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Some well-posedness and general stability results in Timoshenko systems with infinite memory and distributed time delay

Aissa Guesmia[a)]

Elie Cartan Institute of Lorraine, UMR 7502, University of Lorraine, Bat. A, Ile du Saulcy, 57045 Metz Cedex 01, France

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In this paper, we consider a Timoshenko system in one-dimensional bounded domain with infinite memory and distributed time delay both acting on the equation of the rotation angle. Without any restriction on the speeds of wave propagation and under appropriate assumptions on the infinite memory and distributed time delay convolution kernels, we prove, first, the well-posedness and, second, the stability of the system, where we present some decay estimates depending on the equal-speed propagation case and the opposite one. The obtained decay rates depend on the growths of the memory and delay kernels at infinity. In the nonequal-speed case, the decay rate depends also on the regularity of initial data. Our stability results show that the only dissipation resulting from the infinite memory guarantees the asymptotic stability of the system regardless to the speeds of wave propagation and in spite of the presence of a distributed time delay. Applications of our approach to specific coupled Timoshenko-heat and Timoshenko-wave systems as well as the discrete time delay case are also presented. © *2014 AIP Publishing LLC*. [http://dx.doi.org/10.1063/1.4891489]

I. INTRODUCTION

Let $g:\ \mathbb{R}_+ \to \mathbb{R}_+$ and $f:\ \mathbb{R}_+ \to \mathbb{R}$ be given functions. We consider the following Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x(x,t)+\psi(x,t))_x = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t)+\psi(x,t)) \\ \qquad\qquad + \displaystyle\int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds + \int_0^{+\infty} f(s)\psi_t(x,t-s)ds = 0, \\ \varphi(0,t) = \psi(0,t) = \varphi(L,t) = \psi(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t), \quad \psi_t(x,-t) = \psi_1(x,t), \end{cases} \tag{1.1}$$

where $(x,t) \in]0, L[\times\mathbb{R}_+$, $(\varphi_0, \psi_0, \varphi_1, \psi_1)$ are given initial data belonging to a suitable space, (φ, ψ) is the state (unknown) of (1.1), L, ρ_1, ρ_2, k_1, and k_2 are positive constants. A subscript y denotes the derivative with respect to y. We also use the prime notation to denote the derivative when the function has only one variable. The infinite integrals depending on g and f represent, respectively, the infinite memory and the distributed time delay terms. This type of systems has been introduced in Ref. 44. It describes the transverse vibration of a thick beam of length L, where φ is the transverse displacement of the beam, $-\psi$ is the rotation angle of the filament of the beam, and ρ_1, ρ_2, k_1, and k_2 account for some physical properties of the beam (see, for example, Refs. 23 and 24).

a)Also at Department of Mathematics and Statistics, College of Sciences, King Fahd University of Petroleum and Minerals, P. O. Box. 5005, Dhahran 31261, Saudi Arabia. Electronic addresses: guesmia@univ-metz.fr and guesmia@kfupm.edu.sa

Our objective here is to prove the well-posedness and investigate the asymptotic behavior as time goes to infinity of solutions of (1.1) under appropriate assumptions on the convolution kernels g and f.

The questions related to well-posedness and stability/instability of evolution equations with delay and/or memory have attracted considerable attention in recent years and many researchers have shown that the memory plays the role of a damper, whereas the time delay can destabilize a system that was asymptotically stable in the absence of time delay. The main problem concerning the stability in the presence of memory is determining the largest class of kernels g which guarantee the stability and the best relation between the decay rate of g and the asymptotic behavior of solutions of the considered system. Because a small delay time can be a source of instability (see, for example, Ref. 28), to stabilize a hyperbolic system involving input delay terms, additional control terms (like memory or frictional damping) will be necessary. Let us recall some works related to the subject of the present paper.

In the absence of time delay term (i.e., $f \equiv 0$), a large amount of the literature is available on Timoshenko-type systems[44] with (finite or infinite) memory or frictional damping, addressing the issues of the existence, uniqueness, smoothness, and asymptotic behavior in time; see, for example, Refs. 2, 4, 5, 8, 9, 16–19, 25, and 26, and the references cited therein. In these papers, it was shown that the dissipation given by the memory term is strong enough to stabilize the system, and various decay estimates (exponential, polynomial, or others) have been obtained depending on the regularity of the initial data, the growth of g at infinity and the relation

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \tag{1.2}$$

and the opposite one. The equality (1.2) means that the first two equations of (1.1) have the same speeds of wave propagation $\sqrt{\frac{k_1}{\rho_1}}$ and $\sqrt{\frac{k_2}{\rho_2}}$, respectively. Similar stability results are known in the literature for other hyperbolic evolution equations with memory; see, for example, Ref. 14 and the references cited therein. The idea of proof consists in considering some integral and/or differenial inequalities involving g and/or some of its derivatives as a characterization of the growth of g at infinity from which the decay rate of the solution is deduced.

When the second equation of (1.1) is replaced by

$$\rho_2 \psi_{tt}(x,t) - k_2 \psi_{xx}(x,t) + k_1(\varphi_x(x,t) + \psi(x,t)) \tag{1.3}$$

$$+ \int_0^t g(s)\psi_{xx}(x, t-s)ds + \mu_1 \psi_t(x,t) + \mu_2 \psi_t(x, t-\tau) = 0,$$

where μ_1, μ_2, and τ are fixed non-negative constants, the stability of Timoshenko system was proved in Ref. 35 under the assumption $0 \leq \mu_2 \leq \mu_1$. The decay rate of solution obtained in Ref. 35 depends on the one of g. This result shows that the dissipation resulting from both finite memory $\int_0^t g(s)\psi_{xx}(x, t-s)ds$ and frictional damping $\mu_1 \psi_t(x, t)$ is strong enough to stabilize Timoshenko system in presence of a constant discrete time delay $\mu_2 \psi_t(x, t - \tau)$ provided that the coefficient of the delay is smaller or equal than the one of the damping. Similar stability results for various hyperbolic evolution equations with frictional damping and time delay exist in the literature, in this regard, we refer the reader to Refs. 3, 6, 7, 21, 27–31, and 32.

As far as we know, the problem of stability of Timoshenko system with infinite memory and distributed time delay considered in this paper has never been treated in the literature. The stability of the following abstract hyperbolic equation with infinite memory and discrete or distributed time delay:

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Au(t-s)ds + \mu u_t(t-\tau)ds = 0 \tag{1.4}$$

and

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)ds + \int_0^{+\infty} f(s)u_t(t-s)ds = 0 \tag{1.5}$$

was studied in, respectively, Refs. 15 and 20, and several decay estimates were proved depending on the growth of g and f at infinity, and the connection between the operators A and B. But (1.4)(1.5) do not include (1.1), since the operators A and B are supposed to be definite positive in Refs. 15 and 20.

Unlike the discrete time delay models, which ignore the inherent memory effects, the distributed time delay considered in this paper do take into account the whole (infinite) past history of the solution. More precisely, we are in presence of an indefinite frictional damping which depends on all previous states (the past information is stored and used later). This is what makes the present case more realistic. In fact, the discrete case will be a special case which corresponds to the Dirac delta distribution kernel (at some time τ).

According to the known results cited above, one main question naturally arises: is it possible for the memory term, which plays solely the role of dissipation in (1.1), to play the same role as a robust controller against the delay and stabilize (1.1), and is it possible to get the decay rate of solutions explicitly in term of, in particular, the connection between the delay and the memory kernels? As far as we know, this situation has never been considered before in the literature. In this paper, we shall prove, regardless of the speeds of wave propagation and under some appropriate assumptions on g and f, that (1.1) is well-posed in an appropriate underlying space, and that the only dissipation generated by the infinite memory guarantees the asymptotic stability of (1.1) in spite of the presence of a distributed time delay. Moreover, the decay rate of solutions is explicitly found in terms of the growths of g and f at infinity. When (1.2) does not hold, the decay rate depends also on the regularity of initial data and, so, it can be improved by choosing initial data regular enough. The proof is based on the semigroup theory for the well-posedness, and the energy method for the stability. We introduce new functionals to get crucial estimates on the distributed time delay and the infinite memory, and overcome subsequently the difficulties generated by the nondissipativeness character of our system (1.1). Moreover, we will appeal to some ideas and arguments in Refs. 20 and 36–43. These ideas will, in particular, allow us to deal with some arbitrary decaying kernels without assuming explicit conditions on their derivatives. The approach presented in this paper can be applied to the case of finite memory and/or discrete time delay as well as other Timoshenko-type systems.

The plan of the paper is as follows. In Sec. II, we present our assumptions on g and f, and state and prove the well-posedness of (1.1). Section III is devoted to the statement and proof of the asymptotic stability results of (1.1) under some additional assumptions on g and f. Section IV will be devoted to some applications to the coupled Timoshenko-heat and Timoshenko-wave systems as well as to the discrete time delay case. Finally, in Sec. V, we discuss some general comments and issues.

II. WELL-POSEDNESS

In this section, we state our assumptions on g and f, and prove the global existence, uniqueness, and smoothness of the solution of (1.1). We assume that

(A1) The function g is of class $C^1(\mathbb{R}_+, \mathbb{R}_+)$, nonincreasing and satisfies

$$g_0 := \int_0^{+\infty} g(s)ds < k_2. \tag{2.1}$$

Moreover, for some positive constant θ_0,

$$-g'(s) \le \theta_0 g(s), \quad \forall s \in \mathbb{R}_+. \tag{2.2}$$

(A2) The function f is of class $C^1(\mathbb{R}_+, \mathbb{R})$ and satisfies, for some positive constant α,

$$|f(s)| \le \alpha g(s) \quad \text{and} \quad |f'(s)| \le \alpha g(s), \quad \forall s \in \mathbb{R}_+. \tag{2.3}$$

Following a method devised in Ref. 5, we consider a new auxiliary variable η to treat the infinite memory and distributed time delay terms, and formulate the system (1.1) in the following abstract linear first-order system:

$$\begin{cases} \mathscr{U}_t(t) = (\mathscr{A} + \mathscr{B})\mathscr{U}(t), \\ \mathscr{U}(0) = \mathscr{U}_0, \end{cases} \tag{2.4}$$

where

$$\begin{cases} \mathscr{U} = (\varphi, \psi, \varphi_t, \psi_t, \eta)^T, \\ \mathscr{U}_0 = (\varphi_0, \psi_0(\cdot, 0), \varphi_1, \psi_1(\cdot, 0), \eta_0)^T \in \mathscr{H}, \end{cases}$$

$$\mathscr{H} = \left(H_0^1(]0, L[)\right)^2 \times \left(L^2(]0, L[)\right)^2 \times L_g^2(\mathbb{R}_+, H_0^1(]0, L[))$$

and

$$\begin{cases} \eta(x, t, s) = \psi(x, t) - \psi(x, t - s), \\ \eta_0(x, s) = \eta(x, 0, s) = \psi_0(x, 0) - \psi_0(x, s). \end{cases} \tag{2.5}$$

The set $L_g^2(\mathbb{R}_+, H_0^1(]0, L[))$ is the weighted space with respect to the measure $g(s)ds$ defined by

$$L_g^2(\mathbb{R}_+, H_0^1(]0, L[)) = \left\{ w : \mathbb{R}_+ \to H_0^1(]0, L[), \quad \int_0^L \int_0^{+\infty} g(s) w_x^2(x, s) ds dx < +\infty \right\}$$

and endowed with the classical inner product

$$\langle v, w \rangle_{L_g^2(\mathbb{R}_+, H_0^1(]0, L[))} = \int_0^L \int_0^{+\infty} g(s) v_x(x, s) w_x(x, s) ds dx.$$

The operators $\mathscr{B}$ and $\mathscr{A}$ are linear and given by

$$\mathscr{B}(w_1, w_2, w_3, w_4, w_5)^T = \left(0, 0, 0, \frac{\|f\|_\infty}{\rho_2} w_4, \epsilon_0 w_5\right)^T \tag{2.6}$$

and

$$\mathscr{A} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} w_3 \\ w_4 \\ \frac{k_1}{\rho_1}(w_{1x} + w_2)_x \\ \tilde{w}_4 \\ -w_{5s} - \epsilon_0 w_5 + w_4 \end{pmatrix}, \tag{2.7}$$

where

$$\tilde{w}_4 = \frac{1}{\rho_2}(k_2 - g_0) w_{2xx} - \frac{k_1}{\rho_2}(w_{1x} + w_2) - \frac{\|f\|_\infty}{\rho_2} w_4 + \frac{1}{\rho_2} \int_0^{+\infty} g(s) w_{5xx}(s) ds - \frac{1}{\rho_2} \int_0^{+\infty} f(s) w_{5s}(s) ds,$$

$$\epsilon_0 = \frac{\alpha^2 g_0 c_0}{4\|f\|_\infty} \tag{2.8}$$

and c_0 is the smallest positive constant satisfying (Poincaré's inequality)

$$\int_0^L w^2(x) dx \le c_0 \int_0^L w_x^2(x) dx, \quad \forall w \in H_0^1(]0, L[) \tag{2.9}$$

(ϵ_0 is well defined positive constant, since $\|f\|_\infty > 0$. Otherwise, $f \equiv 0$ and then no distributed time delay is considered in (1.1), which is a well studied case in the literature; see the Introduction). The domains $\mathscr{D}(\mathscr{B})$ and $\mathscr{D}(\mathscr{A})$ of $\mathscr{B}$ and $\mathscr{A}$, respectively, are given by $\mathscr{D}(\mathscr{B}) = \mathscr{H}$ and

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{c} W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathscr{H},\ w_5(0) = 0 \\ w_{5s} \in L_g^2(\mathbb{R}_+, H_0^1(]0, L[)),\ w_3,\ w_4 \in H_0^1(]0, L[),\ w_1 \in H^2(]0, L[) \\ (k_2 - g_0) w_{2xx} + \displaystyle\int_0^{+\infty} g(s) w_{5xx}(s) ds \in L^2(]0, L[) \end{array} \right\}, \tag{2.10}$$

since, thanks to the Cauchy-Schwarz inequality and the first inequality in (2.3),

$$w_{5s} \in L_g^2(\mathbb{R}_+, H_0^1(]0, L[)) \Longrightarrow \int_0^{+\infty} f(s) w_{5s}(s) ds \in L^2(]0, L[). \tag{2.11}$$

We use the classical notations $\mathscr{D}(\mathscr{A}^0) = \mathscr{H}$, $\mathscr{D}(\mathscr{A}^1) = \mathscr{D}(\mathscr{A})$ and

$$\mathscr{D}(\mathscr{A}^n) = \{W \in \mathscr{D}(\mathscr{A}^{n-1}), \mathscr{A} W \in \mathscr{D}(\mathscr{A}^{n-1})\}, \quad n = 2, 3, \cdots,$$

endowed with the classical graph norm

$$\|W\|_{\mathscr{D}(\mathscr{A}^n)} = \sum_{k=0}^{n} \|\mathscr{A}^k W\|_{\mathscr{H}}. \tag{2.12}$$

On the other hand, keeping in mind the definition of η in (2.5), we have

$$\begin{cases} \eta_t(x, t, s) + \eta_s(x, t, s) = \psi_t(x, t), \\ \eta(0, t, s) = \eta(L, t, s) = 0, \\ \eta(x, t, 0) = 0. \end{cases} \tag{2.13}$$

Therefore, we conclude from (2.6), (2.7), and (2.13) and the equality

$$\eta_s(x, t, s) = \psi_t(x, t - s) \tag{2.14}$$

that the systems (1.1) and (2.4) are equivalent.

Thanks to (2.1), it is well-known that $\mathscr{H}$ endowed with the inner product, for $W = (w_1, w_2, w_3, w_4, w_5)^T$ and $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4, \tilde{w}_5)^T$,

$$\begin{aligned} \langle W, \tilde{W} \rangle_{\mathscr{H}} &= \int_0^L ((k_2 - g_0) w_{2x}(x) \tilde{w}_{2x}(x) + k_1 (w_{1x}(x) + w_2(x))(\tilde{w}_{1x}(x) + \tilde{w}_2(x)))\, dx \\ &\quad + \int_0^L (\rho_1 w_3(x) \tilde{w}_3(x) + \rho_2 w_4(x) \tilde{w}_4(x)) dx + \langle w_5, \tilde{w}_5 \rangle_{L_g^2(\mathbb{R}_+, H_0^1(]0, L[))} \end{aligned}$$

is a Hilbert space and $\mathscr{D}(\mathscr{A}) \subset \mathscr{H}$ with dense embedding, since, using contradiction arguments, this inner product generates on $\mathscr{H}$ a norm equivalent to the one of

$$\mathscr{H}_0 := (H^1(]0, L[))^2 \times (L^2(]0, L[))^2 \times L_g^2(\mathbb{R}_+, H^1(]0, L[));$$

that is, there exist two positive constants d_1 and d_2 satisfying, for all $W \in \mathscr{H}$,

$$d_1 \|W\|_{\mathscr{H}_0} \leq \|W\|_{\mathscr{H}} \leq d_2 \|W\|_{\mathscr{H}_0}. \tag{2.15}$$

The well-posedness of problem (2.4) is ensured by the following theorem:

Theorem 2.1. *Assume that* **(A1)–(A2)** *hold. Then, for any $n \in \mathbb{N}$ and $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A}^n)$, the system (2.4) has a unique solution satisfying*

$$\mathscr{U} \in \cap_{k=0}^{n} C^k(\mathbb{R}_+, \mathscr{D}(\mathscr{A}^{n-k})). \tag{2.16}$$

Proof. To prove Theorem 2.1, we use the semigroup approach. So, first, we show that the linear operator $\mathscr{A}$ is dissipative. Indeed, let $W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathscr{D}(\mathscr{A})$, then

$$\begin{aligned} &\langle \mathscr{A} W, W\rangle_{\mathscr{H}} \qquad (2.17)\\ &= \int_0^L ((k_2-g_0)w_{4x}(x)w_{2x}(x) + k_1(w_{3x}(x)+w_4(x))(w_{1x}(x)+w_2(x)))\,dx \\ &\quad + \int_0^L (k_1(w_{1x}(x)+w_2(x))_x w_3(x) - k_1(w_{1x}(x)+w_2(x))w_4(x))\,dx \\ &\quad + \int_0^L ((k_2-g_0)w_{2xx} - \|f\|_\infty w_4(x))\, w_4(x)dx \\ &\quad + \int_0^L \left(\int_0^{+\infty} g(s)w_{5xx}(x,s)ds - \int_0^{+\infty} f(s)w_{5s}(x,s)ds\right) w_4(x)dx \\ &\quad + \int_0^L \int_0^{+\infty} g(s)(-w_{5s}(x,s) - \epsilon_0 w_5(x,s) + w_4(x))_x w_{5x}(x,s)dsdx. \end{aligned}$$

It is clear that by integrating by parts with respect to s and using the fact that

$$\lim_{s\to+\infty} g(s)w_{5x}(x,s) = \lim_{s\to+\infty} f(s)w_5(x,s) = 0$$

(due to **(A1)**, (2.3) and (2.9)) and $w_5(x,0) = 0$ (definition of $\mathscr{D}(\mathscr{A})$), we find

$$-\int_0^L \int_0^{+\infty} g(s)w_{5x}(x,s)w_{5xs}(x,s)dsdx = \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)w_{5x}^2(x,s)dsdx. \qquad (2.18)$$

Note that, thanks to (2.2) and the fact that g is nonincreasing and $w_5 \in L_g^2(\mathbb{R}_+, H_0^1(]0,L[))$,

$$\begin{aligned} \left|\int_0^L \int_0^{+\infty} g'(s)w_{5x}^2(x,s)ds\right| &= -\int_0^L \int_0^{+\infty} g'(s)w_{5x}^2(x,s)ds \\ &\le \theta_0 \int_0^L \int_0^{+\infty} g(s)w_{5x}^2(x,s)ds \\ &< +\infty, \end{aligned}$$

so the integral in the right hand side of (2.18) is well defined. Moreover,

$$-\int_0^L w_4(x)\int_0^{+\infty} f(s)w_{5s}(x,s)dsdx = \int_0^L w_4(x)\int_0^{+\infty} f'(s)w_5(x,s)dsdx. \qquad (2.19)$$

Consequently, inserting (2.18) and (2.19) in (2.17) and integrating by parts with respect to x, we get

$$\begin{aligned} \langle \mathscr{A} W, W\rangle_{\mathscr{H}} &= \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)w_{5x}^2(x,s)dsdx \\ &\quad + \int_0^L w_4(x)\int_0^{+\infty} f'(s)w_5(x,s)dsdx \qquad (2.20)\\ &\quad - \|f\|_\infty \int_0^L w_4^2(x)dx - \epsilon_0 \int_0^L \int_0^{+\infty} g(s)w_{5x}^2(x,s)dsdx. \end{aligned}$$

Now, using Young's inequalities, (2.9) and the second inequality in (2.3) imply that (ϵ_0 and c_0 are defined, respectively, in (2.8) and (2.9)),

$$\int_0^L w_4(x)\int_0^{+\infty} f'(s)w_5(x,s)dsdx \qquad (2.21)$$

$$\leq \frac{\alpha}{2}\sqrt{\frac{c_0\|f\|_\infty}{\epsilon_0 g_0}}\int_0^L\int_0^{+\infty} g(s)w_4^2(x)dsdx + \frac{\alpha}{2}\sqrt{\frac{\epsilon_0 g_0}{c_0\|f\|_\infty}}\int_0^L\int_0^{+\infty} g(s)w_5^2(x,s)dsdx$$

$$\leq \frac{\alpha}{2}\sqrt{\frac{c_0 g_0\|f\|_\infty}{\epsilon_0}}\int_0^L w_4^2(x)dx + \frac{\alpha}{2}\sqrt{\frac{\epsilon_0 g_0 c_0}{\|f\|_\infty}}\int_0^L\int_0^{+\infty} g(s)w_{5x}^2(x,s)dsdx$$

$$\leq \|f\|_\infty\int_0^L w_4^2(x)dx + \epsilon_0\int_0^L\int_0^{+\infty} g(s)w_{5x}^2(x,s)dsdx.$$

Finally, combining (2.20) and (2.21), and using the fact that g is nonincreasing, we obtain

$$\langle \mathscr{A}W, W\rangle_{\mathscr{H}} \leq \frac{1}{2}\int_0^L\int_0^{+\infty} g'(s)w_{5x}^2(x,s)dsdx \leq 0, \tag{2.22}$$

which means that $\mathscr{A}$ is dissipative.

Next, we shall prove that $Id - \mathscr{A}$ is surjective. Indeed, let $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathscr{H}$, we show that there exists $W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathscr{D}(\mathscr{A})$ satisfying

$$(Id - \mathscr{A})W = F, \tag{2.23}$$

which is equivalent to

$$\begin{cases} w_3 = w_1 - f_1, \\ w_4 = w_2 - f_2, \\ \rho_1 w_1 - k_1(w_{1x} + w_2)_x = \rho_1(f_1 + f_3), \\ (\rho_2 + \|f\|_\infty)w_2 - (k_2 - g_0)w_{2xx} + k_1(w_{1x} + w_2) - \displaystyle\int_0^{+\infty} g(s)w_{5xx}(s)ds \\ \qquad + \displaystyle\int_0^{+\infty} f(s)w_{5s}(s)ds = (\rho_2 + \|f\|_\infty)f_2 + \rho_2 f_4, \\ w_{5s} + (1+\epsilon_0)w_5 = w_2 + f_5 - f_2. \end{cases} \tag{2.24}$$

We note that the last equation in (2.24) with $w_5(0) = 0$ has the unique solution

$$\begin{aligned} w_5(s) &= e^{-(1+\epsilon_0)s}\int_0^s e^{(1+\epsilon_0)y}(w_2 + f_5(y) - f_2)dy \\ &= (1+\epsilon_0)^{-1}\left(1 - e^{-(1+\epsilon_0)s}\right)w_2 + e^{-(1+\epsilon_0)s}\int_0^s e^{(1+\epsilon_0)y}(f_5(y) - f_2)dy. \end{aligned} \tag{2.25}$$

Next, plugging (2.25) into the fourth equation in (2.24), we get

$$\begin{cases} \rho_1 w_1 - k_1(w_{1x} + w_2)_x = \rho_1(f_1 + f_3), \\ l_2 w_2 - l_1 w_{2xx} + k_1(w_{1x} + w_2) = \tilde{f}, \end{cases} \tag{2.26}$$

where

$$l_1 = k_2 - g_0 + (1+\epsilon_0)^{-1}g_0 - (1+\epsilon_0)^{-1}\int_0^{+\infty} g(s)e^{-(1+\epsilon_0)s}ds,$$

$$l_2 = \rho_2 + \|f\|_\infty + \int_0^{+\infty} f(s)e^{-(1+\epsilon_0)s}ds$$

and

$$\begin{aligned} \tilde{f} = (\rho_2 + \|f\|_\infty)f_2 + \rho_2 f_4 + (1+\epsilon_0)\int_0^{+\infty} f(s)e^{-(1+\epsilon_0)s}\left(\int_0^s e^{(1+\epsilon_0)y}(f_5(y) - f_2)dy\right)ds \\ - \int_0^{+\infty} f(s)(f_5(s) - f_2)ds + \int_0^{+\infty} g(s)e^{-(1+\epsilon_0)s}\left(\int_0^s e^{(1+\epsilon_0)y}(f_5(y) - f_2)_{xx}dy\right)ds. \end{aligned}$$

It remains only to prove that (2.26) has a solution $(w_1, w_2) \in \left(H_0^1(]0, L[)\right)^2$. Then, substituting in (2.25) and the first two equations in (2.24), we obtain $W \in \mathscr{D}(\mathscr{A})$ satisfying (2.23). Since $l_1 \geq k_2 - g_0 > 0$ (according to (2.1)) and

$$l_2 \geq \rho_2 + \|f\|_\infty - \|f\|_\infty \frac{1}{1+\epsilon_0} \geq \rho_2 > 0,$$

we see that the operator

$$\mathscr{K}\begin{pmatrix} w_1 \\ w_2 \end{pmatrix} = \begin{pmatrix} \rho_1 w_1 - k_1(w_{1x} + w_2)_x \\ l_2 w_2 - l_1 w_{2xx} + k_1(w_{1x} + w_2) \end{pmatrix},$$

is self-adjoint linear positive definite. Considering the variational formulation of (2.26), and applying the Lax-Milgram theorem and classical regularity arguments, we conclude that (2.26) has a unique solution $(w_1, w_2) \in \left(H_0^1(]0, L[)\right)^2$ satisfying the third and fourth equations of (2.24), since (2.25). Therefore, using (2.11),

$$(k_2 - g_0)w_{2xx} + \int_0^{+\infty} g(s) w_{5xx}(s) ds \in L^2(]0, L[).$$

This proves that $Id - \mathscr{A}$ is surjective. Finally, we note that (2.22) and (2.23) mean that $-\mathscr{A}$ is a maximal monotone operator. Hence, using Lummer-Phillips theorem (see Ref. 34), we deduce that $\mathscr{A}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$.

On the other hand, as the linear operator $\mathscr{B}$ (defined in (2.6)) is Lipschitz continuous, it follows that $\mathscr{A} + \mathscr{B}$ also is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$ (see Ref. 34: Chap. 3 – Theorem 1.1). Consequently, (2.4) is well-posed in the sense of Theorem 2.1 (see Refs. 22 and 34). ■

III. ASYMPTOTIC STABILITY

In this section, we investigate the asymptotic behavior of the solution of (2.4) by the use of the energy method. We produce suitable Lyapunov functionals and prove some decay estimates depending on the asymptotic behavior of g, the connection between g and f, and the regularity of initial data.

A. Additional assumptions and stability results

Our asymptotic stability results hold under the following additional assumptions:

(A3) The function g satisfies

$$g_0 := \int_0^{+\infty} g(s) ds > 0 \tag{3.1}$$

and one of the following two conditions holds:

$$\exists \theta_1 > 0, \ \ g'(s) \leq -\theta_1 g(s), \quad \forall s \in \mathbb{R}_+ \tag{3.2}$$

or there exists a positive nonincreasing function $\xi : \mathbb{R}_+ \to \mathbb{R}_+^*$ of class $C(\mathbb{R}_+, \mathbb{R}_+^*)$ such that

$$\begin{cases} g(t-s) \geq \xi(t) \displaystyle\int_t^{+\infty} g(\tau - s) d\tau, \ \forall t \in \mathbb{R}_+, \ \ \forall s \in [0, t], \\ g'(s) < 0, \qquad\qquad\qquad\qquad\qquad \forall s \in \mathbb{R}_+. \end{cases} \tag{3.3}$$

(A4) There exists a positive even function $\gamma : \mathbb{R} \to \mathbb{R}_+^*$ of class $C(\mathbb{R}, \mathbb{R}_+^*)$ and nonincreasing on $\mathbb{R}_+$, and a positive function $\beta : \mathbb{R}_+ \to \mathbb{R}_+^*$ of class $C(\mathbb{R}_+, \mathbb{R}_+^*)$ such that

$$\beta_0 := \int_0^{+\infty} \beta(s) ds < +\infty \tag{3.4}$$

and

$$|f(s)| \leq e^{-\tilde{\gamma}(s)}\beta(s)g(s), \quad \forall s \in \mathbb{R}_+, \tag{3.5}$$

where

$$\tilde{\gamma}(s) = 2\int_0^{\frac{s}{2}} \gamma(\tau)d\tau, \quad \forall s \in \mathbb{R}_+. \tag{3.6}$$

Moreover, when (1.2) does not hold, we assume also that *f* is of class $C^3(\mathbb{R}_+, \mathbb{R})$ and satisfies, for some positive constant $\tilde{\alpha}$,

$$|f''(s)| \leq \tilde{\alpha}g(s) \quad \text{and} \quad |f'''(s)| \leq \tilde{\alpha}g(s), \quad \forall s \in \mathbb{R}_+. \tag{3.7}$$

Remark 3.1. The condition (2.2) implies that the decay rate of g is at most of exponential type. The conditions (3.2) and (3.3) include, respectively, the class of functions g which converge to zero at least exponentially or less than exponentially. When

$$\lim_{t\to+\infty} \xi(t) > 0,$$

the first inequality in (3.3), introduced in Refs. 39 and 42, implies that g converges to zero at least exponentially but it does not involve the derivative of g. We distinguish the cases (3.2) and (3.3) because they lead to different kinds of decays.

Theorem 3.2. *Assume that* (A1)–(A4) *hold. Then there exists a positive constant δ_0 independent of f such that, if*

$$\int_0^{+\infty} |f(s)|ds < \delta_0, \tag{3.8}$$

then, we have the following stability results:

(i) Equal speed propagation and exponential decay of g**:** *if (1.2) and (3.2) hold, then, for any $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, there exist positive constants δ_1 and δ_2 such that*

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \delta_2 e^{-\delta_1\phi(t)}, \quad \forall t \in \mathbb{R}_+, \tag{3.9}$$

where

$$\phi(t) = \int_0^t \min\{1, \gamma(s)\}ds. \tag{3.10}$$

(ii) Nonequal speed propagation and exponential decay of g**:** *if (1.2) does not hold and (3.2) holds, then, for any $n = 2, 3, \cdots$ and $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A}^n)$, there exists a positive constant δ_1 such that*

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \frac{\delta_1}{(1+t)^{n-1}}, \quad \forall t \in \mathbb{R}_+. \tag{3.11}$$

(iii) Equal speed propagation and arbitrary decay of g**:** *if (3.2) does not hold, and (1.2) and (3.3) hold, then, for any $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, there exist positive constants δ_1 and δ_2 such that*

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \delta_2 e^{-\delta_1\phi(t)}\left(1 + \int_0^L \int_0^t e^{\delta_1\phi(s)} \int_s^{+\infty} g(\tau)\psi_{0x}^2(x, \tau - s)d\tau ds dx\right), \forall t \in \mathbb{R}_+, \tag{3.12}$$

where

$$\phi(t) = \int_0^t \min\{1, \gamma(s), \xi(s)\}ds. \tag{3.13}$$

B. Examples and comments

Let us illustrate our decay estimates (3.9), (3.11), and (3.12) by the following simple examples (some of them were given in Ref. 20 for (1.5)):

1. Equal speed propagation and exponential decay of g: (1.2) and (3.2) hold

Let us consider the class $g(s) = \alpha_2 e^{-\alpha_1 s}$, with $\alpha_1, \alpha_2 > 0$. Then **(A1)** and (3.1) hold provided that α_2 is small enough so that (2.1) holds. This class satisfies (3.2) with $\theta_1 = \alpha_1$.

3.2.1.1. If

$$|f(s)| \leq \beta_2 e^{-\beta_1 (s+1)^p} g(s), \quad \forall s \in \mathbb{R}_+, \tag{3.14}$$

for some constants β_1, β_2, $p > 0$ with β_2 small enough so that (3.8) holds, then (3.5) is satisfied with $\beta(s) = \beta_2 e^{-\beta_0 (s+1)^p}$,

$$\gamma(s) = q(\beta_1 - \beta_0)(2|s| + 1)^{q-1}, \tag{3.15}$$

any $\beta_0 \in]0, \beta_1[$ and $q = \min\{p, 1\}$ (so γ is positive on $\mathbb{R}$ and nonincreasing on $\mathbb{R}_+$), and therefore, (3.9) gives, for some positive constants c' and c'',

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq c'' e^{-c'(t+1)^q}, \quad \forall t \in \mathbb{R}_+.$$

3.2.1.2. If

$$|f(s)| \leq \beta_2 e^{-\beta_1 (\ln(s+1))^p} g(s), \quad \forall s \in \mathbb{R}_+, \tag{3.16}$$

for some constants $\beta_1, \beta_2 > 0$, and $p > 1$ with β_2 small enough so that (3.8) holds, then (3.5) holds with $\beta(s) = \beta_3 e^{-\beta_0 (\ln(s+1))^p}$,

$$\gamma(s) = \begin{cases} p(\beta_1 - \beta_0)\dfrac{(\ln(2|s|+1))^{p-1}}{2|s|+1} & \text{if } |s| \geq \dfrac{1}{2}(e^{p-1} - 1) := s_0, \\ p(\beta_1 - \beta_0)(p-1)^{p-1} e^{1-p} := \tilde{c} & \text{if } |s| \in [0, s_0], \end{cases} \tag{3.17}$$

$\beta_3 = \beta_2 e^{2\tilde{c} s_0}$ and any $\beta_0 \in]0, \beta_1[$ (so γ is positive and continuous on $\mathbb{R}$, and nonincreasing on $\mathbb{R}_+$), and therefore, (3.9) gives, for some positive constants c' and c'',

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq c'' e^{-c'(\ln(t+1))^p}, \quad \forall t \in \mathbb{R}_+.$$

3.2.1.3. If

$$|f(s)| \leq \frac{\beta_1}{(s+1)^p} g(s), \quad \forall s \in \mathbb{R}_+, \tag{3.18}$$

for some constants $\beta_1 > 0$ and $p > 1$ with β_1 small enough so that (3.8) holds, then (3.5) holds with $\beta(s) = \frac{\beta_1}{(s+1)^{p-\beta_0}}$,

$$\gamma(s) = \frac{\beta_0}{2|s| + 1} \tag{3.19}$$

and any $\beta_0 \in]0, p - 1[$ (so β is integrable on $\mathbb{R}_+$), and therefore, (3.9) gives, for some positive constants c' and c'',

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq c''(t+1)^{-c'}, \quad \forall t \in \mathbb{R}_+.$$

2. Nonequal speed propagation and exponential decay of g: (1.2) does not hold and (3.2) holds

The estimate (3.11) gives a decay rate of polynomial type for the solution of (2.4), where the decay rate depends on the regularity of initial data $\mathscr{U}_0$.

3. Equal speed propagation and arbitrary decay of g: (3.2) does not hold, and (1.2) and (3.3) hold

Let us consider the classes (3.14), (3.16), and (3.18) of f, and the following three classes of g which satisfy (3.3) and do not satisfy (3.2).

3.2.3.1. If

$$g(s) = \alpha_2 (s+1)^{r-1} e^{-\alpha_1 (s+1)^r}, \tag{3.20}$$

for some constants $\alpha_1, \alpha_2 > 0$, and $r \in]0, 1[$. Then **(A1)** and (3.1) hold provided that α_2 is small enough so that (2.1) holds. On the other hand, (3.3) holds with $\xi(s) = \alpha_1 (s+1)^{r-1}$, and therefore, (3.12) holds with

$$\phi(t) = \begin{cases} \int_0^t \xi(s)ds & \text{in case (3.14) with } r \leq p, \\ \int_0^t \gamma(s)ds & \text{in case (3.14) with } r > p, \text{ and in cases (3.16) and (3.18).} \end{cases}$$

If, for example, for some positive constants λ_0 and M_0,

$$\int_0^L \psi_{0x}^2(x,s)dx \leq M_0 e^{(\ln(s+1))^{\lambda_0}}, \quad \forall s \in \mathbb{R}_+, \tag{3.21}$$

then (3.12) implies that, for some positive constants c' and c'', and for all $t \in \mathbb{R}_+$,

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \begin{cases} c'' e^{-c'(t+1)^{\min\{p,r\}}} & \text{in case (3.14),} \\ c'' e^{-c'(\ln(t+1))^p} & \text{in case (3.16),} \\ c''(t+1)^{-c'} & \text{in case (3.18).} \end{cases} \tag{3.22}$$

3.2.3.2. If

$$g(s) = \frac{\alpha_2}{s+e^{r-1}} (\ln(s+e^{r-1}))^{r-1} e^{-\alpha_1 (\ln(s+e^{r-1}))^r}, \tag{3.23}$$

for some constants $\alpha_1, \alpha_2 > 0$, and $r > 1$. Then **(A1)** and (3.1) hold provided that α_2 is small enough so that (2.1) holds. On the other hand, (3.3) holds with $\xi(s) = r\alpha_1 (s+e^{r-1})^{-1} (\ln(s+e^{r-1}))^{r-1}$, and therefore, (3.12) holds with

$$\phi(t) = \begin{cases} \int_0^t \xi(s)ds & \text{in case (3.14), and in case (3.16) with } r \leq p, \\ \int_0^t \gamma(s)ds & \text{in case (3.18), and in case (3.16) with } r > p. \end{cases}$$

If, for example, for some positive constants λ_0 and M_0,

$$\int_0^L \psi_{0x}^2(x,s)dx \leq M_0 (s+1)^{\lambda_0}, \quad \forall s \in \mathbb{R}_+, \tag{3.24}$$

then (3.12) implies that, for some positive constants c' and c'', and for all $t \in \mathbb{R}_+$,

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \begin{cases} c'' e^{-c'(\ln(t+1))^r} & \text{in case (3.14),} \\ c'' e^{-c'(\ln(t+1))^{\min\{p,r\}}} & \text{in case (3.16),} \\ c''(t+1)^{-c'} & \text{in case (3.18).} \end{cases} \tag{3.25}$$

3.2.3.3. If

$$g(s) = \alpha_1 (s+1)^{-r}, \tag{3.26}$$

for some constants $\alpha_1 > 0$ and $r > 1$. Then **(A1)** and (3.1) hold provided that α_1 is small enough so that (2.1) holds. On the other hand, (3.3) is satisfied with $\xi(s) = (r-1)(s+1)^{-1}$, and therefore, (3.12) holds with $\phi(t) = \int_0^t \xi(s)ds$.

If, for example, for some positive constants λ_0 and M_0,

$$\int_0^L \psi_{0x}^2(x,s)dx \le M_0(\ln(s+2))^{\lambda_0}, \quad \forall s \in \mathbb{R}_+, \quad \forall s \in \mathbb{R}_+, \tag{3.27}$$

then (3.12) implies that, for some positive constants c' and c'', and for all $t \in \mathbb{R}_+$,

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \le \begin{cases} c''(t+1)^{-c'} & \text{if } r > 2, \\ c''(t+1)^{c'} & \text{if } r \le 2. \end{cases} \tag{3.28}$$

The estimate (3.28) in case $1 < r \le 2$ does not imply the strong stability of (2.4),

$$\lim_{t\to+\infty} \|\mathscr{U}(t)\|_{\mathscr{H}}^2 = 0. \tag{3.29}$$

The $\lim_{t\to+\infty} \|\mathscr{U}(t)\|_{\mathscr{H}}^2$ in (3.12) depends on the connection between the growths at infinity of g, f, and $\int_0^L \psi_{0x}^2(x,\cdot)dx$.

C. Proof of Theorem 3.2

We start our proof by giving the modified energy functional E associated with any weak solution of (2.4) (corresponding to initial data $\mathscr{U}_0 \in \mathscr{H}$),

$$E(t) := \frac{1}{2}\|\mathscr{U}(t)\|_{\mathscr{H}}^2. \tag{3.30}$$

Let $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A}^{n_0})$, where $n_0 = 1$ if (1.2) holds, and $n_0 = 2$ if (1.2) does not hold, so that all the calculations below are justified. From (2.4), (2.6), (2.14), (2.19), and (2.20) we get

$$E'(t) = \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx - \int_0^L \psi_t(x,t)\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx. \tag{3.31}$$

Note that, in contrast to the situation of absence of delay and/or presence of frictional damping considered in the literature (as in (1.3)), we are unable to determine the sign of E' from (3.31), and therefore, the system (2.4) is not necessarily dissipative with respect to E at this stage.

On the other hand, using Cauchy-Schwarz inequality, the following classical inequalities hold, for all $v \in C(\mathbb{R}_+, \mathbb{R}_+)$ (see, for example, Refs. 16 and 26),

$$\left(\int_0^{+\infty} g(s)v(s)ds\right)^2 \le g_0 \int_0^{+\infty} g(s)v^2(s)ds \tag{3.32}$$

and

$$\left(\int_0^{+\infty} g'(s)v(s)ds\right)^2 \le -g(0)\int_0^{+\infty} g'(s)v^2(s)ds. \tag{3.33}$$

Inequalities (3.32) and (3.33) will be repeatedly used in the proof. Also, we will denote by c_δ a positive constant depending on some parameter δ.

In order to prove (3.9), (3.11), and (3.12) we prove briefly several lemmas. Lemmas 3.3–3.9 and 3.11 are known in case $f \equiv 0$ (see, for example, Refs. 2, 8, 17, 19, 25, and 26), while the ones 3.10 and 3.12–3.14 are introduced in the present paper to cope with the new situation due to the distributed time delay and the nondissipativeness character of (1.1).

Lemma 3.3. The functional

$$I_1(t) := -\rho_2 \int_0^L \psi_t(x,t)\int_0^{+\infty} g(s)\eta(x,t,s)dsdx,$$

satisfies, for all $\delta > 0$,

$$I_1'(t) \leq \int_0^L \left(-\rho_2(g_0-\delta)\psi_t^2(x,t) + \delta\left(\psi_x^2(x,t) + (\varphi_x(x,t)+\psi(x,t))^2\right)\right)dx$$
$$+ c_\delta \int_0^L \int_0^{+\infty} \left(g(s)\eta_x^2(x,t,s) - g'(s)\eta_x^2(x,t,s)\right)dsdx \quad (3.34)$$
$$+ \int_0^L \left(\int_0^{+\infty} f(s)\psi_t(x,t-s)ds\right)\left(\int_0^{+\infty} g(s)\eta(x,t,s)ds\right)dx.$$

Proof. The proof is identical to the one given in Refs. 25 and 26 in case $f \equiv 0$. Indeed, by differentiating I_1, using the second equation in (1.1) and the first one in (2.13), and integrating by parts, we get

$$I_1'(t) = -\rho_2 g_0 \int_0^L \psi_t^2(x,t)dx + \int_0^L \left(\int_0^{+\infty} f(s)\psi_t(x,t-s)ds\right)\left(\int_0^{+\infty} g(s)\eta(x,t,s)ds\right)dx$$
$$+ \int_0^L \left(\int_0^{+\infty} g(s)\eta_x(x,t,s)ds\right)^2 dx - g_0 \int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\eta_x(x,t,s)dsdx$$
$$- \rho_2 \int_0^L \psi_t(x,t)\int_0^{+\infty} g'(s)\eta(x,t,s)dsdx + k_2 \int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\eta_x(x,t,s)dsdx$$
$$+ k_1 \int_0^L (\varphi_x(x,t)+\psi(x,t))\int_0^{+\infty} g(s)\eta(x,t,s)dsdx.$$

Applying Cauchy-Schwarz inequality, (2.9) and Young's inequality

$$ab \leq \frac{d}{2}a^2 + \frac{1}{2d}b^2, \quad \forall a,\ b \in \mathbb{R},\ \forall d > 0,$$

to the last four terms of the above equality, and using (3.33), we get

$$I_1'(t) \leq \int_0^L \left(-\rho_2(g_0-\delta)\psi_t^2(x,t) + \frac{\delta}{2}\psi_x^2(x,t) + \delta(\varphi_x(x,t)+\psi(x,t))^2\right)dx$$
$$+ \int_0^L \left(\int_0^{+\infty} f(s)\psi_t(x,t-s)ds\right)\left(\int_0^{+\infty} g(s)\eta(x,t,s)ds\right)dx$$
$$- c_\delta \int_0^L \int_0^{+\infty} g'(t)\eta_x^2(x,t,s)dsdx + c_\delta \int_0^L \left(\int_0^{+\infty} g(t)\eta_x(x,t,s)ds\right)^2 dx \quad (3.35)$$
$$- g_0 \int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\eta_x(x,t,s)dsdx.$$

Again, applying Cauchy-Schwarz and Young's inequalities to the last term in (3.35), and using (3.32), we find (3.34).

In case (3.3), we will consider another manipulations for the last two terms in (3.35). ■

Lemma 3.4. The functional

$$I_2(t) := -\int_0^L (\rho_1\varphi(x,t)\varphi_t(x,t) + \rho_2\psi(x,t)\psi_t(x,t))\,dx$$

satisfies, for some $c_1 > 0$ (*not depending on* f),

$$I_2'(t) \leq -\int_0^L \left(\rho_1\varphi_t^2(x,t)+\rho_2\psi_t^2(x,t)\right)dx + k_1\int_0^L (\varphi_x(x,t)+\psi(x,t))^2 dx \\ + c_1\int_0^L \psi_x^2(x,t)dx + c_1\int_0^L\int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx \\ + \int_0^L \psi(x,t)\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx. \tag{3.36}$$

Proof. Similar to the proof of Refs. 2, 8, 25, and 26 in case $f \equiv 0$, by differentiating I_2, using the first two equations in (1.1) and integrating by parts, we find

$$I_2'(t) = -\int_0^L (\rho_1\varphi_t^2(x,t)+\rho_2\psi_t^2(x,t))dx + k_1\int_0^L (\varphi_x(x,t)+\psi(x,t))^2 dx \\ + (k_2-g_0)\int_0^L \psi_x^2(x,t)dx + \int_0^L \psi(x,t)\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx \\ + \int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\eta_x(x,t,s)dsdx. \tag{3.37}$$

The use of Young's inequality and (3.32) for the last term in (3.37) leads to (3.36). ■

Lemma 3.5. The functional

$$I_3(t) := \rho_2\int_0^L \psi_t(x,t)\,(\varphi_x(x,t)+\psi(x,t))\,dx + \frac{k_2\rho_1}{k_1}\int_0^L \psi_x(x,t)\varphi_t(x,t)dx \\ - \frac{\rho_1}{k_1}\int_0^L \varphi_t(x,t)\int_0^{+\infty} g(s)\psi_x(x,t-s)dsdx$$

satisfies, for all $\varepsilon > 0$,

$$I_3'(t) \leq \rho_2\int_0^L \psi_t^2(x,t)dx + \frac{1}{2\varepsilon}\left(k_2\psi_x(L,t) - \int_0^{+\infty} g(s)\psi_x(L,t-s)ds\right)^2 \\ + \frac{1}{2\varepsilon}\left(k_2\psi_x(0,t) - \int_0^{+\infty} g(s)\psi_x(0,t-s)ds\right)^2 \\ + \frac{\varepsilon}{2}\left(\varphi_x^2(L,t)+\varphi_x^2(0,t)\right) - k_1\int_0^L (\varphi_x(x,t)+\psi(x,t))^2 dx \\ - c_\varepsilon\int_0^L\int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx + \varepsilon\int_0^L \varphi_t^2(x,t)dx \\ - \int_0^L (\varphi_x(x,t)+\psi(x,t))\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx \\ + \left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx. \tag{3.38}$$

Proof. As in Ref. 2 in case $f \equiv 0$, a simple differentiation of I_3, using the first two equations in (1.1) and the fact that

$$\psi_{xt}(x,t-s) = \eta_{xs}(x,t,s), \tag{3.39}$$

and integration by parts give

$$
\begin{aligned}
I_3'(t) &= -k_1 \int_0^L (\varphi_x(x,t)+\psi(x,t))^2 dx - \frac{\rho_1}{k_1}\int_0^L \varphi_t(x,t)\int_0^{+\infty} g(s)\psi_{xt}(x,t-s)dsdx \\
&\quad + \left(\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx + k_2\int_0^L \psi_x(x,t)(\varphi_x(x,t)+\psi(x,t))_x dx \\
&\quad + \int_0^L \left(k_2\psi_{xx}(x,t) - \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds\right)(\varphi_x(x,t)+\psi(x,t))dx \\
&\quad - \int_0^L \left(\int_0^{+\infty} f(s)\psi_t(x,t-s)ds\right)(\varphi_x(x,t)+\psi(x,t))dx \\
&\quad + \rho_2\int_0^L \psi_t^2(x,t)dx - \int_0^L (\varphi_x(x,t)+\psi(x,t))_x \int_0^{+\infty} g(s)\psi_x(x,t-s)dsdx \\
&= -k_1\int_0^L (\varphi_x(x,t)+\psi(x,t))^2dx + \rho_2\int_0^L \psi_t^2(x,t)dx + \left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx \\
&\quad - \int_0^L (\varphi_x(x,t)+\psi(x,t))\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx \\
&\quad + \left(k_2\psi_x(L,t) - \int_0^{+\infty} g(s)\psi_x(L,t-s)ds\right)\varphi_x(L,t) \\
&\quad - \left(k_2\psi_x(0,t) - \int_0^{+\infty} g(s)\psi_x(0,t-s)ds\right)\varphi_x(0,t) + \frac{\rho_1}{k_1}\int_0^L \varphi_t(x,t)\int_0^{+\infty} g'(s)\eta_x(x,t,s)dsdx.
\end{aligned}
$$

By using (3.33) and Young's inequality for the last three terms in this equality, we get (3.38). ∎

To handle the boundary terms in (3.38), we proceed as in Ref. 2.

Lemma 3.6. Let $m(x) := 2 - \frac{4}{L}x$. *The functionals*

$$
J_1(t) := \rho_2 \int_0^L m(x)\psi_t(x,t)\left(k_2\psi_x(x,t) - \int_0^{+\infty} g(s)\psi_x(x,t-s)ds\right)dx
$$

and

$$
J_2(t) := \rho_1 \int_0^L m(x)\varphi_t(x,t)\varphi_x(x,t)dx
$$

satisfy, for all $\varepsilon > 0$ *and for some* $c_2, c_3 > 0$ *(not depending neither on f nor on* ε*),*

$$
\begin{aligned}
J_1'(t) &\le c_2\left(1+\frac{1}{\varepsilon}\right)\int_0^L \psi_x^2(x,t)dx - \left(k_2\psi_x(L,t) - \int_0^{+\infty} g(s)\psi_x(L,t-s)ds\right)^2 \\
&\quad - \left(k_2\psi_x(0,t) - \int_0^{+\infty} g(s)\psi_x(0,t-s)ds\right)^2 + \varepsilon k_1\int_0^L (\varphi_x(x,t)+\psi(x,t))^2dx \\
&\quad + c_\varepsilon \int_0^L\int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx - c_2\int_0^L\int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx \qquad (3.40) \\
&\quad + c_2\int_0^L \psi_t^2(x,t)dx - k_2\int_0^L m(x)\psi_x(x,t)\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx \\
&\quad + \int_0^L m(x)\left(\int_0^{+\infty} g(s)\psi_x(x,t-s)ds\right)\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx
\end{aligned}
$$

and

$$J_2'(t) \le -k_1 \left(\varphi_x^2(L,t) + \varphi_x^2(0,t)\right) + c_3 \int_0^L \left(\varphi_t^2(x,t) + (\varphi_x(x,t) + \psi(x,t))^2 + \psi_x^2(x,t)\right) dx. \tag{3.41}$$

Proof. As in Ref. 2 in case $f \equiv 0$, differentiating J_1, using the second equation in (1.1) and (3.39), and integrating by parts, we find

$$\begin{aligned}
J_1'(t) &= \int_0^L m(x)\left(k_2\psi_{xx}(x,t) - \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds\right)\left(k_2\psi_x(x,t) - \int_0^{+\infty} g(s)\psi_x(x,t-s)ds\right)dx \\
&\quad - k_1 \int_0^L m(x)(\varphi_x(x,t) + \psi(x,t))\left(k_2\psi_x(x,t) - \int_0^{+\infty} g(s)\psi_x(x,t-s)ds\right)dx \\
&\quad - \int_0^L m(x)\left(\int_0^{+\infty} f(s)\psi_t(x,t-s)ds\right)\left(k_2\psi_x(x,t) - \int_0^{+\infty} g(s)\psi_x(x,t-s)ds\right)dx \\
&\quad + \rho_2 \int_0^L m(x)\psi_t(x,t)\left(k_2\psi_{xt}(x,t) - \int_0^{+\infty} g(s)\psi_{xt}(x,t-s)ds\right)dx \\
&= -\left(k_2\psi_x(L,t) - \int_0^{+\infty} g(s)\psi_x(L,t-s)ds\right)^2 - \left(k_2\psi_x(0,t) - \int_0^{+\infty} g(s)\psi_x(0,t-s)ds\right)^2 \\
&\quad - \int_0^L m(x)\left(\int_0^{+\infty} f(s)\psi_t(x,t-s)ds\right)\left(k_2\psi_x(x,t) - \int_0^{+\infty} g(s)\psi_x(x,t-s)ds\right)dx \\
&\quad + \frac{2k_2\rho_2}{L}\int_0^L \psi_t^2(x,t)dx + \rho_2 \int_0^L m(x)\psi_t(x,t)\int_0^{\infty} g'(s)\eta_x(x,t,s)dsdx \\
&\quad + \frac{2}{L}\int_0^L \left((k_2 - g_0)\psi_x(x,t) + \int_0^{+\infty} g(s)\eta_x(x,t,s)ds\right)^2 dx \\
&\quad - k_1 \int_0^L m(x)(\varphi_x(x,t) + \psi(x,t))\left((k_2 - g_0)\psi_x(x,t) + \int_0^{+\infty} g(s)\eta_x(x,t,s)ds\right)dx.
\end{aligned}$$

Using Young's inequality and the fact that $|m(x)| \le 2$ on $]0, L[$, we get

$$\begin{aligned}
&-k_1 m(x)(\varphi_x(x,t) + \psi(x,t))\left((k_2 - g_0)\psi_x(x,t) + \int_0^{+\infty} g(s)\eta_x(x,t,s)ds\right) \\
&\le \varepsilon k_1(\varphi_x(x,t) + \psi(x,t))^2 + \frac{k_1}{\varepsilon}\left((k_2 - g_0)\psi_x(x,t) + \int_0^{+\infty} g(s)\eta_x(x,t,s)ds\right)^2.
\end{aligned}$$

Developing the last term in this inequality, inserting it in the previous equality and using Young's inequality and (3.33), we find

$$\begin{aligned}
J_1'(t) &\le c_2\left(1 + \frac{1}{\varepsilon}\right)\int_0^L \psi_x^2(x,t)dx - \left(k_2\psi_x(L,t) - \int_0^{+\infty} g(s)\psi_x(L,t-s)ds\right)^2 \\
&\quad - \left(k_2\psi_x(0,t) - \int_0^{+\infty} g(s)\psi_x(0,t-s)ds\right)^2 + c_2\int_0^L \psi_t^2(x,t)dx \\
&\quad + \varepsilon k_1 \int_0^L (\varphi_x(x,t) + \psi(x,t))^2 dx - c_2 \int_0^L \int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx
\end{aligned}$$

$$\begin{aligned}&+c_2\left(1+\frac{1}{\varepsilon}\right)\int_0^L\left(\int_0^{+\infty}g(s)\eta_x(x,t,s)ds\right)^2dx\\&+\int_0^L m(x)\left(\int_0^{+\infty}g(s)\psi_x(x,t-s)ds\right)\int_0^{+\infty}f(s)\psi_t(x,t-s)dsdx\\&-k_2\int_0^L m(x)\psi_x(x,t)\int_0^{+\infty}f(s)\psi_t(x,t-s)dsdx\\&-2g_0\left(\frac{k_1}{\varepsilon}+\frac{2}{L}\right)\int_0^L\psi_x(x,t)\int_0^{+\infty}g(s)\eta_x(x,t,s)dsdx.\end{aligned}\tag{3.42}$$

Applying again Young's inequality to the last term in (3.42), and using (3.32), we get (3.40).

On the other hand, using Poincaré's inequality (2.9) for ψ, we get

$$\int_0^L\varphi_x^2(x,t)dx\le 2\int_0^L(\varphi_x(x,t)+\psi(x,t))^2dx+2c_0\int_0^L\psi_x^2(x,t)dx.\tag{3.43}$$

Then, similarly, differentiating J_2, using the first equation in (1.1), integrating by parts and using Young's inequality and (3.43), we obtain (3.41). ■

Lemma 3.7. For any $\epsilon\in\,]0,1[$, *the functional*

$$I_4(t):=I_3(t)+\frac{1}{2\varepsilon}J_1(t)+\frac{\varepsilon}{2k_1}J_2(t)$$

satisfies, for some $c_4>0$ (*not depending neither on f nor on* ε),

$$\begin{aligned}I_4'(t)\le&-\left(\frac{k_1}{2}-\varepsilon c_4\right)\int_0^L(\varphi_x(x,t)+\psi(x,t))^2dx+\varepsilon c_4\int_0^L\varphi_t^2(x,t)dx\\&+c_\varepsilon\int_0^L\int_0^{+\infty}\left(g(s)\eta_x^2(x,t,s)-g'(s)\eta_x^2(x,t,s)\right)dsdx\\&+\frac{c_4}{\varepsilon^2}\int_0^L\psi_x^2(x,t)dx+\int_0^L J_3(x,t)\int_0^{+\infty}f(s)\psi_t(x,t-s)dsdx\\&+\frac{c_4}{\varepsilon}\int_0^L\psi_t^2(x,t)dx+\left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L\varphi_t(x,t)\psi_{xt}(x,t)dx,\end{aligned}\tag{3.44}$$

where

$$J_3(x,t):=-\varphi_x(x,t)-\psi(x,t)-\frac{1}{2\varepsilon}m(x)\left(k_2\psi_x(x,t)-\int_0^{+\infty}g(s)\psi_xx,t-sds\right).\tag{3.45}$$

Proof. Combining (3.38) and (3.40) and (3.41), we obtain (3.44).

Lemma 3.8. The functional

$$I_5(t):=\frac{1}{8}I_2(t)+I_4(t)$$

satisfies, for some $c_5 > 0$ (*not depending on* f),

$$\begin{aligned} I_5'(t) \le &-\frac{k_1}{4}\int_0^L (\varphi_x(x,t)+\psi(x,t))^2 dx + c_5\int_0^L (\psi_t^2(x,t)+\psi_x^2(x,t))dx \\ &+ c_5\int_0^L\int_0^\infty \left(g(s)\eta_x^2(x,t,s) - g'(s)\eta_x^2(x,t,s)\right) ds dx \\ &- \frac{\rho_1}{16}\int_0^L \varphi_t^2(x,t)dx + \int_0^L J_4(x,t)\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx \\ &+ \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx, \end{aligned} \tag{3.46}$$

where

$$J_4(x,t) := J_3(x,t) + \frac{1}{8}\psi(x,t). \tag{3.47}$$

Proof. Estimates (3.36) and (3.44) with $0 < \varepsilon < \min\left\{\frac{k_1}{8c_4}, \frac{\rho_1}{16c_4}\right\}$ small enough imply (3.46).■

Now, as in Ref. 2, we use a function w to get a crucial estimate.

Lemma 3.9. Let

$$w(x,t) := -\int_0^x \psi(y,t)dy + \frac{1}{L}\left(\int_0^L \psi(y,t)dy\right)x.$$

Then the functional

$$I_6(t) := \int_0^L \left(\rho_1\varphi_t(x,t)w(x,t) + \rho_2\psi_t(x,t)\psi(x,t)\right) dx$$

satisfies, for all $\varepsilon_1 \in]0,1[$ *and for some* $c_6 > 0$ (*not depending neither on* f *nor on* ε_1),

$$\begin{aligned} I_6'(t) \le &-\frac{k_2 - g_0}{2}\int_0^L \psi_x^2(x,t)dx + \frac{c_6}{\varepsilon_1}\int_0^L \psi_t^2(x,t)dx \\ &+ \varepsilon_1\int_0^L \varphi_t^2(x,t)dx + c_6\int_0^L\int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx \\ &- \int_0^L \psi(x,t)\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx. \end{aligned} \tag{3.48}$$

Proof. The fact that $-w_{xx} = \psi_x$ and $w(0,t) = w(L,t) = 0$ imply that

$$\begin{aligned} \int_0^L w_x^2(x,t)dx &= \int_0^L \psi_x(x,t)w(x,t)dx = -\int_0^L \psi(x,t)w_x(x,t)dx \\ &\le \left(\int_0^L \psi^2(x,t)dx\right)^{\frac{1}{2}}\left(\int_0^L w_x^2(x,t)dx\right)^{\frac{1}{2}}, \end{aligned}$$

which gives, using (2.9) for w_t (note that $w_t(0,t) = w_t(L,t) = 0$),

$$\int_0^L w_x^2(x,t)dx \le \int_0^L \psi^2(x,t)dx \quad \text{and} \quad \int_0^L w_t^2(x,t)dx \le c_0\int_0^L \psi_t^2(x,t)dx. \tag{3.49}$$

On the other hand,

$$-k_1\int_0^L (\varphi_x(x,t)+\psi(x,t))(w_x(x,t)+\psi(x,t))dx \tag{3.50}$$

$$= -k_1 \left(\int_0^L (\varphi_x(x,t) + \psi(x,t))dx \right) \left(\frac{1}{L} \int_0^L \psi(y,t)dy \right)$$
$$= \frac{-k_1}{L} \left(\int_0^L \psi(x,t)dx \right)^2$$
$$\leq 0.$$

Therefore, as in Ref. 2 in case $f \equiv 0$, by differentiating I_6, using the first two equations in (1.1), integrating by parts and using (3.49) and (3.50) and Young's inequality for $\varphi_t w_t$, we find

$$I_6'(t) \leq -(k_2 - g_0) \int_0^L \psi_x^2(x,t)dx + \frac{c_6}{\varepsilon_1} \int_0^L \psi_t^2(x,t)dx$$
$$+ \varepsilon_1 \int_0^L \varphi_t^2(x,t)dx - \int_0^L \psi(x,t) \int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx \tag{3.51}$$
$$- \int_0^L \psi_x(x,t) \int_0^{+\infty} g(s)\eta_x(x,t,s)dsdx.$$

The use of Young's inequality and (3.32) for the last term in (3.51) gives (3.48). ■

Now, we introduce a new functional I_7 which plays a crucial role in dealing with the distributed time delay.

Lemma 3.10. Let

$$I_7(t) := e^{-\hat{\gamma}(t)} \int_0^L \int_0^{+\infty} e^{\tilde{\gamma}(s)} |f(s)| \left(\int_{t-s}^t e^{\hat{\gamma}(\tau)} \psi_t^2(x,\tau)d\tau \right) dsdx, \tag{3.52}$$

where $\tilde{\gamma}$ and γ are defined in assumption **(A4)**, *and*

$$\hat{\gamma}(t) := \int_0^t \gamma(s)ds. \tag{3.53}$$

The functional I_7 satisfies

$$I_7'(t) \leq g(0)\beta_0 \int_0^L \psi_t^2(x,t)dx - \gamma(t)I_7(t) - \int_0^L \int_0^{+\infty} |f(s)|\psi_t^2(x,t-s)dsdx, \tag{3.54}$$

where β_0 is defined in (3.4).

Proof. First, thanks to (2.9), (2.14), and (3.5) we have (note also that g is nonincreasing and $\hat{\gamma}$ is increasing)

$$I_7(t) \leq \int_0^L \int_0^{+\infty} \beta(s)g(s) \left(\int_{t-s}^t \psi_t^2(x,\tau)d\tau \right) dsdx$$
$$\leq \int_0^L \int_0^{+\infty} \beta(s) \left(\int_{t-s}^t g(t-\tau)\psi_t^2(x,\tau)d\tau \right) dsdx$$
$$\leq c_0 \int_0^L \int_0^{+\infty} \beta(s) \left(\int_0^s g(\tau)\psi_{xt}^2(x,t-\tau)d\tau \right) dsdx$$
$$\leq c_0 \int_0^L \int_0^{+\infty} \beta(s) \left(\int_0^{+\infty} g(\tau)\eta_{xs}^2(x,t,\tau)d\tau \right) dsdx;$$

thus, due to (3.4) and the fact that $\eta_s \in L_g^2(\mathbb{R}_+, H_0^1(]0,L[))$ (in virtue of (2.16) with $n = 1$),

$$I_7(t) \leq c_0\beta_0 \|\eta_s(x,t,\cdot)\|^2_{L_g^2(\mathbb{R}_+, H_0^1(]0,L[))}, \tag{3.55}$$

and therefore, the functional I_7 is well-defined. Moreover, using again (3.5), a simple and direct differentiation gives

$$\begin{aligned} I_7'(t) &= -\gamma(t) I_7(t) + \left(\int_0^{+\infty} e^{\tilde{\gamma}(s)}|f(s)|ds\right)\int_0^L \psi_t^2(x,t)dx \\ &\quad - \int_0^L\int_0^{+\infty} e^{\tilde{\gamma}(s)+\hat{\gamma}(t-s)-\hat{\gamma}(t)}|f(s)|\psi_t^2(x,t-s)dsdx \\ &\le -\gamma(t)I_7(t) + \left(\int_0^{+\infty}\beta(s)g(s)ds\right)\int_0^L \psi_t^2(x,t)dx \\ &\quad - \int_0^L\int_0^{+\infty} e^{\tilde{\gamma}(s)+\hat{\gamma}(t-s)-\hat{\gamma}(t)}|f(s)|\psi_t^2(x,t-s)dsdx \\ &\le -\gamma(t)I_7(t) + g(0)\beta_0\int_0^L \psi_t^2(x,t)dx \\ &\quad - \int_0^L\int_0^{+\infty} e^{\tilde{\gamma}(s)+\hat{\gamma}(t-s)-\hat{\gamma}(t)}|f(s)|\psi_t^2(x,t-s)dsdx. \end{aligned} \tag{3.56}$$

On the other hand, the function $h(t) := \tilde{\gamma}(s) + \hat{\gamma}(t-s) - \hat{\gamma}(t)$, for $s \ge 0$ fixed and $t \ge 0$, satisfies $h'(t) = \gamma(t-s) - \gamma(t)$. Then h is nondecreasing, for $t \ge \frac{s}{2}$, and it is nonincreasing, for $t \le \frac{s}{2}$, because γ is even and nonincreasing on $\mathbb{R}_+$. Therefore, $h(t) \ge h(\frac{s}{2}) = 0$, and (3.54) follows at once. ∎

Now, let N, N_1, and N_2 be positive constants (which will be fixed latter on). We define the functional

$$\mathscr{L}_1(t) := NE(t) + N_1 I_1(t) + I_5(t) + N_2 I_6(t) + I_7(t). \tag{3.57}$$

At this step, we distinguish three cases depending on (1.2) and (3.2)–(3.3).

1. Equal speed propagation and exponential decay of g: (1.2) and (3.2) hold

By combining (3.31), (3.34), (3.46), (3.48), and (3.54) taking $\delta = \frac{k_1}{8N_1}$ and using (3.2), we get

$$\mathscr{L}_1'(t) \tag{3.58}$$

$$\begin{aligned} &\le -\left(\frac{(k_2-g_0)N_2}{2} - c_5 - \frac{k_1}{8}\right)\int_0^L \psi_x^2(x,t)dx - \left(\frac{\rho_1}{16} - \varepsilon_1 N_2\right)\int_0^L \varphi_t^2(x,t)dx \\ &\quad - \frac{k_1}{8}\int_0^L (\varphi_x(x,t)+\psi(x,t))^2dx - \int_0^L\int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx \\ &\quad - \left(\rho_2(g_0N_1 - \frac{k_1}{8}) - \frac{c_6N_2}{\varepsilon_1} - g(0)\beta_0 - c_5\right)\int_0^L \psi_t^2(x,t)dx - \gamma(t)I_7(t) \\ &\quad + \left(\frac{N}{2} - c_{N_1,N_2}\right)\int_0^L\int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx - \int_0^L\int_0^{+\infty}|f(s)|\psi_t^2(x,t-s)dsdx \\ &\quad + \int_0^L J_5(x,t)\int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx, \end{aligned}$$

where

$$J_5(x,t) := J_4(x,t) - N\psi_t(x,t) - N_2\psi(x,t) + N_1\int_0^{+\infty} g(s)\eta(x,t,s)ds. \tag{3.59}$$

Due to (1.2), the last term in (3.58) vanishes. Now, we choose N_2 large enough such that

$$\frac{(k_2 - g_0)N_2}{2} - c_5 - \frac{k_1}{8} > 0 \tag{3.60}$$

(N_2 exists according to (2.1)), then we take $\varepsilon_1 \in]0, 1[$ small enough so that

$$\frac{\rho_1}{16} - \varepsilon_1 N_2 > 0.$$

Next, we pick N_1 so large such that

$$\rho_2(g_0 N_1 - \frac{k_1}{8}) - \frac{c_6 N_2}{\varepsilon_1} - g(0)\beta_0 - c_5 > 0$$

(N_1 exists thanks to (3.1)). On the other hand, by the definition of the functionals E, $I_1 - I_6$, and $J_1 - J_2$, and the use of (2.15), (2.9) and Young's inequality, there exists a positive constant c_7 (not depending neither on *f* nor on *N*) such that

$$|N_1 I_1(t) + I_5(t) + N_2 I_6(t)| \leq c_7 E(t), \quad \forall t \in \mathbb{R}_+, \tag{3.61}$$

which implies that

$$(N - c_7)E(t) \leq \mathscr{L}_1(t) - I_7(t) \leq (N + c_7)E(t), \quad \forall t \in \mathbb{R}_+. \tag{3.62}$$

Then, we choose N large enough such that

$$N > \max\{c_7, 2c_{N_1,N_2}\}, \tag{3.63}$$

so $E \sim \mathscr{L}_1 - I_7$ holds and, from (3.58) and the definition of E, we obtain, for some $c_8 > 0$ (not depending on *f*),

$$\begin{aligned}\mathscr{L}_1'(t) \leq -c_8 E(t) - \int_0^L \int_0^{+\infty} |f(s)|\psi_t^2(x, t-s)dsdx \\ - \gamma(t)I_7(t) + \int_0^L J_5(x,t) \int_0^{+\infty} f(s)\psi_t(x, t-s)dsdx.\end{aligned} \tag{3.64}$$

Similar to (3.61), from the definition of E and $J_3 - J_5$, and using (2.9) and Young's inequality, we find that there exists a positive constant c_9 (not depending on *f*) such that

$$\int_0^L J_5^2(x,t)dx \leq c_9 E(t).$$

Therefore, applying Cauchy-Schwarz and Young's inequalities, we get, for

$$\epsilon' = 2\left(\int_0^{+\infty} |f(s)|ds\right)^{-1} \tag{3.65}$$

(if $\int_0^{+\infty} |f(s)|ds = 0$, then $f \equiv 0$, and therefore, the two terms in (3.64) depending on *f* vanish),

$$\int_0^L J_5(x,t) \int_0^{+\infty} f(s)\psi_t(x, t-s)dsdx \tag{3.66}$$

$$\begin{aligned}&\leq \left(\int_0^L \left(\int_0^{+\infty} f(s)\psi_t(x,t-s)ds\right)^2 dx\right)^{\frac{1}{2}} \left(\int_0^L J_5^2(x,t)dx\right)^{\frac{1}{2}} \\ &\leq \frac{\epsilon'}{2}\left(\int_0^{+\infty}|f(s)|ds\right)\int_0^L\int_0^{+\infty}|f(s)|\psi_t^2(x,t-s)dsdx + \frac{1}{2\epsilon'}\int_0^L J_5^2(x,t)dx \\ &\leq \int_0^L\int_0^{+\infty}|f(s)|\psi_t^2(x,t-s)dsdx + \frac{c_9}{4}\left(\int_0^{+\infty}|f(s)|ds\right)E(t).\end{aligned}$$

Hence, under condition (3.8) with

$$\delta_0 = \frac{4c_8}{c_9} \tag{3.67}$$

(noting that δ_0 is positive and does not depend on *f*) and by combining (3.64) and (3.66), we find, for some $c_{10} > 0$,

$$\mathscr{L}_1'(t) \leq -c_{10}E(t) - \gamma(t)I_7(t), \quad \forall t \in \mathbb{R}_+. \tag{3.68}$$

By combining (3.62) and (3.68), we obtain

$$\mathscr{L}_1'(t) \leq -\delta_1 \min\{1, \gamma(t)\}\mathscr{L}(t), \quad \forall t \in \mathbb{R}_+, \tag{3.69}$$

where $\delta_1 = \min\{\frac{c_{10}}{N+c_7}, 1\}$. Then, an integration of the differential inequality (3.69) over $[0, t]$ gives

$$\mathscr{L}_1(t) \leq \mathscr{L}_1(0)e^{-\delta_1\phi(t)}, \quad \forall t \in \mathbb{R}_+, \tag{3.70}$$

where ϕ is defined in (3.10). Consequently, the choice (3.63) of N and the relations (3.30)(3.62)(3.70) lead to

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 = 2E(t) \leq \frac{2}{N-c_7}\mathscr{L}_1(t) \leq \frac{2\mathscr{L}_1(0)}{N-c_7}e^{-\delta_1\phi(t)}, \quad \forall t \in \mathbb{R}_+,$$

which is the decay estimate (3.9) with $\delta_2 = \dfrac{2\mathscr{L}_1(0)}{N-c_7}$.

2. Nonequal speed propagation and exponential decay of g: (1.2) does not hold and (3.2) holds

To deal with the last term in (3.58) (which can not be absorbed by E) and get (3.11), we appeal to some ideas of Refs. 1, 8, and 13 based on the energies of high orders defined by

$$E_k(t) = \frac{1}{2}\|\mathscr{U}^{(k)}(t)\|_{\mathscr{H}}^2, \quad \forall \mathscr{U}_0 \in \mathscr{D}(\mathscr{A}^k), \ \ k = 0, \cdots, n \tag{3.71}$$

(so $E_0 = E$). As for (3.31), E_k satisfies

$$\begin{aligned} E_k'(t) &= \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)(\partial_t^k \eta_x)^2(x,t,s)dsdx \\ &\quad - \int_0^L \partial_t^{k+1}\psi(x,t)\int_0^{+\infty} f(s)\partial_t^{k+1}\psi(x,t-s)dsdx. \end{aligned} \tag{3.72}$$

We start by proving two lemmas, where the first one is given in Ref. 8, while the second one is introduced in the present paper to cope with some delay terms.

Lemma 3.11. For any $\varepsilon > 0$, we have

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx \tag{3.73}$$

$$\begin{aligned} &\leq \epsilon\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right|\int_0^L \varphi_t^2(x,t)dx - \frac{1}{\epsilon\theta_1 g_0}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right| E_1'(t) \\ &\quad - \frac{g(0)}{2\epsilon g_0^2}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right|\int_0^L\int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx \\ &\quad - \frac{1}{\epsilon\theta_1 g_0}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right|\int_0^L \psi_{tt}(x,t)\int_0^{+\infty} f(s)\psi_{tt}(x,t-s)dsdx. \end{aligned}$$

Proof. We proceed as in Ref. 8. By recalling that $g_0 = \int_0^{+\infty} g(s)ds$, we have

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_t(x,t)\psi_{xt}(x,t)dx \tag{3.74}$$
$$= \frac{1}{g_0}\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_t(x,t) \int_0^{+\infty} g(s)\eta_{xt}(x,t,s)dsdx$$
$$+ \frac{1}{g_0}\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_t(x,t) \int_0^{+\infty} g(s)\psi_{xt}(x,t-s)dsdx.$$

Using Young's inequality and (3.32) (for $v = \eta_{xt}$), we get, for all $\varepsilon > 0$,

$$\frac{1}{g_0}(\frac{\rho_1 k_2}{k_1} - \rho_2) \int_0^L \varphi_t(x,t) \int_0^{+\infty} g(s)\eta_{xt}(x,t,s)dsdx \tag{3.75}$$
$$\leq \frac{\epsilon}{2}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right| \int_0^L \varphi_t^2(x,t)dx + \frac{1}{2\epsilon g_0}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right| \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2(x,t,s)dsdx.$$

Moreover, using (3.2) and (3.72) for $k = 1$, we get

$$\frac{1}{2\epsilon g_0}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right| \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2(x,t,s)dsdx \tag{3.76}$$
$$\leq -\frac{1}{\epsilon\theta_1 g_0}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right| E_1'(t)$$
$$- \frac{1}{\epsilon\theta_1 g_0}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right| \int_0^L \psi_{tt}(x,t) \int_0^{+\infty} f(s)\psi_{tt}(x,t-s)dsdx.$$

On the other hand, by integrating by parts with respect to s and using (3.39) and (3.33) (for $v = \eta_x$) and Young's inequality, we have, for all $\varepsilon > 0$ (note also that $\eta_x(x, t, 0) = \lim_{s\to+\infty} g(s)\eta_x(x, t, s) = 0$ due to (2.13) and (2.16) for $n \geq 1$),

$$\frac{1}{g_0}\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_t(x,t) \int_0^{+\infty} g(s)\psi_{xt}(x,t-s)dsdx \tag{3.77}$$
$$= \frac{1}{g_0}\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_t(x,t) \int_0^{+\infty} g(s)\eta_{xs}(x,t,s)dsdx$$
$$= \frac{1}{g_0}\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_t(x,t) \int_0^{+\infty} (-g'(s))\eta_x(x,t,s)dsdx$$
$$\leq \frac{\epsilon}{2}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right| \int_0^L \varphi_t^2(x,t)dx - \frac{g(0)}{2\epsilon g_0^2}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right| \int_0^L \int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx.$$

Inserting (3.75)–(3.77) into (3.74), we find (3.73). ∎

Lemma 3.12. The functional

$$I_8(t) := \int_0^L \psi_t(x,t)\left(\frac{f(0)}{2}\psi_t(x,t) - \int_0^{+\infty} f''(s)\eta(x,t,s)ds\right)dx \tag{3.78}$$

satisfies ($\tilde{\alpha}$ is defined in (3.7)),

$$I_8'(t) \leq \int_0^L \psi_{tt}(x,t) \int_0^{+\infty} f(s)\psi_{tt}(x,t-s)dsdx + \left(\alpha g(0) + \frac{1}{2}\right) \int_0^L \psi_t^2(x,t)dsdx$$
$$- \frac{\tilde{\alpha}^2 g_0 c_0}{2\theta_1} \int_0^L \int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx. \tag{3.79}$$

Proof. From (2.14), we conclude that $\eta_{ss}(x, t, s) = -\psi_{tt}(x, t-s)$ (recall that $\eta_{ss} \in L^2_g(\mathbb{R}_+, H^1_0(]0, L[))$, for any $t \in \mathbb{R}_+$ fixed, since $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A}^n)$ with $n \geq 2$), and then

$$\int_0^L \psi_{tt}(x,t)\int_0^{+\infty} f(s)\psi_{tt}(x,t-s)dsdx = -\int_0^L \psi_{tt}(x,t)\int_0^{+\infty} f(s)\eta_{ss}(x,t,s)dsdx.$$

By integrating by parts with respect to s and using the fact that $\eta_s(x, t, 0) = \psi_t(x, t)$ (du to (2.14)), $\eta(x, t, 0) = 0$ (thanks to (2.13)) and $\lim_{s\to+\infty} f(s)\eta_s(x, t, s) = \lim_{s\to+\infty} f'(s)\eta(x, t, s) = 0$ (according to (2.3) and (2.16) with $n \geq 2$), we get

$$\begin{aligned}\int_0^L \psi_{tt}(x,t)\int_0^{+\infty} f(s)\psi_{tt}(x,t-s)dsdx &= \int_0^L \psi_{tt}(x,t)\left(f(0)\psi_t(x,t) - \int_0^{+\infty} f''(s)\eta(x,t,s)ds\right)dx\\ &= I_8'(t) + \int_0^L \psi_t(x,t)\int_0^{+\infty} f''(s)\eta_t(x,t,s)dsdx.\end{aligned}$$

Again, using $\eta(x, t, 0) = \lim_{s\to+\infty} f'(s) = \lim_{s\to+\infty} f'(s)\eta(x, t, s) = 0$ (in virtue of (2.3), (2.13), (3.7), and (2.16) with $n \geq 1$), and integrating by parts with respect to s, we find

$$\begin{aligned}I_8'(t) &= \int_0^L \psi_{tt}(x,t)\int_0^{+\infty} f(s)\psi_{tt}(x,t-s)dsdx\\ &\quad - \int_0^L \psi_t(x,t)\int_0^{+\infty} f''(s)(\psi_t(x,t) - \eta_s(x,t,s))dsdx\\ &= \int_0^L \psi_{tt}(x,t)\int_0^{+\infty} f(s)\psi_{tt}(x,t-s)dsdx + f'(0)\int_0^L \psi_t^2(x,t)dx\\ &\quad - \int_0^L \psi_t(x,t)\int_0^{+\infty} f'''(s)\eta(x,t,s)dsdx.\end{aligned}$$

On the other hand, using (2.3), we have

$$f'(0)\int_0^L \psi_t^2(x,t)dx \leq \alpha g(0)\int_0^L \psi_t^2(x,t)dx.$$

Moreover, using (3.7), Young's inequality, (3.32) (for $v = |\eta|$) and (2.9), we obtain

$$\begin{aligned}&-\int_0^L \psi_t(x,t)\int_0^{+\infty} f'''(s)\eta(x,t,s)dsdx\\ &\leq \frac{1}{2}\int_0^L \psi_t^2(x,t)dx + \frac{\tilde{\alpha}^2 g_0 c_0}{2}\int_0^L\int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx.\end{aligned}$$

Inserting these two inequalities in the previous equality, we arrive at

$$\begin{aligned}I_8'(t) \leq &\int_0^L \psi_{tt}(x,t)\int_0^{+\infty} f(s)\psi_{tt}(x,t-s)dsdx + \left(\alpha g(0) + \frac{1}{2}\right)\int_0^L \psi_t^2(x,t)dsdx\\ &+ \frac{\tilde{\alpha}^2 g_0 c_0}{2}\int_0^L\int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx.\end{aligned} \tag{3.80}$$

Applying (3.2) to the last term of (3.80) gives (3.79). ■

Now, let us consider the functional

$$\mathscr{L}_2(t) := \mathscr{L}_1(t) + \frac{1}{\epsilon g_0\theta_1}\left|\frac{\rho_1 k_2}{k_1} - \rho_2\right|(E_1(t) + I_8(t)). \tag{3.81}$$

By combining (3.58), (3.73), and (3.79) we get

$$\mathscr{L}_2'(t) \tag{3.82}$$

$$\leq -\left(\frac{(k_2-g_0)N_2}{2}-c_5-\frac{k_1}{8}\right)\int_0^L \psi_x^2(x,t)dx-\left(\frac{\rho_1}{16}-\varepsilon_1 N_2-\epsilon\left|\frac{\rho_1 k_2}{k_1}-\rho_2\right|\right)\int_0^L \varphi_t^2(x,t)dx$$
$$-\frac{k_1}{8}\int_0^L(\varphi_x(x,t)+\psi(x,t))^2dx-\int_0^L\int_0^{+\infty}g(s)\eta_x^2(x,t,s)dsdx$$
$$-\left(\rho_2(g_0N_1-\frac{k_1}{8})-\frac{c_6N_2}{\varepsilon_1}-\frac{1}{\epsilon g_0\theta_1}\left(\alpha g(0)+\frac{1}{2}\right)\left|\frac{\rho_1 k_2}{k_1}-\rho_2\right|-g(0)\beta_0-c_5\right)\int_0^L \psi_t^2(x,t)dx$$
$$+\left(\frac{N}{2}-\left(\frac{g(0)}{2\epsilon g_0^2}+\frac{\tilde{\alpha}^2c_0}{2\epsilon\theta_1^2}\right)\left|\frac{\rho_1 k_2}{k_1}-\rho_2\right|-c_{N_1,N_2}\right)\int_0^L\int_0^{+\infty}g'(s)\eta_x^2(x,t,s)dsdx-\gamma(t)I_7(t)$$
$$-\int_0^L\int_0^{+\infty}|f(s)|\psi_t^2(x,t-s)dsdx+\int_0^L J_5(x,t)\int_0^{+\infty}f(s)\psi_t(x,t-s)dsdx.$$

First, we choose N_2 as in (3.60), and then we take $\varepsilon=\varepsilon_1$ and $\varepsilon_1\in\,]0,1[$ small enough so that

$$\frac{\rho_1}{16}-\epsilon_1\left(N_2+\left|\frac{\rho_1k_2}{k_1}-\rho_2\right|\right)>0.$$

Next, we pick N_1 so large such that

$$\rho_2\left(g_0N_1-\frac{k_1}{8}\right)-\frac{1}{\varepsilon_1}\left(c_6N_2+\frac{1}{g_0\theta_1}\left(\alpha g(0)+\frac{1}{2}\right)\left|\frac{\rho_1k_2}{k_1}-\rho_2\right|\right)-g(0)\beta_0-c_5>0$$

(N_1 exists due to (3.1)). On the other hand, using (2.3) and (3.7), Young's inequality, (3.32) (for $v=|\eta|$) and (2.9), we find

$$|I_8(t)|\leq\frac{1+\alpha g(0)}{2}\int_0^L\psi_t^2(x,t)dx+\frac{\tilde{\alpha}^2g_0c_0}{2}\int_0^L\int_0^{+\infty}g(s)\eta_x^2(x,t,s)dsdx.$$

Consequently, in virtue of (2.15) and (3.30), there exists a positive constant $\tilde{c}_7$ (not depending neither on $\int_0^{+\infty}|f(s)|ds$ nor on N) such that

$$\frac{1}{\epsilon g_0\theta_1}\left|\frac{\rho_1k_2}{k_1}-\rho_2\right||I_8(t)|\leq\tilde{c}_7E(t),\quad\forall t\in\mathbb{R}_+,\tag{3.83}$$

which implies that, for all $t\in\mathbb{R}_+$, using (3.62) and (3.81),

$$(N-c_7-\tilde{c}_7)E(t)\leq\mathscr{L}_2(t)-\frac{1}{\epsilon g_0\theta_1}\left|\frac{\rho_1k_2}{k_1}-\rho_2\right|E_1(t)-I_7(t)\leq(N+c_7+\tilde{c}_7)E(t).\tag{3.84}$$

Finally, we choose N large enough such that

$$N>\max\left\{c_7+\tilde{c}_7,2c_{N_1,N_2}+\left(\frac{g(0)}{\epsilon g_0^2}+\frac{\tilde{\alpha}^2c_0}{\epsilon\theta_1^2}\right)\left|\frac{\rho_1k_2}{k_1}-\rho_2\right|\right\}.\tag{3.85}$$

Therefore, from (3.82) and the definition of E, we obtain (3.64) for $\mathscr{L}_2$ instead of $\mathscr{L}_1$, and some $\tilde{c}_8>0$ (not depending on $\int_0^{+\infty}|f(s)|ds$) instead of c_8. Thus, according to (3.66) and under condition (3.8), where δ_0 is defined in (3.67) with $\tilde{c}_8$ instead of c_8, we find, for some $\tilde{c}_{10}>0$,

$$\mathscr{L}_2'(t)\leq-\tilde{c}_{10}E(t)-\gamma(t)I_7(t)\leq-\tilde{c}_{10}E(t),\quad\forall t\in\mathbb{R}_+.\tag{3.86}$$

Before concluding (3.11), we prove this last lemma.

Lemma 3.13. There exist two positive constants a_0 and a_1 such that, for any $\mathscr{U}_0\in\mathscr{D}(\mathscr{A}^2)$,

$$\int_S^T E(t)dt\leq a_1(E(S)+E_1(S)),\quad\forall 0\leq S\leq T\tag{3.87}$$

and

$$E'(t) \leq a_0 E(t), \quad \forall t \in \mathbb{R}_+. \tag{3.88}$$

Proof. By integrating (3.86) over $[S, T]$, we get (note that $\mathscr{L}_2 \geq 0$ in virtue of (3.84) and (3.85))

$$\tilde{c}_{10} \int_S^T E(t)dt \leq \mathscr{L}_2(S) - \mathscr{L}_2(T) \leq \mathscr{L}_2(S), \quad \forall 0 \leq S \leq T. \tag{3.89}$$

Moreover, (3.55) and (2.13) imply that

$$\begin{aligned} I_7(t) &\leq c_0\beta_0 \|\eta_s(x,t,\cdot)\|^2_{L^2_g(\mathbb{R}_+, H^1_0(]0,L[))} \\ &\leq c_0\beta_0 \|\psi_t(x,t) - \eta_t(x,t,\cdot)\|^2_{L^2_g(\mathbb{R}_+, H^1_0(]0,L[))} \\ &\leq 2c_0\beta_0 \left(\|\psi_t(x,t)\|^2_{L^2_g(\mathbb{R}_+, H^1_0(]0,L[))} + \|\eta_t(x,t,\cdot)\|^2_{L^2_g(\mathbb{R}_+, H^1_0(]0,L[))} \right) \\ &\leq 2c_0\beta_0 \left(g_0 \int_0^L \psi_{xt}^2(x,t)dx + \int_0^L \int_0^{+\infty} g(s)\eta_{xt}^2(x,t,s)dsdx \right), \end{aligned}$$

and then, according to (3.71) for $k = 1$,

$$I_7(t) \leq 4c_0\beta_0 \max\left\{ \frac{g_0}{k_2 - g_0}, 1 \right\} E_1(t). \tag{3.90}$$

Consequently, by substituting (3.90) and the right inequality of (3.84) in (3.89), we deduce (3.87) with

$$a_1 = \frac{1}{\tilde{c}_{10}} \max\left\{ N + c_7 + \tilde{c}_7, \frac{1}{\epsilon g_0 \theta_1} \left| \frac{\rho_1 k_2}{k_1} - \rho_2 \right| + 4c_0\beta_0 \max\left\{ \frac{g_0}{k_2 - g_0}, 1 \right\} \right\}.$$

On the other hand, taking (3.31) and (2.14) in consideration, integrating with respect to s and using the fact that $\eta(x, t, 0) = \lim_{s \to +\infty} f(s)\eta(x, t, s) = 0$ (thanks to (2.3), (2.9), (2.13), and (2.16) with $n \geq 1$), we find

$$\begin{aligned} E'(t) &\leq -\int_0^L \psi_t(x,t) \int_0^{+\infty} f(s)\psi_t(x,t-s)dsdx \\ &\leq -\int_0^L \psi_t(x,t) \int_0^{+\infty} f(s)\eta_s(x,t,s)dsdx \\ &\leq \int_0^L \psi_t(x,t) \int_0^{+\infty} f'(s)\eta(x,t,s)dsdx, \end{aligned}$$

hence, applying Young's inequality and using (2.3) and (3.32) (for $v = |\eta|$) and (2.9),

$$E'(t) \leq \frac{1}{2} \int_0^L \psi_t^2(x,t)dx + \frac{\alpha^2 g_0 c_0}{2} \int_0^L \int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx,$$

which gives (3.88) with $a_0 = \max\left\{ \frac{1}{\rho_2}, \alpha^2 g_0 c_0 \right\}$. ■

Lemma 3.13 allows us to apply [Theorem 2.2 of Ref. 13: case $f \equiv 1$, $m = 1$, $a_2 = 0$, and $n \geq 2$] and get (3.11) by continuity of E.

3. Equal speed propagation and arbitrary decay of g: (1.2) and (3.3) hold, and (3.2) does not hold

We prove here the decay estimate (3.12) under condition (3.3), which allows g to have a general decay at infinity that can be arbitrary close to $\frac{1}{t}$.

In the cases of absence of delay and/or presence of frictional damping considered in the literature (like (1.3)), the proof of the known stability estimates when g has an arbitrary decay is based on some differential inequalities on g involving at least its first derivative in order to express

$$\int_0^L \int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx \tag{3.91}$$

in term of

$$\int_0^L \int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx,$$

and then use the nonincreasingness of E. This strategy seems not applicable in our case, since E is not necessarily nonincreasing due to the last term in (3.31) generated by the distributed time delay.

Our proof is based on different manipulations of the term (3.91), the integral inequality in (3.3) introduced and used in Refs. 36–39, and 40, which does not involve any derivative of g, and the use of a new functional J_6 (defined in (3.100) below) that is able to absorb some memory terms without passing by E'.

First, following the idea in Refs. 36–39, and 40, we see that

$$2\int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\psi_x(x,t-s)dsdx \tag{3.92}$$

$$= \int_0^L \int_0^{+\infty} g(s)\left(\psi_x^2(x,t-s)-\eta_x^2(x,t,s)\right)dsdx + g_0\int_0^L \psi_x^2(x,t)dx.$$

Second, for $i\in\mathbb{N}^*$, we consider, as in Ref. 33, the set

$$A_i = \{s\in\mathbb{R}_+,\ g(s)\le -ig'(s)\}, \tag{3.93}$$

and we put

$$g_i = \int_{A_i^c} g(s)ds. \tag{3.94}$$

Note that $g_i > 0$, otherwise, $A_i^c = \emptyset$ and then (3.2) is satisfied for $\theta_1 = \frac{1}{i}$, which is the case of exponential decay of g treated previously. On the other hand, thanks to the second inequality in (3.3), we have $\lim_{i\to+\infty} A_i^c = \cap_{i\in\mathbb{N}^*} A_i^c = \emptyset$, and then

$$\lim_{i\to+\infty} g_i = 0. \tag{3.95}$$

Next, we go back to (3.35), (3.37), (3.42), and (3.51). Clearly, we have

$$\int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\eta_x(x,t,s)dsdx = \int_0^L \psi_x(x,t)\int_{A_i} g(s)\eta_x(x,t,s)dsdx$$
$$+\int_0^L \psi_x(x,t)\int_{A_i^c} g(s)\eta_x(x,t,s)dsdx.$$

Then, using Cauchy-Schwarz and Young's inequalities for the two terms in the right hand side of the above equality and the definition (3.94) of g_i, we have, for any $\varepsilon_2 > 0$,

$$\int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\eta_x(x,t,s)dsdx \le \epsilon_2\int_0^L \psi_x^2(x,t)dx + \frac{g_0}{4\epsilon_2}\int_0^L\int_{A_i} g(s)\eta_x^2(x,t,s)dsdx$$
$$+\frac{\sqrt{g_i}}{2}\left(\int_0^L \psi_x^2(x,t)dx + \int_0^L\int_{A_i^c} g(s)\eta_x^2(x,t,s)dsdx\right).$$

Using the definition (3.93) of A_i, we obtain

$$\int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\eta_x(x,t,s)dsdx \tag{3.96}$$

$$\leq \epsilon_2 \int_0^L \psi_x^2(x,t)dx - \frac{ig_0}{4\epsilon_2}\int_0^L \int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx$$
$$+ \frac{\sqrt{g_i}}{2}\left(\int_0^L \psi_x^2(x,t)dx + \int_0^L \int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx\right).$$

Similarly, we find

$$\int_0^L \left(\int_0^{+\infty} g(s)\eta_x(x,t,s)ds\right)^2 dx$$

$$= \int_0^L \left(\int_{A_i} g(s)\eta_x(x,t,s)ds + \int_{A_i^c} g(s)\eta_x(x,t,s)ds\right)^2 dx$$
$$\leq 2\int_0^L \left(\int_{A_i} g(s)\eta_x(x,t,s)ds\right)^2 dx + 2\int_0^L \left(\int_{A_i^c} g(s)\eta_x(x,t,s)ds\right)^2 dx$$
$$\leq 2g_0 \int_0^L \int_{A_i} g(s)\eta_x^2(x,t,s)dsdx + 2g_i \int_0^L \int_{A_i^c} g(s)\eta_x^2(x,t,s)dsdx.$$

Therefore, using again the definition (3.93) of A_i, we deduce that

$$\int_0^L \left(\int_0^{+\infty} g(s)\eta_x(x,t,s)ds\right)^2 dx \leq -2ig_0 \int_0^L \int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx + 2g_i \int_0^L \int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx. \tag{3.97}$$

The last term we discuss, using (3.92), is

$$-\int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\eta_x(x,t,s)dsdx \tag{3.98}$$

$$= -g_0 \int_0^L \psi_x^2(x,t)dx + \int_0^L \psi_x(x,t)\int_0^{+\infty} g(s)\psi_x(x,t-s)dsdx$$
$$= -\frac{g_0}{2}\int_0^L \psi_x^2(x,t)dx + \frac{1}{2}\int_0^L \int_0^{+\infty} g(s)\left(\psi_x^2(x,t-s) - \eta_x^2(x,t,s)\right)dsdx.$$

After, we insert (3.96), (3.97), and (3.98) into (3.35), (3.37), (3.42), and (3.51) considering the functional $\mathscr{L}_1$ defined in (3.57), using (3.31), (3.38), (3.41), and (3.54) taking (2.1) in consideration, choosing $\delta = \dfrac{k_1}{8N_1}$ and

$$0 < \epsilon < \left\{\frac{k_1^2}{4c_3}, \frac{k_1\rho_1}{8(2k_1 + c_3)}\right\},$$

we obtain, for some $c_{11}, c_{12} > 0$ (not depending on N, N_1, N_2, i, ε_1, ε_2 and f),

$$\mathscr{L}_1'(t) \tag{3.99}$$

$$\leq -\left(\frac{k_2N_2+g_0^2N_1}{2}-c_{11}-\frac{\varepsilon_2}{8}\right)\int_0^L \psi_x^2(x,t)dx-\frac{k_1}{8}\int_0^L(\varphi_x(x,t)+\psi(x,t))^2dx$$

$$-\left(\frac{\rho_1}{16}-\varepsilon_1N_2\right)\int_0^L \varphi_t^2(x,t)dx-\left(\rho_2(g_0N_1-\frac{k_1}{8})-\frac{c_6N_2}{\varepsilon_1}-c_{11}\right)\int_0^L\psi_t^2(x,t)dx$$

$$+\int_0^L\int_0^{+\infty}\left(-c_{N_1,N_2}g(s)\eta_x^2(x,t,s)+\left(\frac{N}{2}-c_{N_1,\varepsilon_2,i}\right)g'(s)\eta_x^2(x,t,s)\right)dsdx$$

$$-\int_0^L\int_0^{+\infty}|f(s)|\psi_t^2(x,t-s)dsdx+\frac{\sqrt{g_i}}{16}\int_0^L\psi_x^2(x,t)dx-\gamma(t)I_7(t)$$

$$+c_{N_1}(g_i+\sqrt{g_i})\int_0^L\int_0^{+\infty}g(s)\eta_x^2(x,t,s)dsdx$$

$$+\left(\frac{g_0N_1+N_2}{2}+c_{12}\right)\int_0^L\int_0^{+\infty}g(s)\psi_x^2(x,t-s)dsdx$$

$$+\int_0^L J_5(x,t)\int_0^{+\infty}f(s)\psi_t(x,t-s)dsdx+\left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L\varphi_t(x,t)\psi_{xt}(x,t)dx.$$

Last, we introduce the functional J_6 and prove a crucial identity on its derivative.

Lemma 3.14. Let

$$J_6(t)=\int_0^L\int_0^t\left(\int_t^{+\infty}g(\tau-s)d\tau\right)\psi_x^2(x,s)dsdx. \tag{3.100}$$

Then, for any $\lambda\in\,]0,1[$,

$$\begin{aligned}J_6'(t)\leq&-(1-\lambda)\xi(t)J_6(t)-\lambda\int_0^L\int_0^{+\infty}g(s)\psi_x^2(x,t-s)dsdx\\&+g_0\int_0^L\psi_x^2(x,t)dx+\lambda\int_0^L\int_t^{+\infty}g(s)\psi_{0x}^2(x,s-t)dsdx.\end{aligned} \tag{3.101}$$

Proof. First, thanks to the first inequality in (3.3) and the fact that $\eta\in L_g^2(\mathbb{R}_+,H_0^1(]0,L[))$ (due to (2.16) for $n=1$), we have

$$\begin{aligned}J_6(t)&\leq\frac{1}{\xi(t)}\int_0^L\int_0^t g(t-s)\psi_x^2(x,s)dsdx\\&\leq\frac{1}{\xi(t)}\int_0^L\int_0^t g(s)\psi_x^2(x,t-s)dsdx\\&\leq\frac{2}{\xi(t)}\left(g_0\int_0^L\psi_x^2(x,t)dx+\int_0^L\int_0^{+\infty}g(s)\eta_x^2(x,t,s)dsdx\right),\end{aligned}$$

hence, according to the definition of E,

$$J_6(t)\leq\frac{4}{\xi(t)}\max\left\{\frac{g_0}{k_2-g_0},1\right\}E(t). \tag{3.102}$$

Consequently, the functional J_6 is well-defined. Moreover, by the first inequality in (3.3), a simple and direct differentiation gives

$$J_6'(t) = \left(\int_t^{+\infty} g(\tau - t)d\tau\right)\int_0^L \psi_x^2(x,t)dx - \int_0^L\int_0^t g(t-s)\psi_x^2(x,s)dsdx$$
$$= g_0\int_0^L \psi_x^2(x,t)dx - (1-\lambda)\int_0^L\int_0^t g(t-s)\psi_x^2(x,s)dsdx$$
$$-\lambda\int_0^L\int_{-\infty}^t g(t-s)\psi_x^2(x,s)dsdx + \lambda\int_0^L\int_{-\infty}^0 g(t-s)\psi_x^2(x,s)dsdx$$
$$\le g_0\int_0^L \psi_x^2(x,t)dx - (1-\lambda)\xi(t)J_6(t)$$
$$-\lambda\int_0^L\int_0^{+\infty} g(s)\psi_x^2(x,t-s)dsdx + \lambda\int_0^L\int_t^{+\infty} g(s)\psi_{0x}^2(x,s-t)dsdx,$$

which is exactly (3.101). ■

Finally, let $N_3 > 0$ and

$$\mathscr{F}_1(t) := \mathscr{L}_1(t) + N_3 J_6(t). \tag{3.103}$$

Due to (1.2), the last term in (3.99) vanishes. Taking into account the relations (3.66), (3.99), and (3.101) we get

$$\mathscr{F}_1'(t) \tag{3.104}$$

$$\le -\left(\frac{k_2N_2 + g_0^2N_1}{2} - c_{11} - g_0N_3 - \frac{\varepsilon_2}{8}\right)\int_0^L \psi_x^2(x,t)dx - \left(\frac{\rho_1}{16} - \varepsilon_1N_2\right)\int_0^L \varphi_t^2(x,t)dx$$
$$-\frac{k_1}{8}\int_0^L(\varphi_x(x,t)+\psi(x,t))^2dx - \left(\rho_2(g_0N_1 - \frac{k_1}{8}) - \frac{c_6N_2}{\varepsilon_1} - c_{11}\right)\int_0^L \psi_t^2(x,t)dx$$
$$-c_{N_1,N_2}\int_0^L\int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx + \left(\frac{N}{2} - c_{N_1,\varepsilon_2,i}\right)\int_0^L\int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx$$
$$-\gamma(t)I_7(t) - (1-\lambda)N_3\xi(t)J_6(t) + \frac{\sqrt{g_i}}{16}\int_0^L \psi_x^2(x,t)dx$$
$$+c_{N_1}(g_i + \sqrt{g_i})\int_0^L\int_0^{+\infty} g(s)\eta_x^2(x,t,s)dsdx + \lambda N_3\int_0^L\int_t^{+\infty} g(s)\psi_{0x}^2(x,s-t)dsdx$$
$$+\frac{c_9}{4}\left(\int_0^{+\infty}|f(s)|ds\right)E(t) - (\lambda N_3 - \frac{g_0N_1 + N_2}{2} - c_{12})\int_0^L\int_0^{+\infty} g(s)\psi_x^2(x,t-s)dsdx.$$

Now, we choose the different constants carefully so as to obtain some desired signs of the coefficients. First, we select

$$N_2 > \frac{2}{k_2 - g_0}(c_{11} + g_0c_{12} + 1)$$

(note that N_2 exists according to (2.1)). Next, we pick ε_1 such that

$$0 < \varepsilon_1 < \min\left\{1, \frac{\rho_1}{16N_2}\right\}.$$

Then, we choose N_1 such that

$$N_1 > \frac{1}{\rho_2 g_0}\left(c_{11} + \frac{c_6N_2}{\varepsilon_1}\right) + \frac{k_1}{8g_0}$$

(N_1 exists due to (3.1)) and

$$0 < \varepsilon_2 < 8\left(\frac{(k_2 - g_0)N_2}{2} - c_{11} - g_0 c_{12} - 1\right)$$

(clearly ε_2 exists by our choice of N_2). Next, we select N_3 and λ such that

$$N_3 = \frac{g_0 N_1 + N_2}{2} + c_{12} + \frac{1}{g_0} \quad \text{and} \quad \lambda = \frac{1}{N_3}\left(\frac{g_0 N_1 + N_2}{2} + c_{12}\right)$$

(noting that N_3 exists thanks to (3.1), and $\lambda \in]0, 1[$ according to the choice of N_3). These choices imply that, for some $c_{13}, c_{14} > 0$ (not depending on N, i and f),

$$\begin{aligned}\mathscr{F}_1'(t) \le& -c_{13}E(t) - \gamma(t)I_7(t) - c_{13}\xi(t)J_6(t)\\ &+ c_{14}\int_0^L \int_t^{+\infty} g(s)\psi_{0x}^2(x, s-t)dsdx + \frac{c_9}{4}\left(\int_0^{+\infty} |f(s)|ds\right)E(t)\\ &+ c_{14}(g_i + \sqrt{g_i})E(t) + \left(\frac{N}{2} - \tilde{c}_i\right)\int_0^L \int_0^{+\infty} g'(s)\eta_x^2(x, t, s)dsdx,\end{aligned} \tag{3.105}$$

where $\tilde{c}_i$ is a positive constant depending on i. Then, under condition (3.8), where δ_0 is defined in (3.67) with c_{13} instead of c_8, and in virtue of (3.95), we can choose i big enough so that

$$g_i + \sqrt{g_i} < \frac{1}{c_{14}}\left(c_{13} - \frac{c_9}{4}\int_0^{+\infty} |f(s)|ds\right).$$

Last, we select N big enough so that

$$N > \max\{2\tilde{c}_i, c_7\},$$

where c_7 is defined in (3.61); so the last term in (3.105) is nonpositive and, using (3.62) and (3.103),

$$(N - c_7)E(t) \le \mathscr{F}_1(t) - I_7(t) - N_3 J_6(t) \le (N + c_7)E(t), \quad \forall t \in \mathbb{R}_+. \tag{3.106}$$

According to our choices of i and N, we deduce from (3.105) that, for some $c_{15} > 0$,

$$\mathscr{F}_1'(t) \le -c_{15}E(t) - \gamma(t)I_7(t) - c_{13}\xi(t)J_6(t) + c_{14}\int_0^L \int_t^{+\infty} g(s)\psi_{0x}^2(x, s-t)dsdx. \tag{3.107}$$

Therefore, using (3.106) and (3.107), we find, for

$$\delta_1 = \min\left\{\frac{c_{15}}{N + c_7}, 1, \frac{c_{13}}{N_3}\right\},$$

$$\mathscr{F}_1'(t) \le -\delta_1 \min\{1, \gamma(t), \xi(t)\}\mathscr{F}_1(t) + c_{14}\int_0^L \int_t^{+\infty} g(s)\psi_{0x}^2(x, s-t)dsdx. \tag{3.108}$$

By integrating the differential inequality (3.108) over $[0, t]$, we get

$$\mathscr{F}_1(t) \le e^{-\delta_1\phi(t)}\left(\mathscr{F}_1(0) + c_{14}\int_0^L \int_0^t e^{\delta_1\phi(s)}\int_s^{+\infty} g(\tau)\psi_{0x}^2(x, \tau - s)d\tau dsdx\right), \tag{3.109}$$

where ϕ is defined in (3.13). Then, from (3.30), (3.106), and (3.109) we find

$$\begin{aligned}\|\mathscr{U}(t)\|_{\mathscr{H}}^2 &= 2E(t)\\ &\le \frac{2}{N - c_7}\mathscr{F}_1(t)\\ &\le \delta_2 e^{-\delta_1\phi(t)}\left(1 + \int_0^L \int_0^t e^{\delta_1\phi(s)}\int_s^{+\infty} g(\tau)\psi_{0x}^2(x, \tau - s)d\tau dsdx\right),\end{aligned} \tag{3.110}$$

which gives (3.12) with $\delta_2 = \dfrac{2}{N - c_7}\max\{c_{14}, \mathscr{F}_1(0)\}$.

IV. APPLICATIONS

Our well-posedness and asymptotic stability results for (1.1) hold for various Timoshenko-type systems. We present here some examples.

A. Timoshenko-heat

Let us consider coupled Timoshenko-heat systems under Fourier's law of heat conduction

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x(x,t)+\psi(x,t))_x = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t)+\psi(x,t)) + k_4\theta_x(x,t) \\ \qquad\qquad + \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds + \int_0^{+\infty} f(s)\psi_t(x,t-s)ds = 0, \\ \rho_3\theta_t(x,t) - k_3\theta_{xx}(x,t) + k_4\psi_{xt}(x,t) = 0, \\ \varphi(0,t) = \psi(0,t) = \theta(0,t) = \varphi(L,t) = \psi(L,t) = \theta(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t), \quad \psi_t(x,-t) = \psi_1(x,t), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{4.1}$$

where θ denotes the temperature difference (see Ref. 9 for more details).

Under **(A1)** and **(A2)**, system (4.1) can be formulated in the abstract form (2.4), where $\mathscr{U} = (\varphi, \psi, \varphi_t, \psi_t, \theta, \eta)^T$, $\mathscr{U}_0 = (\varphi_0, \psi_0(\cdot,0), \varphi_1, \psi_1(\cdot,0), \theta_0, \eta_0)^T \in \mathscr{H}$,

$$\mathscr{H} = (H_0^1(]0,L[))^2 \times (L^2(]0,L[))^3 \times L_g^2(\mathbb{R}_+, H_0^1(]0,L[)),$$

and the operators $\mathscr{A}$ and $\mathscr{B}$ are given by (ε_0 and c_0 are defined, respectively, in (2.8) and (2.9)),

$$\mathscr{B}(w_1,w_2,w_3,w_4,w_5,w_6)^T = \left(0,0,0,\frac{\|f\|_\infty}{\rho_2}w_4, 0, \epsilon_0 w_6\right)^T$$

and

$$\mathscr{A}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{pmatrix} = \begin{pmatrix} w_3 \\ w_4 \\ \frac{k_1}{\rho_1}(w_{1x}+w_2)_x \\ \tilde{w}_4 \\ \frac{1}{\rho_3}(k_3 w_{5x} - k_4 w_4)_x \\ -w_{6s} - \epsilon_0 w_6 + w_4 \end{pmatrix},$$

where

$$\tilde{w}_4 = \frac{1}{\rho_2}(k_2 - g_0)w_{2xx} - \frac{k_1}{\rho_2}(w_{1x}+w_2) - \frac{k_4}{\rho_2}w_{5x} - \frac{\|f\|_\infty}{\rho_2}w_4 + \frac{1}{\rho_2}\int_0^{+\infty} g(s)w_{6xx}(s)ds - \frac{1}{\rho_2}\int_0^{+\infty} f(s)w_{6s}(s)ds.$$

From (2.1), $\mathscr{H}$ endowed with the inner product, for $W = (w_1,w_2,w_3,w_4,w_5,w_6)^T$ and $\tilde{W}^T = (\tilde{w}_1,\tilde{w}_2,\tilde{w}_3,\tilde{w}_4,\tilde{w}_5,\tilde{w}_6)^T$,

$$\langle W, \tilde{W}\rangle_{\mathscr{H}} = \int_0^L ((k_2-g_0)w_{2x}(x)\tilde{w}_{2x}(x) + k_1(w_{1x}(x)+w_2(x))(\tilde{w}_{1x}(x)+\tilde{w}_2(x)))\,dx + \int_0^L (\rho_1 w_3(x)\tilde{w}_3(x) + \rho_2 w_4(x)\tilde{w}_4(x) + \rho_3 w_5(x)\tilde{w}_5(x))dx + \langle w_6, \tilde{w}_6\rangle_{L_g^2(\mathbb{R}_+, H_0^1(]0,L[))}$$

is a Hilbert space. Similar to the proof of Theorem 2.1 for (1.1), we can prove that the linear operator $-\mathscr{A}$ is a maximal monotone operator, and $\mathscr{B}$ is Lipschitz continuous. Then $\mathscr{A}+\mathscr{B}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$. Consequently, (2.4) associated with (4.1) is well-posed in the sense of Theorem 2.1.

On the other hand, under **(A1)–(A4)**, Theorem 3.2 holds for (4.1). We use the same functionals and arguments given in Sec. III (see also Ref. 19 in case $f \equiv 0$).

B. Timoshenko-thermoelasticity

Our approach can be applied to the following Timoshenko-thermoelasticity system of type III,

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x(x,t)+\psi(x,t))_x = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t)+\psi(x,t)) + k_4\theta_{xt}(x,t) \\ \qquad + \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds + \int_0^{+\infty} f(s)\psi_t(x,t-s)ds = 0, \\ \rho_3\theta_{tt}(x,t) - k_3\theta_{xx}(x,t) + k_4\psi_{xt}(x,t) - k_5\theta_{xxt}(x,t) = 0, \\ \varphi(0,t) = \psi(0,t) = \theta_x(0,t) = \varphi(L,t) = \psi(L,t) = \theta_x(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t), \quad \psi_t(x,-t) = \psi_1(x,t), \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \end{cases} \tag{4.2}$$

which models the transverse vibrations of a thick beam, taking into account the heat conduction introduced in Refs. 10–12.

Under **(A1)** and **(A2)**, system (4.2) also can be formulated in the abstract form (2.4), where $\mathscr{U} = (\varphi, \psi, \theta, \varphi_t, \psi_t, \theta_t, \eta)^T$, $\mathscr{U}_0 = (\varphi_0, \psi_0(\cdot, 0), \theta_0, \varphi_1, \psi_1(\cdot, 0), \theta_1, \eta_0)^T \in \mathscr{H}$,

$$\mathscr{H} = (H_0^1(]0,L[))^2 \times H_*^1(]0,L[) \times (L^2(]0,L[))^2 \times L_*^2(]0,L[) \times L_g^2(\mathbb{R}_+, H_0^1(]0,L[)),$$

$$H_*^1(]0,L[) = \left\{ v \in H^1(]0,L[) : \int_0^L v(x)dx = 0 \right\}, \quad L_*^2(]0,L[) = \left\{ v \in L^2(]0,L[) : \int_0^L v(x)dx = 0 \right\}$$

and the operators $\mathscr{A}$ and $\mathscr{B}$ are given by

$$\mathscr{B}(w_1, w_2, w_3, w_4, w_5, w_6, w_7)^T = \left(0, 0, 0, 0, \frac{\|f\|_\infty}{\rho_2} w_5, 0, \epsilon_0 w_7\right)^T$$

and

$$\mathscr{A}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \\ w_7 \end{pmatrix} = \begin{pmatrix} w_4 \\ w_5 \\ w_6 \\ \frac{k_1}{\rho_1}(w_{1x}+w_2)_x \\ \tilde{w}_5 \\ \frac{1}{\rho_3}(k_3 w_{3x} - k_4 w_5 + k_5 w_{6x})_x \\ -w_{7s} - \epsilon_0 w_7 + w_5 \end{pmatrix},$$

where

$$\tilde{w}_5 = \frac{1}{\rho_2}(k_2 - g_0)w_{2xx} - \frac{k_1}{\rho_2}(w_{1x}+w_2) - \frac{k_4}{\rho_2}w_{6x} - \frac{\|f\|_\infty}{\rho_2}w_5 + \frac{1}{\rho_2}\int_0^{+\infty} g(s)w_{7xx}(s)ds - \frac{1}{\rho_2}\int_0^{+\infty} f(s)w_{7s}(s)ds.$$

From (2.1), $\mathscr{H}$ endowed with the inner product, for $W = (w_1, w_2, w_3, w_4, w_5, w_6, w_7)^T$ and $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4, \tilde{w}_5, \tilde{w}_6, \tilde{w}_7)^T$,

$$\begin{aligned}\langle W, \tilde{W}\rangle_{\mathscr{H}} &= \int_0^L ((k_2 - g_0)w_{2x}(x)\tilde{w}_{2x}(x) + k_1(w_{1x}(x) + w_2(x))(\tilde{w}_{1x}(x) + \tilde{w}_2(x)))\,dx \\ &+ \int_0^L (k_3 w_{3x}(x)\tilde{w}_{3x}(x) + \rho_1 w_4(x)\tilde{w}_4(x) + \rho_2 w_5(x)\tilde{w}_5(x) + \rho_3 w_6(x)\tilde{w}_6(x))dx \\ &+ \langle w_7, \tilde{w}_7\rangle_{L^2_g(\mathbb{R}_+, H^1_0(]0,L[))}\end{aligned}$$

is a Hilbert space. Similar to the proof of Theorem 2.1 for (1.1), we can prove that $\mathscr{A} + \mathscr{B}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$ by proving that $-\mathscr{A}$ is maximal monotone and $\mathscr{B}$ is Lipschitz continuous, and then we deduce that (2.4) associated with (4.2) is well-posed in the sense of Theorem 2.1.

On the other hand, under **(A1)–(A4)**, Theorem 3.2 holds for (4.2), where, here, $E(t) = \frac{1}{2}\|(\varphi, \psi, \tilde{\theta}, \varphi_t, \psi_t, \tilde{\theta}_t, \eta)\|^2_{\mathscr{H}}$ and

$$\tilde{\theta}(x, t) = \theta(x, t) - \frac{t}{L}\int_0^L \theta_1(y)dy - \frac{1}{L}\int_0^L \theta_0(y)dy;$$

so

$$\int_0^L \tilde{\theta}(x, t)dx = 0,$$

and then (2.9) is applicable for $\tilde{\theta}$. For the proof, we use the same functionals as in Sec. III and some arguments in Ref. 19 considered in case $f \equiv 0$.

C. Porous thermoelastic

Our approach can also be applied to the following porous thermoelastic system:

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x(x,t) + \psi(x,t))_x + k_4\theta_x(x,t) = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t) + \psi(x,t)) - k_5\theta(x,t) \\ \qquad + \int_0^{+\infty} g(s)\psi_{xx}(x, t-s)ds + \int_0^{+\infty} f(s)\psi_t(x, t-s)ds = 0, \\ \rho_3\theta_t(x,t) - k_3\theta_{xx}(x,t) + k_4\varphi_{xt}(x,t) + k_5\psi_t(x,t) = 0, \\ \varphi(0,t) = \psi(0,t) = \theta(0,t) = \varphi(L,t) = \psi(L,t) = \theta(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t), \quad \psi_t(x,-t) = \psi_1(x,t), \quad \theta(x,0) = \theta_0(x). \end{cases} \tag{4.3}$$

Under **(A1)** and **(A2)**, system (4.3) can be formulated in the abstract form (2.4), where $\mathscr{U}$, $\mathscr{U}_0$, $\mathscr{B}$, and $\langle\cdot,\cdot\rangle_{\mathscr{H}}$ are defined as in case (4.1), and the operator $\mathscr{A}$ is given by

$$\mathscr{A}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{pmatrix} = \begin{pmatrix} w_3 \\ w_4 \\ \frac{k_1}{\rho_1}(w_{1x} + w_2)_x - \frac{k_4}{\rho_1}w_{5x} \\ \tilde{w}_4 \\ \frac{1}{\rho_3}(k_3 w_{5x} - k_4 w_3)_x - \frac{k_5}{\rho_3}w_4 \\ -w_{6s} - \epsilon_0 w_6 + w_4 \end{pmatrix},$$

where

$$\begin{aligned}\tilde{w}_4 &= \frac{1}{\rho_2}(k_2 - g_0)w_{2xx} - \frac{k_1}{\rho_2}(w_{1x} + w_2) + \frac{k_5}{\rho_2}w_5 - \frac{\|f\|_\infty}{\rho_2}w_4 \\ &+ \frac{1}{\rho_2}\int_0^{+\infty} g(s)w_{6xx}(s)ds - \frac{1}{\rho_2}\int_0^{+\infty} f(s)w_{6s}(s)ds.\end{aligned}$$

As in the previous applications, the proof of Theorem 2.1 for (4.3) is similar to the one given in Sec. II for (1.1).

Under **(A1)–(A4)**, Theorem 3.2 holds also for (4.3) with the same proof as in Sec. III.

D. Discrete time delay

Similar well-posedness results to the ones in Theorem 2.1 and, under (1.2), the stability estimates (3.9) and (3.12) hold in the case of discrete time delay for (1.1) as well as for (4.1)–(4.3). Let us discuss the case of (1.1) with discrete time delay (the cases of (4.1)–(4.3) with discrete time delay can be treated similarly)

$$\begin{cases} \rho_1\varphi_{tt}(x,t) - k_1(\varphi_x(x,t)+\psi(x,t))_x = 0, \\ \rho_2\psi_{tt}(x,t) - k_2\psi_{xx}(x,t) + k_1(\varphi_x(x,t)+\psi(x,t)) \\ \qquad\qquad + \int_0^{+\infty} g(s)\psi_{xx}(x,t-s)ds + \mu\psi_t(x,t-\tau) = 0, \\ \varphi(0,t) = \psi(0,t) = \varphi(L,t) = \psi(L,t) = 0, \\ \varphi(x,0) = \varphi_0(x), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,-t) = \psi_0(x,t), \quad \psi_t(x,0) = \psi_1(x), \quad \psi_t(x,t-\tau) = f_0(x,t-\tau) \ (t \in]0,\tau[), \end{cases} \tag{4.4}$$

where $\tau \in]0,+\infty[$, $\mu \in \mathbb{R}^*$ and $(\varphi_0, \psi_0, \varphi_1, \psi_1, f_0)$ are given initial data.

We prove briefly that (4.4) is well posed under the assumption **(A1)**, and it is stable provided that (1.2), **(A1),** and **(A3)** hold and $|\mu|$ is small enough, and we establish decay estimates similar to (3.9) and (3.12).

1. Well-posedness

Following the idea in Ref. 28 (see also Refs. 29–31) to deal with the delay term by considering a new auxiliary variable z, we can formulate the system (4.4) in the abstract form (2.4), where $\mathscr{U} = (\varphi, \psi, \varphi_t, \psi_t, \eta, z)^T$, $\mathscr{U}_0 = (\varphi_0, \psi_0(\cdot,0), \varphi_1, \psi_1, \eta_0, z_0)^T \in \mathscr{H}$,

$$\mathscr{H} = (H_0^1(]0,L[))^2 \times (L^2(]0,L[))^2 \times L_g^2(\mathbb{R}_+, H_0^1(]0,L[)) \times L^2(]0,1[, L^2(]0,L[)),$$

$$L^2(]0,1[, L^2(]0,L[)) = \left\{ w :]0,1[\to L^2(]0,L[), \quad \int_0^L \int_0^1 w^2(x,p)dpdx < +\infty \right\}$$

endowed with the inner product

$$\langle w_1, w_2 \rangle_{L^2(]0,1[,L^2(]0,L[))} = \int_0^L \int_0^1 w_1(x,p)w_2(x,p)dpdx,$$

and

$$\begin{cases} z(x,t,p) = \psi_t(x,t-\tau p), \\ z_0(x,p) = z(x,0,p) = f_0(x,-\tau p). \end{cases} \tag{4.5}$$

The operators $\mathscr{A}$ and $\mathscr{B}$ are linear and given by

$$\mathscr{B}(w_1,w_2,w_3,w_4,w_5,w_6)^T = \frac{|\mu|}{\rho_2}\Big(0,0,0,w_4,0,0\Big)^T \tag{4.6}$$

and

$$\mathscr{A}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \\ w_6 \end{pmatrix} = \begin{pmatrix} w_3 \\ w_4 \\ \frac{k_1}{\rho_1}(w_{1x}+w_2)_x \\ \tilde{w}_4 \\ -w_{5s}+w_4 \\ -\frac{1}{\tau}w_{6p} \end{pmatrix}, \tag{4.7}$$

where

$$\tilde{w}_4 = \frac{1}{\rho_2}(k_2 - g_0)w_{2xx} - \frac{k_1}{\rho_2}(w_{1x} + w_2) - \frac{|\mu|}{\rho_2}w_4 - \frac{\mu}{\rho_2}w_6(1) + \frac{1}{\rho_2}\int_0^{+\infty} g(s)w_{5xx}(s)ds. \tag{4.8}$$

The domains $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}(\mathscr{B})$ of $\mathscr{A}$ and $\mathscr{B}$, respectively, are given by

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{c} (w_1, w_2, w_3, w_4, w_5, w_6)^T \in \mathscr{H},\ w_5(0) = 0,\ w_6(0) = w_4 \\ w_{6p} \in L^2(]0, 1[, L^2(]0, L[)),\ w_{5s} \in L^2_g(\mathbb{R}_+, H^1_0(]0, L[)),\ w_3, w_4 \in H^1_0(]0, L[) \\ w_1 \in H^2(]0, L[),\ (k_2 - g_0)w_{2xx} + \int_0^{+\infty} g(s)w_{5xx}(s)ds \in L^2(]0, L[) \end{array} \right\} \tag{4.9}$$

and $\mathscr{D}(\mathscr{B}) = \mathscr{H}$. Keeping in mind the definition (4.5) of z, we have

$$\begin{cases} \tau z_t(x, t, p) + z_p(x, t, p) = 0, \\ z(x, t, 0) = \psi_t(x, t). \end{cases} \tag{4.10}$$

Therefore, we conclude from (2.13) and (4.10) that the systems (4.4) and (2.4) are equivalent.

Clearly, thanks to (2.1), $\mathscr{H}$ endowed with the inner product, for $W = (w_1, w_2, w_3, w_4, w_5, w_6)^T$ and $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4, \tilde{w}_5, \tilde{w}_6)^T$,

$$\begin{aligned} \langle W, \tilde{W}\rangle_{\mathscr{H}} &= \int_0^L ((k_2 - g_0)w_{2x}(x)\tilde{w}_{2x}(x) + k_1(w_{1x}(x) + w_2(x))(\tilde{w}_{1x}(x) + \tilde{w}_2(x)))\,dx \\ &\quad + \int_0^L (\rho_1 w_3(x)\tilde{w}_3(x) + \rho_2 w_4(x)\tilde{w}_4(x))dx \\ &\quad + \langle w_5, \tilde{w}_5\rangle_{L^2_g(\mathbb{R}_+, H^1_0(]0,L[))} + \tau|\mu|\,\langle w_6, \tilde{w}_6\rangle_{L^2(]0,1[, L^2(]0,L[))} \end{aligned}$$

is a Hilbert space and $\mathscr{D}(\mathscr{A}) \subset \mathscr{H}$ with dense embedding. Similar to the proof of Theorem 2.1 for (1.1), we can prove that the linear operator $-\mathscr{A}$ is a maximal monotone operator, and $\mathscr{B}$ is Lipschitz continuous; the proof is similar to the one given in Ref. 15 for an abstract evolution equation with infinite memory and discrete time delay. Then $\mathscr{A} + \mathscr{B}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$ (see Ref. 34: Chap. 3 – Theorem 1.1). Consequently, (2.4) associated with (4.4) is well-posed in the sense of Theorem 2.1 (see Refs. 22 and 34).

2. Stability

We prove here that the system (2.4) associated with (4.4) is stable under **(A1)**, **(A3),** and (1.2), and provided that $|\mu|$ is small enough. We provide two decay estimates depending on ξ.

Theorem 4.1. *Assume* **(A1)**, **(A3),** *and* (1.2) *hold. Then there exists a positive constant* δ_0 *independent of* μ *such that, if*

$$|\mu| < \delta_0, \tag{4.11}$$

then, for any $\mathscr{U}_0 \in \mathscr{H}$, *there exist positive constants* δ_1 and δ_2 *such that the weak solution of* (2.4) *associated with (4.4) satisfies (3.9) with* $\phi(t) = t$ *if (3.2) holds, and it satisfies (3.12) with* $\phi(t) = \int_0^t \min\{1, \xi(s)\}ds$ *if (3.2) does not hold and (3.3) holds.*

Proof. Let $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, so that all the calculations below are justified. By a simple density argument, the decay estimates in Theorem 4.1 remain valid for any weak solution ($\mathscr{U}_0 \in \mathscr{H}$). First, as in Ref. 15, we provide a bound on the derivative of the energy functional E (defined in (3.30)) associated with the solution of (2.4). We find

$$E'(t) \leq \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\eta_x^2(x,t,s)dsdx + |\mu| \int_0^L \psi_t^2(x,t)dx. \tag{4.12}$$

Note that, as in the distributed delay case, the sign of E' cannot be determined directly from (4.12).

The proof of Theorem 4.1 is identical to the one given for system (1.1). The only modification of the proof given in Sec. III for (3.9) and (3.12) is the use of the following functional J_7, introduced in Ref. 28, instead of I_7 (defined in (3.52)) to get a crucial estimate on the discrete delay term.

Lemma 4.2. The functional

$$J_7(t) = \tau e^{2\tau} \int_0^L \int_0^1 e^{-2\tau p} z^2(x,t,p)dpdx$$

satisfies

$$J_7'(t) = -2J_7(t) + e^{2\tau} \int_0^L \psi_t^2(x,t)dx - \int_0^L z^2(x,t,1)dx. \tag{4.13}$$

Proof. See, for example, Ref. 20. ■

Now, defining $\mathscr{L}_1$ and $\mathscr{F}_1$, respectively, by (3.57) and (3.103) with J_7 instead of I_7, we get, for some positive constants $\delta_1, c_{16} > 0$ (as for (3.69) and (3.108)),

$$\mathscr{L}_1'(t) \leq -\delta_1 \mathscr{L}_1(t), \quad \forall t \in \mathbb{R}_+$$

when (3.2) holds, and

$$\mathscr{F}_1'(t) \leq -\delta_1 \min\{1, \xi(t)\}\mathscr{F}_1(t) + c_{16}\int_0^L \int_t^{+\infty} g(s)\psi_{0x}^2(x,t-s)dsdx, \quad \forall t \in \mathbb{R}_+$$

when (3.2) does not hold and (3.3) holds. The rest of the proof carries out as in the case of distributed delay. ■

V. GENERAL COMMENTS AND ISSUES

1. In the case of distributed time delay (1.1) and (4.1)–(4.3), the decay estimates (3.9) and (3.12) are obtained only for classical solutions (that is, for $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$), since the functional I_7 defined in (3.52) is not well-defined for weak solutions; that is, when $\mathscr{U}_0 \in \mathscr{H}$ (see (3.55)).
2. When (1.2) does not hold (which is more interesting from the physical point of view), proving the stability of (4.1)–(4.3) with discrete delay instead of the distributed one (that is, the infinite integral depending on f is replaced by $\mu\psi_t(x, t-\tau)$) and (4.4), seems a delicate question, since the second energy E_1 (defined in (3.71) with $k = 1$) satisfies

   $$E_1'(t) \leq \frac{1}{2}\int_0^L \int_0^{+\infty} g'(s)\eta_{xt}^2(x,t,s)dsdx + |\mu| \int_0^L \psi_{tt}^2(x,t)dx;$$

   so E_1 is not necessarily nonincreasing due to the term depending on μ, this term cannot be absorbed by E itself even if $|\mu|$ is supposed small enough. In the case of distributed delay, the key of solution was the introduction of the functional I_8 defined in (3.78).
3. In the case of absence of delay (i.e., $f \equiv 0$ and $\mu = 0$), it is well-known that (see Refs. 8, 9, and 19), the systems (1.1), (4.1), (4.2), and (4.3) are not exponentially stable when (1.2) does not hold, but they are dissipative (with respect to E) and the energy satisfies some weaker decay estimates depending on the (exponential or arbitrary) decay of g at infinity and the regularity of the initial data $\mathscr{U}_0$. In the particular case of exponential decay (3.2), the decay rate obtained in Refs. 8, 9, and 19 is of type $\frac{1}{t^n}$, which is stronger than the one given in (3.11). This fact is caused by the nondissipativeness character of (1.1), (4.1), (4.2), and (4.3) generated by the presence of delay.

4. In both distributed and discrete delay cases, the arguments presented in this paper can be adapted to the case where the delay is considered in the first equation; so similar well-posedness and stability results can be obtained. However, proving the stability of (1.1) in the case where the memory is considered in the first equation seems a delicate question.
5. The estimate (3.12) does not imply (3.29); that limit depends on the connection between the growths at infinity of g, f, and $\int_0^L \psi_{0x}^2(x,\cdot)dx$.
   If ψ_{0x} satisfies
$$\int_0^L \int_0^{+\infty} \psi_{0x}^2(x,s)dsdx < +\infty, \tag{5.1}$$
   then (3.9) holds, where ϕ is defined in (3.13). The idea of proof relies on the following functional J_8 instead of J_6 (defined in (3.100)):
$$J_8(t) := \int_0^L \int_0^{+\infty} g(s) \int_{t-s}^t \psi_x^2(x,\tau)d\tau dsdx. \tag{5.2}$$
   The functional J_8 is well-defined and satisfies, for all $\lambda \in]0,1[$,
$$J_8'(t) \le -(1-\lambda)\xi(t)J_8(t) - \lambda \int_0^L \int_0^{+\infty} g(s)\psi_x^2(x,t-s)dsdx + g_0 \int_0^L \psi_x^2(x,t)dx, \tag{5.3}$$
   which is similar to (3.101) but without the last term of (3.101). See Ref. 20 in case (1.5) for more details concerning the idea of proof.
6. If the second initial data ψ_1 satisfies
$$\int_0^L \int_0^{+\infty} \psi_1^2(x,s)dsdx < +\infty, \tag{5.4}$$
   the condition (3.5) is not needed, and the estimates (3.9) and (3.12) hold for $\gamma : \mathbb{R}_+ \to \mathbb{R}_+^*$ of class $C(\mathbb{R}_+, \mathbb{R}_+^*)$ and nonincreasing such that
$$|f(t-s)| \ge \gamma(t) \int_t^{+\infty} |f(\tau - s)|d\tau, \quad \forall t \in \mathbb{R}_+,\ \forall s \in [0,t]. \tag{5.5}$$
   Replacing (3.5) by (5.5) allows f to have more general decay rate at infinity; so the growth of f does not depend on the one of g. The idea of proof (see Ref. 20 for (1.5)) relies on the following functional J_9 instead of I_7 (defined in (3.52)):
$$J_9(t) := 2\int_0^L \int_0^{+\infty} |f(s)| \int_{t-s}^t \psi_t^2(x,\tau)d\tau dsdx. \tag{5.6}$$
   As for (5.3) with $\lambda = \frac{1}{2}$, we find
$$J_9'(t) \le 2\alpha g_0 \int_0^L \psi_t^2(x,t)dx - \gamma(t)J_9(t) - \int_0^L \int_0^{+\infty} |f(s)|\psi_t^2(x,t-s)dsdx, \tag{5.7}$$
   which is similar to (3.54) with $2\alpha g_0$ and J_9 instead of $g(0)\beta_0$ and I_7, respectively. Similarly, if both (5.1) and (5.4) are satisfied, then (3.9) holds under (5.5) instead of (3.5).
7. As in the distributed delay case, if the first initial data ψ_{0x} satisfies (5.1), then (3.9) holds for the discrete time delay case (4.4), where $\phi(t) = \int_0^t \min\{1, \xi(s)\}ds$. The idea of proof consists in replacing J_6 (defined in (3.100)) by J_8 (defined in (5.2)).

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