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Energy Decay for a Damped Nonlinear Coupled System

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This paper proves the well-posedness and uniform stabilization of a nonlinear coupled system. We estimate the energy decay rate by using the multiplier method.

Key Words: uniform stabilization; strong damping; integral inequality; coupled systems.

1. INTRODUCTION AND STATEMENT OF THE MAIN RESULT

In this paper we are concerned with the energy decay of the solution to the initial boundary value problem for the nonlinear coupled wave equation and Petrovsky system

$$u_1'' + \Delta^2 u_1 + a u_2 + g_1(u_1') = 0 \qquad \text{in } \Omega \times \mathbb{R}^+ \tag{1.1}$$

$$u_2'' - \Delta u_2 + a u_1 + g_2(u_2') = 0 \qquad \text{in } \Omega \times \mathbb{R}^+ \tag{1.2}$$

$$\partial_\nu u_1 = u_1 = u_2 = 0 \qquad \text{on } \Gamma \times \mathbb{R}^+ \tag{1.3}$$

$$u_i(x,0) = u_i^0(x) \quad \text{and} \quad u_i'(x,0) = u_i^1(x) \qquad \text{on } \Omega,\ i = 1,2, \tag{1.4}$$

where Ω is a bounded open domain in $\mathbb{R}^n$ with smooth boundary Γ of class C^4; ν is the outward unit normal vector to Γ, $\mathbb{R}^+ = [0,\infty)$; and a: $\Omega \to \mathbb{R}$, g_1, g_2: $\mathbb{R} \to \mathbb{R}$ are some given functions. Under suitable assumptions we shall prove that this system is well posed and dissipative, and we shall obtain explicit decay rate estimates.

Our work was motivated by some recent results of Komornik et Loreti [4]. They proved the observability of system (1.1), (1.2), (1.4) with $g_i = 0$,

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$a \in \mathbb{R}$, and the boundary condition

$$u_1 = u_2 = \Delta u_1 = 0 \qquad \text{on } \Gamma \times \mathbb{R}^+$$

by a new approach based on a generalization of some results of nonharmonic analysis to vector-valued functions. However, this approach forced them to restrict themselves to the case where Ω is an open ball.

We will show that in fact the multiplier method can be adapted to the study of the stability of our problem, in any bounded open domain Ω of class C^4. In fact, we shall give a new approach based on a direct adaptation of the usual multiplier method (cf., e.g., [5]). This method leads to decay rate estimates in the nonlinear case, under the assumption that a is sufficiently small.

Throughout the paper we shall make the following assumptions:

(H1) The function a belongs to $L^\infty(\Omega)$ and

$$\|a\|_{L^\infty(\Omega)} < \frac{1}{\sqrt{c'c''}}, \tag{1.5}$$

where $c', c'' > 0$ (depending only on the geometry of Ω) are the constants such that

$$\|u\|^2_{H^2(\Omega)} \le c' \int_\Omega (\Delta u)^2 \, dx, \qquad \forall u \in H_0^2(\Omega),$$

$$\|u\|^2_{H^1(\Omega)} \le c'' \int_\Omega |\nabla u|^2 \, dx, \qquad \forall u \in H_0^1(\Omega).$$

(H2) The functions g_i are continuous and nondecreasing and $g_i(0) = 0$. Furthermore, there exists a constant $c_i > 0$, $i = 1, 2$, such that

$$|g_i(x)| \le c_i(1 + |x|), \qquad \forall x \in \mathbb{R}. \tag{1.6}$$

Remark. It is possible to weaken the growth assumption (1.6) as was done for the study of the wave equation in [5]. To keep this paper from becoming too long, we consider only the case of (1.6).

If $u = (u_1, u_2)$ is a solution of the problem (1.1)–(1.4), then we define its energy E: $\mathbb{R}^+ \to \mathbb{R}$ by the following formula:

$$E(t) = \tfrac{1}{2}\int_\Omega (u_1')^2 + (u_2')^2 + (\Delta u_1)^2 + |\nabla u_2|^2 \, dx + \int_\Omega a u_1 u_2 \, dx. \tag{1.7}$$

By assumption (1.5) we have

$$\int_\Omega a u_1 u_2\,dx \geq -\frac{1}{2}\|a\|_{L^\infty(\Omega)} \int_\Omega \sqrt{\frac{c''}{c'}}\,(u_1)^2 + \sqrt{\frac{c'}{c''}}\,(u_2)^2\,dx$$
$$\geq -\frac{1}{2}\sqrt{c'c''}\,\|a\|_{L^\infty(\Omega)} \int_\Omega (\Delta u_1)^2 + |\nabla u_2|^2\,dx.$$

Hence

$$E(t) \geq \tfrac{1}{2}\int_\Omega (u_1')^2 + (u_2')^2 + \left(1 - \sqrt{c'c''}\,\|a\|_{L^\infty(\Omega)}\right)\left((\Delta u_1)^2 + |\nabla u_2|^2\right) dx$$
$$\geq 0.$$

Then E is a nonnegative function.

Let us introduce for brevity the Hilbert spaces $H = L^2(\Omega) \times L^2(\Omega)$, $V = H_0^2(\Omega) \times H_0^1(\Omega)$, and $W = (H^4(\Omega) \cap H_0^2(\Omega)) \times (H^2(\Omega) \cap H_0^1(\Omega))$, endowed with the norms defined by

$$\|(u_1, u_2)\|_H^2 = \int_\Omega (u_1)^2 + (u_2)^2\,dx$$

$$\|(u_1, u_2)\|_V^2 = \int_\Omega (\Delta u_1)^2 + |\nabla u_2|^2\,dx + 2\int_\Omega a u_1 u_2\,dx$$

and

$$\|(u_1, u_2)\|_W^2 = \int_\Omega \left(\Delta^2 u_1\right)^2 + |\nabla \Delta u_1|^2 + (\Delta u_2)^2\,dx = \|(u_1, u_2)\|_V^2.$$

Thanks to hypothesis (1.5) the second expression defines in V a norm, equivalent to the norm induced by $H^2(\Omega) \times H^1(\Omega)$, and then the latter expression defines in W a norm, equivalent to the norm induced by $H^4(\Omega) \times H^2(\Omega)$. Therefore we have a dense and compact imbedding $W \subset V \subset H$ by Rellich's theorem. Identifying H with its dual H', we obtain the diagram

$$W \subset V \subset H = H' \subset V' \subset W'$$

with dense and compact imbeddings.

We shall establish a well-posedness and a regularity result:

THEOREM 1. 1. *Given* $(u_1^0, u_2^0) \in V$ *and* $(u_1^1, u_2^1) \in H$ *arbitrarily, the problem* (1.1)–(1.4) *has a unique weak solution satisfying*

$$(u_1, u_2) \in C(\mathbb{R}^+; V) \cap C^1(\mathbb{R}^+; H). \tag{1.8}$$

Furthermore, its energy is nonincreasing.

2. *Assume that* $(u_1^0, u_2^0) \in W$ *and* $(u_1^1, u_2^1) \in V$. *Then the solution of* (1.1)–(1.4) *satisfies*

$$(u_1', u_2') \in L^\infty(\mathbb{R}^+; V), \quad (u_1'', u_2'') \in L^\infty(\mathbb{R}^+; H) \tag{1.9}$$

and

$$(u_1, u_2) \in L^\infty(\mathbb{R}^+; W). \tag{1.10}$$

Turning to the stability of system (1.1)–(1.4), let us assume that there exists a number $p \geq 1$ and four positive constants α_i, β_i, $i = 1, 2$, such that

$$\beta_i \min\{|x|, |x|^p\} \leq |g_i(x)| \leq \alpha_i \max\{|x|, |x|^{1/p}\} \qquad \forall x \in \mathbb{R}. \tag{1.11}$$

THEOREM 2. *Assume* (1.11). *Then every weak solution of* (1.1)–(1.4) *satisfies the estimate*

$$E(t) \leq c_0 E(0) e^{-\omega t}, \qquad \forall t \geq 0, \quad \text{if } p = 1, \tag{1.12}$$

while c_0, ω *are positive constants, independent of the initial data, and*

$$E(t) \leq c_0'(1 + t)^{-2/(p-1)}, \qquad \forall t \geq 0, \quad \text{if } p > 1, \tag{1.13}$$

where c_0' *is a constant depending on the initial energy* $E(0)$.

Remark. Using a technique of [2, 3], we could consider more general growth conditions than (1.11).

2. WELL-POSEDNESS AND REGULARITY

Let us introduce the duality mapping $A: V \to V'$ and define another, nonlinear mapping $B: V \to V'$ by

$$\langle Bu, z\rangle_{V',V} = \int_\Omega g_1(u_1) z_1 + g_2(u_2) z_2 \, dx,$$
$$u = (u_1, u_2), \quad z = (z_1, z_2) \in V.$$

Thanks to assumption (1.6) this definition is correct.

Choose $z = (z_1, z_2) \in V$ arbitrarily. Assume for the moment that (1.1)–(1.4) has a smooth solution $u = (u_1, u_2)$. Multiplying equations (1.1), (1.2), respectively, by z_1, z_2; integrating by parts their sum in Ω; and finally using boundary condition (1.3) we easily obtain

$$\langle u'' + Au + Bu', z\rangle_{V',V} = 0, \qquad \forall z \in V. \tag{2.1}$$

Therefore we deduce from (1.1)–(1.4) that

$$u'' + Au + Bu' = 0 \text{ in } \mathbb{R}^+, \quad u(0) = \left(u_1^0, u_2^0\right), \quad u'(0) = \left(u_1^1, u_2^1\right). \tag{2.2}$$

Putting $U = (u, z) := (u, u')$ and $\mathscr{A}U := (-z, Au + Bz)$, we can rewrite (2.2) as a first-order system:

$$U' + \mathscr{A}U = 0 \text{ in } \mathbb{R}^+, \quad U(0) = (u^0, u^1), \tag{2.3}$$

where $u^0 = (u_1^0, u_2^0)$, $u^1 = (u_1^1, u_2^1)$. It is natural to consider the operator $\mathscr{A}$ in the Hilbert space $\mathscr{H} := V \times H$. Therefore we define its domain by

$$D(\mathscr{A}) := \{U = (u, z) \in V \times V\colon Au + Bz \in H\},$$

and we define the solution of (1.1)–(1.4) as that of (2.3).

LEMMA 2.1. *$\mathscr{A}$ is a maximal monotone operator in $\mathscr{H}$.*

Proof. The monotonicity of $\mathscr{A}$ follows from the nondecreasingness of g_1, g_2. Indeed, given $U = (u, z)$, $\tilde{U} = (\tilde{u}, \tilde{z}) \in D(\mathscr{A})$ arbitrarily, we have

$$\begin{aligned}
&\langle \mathscr{A}U - \mathscr{A}\tilde{U}, U - \tilde{U}\rangle_{\mathscr{H}} \\
&\quad = \langle \tilde{z} - z, u - \tilde{u}\rangle_V + \langle Au - A\tilde{u} + Bz - B\tilde{z}, z - \tilde{z}\rangle_H \\
&\quad = \langle Bz - B\tilde{z}, z - \bar{z}\rangle_{V',V} \\
&\quad = \int_\Omega \sum_{i=1}^{i=2} \left(g_i(z_i) - g_i(\tilde{z}_i)\right)(z_i - \tilde{z}_i)\, dx \geq 0.
\end{aligned}$$

It remains to show that for any given $U^0 = (u^0, z^0) \in \mathscr{H}$ there exists $U = (u, z) \in D(\mathscr{A})$ such that $(I + \mathscr{A})U = U^0$. It suffices to show that the map $I + A + B\colon V \to V'$ is onto. Indeed, then there exists $z \in V$ satisfying

$$(I + A + B)z = z^0 - Au^0.$$

Setting $u = z + u^0$ we conclude easily that $U \in V \times V$, $Au + Bz = z^0 - z \in H$ (hence $U \in D(\mathscr{A})$), and $(I + \mathscr{A})U = U^0$.

To prove the surjectivity of $I + A + B\colon V \to V'$, fix $f \in V'$ arbitrarily, set

$$G_i(t) = \int_0^t g_i(s)\, ds, \qquad t \in \mathbb{R}, \quad i = 1, 2,$$

and consider the map $F\colon V \to \mathbb{R}$ defined by the formula

$$F(u) = \tfrac{1}{2}\|u\|_H^2 + \tfrac{1}{2}\|u\|_V^2 + \int_\Omega G_1(u_1) + G_2(u_2)\,dx - \langle f, u\rangle_{V',V}.$$

Using the growth assumption in (H2) one can readily verify that F is well defined and continuously differentiable and that

$$\langle F'(u), z\rangle_{V',V} = \langle (I + A + B)u - f, z\rangle_{V',V}$$

for all $u, z \in V$. Furthermore, thanks to the nondecreasingness of the function g_i, F is convex and hence lower semicontinuous in V. Finally, we deduce from the inequality

$$F(z) \geq \left(\tfrac{1}{2}\|z\|_V - \|f\|_{V'}\right)\|z\|_V$$

that $F(z) \to +\infty$ if $\|z\|_V \to +\infty$. Hence there is a point $u \in V$ minimizing F. It follows that $F(u) = 0$, i.e., $(I + A + B)u = f$.

LEMMA 2.2. *We have $W \times V = D(\mathscr{A})$, and therefore $D(\mathscr{A})$ is dense in $\mathscr{H}$.*

Proof. Fix $(u, z) \in W \times V$ arbitrarily; to prove that $(u, z) \in D(\mathscr{A})$, it suffices to prove the estimate

$$|\langle Au + Bz, v\rangle_{V',V}| \leq c\|v\|_H, \qquad \forall v \in V, \tag{2.4}$$

with a suitable constant c. Using the definition of A and B we have

$$\langle Au + Bz, v\rangle_{V',V} = \int_\Omega \Delta u_1\,\Delta v_1 + \nabla u_2 \cdot \nabla v_2 + a(u_1 v_2 + u_2 v_1)\,dx$$
$$+ \int_\Omega g_1(z_1)v_1 + g_2(z_2)v_2\,dx. \tag{2.5}$$

Since $(u, z) \in W \times V$ implies $u_1 \in H^4(\Omega)$, $u_2 \in H^2(\Omega)$, we may apply Green's formula to the right hand side and use the boundary condition (1.3). We obtain

$$\langle Au + Bz, v\rangle_{V',V} = \int_\Omega (\Delta^2 u_1)v_1 + (-\Delta u_2)v_2 + a(u_1 v_2 + u_2 v_1)\,dx$$
$$+ \int_\Omega g_1(z_1)v_1 + g_2(z_2)v_2\,dx.$$

Since $u_1 \in H^4(\Omega)$, $u_2 \in H^2(\Omega)$ implies that $(\Delta^2 u_1, \Delta u_2) \in H$, by using (1.6), (2.4) follows. Since the density of $W \times V$ in $\mathscr{H}$ is well known, it follows that $D(\mathscr{A})$ is dense in $\mathscr{H}$. We may now apply the standard theory of nonlinear semigroups [1], and (1.8), (1.9) follow.

Fix $(u, z) \in D(\mathscr{A})$ arbitrarily; to prove that $(u, z) \in W \times V$, set

$$f := Au + Bz, \quad h := -\left(g_1(z_1), g_2(z_2)\right).$$

Observe that Eq. (2.5) may also be written in the form

$$\langle f, v \rangle_{V',V} = \int_\Omega \Delta u_1 \, \Delta v_1 + \nabla u_2 \cdot \nabla v_2 + a(u_1 v_2 + u_2 v_1)\, dx$$

$$- \int_\Omega h \cdot v \, dx, \qquad \forall v \in V.$$

This means that u is the weak solution of the boundary value problem

$$\begin{aligned} \Delta^2 u_1 &= f_1 + h_1 - a u_2 \qquad \text{in } \Omega \\ -\Delta u_2 &= f_2 + h_2 - a u_1 \qquad \text{in } \Omega \\ u_1 = u_2 &= \partial_\nu u_1 = 0, \qquad \text{in } \Gamma. \end{aligned}$$

Since $f \in H$ by assumption and by hypothesis (1.6) we have $h \in H$. Then applying the elliptic regularity theory to this problem we conclude that $u \in W$. Then $D(\mathscr{A}) = W \times V$ and (1.10) follows from the general theory of nonlinear semigroups [1].

We finish this section by giving an explicit formula for the derivative of the energy.

LEMMA 2.3. *The energy of a weak solution of* (1.1)–(1.4) *is nonincreasing and locally absolutely continuous and*

$$E'(t) = -\int_\Omega u_1' g_1(u_1') + u_2' g_2(u_2')\, dx \qquad a.e. \text{ in } \mathbb{R}^+. \tag{2.6}$$

Proof. Multiplying (1.1) by u_1' and (1.2) by u_2' integrating by parts their sum over $\Omega \times (0, T)$, and finally eliminating the normal derivatives by using the boundary condition (1.3), we easily obtain

$$E(0) - E(T) = \int_0^T \int_\Omega u_1' g_1(u_1') + u_2' g_2(u_2')\, dx\, dt$$

for every positive number T. Being the primitive of an integrable function, E is locally absolutely continuous and equality (2.6) is satisfied. Thanks to assumption (H2) the nonincreasingness of the energy follows from (2.6).

3. DECAY ESTIMATES

Using an easy density argument, based on Theorem 1, it is sufficient to prove the estimate of Theorem 2 for strong solutions. Henceforth we assume that u is a strong solution of the system (1.1)–(1.4).

We are going to prove that the energy of this solution satisfies the estimate

$$\int_S^T E(t)^{(p+1)/2}\,dt \le cE(S) \tag{3.1}$$

for all $0 \le S < T < +\infty$. Here and in what follows we shall denote by c diverse positive constants. We recall that if a nonnegative and nonincreasing function $E\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the estimate (3.1), then it also satisfies (1.12)–(1.13) (cf., e.g., [5, Theorem 8.1 and 9.1]). Then Theorem 2 will be proved if we establish inequality (3.1).

LEMMA 3.1. *We have*

$$\int_S^T E^{(p+1)/2}(t)\,dt \le cE^{(p+1)/2}(S) + c\sum_{i=1}^{i=2}\int_S^T E^{(p-1)/2}(t)\int_\Omega (u_i')^2 + g_i(u_i')^2\,dx\,dt \tag{3.2}$$

for all $0 \le S < T < +\infty$.

Proof. Multiplying Eq. (1.1) by $E(t)^{(p-1)/2}u_1$, integrating by parts, and using the boundary condition (1.3), we obtain that

$$\begin{aligned} 0 &= \int_S^T E(t)^{(p-1)/2}\int_\Omega u_1\big(u_1'' + \Delta^2 u_1 + au_2 + g_1(u_1')\big)\,dx\,dt \\ &= \left[E(t)^{(p-1)/2}\int_\Omega u_1u_1'\,dx\right]_S^T - \frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u_1u_1'\,dx\,dt \\ &\quad + \int_S^T E^{(p-1)/2}(t)\int_\Omega (\Delta u_1)^2 - (u_1')^2 + au_1u_2 + u_1g_1(u_1')\,dx\,dt. \end{aligned}$$

Similarly, we have

$$\begin{aligned} 0 &= \int_S^T E(t)^{(p-1)/2}\int_\Omega u_2\big(u_2'' - \Delta u_2 + au_1 + g(u_2')\big)\,dx\,dt \\ &= \left[E(t)^{(p-1)/2}\int_\Omega u_2u_2'\,dx\right]_S^T - \frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u_2u_2'\,dx\,dt \\ &\quad + \int_S^T E^{(p-1)/2}(t)\int_\Omega |\nabla u_2|^2 - (u_2')^2 + au_1u_2 + u_2g_2(u_2')\,dx\,dt. \end{aligned}$$

Taking their sum, we obtain that

$$\int_S^T E^{(p-1)/2}(t)\int_\Omega (u_1')^2 + (u_2')^2 + (\Delta u_1)^2 + |\nabla u_2|^2 + 2au_1u_2\,dx\,dt$$
$$= -\left[E(t)^{(p-1)/2}\int_\Omega u_1u_1' + u_2u_2'\,dx\right]_S^T$$
$$+\frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u_1u_1' + u_2u_2'\,dx\,dt$$
$$+\int_S^T E^{(p-1)/2}(t)\int_\Omega 2(u_1')^2 + 2(u_2')^2$$
$$- u_1g_1(u_1') - u_2g_2(u_2')\,dx\,dt. \tag{3.3}$$

Next we observe that (note that the energy is nonincreasing)

$$\left|\left[E(t)^{(p-1)/2}\int_\Omega u_1u_1' + u_2u_2'\,dx\right]_S^T\right|$$
$$\le cE(S)^{(p-1)/2}(E(T) + E(S)) \le cE(S)^{(p+1)/2};$$
$$\frac{p-1}{2}\int_S^T E^{(p-3)/2}(t)E'(t)\int_\Omega u_1u_1' + u_2u_2'\,dx\,dt$$
$$\le c\int_S^T E^{(p-1)/2}(t)|E'(t)| \le cE(S)^{(p+1)/2},$$

and, for $\epsilon = \frac{1}{2}(1 - \sqrt{c'c''}\|a\|_{L^\infty(\Omega)})$ (thanks to (1.5), $\epsilon > 0$),

$$-\int_\Omega \sum_{i=1}^{i=2} u_ig_i(u_i')\,dx$$
$$\le \int_\Omega \frac{\epsilon}{c'}(u_1)^2 + \frac{\epsilon}{c''}(u_2)^2 + \frac{c'}{4\epsilon}g_1(u_1')^2 + \frac{c''}{4\epsilon}g_2(u_2')^2\,dx$$
$$\le \int_\Omega \epsilon\left((\Delta u_1)^2 + |\nabla u_2|^2\right) + \frac{c'}{4\epsilon}g_1(u_1')^2 + \frac{c''}{4\epsilon}g_2(u_2')^2\,dx$$

(c', c'' are the constants defined in assumption (H1)). Therefore we conclude from (3.3) the estimate (3.2).

LEMMA 3.2. *We have, for all* $t \in \mathbb{R}^+$,

$$\int_\Omega (u_i')^2\,dx \le -cE'(t) + c(-E'(t))^{2/(p+1)}, \qquad i = 1, 2 \tag{3.4}$$

and

$$\int_{\Omega} g_i(u_i')^2\,dx \le -cE'(t) + c(-E'(t))^{2/(p+1)}, \qquad i = 1,2. \quad (3.5)$$

Proof. Fix $t \ge 0$ arbitrarily, and set

$$\Omega_i^- = \{x \in \Omega : |u_i'(x)| \le 1\}, \quad \Omega_i^+ = \{x \in \Omega : |u_i'(x)| > 1\}, \qquad i = 1,2.$$

Using the growth assumption (1.11), we have

$$\int_{\Omega_i^-} (u_i')^2\,dx \le c\int_{\Omega_i^-} (u_i' g_i(u_i'))^{2/(p+1)}\,dx \le c\left(\int_{\Omega_i^-} u_i' g_i(u_i')\,d\right)^{2/(p+1)}$$

$$\le c\left(\int_{\Omega} u_i' g_i(u_i')\,dx\right)^{2/(p+1)} \le c(-E'(t))^{2/(p+1)}$$

(we applied Lemma 2.3 in the last step) and

$$\int_{\Omega_i^+} (u_i')^2\,dx \le c\int_{\Omega_i^+} u_i' g_i(u_i')\,dx \le -cE'(t).$$

Taking their sum, we obtain (3.4). Using assumption (1.11), we may prove in the same way estimate (3.5).

LEMMA 3.3. *The estimate*

$$\int_S^T E(t)^{(p+1)/2}\,dt \le c\left(1 + E(0)^{(p-1)/2}\right)E(S) \quad (3.6)$$

holds for all $0 \le S < T < +\infty$.

Proof. Substituting the estimates (3.4) and (3.5) into the right-hand side of (3.2), we obtain that

$$\int_S^T E(t)^{(p+1)/2}\,dt \le cE(S)^{(p+1)/s} + c\int_S^T E(t)^{(p-1)/2}(-E'(t))$$

$$+ E(t)^{(p-1)/2}(-E'(t))^{2/(p+1)}\,dt$$

$$\le cE(S)^{(p+1)/2} + c\int_S^T E(t)^{(p-1)/2}(-E'(t))^{2/(p+1)}\,dt.$$

Using the Young inequality, for any fixed $\epsilon > 0$ we have

$$cE(t)^{(p-1)/2}(-E'(t))^{2/(p+1)} \le \epsilon E(t)^{(p+1)/2} + c\epsilon^{(1-p)/2}(-E'(t)).$$

Therefore

$$\begin{aligned}(1-\epsilon)\int_S^T E(t)^{(p+1)/2}\,dt \\ &\le cE(S)^{(p+1)/2} + c\epsilon^{(1-p)/2}\int_S^T(-E'(t))\,dt \\ &\le c\left(1+\epsilon^{(1-p)/2}\right)\left(1+E(S)^{(p-1)/2}\right)E(S);\end{aligned}$$

choosing $0 < \epsilon < 1$ and using the nonincreasingness of the energy, (3.6) follows.

The constant c in (3.6) is independent of S, T and $E(0)$; then the estimate (3.1) holds with a constant c independent of S, T, and, if $p = 1$, c is also independent of $E(0)$.

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