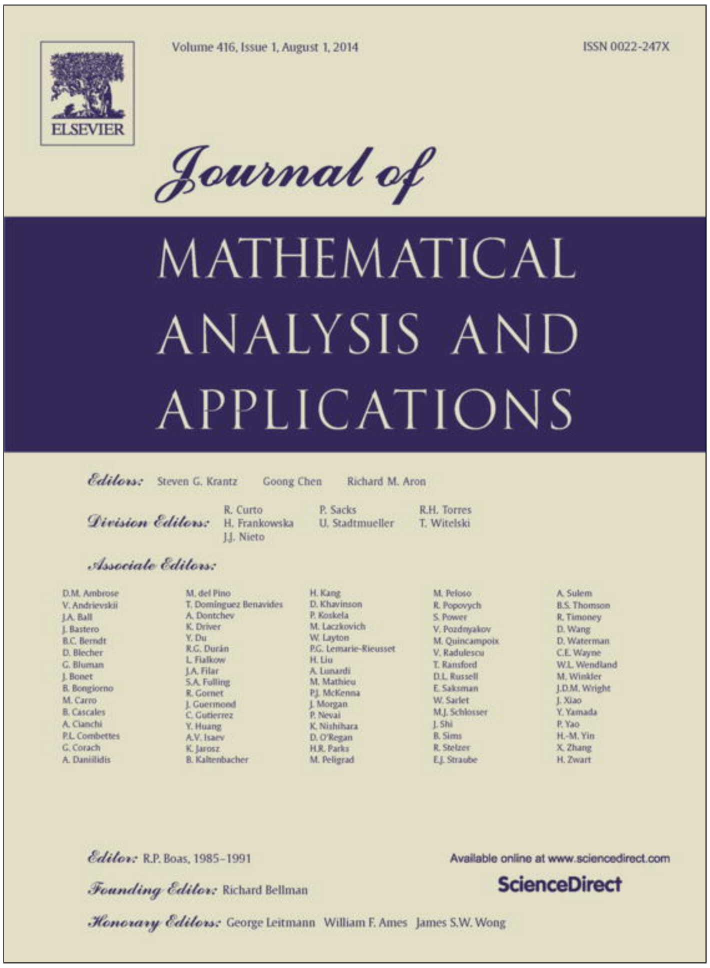
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A new approach to the stability of an abstract system in the presence of infinite history

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ABSTRACT

In this paper, we consider the following problem

$$\begin{cases} u_{tt}(t) + Au(t) - \displaystyle\int_0^{+\infty} g(s)Au(t-s)\,ds = 0, & \forall t > 0 \\ u(-t) = u_0(t), & \forall t \geqslant 0 \\ u_t(0) = u_1, \end{cases}$$

where A is a self-adjoint positive definite operator and g is a positive nonincreasing function. We adopt the method introduced in [19], for finite history, with some modifications imposed by the nature of our problem, to establish a general decay result which depends only on the behavior of the relaxation function. Our result extends the decay result obtained for problems with finite history to those with infinite history. In addition, it improves, in some cases, some decay results obtained earlier in [15].

1. Introduction

Let H be a real Hilbert space with inner product and related norm denoted by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$, respectively. Let $A : D(A) \to H$ be a self-adjoint linear positive definite operator with domain $D(A) \subset H$ such that the embedding is dense and compact. We consider the following class of second-order linear integrodifferential equations:

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$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Au(t-s)\,ds = 0, \quad \forall t > 0 \tag{1.1}$$

with initial conditions

$$\begin{cases} u(-t) = u_0(t), & \forall t \in \mathbb{R}^+ = [0, +\infty[\\ u_t(0) = u_1 \end{cases} \tag{1.2}$$

where $u_{tt} = \frac{\partial^2 u}{\partial t^2}$, $u_t = \frac{\partial u}{\partial t}$, u_0 and u_1 are given history and initial data, and $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a given function.

Since the pioneer work of Dafermos [10], problems related to (1.1)–(1.2) have attracted the attention of many researchers and a large number of papers have appeared. We start by the work of Chepyzhov and Pata [9], where an abstract problem of the form

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)\big(A(t) - Au(t-s)\big)\,ds = 0, & \forall t > 0 \\ u(-t) = u_0(t), & \forall t \geqslant 0 \\ u_t(0) = u_1 \end{cases}$$

was considered in a Hilbert space H. Here A is a strictly positive self-adjoint operator with a domain $D(A) \subset H$ and $0 < \int_0^{+\infty} g(s)\,ds < +\infty$. They proved the well-posedness and showed that the exponential stability holds only for kernels of exponential decay. Also, in a survey paper, Pata [28] discussed the decay properties of the semigroup associated with Eq. (1.1) and established several stability results. In [27], Pata studied the asymptotic behavior of an abstract integrodifferential equation of the form

$$u_{tt}(t) + \alpha Au(t) + \beta u_t - \int_0^{t} g(s)Au(t-s)\,ds = 0, \quad \forall t > 0$$

for $\alpha > 0$, $\beta \geqslant 0$ and g a positive summable kernel, and analyzed the exponential stability of the semigroup associated with the positive operator under some sufficient conditions on the kernel which were not considered before in the literature. He also introduced some new concepts such as the flatness of a kernel. We refer the reader to Fabrizio et al. [13] and Grasselli et al. [14] for more results of this nature.

In all the above mentioned works, the kernels considered were of either exponential or polynomial decay. Recently, Guesmia [15] considered the following problem:

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)\,ds = 0, & \forall t > 0 \\ u(-t) = u_0(t), & \forall t \geqslant 0 \\ u_t(0) = u_1 \end{cases}$$

for A and B two self-adjoint positive definite operators with $D(A) \subset D(B)$ and a kernel of more general decay rate satisfying

(A0) There exist $a_0, a_1 > 0$ such that

$$a_1\|v\|^2 \leqslant \big\|B^{\frac{1}{2}}v\big\|^2 \leqslant a_0\big\|A^{\frac{1}{2}}v\big\|^2, \quad \forall v \in D(A)$$

(A1) $g : \mathbb{R}^+ \to \mathbb{R}^+$is a differentiable nonincreasing function satisfying

$$0 < \int_0^{+\infty} g(s)\,ds < \frac{1}{a_0}$$

(A2) There exists a positive, increasing strictly convex function $G : \mathbb{R}^+ \to \mathbb{R}^+$ of class $C^1(\mathbb{R}^+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0, \qquad \lim_{t\to+\infty} G'(t) = +\infty$$

such that

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))}\,ds + \sup_{s\in\mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty.$$

He established a general decay estimate given in term of the convex function G. His result generalizes the usual exponential and polynomial decay results found in the literature. His proof makes use of some properties of the convex functions and a generalized version of the Young inequality.

For problems with finite history (viscoelasticity), we mention some results related to

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \displaystyle\int_0^t g(t-\tau)\Delta u(x,\tau)\,d\tau = 0, & \text{in } \Omega \times]0, +\infty[\\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega \end{cases} \tag{1.3}$$

where Ω is a bounded domain of $\mathbb{R}^n$ $(n \geqslant 1)$ with a smooth boundary $\partial\Omega$ and g is a positive nonincreasing function defined on $\mathbb{R}^+$. The first work that dealt with uniform decay was by Dassios and Zafiropoulos [11] in which a viscoelastic problem in $\mathbb{R}^3$ was studied and a polynomial decay result was proved for exponentially decaying kernels. After that, a very important contribution by Rivera was introduced. In 1994, Rivera [20] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy a bounded domain or the whole space $\mathbb{R}^n$, with zero boundary and history data and in the absence of body forces. For the bounded domains, he proved an exponential decay result for exponentially decaying relaxation functions. However, for the whole space case, he showed that only the polynomial decay can be obtained even if the kernel is of exponential decay. The rate of the decay was also given. This result was later generalized to a situation where the kernel is decaying algebraically but not exponentially by Cabanillas and Rivera [5]. In their paper, the authors considered both cases the bounded domains and that of a material occupying the entire space and showed that the decay of solutions is algebraic, at a rate which can be determined by the rate of the decay of the relaxation function. Barreto et al. [2] improved this latter result further by considering equations related to linear viscoelastic plates. They showed that the solution energy decays at the same decay rate of the relaxation function. In [22], a class of abstract viscoelastic systems of the form

$$\begin{cases} u_{tt}(t) + \mathcal{A}u(t) + \beta u(t) - \displaystyle\int_0^{+\infty} g(t-s)\mathcal{A}^\alpha u(s)\,ds = 0, \quad \forall t > 0 \\ u(0) = u_0, \qquad u_t(0) = u_1 \end{cases} \tag{1.4}$$

for $0 \leqslant \alpha \leqslant 1$, $\beta \geqslant 0$, were investigated. The main focus was on the case when $0 < \alpha < 1$ and the main result was that solutions for (1.4) decay polynomially even if the kernel g decays exponentially. This result was improved by Rivera and Naso [24], where the authors considered a more general abstract problem than (1.4) and established a necessary and sufficient condition to obtain an exponential decay (see also [21]). In the case of lack of exponential decay, a polynomial decay result has been proved. In the latter case, they showed that the rate of decay can be improved by taking more regular initial data. Application to concrete examples was also presented.

For systems with localized frictional dampings cooperating with the dissipation induced by the viscoelastic term, we mention the work of Cavalcanti et al. [7]. Under the condition

$$-\xi_1 g(t) \leqslant g'(t) \leqslant -\xi_2 g(t), \quad \forall t \geqslant 0$$

with $\|g\|_{L^1(\mathbb{R}^+)}$ small enough, the authors obtained an exponential rate of decay. Berrimi and Messaoudi [3,4] improved Cavalcanti's result by showing that the viscoelastic dissipation alone is strong enough to stabilize the system. Also, Cavalcanti and Oquendo [8] considered

$$u_{tt} - k_0 \Delta u + \int_0^t \operatorname{div}\big(a(x) g(t-\tau) \nabla u(\tau)\big)\, d\tau + b(x) h(u_t) + f(u) = 0$$

under similar conditions on the relaxation function g and $a(x) + b(x) \geqslant \delta > 0$, and improved the result in [7]. They established an exponential stability when g is decaying exponentially and h is linear; and a polynomial stability when g is decaying polynomially and h is nonlinear. A related problem, in a bounded domain, of the form

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau) \Delta u(\tau)\, d\tau - \gamma \Delta u_t = 0$$

for $\rho > 0$ and g decaying exponentially, was also studied by Cavalcanti et al. [6]. A global existence result for $\gamma \geqslant 0$, as well as an exponential decay for $\gamma > 0$, has been established. This latter "exponential decay" result has been extended to a situation, where $\gamma = 0$, by Messaoudi and Tatar [17,18]. Moreover, some polynomial decay results have been established in the absence, as well as in the presence, of a source term, for polynomially decaying relaxation functions.

For viscoelastic systems with oscillating kernels, Rivera and Naso [23] showed that, if the kernel satisfies $g(0) > 0$ and decays exponentially to zero, then the solution decays exponentially to zero. On the other hand, if the kernel decays polynomially, then the corresponding solution also decays polynomially to zero with the same rate of decay.

For more general decaying kernels, Messaoudi [19] considered

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \displaystyle\int_0^t g(t-\tau) \Delta u(x,\tau)\, d\tau = 0, & \text{in } \Omega \times]0, +\infty[\\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega \end{cases} \tag{1.5}$$

where Ω is a bounded domain of $\mathbb{R}^n$ and g is a positive nonincreasing function satisfying

$$g'(t) \leqslant -\xi(t) g(t), \quad \forall t \geqslant 0 \tag{1.6}$$

for ξ a nonincreasing differentiable function. He established a general decay result, from which the usual exponential and polynomial decay results are only special cases. After that, using the idea of [19], a series of papers have appeared. See, for instance, Liu [16], Park and Park [26] and Xiaosen and Mingxing [33].

Very recently, Mustafa and Messaoudi [25] considered (1.5), for relaxation functions satisfying, instead of (1.6), a relation of the form

$$g'(t) \leqslant -H\big(g(t)\big), \quad \forall t \geqslant 0$$

where H is a positive convex function. They used some properties of the convex functions together with the generalized Young inequality and established a general decay result depending on g and H. We should mention here that the result of [25] is established under weaker conditions than those imposed by Alabau-Boussouira and Cannarsa [1]. For more results related to stability of viscoelastic systems, we refer the reader to works by Fabrizio and Polidoro [12] and Tatar [30–32].

In the present work, we study the asymptotic behavior of solutions of (1.1)–(1.2), under the assumption (1.6) instead of (A2), considered in [15]. This work will "relatively" extend the result of Messaoudi [19], known for the finite history case, to the infinite history case. The proof of the current result is easier than the one in [15] since we need no convex function properties or the generalized Young inequality. Moreover, this result gives a better rate of decay in some situations (see Remark 2.2 below).

This paper is organized as follows. In Section 2, we discuss the well-posedness and present our main stability result. In Section 3, the proof of the main result is given. Section 4 is devoted to applications of our main result.

2. Well-posedness and stability results

In order to discuss the semigroup formulation of our problem and state our stability result, we assume that A and g satisfy the following hypotheses:

(H1) There exists a positive constant a such that

$$a\|v\|^2 \leqslant \big\|A^{\frac{1}{2}}v\big\|^2, \quad \forall v \in D\big(A^{\frac{1}{2}}\big) \tag{2.1}$$

(H2) $g : \mathbb{R}^+ \to \mathbb{R}^+$ is of class $C^1(\mathbb{R}^+)$ nonincreasing and satisfies

$$g_0 := \int_0^{+\infty} g(s)\, ds \in \left]0, 1\right[\tag{2.2}$$

(H3) There exists a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$g'(s) \leqslant -\xi(s)g(s), \quad \forall s \in \mathbb{R}^+ \tag{2.3}$$

2.1. Well-posedness

It is well known, following a method introduced by Dafermos [10], that (1.1)–(1.2) can be formulated as an abstract linear first-order system of the form

$$\begin{cases} \mathcal{U}_t(t) = \mathcal{A}\mathcal{U}(t), \quad \forall t > 0 \\ \mathcal{U}(0) = \mathcal{U}_0 \end{cases} \tag{2.4}$$

where $\mathcal{U}_0 = (u_0(0), u_1, \eta_0)^T \in \mathcal{H} = D(A^{\frac{1}{2}}) \times H \times L_g^2$, $\mathcal{U} = (u, u_t, \eta^t)^T$,

$$\begin{cases} \eta^t(s) = u(t) - u(t-s), & \forall t, s \in \mathbb{R}^+ \\ \eta_0(s) = \eta^0(s) = u_0(0) - u_0(s), & \forall s \in \mathbb{R}^+ \end{cases}$$

L_g^2 is defined by

$$L_g^2 = \left\{ z : \mathbb{R}^+ \to D\left(A^{\frac{1}{2}}\right),\ \int_0^{+\infty} g(s) \left\| A^{\frac{1}{2}} z(s) \right\|^2 ds < +\infty \right\}$$

endowed with the inner product

$$\langle z_1, z_2 \rangle_{L_g^2} = \int_0^{+\infty} g(s) \left\langle A^{\frac{1}{2}} z_1(s), A^{\frac{1}{2}} z_2(s) \right\rangle ds$$

$\mathcal{A}$ is the linear operator given by

$$\mathcal{A}(v, w, z)^T = \left(w, -(1 - g_0) A v - \int_0^{+\infty} g(s) A z(s)\, ds, -\frac{\partial z}{\partial s} + w \right)^T$$

and

$$D(\mathcal{A}) = \left\{ (v, w, z)^T \in \mathcal{H},\ w \in D\left(A^{\frac{1}{2}}\right),\ \frac{\partial z}{\partial s} \in L_g^2,\ (1 - g_0) v + \int_0^{+\infty} g(s) z(s)\, ds \in D(A),\ z(0) = 0 \right\}$$

Under the hypotheses (H1) and (H2), it is well known (see [24]) that $\mathcal{H}$ endowed with the inner product

$$\left\langle (v_1, w_1, z_1)^T, (v_2, w_2, z_2)^T \right\rangle_{\mathcal{H}} = (1 - g_0) \left\langle A^{\frac{1}{2}} v_1, A^{\frac{1}{2}} v_2 \right\rangle + \langle w_1, w_2 \rangle + \langle z_1, z_2 \rangle_{L_g^2}$$

is a Hilbert space, $D(\mathcal{A}) \subset \mathcal{H}$ with dense embedding, and $\mathcal{A}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathcal{H}$ (see [24]). Therefore, the classical semigroup theory implies that (see [29]), for any $\mathcal{U}_0 \in \mathcal{H}$, the system (2.4) has a unique weak solution

$$\mathcal{U} \in C\left(\mathbb{R}^+, \mathcal{H}\right) \tag{2.5}$$

Moreover, if $\mathcal{U}_0 \in D(\mathcal{A})$, then the solution of (2.4) satisfies

$$\mathcal{U} \in C^1\left(\mathbb{R}^+, \mathcal{H}\right) \cap C\left(\mathbb{R}^+, D(\mathcal{A})\right) \tag{2.6}$$

2.2. Asymptotic behavior

Our main concern in this paper is the asymptotic stability of (2.4). We have the following result:

Theorem 2.1. *Assume that* (H1)–(H3) *hold. Then, for any $\mathcal{U}_0 \in \mathcal{H}$ satisfying, for some $m_0 \geqslant 0$,*

$$\left\| A^{\frac{1}{2}} u_0(s) \right\| \leqslant m_0, \quad \forall s > 0 \tag{2.7}$$

there exist constants $\gamma_0 \in]0,1[$ *and* $\delta_1 > 0$ *such that, for all* $t \in \mathbb{R}^+$ *and for all* $\delta_0 \in]0,\gamma_0]$,

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant \delta_1 \left(1 + \int_0^t \big(g(s)\big)^{1-\delta_0}\,ds\right) e^{-\delta_0 \int_0^t \xi(s)\,ds} + \delta_1 \int_t^{+\infty} g(s)\,ds \tag{2.8}$$

Remark 2.2.

1. Our decay estimate (2.8) still holds for the following "little" more general form considered in [15]:

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)\,ds = 0, \quad \forall t > 0 \tag{2.9}$$

   where $B : D(B) \to H$ is a self-adjoint linear positive definite operator with domain $D(A) \subset D(B) \subset H$ with dense and compact embeddings such that, for positive constants a_0, a_1 and a_2,

$$\|v\|^2 \leqslant a_0 \big\|B^{\frac{1}{2}}v\big\|^2 \leqslant a_1 \big\|A^{\frac{1}{2}}v\big\|^2 \leqslant a_2 \big\|B^{\frac{1}{2}}v\big\|^2, \quad \forall v \in D\big(A^{\frac{1}{2}}\big) \tag{2.10}$$

2. If there exists $\varepsilon_0 \in]0,1[$, for which

$$\int_0^{+\infty} \big(g(s)\big)^{1-\varepsilon_0}\,ds < +\infty \tag{2.11}$$

   then we can choose $\delta_0 \in]0,\gamma_1]$, $\gamma_1 = \min\{\varepsilon_0, \gamma_0\}$ such that

$$\int_0^{+\infty} \big(g(s)\big)^{1-\delta_0}\,ds < +\infty$$

   and consequently, (2.8) takes the form

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant \delta_1 \left(e^{-\delta_0 \int_0^t \xi(s)\,ds} + \int_t^{+\infty} g(s)\,ds\right) \tag{2.12}$$

   for some $\delta_1 > 0$.
3. Let us compare our estimates (2.8) and (2.12) with the one of [15] obtained for (2.9) under the assumptions (2.10) and (A2).
   i) Our estimate (2.12) improves, in some particular cases, the decay rate given in [15]. Indeed. Let $g(t) = de^{-(1+t)^q}$ with $0 < q < 1$, and $d > 0$ small enough so that (2.2) and (2.3), with $\xi(t) = q(1+t)^{q-1}$, hold. Then, (2.11) is satisfied and consequently, (2.12) gives, for two positive constants c_1 and c_2,

$$\big\|\mathcal{U}(t)\big\|_{\mathcal{H}}^2 \leqslant c_1 e^{-c_2(1+t)^q}, \quad \forall t \in \mathbb{R}^+ \tag{2.13}$$

   which implies that $\|\mathcal{U}(t)\|_{\mathcal{H}}^2$ has the same decay rate as g, and improves the following decay rate obtained in [15]:

$$\big\|\mathcal{U}(t)\big\|_{\mathcal{H}}^2 \leqslant c_1 e^{-c_2 t^p}, \quad \forall t \in \mathbb{R}^+,\ \forall p \in \left]0, \frac{q}{2}\right[\tag{2.14}$$

Similarly, if $g(t) = de^{-(\ln(2+t))^q}$ with $q > 1$, and $d > 0$ small enough so that (2.2) and (2.3), with $\xi(t) = \frac{q(\ln(2+t))^{q-1}}{2+t}$, hold. Then, (2.11) is satisfied and, hence, (2.12) yields

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant c_1 e^{-c_2(\ln(1+t))^q}, \quad \forall t \in \mathbb{R}^+ \tag{2.15}$$

Estimate (2.15) is little better than the following one obtained in [15]:

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant c_1 e^{-c_2(\ln(1+t))^p}, \quad \forall t \in \mathbb{R}^+, \ \forall p \in]1, q[\tag{2.16}$$

ii) When g has at most a polynomial decay, for example $g(t) = \frac{d}{(1+t)^q}$ with $q > 1$, and $d > 0$ small enough so that (2.2) and (2.3), with $\xi(t) = q(1+t)^{-1}$ hold, condition (2.11) is satisfied and, hence, (2.12) gives

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant \frac{c_1}{(t+1)^{c_2}}, \quad \forall t \in \mathbb{R}^+ \tag{2.17}$$

Here c_2, generated by the calculations, is generally small. However, the approach of [15] gives, in this case, the following stronger and precise decay rate:

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant \frac{c_1}{(t+1)^p}, \quad \forall t \in \mathbb{R}^+, \ \forall p \in \left]0, \frac{q-1}{2}\right[\tag{2.18}$$

iii) Let us consider an example where (2.11) is never satisfied.
If $g(t) = \frac{a}{(2+t)(\ln(2+t))^q}$, with $q > 1$, and $a > 0$ small enough so that (2.2) holds, then, simple calculations show that $\xi(t) = \frac{q+\ln(2+t)}{(2+t)\ln(2+t)}$,

$$\int_0^t \xi(s)\,ds = \ln(2+t) - \ln 2 + q\big(\ln\big(\ln(2+t)\big) - \ln(\ln 2)\big)$$

and

$$\int_t^{+\infty} g(s)\,ds = \frac{a}{q-1}\big(\ln(2+t)\big)^{1-q}$$

In this case, we apply estimate (2.8), which gives, for $\delta_0 \in]0, \gamma_0]$ small enough so that $(1-\delta_0)q > 1$ (that is $q\delta_0 < q - 1$),

$$\begin{aligned}
\|\mathcal{U}(t)\|_{\mathcal{H}}^2 &\leqslant \delta_1\left(1 + \int_0^t \frac{a^{1-\delta_0}}{(2+s)^{1-\delta_0}(\ln(2+s))^{(1-\delta_0)q}}\,ds\right)\left(\frac{2^{\delta_0}(\ln 2)^{q\delta_0}}{(2+t)^{\delta_0}(\ln(2+t))^{q\delta_0}}\right) \\
&\quad + \frac{a\delta_1}{q-1}\big(\ln(2+t)\big)^{1-q} \\
&\leqslant \delta_1\left(1 + \int_0^t \frac{a^{1-\delta_0}}{(2+s)(\ln(2+s))^{(1-\delta_0)q}}\,ds\right)\frac{2^{\delta_0}(\ln 2)^{q\delta_0}}{(\ln(2+t))^{q\delta_0}} \\
&\quad + \frac{a\delta_1}{q-1}\big(\ln(2+t)\big)^{1-q} \\
&\leqslant c\left(\frac{1}{(\ln(2+t))^{q-1}} + \frac{1}{(\ln(2+t))^{q\delta_0}}\right)
\end{aligned}$$

for some positive constant c. Then

$$\left\|\mathcal{U}(t)\right\|_{\mathcal{H}}^2 \leqslant \frac{c_1}{(\ln(2+t))^{q\delta_0}}, \quad \forall t \in \mathbb{R}^+$$

Clearly, this decay result is weaker and less precise than the one obtained in [15], which is

$$\left\|\mathcal{U}(t)\right\|_{\mathcal{H}}^2 \leqslant \frac{c}{(\ln(2+t))^{q-1}}, \quad \forall t \in \mathbb{R}^+$$

4. According to the above particular examples, it seems that our approach gives a better decay rate than the one of [15] when g converges to zero faster than $\frac{1}{t^q}$, for any $q > 0$, and the approach of [15] gives a better decay rate than ours when g converges to zero at most polynomially.
5. It is well known that (see [24]), if g satisfies (2.3) with a constant function ξ (hence g decays at least exponentially to zero), then (without the restriction (2.7))

$$\left\|\mathcal{U}(t)\right\|_{\mathcal{H}}^2 \leqslant \delta_1 e^{-\delta_2 t}, \quad \forall t \in \mathbb{R}^+ \tag{2.19}$$

which is the best decay rate known in the literature.

3. Proof of the stability estimate

In order to justify the calculations, we establish (2.8) for initial data $\mathcal{U}_0 \in D(A)$. The estimate, then, remains valid for $\mathcal{U}_0 \in \mathcal{H}$ by a simple density argument. We consider the energy functional E associated with the solution of (2.4), corresponding to $\mathcal{U}_0 \in \mathcal{H}$,

$$\begin{aligned} E(t) &= \frac{1}{2}\left\|\mathcal{U}(t)\right\|_{\mathcal{H}}^2 \\ &= \frac{1}{2}\left((1-g_0)\left\|A^{\frac{1}{2}}u(t)\right\|^2 + \left\|u_t(t)\right\|^2 + \int_0^{+\infty} g(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right) \end{aligned} \tag{3.1}$$

where g_0 is given in (2.2).

Multiplying (1.1) by $u_t(t)$ "scalarly", we get

$$E'(t) = \frac{1}{2}\int_0^{+\infty} g'(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds, \quad \forall t \in \mathbb{R}^+ \tag{3.2}$$

Since g is nonincreasing, then E is nonincreasing and, consequently, (2.4) is dissipative. Now, we recall the following three lemmas of [15] (see also [24]).

Lemma 3.1. *Assume that* (H1) *and* (H2) *are satisfied. Then the functional*

$$I_1(t) = -\left\langle u_t(t), \int_0^{+\infty} g(s)\eta^t(s)\, ds\right\rangle$$

satisfies, for any $\epsilon > 0$ and for all $t \in \mathbb{R}^+$,

$$I_1'(t) \leqslant -(g_0-\epsilon)\|u_t(t)\|^2 + \epsilon\|A^{\frac{1}{2}}u(t)\|^2 + c_\epsilon\left(\int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds - \int_0^{+\infty} g'(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right), \tag{3.3}$$

where $c_\epsilon > 0$ is a constant depending on ϵ.

Proof. Multiplying "scalarly" (1.1) by $\int_0^{+\infty} g(s)\eta^t(s)\,ds$, we get

$$0 = \left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta^t(s)\,ds\right\rangle + (1-g_0)\left\langle Au(t), \int_0^{+\infty} g(s)\eta^t(s)\,ds\right\rangle + \left\langle \int_0^{+\infty} g(s)A\eta^t(s)\,ds, \int_0^{+\infty} g(s)\eta^t(s)\,ds\right\rangle$$

Using the definition of $A^{\frac{1}{2}}$, we get

$$0 = \left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta^t(s)\,ds\right\rangle + (1-g_0)\left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle + \left\langle \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta^t(s)\,ds, \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle$$

On the other hand, by using the fact that $\frac{\partial\eta^t}{\partial t}(s) = -\frac{\partial\eta^t}{\partial s}(s) + u_t(t)$, we find

$$\left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta^t(s)\,ds\right\rangle = \frac{\partial}{\partial t}\left\langle u_t(t), \int_0^{+\infty} g(s)\eta^t(s)\,ds\right\rangle - \left\langle u_t(t), \int_0^{+\infty} g(s)\frac{\partial}{\partial t}\eta^t(s)\,ds\right\rangle$$
$$= -I_1'(t) - g_0\|u_t(t)\|^2 + \left\langle u_t(t), \int_0^{+\infty} g(s)\partial_s\eta^t(s)\,ds\right\rangle$$

By integrating by parts with respect to s in the infinite integral, we get

$$\left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta^t(s)\,ds\right\rangle = -I_1'(t) - g_0\|u_t(t)\|^2 - \left\langle u_t(t), \int_0^{+\infty} g'(s)\eta^t(s)\,ds\right\rangle$$

By combining these equalities, we deduce that

$$I_1'(t) = -g_0\|u_t(t)\|^2 + (1-g_0)\left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle - \left\langle u_t(t), \int_0^{+\infty} g'(s)\eta^t(s)\,ds\right\rangle + \left\langle \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta^t(s)\,ds, \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle$$

By using Cauchy–Schwarz and Young's inequalities for the last three terms and recall (2.1) to estimate $\|\eta^t(s)\|^2$ by $\frac{1}{a}\|A^{\frac{1}{2}}\eta^t(s)\|^2$, (3.3) follows with $c_\epsilon = (1+\frac{1}{\epsilon})c$, where c is a positive constant depending only on g and a. □

Lemma 3.2. *Assume that* (H1) *and* (H2) *are satisfied. Then the functional*

$$I_2(t) = \langle u_t(t), u(t)\rangle$$

satisfies, for any $\epsilon > 0$ and for all $t \in \mathbb{R}^+$,

$$I_2'(t) \leqslant \|u_t(t)\|^2 - (1-g_0-\epsilon)\|A^{\frac{1}{2}}u(t)\|^2 + \tilde{c}_\epsilon \int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds \tag{3.4}$$

where $\tilde{c}_\epsilon > 0$ is a constant depending on ϵ.

Proof. Multiplying (1.1) "scalarly" by u, we find

$$0 = \langle u_{tt}(t), u(t)\rangle + (1-g_0)\langle Au(t), u(t)\rangle + \left\langle \int_0^{+\infty} g(s)A\eta^t(s)\,ds, u(t)\right\rangle$$

Consequently, using the definition of $A^{\frac{1}{2}}$, we arrive at

$$0 = \frac{\partial}{\partial t}\langle u_t(t), u(t)\rangle - \|u_t(t)\|^2 + (1-g_0)\|A^{\frac{1}{2}}u(t)\|^2 + \left\langle \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta^t(s)\,ds, A^{\frac{1}{2}}u(t)\right\rangle$$

By using Cauchy–Schwarz and Young's inequalities for the last term, (3.4) holds with $\tilde{c}_\epsilon = \frac{\tilde{c}}{\epsilon}$, where $\tilde{c}$ is a positive constant depending only on g. □

Lemma 3.3. *Assume that* (H1) *and* (H2) *are satisfied. Then there exist constants α_0, α_1, $\alpha_2 > 0$ such that the functional*

$$I_3 = I_1 + \frac{g_0}{2}I_2 + \alpha_0 E$$

satisfies, for all $t \in \mathbb{R}^+$,

$$I_3'(t) \leqslant -\alpha_1 E(t) + \alpha_2 \int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds \tag{3.5}$$

Proof. Multiplying (3.4) by $\frac{g_0}{2}$, adding (3.3), choosing $\epsilon = \frac{g_0(1-g_0)}{2(2+g_0)}$ (note that $\epsilon > 0$ thanks to (2.2)), using (3.2) to replace $\int_0^{+\infty} g'(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds$ with $2E'(t)$, choosing $\alpha_0 = 2c_\epsilon$, and noting that (thanks to (2.2) and (3.1)),

$$E(t) \leqslant \frac{1}{2}\left(\|A^{\frac{1}{2}}u(t)\|^2 + \|u_t(t)\|^2 + \int_0^{+\infty} g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right), \quad \forall t \in \mathbb{R}^+$$

we get (3.5) with $\alpha_1 = \min\{g_0 - 2\epsilon, g_0(1 - g_0 - \epsilon) - 2\epsilon\}$ ($\alpha_1 > 0$ thanks to (2.2) and the choice of ϵ) and $\alpha_2 = c_\epsilon + \frac{g_0}{2}\tilde{c}_\epsilon + \frac{\alpha_1}{2}$. □

Now, let M be a positive number and

$$I_4 = ME + I_3$$

Keeping in mind (2.2) and (3.1), we have, for all $t \in \mathbb{R}^+$,

$$E(t) \geqslant \frac{1-g_0}{2}\left(\left\|A^{\frac{1}{2}}u(t)\right\|^2 + \left\|u_t(t)\right\|^2 + \int_0^{+\infty} g(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right) \tag{3.6}$$

By using (2.1), (3.6) and the definitions of I_1 and I_2, we easily conclude that there exist two positive constants d_1 and d_2 depending only on g and a such that $|I_1| \leqslant d_1 E$ and $|I_2| \leqslant d_2 E$. Thus, $|I_3| \leqslant M_0 E$ with $M_0 = d_1 + \frac{g_0}{2}d_2 + \alpha_0$.

Therefore, for $M = 2M_0$, we get

$$M_0 E \leqslant I_4 \leqslant 3M_0 E \tag{3.7}$$

Thanks to (3.5) and the fact that E is nonincreasing, we have, for all $t \in \mathbb{R}^+$,

$$I_4'(t) \leqslant -\alpha_1 E(t) + \alpha_2 \int_0^{+\infty} g(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \tag{3.8}$$

Now, we estimate the integral term of (3.8), which represents the main difficulty in the proof of the stability estimate. To achieve this goal, we adapt, with some necessary modifications, the approach introduced in [19], for the wave equation with finite history.

Lemma 3.4. *Assume that* (H1)–(H3) *and* (2.7) *are satisfied. Then there exist positive constants β_1 and β_2 such that, for all $t \in \mathbb{R}^+$,*

$$\xi(t)I_4'(t) + \beta_1 E'(t) \leqslant -\alpha_1 \xi(t)E(t) + \beta_2 \xi(t) \int_t^{+\infty} g(s)\, ds \tag{3.9}$$

Proof. Using (2.3) and the fact that ξ is nonincreasing, we get, for all $t \in \mathbb{R}^+$,

$$\xi(t)\int_0^t g(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \leqslant \int_0^t \xi(s)g(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \leqslant -\int_0^t g'(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds$$

then, using (3.2) and the fact that g is nonincreasing, to obtain

$$\xi(t)\int_0^t g(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \leqslant -2E'(t), \quad \forall t \in \mathbb{R}^+ \tag{3.10}$$

On the other hand, (3.6) and the fact that E is nonincreasing imply that

$$\left\|A^{\frac{1}{2}}u(t)\right\|^2 \leqslant \frac{2}{1-g_0}E(t) \leqslant \frac{2}{1-g_0}E(0), \quad \forall t \in \mathbb{R}^+$$

Therefore, for all $s > t$,

$$\begin{aligned}\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 &\leqslant 2\left\|A^{\frac{1}{2}}u(t)\right\|^2 + 2\left\|A^{\frac{1}{2}}u(t-s)\right\|^2 \\ &\leqslant 2\sup_{\tau>0}\left\|A^{\frac{1}{2}}u(\tau)\right\|^2 + 2\sup_{\tau<0}\left\|A^{\frac{1}{2}}u(\tau)\right\|^2 \\ &\leqslant \frac{4}{1-g_0}E(0) + 2\sup_{\tau>0}\left\|A^{\frac{1}{2}}u_0(\tau)\right\|^2, \quad \forall t, s \in \mathbb{R}^+\end{aligned}$$

Then we deduce from (2.7) that, for all $t \in \mathbb{R}^+$,

$$\xi(t)\int_t^{+\infty} g(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \leqslant \left(\frac{4}{1-g_0}E(0) + 2m_0^2\right)\xi(t)\int_t^{+\infty} g(s)\, ds \tag{3.11}$$

Finally, multiplying (3.8) by $\xi(t)$ and combining with (3.10) and (3.11), we get (3.9) with $\beta_1 = 2\alpha_2$ and $\beta_2 = \alpha_2(\frac{4}{1-g_0}E(0) + 2m_0^2)$. $\square$

Now, let

$$F = \xi I_4 + \beta_1 E \quad \text{and} \quad h(t) = \xi(t)\int_t^{+\infty} g(s)\, ds$$

Thanks to (3.7) and the fact that ξ is nonnegative and nonincreasing, we have

$$\beta_1 E \leqslant F \leqslant \left(3\xi(0)M_0 + \beta_1\right)E \tag{3.12}$$

Then, using (3.9) and again the fact that ξ is nonincreasing,

$$F'(t) \leqslant -\gamma_0\xi(t)F(t) + \beta_2 h(t), \quad \forall t \in \mathbb{R}^+$$

with $\gamma_0 = \frac{\alpha_1}{3\xi(0)M_0+\beta_1}$ (note that $\beta_1 = 2\alpha_2 = 2c_\epsilon + g_0\tilde{c}_\epsilon + \alpha_1 > \alpha_1$, hence $\gamma_0 \in]0,1[$). This last inequality still holds for any $\delta_0 \in]0,\gamma_0]$; that is

$$F'(t) \leqslant -\delta_0\xi(t)F(t) + \beta_2 h(t), \quad \forall t \in \mathbb{R}^+ \tag{3.13}$$

Then (3.13) implies that, for all $t \in \mathbb{R}^+$,

$$\left(e^{\delta_0\int_0^t \xi(s)\, ds}F(t)\right)' \leqslant \beta_2 e^{\delta_0\int_0^t \xi(s)\, ds}h(t)$$

Therefore, by integrating over $[0,T]$ with $T \geqslant 0$,

$$F(T) \leqslant e^{-\delta_0\int_0^T \xi(s)\, ds}\left(F(0) + \beta_2\int_0^T e^{\delta_0\int_0^t \xi(s)\, ds}h(t)\, dt\right)$$

which implies that, thanks to (3.12),

$$E(T) \leqslant \frac{1}{\beta_1} e^{-\delta_0 \int_0^T \xi(s)\,ds} \left(F(0) + \beta_2 \int_0^T e^{\delta_0 \int_0^t \xi(s)\,ds} h(t)\,dt \right) \tag{3.14}$$

Because

$$e^{\delta_0 \int_0^t \xi(s)\,ds} h(t) = \frac{1}{\delta_0} \left(e^{\delta_0 \int_0^t \xi(s)\,ds}\right)' \int_t^{+\infty} g(s)\,ds, \quad \forall t \in \mathbb{R}^+$$

then, by integration by parts,

$$\int_0^T e^{\delta_0 \int_0^t \xi(s)\,ds} h(t)\,dt = \frac{1}{\delta_0} \left(e^{\delta_0 \int_0^T \xi(s)\,ds} \int_T^{+\infty} g(s)\,ds - \int_0^{+\infty} g(s)\,ds + \int_0^T e^{\delta_0 \int_0^t \xi(s)\,ds} g(t)\,dt \right)$$

Consequently, combining with (3.14),

$$\begin{aligned} E(T) \leqslant{} & \frac{1}{\beta_1} \left(F(0) e^{-\delta_0 \int_0^T \xi(s)\,ds} + \frac{\beta_2}{\delta_0} \int_T^{+\infty} g(s)\,ds \right) \\ & + \frac{\beta_2}{\beta_1 \delta_0} e^{-\delta_0 \int_0^T \xi(s)\,ds} \int_0^T e^{\delta_0 \int_0^t \xi(s)\,ds} g(t)\,dt \end{aligned} \tag{3.15}$$

On the other hand, (2.3) implies that $(e^{\int_0^t \xi(s)\,ds} g(t))' \leqslant 0$, for all $t \in \mathbb{R}^+$, and then $e^{\int_0^t \xi(s)\,ds} g(t) \leqslant g(0)$. Therefore,

$$\int_0^T e^{\delta_0 \int_0^t \xi(s)\,ds} g(t)\,dt \leqslant \big(g(0)\big)^{\delta_0} \int_0^T \big(g(t)\big)^{1-\delta_0}\,dt \tag{3.16}$$

Finally, (3.1), (3.15) and (3.16) give (2.8) with

$$\delta_1 = \frac{2}{\beta_1} \max\left\{ F(0), \frac{\beta_2}{\delta_0}, \frac{\beta_2}{\delta_0} \big(g(0)\big)^{\delta_0} \right\}$$

4. Applications

In this section, we discuss some particular problems that fall in the framework of our abstract model (1.1).

4.1. Finite memory

When $u_0(t) = 0$, $\forall t > 0$, (1.1) takes the form

$$\begin{cases} u_{tt}(t) + Au(t) - \int_0^t g(s) Au(t-s)\,ds = 0, & \forall t > 0 \\ u(0) = u_0, \qquad u_t(0) = u_1 \end{cases}$$

A close look at the proof of Theorem 2.1 shows that the decay estimate (2.8) becomes

$$\left\|\mathcal{U}(t)\right\|_{\mathcal{H}}^2 \leqslant \delta_1 e^{-\delta_0 \int_0^t \xi(s)\,ds}$$

which is the result obtained in [19].

4.2. Wave equation

Our result (2.8) holds for the following problem:

$$\begin{cases} u_{tt}(x,t) - \Delta u(x,t) + \int\limits_0^{+\infty} g(s)\Delta u(x,t-s)\,ds = 0, & \text{in } \Omega \times]0,+\infty[\\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+ \end{cases}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^n$. This is a particular case of (1.1), with $A = -\Delta$, $H = L^2(\Omega)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

4.3. Elastic system

Our result (2.8) holds for the following problem:

$$\begin{cases} u_{tt}(t) - \sum\limits_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u(x,t)}{\partial x_k}\right) + \int\limits_0^{+\infty} g(s) \sum\limits_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u(x,t-s)}{\partial x_k}\right) ds = 0, & \text{in } \Omega \times]0,+\infty[\\ u(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), & \text{in } \Omega \times \mathbb{R}^+ \end{cases}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^n$, $a_{jk} \in C^1(\overline{\Omega})$, $j,k = 1,\dots,n$ satisfying some smoothness, symmetry and coercivity conditions. This is a particular case of (1.1), with

$$A = -\sum_{j,k=1}^{n} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial}{\partial x_k}\right)$$

$H = L^2(\Omega)$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

4.4. Petrovsky system

Our result (2.8), also, holds for the following problem:

$$\begin{cases} u_{tt}(t) + \Delta^2 u(t) - \int\limits_0^{+\infty} g(s)\Delta^2 u(t-s)\,ds = 0, & \text{in } \Omega \times]0,+\infty[\\ u(x,t) = \frac{\partial u}{\partial \nu}(x,t) = 0, & \text{on } \partial\Omega \times \mathbb{R}^+ \\ u(x,-t) = u_0(x,t), \qquad u_t(x,0) = u_1(x), & \text{on } \partial\Omega \times \mathbb{R}^+ \end{cases}$$

where Ω is a bounded and smooth domain of $\mathbb{R}^n$ and ν is the unit outer normal to Ω. This is a particular case of (1.1), with $A = \Delta^2$, $H = L^2(\Omega)$ and $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$.

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