(t-t_\varepsilon)} \leq C\varepsilon^2 \mathrm{e}^{-r(t-t_\varepsilon)} \leq C\varepsilon^{-2} \mathrm{e}^{-rt},$$

since $\mathrm{e}^{rt_\varepsilon} \leq \mathrm{e}^{mt_\varepsilon} = \varepsilon^{-4}$. Denote by χ the indicator of the interval $[t_\varepsilon/2, t_\varepsilon]$. As $Z + h$ is small on $[t_\varepsilon/2, t_\varepsilon]$ we deduce from (4.31) that

$$(4.35) \qquad \|\underline{\mathcal{S}}(t)\| \leq \frac{C}{t_\varepsilon}\big(\|Z(t)\| + \|h(t)\|\big)\chi(t) + C\big(\|Z(t)\| + \|h(t)\|\big)\chi(t).$$

As a result, from (4.34) and (1.13) we infer

$$(4.36) \qquad \|\underline{\mathcal{S}}(t)\| \leq C(\varepsilon^{-2}\mathrm{e}^{-rt} + \mathrm{e}^{-mt})\chi(t) \leq C\varepsilon^{-2}\mathrm{e}^{-rt}\chi(t).$$

The inequality (4.35) together with (4.33) also gives

$$(4.37) \qquad \|\underline{\mathcal{S}}(t)\| \leq C\frac{\varepsilon^2}{|\ln\varepsilon|}\chi(t).$$

• We have $|Y_h| \leq \delta$. For $\Delta \in \mathcal{B}_\delta^-$ we can apply (4.8) to deduce

$$\big\|\big(F(Y) - F(Y - \Delta)\big)(\tau)\big\| \leq C\delta^2\|\Delta(\tau)\| \leq C\delta^2\|\Delta\|_r\mathrm{e}^{-r\tau}.$$

Thanks to this latter inequality and (4.36), we obtain

$$
\begin{aligned}
\|\mathcal{J}\Delta(t)\|\mathrm{e}^{rt} &\leq & C\int_{-\infty}^t \big(\delta^2\|\Delta\|_r + \varepsilon^{-2}\chi(\tau)\big)\mathrm{e}^{(m-r)\tau}\mathrm{d}\tau\,\mathrm{e}^{(r-m)t} \\
&& +C\int_t^{+\infty} \big(\delta^2\|\Delta\|_r + \varepsilon^{-2}\chi(\tau)\big)\mathrm{e}^{-(m+r)\tau}\mathrm{d}\tau\,\mathrm{e}^{(m+r)t} \\
&& +C\int_t^{+\infty} \big(\delta^2\|\Delta\|_r + \varepsilon^{-2}\chi(\tau)\big)\mathrm{e}^{-r\tau}\mathrm{d}\tau\,\mathrm{e}^{rt} \\
&\leq & C(\delta^2\|\Delta\|_r + \varepsilon^{-2}),
\end{aligned}
$$

and therefore

$$(4.38) \qquad \|\mathcal{J}\Delta\|_r \leq C(\delta^2\|\Delta\|_r + \varepsilon^{-2}).$$

Next, by (4.37), for all $t \in \mathbb{R}$

$$
\begin{aligned}
|(\mathcal{J}\Delta)_h(t)| &\leq & C\int_{-\infty}^t \big(\delta^3 + \frac{\varepsilon^2}{|\ln\varepsilon|}\chi(\tau)\big)\mathrm{e}^{m\tau}\mathrm{d}\tau\,\mathrm{e}^{-mt} \\
&& +C\int_t^{+\infty} \big(\delta^3 + \frac{\varepsilon^2}{|\ln\varepsilon|}\chi(\tau)\big)\mathrm{e}^{-m\tau}\mathrm{d}\tau\,\mathrm{e}^{mt} \\
&\leq & C(\delta^3 + \varepsilon^2).
\end{aligned}
$$

$$(4.39)$$

The bounds (4.38) and (4.39) show that $\mathcal{J} : \mathcal{B}_\delta^- \longrightarrow \mathcal{B}_\delta^-$.

• Let $\Delta_1, \Delta_2 \in \mathcal{B}_\delta^-$. By (4.8), we have

(4.40) $$\left\|\left(F(Y - \Delta_2) - F(Y - \Delta_1)\right)(\tau)\right\| \leq C\delta^2 \|\Delta_1 - \Delta_2\|_r e^{-r\tau}.$$

Clearly,

$$\mathcal{J}\Delta_1(t) - \mathcal{J}\Delta_2(t) = \int_{-\infty}^t \langle P\left(\underline{F}(Y - \Delta_2) - \underline{F}(Y - \Delta_1), \sigma^\star e^{m\tau}\right\rangle d\tau \, \sigma e^{-mt}$$
$$- \int_t^{+\infty} \langle P\left(\underline{F}(Y - \Delta_2) - \underline{F}(Y - \Delta_1)\right)(\tau), \rho^\star e^{-m\tau}\rangle d\tau \, \rho e^{mt}$$
$$- \int_t^{+\infty} K(t-\tau)Q\left(\underline{F}(Y - \Delta_2) - \underline{F}(Y - \Delta_1)\right)(\tau)d\tau,$$

and by (4.40), for all $t \in \mathbb{R}$

$$\|\mathcal{J}\Delta_1(t) - \mathcal{J}\Delta_2(t)\|e^{rt} \leq C \int_{-\infty}^t \left(\delta^2\|\Delta_1 - \Delta_2\|_r\right)e^{(m-r)\tau}d\tau \, e^{(r-m)t}$$
$$+C \int_t^{+\infty} \left(\delta^2\|\Delta_1 - \Delta_2\|_r\right)e^{-(m+r)\tau}d\tau \, e^{(m+r)t}$$
$$+C \int_t^{+\infty} \left(\delta^2\|\Delta_1 - \Delta_2\|_r\right)e^{-r\tau}d\tau \, e^{rt}$$
$$\leq C\delta^2\|\Delta_1 - \Delta_2\|_r.$$

As a consequence, for $\delta > 0$ small enough, the map $\mathcal{J} : \mathcal{B}_\delta^- \longrightarrow \mathcal{B}_\delta^-$ is a contraction, thus there exists a unique fixed point $\Delta \in \mathcal{B}_\delta^-$. By (4.39) and choosing again $\delta^2 = \varepsilon^3$, we get $|\Delta_h(t)| \leq \varepsilon^2$. Furthermore, as $\Delta = \mathcal{J}(\Delta)$, (4.38) leads to $\|\Delta\|_r \leq C\varepsilon^{-2}$, which in turn implies $\|\Delta(t)\| \leq C\varepsilon^{-2}e^{-rt}$ for all $t \in \mathbb{R}$.

• We now define $\widetilde{X} = Y - \Delta$ and it remains to show that $\widetilde{X} \in W^c$. By definition, it is sufficient to prove that $|\widetilde{X}_h(t)| \leq \delta$ for all t. Write $Y = \Theta(Z + h)$. By (4.12) and the choice of t_ε we get that $|Y_h(t)| \leq C\varepsilon^2$ for all $t \in \mathbb{R}$ and then

$$|\widetilde{X}_h(t)| \leq |Y_h(t)| + |\Delta_h(t)| \leq C\varepsilon^2 \leq \delta, \quad \text{for all} \quad t \in \mathbb{R},$$

which implies the result, for $\varepsilon > 0$ small enough. □

Proposition 4.9. — *Let $|V_s|, \|V_c\| \leq \varepsilon^2$ and ε small enough. The solution Z of (4.10) with $Z_c(0) = V_c$ and $Z_s(0) = V_s$ constructed in Proposition 4.3 satisfies $\|Z_c(t)\| \leq \delta$ for all $t \geq 0$. In particular $X = Z + h$ is a solution of (4.2) for $t \geq 0$.*

Proof. — (see Figure 3 on page 14) For such a Z, consider Y defined by (4.29) and $\widetilde{X}$ defined in Proposition 4.8. For $t \geq t_\varepsilon$, we have $Y = Z + h$, thus by (4.30)

$$\|\widetilde{X}_c(t) - Z_c(t)\| = \|\widetilde{X}_c(t) - Y_c(t)\| \leq \|\widetilde{X}(t) - Y(t)\| \leq C\varepsilon^{-2}e^{-rt}, \quad \text{for} \quad t \geq t_\varepsilon.$$

Recall that by (4.28), $t_\varepsilon := \dfrac{4}{m}\ln\dfrac{1}{\varepsilon}$. Thus, for $\varepsilon > 0$ small enough $t_\varepsilon \leq \varepsilon^{-1}$ and the previous inequality implies

(4.41) $$\|\widetilde{X}_c(t) - Z_c(t)\| \leq C\varepsilon^{-2}e^{-r\varepsilon^{-1}} \leq C\varepsilon^2, \quad \text{for} \quad t \geq \varepsilon^{-1}.$$

Now, take $t^\star = \varepsilon^{-1}$, by (4.13) we have $\|Z_c(t^\star)\| \leq C\varepsilon^2$, and then (4.41) implies $\|\widetilde{X}_c(t^\star)\| \leq C\varepsilon^2$. By Lemma 4.7, we deduce that $\|\widetilde{X}_c(t)\| \leq C\varepsilon^2$ for all $t \geq \varepsilon^{-1}$, and coming back to (4.41), we infer

$$\|Z_c(t)\| \leq C\varepsilon^2 \leq \delta, \quad \text{for all} \quad t \geq \varepsilon^{-1}.$$

This bound together with (4.13) conclude the proof. $\qquad\square$

Gathering the results of the previous propositions, we get the following Theorem, which ensures that W^{cs} is the standard center-stable manifold of 0 for equation (4.2).

Theorem 4.10. — *For ε sufficiently small, for all $\|V_c\|, |V_s| \leq \varepsilon^2$, the functions*

$$X_{V_c,V_s} = Z_{V_c,V_s} + h,$$

(where Z_{V_c,V_s} is given by Proposition 4.3) defined for $t \geq 0$ are solutions of (4.2). Moreover, Z_{V_c,V_s} belongs to $\mathcal{E}_r^+$ and satisfy

$$\|Z_c(t)\| \leq C\varepsilon^2 \leq \delta, \quad \|Z_h(t)\| \leq C\varepsilon^2, \quad \text{for all } t \in \mathbb{R}^+;$$

and there exists $\widetilde{X} \in W^c$ such that

$$\|(X_{V_c,V_s} - \widetilde{X})(t)\| \leq C\varepsilon^{-2} e^{-rt}.$$

This completes the proof of Theorem 1.1.

4.5. Construction of reversible heteroclinic solutions: proof of Theorem 1.2. — Recall that the vector field is reversible, and thus if the initial condition of a solution X satisfies

$$X(0) = SX(0),$$

then the solution is reversible, *i.e.*

$$X(-t) = SX(t), \quad \text{for all } t \in \mathbb{R}.$$

Lemma 4.11. — *For ε sufficiently small, for all $\|V_c\| \leq \varepsilon^2$ verifying*

$$V_c = SV_c,$$

X_{V_c,V_s} defined in Theorem 4.10 satisfies

$$X_{V_c,0}(0) = SX_{V_c,0}(0).$$

Proof. — On the one hand, from the explicit form (1.12) of h, we know that $h(0) = Sh(0)$. On the other hand, given that Z is a fixed point of the map $\mathcal{F}$ defined in the proof of Proposition 4.3, we get

$$Z_{V_c,0}(0) = V_c - \int_0^{+\infty} \langle\, P\underline{\mathcal{N}}(Z)(\tau), \rho^\star(\tau)\,\rangle \mathrm{d}\tau\, \rho(0).$$

And since $\rho(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ (see Lemma 3.5), we have $S\rho(0) = \rho(0)$. So, if $V_c = SV_c$, then $Z_{V_c,0}(0) = SZ_{V_c,0}(0)$. $\qquad\square$

As a consequence of the previous lemma, $X_{V_c,0}$ is a reversible solution. From Theorem 4.10, we then get that on the one hand, $Z_{V_c,0} = X_{V_c,0} - h$ belongs to $\mathcal{E}_r$ and satisfy

$$\|(Z_{V_c,0})_c(t)\| \leq \delta, \quad \|(Z_{V_c,0})_h(t)\| \leq C\varepsilon^2, \quad \text{for all } t \in \mathbb{R};$$

and on the other hand, there exists $X \in W^c$ such that

$$\|(X_{V_c,0} - X)(t)\| \leq C\varepsilon^{-2}e^{-rt} \quad \text{for all} \quad t \geq 0,$$

and

$$\|(X_{V_c,0} - SX)(t)\| \leq C\varepsilon^{-2}e^{rt} \quad \text{for all} \quad t \leq 0.$$

This means that $X_{V_c,0}$ is an heteroclinic connection between the two solutions X and SX of the center manifold W^c.

Then Theorem 1.2 follows since the condition $V_c = SV_c$ and $V_s = 0$ is equivalent to (1.16).

Appendix A

In this appendix we recall a result concerning the long time behaviour of the solution of the ordinary differential equation

$$\ddot{x} = (-\alpha^2 + q(t))x, \quad t \in \mathbb{R}$$

where α is a real constant and $t \mapsto q(t)$ is a continuous function. Of course the solution of such linear equation are globally defined, but we would like to know whether the solution are bounded or not. It turns out that even if $\|q\|_\infty < \alpha^2$ and $q(t) \longrightarrow 0$ when $t \longrightarrow \pm\infty$, the solutions may grow indefinitely [1]. The good condition concerns the integrability of $|q|$:

Lemma A.1. — *Assume that* $\displaystyle\int_{\mathbb{R}} |q(t)|\,dt < +\infty$ *then all the solutions of the Cauchy problem*

$$(A.1) \qquad \begin{cases} \dot{x} = y \\ \dot{y} = (-\alpha^2 + q(t))x, \end{cases}$$

are bounded on $\mathbb{R}$. *More precisely, there exists* $C > 0$ *which only depends on* q *so that*

$$\left(\alpha^2|x(t)|^2 + |y(t)|^2\right) \leq C\left(\alpha^2|x(0)|^2 + |y(0)|^2\right) \quad \text{for all } t \in \mathbb{R}.$$

The proof is classical (see for instance [**16**, page 212]) but does not precise the bound on the solution in term of the initial datum. The following argument is more explicit.

Proof. — Denote by S the flow of (A.1) for $q \equiv 0$, and introduce the norm $\|x, y\| = (\alpha^2 x^2 + y^2)^{1/2}$ (observe that S preserves this norm). For $T \geq 0$, the Duhamel formula reads

$$(A.2) \qquad \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = S(t - T)\begin{pmatrix} x(T) \\ y(T) \end{pmatrix} + \int_T^t S(t - T - s)\begin{pmatrix} 0 \\ q(s)x(s) \end{pmatrix} ds.$$

---

1. Consider the example induced by the solution $x(t) = \sin t(1 + (t - \sin t \cos t)^2)$ (see O. Perron [**25**]).

There exists T_1 so that $\int_0^{T_1} |q(t)| \mathrm{d}t \leq 1/2$. Therefore from (A.2) with $T = 0$ we deduce that for all $t \in [0, T_1]$

$$\|x(t), y(t)\| \leq \|x(0), y(0)\| + \frac{1}{2} \max_{s \in [0, T_1]} \|x(s), y(s)\|,$$

and thus $\max_{s \in [0, T_1]} \|x(s), y(s)\| \leq 2\|x(0), y(0)\|$.

By induction, we can define a finite number of times $T_2, \ldots, T_k$ so that $\int_{T_j}^{T_{j+1}} |q(t)| \mathrm{d}t = 1/2$ and $\int_{T_k}^{+\infty} |q(t)| \mathrm{d}t \leq 1/2$. Then we apply (A.2) with $T = T_j$ and we show $\max_{s \in [T_j, T_{j+1}]} \|x(s), y(s)\| \leq 2\|x(T_j), y(T_j)\|$ which in turn implies

$$\|x(t), y(t)\| \leq 2^{k+1}\|x(0), y(0)\| \quad \text{for all } t \geq 0.$$

$\square$

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Benoît Grébert, Tiphaine Jézéquel and Laurent Thomann, Laboratoire de Mathématiques J. Leray, Université de Nantes, UMR CNRS 6629, 2, rue de la Houssinière, 44322 Nantes Cedex 03, France.
E-mail : benoit.grebert@univ-nantes.fr,tiphaine.jezequel@univ-nantes.fr,laurent.thomann@univ-nantes.fr