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Asymptotic stability of abstract dissipative systems with infinite memory

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ABSTRACT

We consider in this paper the problem of asymptotic behavior of solutions to an abstract linear dissipative integrodifferential equation with infinite memory (past history) modeling linear viscoelasticity. We show that the stability of the system holds for a much larger class of the convolution kernels than the one considered in the literature, and we provide a relation between the decay rate of the solutions and the growth of the kernel at infinity. Some applications are also given.

1. Introduction

Let H be a Hilbert space with inner product and related norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $A : D(A) \to H$ and $B : D(B) \to H$ be self-adjoint linear positive definite operators with domains $D(A) \subset D(B) \subset H$ such that the embeddings are dense and compact. We consider the following class of second-order linear integrodifferential equation:

$$u''(t) + Au(t) - \int_0^\infty g(s)Bu(t-s)\,ds = 0, \quad \forall t > 0, \tag{1.1}$$

where $' = \frac{\partial}{\partial t}$, with initial conditions

$$\begin{cases} u(-t) = u_0(t), & \forall t \in \mathbb{R}_+ = [0, \infty[, \\ u'(0) = u_1, \end{cases} \tag{1.2}$$

where u_0 and u_1 are given history and initial data, and the convolution kernel $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a given function which represents the term of dissipation.

In the particular case $A = B = -\Delta$ (the negative Laplacian operator with respect to the space variable) on $L^2(\Omega)$ (where $\Omega \subset \mathbb{R}^n$ is a given domain) with Dirichlet boundary conditions, Eq. (1.1) describes the dynamics of linear viscoelastic solids (see [16] for example). Eq. (1.1) can also used to formulate a generalized Kirchhoff viscoelastic beam with memory (see [15] and the references therein). For more details concerning the physical phenomena which are modeled by differential equations with memory, as well as the problem of the modeling of materials with memory, we refer the reader to the recent and interesting paper [18].

It is well known, following a method devised in the pioneering paper [5] (see also [13,15,16]), that the system (1.1)–(1.2) can be formulated as the following abstract linear first-order system:

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$$\begin{cases} \mathcal{U}'(t) + \mathcal{A}\mathcal{U}(t) = 0, & \forall t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{1.3}$$

where $\mathcal{U}_0 = (u_0(0), u_1, \eta_0)^T \in \mathcal{H} = D(A^{\frac{1}{2}}) \times H \times L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))$, $\mathcal{U} = (u, u', \eta^t)^T$,

$$\begin{cases} \eta^t(s) = u(t) - u(t-s), & \forall t, s \in \mathbb{R}_+, \\ \eta_0(s) = \eta^0(s) = u_0(0) - u_0(s), & \forall s \in \mathbb{R}_+ \end{cases}$$

(η^t is the relative history of u, and it was introduced first in [5]), $L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))$ is the weighted space with respect to the measure $g(s)\,ds$ defined by

$$L^2_g\big(\mathbb{R}_+, D\big(B^{\frac{1}{2}}\big)\big) = \left\{ z\colon \mathbb{R}_+ \to D\big(B^{\frac{1}{2}}\big),\ \int_0^\infty g(s) \big\| B^{\frac{1}{2}} z(s) \big\|^2 \,ds < \infty \right\}$$

endowed with the inner product

$$\langle z_1, z_2 \rangle_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))} = \int_0^\infty g(s) \big\langle B^{\frac{1}{2}} z_1(s), B^{\frac{1}{2}} z_2(s) \big\rangle \,ds,$$

and $\mathcal{A}$ is the linear operator given by

$$\mathcal{A}(v, w, z)^T = \left(-w, Av - g_0 Bv + \int_0^\infty g(s) Bz(s)\,ds, \frac{\partial z}{\partial s} - w \right)^T,$$

where $g_0 = \int_0^\infty g(s)\,ds$,

$$D(\mathcal{A}) = \left\{ (v, w, z)^T \in \mathcal{H},\ v \in D(A),\ w \in D\big(A^{\frac{1}{2}}\big),\ z \in \mathcal{L}_g,\ \int_0^\infty g(s) Bz(s)\,ds \in H \right\}$$

and $\mathcal{L}_g = \{z \in L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}})),\ \partial_s z \in L_g,\ z(0) = 0\}$. Under the following assumptions on A, B and g:

(A0) there exist positive constants a_0 and a_1 such that

$$a_1 \|v\|^2 \leqslant \big\| B^{\frac{1}{2}} v \big\|^2 \leqslant a_0 \big\| A^{\frac{1}{2}} v \big\|^2, \quad \forall v \in D\big(A^{\frac{1}{2}}\big),$$

(A1) $g : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+)$ nonincreasing and satisfies

$$0 < g_0 < \frac{1}{a_0}, \tag{1.4}$$

it is well known (see [13] for example) that $\mathcal{H}$ endowed with the inner product

$$\big\langle (v_1, w_1, z_1)^T, (v_2, w_2, z_2)^T \big\rangle_{\mathcal{H}} = \big\langle A^{\frac{1}{2}} v_1, A^{\frac{1}{2}} v_2 \big\rangle - g_0 \big\langle B^{\frac{1}{2}} v_1, B^{\frac{1}{2}} v_2 \big\rangle + \langle w_1, w_2 \rangle + \langle z_1, z_2 \rangle_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))}$$

is a Hilbert space, $D(\mathcal{A}) \subset \mathcal{H}$ with dense embedding, and $\mathcal{A}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathcal{H}$. Therefore, the classical semigroup theory implies that (see [19]), for any $\mathcal{U}_0 \in \mathcal{H}$, the system (1.3) has a unique weak solution

$$\mathcal{U} \in C(\mathbb{R}_+, \mathcal{H}).$$

Moreover, if $\mathcal{U}_0 \in D(\mathcal{A})$, then the solution of (1.3) is classical; that is

$$\mathcal{U} \in C^1(\mathbb{R}_+, \mathcal{H}) \cap C\big(\mathbb{R}_+, D(\mathcal{A})\big).$$

The question we consider in this paper concerns the asymptotic stability of (1.3). In other words, for which class of kernels g we have (strong stability)

$$\lim_{t \to \infty} \big\| \mathcal{U}(t) \big\|^2_{\mathcal{H}} = 0, \tag{1.5}$$

and is it possible to get a decay estimation on $\|\mathcal{U}\|^2_{\mathcal{H}}$ in function of g?

This question was the subject of several works appeared in the last few years (see [3,6–9,13,15–18], and the references therein). To focus on our motivation, let us mention some known results in the literature related to the stabilization of abstract systems with past history (for further results of stabilization, we refer the reader to the list of references of this paper, which is not exhaustive, and the references therein).

Under the condition

$$\exists \delta > 0: \quad g'(s) \leqslant -\delta g(s), \quad \forall s \in \mathbb{R}_+, \tag{1.6}$$

the authors in [7] (using Laplace transform method), [8] (using energy estimates) and [9] (using semigroups theory) have proved the exponential (uniform) stability of (1.3); that is

$$\exists m, M > 0: \quad \|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant M e^{-mt}, \quad \forall t \in \mathbb{R}_+. \tag{1.7}$$

The authors in [13] considered (for given operators D, A, B and C) the more general abstract equation than (1.1), namely

$$Du''(t) + Au(t) - \int_0^\infty g(s) Bu(t-s)\, ds + Cu'(t) = 0, \quad \forall t > 0, \tag{1.8}$$

and proved that (1.7) holds if the operators satisfy some conditions and g satisfies

$$\exists \delta_1, \delta_2 > 0: \quad -\delta_1 g(s) \leqslant g'(s) \leqslant -\delta_2 g(s), \quad \forall s \in \mathbb{R}_+. \tag{1.9}$$

The dissipation of (1.8) is given by the infinite memory integral and the damping Cu', and then it is stronger than the one of (1.1).

Condition (1.9) was also considered in [14] to prove (1.7) for Timoshenko systems, and in [12] it was proved that these Timoshenko systems are polynomially stable if, in some how, g converges to zero faster than $\frac{1}{t^2}$.

Eq. (1.1) in the particular case $A = \alpha B$ with $\alpha > 0$ was considered in [16], and (1.7) was proved for g equal to a negative exponential except on a sufficiently small set where g is flat. This condition allows g to have horizontal inflection points or even flat zones. The case $A = \alpha B$ ($\alpha > 0$) was also considered in [18] with general memory; that is the infinite integral in (1.1) is replaced with $\int_0^l$ ($l \in]0, \infty]$), and the exponential stability (1.7) was proved under the condition (1.6).

In [4], it was proved that the weaker condition

$$\exists \delta_1 \geqslant 1,\ \exists \delta_2 > 0: \quad g(t+s) \leqslant \delta_1 e^{-\delta_2 t} g(s), \quad \forall t \in \mathbb{R}_+,\ \text{for a.e. } s \in \mathbb{R}_+ \tag{1.10}$$

is a *necessary* condition for (1.3) to be exponentially stable; that is (1.7) holds. In the particular case $A = -(1+g_0)\Delta$ and $B = -\Delta$ (with homogeneous Dirichlet boundary condition), the exponential stability (1.7) was discussed in [17], establishing a necessary and sufficient condition involving the kernel g, this condition implies (1.10) but allows g to be almost flat.

It was also proved in [13], for some operators D, A, B and C, and under condition (1.9), that (1.8) is not exponentially stable and, for any $\mathcal{U}_0 \in D(\mathcal{A})$, the following polynomial rate of decay was obtained:

$$\exists M > 0: \quad \|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant \frac{M}{t}, \quad \forall t > 0. \tag{1.11}$$

Eq. (1.1) with $B = A^\alpha$, $\alpha \in [0, 1[$ and g satisfies (1.6) was recently considered in [15], where the authors proved that for any $\mathcal{U}_0 \in D(\mathcal{A})$,

$$\exists M > 0: \quad \|\mathcal{U}(t)\|_{\mathcal{H}} \leqslant M \left(\frac{\ln t}{t}\right)^{\frac{1}{2-2\alpha}} \ln t, \quad \forall t > 0,$$

and the decay rate is optimal in the sense that $t^{\frac{-1}{2-2\alpha}}$ cannot be improved on $D(\mathcal{A})$.

According to the results cited above, the problem of stability of (1.3) is well solved when the kernel g converges exponentially to zero at infinity, and then there is almost nothing more to say in this case. The main question now is the following: is (1.3) still stable (that is (1.5) holds) if g does not satisfy (1.10), and if yes, is it possible to get a decay estimate on $\|\mathcal{U}\|_{\mathcal{H}}^2$? In other words, when g *does not converge exponentially* to zero at infinity (then (1.7) is not satisfied), what kind of decay estimates we have? The aim of the present paper is to give a positive answer to these questions by proving that (1.3) is stable for much larger class of kernels g than the one satisfying (1.10), and providing a general decay estimate on $\|\mathcal{U}\|_{\mathcal{H}}^2$ in function of g. Our decay estimate is necessarily weaker than (1.7). We consider two cases corresponding to the following two conditions on A and B: there exists a positive constant a_2 such that

$$\|A^{\frac{1}{2}} v\|^2 \leqslant a_2 \|B^{\frac{1}{2}} v\|^2, \quad \forall v \in D(A^{\frac{1}{2}}) \tag{1.12}$$

or

$$\|A^{\frac{1}{2}} v\|^2 \leqslant a_2 \|A^{\frac{1}{2}} B^{\frac{1}{2}} v\|^2, \quad \forall v \in D(A^{\frac{1}{2}} B^{\frac{1}{2}}). \tag{1.13}$$

The decay estimate (2.4) we get under condition (1.12) is stronger than (2.6) we get by assuming (1.13) (see examples below).

The plan of the present paper is as follows: in Section 2, we specify the assumption on g and formulate our main stability results. Section 3 is devoted to the proof of the main results. Finally, in Section 4, we discuss some applications, give some general comments and state some open problems.

2. The main results

In addition to (1.12) or (1.13), (A0) and (A1), assuming that g satisfies the following assumption:

(A2) There exists an increasing strictly convex function $G:\mathbb{R}_+\to\mathbb{R}_+$ of class $C^1(\mathbb{R}_+)\cap C^2(]0,\infty[)$ satisfying

$$G(0)=G'(0)=0 \quad\text{and}\quad \lim_{t\to\infty} G'(t)=\infty \tag{2.1}$$

such that

$$\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))}\,ds+\sup_{s\in\mathbb{R}_+}\frac{g(s)}{G^{-1}(-g'(s))}<\infty. \tag{2.2}$$

Remark 2.1. The class of kernels satisfying (A1)–(A2) and do not satisfy (1.10) is very large; for example

$$g(t)=\frac{d}{(t+2)(\ln(t+2))^q}$$

with $d>0$ small enough such that (1.4) holds, and $q>1$, where we can take $G(t)=e^{-t^{-p}}$ for $p>\frac{1}{q-1}$ and t near zero (see also the examples below). In general, all positive function g of class $C^1(\mathbb{R}_+)$ with $g'<0$ satisfies (A2) if it is integrable on $\mathbb{R}_+$, and it does not satisfy (1.10) if it does not converge exponentially to zero at infinity.

Now, we are in position to state our main results.

Theorem 2.1. *Assume that* (A0)–(A2) *hold.*

(1) *If* (1.12) *holds, then for any* $\mathcal{U}_0\in\mathcal{H}$ *satisfying*

$$\exists m_0\geqslant 0:\quad \left\|B^{\frac12}u_0(s)\right\|\leqslant m_0,\quad \forall s>0, \tag{2.3}$$

there exist positive constants δ_0, δ_1 *and* δ_2 (*depending continuously on* $\|\mathcal{U}_0\|^2_{\mathcal{H}}$) *such that*

$$\left\|\mathcal{U}(t)\right\|^2_{\mathcal{H}}\leqslant \delta_1 G_1^{-1}(\delta_2 t),\quad \forall t\in\mathbb{R}_+, \tag{2.4}$$

where $G_1(t)=\int_t^1\frac{1}{sG'(\delta_0 s)}\,ds$ $(t\in\,]0,1])$.

(2) *If* (1.13) *holds, then for any* $\mathcal{U}_0\in D(A)\times D(A^{\frac12})\times L^2_g(\mathbb{R}_+,D(A^{\frac12}B^{\frac12}))$ *satisfying*

$$\exists m_0\geqslant 0:\quad \left\|A^{\frac12}B^{\frac12}u_0(s)\right\|\leqslant m_0,\quad \forall s>0, \tag{2.5}$$

there exist positive constants δ_0, δ_1 *and* δ_2 (*depending continuously on* $\|A^{\frac12}\mathcal{U}_0\|^2_{\mathcal{H}}$) *such that*

$$\left\|\mathcal{U}(t)\right\|^2_{\mathcal{H}}\leqslant \delta_1 G_0^{-1}\left(\frac{\delta_2}{t}\right),\quad \forall t>0, \tag{2.6}$$

where $G_0(t)=tG'(\delta_0 t)$ $(t\in\mathbb{R}_+)$.

Remark 2.2. Before going on, let us first give some comments on our results. Our estimates (2.4) and (2.6) imply (1.5) (since $G_0^{-1}(0)=0$ and $\lim_{s\to 0^+}G_1(s)=\infty$) and they are weaker than (1.7) and (1.11) in general, and coincide with (1.7) and (1.11) when $G=Id$, respectively. But the class of kernels g satisfying (A1)–(A2) and does not satisfy (1.10) (which is a *necessary* condition to get (1.7)) is very huge (see Remark 2.1 above), and on the other hand, estimates (2.4) and (2.6) give precise and general informations on the decay rate of $\|\mathcal{U}\|^2_{\mathcal{H}}$ in function of g.

Now, let us give just three simple examples to illustrate our results. In these examples (where (1.10) does not hold), we see that (2.4) implies that $\|\mathcal{U}\|^2_{\mathcal{H}}$ has at least a similar decay to the one of g when (1.12) holds, but the decay rate of $\|\mathcal{U}\|^2_{\mathcal{H}}$ is smaller than the one of g (nevertheless, it is arbitrary close to the one of g in (2.9)). When the weaker condition (1.13) holds, (2.6) is much weaker than (2.4) (nevertheless, it is arbitrary close to (1.11) in (2.10) and (2.12)).

Example 2.1. Let $g(t) = \frac{d}{(1+t)^q}$ with $q > 1$, and $d > 0$ small enough so that (1.4) holds. Assumption (A2) is satisfied with $G(t) = t^{\frac{1}{p}+1}$ for any $p \in]0, \frac{q-1}{2}[$. Then (2.4) and (2.6) imply, respectively (for some $C > 0$),

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant \frac{C}{(t+1)^p}, \quad \forall t \in \mathbb{R}_+,\ \forall p \in \left]0, \frac{q-1}{2}\right[, \tag{2.7}$$

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant \frac{C}{(t+1)^{\frac{p}{p+1}}}, \quad \forall t \in \mathbb{R}_+,\ \forall p \in \left]0, \frac{q-1}{2}\right[. \tag{2.8}$$

Example 2.2. Let $g(t) = de^{-(\ln(2+t))^q}$ with $q > 1$, and $d > 0$ small enough so that (1.4) holds. For

$$G(t) = \int_0^t (-\ln s)^{1-\frac{1}{p}} e^{-(-\ln s)^{\frac{1}{p}}}\, ds$$

when t is near zero, assumption (A2) is satisfied for any $p \in]1, q[$ (note that for this example, (2.2) depends only on the growth of G at zero, and for any $r > 1$, $G(t^r g(t)) \leqslant -g'(t)$ for t near infinity). Then (2.4) implies (for some $C_1, C_2 > 0$)

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant C_1 e^{-C_2(\ln(1+t))^p}, \quad \forall t \in \mathbb{R}_+,\ \forall p \in]1, q[. \tag{2.9}$$

Assumption (A2) holds also with $G(t) = t^p$ for any $p > 1$. Then (2.6) implies (for some $C > 0$)

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant \frac{C}{(t+1)^{\frac{1}{p}}}, \quad \forall t \in \mathbb{R}_+,\ \forall p > 1. \tag{2.10}$$

Example 2.3. Let $g(t) = de^{-(1+t)^q}$ with $0 < q < 1$, and $d > 0$ small enough so that (1.4) holds. Assumption (A2) is satisfied with

$$G(t) = \int_0^t (-\ln s)^{1-\frac{1}{p}}\, ds$$

for t near zero and for any $p \in]0, \frac{q}{2}[$ (we can see that $G(t^r G(t)) \leqslant -g'(t)$ for t near infinity and for any $r \in]1, \frac{q}{p} - 1[$). Then (2.4) implies (for some $C_1, C_2 > 0$)

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant C_1 e^{-C_2 t^p}, \quad \forall t \in \mathbb{R}_+,\ \forall p \in \left]0, \frac{q}{2}\right[. \tag{2.11}$$

Assumption (A2) holds also with $G(t) = t^p$ for any $p > 1$. Then (2.6) implies (for some $C > 0$)

$$\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \leqslant \frac{C}{(t+1)^{\frac{1}{p}}}, \quad \forall t \in \mathbb{R}_+,\ \forall p > 1. \tag{2.12}$$

3. Proof of Theorem 2.1

In order to prove (2.4) and (2.6), first, we assume that (A0) and (A1) are satisfied and we consider the energy functional E associated with the solution of (1.3) corresponding to $\mathcal{U}_0 \in \mathcal{H}$,

$$\begin{aligned} E(t) &= \frac{1}{2}\|\mathcal{U}(t)\|_{\mathcal{H}}^2 \\ &= \frac{1}{2}\left(\|A^{\frac{1}{2}}u(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2 + \|u'(t)\|^2\right) + \frac{1}{2}\int_0^\infty g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\, ds, \quad \forall t \in \mathbb{R}_+. \end{aligned} \tag{3.1}$$

Using simple computations (multiplying (1.1) by $u'(t)$ and integrating by parts, see [13]), we get

$$E'(t) = \frac{1}{2}\int_0^\infty g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\, ds \leqslant 0, \quad \forall t \in \mathbb{R}_+. \tag{3.2}$$

Therefore, thanks to the nonincreasingness of g, E is nonincreasing, and consequently (1.3) is dissipative, where the total dissipation is given by the infinite memory integral.

Now, we proceed as in [13] to prove the following three lemmas for classical solutions, so all the calculations are justified. By a simple density argument, these lemmas remain valid for any weak solution. On the other hand, if $E(t_0)=0$ for some $t_0 \geqslant 0$, then $E(t)=0$ for all $t \geqslant t_0$ (thanks to (3.2)) and thus (2.4) and (2.6) are satisfied. Then, without loss of generality, we assume that $E(t)>0$ for all $t \in \mathbb{R}_+$.

Lemma 3.1. *Assume that* (A0) *and* (A1) *are satisfied. Then the functional*

$$I_1(t) = -\left\langle u'(t), \int_0^\infty g(s)\eta^t(s)\,ds \right\rangle$$

satisfies, for any $\epsilon > 0$, *there exists* $c_\epsilon > 0$ *such that for all* $t \in \mathbb{R}_+$,

$$I_1'(t) \leqslant -(g_0-\epsilon)\left\|u'(t)\right\|^2 + \epsilon\left\|A^{\frac{1}{2}}u(t)\right\|^2 + c_\epsilon \int_0^\infty g(s)\left\|A^{\frac{1}{2}}\eta^t(s)\right\|^2 ds$$
$$+ c_\epsilon\left(\int_0^\infty g(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds - \int_0^\infty g'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds\right). \tag{3.3}$$

Proof. Multiplying (1.1) by $\int_0^\infty g(s)\eta^t(s)\,ds$, we get

$$0 = \left\langle u'', \int_0^\infty g(s)\eta^t(s)\,ds\right\rangle + \left\langle Au, \int_0^\infty g(s)\eta^t(s)\,ds\right\rangle - g_0\left\langle Bu, \int_0^\infty g(s)\eta^t(s)\,ds\right\rangle + \left\langle \int_0^\infty g(s)B\eta^t(s)\,ds, \int_0^\infty g(s)\eta^t(s)\,ds\right\rangle.$$

Using the definition of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we get

$$0 = \left\langle u'', \int_0^\infty g(s)\eta^t(s)\,ds\right\rangle + \left\langle A^{\frac{1}{2}}u, \int_0^\infty g(s)A^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle - g_0\left\langle B^{\frac{1}{2}}u, \int_0^\infty g(s)B^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle$$
$$+ \left\langle \int_0^\infty g(s)B^{\frac{1}{2}}\eta^t(s)\,ds, \int_0^\infty g(s)B^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle.$$

By using the fact that $\partial_t\eta^t = -\partial_s\eta^t + u'$ (we note $\partial_\mu = \frac{\partial}{\partial\mu}$), we find

$$\left\langle u'', \int_0^\infty g(s)\eta^t(s)\,ds\right\rangle = \partial_t\left\langle u', \int_0^\infty g(s)\eta^t(s)\,ds\right\rangle - \left\langle u', \int_0^\infty g(s)\partial_t\eta^t(s)\,ds\right\rangle$$
$$= -I_1' - g_0\left\|u'\right\|^2 + \left\langle u', \int_0^\infty g(s)\partial_s\eta^t(s)\,ds\right\rangle.$$

By integrating by parts with respect to s in the infinite memory integral, we get

$$\left\langle u'', \int_0^\infty g(s)\eta^t(s)\,ds\right\rangle = -I_1' - g_0\left\|u'\right\|^2 - \left\langle u', \int_0^\infty g'(s)\eta^t(s)\,ds\right\rangle.$$

By exploiting these equalities, we deduce

$$I_1' = -g_0\left\|u'\right\|^2 - \left\langle u', \int_0^\infty g'(s)\eta^t(s)\,ds\right\rangle + \left\langle A^{\frac{1}{2}}u, \int_0^\infty g(s)A^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle - g_0\left\langle B^{\frac{1}{2}}u, \int_0^\infty g(s)B^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle$$
$$+ \left\langle \int_0^\infty g(s)B^{\frac{1}{2}}\eta^t(s)\,ds, \int_0^\infty g(s)B^{\frac{1}{2}}\eta^t(s)\,ds\right\rangle.$$

By using Cauchy–Schwarz inequality and Young's inequality for the last four terms of this equality, and (A0) to estimate $\|B^{\frac{1}{2}}u\|^2$ by $a_0\|A^{\frac{1}{2}}u\|^2$, and $\|\eta^t(s)\|^2$ by $\frac{1}{a_1}\|B^{\frac{1}{2}}\eta^t(s)\|^2$, (3.3) follows. □

Lemma 3.2. *Assume that* (A0) *and* (A1) *are satisfied. Then the functional*

$$I_2(t) = \langle u'(t), u(t)\rangle$$

satisfies, for any $\epsilon > 0$, there exists $\tilde{c}_\epsilon > 0$ such that for all $t \in \mathbb{R}_+$,

$$I_2'(t) \leqslant \|u'(t)\|^2 - (1 - g_0 a_0 - \epsilon)\|A^{\frac{1}{2}} u(t)\|^2 + \tilde{c}_\epsilon \int_0^\infty g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds. \tag{3.4}$$

Proof. Multiplying (1.1) by u we find

$$0 = \langle u'', u\rangle + \langle Au, u\rangle - g_0\langle Bu, u\rangle + \left\langle \int_0^\infty g(s)B\eta^t(s)\,ds, u\right\rangle.$$

Consequently, using the definition of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we have

$$0 = \partial_t\langle u', u\rangle - \|u'\|^2 + \|A^{\frac{1}{2}}u\|^2 - g_0\|B^{\frac{1}{2}}u\|^2 + \left\langle \int_0^\infty g(s)B^{\frac{1}{2}}\eta^t(s)\,ds, B^{\frac{1}{2}}u\right\rangle.$$

By using Cauchy–Schwarz inequality and Young's inequality for the last term of this equality, and (A0) to estimate $\|B^{\frac{1}{2}}u\|^2$ by $a_0\|A^{\frac{1}{2}}u\|^2$, (3.4) holds. □

Lemma 3.3. *Assume that* (A0) *and* (A1) *are satisfied. Then there exist $\alpha_0, \alpha_1, \alpha_2 > 0$ such that the functional*

$$I_3 = I_1 + \frac{g_0}{2} I_2 + \alpha_0 E$$

satisfies for all $t \in \mathbb{R}_+$,

$$I_3'(t) \leqslant -\alpha_1 E(t) + \alpha_2\left(\int_0^\infty g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds + \int_0^\infty g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right). \tag{3.5}$$

Proof. Multiplying (3.4) by $\frac{g_0}{2}$, adding (3.3), choosing $\epsilon = \frac{g_0(1-a_0 g_0)}{2(2+g_0)}$ ($\epsilon > 0$ thanks to (1.4)), using (3.2) to replace $\int_0^\infty g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds$ with $2E'(t)$, and noting that (thanks to (3.1)),

$$E(t) \leqslant \frac{1}{2}\left(\|A^{\frac{1}{2}}u(t)\|^2 + \|u'(t)\|^2 + \int_0^\infty g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right), \quad \forall t \in \mathbb{R}_+,$$

we get (3.5). □

Now, let

$$I_4 = ME + I_3$$

for positive constant M (to be chosen later). Thanks to (A0) and (1.4), we see that for all $t \in \mathbb{R}_+$,

$$E(t) \geqslant \frac{1 - a_0 g_0}{2}\left(\|A^{\frac{1}{2}}u(t)\|^2 + \|u'(t)\|^2 + \int_0^\infty g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right), \tag{3.6}$$

and then, by definition of I_1 and I_2, there exist two positive constants d_1 and d_2 such that $|I_1| \leqslant d_1 E$ and $|I_2| \leqslant d_2 E$. Therefore, there exists $M_0 > 0$ such that $|I_3| \leqslant M_0 E$. Then we choose $M > M_0$ and we get $I_4 \sim E$; that is

$$\exists M_1, M_2 > 0: \quad M_1 E \leqslant I_4 \leqslant M_2 E.$$

Thanks to (3.5) and the nonincreasingness of E, we have for all $t \in \mathbb{R}_+$,

$$I_4'(t) \leqslant -\alpha_1 E(t) + \alpha_2\left(\int_0^\infty g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2\,ds + \int_0^\infty g(s)\|A^{\frac{1}{2}}\eta^t(s)\|^2\,ds\right). \tag{3.7}$$

Now, we estimate the first infinite memory integral of (3.7) in function of E'. This is the main difficulty in the proof of the stability of (1.3). When g satisfies condition (1.6), the classical conclusion is immediate (using (3.2)):

$$\int_0^\infty g(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \leqslant \frac{-2}{\delta}E'(t), \quad \forall t \in \mathbb{R}_+.$$

This inequality coincides with (3.8) below when $G = Id$. Here, by proving (3.8), we introduce a new approach to estimate this term under (2.3) and the weaker assumption (A2).

Lemma 3.4. *Assume that* (A0)–(A2) *and* (2.3) *are satisfied. Then there exists* $\beta_1 > 0$ *such that for all* $\delta_0 > 0$ *and* $t \in \mathbb{R}_+$,

$$G'\big(\delta_0 E(t)\big)\int_0^\infty g(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \leqslant -\beta_1 E'(t) + \beta_1\delta_0 E(t)G'\big(\delta_0 E(t)\big). \tag{3.8}$$

Proof. First, we note that, if $g'(s_0) = 0$ for some $s_0 \geqslant 0$, then $g(s_0) = 0$ because $G^{-1}(0) = 0$ and $s \mapsto \frac{g(s)}{G^{-1}(-g'(s))}$ is bounded (thanks to (A2)), and therefore, $g(s) = 0$ for all $s \geqslant s_0$ because g is nonnegative and nonincreasing. This implies that the infinite integrals in (3.7) are effective only on $[0, s_0]$. Thus, without loss of generality, we can assume that $g' < 0$.

Let $G^*(t) = \sup_{s\in\mathbb{R}_+}\{ts - G(s)\}$ for $t \in \mathbb{R}_+$ denote the dual function of G. Thanks to (A2), G' is increasing and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, and then for any $t \in \mathbb{R}_+$, the function $s \mapsto ts - G(s)$ reaches its maximum on $\mathbb{R}_+$ at the unique point $(G')^{-1}(t)$. Therefore

$$G^*(t) = t\big(G'\big)^{-1}(t) - G\big(\big(G'\big)^{-1}(t)\big), \quad \forall t \in \mathbb{R}_+.$$

Let $\delta_0, \tau_1, \tau_2 > 0$. Using the general Young's inequality: $t_1 t_2 \leqslant G(t_1) + G^*(t_2)$ for

$$t_1 = G^{-1}\big(-\tau_2 g'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2\big), \qquad t_2 = \frac{\tau_1 G'(\delta_0 E(t))g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2}{G^{-1}(-\tau_2 g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2)},$$

we get for all $t \in \mathbb{R}_+$,

$$\begin{aligned}\int_0^\infty g(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds &= \frac{1}{\tau_1 G'(\delta_0 E(t))}\int_0^\infty G^{-1}\big(-\tau_2 g'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2\big)\frac{\tau_1 G'(\delta_0 E(t))g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2}{G^{-1}(-\tau_2 g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2)}\,ds \\ &\leqslant -\frac{\tau_2}{\tau_1 G'(\delta_0 E(t))}\int_0^\infty g'(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds \\ &\quad + \frac{1}{\tau_1 G'(\delta_0 E(t))}\int_0^\infty G^*\left(\frac{\tau_1 G'(\delta_0 E(t))g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2}{G^{-1}(-\tau_2 g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2)}\right)ds.\end{aligned}$$

Using (3.2) and the fact that $G^*(s) \leqslant s(G')^{-1}(s)$, we get for all $t \in \mathbb{R}_+$,

$$\begin{aligned}\int_0^\infty g(s)\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 ds &\leqslant -\frac{2\tau_2}{\tau_1 G'(\delta_0 E(t))}E'(t) \\ &\quad + \int_0^\infty \frac{g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2}{G^{-1}(-\tau_2 g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2)}\big(G'\big)^{-1}\left(\frac{\tau_1 G'(\delta_0 E(t))g(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2}{G^{-1}(-\tau_2 g'(s)\|B^{\frac{1}{2}}\eta^t(s)\|^2)}\right)ds.\end{aligned} \tag{3.9}$$

Now, assumption (A0), (3.6) and the nonincreasingness of E imply that

$$\left\|B^{\frac{1}{2}}u(t)\right\|^2 \leqslant a_0\left\|A^{\frac{1}{2}}u(t)\right\|^2 \leqslant \frac{2a_0}{1-a_0g_0}E(t) \leqslant \frac{2a_0}{1-a_0g_0}E(0), \quad \forall t \in \mathbb{R}_+.$$

Therefore

$$\left\|B^{\frac{1}{2}}\eta^t(s)\right\|^2 \leqslant 2\left\|B^{\frac{1}{2}}u(t)\right\|^2 + 2\left\|B^{\frac{1}{2}}u(t-s)\right\|^2 \leqslant \frac{8a_0}{1-a_0g_0}E(0) + 2\sup_{\tau>0}\left\|B^{\frac{1}{2}}u_0(\tau)\right\|^2, \quad \forall t,\ s \in \mathbb{R}_+.$$

Then, in both cases (2.3) and (2.5) (note that (2.5) implies (2.3) thanks to (A0)), we deduce that there exists a positive constant N_1 satisfying

$$\left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 \leqslant N_1, \quad \forall t, s \in \mathbb{R}_+. \tag{3.10}$$

On the other hand, let $K(s) = \frac{s}{G^{-1}(s)}$ for $s \in \mathbb{R}_+$ ($K(0) = 0$ because, thanks to (A2), $\lim_{s\to 0^+} \frac{s}{G^{-1}(s)} = \lim_{t\to 0^+} \frac{G(t)}{t} = G'(0) = 0$). The function K is nondecreasing. Indeed, the fact that G^{-1} is concave and $G^{-1}(0) = 0$ (thanks to (A2)) implies that for any $0 \leqslant s_1 < s_2$

$$K(s_1) = \frac{s_1}{G^{-1}(\frac{s_1}{s_2} s_2 + (1 - \frac{s_1}{s_2})0)} \leqslant \frac{s_1}{\frac{s_1}{s_2} G^{-1}(s_2) + (1 - \frac{s_1}{s_2}) G^{-1}(0)} = \frac{s_2}{G^{-1}(s_2)} = K(s_2).$$

Therefore, using (3.10) and the fact that $(G')^{-1}$ is nondecreasing,

$$\begin{aligned}
(G')^{-1}\left(\frac{\tau_1 G'(\delta_0 E(t)) g(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2}{G^{-1}(-\tau_2 g'(s) \|B^{\frac{1}{2}} \eta^t(s)\|^2)} \right) &= (G')^{-1}\left(\frac{\tau_1 G'(\delta_0 E(t)) g(s)}{-\tau_2 g'(s)} K\left(-\tau_2 g'(s) \left\|B^{\frac{1}{2}} \eta^t(s)\right\|^2\right) \right) \\
&\leqslant (G')^{-1}\left(\frac{\tau_1 G'(\delta_0 E(t)) g(s)}{-\tau_2 g'(s)} K\left(-\tau_2 N_1 g'(s)\right) \right) \\
&\leqslant (G')^{-1}\left(\frac{\tau_1 N_1 G'(\delta_0 E(t)) g(s)}{G^{-1}(-\tau_2 N_1 g'(s))} \right).
\end{aligned}$$

Then we get from (3.9) and (3.10) that for all $t \in \mathbb{R}_+$,

$$\int_0^\infty g(s) \left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 ds \leqslant -\frac{2\tau_2}{\tau_1 G'(\delta_0 E(t))} E'(t) + N_1 \int_0^\infty \frac{g(s)}{G^{-1}(-\tau_2 N_1 g'(s))} (G')^{-1}\left(\frac{\tau_1 N_1 G'(\delta_0 E(t)) g(s)}{G^{-1}(-\tau_2 N_1 g'(s))} \right) ds.$$

Condition (2.2) implies that $\sup_{s\in\mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} = N_2 < \infty$. Then, choosing $\tau_2 = \frac{1}{N_1}$ and using again the fact that $(G')^{-1}$ is nondecreasing, we find for all $t \in \mathbb{R}_+$,

$$\int_0^\infty g(s) \left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 ds \leqslant -\frac{2}{\tau_1 N_1 G'(\delta_0 E(t))} E'(t) + N_1 (G')^{-1}\left(\tau_1 N_1 N_2 G'\left(\delta_0 E(t)\right)\right) \int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))} ds.$$

Similarly, thanks to (2.2), $\int_0^\infty \frac{g(s)}{G^{-1}(-g'(s))} ds = N_3 < \infty$. Then, choosing $\tau_1 = \frac{1}{N_1 N_2}$,

$$\int_0^\infty g(s) \left\| B^{\frac{1}{2}} \eta^t(s) \right\|^2 ds \leqslant -\frac{2N_2}{G'(\delta_0 E(t))} E'(t) + N_1 N_3 \delta_0 E(t), \quad \forall t \in \mathbb{R}_+$$

which gives (3.8) with $\beta_1 = \max\{2N_2, N_1 N_3\}$. □

Now, following the two assumptions (1.12) and (1.13), we have the following two cases:

Case 1: (1.12) holds. Using (1.12) to estimate $\|A^{\frac{1}{2}} \eta^t(s)\|^2$ by $a_2 \|B^{\frac{1}{2}} \eta^t(s)\|^2$ in (3.7), multiplying then by $G'(\delta_0 E(t))$ and using (3.8), we get

$$G'(\delta_0 E) I_4' + \beta_1 \alpha_2 (1 + a_2) E' \leqslant -\left(\alpha_1 - \beta_1 \alpha_2 (1 + a_2) \delta_0\right) E G'(\delta_0 E).$$

Choosing δ_0 small enough so that $\beta_2 = \alpha_1 - \beta_1 \alpha_2 (1 + a_2) \delta_0 > 0$ and put

$$F = \tau\left(G'(\delta_0 E) I_4 + \beta_1 \alpha_2 (1 + a_2) E\right)$$

with $\tau > 0$, we deduce (note that $G'(\delta_0 E)$ is nonincreasing)

$$F' \leqslant -\tau \beta_2 E G'(\delta_0 E). \tag{3.11}$$

Thanks to the fact that $I_4 \sim E$ and the nonincreasingness of $G'(\delta_0 E)$, we have $F \sim E$. Choosing $\tau > 0$ small enough so that

$$F \leqslant E \quad \text{and} \quad F(0) \leqslant 1, \tag{3.12}$$

we deduce from (3.11) that (note that $s \mapsto s G'(\delta_0 s)$ is nondecreasing)

$$F' \leqslant -\delta_2 F G'(\delta_0 F), \tag{3.13}$$

where $\delta_2 = \tau\beta_2$. Inequality (3.13) implies that $(G_1(F))' \geqslant \delta_2$, where $G_1(t) = \int_t^1 \frac{1}{sG'(\delta_0 s)}\,ds$ for $t \in]0,1]$. Then, by integrating over $[0,t]$, we get

$$G_1\bigl(F(t)\bigr) \geqslant \delta_2 t + G_1\bigl(F(0)\bigr) \geqslant \delta_2 t, \quad \forall t \in \mathbb{R}_+$$

since G_1 is nonincreasing, $F(0) \leqslant 1$ and $G_1(1) = 0$. Therefore,

$$F(t) \leqslant G_1^{-1}(\delta_2 t), \quad \forall t \in \mathbb{R}_+.$$

The equivalence $F \sim E$ and the definition (3.1) of E give (2.4).

Case 2: (1.13) holds. Multiplying (3.7) by $G'(\delta_0 E(t))$ and using (3.8), we get for all $\delta_0 > 0$ and $t \in \mathbb{R}_+$,

$$\begin{aligned} &G'\bigl(\delta_0 E(t)\bigr) I_4'(t) + \beta_1\alpha_2 E'(t) \\ &\quad \leqslant -(\alpha_1 - \beta_1\alpha_2\delta_0) E(t) G'\bigl(\delta_0 E(t)\bigr) + \alpha_2 G'\bigl(\delta_0 E(t)\bigr) \int_0^\infty g(s) \bigl\| A^{\frac{1}{2}} \eta^t(s) \bigr\|^2 ds. \end{aligned} \tag{3.14}$$

We follow an approach of [13] and we consider the energy E_2 associated with $A^{\frac{1}{2}}\mathcal{U}$ corresponding to $\mathcal{U}_0 \in D(A) \times D(A^{\frac{1}{2}}) \times L_g^2(\mathbb{R}_+, D(A^{\frac{1}{2}} B^{\frac{1}{2}}))$ defined on $\mathbb{R}_+$ by

$$\begin{aligned} E_2(t) &= \frac{1}{2} \bigl\| A^{\frac{1}{2}} \mathcal{U}(t) \bigr\|_{\mathcal{H}}^2 \\ &= \frac{1}{2} \bigl(\| Au(t) \|^2 - g_0 \bigl\| A^{\frac{1}{2}} B^{\frac{1}{2}} u(t) \bigr\|^2 + \bigl\| A^{\frac{1}{2}} u'(t) \bigr\|^2 \bigr) + \frac{1}{2} \int_0^\infty g(s) \bigl\| A^{\frac{1}{2}} B^{\frac{1}{2}} \eta^t(s) \bigr\|^2 ds. \end{aligned} \tag{3.15}$$

As for (3.2) (applying $A^{\frac{1}{2}}$ to (1.1), multiplying by $A^{\frac{1}{2}} u'(t)$ and integrating by parts), we get

$$E_2'(t) = \frac{1}{2} \int_0^\infty g'(s) \bigl\| A^{\frac{1}{2}} B^{\frac{1}{2}} \eta^t(s) \bigr\|^2 ds \leqslant 0, \quad \forall t \in \mathbb{R}_+. \tag{3.16}$$

Now, similarly to (3.8) (in the proof of (3.8), we replace $B^{\frac{1}{2}}$ with $A^{\frac{1}{2}} B^{\frac{1}{2}}$ and use (2.5) and (3.16)), we deduce that there exists $\lambda_1 > 0$ such that for all $\delta_0 > 0$ and $t \in \mathbb{R}_+$,

$$G'\bigl(\delta_0 E(t)\bigr) \int_0^\infty g(s) \bigl\| A^{\frac{1}{2}} B^{\frac{1}{2}} \eta^t(s) \bigr\|^2 ds \leqslant -\lambda_1 E_2'(t) + \lambda_1 \delta_0 E(t) G'\bigl(\delta_0 E(t)\bigr). \tag{3.17}$$

Using (1.13) to estimate $\|A^{\frac{1}{2}} \eta^t(s)\|^2$ by $a_2 \|A^{\frac{1}{2}} B^{\frac{1}{2}} \eta^t(s)\|^2$ in (3.14) and using (3.17), we get

$$G'(\delta_0 E) I_4' + \beta_1\alpha_2 E' + \lambda_1\alpha_2 a_2 E_2' \leqslant -\bigl(\alpha_1 - \alpha_2(\beta_1 + a_2\lambda_1)\delta_0\bigr) E G'(\delta_0 E).$$

Choosing δ_0 small enough so that $\beta_2 = \alpha_1 - \alpha_2(\beta_1 + a_2\lambda_1)\delta_0 > 0$ and put

$$F = G'(\delta_0 E) I_4 + \beta_1\alpha_2 E + \lambda_1\alpha_2 a_2 E_2,$$

we deduce (note that $G'(\delta_0 E)$ is nonincreasing)

$$F' \leqslant -\beta_2 G_0(E), \tag{3.18}$$

where $G_0(t) = tG'(\delta_0 t)$ for $t \in \mathbb{R}_+$. Thanks to the fact that $G_0(E)$ is nonincreasing, and by integrating (3.18) over $[0,t]$, we get for all $t \in \mathbb{R}_+$,

$$tG_0\bigl(E(t)\bigr) \leqslant \int_0^t G_0\bigl(E(s)\bigr) ds \leqslant \frac{-1}{\beta_2} \int_0^t F'(s)\, ds = \frac{1}{\beta_2}\bigl(F(0) - F(t)\bigr) \leqslant \frac{1}{\beta_2} F(0) = \delta_2.$$

Then, using the nondecreasingness of G_0 and the definition (3.1) of E, we get (2.6).

4. Applications

We present in this section some applications for the stability results of our abstract Eq. (1.1). In the first three applications which concern the wave equation, Petrovsky type system and elasticity model, we consider an open bounded domain $\Omega \subset \mathbb{R}^n$, where $n \in \mathbb{N}^*$, with smooth boundary Γ.

4.1. Wave equations

Let $a_{ij}, b_{ij} \in C^1(\bar{\Omega})$, $i, j = 1, \dots, n$, such that

$$a_{ij}(x) = a_{ji}(x), \qquad b_{ij}(x) = b_{ji}(x), \quad \forall i, j = 1, \dots, n,\ \forall x \in \Omega$$

and there exist $a, b > 0$ satisfying

$$\sum_{i,j=1}^{n} a_{ij}(x)\epsilon_i\epsilon_j \geqslant a \sum_{i=1}^{n} \epsilon_i^2, \qquad \sum_{i,j=1}^{n} b_{ij}(x)\epsilon_i\epsilon_j \geqslant b \sum_{i=1}^{n} \epsilon_i^2, \quad \forall \epsilon_1, \dots, \epsilon_n \in \mathbb{R},\ \forall x \in \Omega.$$

Let $A = -\sum_{i,j=1}^{n} \partial_i(a_{ij}\partial_j)$ and $B = -\sum_{i,j=1}^{n} \partial_i(b_{ij}\partial_j)$, where we note $\partial_k = \frac{\partial}{\partial x_k}$, and let us consider the problem

$$\begin{cases} u'' + Au - \displaystyle\int_0^\infty g(s)Bu(t-s)\,ds = 0, & \forall (x,t) \in \Omega \times \mathbb{R}_+, \\ u = 0, & \forall (x,t) \in \Gamma \times \mathbb{R}_+, \\ u(x,-t) = u_0(x,t), \qquad u'(x,0) = u_1(x), & \forall (x,t) \in \Omega \times \mathbb{R}_+. \end{cases} \tag{4.1}$$

The particular case $A = -\Delta$ (corresponds to $a_{ij} = \delta_{ij}$ the Kronecker's symbol) represents the classical wave equation. Problem (4.1) can be rewritten in the abstract form (1.3), where $H = L^2(\Omega)$ endowed with its natural inner product $\langle v_1, v_2 \rangle = \int_\Omega v_1 v_2\,dx$, $D(A^{\frac{1}{2}}) = H_0^1(\Omega) = \{v \in H^1(\Omega),\ v = 0 \text{ on } \Gamma\}$ and $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$. It is well known that A and B satisfy (A0) and (1.12), and then (2.4) holds under assumptions (A1), (A2) and (2.3).

If we consider in (4.1) $Bu = bu$ with $b > 0$, then (1.12) is not satisfied but (1.13) is, and then (2.6) holds under assumptions (A1), (A2) and (2.5).

4.2. Petrovsky type system

Let us consider the problem

$$\begin{cases} u'' + a\Delta^2 u - b\displaystyle\int_0^\infty g(s)\Delta^2 u(t-s)\,ds = 0, & \forall (x,t) \in \Omega \times \mathbb{R}_+, \\ u = \partial_\nu u = 0, & \forall (x,t) \in \Gamma \times \mathbb{R}_+, \\ u(x,-t) = u_0(x,t), \qquad u'(x,0) = u_1(x), & \forall (x,t) \in \Omega \times \mathbb{R}_+, \end{cases} \tag{4.2}$$

where $a, b > 0$ and ∂_ν is the outer normal derivative. For $A = a\Delta^2$, $B = b\Delta^2$, $H = L^2(\Omega)$, $D(A^{\frac{1}{2}}) = H_0^2(\Omega) = \{v \in H^2(\Omega),\ v = \partial_\nu v = 0 \text{ on } \Gamma\}$ and $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$, (A0) and (1.12) are satisfied and (4.2) is equivalent to (1.3). Then, under assumptions (A1), (A2) and (2.3), (2.4) holds.

If we consider in (4.2) $Bu = bu$ with $b > 0$ (instead of $Bu = b\Delta^2 u$), then (1.13) holds but (1.12) does not, and then (2.6) holds under assumptions (A1), (A2) and (2.5).

In this application, the following boundary conditions can also be considered (as in [13]):

$$u = \Delta u = 0, \quad \forall (x,t) \in \Gamma \times \mathbb{R}_+.$$

Under these boundary conditions, we can consider also $Bu = -b\Delta u$ with $b > 0$, then (1.13) holds but (1.12) does not, and then (2.6) holds under assumptions (A1), (A2) and (2.5).

4.3. Elasticity model

Let $a_{ijkl}, b_{ijkl} \in C^1(\bar{\Omega})$, $i, j, k, l = 1, \dots, n$, such that for all $i, j, k, l = 1, \dots, n$,

$$a_{ijkl}(x) = a_{jikl}(x) = a_{klij}(x), \qquad b_{ijkl}(x) = b_{jikl}(x) = b_{klij}(x), \quad \forall x \in \Omega,$$

and there exist $a, b > 0$ satisfying for all symmetric matrix $(\epsilon_{ij})_{ij}$,

$$\sum_{i,j,k,l=1}^{n} a_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geqslant a \sum_{i,j=1}^{n} \epsilon_{ij}^2, \qquad \sum_{i,j,k,l=1}^{n} b_{ijkl}(x)\epsilon_{ij}\epsilon_{kl} \geqslant b \sum_{i,j=1}^{n} \epsilon_{ij}^2, \quad \forall x \in \Omega.$$

Let $u = (u_1, \ldots, u_n)^T$, $Au = \frac{-1}{2}(\sum_{j,k,l=1}^{n} \partial_j (a_{ijkl}(\partial_k u_l + \partial_l u_k)))_i^T$ and $Bu = \frac{-1}{2}(\sum_{j,k,l=1}^{n} \partial_j (b_{ijkl}(\partial_k u_l + \partial_l u_k)))_i^T$. We consider the problem

$$\begin{cases} u'' + Au - \int_0^\infty g(s)Bu(t-s)\,ds = 0, & \forall (x,t) \in \Omega \times \mathbb{R}_+, \\ u = 0, & \forall (x,t) \in \Gamma \times \mathbb{R}_+, \\ u(x,-t) = u_0(x,t), \quad u'(x,0) = u_1(x), & \forall (x,t) \in \Omega \times \mathbb{R}_+. \end{cases} \tag{4.3}$$

The case $a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $\lambda, \mu > 0$ (the Lamé's coefficients) represents the isotropic elasticity model. Assumptions (1.12) and (A0), and the reformulation of (4.3) in the abstract form (1.3) hold with $H = (L^2(\Omega))^n$ endowed with the natural inner product $\langle v, w\rangle = \int_\Omega \sum_{i=1}^{n} v_i w_i\,dx$, $D(A^{\frac{1}{2}}) = (H_0^1(\Omega))^n$ and $D(A) = (H^2(\Omega) \cap H_0^1(\Omega))^n$. Then, under assumptions (A1), (A2) and (2.3), (2.4) holds.

If we consider in (4.3) $Bu = (b_1 u_1, \ldots, b_n u_n)^T$ with $b_i > 0$ $(i = 1, \ldots, n)$, then (1.12) is not satisfied but (1.13) is, and then (2.6) holds under assumptions (A1), (A2) and (2.5).

4.4. Infinite memory and internal damping

The well-possedness and stability results of this paper remain valid if we add to the Eq. (1.1) a linear damping $\beta u'$ $(\beta > 0)$:

$$u''(t) + Au(t) + \beta u' - \int_0^\infty g(s)Bu(t-s)\,ds = 0, \quad \forall t > 0. \tag{4.4}$$

Eq. (4.4) includes the one considered in [16] $(A = \alpha B,\ \alpha > 0)$. The dissipation in (4.4) is stronger than the one in (1.1) because it is induced by both past history and damping, and then the proof in case (4.4) is simpler; we do not need the functional I_1 because the term $\|u'\|^2$ can be directly estimated using the derivative of the energy of (4.4).

4.5. Finite memory

Our stability results (2.4) and (2.6) remain valid if we consider a finite memory; that is the infinite integral $\int_0^\infty$ in (1.1) (and then in particular in (4.1)–(4.4)) is replaced with the finite one $\int_0^t$:

$$u''(t) + Au(t) - \int_0^t g(s)Bu(t-s)\,ds = 0, \quad \forall t > 0. \tag{4.5}$$

Eq. (4.5) is in fact a particular case of (1.1) corresponding to a null past history ($u_0(t) = 0$ for all $t > 0$), and then the restrictions (2.3) and (2.5) are not needed in this case. Eq. (4.5) was considered in [1] in the particular case $A = B$ (with a semilinear source term) and g satisfies (A1) and a nonlinear differential inequality which implies that g converges to zero faster than $\frac{1}{t^2}$. On the other hand, in case (1.12) and for some particular kernels, our decay estimate (2.4) is stronger than the one of [1] (see (2.9) and (2.11)), where only the polynomial decay was obtained in [1].

In the particular case $A = B = -\Delta$, (4.5) was considered in [10] and [11] (with a nonlinear source term), and a general decay result (not necessarily of exponential or polynomial type) was established under condition (1.6) with positive and nonincreasing function $\delta = \delta(s)$. In the case $A = B$, Eq. (4.5) was considered in [2] with g satisfying

$$g'(s) \leqslant -\delta(s)K\big(g(s)\big), \quad \forall s \in \mathbb{R}_+,$$

where K is a nonnegative function satisfying some hypotheses, and a general decay estimate was proved.

On the other hand, the approach presented in this paper can be applied (with some small adaptations) to the case where the infinite integral in (1.1) is replaced with the finite one $\int_0^l$, where $l \in]0, \infty[$, in objective to get the decay estimates (2.4) and (2.6) under the assumption (A2). This application gives an extension to the exponential stability result proved in [18] under the condition (1.6).

Remark 4.1. The semigroup theory implies that (see [19]), under assumptions (A0) and (A1), and for any $n \in \mathbb{N}^*$ and $\mathcal{U}_0 \in D(\mathcal{A}^n)$, the solution of (1.3) has the regularity

$$\mathcal{U} \in \bigcap_{k=0}^{n} C^{n-k}\big(\mathbb{R}_+, D\big(\mathcal{A}^k\big)\big).$$

A more general decay estimate (depending on n) than (2.6) can be proved in case (1.13) (see [13]). To keep this paper short, we do not discuss this point.

Open problems. 1. For technical reasons (proof of (3.8) and (3.17)), the stability estimates (2.4) and (2.6) are proved under the restrictions (2.3) and (2.5), respectively. Proving (2.4) and (2.6) for arbitrary $\eta_0 \in L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))$ and $\eta_0 \in L^2_g(\mathbb{R}_+, D(A^{\frac{1}{2}} B^{\frac{1}{2}}))$, respectively, is open.

2. The damping Cu' and the assumption (1.9) played an important role in the proof of (1.7) for (1.8) considered in [13] with operators A and B satisfying more weaker conditions than ours (A0), (1.12) and (1.13). It would be interesting to get a stability estimate of (1.8) (in particular with $C = 0$) under the weaker assumption (A2).

3. The kernel g converges to zero at infinity faster than G_1^{-1} given in (2.4), and (2.4) is very probably not optimal. Recently, the authors in [2] considered the case of finite memory (4.5) (with $A = B$) and presented a general and sufficient condition under which the energy converges to zero at least as fast as g at infinity. Getting the optimal decay rate of the energy of (1.1) with g satisfying (A2) is a very interesting and open problem.

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