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GENERAL STABILITY RESULTS FOR THE TRANSLATIONAL PROBLEM OF MEMORY-TYPE IN POROUS THERMOELASTICITY OF TYPE III

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Abstract. A beam modelled by a Timoshenko system with a viscoelastic damping on one component is considered. The system is coupled with a hyperbolic heat equation. One end of the structure is fixed to a platform in a translational movement and the other one is attached to a non-negligble mass. The well-posedness and asymptotic stability results for the system under some conditions on the initial and the boundary data are established.

Keywords. Translational problem; Uniform and weak decay; Viscoelastic damping; Porous thermoelastic system.

1. INTRODUCTION

Strings, beams and plates in translational movement have been the subject of extensive studies due to their numerous applications in industry (see, for example, [1, 2, 3, 4, 5, 6]). Unlike the immobile case, moving structures requires more attention as they are more delicate to handle. We refer to [7, 8, 9] and the references therein where Euler-Bernoulli beams and Timoshenko type beams were discussed. In [10], Kafini and Tatar considered the stabilization of the system by a thermal effect and a feedback control. Here we consider the beam of Timoshenko type coupled with a thermal equation but in the context of porous thermo-elasticity. The beam is fixed to a small platform in translational displacement at one end and a dynamic mass is attached to the other end. This mass together with the translation of the base brings considerable complications when it comes to the stability issue. The objective is to control the structure through the platform, which is an easily accessible location. The problem appears in many engineering

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applications as mentioned above. Namely, we investigate the problem

$$\begin{cases} m\left(S_{tt}(t)+v_{tt}(x,0)\right)+\rho_1\int_0^L\left(S_{tt}(t)+v_{tt}(x,t)\right)dx+m_E\left(S_{tt}(t)+v_{tt}(L,t)\right)=\tau(t) \\ \rho_1\left(S_{tt}(t)+v_{tt}(x,t)\right)-k_1\left(v_x+\psi\right)_x+\theta_{tx}=0, \\ \rho_2\psi_{tt}-k_2\psi_{xx}+k_1\left(v_x+\psi\right)-\theta_t+\int_0^{+\infty}g(s)\,\psi_{xx}(x,t-s)\,ds=0, \\ \rho_3\theta_{tt}-k_3\theta_{xx}-\alpha\theta_{xxt}+\beta v_{xt}+\beta\psi_t=0, \end{cases} \tag{1.1}$$

where $x\in[0,L]$, $t>0$, and L, α,β,m,m_E, ρ_i and k_i are positive constants, for $i=1,2,3$, with the boundary conditions

$$\begin{cases} v_x(0,t)=\psi(0,t)=\theta(0,t)=\psi(L,t)=\theta(L,t)=0, \\ k_1v_x(L,t)+m_E\left(S_{tt}(t)+v_{tt}(L,t)\right)=0, \end{cases} \tag{1.2}$$

and the initial data

$$\begin{cases} \psi(x,-t)=\psi_0(x,t),\psi_t(x,0)=\psi_1(x),\theta(x,0)=\theta_0(x),\theta_t(x,0)=\theta_1(x), \\ S(0)=S_0,S_t(0)=S_1,v(x,0)=v_0(x),v_t(x,0)=v_1(x). \end{cases} \tag{1.3}$$

where φ is the beam transversal displacement, ψ is the rotational angle of the beam, θ is the temperature difference, S is the base motion and τ is the control.

The models of structures fixed on a translational base have been derived in several works. In [11], a model was derived for an Euler-Bernoulli beam. A Timoshenko beam in translational displacement was considered in [7]. Here, it is rather a porous thermoelastic Timoshenko system of type III which is fixed to a base in translational motion [12, 13]. The main objective of Goodman and Cowin [14] was to extend the classical elasticity theory to porous media. This idea was extended further to materials by involving temperature and microtemperate elements in [12, 15, 16, 17, 18]. Timoshenko systems are by now well-known as they have been extensively studied for a long time (see, for instance, [19, 20] and references therein). There exists a well-established theory and a huge number of applications. Porous elastic problems can be traced back to the early seventies [14]. The basic model is

$$\begin{cases} \rho_0u_{tt}=\mu u_{xx}+\beta\varphi_x, \\ \rho_0\kappa\varphi_{tt}=\alpha\varphi_{xx}-\beta u_x-\tau\varphi_t-\xi\varphi, \end{cases}$$

where u is the displacement of the solid elastic material, φ is the volume fraction and all the coefficients are positive constants.

As mentioned above, all these systems were considered with $S\equiv 0$ and without end mass. In our present paper, the situation is different. Both ends of the structure are dynamic. This gives rise to some complications. The dynamic at the end points is modelled by some complex boundary conditions which would to be handled carefully. In addition, searching for an appropriate reasonable control to be implemented at the base, one needs to deal with some boundary terms. For more ideas and results related in this regard, we refer to [7, 8, 9, 10] and the references therein.

In this paper, we first establish an existence and uniqueness result in an appropriate space. Then, we come up with a suitable control at the fixed end to the base. It is shown that this control is capable of stabilizing the structure in case of equal speed of propagation as well as the more reasonable case of non-equal speed of propagation. However, the type of stability is different as the latter case is less favorable from the mathematics point of view. In Section 2, we transform the problem into a simpler one, determine a feedback control, introduce the energy functional and compute its derivative. In Section 3, we show that the system is well posed. Our uniform and weak stability results will be proved in the last section.

2. Control and energy

We start by defining the total deflection of the beam as follows:

$$\varphi(x,t) = S(t) + v(x,t).$$

Hence,

$$\begin{cases} \varphi(0,t) = S(t) + v(0,t), \ \varphi_{tt}(0,t) = S_{tt}(t) + v_{tt}(0,t), \\ \varphi_t(x,t) = S_t(t) + v_t(x,t), \ \varphi_{tt}(x,t) = S_{tt}(t) + v_{tt}(x,t), \\ \varphi_x(x,t) = v_x(x,t), \ \varphi_{xx}(x,t) = v_{xx}(x,t). \end{cases}$$

By using the second equation in (1.1) and boundary conditions (1.2), we get

$$\rho_1 \int_0^L (S_{tt}(t) + v_{tt}(x,t))\, dx + m_E (S_{tt}(t) + v_{tt}(L,t)) = 0.$$

Thus, problem (1.1) is equivalent to

$$\begin{cases} m\varphi_{tt}(0,t) = \tau(t), \\ \rho_1 \varphi_{tt} - k_1 (\varphi_x + \psi)_x + \theta_{tx} = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1 (\varphi_x + \psi) - \theta_t + \int_0^{+\infty} g(s)\, \psi_{xx}(x,t-s)\, ds = 0, \\ \rho_3 \theta_{tt} - k_3 \theta_{xx} - \alpha \theta_{xxt} + \beta \varphi_{xt} + \beta \psi_t = 0, \\ \varphi(x,0) = \varphi_0(x), \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t), \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x,t), \theta_t(x,0) = \theta_1(x) \end{cases} \tag{2.1}$$

with the boundary conditions

$$\begin{cases} \varphi_x(0,t) = \psi(0,t) = \theta(0,t) = \psi(L,t) = \theta(L,t) = 0, \\ k_1 \varphi_x(L,t) + m_E \varphi_{tt}(L,t) = 0. \end{cases} \tag{2.2}$$

For the relaxation function g, we assume the following:

(A1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a C^1 function satisfying

$$0 < g_0 := \int_0^{+\infty} g(s) ds < k_2. \tag{2.3}$$

(A2) There exist a positive constant ξ_1 and a nonincreasing differentiable function $\xi_0 : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_0(t) g(t), \quad \forall t \in \mathbb{R}_+. \tag{2.4}$$

The associated energy E to (2.1) is given by

$$\begin{aligned} E(t) = & \frac{1}{2} \left[\varphi^2(0,t) + m\varphi_t^2(0,t) + \beta m_E \varphi_t^2(L,t) + \rho_3 \|\theta_t\|_2^2 + k_3 \|\theta_x\|_2^2 \right] \\ & + \frac{\beta}{2} \left[\rho_1 \|\varphi_t\|_2^2 + \rho_2 \|\psi_t\|_2^2 + l \|\psi_x\|_2^2 + k_1 \|\varphi_x + \psi\|_2^2 + (g \circ \psi_x)(t) \right], \end{aligned} \tag{2.5}$$

where

$$(g\circ v)(t)=\int_0^{+\infty}g(s)\|v(t)-v(t-s)\|_2^2ds,$$

where $\|\cdot\|_2$ denotes the classical norm of $L^2(0,L)$ and $l:=k_2-g_0$ ($l>0$ according to (2.3)). The expression in (2.5) consists of the total energy (kinetic and potential) of the structure modified by a quadratic form (related to the viscoelastic term) so as to exhibit the dissipative character of the system. The derivative of E is equal to

$$E'(t)=\varphi(0,t)\varphi_t(0,t)+\tau(t)\varphi_t(0,t)-\alpha\|\theta_{xt}\|_2^2+\frac{\beta}{2}\left(g'\circ\psi_x\right). \tag{2.6}$$

This equation can be achieved by, first, multiplying $(2.1)_1$ by $\varphi_t(0,t)$ to get

$$\frac{m}{2}\frac{d}{dt}\varphi_t^2(0,t)=\tau(t)\varphi_t(0,t), \tag{2.7}$$

and then multiplying $(2.1)_2$ by $\beta\varphi_t(x,t)$ and integrating over $[0,L]$ to get

$$\begin{aligned}&\beta\frac{\rho_1}{2}\frac{d}{dt}\|\varphi_t\|_2^2+\beta\left[k_1\int_0^L\varphi_{xt}(\varphi_x+\psi)dx-k_1\varphi_t(L,t)(\varphi_x(L,t)+\psi(L,t))\right]\\&\quad+\beta\left[k_1\varphi_t(0,t)(\varphi_x(0,t)+\psi(0,t))+\varphi_t(L,t)\theta_t(L,t)-\varphi_t(0,t)\theta_t(0,t)-\int_0^L\varphi_{tx}\theta_t dx\right]\\&=0.\end{aligned}$$

Using (2.2), we obtain

$$\beta\frac{d}{dt}\left[\frac{\rho_1}{2}\|\varphi_t\|_2^2+\frac{m_E}{2}\varphi_t^2(L,t)\right]+\beta k_1\int_0^L\varphi_{xt}(\varphi_x+\psi)dx-\beta\int_0^L\varphi_{tx}\theta_t dx=0. \tag{2.8}$$

Also multiplying $(2.1)_3$ by $\beta\psi_t(x,t)$ and integrating over $[0,L]$, we entail

$$\begin{aligned}&\beta\frac{d}{dt}\left[\frac{\rho_2}{2}\|\psi_t\|_2^2+\frac{k_2}{2}\|\psi_x\|_2^2\right]-\beta\left[k_2\psi_t(L,t)\psi_x(L,t)-k_2\psi_t(0,t)\psi_x(0,t)\right]\\&\quad+\beta\left[k_1\int_0^L\psi_t(\varphi_x+\psi)dx-\int_0^L\psi_t\theta_t dx+\int_0^L\psi_t\int_0^{+\infty}g(s)\psi_{xx}(x,t-s)dsdx\right]=0.\end{aligned}$$

Next, using (2.2), we arrive at

$$\begin{aligned}&\beta\frac{d}{dt}\left[\frac{\rho_2}{2}\|\psi_t\|_2^2+\frac{k_2}{2}\|\psi_x\|_2^2\right]+\beta k_1\int_0^L\psi_t(\varphi_x+\psi)dx-\beta\int_0^L\psi_t\theta_t dx\\&\quad-\beta\int_0^L\psi_{tx}\int_0^{+\infty}g(s)\psi_x(x,t-s)dsdx=0.\end{aligned} \tag{2.9}$$

Moreover, multiplying $(2.1)_4$ by $\theta_t(x,t)$ and integrating over $[0,L]$, we find

$$\begin{aligned}&\frac{d}{dt}\left[\frac{\rho_3}{2}\|\theta_t\|_2^2+\frac{k_3}{2}\|\theta_x\|_2^2\right]-k_3\theta_t(L,t)\theta_x(L,t)+k_3\theta_t(0,t)\theta_x(0,t)\\&\quad+\alpha\|\theta_{xt}\|_2^2+\beta\int_0^L\theta_t\varphi_{xt}dx+\beta\int_0^L\theta_t\psi_t dx=0.\end{aligned}$$

Therefore,

$$\frac{d}{dt}\left[\frac{\rho_3}{2}\|\theta_t\|_2^2+\frac{k_3}{2}\|\theta_x\|_2^2\right]+\alpha\|\theta_{xt}\|_2^2+\beta\int_0^L\theta_t\varphi_{xt}dx+\beta\int_0^L\theta_t\psi_t dx=0. \tag{2.10}$$

Summing up (2.7)-(2.10), taking into account the equation

$$\int_0^L \psi_{tx} \int_0^{+\infty} g(s)\, \psi_x(x,t-s) ds dx = \frac{1}{2}\frac{d}{dt}\left[g_0 \left\|\psi_x\right\|_2^2 - (g\circ\psi_x)\right] + \frac{1}{2}\left(g'\circ\psi_x\right),$$

we obtain (2.6) immediately. Observe that the derivative of E is not readily seen to be non-positive. It would be the case in the absence of an external force. The suggested feedback control force $\tau(t)$ is

$$\tau(t) = -K\varphi_t(0,t) - \varphi(0,t), \tag{2.11}$$

where K is a positive 'control gain'. Consequently, we see that

$$\begin{aligned} E'(t) &= \varphi(0,t)\varphi_t(0,t) + \varphi_t(0,t)\left[-K\varphi_t(0,t) - \varphi(0,t)\right] - \alpha\left\|\theta_{xt}\right\|_2^2 + \frac{\beta}{2}\left(g'\circ\psi_x\right) \\ &= -K\varphi_t^2(0,t) - \alpha\left\|\theta_{xt}\right\|_2^2 + \frac{\beta}{2}\left(g'\circ\psi_x\right). \end{aligned} \tag{2.12}$$

This means that $E' \leq 0$. Hence, (2.1) is dissipative.

3. Well-posedness

In this section, we discuss the well-posedness of (2.1)-(2.2) using the semigroup approach. Following the method in [21], we introduce the functional η by

$$\eta(x,t,s) = \psi(x,t) - \psi(x,t-s) \quad \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+. \tag{3.1}$$

Let $\eta_0(x,s) = \eta(x,0,s)$. The functional η satisfies

$$\begin{cases} \eta_t + \eta_s - \psi_t = 0, & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+, \\ \eta(0,t,s) = \eta(L,t,s) = 0, & \text{in } \mathbb{R}_+\times\mathbb{R}_+, \\ \eta(x,t,0) = 0, & \text{in }]0,L[\times\mathbb{R}_+. \end{cases} \tag{3.2}$$

The problem (2.1) with the feedback control (2.11) can be written in the form

$$\begin{cases} \Psi_t = \mathscr{B}\Psi, \\ \Psi(t=0) = \Psi_0, \end{cases} \tag{3.3}$$

for $\Psi := (\varphi, w, \psi, z, \theta, \sigma, \xi, y, \eta)^T$ and $\Psi_0 := (\varphi_0, \varphi_1, \psi_0, \psi_1, \theta_0, \theta_1, \varphi_1(0,\cdot), \varphi_1(L,\cdot), \eta_0)^T$ with $w = \varphi_t$, $z = \psi_t$, $\sigma = \theta_t$, $\xi = \varphi_t(0,\cdot)$, $y = \varphi_t(L,\cdot)$ and

$$\mathscr{B}\Psi = \begin{pmatrix} w \\ \frac{1}{\rho_1}\left[k_1\left(\varphi_x+\psi\right)_x - \sigma_x\right] \\ z \\ \frac{1}{\rho_2}\left[l\psi_{xx} - k_1\left(\varphi_x+\psi\right) + \sigma + \int_0^{+\infty} g\eta_{xx} ds\right] \\ \sigma \\ \frac{1}{\rho_3}\left[k_3\theta_{xx} + \alpha\sigma_{xx} - \beta w_x - \beta z\right] \\ -\frac{1}{m}\left(K\xi + \varphi(0,\cdot)\right) \\ -\frac{k_1}{m_E}\varphi_x(L,\cdot) \\ -\eta_s + z \end{pmatrix}.$$

Bearing in mind the Dirichlet boundary conditions in (2.2), we introduce the spaces

$$H_0^1(0,L) := \left\{f \in H^1(0,L) : f(0) = f(L) = 0\right\},$$

$$L_g = \left\{ f : \mathbb{R}_+ \to H_0^1(]0,L[), \int_0^{+\infty} g\|f_x\|_2^2 ds < +\infty \right\}$$

and

$$\mathscr{H} := H^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times H_0^1(0,L) \times L^2(0,L) \times \mathbb{R}^2 \times L_g.$$

The Neumann boundary conditions in (2.2) are included in the definition of the domain of $\mathscr{B}$ given by

$$D(\mathscr{B}) := \Big\{ \Psi \in \mathscr{H} : \varphi \in H^2(0,L), w \in H^1(0,L), z \in H_0^1(0,L), \sigma \in H_0^1(0,L), \eta_s \in L_g, \varphi_x(0) = 0,$$

$$l\psi + \int_0^{+\infty} g\eta ds \in H^2(0,L), k_3\theta + \alpha\sigma \in H^2(0,L), \xi = w(0,\cdot), y = w(L,\cdot), \eta(\cdot,0) = 0 \Big\}.$$

Thus (3.3) is an abstract formulation of (2.1)-(2.2). The space $\mathscr{H}$ is a Hilbert space when endowed with the inner product

$$\begin{aligned}\langle \Psi, \tilde{\Psi} \rangle_{\mathscr{H}} = {} & k_1\beta \langle \varphi_x + \psi, \tilde{\varphi}_x + \tilde{\psi} \rangle + \beta l \langle \psi_x, \tilde{\psi}_x \rangle + k_3 \langle \theta_x, \tilde{\theta}_x \rangle + \rho_1\beta \langle w, \tilde{w} \rangle + \rho_2\beta \langle z, \tilde{z} \rangle + \rho_3 \langle \theta, \tilde{\theta} \rangle \\ & + m\xi\tilde{\xi} + \beta m_E y\tilde{y} + \varphi(0)\tilde{\varphi}(0) + \beta \int_0^{+\infty} g \langle \eta_x, \tilde{\eta}_x \rangle ds,\end{aligned}$$

where

$$\Psi = (\varphi, w, \psi, z, \theta, \sigma, \xi, y, \eta)^T, \quad \tilde{\Psi} = \left(\tilde{\varphi}, \tilde{w}, \tilde{\psi}, \tilde{z}, \tilde{\theta}, \tilde{\sigma}, \tilde{\xi}, \tilde{y}, \tilde{\eta} \right)^T$$

and $\langle \cdot, \cdot \rangle$ is the standard inner product of $L^2(0,L)$. The associated energy (2.5) to (2.1)-(2.2) can be written in the form

$$E(t) := \frac{1}{2} \|\Psi\|_{\mathscr{H}}^2. \tag{3.4}$$

Theorem 3.1. *Assume that (A1) and (A2) are satisfied and $\Psi_0 \in \mathscr{H}$. Then, there exists a unique solution of* (3.3) *satisfying*

$$\Psi \in C\left(\mathbb{R}^+, \mathscr{H}\right).$$

In case $\Psi_0 \in D(\mathscr{B})$, the solution is strong:

$$\Psi \in C\left(\mathbb{R}^+, D(\mathscr{B})\right) \cap C^1\left(\mathbb{R}^+, \mathscr{H}\right).$$

Proof. The well-posedness may be derived easily by using the classical semigroup approach. We here give a brief sketch of the proof. First, (2.12), (3.3) and (3.4) lead to

$$\langle \mathscr{B}\Psi, \Psi \rangle_{\mathscr{H}} = -K\varphi_t^2(0,t) - \alpha \|\theta_{xt}\|_2^2 + \frac{\beta}{2} \left(g' \circ \psi_x \right) \leq 0, \tag{3.5}$$

for any $\Psi \in D(\mathscr{B})$. Hence $\mathscr{B}$ is a dissipative operator. Notice that the left inequality in (2.4) guarantees the existence of $(g' \circ \psi_x)$ due to $\eta \in L_g$.

Second, we show that $I - \mathscr{B}$ is surjective, where I denotes the identity operator. Let $F = (f_1, \cdots, f_9) \in \mathscr{H}$. We need to prove that there exists $\Psi \in D(\mathscr{B})$ satisfying

$$(I - \mathscr{B})\Psi = F. \tag{3.6}$$

Using the definition of $\mathscr{B}$, the quations $(3.6)_1$, $(3.6)_3$, $(3.6)_5$, $(3.6)_7$ and $(3.6)_8$ are equivalent to

$$\begin{cases} w=\varphi-f_1, \\ z=\psi-f_3, \\ \sigma=\theta-f_5, \\ \xi=\frac{1}{K+m}(mf_7-\varphi(0)), \\ y=f_8-\frac{k_1}{m_E}\varphi_x(L). \end{cases} \tag{3.7}$$

Consequently, if $\varphi\in H^2(0,L)$ and ψ, $\theta\in H_0^1(0,L)$, then w, z, σ, ξ and y exist and

$$(w,z,\sigma)\in H^1(0,L)\times H_0^1(0,L)\times H_0^1(0,L).$$

Moreover, $\xi=w(0)$ and $y=w(L)$ if

$$\begin{cases} \varphi(0)=\frac{1}{K+m+1}\left[(K+m)f_1(0)+mf_7\right]:=g_1, \\ \frac{k_1}{m_E}\varphi_x(L)+\varphi(L)=f_1(L)+f_8:=g_2. \end{cases} \tag{3.8}$$

On the other hand, using $(3.7)_2$, the last equation in (3.6) is equivalent to

$$\eta_s+\eta=\psi+f_9-f_3. \tag{3.9}$$

By integrating (3.9) and using the fact that $\eta(0)=0$ (the definition of $D(\mathscr{B})$), we get

$$\eta(\cdot,s)=(1-e^{-s})(\psi-f_3)+e^{-s}\int_0^s e^p f_9(\cdot,p)\,dp. \tag{3.10}$$

Using Fubini theorem, Hölder's inequality and noticing that $f_9\in L_g$, it appears that

$$\begin{aligned}
&\int_0^L\int_0^{+\infty}g(s)\left(e^{-s}\int_0^s e^p f_{9x}(x,p)\,dp\right)^2 ds\,dx \\
&\le \int_0^L\int_0^{+\infty}e^{-2s}g(s)\left(\int_0^s e^p\,dp\right)\int_0^s e^p f_{9x}^2(x,p)\,dp\,ds\,dx \\
&\le \int_0^L\int_0^{+\infty}e^{-s}(1-e^{-s})g(s)\int_0^s e^p f_{9x}^2(x,p)\,dp\,ds\,dx \\
&\le \int_0^L\int_0^{+\infty}e^{-s}g(s)\int_0^s e^p f_{9x}^2(x,p)\,dp\,ds\,dx \\
&\le \int_0^L\int_0^{+\infty}e^p f_{9x}^2(x,p)\int_p^{+\infty}e^{-s}g(s)\,ds\,dp\,dx \\
&\le \int_0^L\int_0^{+\infty}e^p g(p) f_{9x}^2(x,p)\int_p^{+\infty}e^{-s}\,ds\,dp\,dx \\
&\le \int_0^L\int_0^{+\infty}g(p) f_{9x}^2(x,p)\,dp\,dx=\|f_9\|_{L_g}^2<+\infty.
\end{aligned}$$

Hence, $s\mapsto e^{-s}\int_0^s e^p f_p(p)\,dp\in L_g$, and (3.10) implies that $\eta\in L_g$. Moreover, one has $\eta_s\in L_g$ by (3.9).

Now, we put $\hat{\varphi}=\varphi-g_1$. We see that, if $\hat{\varphi}\in H^2(0,L)$ satsifies $\hat{\varphi}(0)=\hat{\varphi}_x(0)=0$, then $\varphi\in H^2(0,L)$ satsifies $\varphi(0)=g_1$ and $\varphi_x(0)=0$ (notice that g_1 is a constant). According to

(3.7), we see that equations $(3.6)_2$, $(3.6)_4$ and $(3.6)_6$ can be reduced to

$$\begin{cases} \rho_1\hat{\varphi} - k_1\left(\hat{\varphi}_x + \psi\right)_x + \theta_x = h_1, \\ \rho_2\psi - \left(l\psi + \int_0^{+\infty} g\eta\right)_{xx} ds + k_1\left(\hat{\varphi}_x + \psi\right) - \theta = h_2, \\ \rho_3\theta - (k_3\theta + \alpha\sigma)_{xx} + \beta\left(\hat{\varphi}_x + \psi\right) = h_3, \end{cases} \tag{3.11}$$

where

$$h_1 = \rho_1(f_1 + f_2 - g_1) - f_{5x}, \quad h_2 = \rho_2(f_3 + f_4) - f_5 \quad \text{and} \quad h_3 = \beta(f_{1x} + f_3) + \rho_3(f_5 + f_6).$$

We deduce that (3.6) has a solution $\Psi \in D(\mathscr{B})$ if (3.11) has a solution

$$(\hat{\varphi}, \psi, \theta) \in (H^2(0,L) \cap H_*^1(0,L)) \times H_0^1(0,L) \times H_0^1(0,L) \tag{3.12}$$

satisfying $(3.7)_3$, (3.10),

$$\begin{cases} \hat{\varphi}_x(0) = 0, \\ \frac{k_1}{m_E}\hat{\varphi}_x(L) + \hat{\varphi}(L) = g_2 - g_1 := g_3, \end{cases} \tag{3.13}$$

$$l\psi + \int_0^{+\infty} g\eta ds \in H^2(0,L) \quad \text{and} \quad k_3\theta + \alpha\sigma \in H^2(0,L), \tag{3.14}$$

where

$$H_*^1(0,L) = \{f \in H^2(0,L) : f(0) = 0\}.$$

To this end, we consider the variational formulation of (3.11) in

$$\bar{H} := H_*^1(0,L) \times H_0^1(0,L) \times H_0^1(0,L)$$

by multiplying the equations in (3.11) by $\beta\tilde{\varphi}$, $\beta\tilde{\psi}$ and $\tilde{\theta}$, respectively, where $(\tilde{\varphi}, \tilde{\psi}, \tilde{\theta}) \in \bar{H}$ are test functions. Using $(3.7)_3$, (3.10) and (3.13), we integrate by parts and sum up the obtained formulas to obtain

$$a_1((\hat{\varphi}, \psi, \theta), (\tilde{\varphi}, \tilde{\psi}, \tilde{\theta})) = a_2(\tilde{\varphi}, \tilde{\psi}, \tilde{\sigma}), \quad \forall (\tilde{\varphi}, \tilde{\psi}, \tilde{\theta}) \in \bar{H}, \tag{3.15}$$

where

$$\begin{aligned} &a_1((\hat{\varphi}, \psi, \theta), (\tilde{\varphi}, \tilde{\psi}, \tilde{\theta})) \\ &= \beta \int_0^L [k_1(\hat{\varphi}_x + \psi)(\tilde{\varphi}_x + \tilde{\psi}) + (k_2 - \tilde{g}_0)\psi_x\tilde{\psi}_x + \rho_1\hat{\varphi}\tilde{\varphi} + \rho_2\psi\tilde{\psi}]\, dx \\ &\quad + \int_0^L \left[(k_3 + \alpha)\theta_x\tilde{\theta}_x + \beta(\hat{\varphi}_x\tilde{\theta} - \theta\tilde{\varphi}_x) + \beta(\psi\tilde{\theta} - \theta\tilde{\psi}) + \rho_3\theta\tilde{\theta}\right] dx \\ &\quad + \beta m_E\hat{\varphi}(L)\tilde{\varphi}(L), \end{aligned}$$

$$a_2(\tilde{\varphi}, \tilde{\psi}, \tilde{\sigma}) = \beta \int_0^L \left[h_1\tilde{\varphi} + h_2\tilde{\psi} + h_3\tilde{\theta} + h_4\tilde{\psi}_x\right] dx + \alpha f_{5x}\tilde{\theta} + m_E g_3\tilde{\varphi}(L),$$

and

$$h_4 = (g_0 - \tilde{g}_0)f_{3x} - \int_0^{+\infty} e^{-s}g(s)\int_0^s e^p f_{9x}(\cdot, p)\, dp\, ds \quad \text{and} \quad \tilde{g}_0 = \int_0^{+\infty} e^{-s}g(s)ds.$$

In view of $F \in \mathscr{H}$ and (2.3),

$$h_1, h_2, h_3, h_4 \in L^2(0,L) \quad \text{and} \quad k_2 - \tilde{g}_0 \geq k_2 - g_0 > 0,$$

we realize that a_1 is a bilinear, continuous and coercive form on $\bar{H}\times\bar{H}$, and a_2 is a linear and continuous form on $\bar{H}$. Lax-Milgram theorem implies that (3.15) admits a unique solution $(\hat{\varphi},\psi,\theta)\in\bar{H}$. By classical regularity arguments, we deduce that $(\hat{\varphi},\psi,\theta)$ solves (3.11) and fulfills (3.12), (3.13) and (3.14). This proves that (3.6) has a solution $\Psi\in D(\mathscr{B})$. Finally, since the linear operator $\mathscr{B}$ is maximal monotone, it generates a linear C_0 semigroup of contractions on $\mathscr{H}$ and $D(\mathscr{B})$ is dense in $\mathscr{H}$. So, this theorem holds thanks to Hille-Yosida theorem. □

4. Stability

In this section, we prove the stability of (3.3) and specify explicitly the decay rate of solutions. To this end, we need to prove some lemmas. We use c to denote a generic positive constant which can be different from line to line.

Lemma 4.1. *The functional I_1 defined by*

$$I_1(t)=-\int_0^L(\rho_1\varphi_t\varphi+\rho_2\psi_t\psi)\,dx+m\varphi_t(0,t)\varphi(0,t)-m_E\varphi_t(L,t)\varphi(L,t)$$

fulfills, for any $\varepsilon_1>0$,

$$\begin{aligned}\frac{dI_1(t)}{dt} \leq& \left(m+\frac{K^2}{2}\right)\varphi_t^2(0,t)-m_E\varphi_t^2(L,t)-\frac{1}{2}\varphi^2(0,t)-\rho_1\|\varphi_t\|_2^2-\rho_2\|\psi_t\|_2^2\\&+\frac{1}{\varepsilon_1}\|\theta_t\|_2^2+(c+\varepsilon_1)\|\psi_x\|_2^2+(\varepsilon_1+k_1)\|\varphi_x+\psi\|_2^2+c\,(g\circ\psi_x).\end{aligned}$$

Proof. A direct differentiation of $I_1(t)$ yields

$$\begin{aligned}\frac{dI_1(t)}{dt} =& -k_1\int_0^L\varphi\,(\varphi_x+\psi)_x\,dx+k_1\int_0^L\psi\,(\varphi_x+\psi)\,dx-\int_0^L\varphi_x\theta_t dx-\int_0^L\psi\theta_t dx\\&+k_2\|\psi_x\|_2^2-\int_0^L\psi_x\int_0^{+\infty}g\,(t-s)\,\psi_x\,(x,s)\,dsdx+m\varphi_t^2(0,t)-m_E\varphi_t^2(L,t)\\&-\rho_1\|\varphi_t\|_2^2-\rho_2\|\psi_t\|_2^2-\psi^2(0,t)+m\varphi_{tt}(0,t)\varphi(0,t)-m_E\varphi_{tt}(L,t)\varphi(L,t).\end{aligned}$$

Then

$$\begin{aligned}-k_1\int_0^L\varphi\,(\varphi_x+\psi)_x\,dx =& -k_1\,(\varphi_x\,(L,t)+\psi\,(L,t))\,\varphi\,(L,t)+k_1\int_0^L\varphi_x\,(\varphi_x+\psi)\,dx\\=& \;m_E\varphi_{tt}(L,t)\varphi\,(L,t)+k_1\int_0^L\varphi_x\,(\varphi_x+\psi)\,dx,\end{aligned}$$

$$-\int_0^L\varphi_x\theta_t dx\leq\frac{\varepsilon_1}{2}\|\varphi_x\|_2^2+\frac{1}{2\varepsilon_1}\|\theta_t\|_2^2\leq\frac{\varepsilon_1}{2}\left(2\|\varphi_x+\psi\|_2^2+2c\|\psi_x\|_2^2\right)+\frac{1}{2\varepsilon_1}\|\theta_t\|_2^2,$$

$$-\int_0^L\psi\theta_t dx\leq\frac{\varepsilon_1}{2}\|\psi\|_2^2+\frac{1}{2\varepsilon_1}\|\theta_t\|_2^2\leq\frac{c\varepsilon_1}{2}\|\psi_x\|_2^2+\frac{1}{2\varepsilon_1}\|\theta_t\|_2^2,$$

$$m\varphi_{tt}(0,t)\varphi(0,t)=\tau(t)\varphi(0,t)=-K\varphi_t(0,t)\varphi(0,t)-\varphi^2(0,t)\leq\frac{K^2}{2}\varphi_t^2(0,t)-\frac{1}{2}\varphi^2(0,t)$$

and

$$\begin{aligned}
& -\int_0^L \psi_x \int_0^{+\infty} g(s)\,\psi_x(x,t-s)\,dsdx \\
= & \int_0^L \psi_x \int_0^{\infty} g(s)\left(\psi_x(x,t)-\psi_x(x,t-s)\right)dsdx - g_0\left\|\psi_x\right\|_2^2 \\
\leq & \frac{\varepsilon_1}{2}\left\|\psi_x\right\|_2^2 + \frac{1}{4\varepsilon_1}\int_0^L \left(\int_0^{+\infty} g(s)\left(\psi_x(t-s)-\psi_x(t)\right)ds\right)^2 dx - g_0\left\|\psi_x\right\|_2^2 \\
\leq & \frac{\varepsilon_1}{2}\left\|\psi_x\right\|_2^2 + \frac{g_0}{4\varepsilon_1}\left(g\circ\psi_x\right).
\end{aligned}$$

By combining the above six relations, we obtain the desired conclusion immediately. □

Lemma 4.2. *The functional I_2 defined by*

$$I_2(t) = \rho_2 \int_0^L \psi_t \int_0^{+\infty} g(s)\left(\psi(t)-\psi(t-s)\right)dsdx$$

fulfills, for any $\varepsilon_2 > 0$,

$$\begin{aligned}
\frac{dI_2(t)}{dt} \leq & -\left[\rho_2 g_0 - \varepsilon_2\right]\left\|\psi_t\right\|_2^2 + c\varepsilon_2\left\|\psi_x\right\|_2^2 + \varepsilon_2\left\|\varphi_x+\psi\right\|_2^2 + \varepsilon_2\left\|\theta_t\right\|_2^2 \\
& - c\left(g'\circ\psi_x\right) + c\left(g\circ\psi_x\right).
\end{aligned}$$

Proof. A direct differentiation of $I_2(t)$ yields

$$\begin{aligned}
\frac{dI_2(t)}{dt} = & -\rho_2 g_0\left\|\psi_t\right\|_2^2 - \rho_2\int_0^L \psi_t \int_0^{+\infty} g'(s)\left(\psi(t)-\psi(t-s)\right)dsdx \\
& - k_2\int_0^L \psi_x \int_0^{+\infty} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)dsdx \\
& + k_1\int_0^L \left(\varphi_x+\psi\right)\int_0^{+\infty} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)dsdx \\
& + \int_0^L \theta_t \int_0^{+\infty} g(s)\left(\psi(t)-\psi(t-s)\right)dsdx \\
& + \int_0^L \left(\int_0^{+\infty} g(s)\,\psi_x(x,t-s)\,ds\right)\int_0^{+\infty} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)dsdx,
\end{aligned}$$

therefore

$$\begin{aligned}
\frac{dI_2(t)}{dt} \leq & -\left[\rho_2 g_0 - \varepsilon_2\right]\left\|\psi_t\right\|_2^2 + \varepsilon_2\left\|\psi_x\right\|_2^2 + \varepsilon_2\left\|\varphi_x+\psi\right\|_2^2 + \varepsilon_2\left\|\theta_t\right\|_2^2 \\
& + \varepsilon_2\int_0^L \left(\int_0^{+\infty} g(s)\,\psi_x(x,t-s)\,ds\right)^2 dx \\
& + \frac{1}{4\varepsilon_2}\int_0^L \left(\int_0^{+\infty} g'(s)\left(\psi(t)-\psi(t-s)\right)ds\right)^2 dx \\
& + \frac{1}{4\varepsilon_2}\int_0^L \left(\int_0^{+\infty} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)ds\right)^2 dx \\
& + \frac{1}{4\varepsilon_2}\int_0^L \left(\int_0^{+\infty} g(s)\left(\psi(t)-\psi(t-s)\right)ds\right)^2 dx.
\end{aligned}$$

By virtue of the estimations

$$\begin{aligned}
&\int_0^L \left(\int_0^{+\infty} g(s)\left(\psi(t)-\psi(t-s)\right)ds\right)^2 dx \\
\leq\ & g_0 \int_0^L \int_0^{+\infty} g(s)\left(\left(\psi(t)-\psi(t-s)\right)\right)^2 dsdx \leq g_0 c_p \left(g\circ\psi_x\right), \\
&\int_0^L \left(\int_0^{+\infty} g(s)\left(\psi_x(t)-\psi_x(t-s)\right)ds\right)^2 dx \leq g_0\left(g\circ\psi_x\right)
\end{aligned}$$

and

$$\int_0^L \left(\int_0^{+\infty} g'(s)\left(\psi(t)-\psi(t-s)\right)ds\right)^2 dx \leq -g(0)\left(g'\circ\psi_x\right),$$

the desired result follows. □

Lemma 4.3. *The functional I_3 defined by*

$$I_3(t) = \rho_3 \int_0^L \theta_t \theta dx + \beta \int_0^L \varphi_x \theta dx + \frac{\alpha}{2}\int_0^L \theta_x^2 dx$$

fulfills, for any $\varepsilon_1 > 0$,

$$\begin{aligned}
\frac{dI_3(t)}{dt} \leq & -\frac{k_3}{2}\int_0^L \theta_x^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^L \theta_t^2 dx + \varepsilon_1 \int_0^L \left(\varphi_x+\psi\right)^2 dx \\
& + c\varepsilon_1 \int_0^L \psi_x^2 dx + c\int_0^L \psi_t^2 dx.
\end{aligned}$$

Proof. A direct differentiation of $I_3(t)$ yields

$$\begin{aligned}
\frac{dI_3(t)}{dt} &= \rho_3 \int_0^L \theta_{tt}\theta dx + \rho_3 \int_0^L \theta_t^2 dx + \beta \int_0^L \varphi_{xt}\theta dx + \beta \int_0^L \varphi_x \theta_t dx + \alpha \int_0^L \theta_x \theta_{xt} dx \\
&= -k_3 \int_0^L \theta_x^2 dx + \rho_3 \int_0^L \theta_t^2 dx - \beta \int_0^L \psi_t \theta dx + \beta \int_0^L \varphi_x \theta_t dx.
\end{aligned}$$

We see that (c_p is Poincaré's constant)

$$\begin{aligned}
\beta \int_0^L \varphi_x \theta_t dx &\leq \frac{\varepsilon_1}{2}\int_0^L \varphi_x^2 dx + \frac{\beta^2}{2\varepsilon_1}\int_0^L \theta_t^2 dx \\
&\leq \frac{\varepsilon_1}{2}\left(2\int_0^L \left(\varphi_x+\psi\right)^2 dx + 2c_p \int_0^L \psi_x^2 dx\right) + \frac{\beta^2}{2\varepsilon_1}\int_0^L \theta_t^2 dx
\end{aligned}$$

and

$$-\beta \int_0^L \psi_t \theta dx \leq \frac{\beta^2}{2k_3 c_p}\int_0^L \psi_t^2 dx + \frac{k_3}{2}\int_0^L \theta_x^2 dx.$$

Then, the result of Lemma 4.3 is proved. □

We consider as multiplier w, the solution of

$$-w_{xx} = \psi_x, \quad w(0) = w(L) = 0.$$

It can be shown easily that the solution of this equation is explicitly given by

$$w(x,t) = -\int_0^x \psi(s,t)dx + \frac{x}{L}\int_0^L \psi(s,t)dx.$$

It follows that

$$\|w_x\|_2^2 \leq \|\psi\|_2^2 \leq c_p\|\psi_x\|_2^2 \quad \text{and} \quad \|w_t\|_2^2 \leq c_p\|w_{tx}\|_2^2 \leq c_p\|\psi_t\|_2^2.$$

Lemma 4.4. *The functional I_4 defined by*

$$I_4(t) = \rho_1 \int_0^L \varphi_t w dx + \rho_2 \int_0^L \psi_t \psi dx$$

fulfills, for any $\varepsilon_4 > 0$,

$$\frac{dI_4(t)}{dt} \leq -\frac{l}{2}\int_0^L \psi_x^2 dx + c\int_0^L \theta_t^2 dx + c\left(1+\frac{1}{\varepsilon_4}\right)\int_0^L \psi_t^2 dx + \varepsilon_4\int_0^L \varphi_t^2 dx + c\,(g\circ\psi_x).$$

Proof. A direct differentiation of $I_4(t)$ yields

$$\begin{aligned}
\frac{dI_4(t)}{dt} &= \rho_1\int_0^L \varphi_{tt} w dx + \rho_1\int_0^L \varphi_t w_t dx + \rho_2\int_0^L \psi_{tt}\psi dx + \rho_2\int_0^L \psi_t^2 dx \\
&= k_1\int_0^L w_x^2 dx - k_1\int_0^L \psi^2 dx + \int_0^L w_x\theta_t dx - k_2\int_0^L \psi_x^2 dx + \int_0^L \theta_t\psi dx \\
&\quad +\rho_2\int_0^L \psi_t^2 dx + \rho_1\int_0^L \varphi_t w_t dx + \int_0^L \psi_x\int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds.
\end{aligned}$$

We remark that, for any ε_3, $\varepsilon_4 > 0$,

$$\rho_1\int_0^L \varphi_t w_t dx \leq \frac{\rho_1^2}{4\varepsilon_4}\int_0^L w_t^2 dx + \varepsilon_4\int_0^L \varphi_t^2 dx \leq \frac{\rho_1^2}{4\varepsilon_4}\int_0^L \psi_t^2 dx + \varepsilon_4\int_0^L \varphi_t^2 dx,$$

$$\int_0^L w_x\theta_t dx \leq \varepsilon_3\int_0^L w_x^2 dx + \frac{1}{4\varepsilon_3}\int_0^L \theta_t^2 dx \leq \varepsilon_3\int_0^L \psi_x^2 dx + \frac{1}{4\varepsilon_3}\int_0^L \theta_t^2 dx,$$

$$\int_0^L \theta_t\psi dx \leq \varepsilon_3\int_0^L \psi_x^2 dx + \frac{1}{4\varepsilon_3}\int_0^L \theta_t^2 dx$$

and

$$\begin{aligned}
&\int_0^L \psi_x\int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds \\
=\ & \int_0^L \psi_x\int_0^{+\infty} g(s)\,(\psi_x(t-s)-\psi_x(t))\,dsdx + g_0\int_0^L \psi_x^2 dx \\
\leq\ & (\varepsilon_3+g_0)\int_0^L \psi_x^2 dx + \frac{g_0}{4\varepsilon_3}(g\circ\psi_x).
\end{aligned}$$

By choosing $\varepsilon_3 = l/6$, the result follows. □

Lemma 4.5. *The functional I_5 defined by*

$$I_5(t) = \rho_2\int_0^L \psi_t(\varphi_x+\psi)\,dx + \frac{k_2\rho_1}{k_1}\int_0^L \varphi_t\psi_x dx - \frac{\rho_1}{k_1}\int_0^L \varphi_t\int_0^{+\infty} g(s)\,\psi_x(t-s)\,dsdx$$

fulfills, for any $\varepsilon_1 > 0$,

$$\frac{dI_5(t)}{dt} \leq \varphi_x(L)\left(k_2\psi_x(L) - \int_0^{+\infty} g(s)\,\psi_x(L,t-s)\,ds\right) - \frac{k_1}{2}\int_0^L (\varphi_x+\psi)^2 dx \tag{4.1}$$

$$
\begin{aligned}
&+ \quad c\varepsilon_1 \int_0^L \varphi_t^2 dx + c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^L \psi_x^2 dx + \rho_2 \int_0^L \psi_t^2 dx \\
&+ \quad c\left(1+\frac{1}{\varepsilon_1}\right)\int_0^L \theta_{tx}^2 dx + c\varepsilon_1 \left(g\circ\psi_x\right) - \frac{c}{\varepsilon_1}\left(g'\circ\psi_x\right) + \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_{tx}\psi_t dx.
\end{aligned}
$$

Proof. A direct differentiation of $I_5(t)$ yields

$$
\begin{aligned}
\frac{dI_5(t)}{dt} = \; & k_2 \int_0^L \psi_{xx}\left(\varphi_x+\psi\right)dx - k_1\int_0^L \left(\varphi_x+\psi\right)^2 dx + \int_0^L \left(\varphi_x+\psi\right)\theta_t dx \\
& - \int_0^L \left(\varphi_x+\psi\right)\int_0^{+\infty} g(s)\,\psi_{xx}(x,t-s)\,ds + \rho_2 \int_0^L \psi_t \left(\varphi_x+\psi\right)_t dx \\
& + k_2 \int_0^L \left(\varphi_x+\psi\right)_x \psi_x dx - \frac{k_2}{k_1}\int_0^L \psi_x\theta_{tx} dx \\
& - \int_0^L \left(\varphi_x+\psi\right)_x \int_0^{+\infty} g(s)\,\psi_x(t-s)\,dsdx + \frac{k_2\rho_1}{k_1}\int_0^L \varphi_t\psi_{tx} dx \\
& + \frac{1}{k_1}\int_0^L \theta_{tx}\int_0^{+\infty} g(s)\,\psi_x(t-s)\,dsdx - \frac{\rho_1}{k_1}\int_0^L \varphi_t \int_0^{+\infty} g'(s)\,\psi_x(t-s)\,dsdx,
\end{aligned}
$$

therefore

$$
\begin{aligned}
\frac{dI_5(t)}{dt} = \; & \varphi_x(L)\left(k_2\psi_x(L) - \int_0^{+\infty} g(s)\,\psi_x(L,t-s)\,ds\right) - k_1 \int_0^L \left(\varphi_x+\psi\right)^2 dx \\
& + \int_0^L \left(\varphi_x+\psi\right)\theta_t dx + \rho_2\int_0^L \psi_t^2 dx - \frac{k_2}{k_1}\int_0^L \psi_x\theta_{tx} dx \\
& + \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_{tx}\psi_t dx + \frac{1}{k_1}\int_0^L \theta_{tx}\int_0^{+\infty} g(s)\,\psi_x(t-s)\,dsdx \\
& - \frac{\rho_1}{k_1}\int_0^L \varphi_t \int_0^{+\infty} g'(s)\,\psi_x(t-s)\,dsdx.
\end{aligned}
$$

On the other hand, we have

$$
\int_0^L \left(\varphi_x+\psi\right)\theta_t dx \le \frac{k_1}{2}\int_0^L \left(\varphi_x+\psi\right)^2 dx + \frac{1}{2k_1}\int_0^L \theta_t^2 dx,
$$

$$
-\frac{k_2}{k_1}\int_0^L \psi_x\theta_{tx} dx \le \varepsilon_1 \int_0^L \psi_x^2 dx + \frac{k_2^2}{4\varepsilon_1 k_1^2}\int_0^L \theta_{tx}^2 dx,
$$

$$
\begin{aligned}
& \frac{1}{k_1}\int_0^L \theta_{tx}\int_0^{+\infty} g(s)\,\psi_x(t-s)\,dsdx \\
= \; & \frac{1}{k_1}\int_0^L \theta_{tx}\int_0^{+\infty} g(s)\left(\psi_x(t-s)-\psi_x(t)\right)dsdx + \frac{1}{k_1}\int_0^L \theta_{tx}\int_0^{+\infty} g(s)\,\psi_x(t)\,dsdx \\
\le \; & \varepsilon_1\left(g\circ\psi_x\right) + \frac{g_0}{4\varepsilon_1 k_1^2}\int_0^L \theta_{tx}^2 dx + \frac{1}{k_1}\int_0^{+\infty} g(s)\,ds \int_0^L \theta_{tx}\psi_x(t)\,dx \\
\le \; & \varepsilon_1\left(g\circ\psi_x\right) + \frac{g_0}{k_1^2}\left(g_0 + \frac{1}{4\varepsilon_1}\right)\int_0^L \theta_{tx}^2 dx + \varepsilon_1 \int_0^L \psi_x^2(t)\,dx
\end{aligned}
$$

and

$$\begin{aligned}
& -\frac{\rho_1}{k_1}\int_0^L \varphi_t \int_0^{+\infty} g'(s)\,\psi_x(t-s)\,dsdx \\
= & -\frac{\rho_1}{k_1}\int_0^L \varphi_t \int_0^{+\infty} g'(s)\left(\psi_x(t-s)-\psi_x(t)\right)dsdx+\frac{\rho_1 g(0)}{k_1}\int_0^L \varphi_t\psi_x dx \\
\leq & -\frac{1}{4\varepsilon_1}\left(\frac{g(0)\rho_1}{k_1}\right)^2\left(g'\circ\psi_x\right)+2\varepsilon_1\int_0^L \varphi_t^2 dx+\frac{1}{4\varepsilon_1}\left(\frac{g(0)\rho_1}{k_1}\right)^2\int_0^L \psi_x^2(t)\,dx.
\end{aligned}$$

A combination of the above five relations leads to the result of Lemma 4.5. □

To deal with the boundary terms appearing in the above lemma, we introduce the function

$$m(x)=2-\frac{4}{L}x, \quad x\in[0,L]$$

and prove the following lemma.

Lemma 4.6. *The functional I_6 defined, for $\varepsilon_1>0$, by*

$$I_6(t)=\frac{\varepsilon_1\rho_1}{k_1}\int_0^L m(x)\,\varphi_t\varphi_x dx-\frac{\rho_2}{4\varepsilon_1}\int_0^L m(x)\,\psi_t\left(k_2\psi_x-\int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)dx$$

fulfills

$$\begin{aligned}
\frac{dI_6(t)}{dt} \leq & -\varphi_x(L)\left(k_2\psi_x(L)-\int_0^t g(s)\,\psi_x(L,t-s)\,ds\right) \\
& +c\varepsilon_1\int_0^L \varphi_t^2 dx+c\varepsilon_1\int_0^L \theta_{xt}^2 dx+\frac{c}{\varepsilon_1}\int_0^L \psi_t^2 dx+c\varepsilon_1\int_0^L (\varphi_x+\psi)^2 dx \\
& +c\left(\varepsilon_1+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_1^3}\right)\int_0^L \psi_x^2 dx-\frac{c}{\varepsilon_1}\left(g'\circ\psi_x\right)+c\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_1^3}\right)\left(g\circ\psi_x\right).
\end{aligned}$$

Proof. A direct differentiation of $I_6(t)$ yields

$$\begin{aligned}
\frac{dI_6(t)}{dt} = & \frac{\varepsilon_1\rho_1}{k_1}\int_0^L m(x)\,\varphi_{tt}\varphi_x dx+\frac{\varepsilon_1\rho_1}{k_1}\int_0^L m(x)\,\varphi_t\varphi_{tx} dx \\
& -\frac{\rho_2}{4\varepsilon_1}\int_0^L m(x)\,\psi_{tt}\left(k_2\psi_x-\int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)dx \\
& -\frac{\rho_2}{4\varepsilon_1}\int_0^L m(x)\,\psi_t\left(k_2\psi_x-g(0)\,\psi_x-\int_0^{+\infty} g'(s)\,\psi_x(t-s)\,ds\right)dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\frac{dI_6(t)}{dt} &= \varepsilon_1 \int_0^L m(x)\,(\varphi_x+\psi)_x\,\varphi_x dx - \frac{\varepsilon_1}{k_1}\int_0^L m(x)\,\theta_{tx}\varphi_x dx \\
&+\frac{\varepsilon_1\rho_1}{k_1}\int_0^L m(x)\,\varphi_t\varphi_{tx}dx + \frac{1}{4\varepsilon_1}\int_0^L m(x)\left(k_2\psi_{xx} - \int_0^{+\infty} g(s)\,\psi_{xx}(t-s)\,ds\right) \\
&\times\left(k_2\psi_x - \int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)dx \\
&-\frac{k_1}{4\varepsilon_1}\int_0^L m(x)\,(\varphi_x+\psi)\left(k_2\psi_x - \int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)dx \\
&+\frac{1}{4\varepsilon_1}\int_0^L m(x)\,\theta_t\left(k_2\psi_x - \int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)dx \\
&+\frac{\rho_2}{4\varepsilon_1}\int_0^L m(x)\,\psi_t\left(k_2\psi_{xt} - g(0)\psi_x - \int_0^{+\infty} g'(s)\,\psi_x(t-s)\,ds\right)dx.
\end{aligned}$$

Integrating by parts, one sees that

$$\begin{aligned}
\frac{dI_6(t)}{dt} &= -\varepsilon_1\left(\varphi_x^2(L)+\varphi_x^2(0)\right) - \frac{\varepsilon_1}{k_1}\int_0^L m(x)\,\theta_{tx}\varphi_x dx + \frac{2\varepsilon_1\rho_1}{k_1 L}\int_0^L \varphi_t^2 dx \\
&+\frac{2\varepsilon_1}{L}\int_0^L \varphi_x^2 dx + \varepsilon_1\int_0^L m(x)\varphi_x\psi_x dx - \frac{\varepsilon_1\rho_1}{k_1}(\varphi_t^2(L)+\varphi_t^2(0)) \\
&-\frac{1}{4\varepsilon_1}\Big[\big(k_2\psi_x(L) - \int_0^{+\infty} g(s)\,\psi_x(L,t-s)ds\big)^2 \\
&+\big(k_2\psi_x(0) - \int_0^{+\infty} g(s)\,\psi_x(0,t-s)ds\big)^2\Big] \\
&+\frac{1}{2\varepsilon_1 L}\int_0^L\left(k_2\psi_x - \int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)^2 dx \\
&-\frac{k_1}{4\varepsilon_1}\int_0^L m(x)\,(\varphi_x+\psi)\left(k_2\psi_x - \int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)dx \\
&+\frac{1}{4\varepsilon_1}\int_0^L m(x)\,\theta_t\left(k_2\psi_x - \int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)dx \\
&+\frac{k_2\rho_2}{2\varepsilon_1 L}\int_0^L \psi_t^2 dx - \frac{\rho_2}{4\varepsilon_1}\int_0^L m(x)\,\psi_t\left(g(0)\psi_x + \int_0^{+\infty} g'(s)\,\psi_x(t-s)\,ds\right)dx.
\end{aligned}$$

Using Young's and Poincaré's inequalities, we see that, for any ε_1, $\varepsilon_5 > 0$,

$$-\varepsilon_1(\varphi_x^2(L) - \frac{1}{4\varepsilon_1}\left(k_2\psi_x(L) - \int_0^{+\infty} g(s)\,\psi_x(L,t-s)ds\right)^2$$

$$\geq -\varphi_x(L)\left(k_2\psi_x(L) - \int_0^{+\infty} g(s)\,\psi_x(L,t-s)\,ds\right),$$

$$-\frac{\varepsilon_1}{k_1}\int_0^L m(x)\,\theta_{tx}\varphi_x dx \leq c\varepsilon_1\int_0^L((\varphi_x+\psi)^2+\psi_x^2)dx + c\varepsilon_1\int_0^L \theta_{tx}^2 dx,$$

$$\frac{2\varepsilon_1}{L}\int_0^L \varphi_x^2 dx + \varepsilon_1\int_0^L m(x)\varphi_x\psi_x dx \leq c\varepsilon_1\int_0^L((\varphi_x+\psi)^2+\psi_x^2)dx,$$

$$\frac{1}{2\varepsilon_1 L}\int_0^L \left(k_2\psi_x - \int_0^\infty g(s)\,\psi_x(t-s)\,ds\right)^2 dx \le \frac{c}{\varepsilon_1}\int_0^L \psi_x^2 dx + \frac{c}{\varepsilon_1}(g\circ\psi_x),$$

$$-\frac{k_1}{4\varepsilon_1}\int_0^L m(x)(\varphi_x+\psi)\left(k_2\psi_x - \int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)dx$$

$$\le \frac{c}{\varepsilon_1}\left[\varepsilon_5\int_0^L ((\varphi_x+\psi)^2+\psi_x^2)dx + \frac{c}{\varepsilon_5}(g\circ\psi_x)\right],$$

$$\frac{1}{4\varepsilon_1}\int_0^L m(x)\,\theta_t\left(k_2\psi_x - \int_0^{+\infty} g(s)\,\psi_x(t-s)\,ds\right)dx$$
$$\le \frac{c}{\varepsilon_1}\left[\varepsilon_5\int_0^L \theta_{xt}^2 dx + \frac{c}{\varepsilon_5}\int_0^L \psi_x^2 dx + \frac{c}{\varepsilon_5}(g\circ\psi_x)\right],$$

and

$$-\frac{\rho_2}{4\varepsilon_1}\int_0^L m(x)\,\psi_t\left(g(0)\psi_x + \int_0^{+\infty} g'(s)\,\psi_x(t-s)\,ds\right)dx$$
$$\le \frac{c}{\varepsilon_1}\int_0^L (\psi_x^2+\psi_t^2)dx - \frac{c}{\varepsilon_1}(g'\circ\psi_x).$$

Taking $\varepsilon_5 = \varepsilon_1^2$, the sum of the above inequalities leads to the desired result. □

Next, we define a Lyapunov functional F, which is equivalent to the first-order energy functional E. For positive constants N, N_1 and N_2, to be chosen appropriately later, we let

$$F = NE + I_1 + N_1 I_2 + I_3 + N_2 I_4 + I_5 + I_6.$$

It easy to show that, for N large enough, there exist two positive constants μ_1 and μ_2 such that

$$\mu_1 E \le F \le \mu_2 E. \tag{4.2}$$

4.1. **Uniform stability.** Here, we consider the case

$$\frac{\rho_2}{\rho_1} = \frac{k_2}{k_1}. \tag{4.3}$$

and prove the following uniform stability result.

Theorem 4.7. *Assume that (A1), (A2) and (4.3) are satisfied. Let $\Psi_0 \in \mathscr{H}$ be given and*

$$\xi_0 \equiv \textit{constant} \quad \textit{or} \quad \sup_{s\in\mathbb{R}_+} \|\eta_{0x}\|_2 < +\infty. \tag{4.4}$$

Then, there exist constants $\beta_0 \in]0,1]$ and $\alpha_1 > 0$ such that, for all $\alpha_0 \in]0,\beta_0[$,

$$E(t) \le \alpha_1\left(1+\int_0^t (g(s))^{1-\alpha_0}ds\right)e^{-\alpha_0\int_0^t \xi_0(s)\,ds} + \alpha_1\int_t^{+\infty} g(s)\,ds, \quad \forall t\in\mathbb{R}_+. \tag{4.5}$$

Proof. Assumption (4.3) implies that the last term in (4.1) vanishes. By differentiating F, using (2.12) and the previous lemmas and taking $\varepsilon_2 = \frac{k_1}{4N_1}$, we obtain

$$\begin{aligned}
F'(t) \leq & -[\rho_1 - N_2\varepsilon_4 - 2c\varepsilon_1]\|\varphi_t\|_2^2 - \left[N_1\rho_2 g_0 - \frac{k_1}{4} - c - N_2 c\left(1+\frac{1}{\varepsilon_4}\right) - \frac{c}{\varepsilon_1}\right]\|\psi_t\|_2^2 \\
& - \left[N\alpha - \frac{1}{\varepsilon_1} - N_1\varepsilon_2 - c\left(1+\frac{1}{\varepsilon_1}\right) - N_2 c - c\left(1+\frac{1}{\varepsilon_1}\right) - c\varepsilon_1\right]\|\theta_{tx}\|_2^2 \\
& - \frac{k_3}{2}\|\theta_x\|_2^2 - \left[\frac{N_2 l}{2} - 2c\varepsilon_1 - \frac{ck_1}{4} - c\left(1+\frac{1}{\varepsilon_1}\right) - c\left(1+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_1^3}\right)\right]\|\psi_x\|_2^2 \\
& - \left[\frac{k_1}{4} - \varepsilon_1(2c+1)\right]\|\varphi_x+\psi\|_2^2 + \left[\frac{c}{\varepsilon_1} + c(N_1+N_2) + c\varepsilon_1 + c\left(\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_1^3}\right)\right](g\circ\psi_x) \\
& + \left[N\frac{\beta}{2} - N_1 c - \frac{c}{\varepsilon_1}\right](g'\circ\psi_x) - \left(NK - m - \frac{K^2}{2}\right)\varphi_t^2(0,t) - m_E\varphi_t^2(L,t) - \frac{1}{2}\varphi^2(0,t).
\end{aligned}$$

We choose the parameters as follows: ε_1 small enough so that

$$\gamma_1 := \rho_1 - 2c\varepsilon_1 > 0 \quad \text{and} \quad \frac{k_1}{4} - \varepsilon_1(2c+1) > 0,$$

N_2 large enough so that

$$\frac{N_2 l}{2} - 2c\varepsilon_1 - \frac{ck_1}{4} - c\left(1+\frac{1}{\varepsilon_1}\right) - c\left(1+\frac{1}{\varepsilon_1}+\frac{1}{\varepsilon_1^3}\right) > 0$$

(N_2 exists because $l > 0$ according to (2.3)), ε_4 small enough so that

$$\gamma_1 - N_2\varepsilon_4 > 0,$$

N_1 large enough so that (notice that $g_0 > 0$ according to (2.3))

$$N_1\rho_2 g_0 - c - N_2 c\left(1+\frac{1}{\varepsilon_4}\right) - \frac{c}{\varepsilon_1} > 0,$$

and N large enough so that

$$N\alpha - \frac{1}{\varepsilon_1} - N_1\varepsilon_2 - c\left(1+\frac{1}{\varepsilon_1}\right) - N_2 c - c\left(1+\frac{1}{\varepsilon_1}\right) - c\varepsilon_1 > 0,$$

$$N\frac{\beta}{2} - N_1 c - \frac{c}{\varepsilon_1} > 0 \quad \text{and} \quad NK - m - \frac{K^2}{2} > 0.$$

Consequently, from the definition of the energy functional E, we obtain, for some $c_1 > 0$,

$$F'(t) \leq -c_1 E(t) + c(g\circ\psi_x). \tag{4.6}$$

In order to solve this differential inequality, we have first to distinguish two cases related to (4.4).

The case, $\xi \equiv constant$. From the right inequality in (2.4), we have

$$\xi_0(t)(g\circ\psi_x) = ((\xi_0 g)\circ\psi_x) \leq -(g'\circ\psi_x),$$

then, using (2.12), we find

$$\xi(t)(g\circ\psi_x) \leq -\frac{2}{\beta}E'(t). \tag{4.7}$$

The case ,$\xi_0 \neq$ constant. Following the arguments in [20] and [22], and using the right inequality in (2.4), we get

$$\xi_0(t)\int_0^t g(s)\|\eta_x\|_2^2\,ds \le \int_0^t \xi_0(s)g(s)\|\eta_x\|_2^2\,ds \le -\int_0^t g'(s)\|\eta_x\|_2^2\,ds \le -(g'\circ\psi_x).$$

Next, recalling (2.12), we obtain

$$\xi_0(t)\int_0^t g(s)\|\eta_x\|_2^2\,ds \le -\frac{2}{\beta}E'(t). \tag{4.8}$$

On the other hand, the definition of E and the fact that E is nonincreasing imply that

$$\|\psi_x\|_2^2 \le \frac{2}{\beta l}E(t) \le \frac{2}{\beta l}E(0).$$

Therefore, for $s \ge t$,

$$\|\eta_x\|_2^2 = \|\eta_{0x}(\cdot,s-t)+\psi_x(\cdot,t)-\psi_x(\cdot,0)\|_2^2 \le c\left(E(0)+\sup_{s\in\mathbb{R}_+}\|\eta_{0x}\|_2^2\right).$$

In view of the boundedness condition on η_0 in (4.4), we deduce that

$$\xi_0(t)\int_t^{+\infty} g(s)\|\eta_x\|_2^2\,ds \le c\xi_0(t)\int_t^{+\infty} g(s)ds. \tag{4.9}$$

Hence, the combination of (4.8) and (4.9) yields

$$\xi_0(t)\int_0^{+\infty} g(s)\|\eta_x\|_2^2\,ds \le -\frac{2}{\beta}E'(t)+c\xi_0(t)\int_t^{+\infty} g(s)ds. \tag{4.10}$$

Finally, multiplying (4.6) by $\xi_0(t)$ and combining with (4.7) and (4.10), we get for the two previous cases and for some $c_2>0$,

$$\xi_0(t)F'(t) \le -c_1\xi_0(t)E(t)+c\xi_0(t)\int_t^{+\infty} g(s)ds - c_2E'(t). \tag{4.11}$$

Let

$$\tilde{F} = \xi_0F + c_2E \quad \text{and} \quad h(t) = \xi_0(t)\int_t^{+\infty} g(s)ds.$$

Noticing that ξ_0 is nonincreasing and using (4.2) and (4.11), we deduce that

$$\tilde{F}'(t) \le -c_1\xi_0(t)E(t)+ch(t)+\xi_0'(t)F(t) \le -c_1\xi_0(t)E(t)+ch(t). \tag{4.12}$$

Thanks to (4.3) and because, again, ξ_0 is nonincreasing, we see that, for some positive constants $\tilde{\mu}_1$ and $\tilde{\mu}_2$

$$\tilde{\mu}_1E \le \tilde{F} \le \tilde{\mu}_2E. \tag{4.13}$$

Therefore, (4.12) implies that, for any $\alpha_0 \in]0,\beta_0[$, where $\beta_0 = \min\left\{1,\frac{c_1}{\tilde{\mu}_2}\right\}$,

$$\tilde{F}'(t) \le -\alpha_0\xi_0(t)\tilde{F}(t)+ch(t). \tag{4.14}$$

Then, (4.14) implies that

$$\frac{\partial}{\partial t}\left(e^{\alpha_0\int_0^t \xi_0(s)\,ds}\tilde{F}(t)\right) \le ce^{\alpha_0\int_0^t \xi_0(s)\,ds}h(t).$$

It follows, by integrating over $[0,T]$ with $T \geq 0$, that

$$\tilde{F}(T) \leq e^{-\alpha_0 \int_0^T \xi_0(s)\,ds} \left(\tilde{F}(0) + c \int_0^T e^{\alpha_0 \int_0^t \xi_0(s)\,ds} h(t)dt \right),$$

which implies, according to (4.13), that

$$E(T) \leq c e^{-\alpha_0 \int_0^T \xi_0(s)\,ds} \left(1 + \int_0^T e^{\alpha_0 \int_0^t \xi_0(s)\,ds} h(t)\,dt \right). \tag{4.15}$$

Since

$$e^{\alpha_0 \int_0^t \xi_0(s)\,ds} h(t) = \frac{1}{\alpha_0} \frac{\partial}{\partial t} \left(e^{\alpha_0 \int_0^t \xi_0(s)\,ds} \right) \int_t^{+\infty} g(s)\,ds,$$

we may write

$$\begin{aligned} &\int_0^T e^{\alpha_0 \int_0^t \xi_0(s)\,ds} h(t)\,dt \\ &= \frac{1}{\alpha_0} \left(e^{\alpha_0 \int_0^T \xi_0(s)\,ds} \int_T^{+\infty} g(s)\,ds - \int_0^{+\infty} g(s)\,ds + \int_0^T e^{\alpha_0 \int_0^t \xi_0(s)\,ds} g(t)\,dt \right). \end{aligned}$$

Consequently, combining with (4.15), we arrive at

$$\begin{aligned} E(T) \leq c &\left(e^{-\alpha_0 \int_0^T \xi_0(s)\,ds} + \int_T^{+\infty} g(s)\,ds \right) \\ &+ c e^{-\alpha_0 \int_0^T \xi_0(s)\,ds} \int_0^T e^{\alpha_0 \int_0^t \xi_0(s)\,ds} g(t)\,dt. \end{aligned} \tag{4.16}$$

On the other hand, the right inequality in (2.4) implies that

$$\frac{\partial}{\partial t} \left(e^{\alpha_0 \int_0^t \xi_0(s)\,ds} (g(t))^{\alpha_0} \right) = \alpha_0 (g(t))^{\alpha_0 - 1} (\xi_0(t) g(t) + g'(t)) e^{\alpha_0 \int_0^t \xi_0(s))\,ds} \leq 0,$$

and, hence,

$$e^{\alpha_0 \int_0^t \xi_0(s)\,ds} (g(t))^{\alpha_0} \leq (g(0))^{\alpha_0}.$$

We end up with

$$\int_0^T e^{\alpha_0 \int_0^t \xi_0(s)\,ds} g(t)\,dt \leq (g(0))^{\alpha_0} \int_0^T (g(t))^{1-\alpha_0}\,dt. \tag{4.17}$$

Finally, (4.16) and (4.17) give (4.5). □

4.2. **Weak stability.** Here we consider the case

$$\frac{\rho_2}{\rho_1} \neq \frac{k_2}{k_1}. \tag{4.18}$$

and prove the following weak stability result.

Theorem 4.8. *Assume that (A1), (A2) and (4.18) are satisfied. Let $\Psi_0 \in D(\mathscr{B})$ be given and*

$$\xi_0 \equiv \; constant \quad or \quad \sup_{s\in\mathbb{R}_+} \max_{k=0,1} \left\| \frac{\partial^k}{\partial s^k} \eta_{0x} \right\|_2 < +\infty. \tag{4.19}$$

Then, there exists a positive constant $\alpha_1 > 0$ such that

$$E(t) \leq \alpha_1 \left(1 + \int_0^t \xi_0(s) \int_s^{+\infty} g(p)\,dp\,ds\right) \left(\int_0^t \xi_0(s)\,ds\right)^{-1}, \quad \forall t > 0. \tag{4.20}$$

Remark 4.9. 1. If (A2) holds with $\xi_0 \equiv$ constant (which implies that g converges exponentially to zero at infinity), then (4.5) leads to, for some positive constants d_1 and d_2,

$$E(t) \leq d_1 e^{-d_2 t}, \quad \forall t \in \mathbb{R}_+, \tag{4.21}$$

and (4.20) implies that

$$E(t) \leq \frac{d_1}{t}, \quad \forall t > 0. \tag{4.22}$$

The decay rates of E in (4.21) and (4.22) are the best ones which can be obtained from (4.5) and (4.20), respectively.

2. If $\xi_0 \equiv$ constant, then (A2) implies that g converges exponentially to zero at infinity. However, when $\xi_0 \neq$ constant, condition (A2) allows $s \mapsto g(s)$ to have a decay rate arbitrarly close to $\frac{1}{s}$ at infinity, which represents the critical limit, since g is integrable on $\mathbb{R}_+$. For specific examples of g satisfying (A1)-(A2), and the corresponding decay rates given by (4.5) and (4.20), we refer to [20] and [22].

3. Using the arguments of [23] introduced for wave equations with infinite memories, the boundedness restrictions (4.4) and (4.19) on the initial data η_0 can be removed, and the admissible class of kernels g can be widened by replacing the right inequality in (2.4) by the following weaker one:

$$g'(t) \leq -\xi_0(t) G(g(t)), \quad \forall t \in \mathbb{R}_+,$$

where G is a given function satisfying some hypotheses. Instead (4.5) and (4.20), we will get more general stability estimates which take into consideration the size of η_0. We do not apply the arguments of [23] in the present paper because seeking the largest class possible of η_0 and g is not among our objectives and, moreover, we want to keep our paper not too long.

Proof of Theorem 4.8. We consider the energy of second order defined by

$$\begin{aligned} E_1(t) &= E(\varphi_t, \psi_t, \theta_t) \\ &= \frac{1}{2}\left[\varphi_t^2(0,t) + m\varphi_{tt}^2(0,t) + \beta m_E \varphi_{tt}^2(L,t) + \rho_3 \|\theta_{tt}\|_2^2 + k_3 \|\theta_{xt}\|_2^2\right] \\ &\quad + \frac{\beta}{2}\left[\rho_1 \|\varphi_{tt}\|_2^2 + \rho_2 \|\psi_{tt}\|_2^2 + l \|\psi_{xt}\|_2^2 + k_1 \|(\varphi_x + \psi)_t\|_2^2 + (g \circ \psi_{xt})(t)\right]. \end{aligned}$$

Clearly,

$$E_1'(t) = -K\varphi_{tt}^2(0,t) - \alpha\,\|\theta_{xtt}\|_2^2 + \frac{\beta}{2}\left(g' \circ \psi_{xt}\right). \tag{4.23}$$

On the other hand, because the last term in (4.1) does not necessarily vanish (in view of (4.18)), we find, instead of (4.6),

$$F'(t) \le -c_1 E(t) + c(g \circ \psi_x) + \left(\rho_2 - \frac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_{xt}\psi_t dx. \tag{4.24}$$

Therefore, instead of (4.12), in this case

$$\tilde{F}'(t) \le -\alpha_0\xi_0(t)E(t) + ch(t) + \xi_0(t)\left(\rho_2 - \frac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_{xt}\psi_t dx. \tag{4.25}$$

The following lemma gives an estimation of the last term in (4.25).

Lemma 4.10. *For any $\varepsilon > 0$, there exists $c_\varepsilon > 0$ such that*

$$\left|\left(\rho_2 - \frac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_{xt}\psi_t\,dx\right| \le c_\varepsilon \int_0^{+\infty} g(s)\|\eta_{xt}\|_2^2\,ds + \varepsilon E(t) - c_\varepsilon E'(t). \tag{4.26}$$

Proof. We have, by integrating by parts and using the definition of η,

$$\begin{aligned}\left(\rho_2 - \tfrac{k_2\rho_1}{k_1}\right)\int_0^L \varphi_{xt}\psi_t\,dx &= \left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \psi_{xt}\varphi_t\,dx \\ &= \tfrac{1}{g_0}\left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\eta_{xt}\,ds\,dx \\ &\quad + \tfrac{1}{g_0}\left(\tfrac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_{xt}(t-s)\,ds\,dx.\end{aligned} \tag{4.27}$$

Using the definition of E, it is to see that, for all $\varepsilon > 0$,

$$\left|\frac{1}{g_0}\left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\eta_{xt}\,ds\,dx\right| \le \frac{\varepsilon}{2}E(t) + c_\varepsilon\int_0^{+\infty} g(s)\|\eta_{xt}\|_2^2\,ds. \tag{4.28}$$

On the other hand, an integration with respect to s and the use of the definition of η allow us to write

$$\begin{aligned}\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_{xt}(t-s)\,ds\,dx &= -\int_0^L \varphi_t \int_0^{+\infty} g(s)\partial_s(\psi_x(t-s))\,ds\,dx \\ &= \int_0^L \varphi_t\left(g(0)\psi_x(t) + \int_0^{+\infty} g'(s)\psi_x(t-s)\,ds\right)dx \\ &= -\int_0^L \varphi_t \int_0^{+\infty} g'(s)\eta_x\,ds\,dx.\end{aligned}$$

Therefore, in view of (2.12), we have

$$\left|\frac{1}{g_0}\left(\frac{k_2\rho_1}{k_1} - \rho_2\right)\int_0^L \varphi_t \int_0^{+\infty} g(s)\psi_{xt}(t-s)\,ds\,dx\right| \le \frac{\varepsilon}{2}E(t) - c_\varepsilon E'(t). \tag{4.29}$$

Inserting (4.28) and (4.29) into (4.27), we obtain (4.26).

Now, combining (4.25) and (4.26), and choosing ε small enough, we find, for some positive constant $\tilde{c}$,

$$\tilde{F}'(t) \le -\tilde{c}\xi_0(t)E(t) + ch(t) - c\xi_0(t)E'(t) + c\xi_0(t)\int_0^{+\infty} g(s)\|\eta_{xt}\|_2^2\,ds. \tag{4.30}$$

On the other hand, using the boundedness condition on η_0 in (4.19), we have (as for (4.7) and (4.10))

$$\xi_0(t)\int_0^{+\infty} g(s)\|\eta_{xt}\|_2^2\,ds \le -cE_1'(t)+ch(t). \tag{4.31}$$

Hence, (4.30) and (4.31) lead to

$$\left(\tilde{F}(t)+cE_1(t)+c\xi_0(t)E(t)\right)' \le -c\xi_0(t)E(t)+ch(t)+c\xi_0'(t)E(t) \le -c\xi_0(t)E(t)+ch(t), \tag{4.32}$$

since ξ_0 is nonincreasing. Finally, by integrating over $[0,T]$ and having in mind that E is nonincreasing, we end up with

$$cE(T)\int_0^T \xi_0(t)\,dt \le \tilde{F}(0)+cE_1(0)+c\xi_0(0)E(0)+c\int_0^T h(t)\,dt,$$

which leads to the desired result (4.20). □

Remark 4.11. Here it is rather porous thermoelastic Timoshenko system of type III which is fixed to a base in translational motion at one end and a dynamic mass is attached to the other end. For such a system and that complications, stability results were achieved by controlling the structure through the base platform in both cases, equal and nonequal of wave speeds of propagation.

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