AUTHOR QUERY FORM

ELSEVIER	**Journal:** YJDEQ **Article Number:** 7989	**Please e-mail your responses and any corrections to:** **E-mail:** corrections.esch@elsevier.vtex.lt

Dear Author,

Please check your proof carefully and mark all corrections at the appropriate place in the proof (e.g., by using on-screen annotation in the PDF file) or compile them in a separate list. Note: if you opt to annotate the file with software other than Adobe Reader then please also highlight the appropriate place in the PDF file. To ensure fast publication of your paper please return your corrections within 48 hours.

For correction or revision of any artwork, please consult *http://www.elsevier.com/artworkinstructions*

Any queries or remarks that have arisen during the processing of your manuscript are listed below and highlighted by flags in the proof. Click on the 'Q' link to go to the location in the proof.

Location in article	**Query / Remark: click on the Q link to go** **Please insert your reply or correction at the corresponding line in the proof**
Q1	Your article is registered as a regular item and is being processed for inclusion in a regular issue of the journal. If this is NOT correct and your article belongs to a Special Issue/Collection please contact <p.kanagaraj@elsevier.com> immediately prior to returning your corrections. (p. 1/ line 1)
Q2	Please confirm that given names and surnames have been identified correctly and are presented in the desired order. (p. 1/ line 14)
Q3	Please complete and update the reference(s) given here (preferably with a DOI if the publication data are not known): [7]. (p. 37/ line 17)
Q4	Please provide a grant number for the sponsor "State University of Maring\'a" (if available) and include this number into the main text where appropriate. (p. 37/ line 3)
	Please check this box if you have no corrections to make to the PDF file ☐

Thank you for your assistance.

Available online at www.sciencedirect.com
ScienceDirect
ELSEVIER
J. Differential Equations ●●● (●●●●) ●●●–●●●
Journal of Differential Equations
www.elsevier.com/locate/jde

Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory

M.M. Cavalcanti [a], V.N. Domingos Cavalcanti [a], A. Guesmia [b,*]

[a] *Department of Mathematics, State University of Maringá, Maringá, PR 87020-900, Brazil*
[b] *Institut Elie Cartan de Lorraine, UMR 7502, Université de Lorraine, Bat. A, Ile du Saulcy, 57045 Metz Cedex 01, France*

Received 1 June 2015

Abstract

In this paper, we consider coupled wave–wave, Petrovsky–Petrovsky and wave–Petrovsky systems in N-dimensional open bounded domain with complementary frictional damping and infinite memory acting on the first equation. We prove that these systems are well-posed in the sense of semigroups theory and provide a weak stability estimate of solutions, where the decay rate is given in terms of the general growth of the convolution kernel at infinity and the arbitrary regularity of the initial data. We finish our paper by considering the uncoupled wave and Petrovsky equations with complementary frictional damping and infinite memory, and showing a strong stability estimate depending only on the general growth of the convolution kernel at infinity.

MSC: 35L05; 35L15; 35L70; 93D15

Keywords: Well-posedness; Asymptotic behavior; Infinite memory; Frictional damping; Waves; Petrovsky

* Corresponding author.
E-mail addresses: mmcavalcanti@uem.br (M.M. Cavalcanti), vncavalcanti@uem.br (V.N. Domingos Cavalcanti), aissa.guesmia@univ-lorraine.fr (A. Guesmia).

http://dx.doi.org/10.1016/j.jde.2015.08.028
0022-0396/

1. Introduction

Let $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function, $N \in \mathbb{N}^*$, $\Omega \subset \mathbb{R}^N$ be an open bounded domain with smooth boundary Γ, and $H = L^2(\Omega)$ be endowed with its natural inner product and corresponding norm denoted, respectively, by $\langle \cdot, \cdot \rangle$ and $\| \cdot \|$. Let a, $\tilde{a}$, b, $\tilde{b}$ and d be variable coefficients depending only on the space variable such that $d, \tilde{b} \in L^\infty(\Omega)$,

$$(a, \tilde{a}, b) \in \begin{cases} W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) \times W^{1,\infty}(\Omega) : & \text{wave–wave}, \\ W^{2,\infty}(\Omega) \times W^{2,\infty}(\Omega) \times W^{2,\infty}(\Omega) : & \text{Petrovsky–Petrovsky}, \\ W^{1,\infty}(\Omega) \times W^{2,\infty}(\Omega) \times W^{1,\infty}(\Omega) : & \text{wave–Petrovsky}, \end{cases}$$

$$\inf_\Omega a > 0, \quad \inf_\Omega \tilde{a} > 0, \quad \inf_\Omega b \geq 0 \quad \text{and} \quad \inf_\Omega d \geq 0.$$

We consider the linear bounded self-adjoint operators $D = d\, Id$ and $\tilde{B} = \tilde{b}\, Id$ (Id is the identity operator), and the linear unbounded self-adjoint ones

$$(A, \tilde{A}, B) = \begin{cases} (-div(a\nabla), -div(\tilde{a}\nabla), -div(b\nabla)) : & \text{wave–wave}, \\ (\Delta(a\Delta), \Delta(\tilde{a}\Delta), \Delta(b\Delta)) : & \text{Petrovsky–Petrovsky}, \\ (-div(a\nabla), \Delta(\tilde{a}\Delta), -div(b\nabla)) : & \text{wave–Petrovsky} \end{cases}$$

with domains $D(D) = D(\tilde{B}) = H$ and

$$\begin{aligned} &(D(A), D(\tilde{A}), D(B)) \\ &= \begin{cases} \left(H^2(\Omega) \cap H_0^1(\Omega), H^2(\Omega) \cap H_0^1(\Omega), H^2(\Omega) \cap H_0^1(\Omega)\right) : & \text{wave–wave}, \\ \left(H^4(\Omega) \cap H_0^2(\Omega), H^4(\Omega) \cap H_0^2(\Omega), H^4(\Omega) \cap H_0^2(\Omega)\right) : & \text{Petrovsky–Petrovsky}, \\ \left(H^2(\Omega) \cap H_0^1(\Omega), H^4(\Omega) \cap H_0^2(\Omega), H^2(\Omega) \cap H_0^1(\Omega)\right) : & \text{wave–Petrovsky}. \end{cases} \end{aligned}$$

Also

$$\begin{aligned} &\left(D(A^{\frac{1}{2}}), D(\tilde{A}^{\frac{1}{2}}), D(B^{\frac{1}{2}}), D(\tilde{B}^{\frac{1}{2}}), D(D^{\frac{1}{2}})\right) \\ &= \begin{cases} \left(H_0^1(\Omega), H_0^1(\Omega), H_0^1(\Omega), H, H\right) : & \text{wave–wave}, \\ \left(H_0^2(\Omega), H_0^2(\Omega), H_0^2(\Omega), H, H\right) : & \text{Petrovsky–Petrovsky}, \\ \left(H_0^1(\Omega), H_0^2(\Omega), H_0^1(\Omega), H, H\right) : & \text{wave–Petrovsky}. \end{cases} \end{aligned}$$

The aim of this paper is the study of the well-posedness and asymptotic behavior when time goes to infinity of solutions of the following coupled wave–wave, Petrovsky–Petrovsky and wave–Petrovsky system:

$$\begin{cases} u_{tt}(t) + Au(t) + Du_t(t) - \displaystyle\int_0^{+\infty} g(s) Bu(t-s)\, ds + \tilde{B}v(t) = 0, & \forall t > 0, \\ v_{tt}(t) + \tilde{A}v(t) + \tilde{B}u(t) = 0, & \forall t > 0 \end{cases} \tag{1.1}$$

with initial conditions

$$\begin{cases} u(-t) = u_0(t), \quad \forall t \in \mathbb{R}_+, \\ v(0) = v_0, \quad u_t(0) = u_1, \quad v_t(0) = v_1 \end{cases} \tag{1.2}$$

and homogeneous Dirichlet–Dirichlet, Dirichlet–Dirichlet–Neumann–Neumann and Dirichlet–Dirichlet–Neumann boundary conditions on $\Gamma \times \mathbb{R}_+$

$$\begin{cases} u = v = 0: & \text{wave–wave}, \\ u = v = \frac{\partial u}{\partial \nu} = \frac{\partial v}{\partial \nu} = 0: & \text{Petrovsky–Petrovsky}, \\ u = v = \frac{\partial v}{\partial \nu} = 0: & \text{wave–Petrovsky}, \end{cases} \tag{1.3}$$

where $\frac{\partial}{\partial \nu}$ is the outer normal derivative. The unknown $(u, v) : \mathbb{R}_+ \to H \times H$ is the state of the system (1.1)–(1.3) corresponding to the initial data (u_0, v_0, u_1, v_1). The term Du_t and the infinite integral in (1.1) represent, respectively, the frictional damping and the infinite memory, which play, in a complementary way, the role of dissipation for the whole system (1.1)–(1.3) via only the first equation in (1.1).

The problem of well-posedness and stability of (1.1)–(1.3) has attracted considerable attention in recent years and an important amount of research has been devoted in this direction, where diverse types of dissipative mechanisms have been introduced and several stability results have been obtained.

In the uncoupled case: $\tilde{B} \equiv 0$, it is well-known that the second equation of (1.1):

$$v_{tt}(t) + \tilde{A}v(t) = 0, \quad \forall t > 0 \tag{1.4}$$

is well-posed and it is a conservative equation.

Concerning the first equation in (1.1) with $\tilde{B} \equiv 0$:

$$u_{tt}(t) + Au(t) + Du_t(t) - \int_0^{+\infty} g(s)Bu(t-s)ds = 0, \quad \forall t > 0, \tag{1.5}$$

a large amount of literature is available for this model, addressing problems of existence, uniqueness and asymptotic behavior in time; see, for example, [22] (and the references therein) in case $B \equiv 0$, [9–13,24,26,27] in case $D \equiv 0$ and g converges exponentially to zero at infinity, and [15] and [20] in case $D \equiv 0$ and g having a general growth at infinity. Also, for the particular case of a single wave equation or Timoshenko-type systems with complementary frictional damping and memory or two memories, we refer the reader to [6–8,16,18,19].

In the coupled case: $\tilde{B} \neq 0$, the stability of (1.1) is more complicated since only the first equation in (1.1) is directly controlled, whereas the second one is partially and indirectly controlled via the behavior of the first one. The concept of indirect stability for coupled systems was introduced, as far as we know, in [29], where the controlled equation plays the role of stabilizer for the second one via the coupling terms. See [21] for further related stability results for coupled systems.

When $B \equiv 0$, it has been proved in [1] that (1.1) is not exponentially stable and the asymptotic behavior of solutions is at least of polynomial type with decay rates depending on the smoothness

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

of initial data. Some extensions of the results of [1] to the non-linear and non-dissipative cases are given in [14].

The stability of (1.1) in case $D \equiv 0$ was proved in [17] by providing a stability estimate depending in terms of the growth of g at infinity and the regularity of the initial data.

Our main objective in this paper is showing that the dissipation generated by the complementary frictional damping and infinite memory controls guarantees the stability of (1.1)–(1.3), and investigating the effect of each control on the asymptotic behavior of the solutions, where each control can vanish in a part of the domain. The general idea of the indirect decay estimate (3.11) below lies in the fact that the term v_t, which could be regarded as the viscous damping for the second equation of (1.1), can be expressed via higher derivatives of u through the weak coupling

$$-\tilde{B}v(t) = u_{tt}(t) + Au(t) + Du_t(t) - \int_0^{+\infty} g(s)Bu(t-s)ds.$$

This higher-energy decay estimate on the u-equation provides some control over the terms for the energy of the v-equation. We provide an explicit and general characterization of the decay rate depending on the growth of g at infinity and the regularity of the initial data. This includes the particular two cases $B \equiv 0$ and $D \equiv 0$ (only one control is considered) treated in [1] and [17], respectively. At the end, we consider the uncoupled wave and Petrovsky equations (1.5) and prove a strong stability estimate depending only on the growth of g at infinity. This particular case gives a generalization of some results of [7,9–11,13,15,20,24,26,27] concerning the cases $D \equiv 0$ and g converges exponentially to zero.

The paper is organized as follows. In Section 2, we consider some hypotheses and prove the well-posedness of (1.1)–(1.3). Section 3 is devoted to the statement and proof of the asymptotic stability of (1.1)–(1.3). Finally, in Section 4, we treat the uncoupled case (1.5).

2. Well-posedness

We state in this section some assumptions on B, $\tilde{B}$ and g and give a brief proof of the global existence, uniqueness and smoothness of solutions of (1.1)–(1.3).

First, thanks to the properties of the coefficients a, $\tilde{a}$, b and $\tilde{b}$, there exist positive constants a_0, a_1, $\tilde{a}_1$ and b_1 satisfying

$$\max\left\{a_1\|w\|^2, \|B^{\frac{1}{2}}w\|^2\right\} \le a_0\|A^{\frac{1}{2}}w\|^2, \quad \forall w \in D(A^{\frac{1}{2}}), \tag{2.1}$$

$$\tilde{a}_1\|w\|^2 \le \|\tilde{A}^{\frac{1}{2}}w\|^2, \quad \forall w \in D(\tilde{A}^{\frac{1}{2}}) \tag{2.2}$$

and

$$\|\tilde{B}w\|^2 \le b_1\|w\|^2, \quad \forall w \in H. \tag{2.3}$$

On the other hand, we assume that

(A0) The space

$$L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}})) := \left\{ w : \mathbb{R}_+ \to D(B^{\frac{1}{2}}), \quad \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}w(s)\|^2 ds < +\infty \right\}$$

endowed with the inner product

$$\langle w_1, w_2 \rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))} := \int_0^{+\infty} g(s) \left\langle B^{\frac{1}{2}} w_1(s), B^{\frac{1}{2}} w_2(s) \right\rangle ds$$

is a Hilbert space.

(A1) The kernel g is of class $C^1(\mathbb{R}_+) \cap L^1(\mathbb{R}_+)$, non-increasing and satisfies

$$g_0 := \int_0^{+\infty} g(s) ds < \frac{1}{a_0}. \tag{2.4}$$

Moreover, there exists a positive constant δ_0 such that

$$-g'(s) \le \delta_0 g(s), \quad \forall s \in \mathbb{R}_+. \tag{2.5}$$

(A2) The positive constant b_1 in (2.3) satisfies

$$b_1 < \sqrt{\frac{a_1 \tilde{a}_1 (1 - a_0 g_0)}{a_0}}. \tag{2.6}$$

Remark 2.1. Some interesting examples of a function b, where the assumption **(A0)** holds, can be found in [2–5,23].

Now, following a method devised in [10] to treat the memory term, we formulate the system (1.1)–(1.3) in the following abstract linear first-order system:

$$\begin{cases} \mathscr{U}_t(t) = \mathscr{A}\mathscr{U}(t), & \forall t > 0, \\ \mathscr{U}(0) = \mathscr{U}_0, \end{cases} \tag{2.7}$$

where $\mathscr{U} = (u, v, u_t, v_t, \eta)^T$, $\mathscr{U}_0 = (u_0(0), v_0, u_1, v_1, \eta_0)^T \in \mathscr{H}$,

$$\mathscr{H} = D(A^{\frac{1}{2}}) \times D(\tilde{A}^{\frac{1}{2}}) \times H \times H \times L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}})),$$

$$\begin{cases} \eta(t,s) = u(t) - u(t-s), & \forall t, s \in \mathbb{R}_+, \\ \eta_0(s) = \eta(0,s) = u_0(0) - u_0(s), & \forall s \in \mathbb{R}_+ \end{cases} \tag{2.8}$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

and $\mathscr{A}$ is a linear operator given by

$$\mathscr{A}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \\ w_5 \end{pmatrix} = \begin{pmatrix} w_3 \\ w_4 \\ (-A+g_0B)w_1 - Dw_3 - \int_0^{+\infty} g(s)Bw_5(s)ds - \tilde{B}w_2 \\ -\tilde{A}w_2 - \tilde{B}w_1 \\ -\partial_s w_5 + w_3 \end{pmatrix} \tag{2.9}$$

with domain $\mathscr{D}(\mathscr{A})$ given by

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{c} W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathscr{H},\ \partial_s w_5 \in L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}})),\ w_4 \in D(\tilde{A}^{\frac{1}{2}}), \\ w_3 \in D(A^{\frac{1}{2}}),\ w_2 \in D(\tilde{A}),\ (A - g_0B)w_1 + \int_0^{+\infty} g(s)Bw_5(s)ds \in H,\ w_5(0) = 0 \end{array} \right\}. \tag{2.10}$$

We use the classical notations $\mathscr{D}(\mathscr{A}^0) = \mathscr{H}$, $\mathscr{D}(\mathscr{A}^1) = \mathscr{D}(\mathscr{A})$ and

$$\mathscr{D}(\mathscr{A}^{n+1}) = \left\{ W \in \mathscr{D}(\mathscr{A}^n) : \mathscr{A}W \in \mathscr{D}(\mathscr{A}^n) \right\}, \quad n = 1, 2, \ldots,$$

endowed with the graph norm

$$\|w\|_{\mathscr{D}(\mathscr{A}^n)} = \sum_{k=0}^{n} \|\mathscr{A}^k w\|.$$

The space $\mathscr{H}$ is endowed with the inner product, for $W = (w_1, w_2, w_3, w_4, w_5)^T$ and $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4, \tilde{w}_5)^T$,

$$\begin{aligned} \left\langle W, \tilde{W} \right\rangle_{\mathscr{H}} &= \left\langle A^{\frac{1}{2}} w_1, A^{\frac{1}{2}} \tilde{w}_1 \right\rangle - g_0 \left\langle B^{\frac{1}{2}} w_1, B^{\frac{1}{2}} \tilde{w}_1 \right\rangle \\ &+ \left\langle \tilde{A}^{\frac{1}{2}} w_2, \tilde{A}^{\frac{1}{2}} \tilde{w}_2 \right\rangle + \left\langle \tilde{B} w_2, \tilde{w}_1 \right\rangle + \left\langle \tilde{B} w_1, \tilde{w}_2 \right\rangle \\ &+ \langle w_3, \tilde{w}_3 \rangle + \langle w_4, \tilde{w}_4 \rangle + \langle w_5, \tilde{w}_5 \rangle_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))}. \end{aligned}$$

Note that, from (2.4) and (2.6), one can choose $\epsilon_0 \in]\frac{b_1}{\tilde{a}_1}, \frac{(1-a_0g_0)a_1}{b_1a_0}[$ and, consequently,

$$c_0 := \min\left\{ 1 - a_0g_0 - \frac{\epsilon_0 b_1 a_0}{a_1}, 1 - \frac{b_1}{\epsilon_0 \tilde{a}_1} \right\} > 0. \tag{2.11}$$

Then, thanks to (2.1), (2.2), (2.3) and the Cauchy–Schwarz and Young's inequalities, we have, for any $(w_1, w_2) \in D(A^{\frac{1}{2}}) \times D(\tilde{A}^{\frac{1}{2}})$,

$$\begin{aligned} -g_0\|B^{\frac12}w_1\|^2+\left\langle \tilde B w_2, w_1\right\rangle+\left\langle \tilde B w_1, w_2\right\rangle &\geq -a_0 g_0\|A^{\frac12}w_1\|^2-2\sqrt{b_1}\|w_1\|\|w_2\| \\ &\geq -a_0g_0\|A^{\frac12}w_1\|^2-b_1\epsilon_0\|w_1\|^2-\frac{b_1}{\epsilon_0}\|w_2\|^2 \\ &\geq -\left(a_0g_0+\frac{\epsilon_0 b_1 a_0}{a_1}\right)\|A^{\frac12}w_1\|^2-\frac{b_1}{\epsilon_0\tilde a_1}\|\tilde A^{\frac12}w_2\|. \end{aligned}$$

Therefore

$$\begin{aligned} c_0\left(\|A^{\frac12}w_1\|^2+\|\tilde A^{\frac12}w_2\|^2\right) &\leq \|A^{\frac12}w_1\|^2-g_0\|B^{\frac12}w_1\|^2+\|\tilde A^{\frac12}w_2\|^2 \\ &\quad+\left\langle \tilde B w_2, w_1\right\rangle+\left\langle \tilde B w_1, w_2\right\rangle. \end{aligned} \tag{2.12}$$

Consequently, in virtue of **(A0)**, $(\mathscr{H}, \langle\cdot,\cdot\rangle_{\mathscr{H}})$ is a Hilbert space and $\mathscr{D}(\mathscr{A})\subset\mathscr{H}$ with dense embedding.

Now, keeping (2.8) in mind, we find

$$\begin{cases} \partial_t\eta(t,s)+\partial_s\eta(t,s)=u_t(t), & \forall t,s\in\mathbb{R}_+, \\ \eta(t,0)=0, & \forall t\in\mathbb{R}_+ \end{cases} \tag{2.13}$$

and

$$\begin{cases} \eta=0 \quad \text{on } \Gamma\times\mathbb{R}_+\times\mathbb{R}_+: & \text{wave–wave and wave–Petrovsky.} \\ \eta=\frac{\partial\eta}{\partial\nu}=0 \quad \text{on } \Gamma\times\mathbb{R}_+\times\mathbb{R}_+: & \text{Petrovsky–Petrovsky.} \end{cases} \tag{2.14}$$

Therefore, we deduce from (2.9), (2.13) and (2.14) that (1.1)–(1.3) is equivalent to (2.7), where the well-posedness is ensured by the following theorem:

Theorem 2.2. *Assume that* **(A0)–(A2)** *hold. Then, for any $n\in\mathbb{N}$ and $\mathscr{U}_0\in\mathscr{D}(\mathscr{A}^n)$, the system (2.7) has a unique solution*

$$\mathscr{U}\in\cap_{k=0}^n C^k\left(\mathbb{R}_+,\mathscr{D}\left(\mathscr{A}^{n-k}\right)\right). \tag{2.15}$$

Proof. By proving that the operator $-\mathscr{A}$ is maximal monotone, semigroups theory gives Theorem 2.2. So, for any $W=(w_1,w_2,w_3,w_4,w_5)\in\mathscr{D}(\mathscr{A})$, we have

$$\begin{aligned} \langle\mathscr{A}W,W\rangle_{\mathscr{H}} &= \left\langle A^{\frac12}w_3, A^{\frac12}w_1\right\rangle-g_0\left\langle B^{\frac12}w_3, B^{\frac12}w_1\right\rangle+\left\langle \tilde A^{\frac12}w_4, \tilde A^{\frac12}w_2\right\rangle \\ &\quad+\left\langle \tilde B w_4, w_1\right\rangle+\left\langle \tilde B w_3, w_2\right\rangle+\left\langle -\tilde A w_2-\tilde B w_1, w_4\right\rangle \\ &\quad+\left\langle (-A+g_0B)w_1-Dw_3-\int_0^{+\infty}g(s)Bw_5(s)ds-\tilde B w_2, w_3\right\rangle \\ &\quad+\langle -\partial_s w_5+w_3, w_5\rangle_{L^2_g(\mathbb{R}_+,D(B^{\frac12}))}. \end{aligned} \tag{2.16}$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

By the definitions of $A^{\frac{1}{2}}$, $\tilde{A}^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, and the fact that H is a real Hilbert space,

$$\langle(-A+g_0B)w_1, w_3\rangle = -\left\langle A^{\frac{1}{2}}w_3, A^{\frac{1}{2}}w_1\right\rangle + g_0\left\langle B^{\frac{1}{2}}w_3, B^{\frac{1}{2}}w_1\right\rangle,$$
$$\left\langle -\tilde{A}w_2, w_4\right\rangle = -\left\langle \tilde{A}^{\frac{1}{2}}w_4, \tilde{A}^{\frac{1}{2}}w_2\right\rangle$$

and

$$\left\langle -\int_0^{+\infty} g(s)Bw_5(s)ds, w_3\right\rangle = -\left\langle w_3, w_5\right\rangle_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))}.$$

On the other hand, integrating by parts and using the fact that $\lim_{s\to+\infty} g(s)B^{\frac{1}{2}}w_5(s) = 0$ (due to **(A1)**) and $w_5(0) = 0$ (definition of $\mathscr{D}(\mathscr{A})$), we find

$$\left\langle -\frac{\partial w_5}{\partial s}, w_5\right\rangle_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))} = \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}w_5(s)\|^2 ds.$$

Consequently, inserting these four equalities in (2.16), we get

$$\langle\mathscr{A}W, W\rangle_{\mathscr{H}} = -\|D^{\frac{1}{2}}w_3\|^2 + \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}w_5(s)\|^2 ds, \tag{2.17}$$

which implies that

$$\langle\mathscr{A}W, W\rangle_{\mathscr{H}} \le 0, \tag{2.18}$$

since g is non-increasing. This means that $\mathscr{A}$ is dissipative. Note that, thanks to (2.5) and the fact that $w_5 \in L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))$,

$$\begin{aligned}\left|\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}w_5(s)\|^2 ds\right| &= -\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}w_5(s)\|^2 ds \\ &\le \delta_0\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}w_5(s)\|^2 ds \\ &< +\infty,\end{aligned} \tag{2.19}$$

so the infinite integral in (2.17) is well defined.

Next, we shall prove that $Id - \mathscr{A}$ is surjective. Indeed, let $F = (f_1, f_2, f_3, f_4, f_5)^T \in \mathscr{H}$, we show that there exists $W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathscr{D}(\mathscr{A})$ satisfying

$$(Id - \mathscr{A})W = F. \tag{2.20}$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

We note that the first and second equations in (2.20) give

$$w_3 = w_1 - f_1 \quad \text{and} \quad w_4 = w_2 - f_2. \tag{2.21}$$

The last equation in (2.20) with $w_5(0) = 0$ has a unique solution

$$w_5(s) = \left(\int_0^s e^y (f_5(y) + w_1 - f_1) dy \right) e^{-s}. \tag{2.22}$$

On the other hand, multiplying the third and fourth equations in (2.20) by $\varphi_1 \in D(A^{\frac{1}{2}})$ and $\varphi_2 \in D(\tilde{A}^{\frac{1}{2}})$, respectively, and plugging (2.21) and (2.22), we get

$$\begin{cases} \left\langle A^{\frac{1}{2}} w_1, A^{\frac{1}{2}} \varphi_1 \right\rangle - g_1 \left\langle B^{\frac{1}{2}} w_1, B^{\frac{1}{2}} \varphi_1 \right\rangle + \left\langle D^{\frac{1}{2}} w_1, D^{\frac{1}{2}} \varphi_1 \right\rangle + \left\langle w_1 + \tilde{B} w_2, \varphi_1 \right\rangle \\ \quad = \left\langle B^{\frac{1}{2}} \varphi_1, B^{\frac{1}{2}} \tilde{f}_1 \right\rangle + \left\langle D^{\frac{1}{2}} \varphi_1, D^{\frac{1}{2}} f_1 \right\rangle + \langle \varphi_1, f_1 + f_3 \rangle, \\ \left\langle \tilde{B} w_1, \varphi_2 \right\rangle + \left\langle \tilde{A}^{\frac{1}{2}} w_2, \tilde{A}^{\frac{1}{2}} \varphi_2 \right\rangle + \langle w_2, \varphi_2 \rangle = \langle \varphi_2, f_2 + f_4 \rangle, \end{cases} \tag{2.23}$$

where $g_1 = \int_0^{+\infty} g(s) e^{-s} ds$,

$$\tilde{f}_1 = \int_0^{+\infty} g(s) e^{-s} \left(\int_0^s e^y (f_1 - f_5(y)) dy \right) ds.$$

Notice that

$$f_2 + f_4 \in D(\tilde{A}^{\frac{1}{2}}) + H \subset H, \quad f_1 + f_3 \in D(A^{\frac{1}{2}}) + H \subset H, \quad D^{\frac{1}{2}} f_1 \in H$$

and $g_1 < g_0$. On the other hand,

$$\begin{aligned} \|B^{\frac{1}{2}} \tilde{f}_1\| &\le \|f_1\| \int_0^{+\infty} g(s) e^{-s} \left(\int_0^s e^y dy \right) ds + \int_0^{+\infty} g(s) e^{-s} \left(\int_0^s e^y \|f_5(y)\| dy \right) ds \\ &\le \|f_1\| \int_0^{+\infty} g(s)(1 - e^{-s}) ds + \int_0^{+\infty} e^y \|f_5(y)\| \left(\int_y^{+\infty} g(s) e^{-s} ds \right) dy \\ &\le \|f_1\| \int_0^{+\infty} g(s) ds + \int_0^{+\infty} g(y) e^y \|f_5(y)\| \left(\int_y^{+\infty} e^{-s} ds \right) dy \end{aligned}$$

$$
\begin{aligned}
&\leq g_0\|f_1\| + \int_0^{+\infty} g(y)\|f_5(y)\|dy \\
&\leq g_0\|f_1\| + \left(\int_0^{+\infty} g(y)dy\right)^{\frac{1}{2}} \left(\int_0^{+\infty} g(y)\|f_5(y)\|^2 dy\right)^{\frac{1}{2}} \\
&\leq g_0\|f_1\| + \sqrt{g_0}\|f_5\|_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))} \\
&< +\infty,
\end{aligned}
$$

so, $B^{\frac{1}{2}}\tilde{f}_1 \in H$. Then, using (2.12) and Lax–Milgram theorem, we deduce that (2.23) has a unique solution

$$(w_1, w_2)^T \in D(A^{\frac{1}{2}}) \times D(\tilde{A}^{\frac{1}{2}}).$$

Furthermore, coming back to (2.20), using classical regularity arguments and recalling (2.21) and (2.22), we see that $W \in \mathscr{D}(\mathscr{A})$ satisfying

$$(A - g_0 B)w_1 + \int_0^{+\infty} g(s)Bw_5(s)ds \in H.$$

Hence $Id - \mathscr{A}$ is surjective. Finally, (2.18) and (2.20) mean that $-\mathscr{A}$ is a maximal monotone operator. Therefore, using Lummer–Phillips theorem (see [28]), we deduce that $\mathscr{A}$ is the infinitesimal generator of a linear contraction C_0-semigroup on $\mathscr{H}$, and then the result of Theorem 2.2 is ensured by the semigroup theory (see [22,25,28]). □

3. Asymptotic behavior

This section is devoted to the study of the asymptotic behavior of solutions of (2.7). According to the definitions of A, B and $\tilde{A}$, there exist constants $a_2, \tilde{a}_2 > 0$ such that

$$\|\sqrt{b}A^{\frac{1}{2}}w\|^2 \leq a_2\|B^{\frac{1}{2}}w\|^2, \quad \forall w \in D(A^{\frac{1}{2}}) \tag{3.1}$$

and

$$\|A^{\frac{1}{2}}w\|^2 \leq \tilde{a}_2\|\tilde{A}^{\frac{1}{2}}w\|^2, \quad \forall w \in D(\tilde{A}^{\frac{1}{2}}). \tag{3.2}$$

On the other hand, we consider the following additional assumptions:

(A3) There exist positive constants α_1 and α_2, and $\Gamma_0 \subset \Gamma$ with positive Lebesgue measure $|\Gamma_0| > 0$ such that

$$\inf_{\Omega}(b + d) \geq \alpha_1 \quad \text{and} \quad \inf_{\Gamma_0} b \geq 2\alpha_2. \tag{3.3}$$

(A4) The constant b_1 defined in (2.3) satisfies

$$b_1 < \frac{a_1 \tilde{a}_1 (1 - a_0 g_0)}{a_0}. \tag{3.4}$$

Moreover

$$\inf_\Omega \tilde{b} > 0 \quad \text{or} \quad \sup_\Omega \tilde{b} < 0,$$

which is equivalent to the fact that there exists a positive constant b_0 satisfying

$$\left\langle \tilde{B} w, w \right\rangle \geq b_0 \|w\|^2, \quad \forall w \in H \tag{3.5}$$

or

$$\left\langle \tilde{B} w, w \right\rangle \leq -b_0 \|w\|^2, \quad \forall w \in H. \tag{3.6}$$

(A5) The function g satisfies $g(0) > 0$ and there exists a positive constant δ such that

$$g'(s) \leq -\delta g(s), \quad \forall s \in \mathbb{R}_+ \tag{3.7}$$

or there exists an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying $G(0) = G'(0) = 0$, $\lim_{t \to +\infty} G'(t) = +\infty$ and

$$\int_0^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} ds + \sup_{s \in \mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} < +\infty. \tag{3.8}$$

Now, we introduce two sets of initial data $\mathcal{U}_0$ for which our stability estimate holds. Let, for $n \in \mathbb{N}$,

$$\mathcal{K}_n = \mathcal{D}(\mathcal{A}^n) \tag{3.9}$$

when (3.7) holds, and

$$\mathcal{K}_n = \left\{ \mathcal{U}_0 \in \mathcal{D}(\mathcal{A}^n) : \sup_{t \in \mathbb{R}_+} \max_{k=0,\dots,n} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \left\| B^{\frac{1}{2}} \partial_s^k u_0(s-t) \right\|^2 ds < +\infty \right\} \tag{3.10}$$

when (3.8) holds and (3.7) does not hold.

Theorem 3.1. *Assume that* **(A0)–(A5)** *hold and let $n \in \mathbb{N}^*$ and $\mathcal{U}_0 \in \mathcal{K}_{i_0 n}$, where $i_0 = 2$ if $\tilde{A} = A - g_0 B$, and $i_0 = 3$ if $\tilde{A} \neq A - g_0 B$. Then there exists a positive constant c_n such that*

$$\|\mathcal{U}(t)\|^2_{\mathcal{H}} \leq c_n G_n\left(\frac{c_n}{t}\right), \quad \forall t > 0, \tag{3.11}$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

where $G_1(s) = G_0^{-1}(s)$, $G_m(s) = G_1(sG_{m-1}(s))$, *for* $m = 2, 3, \ldots, n$ *and* $s \in \mathbb{R}_+$, *and*

$$G_0(s) = \begin{cases} s & \text{if (3.7) holds,} \\ sG'(s) & \text{if (3.8) holds and (3.7) does not hold.} \end{cases} \tag{3.12}$$

Remark 3.2. Notice that, because $G_n(0) = 0$,

$$\lim_{t \to +\infty} \|\mathscr{U}(t)\|_{\mathscr{H}}^2 = 0. \tag{3.13}$$

On the other hand, the class of functions satisfying **(A1)** and **(A5)** is very wide and contains the ones which converge to zero exponentially (conditions (2.5) and (3.7)) or at a slower rate (conditions (2.5) and (3.8)) like, respectively,

$$g_1(s) = d_1 e^{-q_1 s} \quad \text{and} \quad g_2(s) = \frac{d_2}{(1+s)^{q_2}}, \tag{3.14}$$

where $d_1, q_1, d_2 > 0$ and $q_2 > 1$. We see that g_1 and g_2 satisfy **(A1)** provided that d_1 and d_2 are small enough so that $d_1 < \frac{q_1}{a_0}$ and $d_2 < \frac{q_2-1}{a_0}$. On the other hand, g_1 satisfies (3.7) with $\delta = q_1$. Then $G_n(s) = s^n$, and therefore, (3.11) gives, for any $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A}^{i_0 n})$,

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \le c_n^{n+1} t^{-n}, \quad \forall t > 0. \tag{3.15}$$

However, g_2 does not satisfy (3.7) but it satisfies (3.8) with $G(s) = s^p$, for any $p > \dfrac{q_2+1}{q_2-1}$. Then $G_n(s) = p^{-p_n} s^{p_n}$, where $p_n = \sum_{m=1}^n p^{-m}$, and therefore, (3.11) gives

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \le c_n^{1+p_n} p^{-p_n} t^{-p_n}, \quad \forall t > 0, \quad \forall p > \frac{q_2+1}{q_2-1}. \tag{3.16}$$

Notice that t^{-p_n} approaches t^{-n} (which is the decay rate in (3.15)) as p goes to 1^+ (that is, when q_2 converges to $+\infty$). Estimate (3.16) holds for initial data satisfying, for example,

$$\mathscr{U}_0 \in \mathscr{D}(\mathscr{A}^{i_0 n}) \quad \text{and} \quad \max_{k=0,\ldots,i_0 n} \left\| B^{\frac{1}{2}} \partial_s^k u_0(s) \right\|^2 \le d_3 (1+s)^{q_3}, \quad \forall s \in \mathbb{R}_+, \tag{3.17}$$

where d_3 is a positive constant and $q_3 < \dfrac{p(q_2-1)-(q_2+1)}{p}$; so $\mathscr{U}_0 \in \mathscr{K}_{i_0 n}$.

For more examples, see [17], where (1.1) is considered in the case $D \equiv 0$.

Proof of Theorem 3.1. The proof of (3.11) focuses on the case $n = 1$ and it is based on the multipliers method by considering some appropriate functionals and adapting to our model (1.1) some arguments of [1,8,17]. The general case (3.11), for any $n \in \mathbb{N}^*$, is then deduced by induction on n.

Now, assume that **(A0)–(A5)** hold and let $\mathscr{U}_0 \in \mathscr{K}_{i_0}$ and E be the associated energy functional with the solution of (2.7) giving by

$$E(t)=\frac{1}{2}\|\mathscr{U}(t)\|^2_{\mathscr{H}}. \tag{3.18}$$

We start our proof, first, by noting that, using (2.7) and (2.17),

$$E'(t)=-\|D^{\frac{1}{2}}u_t(t)\|^2+\frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds, \quad \forall t\in\mathbb{R}_+. \tag{3.19}$$

Recalling that g is non-increasing, (3.19) implies that E is non-increasing, and consequently, (2.7) is dissipative. If $D\equiv 0$ and $g\equiv 0$, then $E'\equiv 0$; thus (2.7) is a conservative system. This fact shows that the unique dissipation considered in (1.1) is the one resulting from the complementary frictional damping and infinite memory controls. On the other hand, if $E(t_0)=0$, for some $t_0\in\mathbb{R}_+$, then $E(t)=0$, for all $t\geq t_0$, and therefore, (3.11) holds. Then, without loss of generality, we assume that $E(t)>0$, for all $t\in\mathbb{R}_+$.

Second, we consider a function α introduced in [8] to establish some needed estimates.

Lemma 3.3. *Let $\alpha_0=\min\{\alpha_1,\alpha_2\}$ and $\alpha\in C^2(\bar\Omega)$ such that*

$$\begin{cases} 0\leq\alpha\leq b & \text{on } \Omega,\\ \alpha=0 & \text{if } b\leq\frac{\alpha_0}{4},\\ \alpha=b & \text{if } b\geq\frac{\alpha_0}{2}.\end{cases} \tag{3.20}$$

Then the function α is not identically zero and satisfies

$$\inf_{\Omega}(\alpha+d)\geq\frac{\alpha_0}{2}. \tag{3.21}$$

Proof. In yirtue of the second inequality of (3.3) and the regularity of b, there exists a neighborhood Ω_0 of Γ_0 such that

$$\inf_{\bar\Omega\cap\bar\Omega_0} b\geq\alpha_2. \tag{3.22}$$

Then, for $x\in\Omega\cap\Omega_0$, we have $b(x)\geq\alpha_0$, which implies, by (3.20), that $\alpha=b\geq\alpha_0$ on $\Omega\cap\Omega_0$. Thus α is not identically zero.

On the other hand, if $b(x)\geq\frac{\alpha_0}{2}$, then (3.20) implies that $\alpha(x)+d(x)\geq\alpha(x)=b(x)\geq\frac{\alpha_0}{2}$. If $b(x)<\frac{\alpha_0}{2}$, then, according to the first inequality of (3.3) and the fact that $\alpha_0\leq\alpha_1$, $d(x)>\frac{\alpha_1}{2}\geq\frac{\alpha_0}{2}$, which implies that $\alpha(x)+d(x)\geq d(x)>\frac{\alpha_0}{2}$. Consequently, (3.21) holds. □

Now, we apply the multipliers method to get some useful inequalities.

Lemma 3.4. *Let us define the functionals*

$$I_1(t)=-\left\langle u_t(t),\alpha\int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

and

$$I_2(t) = -\left\langle u_{ttt}(t), \alpha \int_0^{+\infty} g(s)\eta_{tt}(t,s)ds \right\rangle - \left\langle \tilde{B}v_t(t), \alpha \int_0^{+\infty} g(s)\eta_{tt}(t,s)ds \right\rangle.$$

Then, for any $\epsilon_1, \delta_1 > 0$, *there exist* $c_{\epsilon_1}, c_{\delta_1} > 0$ *such that, for all* $t \in \mathbb{R}_+$,

$$I_1'(t) \le -(\frac{\alpha_0 g_0}{2} - \epsilon_1)\|u_t(t)\|^2 + \epsilon_1 \|A^{\frac{1}{2}}u(t)\|^2 + \epsilon_1 \|\tilde{A}^{\frac{1}{2}}v(t)\|^2 + c_{\epsilon_1}\|D^{\frac{1}{2}}u_t(t)\|^2 + c_{\epsilon_1} \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \tag{3.23}$$

and

$$I_2'(t) \le -(\frac{\alpha_0 g_0}{2} - \delta_1)\|u_{ttt}(t)\|^2 + \delta_1 \|A^{\frac{1}{2}}u_{tt}(t)\|^2 + \delta_1 \|v_t(t)\|^2 + c_{\delta_1}\|D^{\frac{1}{2}}u_{ttt}(t)\|^2 + c_{\delta_1} \int_0^{+\infty} g(s)\left(\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 + \|B^{\frac{1}{2}}\eta_{ttt}(t,s)\|^2\right) ds. \tag{3.24}$$

Proof. As in [19], multiplying the first equation of (1.1) by $\alpha \int_0^{+\infty} g(s)\eta(t,s)ds$, we get

$$0 = \left\langle u_{tt}(t), \alpha \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle + \left\langle (A - g_0 B)u(t) + Du_t(t), \alpha \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle + \left\langle \int_0^{+\infty} g(s)B\eta(t,s)ds, \alpha \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle + \left\langle \tilde{B}v(t), \alpha \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle.$$

Using the definition of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we obtain

$$0 = \left\langle u_{tt}(t) + Du_t(t), \alpha \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle + \left\langle A^{\frac{1}{2}}u(t), \alpha \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds \right\rangle - g_0 \left\langle B^{\frac{1}{2}}u(t)ds, \alpha \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds + (B^{\frac{1}{2}}\alpha) \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

$$+\left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, \alpha \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds + (B^{\frac{1}{2}}\alpha)\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle$$

$$+\left\langle A^{\frac{1}{2}}u(t), (A^{\frac{1}{2}}\alpha)\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle + \left\langle \tilde{B}v(t), \alpha\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle. \tag{3.25}$$

On the other hand, by using $\partial_t\eta(t,s) = -\partial_s\eta(t,s) + u_t(t)$ (according to (2.13)), we find

$$\left\langle u_{tt}(t), \alpha\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle = \partial_t \left\langle u_t(t), \alpha\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle - \left\langle u_t(t), \alpha\int_0^{+\infty} g(s)\eta_t(t,s)ds \right\rangle$$
$$= -I_1'(t) - g_0\|\sqrt{\alpha}u_t(t)\|^2 + \left\langle u_t(t), \alpha\int_0^{+\infty} g(s)\eta_s(t,s)ds \right\rangle.$$

Integrating by parts with respect to s in the infinite memory integral, and using the fact that $\lim_{s\to+\infty} g(s)\eta(t,s) = 0$ and $\eta(t,0) = 0$ (according to **(A1)** and (2.13)), we get

$$\left\langle u_{tt}(t), \alpha\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle = -I_1'(t) - g_0\|\sqrt{\alpha}u_t(t)\|^2 - \left\langle u_t(t), \alpha\int_0^{+\infty} g'(s)\eta(t,s)ds \right\rangle. \tag{3.26}$$

Exploiting (3.25) and (3.26), we deduce

$$I_1'(t) = -g_0\|\sqrt{\alpha}u_t(t)\|^2 - \left\langle u_t(t), \alpha\int_0^{+\infty} g'(s)\eta(t,s)ds \right\rangle$$
$$+\left\langle \tilde{B}v(t) + Du_t(t), \alpha\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle + \left\langle A^{\frac{1}{2}}u(t), \alpha\int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds \right\rangle$$
$$-g_0\left\langle B^{\frac{1}{2}}u(t), \alpha\int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds + (B^{\frac{1}{2}}\alpha)\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle$$
$$+\left\langle A^{\frac{1}{2}}u(t), (A^{\frac{1}{2}}\alpha)\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle$$
$$+\left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, \alpha\int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds + (B^{\frac{1}{2}}\alpha)\int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle. \tag{3.27}$$

Thanks to (3.21), we see that

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

$$
\begin{aligned}
-g_0\|\sqrt{\alpha}u_t(t)\|^2 &= -g_0\|\sqrt{\alpha+d}u_t(t)\|^2 + g_0\|\sqrt{d}u_t(t)\|^2\\
&\le \frac{-\alpha_0 g_0}{2}\|u_t(t)\|^2 + g_0\|D^{\frac{1}{2}}u_t(t)\|^2. \qquad (3.28)
\end{aligned}
$$

On the other hand, since (2.14) and $\Gamma_0 \subset \bar{\Omega} \cap \bar{\Omega}_0 \subset supp\,\alpha$ (in yirtue of (3.20) and (3.22)), then

$$
\begin{aligned}
\|\alpha\eta(t,s)\|^2 &= \int_{supp\,\alpha} \alpha^2\eta^2(t,s)dx\\
&\le (\sup_{\Omega}\alpha^2)\int_{supp\,\alpha}\eta^2(t,s)dx.
\end{aligned}
$$

Hence, using a version of Poincaré's inequality [8], there exists a positive constant c^* such that

$$
\|\alpha\eta(t,s)\|^2 \le c^*(\sup_{\Omega}\alpha^2)\begin{cases}\displaystyle\int_{supp\,\alpha}|\nabla\eta(t,s)|^2dx: & \text{wave–wave and wave–Petrovsky}\\ \displaystyle\int_{supp\,\alpha}|\Delta\eta(t,s)|^2dx: & \text{Petrovsky–Petrovsky,}\end{cases}
$$

thus, using (3.20),

$$
\|\alpha\eta(t,s)\|^2 \le \frac{4c^*}{\alpha_0}(\sup_{\Omega}\alpha^2)\|B^{\frac{1}{2}}\eta(t,s)\|^2. \qquad (3.29)
$$

Similarly,

$$
\|(A^{\frac{1}{2}}\alpha)\eta(t,s)\|^2 \le \frac{4c^*}{\alpha_0}(\sup_{\Omega}|A^{\frac{1}{2}}\alpha|^2)\|B^{\frac{1}{2}}\eta(t,s)\|^2 \qquad (3.30)
$$

and

$$
\|(B^{\frac{1}{2}}\alpha)\eta(t,s)\|^2 \le \frac{4c^*}{\alpha_0}(\sup_{\Omega}|B^{\frac{1}{2}}\alpha|^2)\|B^{\frac{1}{2}}\eta(t,s)\|^2. \qquad (3.31)
$$

Also, using (3.1) and (3.20),

$$
\|\alpha A^{\frac{1}{2}}\eta(t,s)\|^2 \le \frac{4a_2}{\alpha_0}(\sup_{\Omega}\alpha^2)\|B^{\frac{1}{2}}\eta(t,s)\|^2. \qquad (3.32)
$$

Inserting (3.28) into (3.27), applying Cauchy–Schwarz and Young's inequalities to the last six terms of (3.27), using (2.1), (2.2), (2.3) and (2.5) to estimate $\|B^{\frac{1}{2}}u(t)\|^2$, $\|\tilde{B}v(t)\|^2$ and $-g'(s)$ by $a_0\|A^{\frac{1}{2}}u(t)\|^2$, $\frac{b_1}{\tilde{a}_1}\|\tilde{A}^{\frac{1}{2}}v(t)\|^2$ and $\delta_0 g(s)$, respectively, and exploiting (3.29)–(3.32), we get (3.23).

Using the system obtained by differentiating two times the first equation of (1.1) with respect to time t; that is,

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

$$u_{tttt}(t) + Au_{tt}(t) + Du_{ttt}(t) - \int_0^{+\infty} g(s)Bu_{tt}(t-s)ds + \tilde{B}v_{tt}(t) = 0, \quad \forall t > 0, \tag{3.33}$$

multiplying (3.33) by $\alpha \int_0^{+\infty} g(s)\eta_{tt}(t,s)ds$, we find (as for I_1')

$$\begin{aligned}
I_2'(t) = &-g_0\|\sqrt{\alpha}u_{ttt}(t)\|^2 - \left\langle u_{ttt}(t), \alpha \int_0^{+\infty} g'(s)\eta_{tt}(t,s)ds \right\rangle \\
&+ \left\langle A^{\frac{1}{2}}u_{tt}(t), \alpha \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta_{tt}(t,s)ds + (A^{\frac{1}{2}}\alpha) \int_0^{+\infty} g(s)\eta_{tt}(t,s)ds \right\rangle \\
&- g_0 \left\langle B^{\frac{1}{2}}u_{tt}(t), \alpha \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_{tt}(t,s)ds + (B^{\frac{1}{2}}\alpha) \int_0^{+\infty} g(s)\eta_{tt}(t,s)ds \right\rangle \\
&+ \left\langle Du_{ttt}(t), \alpha \int_0^{+\infty} g(s)\eta_{tt}(t,s)ds \right\rangle - \left\langle \tilde{B}v_t(t), \alpha \int_0^{+\infty} g(s)\eta_{ttt}(t,s)ds \right\rangle \\
&+ \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_{tt}(t,s)ds, \alpha \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_{tt}(t,s)ds + (B^{\frac{1}{2}}\alpha) \int_0^{+\infty} g(s)\eta_{tt}(t,s)ds \right\rangle.
\end{aligned}$$

Then, following the same procedure as before, we get (3.24). □

Lemma 3.5. *Define the functionals*

$$J_1(t) = \langle u_t(t), u(t)\rangle, \quad J_2(t) = \langle u_{ttt}(t), u_{tt}(t)\rangle + \left\langle \tilde{B}v_t, u_{tt}(t) \right\rangle \quad \text{and} \quad R_1(t) = \langle v_t(t), v(t)\rangle.$$

Then, for any $\lambda_1, \lambda_2, \lambda_3, \epsilon_2, \delta_2 > 0$, there exist $c_{\epsilon_2}, c_{\delta_2} > 0$ such that

$$\begin{aligned}
J_1'(t) \le &\|u_t(t)\|^2 - (1 - a_0g_0 - \epsilon_2 - \lambda_1)\|A^{\frac{1}{2}}u(t)\|^2 + \frac{b_1a_0}{4\lambda_1 a_1 \tilde{a}_1}\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 \\
&+ c_{\epsilon_2}\|D^{\frac{1}{2}}u_t(t)\|^2 + c_{\epsilon_2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+,
\end{aligned} \tag{3.34}$$

$$\begin{aligned}
J_2'(t) \le &(1+\lambda_2)\|u_{ttt}(t)\|^2 - (1 - a_0g_0 - \delta_2)\|A^{\frac{1}{2}}u_{tt}(t)\|^2 + \frac{b_1}{4\lambda_2}\|v_t(t)\|^2 \\
&+ c_{\delta_2}\|D^{\frac{1}{2}}u_{ttt}(t)\|^2 + c_{\delta_2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+
\end{aligned} \tag{3.35}$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

and

$$R_1'(t) \le \|v_t(t)\|^2 + \frac{a_0 b_1}{4\lambda_3 a_1 \tilde{a}_1}\|A^{\frac{1}{2}}u(t)\|^2 - (1-\lambda_3)\|\tilde{A}^{\frac{1}{2}}v(t)\|^2, \quad \forall t \in \mathbb{R}_+. \tag{3.36}$$

Proof. Multiplying the first equation of (1.1) by $u(t)$, we find

$$0 = \langle u_{tt}(t), u(t)\rangle + \langle (A - g_0 B)u(t) + Du_t(t), u(t)\rangle + \left\langle \int_0^{+\infty} g(s)B\eta(t,s)ds, u(t)\right\rangle + \left\langle \tilde{B}v(t), u(t)\right\rangle.$$

Consequently, using the definition of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we have

$$0 = \partial_t \langle u_t(t), u(t)\rangle - \|u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2 + \langle Du_t(t), u(t)\rangle + \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, B^{\frac{1}{2}}u(t)\right\rangle + \left\langle \tilde{B}v(t), u(t)\right\rangle,$$

which implies that

$$J_1'(t) = \|u_t(t)\|^2 - \|A^{\frac{1}{2}}u(t)\|^2 + g_0\|B^{\frac{1}{2}}u(t)\|^2 - \left\langle \tilde{B}v(t), u(t)\right\rangle - \langle Du_t(t), u(t)\rangle - \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, B^{\frac{1}{2}}u(t)\right\rangle. \tag{3.37}$$

By applying Cauchy–Schwarz and Young's inequalities for the last three terms of (3.37), and exploiting (2.1), (2.2) and (2.3) to estimate $\|u(t)\|^2$, $\|B^{\frac{1}{2}}u(t)\|^2$ and $\|\tilde{B}v(t)\|^2$ by $\frac{a_0}{a_1}\|A^{\frac{1}{2}}u(t)\|^2$, $a_0\|A^{\frac{1}{2}}u(t)\|^2$ and $\frac{b_1}{\tilde{a}_1}\|\tilde{A}^{\frac{1}{2}}v(t)\|^2$, respectively, inequality (3.34) holds. Similarly, multiplying the second equation of (1.1) by $v(t)$ and following the same procedure as for (3.34), we get (3.36).

On the other hand, multiplying (3.33) by $u_{tt}(t)$, we have (as for J_1')

$$J_2'(t) = \|u_{ttt}(t)\|^2 - \|A^{\frac{1}{2}}u_{tt}(t)\|^2 + g_0\|B^{\frac{1}{2}}u_{tt}(t)\|^2 + \left\langle \tilde{B}v_t(t), u_{ttt}(t)\right\rangle - \langle Du_{ttt}(t), u_{tt}(t)\rangle - \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_{tt}(t,s)ds, B^{\frac{1}{2}}u_{tt}(t)\right\rangle.$$

Then, following the same procedure as in the proof of (3.34), (3.35) holds. □

Now, we adapt the approach of [1] to our system (1.1) in objective to get a crucial estimate.

Lemma 3.6. *Let R_2 be the functional defined by*

$$R_2(t) = \langle u_{tt}(t), v_t(t)\rangle - \langle u_t(t), v_{tt}(t)\rangle + \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_t(t,s)ds,\, B^{\frac{1}{2}}v(t)\right\rangle$$

when (3.5) holds and $\tilde{A} = A - g_0 B$,

$$R_2(t) = -\langle u_{tt}(t), v_t(t)\rangle + \langle u_t(t), v_{tt}(t)\rangle - \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_t(t,s)ds,\, B^{\frac{1}{2}}v(t)\right\rangle$$

when (3.6) holds and $\tilde{A} = A - g_0 B$,

$$R_2(t) = \left\langle \tilde{A}^{-1}Au_{tt}(t), v_t(t)\right\rangle - \left\langle \tilde{A}^{-1}Au_t(t), v_{tt}(t)\right\rangle - g_0\left\langle B^{\frac{1}{2}}u_t(t), B^{\frac{1}{2}}v(t)\right\rangle + \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_t(t,s)ds,\, B^{\frac{1}{2}}v(t)\right\rangle$$

when (3.5) holds and $\tilde{A} \neq A - g_0 B$, and

$$R_2(t) = -\left\langle \tilde{A}^{-1}Au_{tt}(t), v_t(t)\right\rangle + \left\langle \tilde{A}^{-1}Au_t(t), v_{tt}(t)\right\rangle + g_0\left\langle B^{\frac{1}{2}}u_t(t), B^{\frac{1}{2}}v(t)\right\rangle - \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta_t(t,s)ds,\, B^{\frac{1}{2}}v(t)\right\rangle$$

when (3.6) holds and $\tilde{A} \neq A - g_0 B$. Then, for any $\delta_3, \epsilon_3, \epsilon_4 > 0$, there exist $c_{\epsilon_3}, c_{\epsilon_4} > 0$ such that

$$R_2'(t) \le -(b_0 - \epsilon_3)\|v_t(t)\|^2 + \sqrt{b_1}\|u_t(t)\|^2 + \epsilon_3\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 + c_{\epsilon_3}\|D^{\frac{1}{2}}u_{tt}(t)\|^2 + c_{\epsilon_3}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2ds, \quad \forall t \in \mathbb{R}_+ \tag{3.38}$$

in case $\tilde{A} = A - g_0 B$, and

$$R_2'(t) \le -(b_0 - \epsilon_3)\|v_t(t)\|^2 + \sqrt{b_1 d_0}\|u_t(t)\|^2 + \frac{d_0+1}{\epsilon_3}\|u_{ttt}(t)\|^2 + \frac{g_0^2 a_0^2 \tilde{a}_2}{4\delta_3}\|A^{\frac{1}{2}}u_{tt}(t)\|^2 + (\delta_3+\epsilon_4)\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 + c_{\epsilon_3}\|D^{\frac{1}{2}}u_{tt}(t)\|^2 + c_{\epsilon_4}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2ds, \quad \forall t \in \mathbb{R}_+ \tag{3.39}$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

JID:YJDEQ AID:7989 /FLA [m1+; v1.211; Prn:1/09/2015; 15:08] P.20 (1-38)

in case $\tilde{A} \neq A - g_0 B$*, where* d_0 *is the smallest positive constant satisfying*

$$\|\tilde{A}^{-1} A w\|^2 \leq d_0 \|w\|^2, \quad \forall w \in D(A) \tag{3.40}$$

(since $\tilde{A}^{-1} A$ *is bounded thanks to (3.2)).*

Proof. 1. *Case* $\tilde{A} = A - g_0 B$: considering the equations obtained by differentiating the equations of (1.1) with respect to time t; that is

$$u_{ttt}(t) + A u_t(t) + D u_{tt}(t) - \int_0^{+\infty} g(s) B u_t(t-s)\,ds + \tilde{B} v_t(t) = 0, \quad \forall t > 0 \tag{3.41}$$

and

$$v_{ttt}(t) + \tilde{A} v_t(t) + \tilde{B} u_t(t) = 0, \quad \forall t > 0, \tag{3.42}$$

and multiplying (3.41) and (3.42) by $v_t(t)$ and $u_t(t)$, respectively, we get

$$\begin{aligned}
0 &= \langle u_{ttt}(t), v_t(t)\rangle + \langle (A - g_0 B) u_t(t) + D u_{tt}(t), v_t(t)\rangle + \left\langle \int_0^{+\infty} g(s) B \eta_t(t,s)\,ds, v_t(t) \right\rangle \\
&\quad - \langle v_{ttt}(t), u_t(t)\rangle - \left\langle \tilde{A} v_t(t), u_t(t) \right\rangle - \left\langle \tilde{B} u_t(t), u_t(t) \right\rangle + \left\langle \tilde{B} v_t(t), v_t(t) \right\rangle \\
&= \partial_t \left(\langle u_{tt}(t), v_t(t)\rangle - \langle u_t(t), v_{tt}(t)\rangle + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_t(t,s)\,ds, B^{\frac{1}{2}} v(t) \right\rangle \right) + \langle D u_{tt}(t), v_t(t)\rangle \\
&\quad - \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_{tt}(t,s)\,ds, B^{\frac{1}{2}} v(t) \right\rangle - \left\langle \tilde{B} u_t(t), u_t(t) \right\rangle + \left\langle \tilde{B} v_t(t), v_t(t) \right\rangle,
\end{aligned}$$

since $\langle (A - g_0 B) u_t(t), v_t(t)\rangle - \left\langle \tilde{A} v_t(t), u_t(t) \right\rangle = 0$ (because $\tilde{A} = A - g_0 B$ and $\tilde{A}$ is self-adjoint). Therefore

$$\begin{aligned}
&\partial_t \left(\langle u_{tt}(t), v_t(t)\rangle - \langle u_t(t), v_{tt}(t)\rangle + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_t(t,s)\,ds, B^{\frac{1}{2}} v(t) \right\rangle \right) \\
&\quad = -\langle D u_{tt}(t), v_t(t)\rangle + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_{tt}(t,s)\,ds, B^{\frac{1}{2}} v(t) \right\rangle + \left\langle \tilde{B} u_t(t), u_t(t) \right\rangle - \left\langle \tilde{B} v_t(t), v_t(t) \right\rangle.
\end{aligned} \tag{3.43}$$

Consequently, using Cauchy–Schwarz and Young's inequalities for the first two terms of the right hand side of (3.43), and using (2.1) and (3.2) to estimate $\|B^{\frac{1}{2}} v(t)\|^2$ by $a_0 \tilde{a}_2 \|\tilde{A}^{\frac{1}{2}} v(t)\|^2$,

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

and using (2.3) and (3.5) to estimate the last two terms of the right hand side of (3.43), we get (3.38) when $\tilde{A} = A - g_0 B$ and (3.5) holds.

Similarly, multiplying (3.43) by -1 and following the same procedure, we find (3.38) when $\tilde{A} = A - g_0 B$ and (3.6) holds.

2. *Case* $\tilde{A} \neq A - g_0 B$: multiplying (3.41) and (3.42) by $v_t(t)$ and $\tilde{A}^{-1} A u_t(t)$, respectively, and noting that $\langle A u_t(t), v_t(t)\rangle - \left\langle \tilde{A}^{-1} A u_t(t), \tilde{A} v_t(t)\right\rangle = 0$ (because $\tilde{A}$ is self-adjoint), we get

$$
\begin{aligned}
0 &= \langle u_{ttt}(t), v_t(t)\rangle + \langle (A - g_0 B) u_t(t) + D u_{tt}(t), v_t(t)\rangle + \left\langle \int_0^{+\infty} g(s) B \eta_t(t,s) ds, v_t(t) \right\rangle \\
&\quad + \left\langle \tilde{B} v_t(t), v_t(t)\right\rangle - \left\langle v_{ttt}(t), \tilde{A}^{-1} A u_t(t)\right\rangle - \left\langle \tilde{A} v_t(t), \tilde{A}^{-1} A u_t(t)\right\rangle - \left\langle \tilde{B} u_t(t), \tilde{A}^{-1} A u_t(t)\right\rangle \\
&= \partial_t \Bigg(\left\langle \tilde{A}^{-1} A u_{tt}(t), v_t(t)\right\rangle - \left\langle \tilde{A}^{-1} A u_t(t), v_{tt}(t)\right\rangle - g_0 \left\langle B^{\frac{1}{2}} u_t(t), B^{\frac{1}{2}} v(t)\right\rangle \\
&\quad + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_t(t,s) ds, B^{\frac{1}{2}} v(t) \right\rangle \Bigg) \\
&\quad - \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_{tt}(t,s) ds, B^{\frac{1}{2}} v(t) \right\rangle - \left\langle \tilde{B} u_t(t), \tilde{A}^{-1} A u_t(t)\right\rangle + g_0 \left\langle B^{\frac{1}{2}} u_{tt}(t), B^{\frac{1}{2}} v(t)\right\rangle \\
&\quad + \langle u_{ttt}(t), v_t(t)\rangle - \left\langle \tilde{A}^{-1} A u_{ttt}(t), v_t(t)\right\rangle + \langle D u_{tt}(t), v_t(t)\rangle + \left\langle \tilde{B} v_t(t), v_t(t)\right\rangle.
\end{aligned}
$$

Therefore

$$
\begin{aligned}
&\partial_t \Bigg(\left\langle \tilde{A}^{-1} A u_{tt}(t), v_t(t)\right\rangle - \left\langle \tilde{A}^{-1} A u_t(t), v_{tt}(t)\right\rangle - g_0 \left\langle B^{\frac{1}{2}} u_t(t), B^{\frac{1}{2}} v(t)\right\rangle \\
&\qquad + \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_t(t,s) ds, B^{\frac{1}{2}} v(t) \right\rangle \Bigg) \\
&= \left\langle \int_0^{+\infty} g(s) B^{\frac{1}{2}} \eta_{tt}(t,s) ds, B^{\frac{1}{2}} v(t) \right\rangle + \left\langle \tilde{B} u_t(t), \tilde{A}^{-1} A u_t(t)\right\rangle \\
&\quad - g_0 \left\langle B^{\frac{1}{2}} u_{tt}(t), B^{\frac{1}{2}} v(t)\right\rangle - \langle u_{ttt}(t), v_t(t)\rangle - \langle D u_{tt}(t), v_t(t)\rangle \\
&\quad + \left\langle \tilde{A}^{-1} A u_{ttt}(t), v_t(t)\right\rangle - \left\langle \tilde{B} v_t(t), v_t(t)\right\rangle. \qquad (3.44)
\end{aligned}
$$

Consequently, using Cauchy–Schwarz and Young's inequalities, applying (3.2), (2.3) and (2.1) to estimate $\|B^{\frac{1}{2}} v(t)\|^2$, $\|\tilde{B} u_t(t)\|$ and $\|B^{\frac{1}{2}} u_{tt}(t)\|^2$ by $a_0 \tilde{a}_2 \|\tilde{A}^{\frac{1}{2}} v(t)\|^2$, $\sqrt{b_1}\|u_t(t)\|$ and $a_0 \|A^{\frac{1}{2}} u_{tt}(t)\|^2$, respectively, and using (3.40) and (3.5) to estimate $\|\tilde{A}^{-1} A u_{ttt}(t)\|$,

$\|\tilde{A}^{-1}Au_t(t)\|^2$ and the last term of the right hand side of (3.44), we get (3.39) when $\tilde{A} \neq A - g_0 B$ and (3.5) holds.

Similarly, multiplying (3.44) by -1 and following the same procedure, we find (3.39) when $\tilde{A} \neq A - g_0 B$ and (3.6) holds. □

Before proving the next lemma, let us consider, for $k = 1, 2, 3$,

$$E_k(t) = \frac{1}{2}\|\partial_t^k \mathscr{U}(t)\|^2_{\mathscr{H}}, \quad \forall t \in \mathbb{R}_+. \tag{3.45}$$

Similarly to (3.19), we have

$$E_k'(t) = -\|D^{\frac{1}{2}}\partial_t^k u_t(t)\|^2 + \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\partial_t^k \eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \tag{3.46}$$

Lemma 3.7. *There exist positive constants N_i, M_i $(i = 0, 1, 2)$ and C_i $(i = 0, 1)$ such that the functional*

$$F(t) = N_0(E(t) + E_1(t)) + N_1 I_1(t) + M_1 J_1(t) + C_1 R_1(t) + R_2(t) \tag{3.47}$$

if $\tilde{A} = A - g_0 B$, and

$$\begin{aligned} F(t) = {} & N_0(E(t) + E_1(t) + E_2(t)) + N_1 I_1(t) + N_2 I_2(t) \\ & + M_1 J_1(t) + M_2 J_2(t) + C_1 R_1(t) + R_2(t) \end{aligned} \tag{3.48}$$

if $\tilde{A} \neq A - g_0 B$, satisfies, for all $t \in \mathbb{R}_+$,

$$F(t) \geq M_0(E(t) + E_1(t)) \tag{3.49}$$

and

$$F'(t) \leq -M_0 E(t) + C_0 \int_0^{+\infty} g(s)\left(\|B^{\frac{1}{2}}\eta(t,s)\|^2 + \|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2\right) ds \tag{3.50}$$

if $\tilde{A} = A - g_0 B$, and

$$F(t) \geq M_0(E(t) + E_1(t) + E_2(t)) \tag{3.51}$$

and

$$F'(t) \leq -M_0 E(t) + C_0 \int_0^{+\infty} g(s)\left(\|B^{\frac{1}{2}}\eta(t,s)\|^2 + \|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 + \|B^{\frac{1}{2}}\eta_{ttt}(t,s)\|^2\right) ds \tag{3.52}$$

if $\tilde{A} \neq A - g_0 B$.

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

Proof. First, we prove (3.49) and (3.51). Using (2.12), (3.18) and the fact that $c_0 < 1$ (c_0 is defined in (2.11)), we find, for all $t \in \mathbb{R}_+$,

$$E(t) \geq \frac{c_0}{2}\left(\|u_t(t)\|^2 + \|v_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 + \|\tilde{A}^{\frac{1}{2}}v(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds\right). \tag{3.53}$$

Similarly,

$$E_1(t) \geq \frac{c_0}{2}\left(\|u_{tt}(t)\|^2 + \|v_{tt}(t)\|^2 + \|A^{\frac{1}{2}}u_t(t)\|^2 + \|\tilde{A}^{\frac{1}{2}}v_t(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_t(t,s)\|^2 ds\right) \tag{3.54}$$

and

$$E_2(t) \geq \frac{c_0}{2}\left(\|u_{ttt}(t)\|^2 + \|v_{ttt}(t)\|^2 + \|A^{\frac{1}{2}}u_{tt}(t)\|^2 + \|\tilde{A}^{\frac{1}{2}}v_{tt}(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2 ds\right). \tag{3.55}$$

From (3.53)–(3.55) and the definition of I_i, J_i and R_i ($i = 1, 2$), we see that there exists a positive constant L (not depending on N_0) such that

$$F(t) \geq (N_0 - L)(E(t) + E_1(t)), \quad \forall t \in \mathbb{R}_+$$

in case $\tilde{A} = A - g_0 B$, and

$$F(t) \geq (N_0 - L)(E(t) + E_1(t) + E_2(t)), \quad \forall t \in \mathbb{R}_+$$

in case $\tilde{A} \neq A - g_0 B$. Thus, for any $N_0 > L$, (3.49) and (3.51) hold, for any

$$0 < M_0 \leq N_0 - L. \tag{3.56}$$

Second, we prove (3.50) and (3.52) by distinguishing two cases.

1. *Case* $\tilde{A} = A - g_0 B$: by combining (3.23), (3.34), (3.36) and (3.38), taking in consideration (3.19) and (3.46), and noting that $g' \leq 0$, we get

$$\begin{aligned} F'(t) \leq & -L_1\|u_t(t)\|^2 - L_2\|v_t(t)\|^2 - L_3\|A^{\frac{1}{2}}u(t)\|^2 - L_4\|\tilde{A}^{\frac{1}{2}}v(t)\|^2 \\ & + \int_0^{+\infty} g(s)\left((N_1 c_{\epsilon_1} + M_1 c_{\epsilon_2})\|B^{\frac{1}{2}}\eta(t,s)\|^2 + c_{\epsilon_3}\|B^{\frac{1}{2}}\eta_{tt}(t,s)\|^2\right) ds \\ & - (N_0 - N_1 c_{\epsilon_1} - M_1 c_{\epsilon_2})\|D^{\frac{1}{2}}u_t(t)\|^2 - (N_0 - c_{\epsilon_3})\|D^{\frac{1}{2}}u_{tt}(t)\|^2, \end{aligned} \tag{3.57}$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

where

$$\begin{cases} L_1 = \frac{\alpha_0 g_0}{2} N_1 - M_1 - \sqrt{b_1} - \epsilon_1 N_1, \\ L_2 = b_0 - C_1 - \epsilon_3, \\ L_3 = (1 - a_0 g_0 - \lambda_1) M_1 - \dfrac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_3} C_1 - \epsilon_1 N_1 - \epsilon_2 M_1, \\ L_4 = (1 - \lambda_3) C_1 - \dfrac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_1} M_1 - \epsilon_1 N_1 - \epsilon_3. \end{cases}$$

We choose $\lambda_1 = \frac{1}{2}(1 - a_0 g_0)$ (which is positive since (2.4)), $\lambda_3 = \frac{1}{2}$, $0 < C_1 < b_0$ and

$$N_1 > \frac{2 a_0 b_1 C_1}{\alpha_0 g_0 a_1 \tilde{a}_1 (1 - a_0 g_0)} + \frac{2\sqrt{b_1}}{\alpha_0 g_0}$$

(note that, because $g(0) > 0$ according to **(A5)**, then $g_0 > 0$). After, we take

$$\frac{a_0 b_1 C_1}{a_1 \tilde{a}_1 (1 - a_0 g_0)} < M_1 < \min \left\{ \frac{\alpha_0 g_0 N_1}{2} - \sqrt{b_1}, \frac{a_1 \tilde{a}_1 (1 - a_0 g_0) C_1}{a_0 b_1} \right\}$$

(M_1 exists due to (3.4) and the definition of N_1). These choices imply that

$$\frac{\alpha_0 g_0 N_1}{2} - M_1 - \sqrt{b_1} > 0, \quad b_0 - C_1 > 0, \quad (1 - a_0 g_0 - \lambda_1) M_1 - \frac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_3} C_1 > 0$$

and

$$(1 - \lambda_3) C_1 - \frac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_1} M_1 > 0.$$

Next, we choose $\epsilon_3 = \epsilon_2 = \epsilon_1$ and ϵ_1 small enough such that $L_i > 0$, $i = 1, \ldots, 4$. Finally, we choose

$$N_0 > \max\{L, N_1 c_{\epsilon_1} + M_1 c_{\epsilon_2}, c_{\epsilon_3}\}$$

(so M_0 in (3.56) exists and the last two terms of (3.57) are negative). On the other hand, using Young inequality, (2.1), (2.2) and (2.3), we find

$$\begin{aligned} \frac{1}{2} \left(\left\langle \tilde{B} v(t), u(t) \right\rangle + \left\langle \tilde{B} u(t), v(t) \right\rangle \right) &= \left\langle \tilde{B} u(t), v(t) \right\rangle \\ &\le \frac{1}{2} \left(\|\tilde{B} u(t)\|^2 + \|v(t)\|^2 \right) \\ &\le \frac{1}{2} \left(\frac{b_1 a_0}{a_1} \|A^{\frac{1}{2}} u(t)\|^2 + \frac{1}{\tilde{a}_1} \|\tilde{A}^{\frac{1}{2}} v(t)\|^2 \right), \quad \forall t \in \mathbb{R}_+, \end{aligned}$$

therefore, from (3.18),

$$E(t) \leq C_2 \Bigg(\|u_t(t)\|^2 + \|v_t(t)\|^2 + \|A^{\frac{1}{2}} u(t)\|^2 + \|\tilde{A}^{\frac{1}{2}} v(t)\|^2 + \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2 ds \Bigg), \tag{3.58}$$

where

$$C_2 = \frac{1}{2} \max \left\{ 1 + \frac{b_1 a_0}{a_1}, 1 + \frac{1}{\tilde{a}_1} \right\}.$$

By combining (3.57) and (3.58), (3.50) holds, for any

$$0 < M_0 \leq \frac{1}{C_2} \min\{L_1, L_2, L_3, L_4\} \tag{3.59}$$

and

$$C_0 = \max \left\{ c_{\epsilon_3}, N_1 c_{\epsilon_1} + M_1 c_{\epsilon_2} + \min\{L_1, L_2, L_3, L_4\} \right\}.$$

So, (3.49) and (3.50) hold, for any $M_0 > 0$ satisfying (3.56) and (3.59).

2. *Case* $\tilde{A} \neq A - g_0 B$: similarly to the proof in case $\tilde{A} = A - g_0 B$, by combining (3.23), (3.24), (3.34)–(3.36) and (3.39), taking in consideration (3.19) and (3.46), and noting that $g' \leq 0$, we get

$$\begin{aligned} F'(t) \leq & -L_1 \|u_t(t)\|^2 - L_2 \|v_t(t)\|^2 - L_3 \|A^{\frac{1}{2}} u(t)\|^2 - L_4 \|\tilde{A}^{\frac{1}{2}} v(t)\|^2 - L_5 \|u_{ttt}(t)\|^2 \\ & - L_6 \|A^{\frac{1}{2}} u_{tt}(t)\|^2 + (N_1 c_{\epsilon_1} + M_1 c_{\epsilon_2}) \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2 ds \\ & + \int_0^{+\infty} g(s) \Big((N_2 c_{\delta_1} + M_2 c_{\delta_2} + c_{\epsilon_4}) \|B^{\frac{1}{2}} \eta_{tt}(t,s)\|^2 + N_2 c_{\delta_1} \|B^{\frac{1}{2}} \eta_{ttt}(t,s)\|^2 \Big) ds \\ & - (N_0 - N_1 c_{\epsilon_1} - M_1 c_{\epsilon_2}) \|D^{\frac{1}{2}} u_t(t)\|^2 - (N_0 - c_{\epsilon_3}) \|D^{\frac{1}{2}} u_{tt}(t)\|^2 \\ & - (N_0 - N_2 c_{\delta_1} - M_2 c_{\delta_2}) \|D^{\frac{1}{2}} u_{ttt}(t)\|^2, \quad \forall t \in \mathbb{R}_+, \end{aligned} \tag{3.60}$$

where

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

$$\begin{cases} L_1 = \frac{\alpha_0 g_0}{2} N_1 - M_1 - \sqrt{b_1 d_0} - \epsilon_1 N_1, \\ L_2 = b_0 - C_1 - \epsilon_3 - \dfrac{b_1 M_2}{4\lambda_2} - \delta_1 N_2, \\ L_3 = (1 - a_0 g_0 - \lambda_1) M_1 - \dfrac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_3} C_1 - \epsilon_1 N_1 - \epsilon_2 M_1, \\ L_4 = (1 - \lambda_3) C_1 - \dfrac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_1} M_1 - \delta_3 - \epsilon_1 N_1 - \epsilon_4, \\ L_5 = \frac{\alpha_0 g_0}{2} N_2 - (1 + \lambda_2) M_2 - \dfrac{d_0 + 1}{\epsilon_3} - \delta_1 N_2, \\ L_6 = (1 - a_0 g_0) M_2 - \dfrac{a_0^2 g_0^2 \tilde{a}_2}{4\delta_3} - \delta_1 N_2 - \delta_2 M_2. \end{cases}$$

We choose

$$\lambda_1 = \frac{1}{2}(1 - a_0 g_0), \quad \lambda_3 = \frac{1}{2} \quad \text{and} \quad 0 < \delta_3 < \frac{b_0 \left((a_1 \tilde{a}_1)^2 (1 - a_0 g_0)^2 - (a_0 b_1)^2 \right)}{2 (a_1 \tilde{a}_1)^2 (1 - a_0 g_0)^2}$$

(λ_1 and δ_3 exist thanks to (2.4) and (3.4)). Next, we take

$$\frac{2\delta_3 (a_1 \tilde{a}_1)^2 (1 - a_0 g_0)^2}{(a_1 \tilde{a}_1)^2 (1 - a_0 g_0)^2 - (a_0 b_1)^2} < C_1 < b_0 \quad \text{and} \quad M_2 > \frac{a_0^2 g_0^2 \tilde{a}_2}{4 (1 - a_0 g_0) \delta_3}$$

(C_1 exists in virtue of the choice of δ_3). After, we pick

$$0 < \epsilon_3 < b_0 - C_1, \quad \frac{a_0 b_1 C_1}{a_1 \tilde{a}_1 (1 - a_0 g_0)} < M_1 < \frac{a_1 \tilde{a}_1 (1 - a_0 g_0)(C_1 - 2\delta_3)}{a_0 b_1}$$

(ϵ_3 and M_1 exist according to the choice of C_1) and

$$\lambda_2 > \frac{b_1 M_2}{4 (b_0 - C_1 - \epsilon_3)}$$

(λ_2 exists due to the choice of ϵ_3). Next, we choose

$$N_1 > \frac{2(\sqrt{b_1 d_0} + M_1)}{\alpha_0 g_0} \quad \text{and} \quad N_2 > \frac{2}{\alpha_0 g_0} \left((1 + \lambda_2) M_2 + \frac{d_0 + 1}{\epsilon_3} \right).$$

These choices imply that

$$\frac{\alpha_0 g_0}{2} N_1 - M_1 - \sqrt{b_1 d_0} > 0, \quad b_0 - C_1 - \epsilon_3 - \frac{b_1 M_2}{4\lambda_2} > 0,$$

$$(1 - a_0 g_0 - \lambda_1) M_1 - \frac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_3} C_1 > 0, \quad (1 - \lambda_3) C_1 - \frac{a_0 b_1}{4 a_1 \tilde{a}_1 \lambda_1} M_1 - \delta_3 > 0,$$

$$\frac{\alpha_0 g_0}{2} N_2 - (1 + \lambda_2) M_2 - \frac{d_0 + 1}{\epsilon_3} > 0 \quad \text{and} \quad (1 - a_0 g_0) M_2 - \frac{a_0^2 g_0^2 \tilde{a}_2}{4\delta_3} > 0.$$

At the end, we take $\epsilon_4 = \epsilon_2 = \delta_2 = \delta_1 = \epsilon_1$ and ϵ_1 small enough such that $L_i > 0$, $i = 1, \ldots, 6$. Finally, we choose

$$N_0 > \max\{L, N_1 c_{\epsilon_1} + M_1 c_{\epsilon_2}, c_{\epsilon_3}, N_2 c_{\delta_1} + M_2 c_{\delta_2}\}$$

(so M_0 in (3.56) exists and the last three terms of (3.60) are negative). By combining (3.58) and (3.60), we find (3.52), for any M_0 satisfying (3.59), and

$$C_0 = \max\left\{N_1 c_{\epsilon_1} + M_1 c_{\epsilon_2} + \min\{L_1, L_2, L_3, L_4\}, N_2 c_{\delta_1} + M_2 c_{\delta_2} + c_{\epsilon_4}, N_2 c_{\delta_1}\right\}.$$

So, (3.51) and (3.52) hold, for any $M_0 > 0$ satisfying (3.56) and (3.59). □

Now, we estimate the integral terms in (3.50) and (3.52). Under the condition (3.7) and using (3.19), we have

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \le \frac{-2}{\delta} E'(t), \quad \forall t \in \mathbb{R}_+. \tag{3.61}$$

In case where (3.8) holds and (3.7) does not hold, we apply this lemma given in [15] and [17] under, respectively, the condition

$$\sup_{s\in\mathbb{R}_+} \|B^{\frac{1}{2}} u_0(s)\|^2 < +\infty$$

and the weaker one

$$\sup_{t\in\mathbb{R}_+} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \|B^{\frac{1}{2}} u_0(s-t)\|^2 ds < +\infty.$$

Lemma 3.8. *There exists a positive constant C_3 such that, for any $\epsilon_0 > 0$, the following inequality holds:*

$$G'(\epsilon_0 E(t)) \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \le -C_3 E'(t) + C_3 \epsilon_0 E(t) G'(\epsilon_0 E(t)), \quad \forall t \in \mathbb{R}_+. \tag{3.62}$$

Proof. For the convenience of the reader, we give a brief proof of this lemma (see [17]).

First, we note that, if $g'(s_0) = 0$, for some $s_0 \ge 0$, then $g(s_0) = 0$ because $G^{-1}(0) = 0$ and $s \mapsto \frac{g(s)}{G^{-1}(-g'(s))}$ is bounded (thanks to (3.8)), and therefore, $g(s) = 0$, for all $s \ge s_0$ because g is non-negative and non-increasing. This implies that the infinite integral in (3.62) is effective only on $[0, s_0]$. Thus, without loss of generality, we can assume that $g' < 0$.

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

Let $t \in \mathbb{R}_+$. Because E is non-increasing and using (2.1), (3.53) implies that

$$\begin{aligned}\|B^{\frac{1}{2}}\eta(t,s)\|^2 &\le 2\left(\|B^{\frac{1}{2}}u(t)\|^2 + \|B^{\frac{1}{2}}u(t-s)\|^2\right)\\ &\le \frac{4a_0}{c_0}E(0) + 2\|B^{\frac{1}{2}}u(t-s)\|^2, \quad \forall s \in \mathbb{R}_+.\end{aligned}$$

Then, for

$$M(t,s) := \begin{cases} \dfrac{8a_0}{c_0}E(0) & \text{if } 0 \le s \le t, \\ \dfrac{4a_0}{c_0}E(0) + 2\|B^{\frac{1}{2}}u_0(s-t)\|^2 & \text{if } s > t, \end{cases} \tag{3.63}$$

we conclude that

$$\|B^{\frac{1}{2}}\eta(t,s)\|^2 \le M(t,s), \quad \forall t, s \in \mathbb{R}_+. \tag{3.64}$$

Let $\tau_1(t,s), \tau_2(t,s) > 0$ (which will be fixed later on), $\epsilon_0 > 0$ and $K(s) = \frac{s}{G^{-1}(s)}$, for $s \in \mathbb{R}_+$. Thanks to **(A5)**, the function K is non-decreasing, then, using (3.64),

$$K\left(-\tau_2(t,s)g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2\right) \le K\left(-M(t,s)\tau_2(t,s)g'(s)\right), \quad \forall s \in \mathbb{R}_+.$$

Using this inequality, we arrive at

$$\begin{aligned}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds &= \frac{1}{G'(\epsilon_0 E(t))}\int_0^{+\infty}\frac{1}{\tau_1(t,s)}G^{-1}\left(-\tau_2(t,s)g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2\right)\\ &\quad \times \frac{\tau_1(t,s)G'(\epsilon_0 E(t))g(s)}{-\tau_2(t,s)g'(s)}K\left(-\tau_2(t,s)g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2\right)ds\\ &\le \frac{1}{G'(\epsilon_0 E(t))}\int_0^{+\infty}\frac{1}{\tau_1(t,s)}G^{-1}\left(-\tau_2(t,s)g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2\right)\\ &\quad \times \frac{\tau_1(t,s)G'(\epsilon_0 E(t))g(s)}{-\tau_2(t,s)g'(s)}K\left(-M(t,s)\tau_2(t,s)g'(s)\right)ds\\ &\le \frac{1}{G'(\epsilon_0 E(t))}\int_0^{+\infty}\frac{1}{\tau_1(t,s)}G^{-1}\left(-\tau_2(t,s)g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2\right)\\ &\quad \times \frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}ds.\end{aligned}$$

Let $G^*(s) = \sup_{\tau\in\mathbb{R}_+}\{s\tau - G(\tau)\}$, $s \in \mathbb{R}_+$, denote the dual function of G. According to **(A5)**,

$$G^*(s) = s(G')^{-1}(s) - G((G')^{-1}(s)), \quad \forall s \in \mathbb{R}_+.$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

Using Young's inequality: $s_1 s_2 \le G(s_1) + G^*(s_2)$, for

$$s_1 = G^{-1}\left(-\tau_2(t,s)g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2\right) \quad \text{and} \quad s_2 = \frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))},$$

we get

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \le \frac{1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} \frac{-\tau_2(t,s)}{\tau_1(t,s)} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds$$
$$+ \frac{1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} \frac{1}{\tau_1(t,s)} G^*\left(\frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}\right) ds.$$

Using the fact that $G^*(s) \le s(G')^{-1}(s)$, we get

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds$$
$$\le \frac{-1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} \frac{\tau_2(t,s)}{\tau_1(t,s)} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds$$
$$+ \int_0^{+\infty} \frac{M(t,s)g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}(G')^{-1}\left(\frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 E(t))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}\right) ds.$$

Thanks to (3.8), $\sup_{s\in\mathbb{R}_+} \frac{g(s)}{G^{-1}(-g'(s))} := m_1 < +\infty$. Then, using the fact that $(G')^{-1}$ is non-decreasing and choosing $\tau_2(t,s) = \frac{1}{M(t,s)}$, we get

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \le \frac{-1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} \frac{1}{M(t,s)\tau_1(t,s)} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds$$
$$+ \int_0^{+\infty} \frac{M(t,s)g(s)}{G^{-1}(-g'(s))}(G')^{-1}\left(m_1 M(t,s)\tau_1(t,s)G'(\epsilon_0 E(t))\right) ds.$$

Choosing $\tau_1(t,s) = \frac{1}{m_1 M(t,s)}$ and using (3.19) and the fact that

$$\sup_{t\in\mathbb{R}_+}\int_0^{+\infty} \frac{M(t,s)g(s)}{G^{-1}(-g'(s))} ds =: m_2 < +\infty$$

(due to (3.8) and (3.10)), we obtain

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \le \frac{-m_1}{G'(\epsilon_0 E(t))}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds + \epsilon_0 E(t)\int_0^{+\infty}\frac{M(t,s)g(s)}{G^{-1}(-g'(s))}ds$$
$$\le \frac{-2m_1}{G'(\epsilon_0 E(t))}E'(t) + \epsilon_0 m_2 E(t),$$

which gives (3.62) with $C_3 = \max\{2m_1, m_2\}$. □

Now, we go back to (3.50) and (3.52). Similarly to (3.61) and (3.62), and using (3.46), we find, for $k = 1, 2, 3$,

$$\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\partial_t^k\eta(t,s)\|^2 ds \le \frac{-2}{\delta}E_k'(t), \quad \forall t \in \mathbb{R}_+ \tag{3.65}$$

if (3.7) holds. Otherwise, when (3.8) holds and (3.7) does not hold, we get, for any $\epsilon_0 > 0$,

$$G'(\epsilon_0 E(t))\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\partial_t^k\eta(t,s)\|^2 ds \le C_{3+k}\left(-E_k'(t) + \epsilon_0 E(t)G'(\epsilon_0 E(t))\right), \quad \forall t \in \mathbb{R}_+, \tag{3.66}$$

where C_4, C_5 and C_6 are defined as C_3 with, respectively, $\|B^{\frac{1}{2}}\partial_s u_0(s-t)\|^2$, $\|B^{\frac{1}{2}}\partial_s^2 u_0(s-t)\|^2$ and $\|B^{\frac{1}{2}}\partial_s^3 u_0(s-t)\|^2$ instead of $\|B^{\frac{1}{2}}u_0(s-t)\|^2$, and $E_1(0)$, $E_2(0)$ and $E_3(0)$ instead of $E(0)$ in the definition (3.63) of $M(t,s)$. Therefore, from (3.50) and (3.52), we get, for some positive constants C_7 and C_8 (do not depending on ϵ_0),

$$F'(t) \le -M_0 E(t) - C_7(E'(t) + E_2'(t) + \xi E_3'(t)), \quad \forall t \in \mathbb{R}_+$$

if (3.7) holds, and

$$G'(\epsilon_0 E(t))F'(t) \le -(M_0 - C_8\epsilon_0)E(t)G'(\epsilon_0 E(t)) - C_8(E'(t) + E_2'(t) + \xi E_3'(t)), \quad \forall t \in \mathbb{R}_+$$

if (3.8) holds and (3.7) does not hold, where $\xi = 0$ if $\tilde{A} = A - g_0 B$, and $\xi = 1$ if $\tilde{A} \neq A - g_0 B$. By choosing $0 < \epsilon_0 < \dfrac{M_0}{C_8}$, we see that, for some positive constants C_9 and C_{10},

$$G_0(\epsilon_0 E(t)) \le -C_9\frac{G_0(\epsilon_0 E(t))}{E(t)}F'(t) - C_{10}(E'(t) + E_2'(t) + \xi E_3'(t)), \quad \forall t \in \mathbb{R}_+, \tag{3.67}$$

where G_0 is defined in (3.12). By integrating (3.67) over $[0, T]$, for $T > 0$, and using the fact that $F, E, E_2, E_3 > 0$ (due to (3.49) and (3.51)), $G_0(\epsilon_0 E)$ and $\dfrac{G_0(\epsilon_0 E)}{E}$ are non-increasing (according to **(A5)** and the fact that E is non-increasing), we find

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

$$G_0(\epsilon_0 E(T))T \le \int_0^T G_0(\epsilon_0 E(t))dt$$
$$\le C_9 \frac{G_0(\epsilon_0 E(0))}{E(0)} F(0) + C_{10}(E(0) + E_2(0) + \xi E_3(0)).$$

This implies (3.11) for $n = 1$ and

$$c_1 = \max\left\{\frac{1}{\epsilon_0}, C_9 \frac{G_0(\epsilon_0 E(0))}{E(0)} F(0) + C_{10}\left(E(0) + E_2(0) + \xi E_3(0)\right)\right\}.$$

Now, suppose that (3.11) holds and let $\mathscr{U}_0 \in \mathscr{K}_{i_0(n+1)}$. We have $\partial_t^k \mathscr{U}(0) \in \mathscr{K}_{i_0 n}$, for $k = 0, 1, 2$ if $i_0 = 2$, and $k = 0, 1, 2, 3$ if $i_0 = 3$ (thanks to Theorem 2.2 and the definition of $\mathscr{K}_n$), and then (3.11) implies that, for $k = 1, \dots, i_0$,

$$E(t) \le c_n G_n(\frac{c_n}{t}) \quad \text{and} \quad E_k(t) \le \theta_n^k G_n(\frac{\theta_n^k}{t}), \tag{3.68}$$

where θ_n^k is a positive constant. On the other hand, for some positive constant C_{11} (according to the definition of F, E, E_i, I_i, J_i and R_i),

$$F(t) \le C_{11}(E(t) + E_1(t)), \quad \forall t \in \mathbb{R}_+ \tag{3.69}$$

if $\tilde{A} = A - g_0 B$, and

$$F(t) \le C_{11}(E(t) + E_1(t) + E_2(t)), \quad \forall t \in \mathbb{R}_+ \tag{3.70}$$

if $\tilde{A} \ne A - g_0 B$. Integrating (3.67) over $[T, 2T]$, for $T > 0$, and using (3.69), (3.70) and the fact that $G_0(\epsilon_0 E)$ and $\dfrac{G_0(\epsilon_0 E)}{E}$ are non-increasing, we deduce that, for all $T > 0$,

$$G_0(\epsilon_0 E(2T))T \le \int_T^{2T} G_0(\epsilon_0 E(t))dt$$
$$\le C_{12}(E(T) + E_1(T) + E_2(T) + \xi E_3(T)), \tag{3.71}$$

where

$$C_{12} = C_{10} + C_9 C_{11} \frac{G_0(\epsilon_0 E(0))}{E(0)}.$$

From (3.68) and (3.71), we get, for all $T > 0$,

$$E(2T) \le \frac{1}{\epsilon_0} G_0^{-1}\left(\frac{2C_{12}}{2T}\left(c_n G_n(\frac{2c_n}{2T}) + \theta_n^1 G_n(\frac{2\theta_n^1}{2T}) + \theta_n^2 G_n(\frac{2\theta_n^2}{2T}) + \xi \theta_n^3 G_n(\frac{2\theta_n^3}{2T})\right)\right).$$

This implies (note that G_0^{-1} and G_n are non-decreasing), for $t = 2T$,

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

$$E(t) \leq c_{n+1} G_0^{-1} \left(\frac{c_{n+1}}{t} G_n(\frac{c_{n+1}}{t}) \right) = c_{n+1} G_{n+1}(\frac{c_{n+1}}{t}), \quad \forall t > 0,$$

where

$$c_{n+1} = \max \left\{ \frac{1}{\epsilon_0}, 2c_n, 2\theta_n^1, 2\theta_n^2, 2\theta_n^3, 2\left(c_n + \theta_n^1 + \theta_n^2 + \xi\theta_n^3\right) C_{12} \right\}.$$

This proves (3.11), for $n+1$. The proof of Theorem 3.1 is completed.

4. Uncoupled wave and Petrovsky equations

As it was mentioned in the introduction, in the uncoupled case: $\tilde{B} \equiv 0$, the second equation (1.4) of (1.1) is conservative; that is, its classical energy

$$E_v(t) = \frac{1}{2}\left(\|v_t(t)\|^2 + \|\tilde{A}^{\frac{1}{2}} v(t)\|^2 \right)$$

is a constant function. In this section, we consider the first equation (1.5) of (1.1); that is

$$\begin{cases} u_{tt}(t) + Au(t) + Du_t(t) - \int_0^{+\infty} g(s)Bu(t-s)ds = 0, & \forall t > 0, \\ u(-t) = u_0(t), \quad u_t(0) = u_1, & \forall t \in \mathbb{R}_+ \end{cases} \tag{4.1}$$

with homogeneous Dirichlet and Dirichlet–Neumann boundary conditions on $\Gamma \times \mathbb{R}_+$

$$\begin{cases} u = 0: & \text{wave}, \\ u = \frac{\partial u}{\partial \nu} = 0: & \text{Petrovsky}, \end{cases} \tag{4.2}$$

where the coefficients a, b and d are as before and

$$(A, B, D) = \begin{cases} (-div(a\nabla), -div(b\nabla), d\, Id): & \text{wave}, \\ (\Delta(a\Delta), \Delta(b\Delta), d\, Id): & \text{Petrovsky}. \end{cases}$$

4.1. Well-posedness

System (4.1)–(4.2) can be written in the abstract form

$$\begin{cases} \mathcal{U}_t(t) = \mathcal{A}\mathcal{U}(t), & \forall t > 0, \\ \mathcal{U}(0) = \mathcal{U}_0, \end{cases} \tag{4.3}$$

where $\mathcal{U} = (u, u_t, \eta)^T$, $\mathcal{U}_0 = (u_0(0), u_1, \eta_0)^T \in \mathcal{H}$,

$$\mathcal{H} = D(A^{\frac{1}{2}}) \times H \times L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))$$

endowed with the inner product, for $W = (w_1, w_2, w_3)^T$ and $\tilde{W} = (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)^T$,

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

$$\left\langle W, \tilde{W} \right\rangle_{\mathscr{H}} = \left\langle A^{\frac{1}{2}} w_1, A^{\frac{1}{2}} \tilde{w}_1 \right\rangle - g_0 \left\langle B^{\frac{1}{2}} w_1, B^{\frac{1}{2}} \tilde{w}_1 \right\rangle + \langle w_2, \tilde{w}_2 \rangle + \langle w_3, \tilde{w}_3 \rangle_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))},$$

and

$$\mathscr{A} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_2 \\ (-A + g_0 B) w_1 - D w_2 - \int_0^{+\infty} g(s) B w_3(s) ds \\ -\partial_s w_3 + w_2 \end{pmatrix}$$

with domain

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{c} W = (w_1, w_2, w_3)^T \in \mathscr{H},\ \partial_s w_3 \in L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}})), \\ w_2 \in D(A^{\frac{1}{2}}),\ (A - g_0 B) w_1 + \int_0^{+\infty} g(s) B w_3(s) ds \in H,\ w_3(0) = 0 \end{array} \right\}.$$

Then, applying the same arguments as in Section 2, we deduce that, under assumptions **(A0)**–**(A1)**, for any $n \in \mathbb{N}$ and $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A}^n)$, the system (4.3) has a unique solution satisfying (2.15).

4.2. Strong stability

We prove here the following strong stability estimate for (4.3):

Theorem 4.1. *Assume that* **(A0)**, **(A1)**, **(A3)** *and* **(A5)** *hold and let $\mathscr{U}_0 \in \mathscr{K}_0$, where $\mathscr{K}_0$ is defined in (3.9) and (3.10) (for $n = 0$). Then there exist positive constants c_1 and c_2 such that*

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} \leq c_1 \tilde{G}^{-1}(c_2 t), \quad \forall t > 0, \tag{4.4}$$

where $\tilde{G}(s) = \int_s^1 \frac{1}{G_0(\tau)} d\tau$, for $s \in]0, 1]$, and G_0 is defined in (3.12).

Remark 4.2. Because $\lim_{s \to 0^+} \tilde{G}(s) = +\infty$, then (3.13) holds. On the other hand, in the case of the particular examples (3.14) considered in Section 3, we deduce from (4.4) that

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} \leq c_1 e^{-c_2 t}, \quad \forall t \in \mathbb{R}_+ \tag{4.5}$$

in case g_1, and

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} \leq c_1 \left(p(p-1) c_2 t + 1 \right)^{-\frac{1}{p-1}}, \quad \forall t \in \mathbb{R}_+, \quad \forall p > \frac{q_2 + 1}{q_2 - 1}. \tag{4.6}$$

in case g_2. Notice that (4.5) and (4.6) are stronger than, respectively, (3.15) and (3.16), for any $n \in \mathbb{N}^*$.

For more examples, see [15], where (4.1) is considered in the case $D \equiv 0$.

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

Proof of Theorem 4.1. Let E be the associated energy functional with the solution of (4.3) given by (3.18). As for (2.7) (by taking $\tilde{A} \equiv 0$ and $\tilde{B} \equiv 0$), E satisfies (3.19).

Now, we consider the functionals I_1 and J_1 defined, respectively, in Lemma 3.4 and Lemma 3.5. We deduce from (3.23) and (3.34) (by taking $\tilde{A} \equiv 0$ and $\tilde{B} \equiv 0$) that, for any $\epsilon_1, \epsilon_2 > 0$, there exist $c_{\epsilon_1}, c_{\epsilon_2}$ such that

$$\begin{aligned} I_1'(t) \leq &-(\frac{\alpha_0 g_0}{2} - \epsilon_1)\|u_t(t)\|^2 + \epsilon_1 \|A^{\frac{1}{2}}u(t)\|^2 + c_{\epsilon_1}\|D^{\frac{1}{2}}u_t(t)\|^2 \\ &+ c_{\epsilon_1} \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+ \end{aligned} \tag{4.7}$$

and

$$\begin{aligned} J_1'(t) \leq &\|u_t(t)\|^2 - (1 - a_0 g_0 - \epsilon_2)\|A^{\frac{1}{2}}u(t)\|^2 + c_{\epsilon_2}\|D^{\frac{1}{2}}u_t(t)\|^2 \\ &+ c_{\epsilon_2} \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \end{aligned} \tag{4.8}$$

Notice that the estimates (4.7) and (4.8) hold, for any $\mathscr{U}_0 \in \mathscr{H}_0$.

Lemma 4.3. *There exist positive constants N_i, M_i $(i = 0, 1)$ and C_0 such that the functional*

$$F(t) = N_0 E(t) + N_1 I_1(t) + J_1(t), \quad \forall t \in \mathbb{R}_+ \tag{4.9}$$

satisfies

$$M_0 E(t) \leq F(t) \leq M_1 E(t), \quad \forall t \in \mathbb{R}_+ \tag{4.10}$$

and

$$F'(t) \leq -M_0 E(t) + C_0 \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \tag{4.11}$$

Proof. Similarly to (3.53), we have

$$E(t) \geq \frac{c_0}{2}\left(\|u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds\right), \quad \forall t \in \mathbb{R}_+. \tag{4.12}$$

Then, from the definition of I_1 and J_1, there exists a positive constant L (not depending on N_0) such that

$$(N_0 - L)E(t) \leq F(t) \leq (N_0 + L)E(t), \quad \forall t \in \mathbb{R}_+.$$

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

Thus, for any $N_0 > L$, (4.10) holds, for any

$$0 < M_0 \leq N_0 - L \quad \text{and} \quad M_1 \geq N_0 + L. \tag{4.13}$$

On the other hand, by combining (3.19), (4.7) and (4.8), and noting that $g' \leq 0$, we get

$$F'(t) \leq -L_1\|u_t(t)\|^2 - L_2\|A^{\frac{1}{2}}u(t)\|^2 + (N_1 c_{\epsilon_1} + c_{\epsilon_2}) \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds$$
$$- (N_0 - N_1 c_{\epsilon_1} - c_{\epsilon_2})\|D^{\frac{1}{2}}u_t(t)\|^2, \tag{4.14}$$

where

$$\begin{cases} L_1 = \frac{\alpha_0 g_0}{2} N_1 - 1 - \epsilon_1 N_1, \\ L_2 = 1 - a_0 g_0 - \epsilon_1 N_1 - \epsilon_2. \end{cases}$$

We choose $N_1 > \dfrac{2}{\alpha_0 g_0}$, $\epsilon_2 = \epsilon_1$, ϵ_1 small enough such that $L_1, L_2 > 0$ (since $1 - a_0 g_0 > 0$ according to (2.4)) and

$$N_0 > \max\{L, N_1 c_{\epsilon_1} + c_{\epsilon_2}\}$$

(so M_0 in (4.13) exists and the last term of (4.14) is negative). On the other hand, from (3.18), we have

$$E(t) \leq \frac{1}{2}\left(\|u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds\right). \tag{4.15}$$

By combining (4.14) and (4.15), (4.11) holds, for any

$$0 < M_0 \leq 2\min\{L_1, L_2\} \tag{4.16}$$

and

$$C_0 = \max\left\{N_1 c_{\epsilon_1} + c_{\epsilon_2} + \min\{L_1, L_2\}\right\}.$$

So, (4.10) and (4.11) hold, for any $M_0 > 0$ satisfying (4.13) and (4.16). □

By combining (3.61), (3.62) and (4.11) (which are satisfied, for any $\mathscr{U}_0 \in \mathscr{K}_0$), we find, for some positive constants C_4 and C_5 (do not depending on ϵ_0),

$$F'(t) \leq -M_0 E(t) - C_4 E'(t) \quad \forall t \in \mathbb{R}_+$$

if (3.7) holds, and

$$G'(\epsilon_0 E(t))F'(t) \leq -(M_0 - C_5\epsilon_0)E(t)G'(\epsilon_0 E(t)) - C_5 E'(t), \quad \forall t \in \mathbb{R}_+$$

if (3.8) holds and (3.7) does not hold. By choosing $0 < \epsilon_0 < \dfrac{M_0}{C_5}$, we see that, for some positive constants C_6 and C_7,

$$G_0(\epsilon_0 E(t)) \leq -C_6 \frac{G_0(\epsilon_0 E(t))}{E(t)} F'(t) - C_7 E'(t), \quad \forall t \in \mathbb{R}_+, \tag{4.17}$$

where G_0 is defined in (3.12). Let $c_2 > 0$ and

$$\tilde{F} = c_2 \left(C_6 \frac{G_0(\epsilon_0 E)}{E} F + C_7 E \right). \tag{4.18}$$

We have $\tilde{F} \sim E$ (because $\frac{G_0(\epsilon_0 E)}{E}$ is non-increasing and $F \sim E$ according to (4.10)), and, using (4.17),

$$\tilde{F}' \leq -c_2 G_0(\epsilon_0 E). \tag{4.19}$$

Then, for $c_2 > 0$ such that

$$\tilde{F} \leq \epsilon_0 E \quad \text{and} \quad \tilde{F}(0) \leq 1, \tag{4.20}$$

we get (since G_0 is increasing)

$$\tilde{F}' \leq -c_2 G_0(\tilde{F}). \tag{4.21}$$

Then (4.21) implies that

$$(\tilde{G}(\tilde{F}))' \geq c_2, \tag{4.22}$$

where $\tilde{G}(s) = \displaystyle\int_s^1 \frac{1}{G_0(\tau)} d\tau$, for $s \in]0, 1]$. Integrating (4.22) over $[0, t]$ yields

$$\tilde{G}(\tilde{F}(t)) \geq c_2 t + \tilde{G}(\tilde{F}(0)), \quad \forall t \in \mathbb{R}_+. \tag{4.23}$$

Because $\tilde{F}(0) \leq 1$, $\tilde{G}(1) = 0$ and $\tilde{G}$ is decreasing, we obtain from (4.23) that

$$\tilde{G}(\tilde{F}(t)) \geq c_2 t, \quad \forall t \in \mathbb{R}_+,$$

which implies that

$$\tilde{F}(t) \leq \tilde{G}^{-1}(c_2 t), \quad \forall t \in \mathbb{R}_+.$$

The fact that $\tilde{F} \sim E$ gives (4.4). This completes the proof of Theorem 4.1.

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

Acknowledgments

Q4 This work was initiated during the visit of the third author to the State University of Maringá in March 2014. The third author wishes to thank this university for its kind support and hospitality.

References

[1] F. Alabau-Boussouira, P. Cannarsa, V. Komornik, Indirect internal stabilization of weakly coupled evolution equations, J. Evol. Equ. 2 (2002) 127–150.
[2] P. Caldiroli, R. Musina, On a variational degenerate elliptic problem, NoDEA 7 (2000) 187–199.
[3] M. Campiti, G. Metafune, D. Pallara, Degenerate self-adjoint evolution equations on the unit interval, Semigroup Forum 57 (1998) 1–36.
[4] P. Cannarsa, D. Rocchetti, J. Vancostenoble, Generation of analytic semi-groups in L^2 for a class of second order degenerate elliptic operators, Control Cybernet. 37 (2008) 831–878.
[5] P. Cannarsa, P. Martinez, J. Vancostenoble, Carleman estimates for a class of degenerate parabolic operators, SIAM J. Control Optim. 47 (2008) 1–19.
[6] M.M. Cavalcanti, V.N. Domingos, J.A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, Electron. J. Differential Equations 44 (2002) 1–14.
Q3 [7] M.M. Cavalcanti, L.H. Fatori, T.F. Ma, Attractors for the semilinear wave equation with combined frictional-viscoelastic dissipation, submitted for publication.
[8] M.M. Cavalcanti, H.P. Oquendo, Frictional versus viscoelastic damping in a semilinear wave equation, SIAM J. Control Optim. 42 (2003) 1310–1324.
[9] V.V. Chepyzhov, V. Pata, Some remarks on stability of semigroups arising from linear viscoelasticity, Asymptot. Anal. 46 (2006) 251–273.
[10] C.M. Dafermos, Asymptotic stability in viscoelasticity, Arch. Ration. Mech. Anal. 37 (1970) 297–308.
[11] M. Fabrizio, B. Lazzari, On the existence and asymptotic stability of solutions for linear viscoelastic solids, Arch. Ration. Mech. Anal. 116 (1991) 139–152.
[12] M. Fabrizio, C. Giorgi, V. Pata, A new approach to equations with memory, Arch. Ration. Mech. Anal. 198 (2010) 189–232.
[13] C. Giorgi, J.E. Muñoz Rivera, V. Pata, Global attractors for a semilinear hyperbolic equation in viscoelasticity, J. Math. Anal. Appl. 260 (2001) 83–99.
[14] A. Guesmia, Quelques résultats de stabilisation indirecte des systèmes couplés non dissipatifs, Bull. Belg. Math. Soc. 15 (2008) 479–497.
[15] A. Guesmia, Asymptotic stability of abstract dissipative systems with infinite memory, J. Math. Anal. Appl. 382 (2011) 748–760.
[16] A. Guesmia, On the stabilization for Timoshenko system with past history and frictional damping controls, Palest. J. Math. 2 (2013) 187–214.
[17] A. Guesmia, Asymptotic behavior for coupled abstract evolution equations with one infinite memory, Appl. Anal. 94 (2015) 184–217.
[18] A. Guesmia, S. Messaoudi, General energy decay estimates of Timoshenko systems with frictional versus viscoelastic damping, Math. Methods Appl. Sci. 32 (2009) 2102–2122.
[19] A. Guesmia, S.A. Messaoudi, A general decay result for a viscoelastic equation in the presence of past and finite history memories, Nonlinear Anal. 13 (2012) 476–485.
[20] A. Guesmia, S.A. Messaoudi, A new approach to the stability of an abstract hyperbolic system in the presence of infinite history, J. Math. Anal. Appl. 416 (2014) 212–228.
[21] B.V. Kapitonov, Uniform stabilization and exact controllability for a class of coupled hyperbolic systems, Comput. Appl. Math. 15 (1996) 199–212.
[22] V. Komornik, Exact Controllability and Stabilization. The Multiplier Method, Masson–John Wiley, Paris, 1994.
[23] K. Liu, Z. Liu, Exponential decay of energy of vibrating strings with local viscoelasticity, Z. Angew. Math. Phys. 53 (2002) 265–280.
[24] Z. Liu, S. Zheng, On the exponential stability of linear viscoelasticity and thermoviscoelasticity, Quart. Appl. Math. 54 (1996) 21–31.
[25] Z. Liu, S. Zheng, Semigroups Associated with Dissipative Systems, Chapman Hall/CRC, 1999.
[26] J.E. Muñoz Rivera, M.G. Naso, Asymptotic stability of semigroups associated with linear weak dissipative systems with memory, J. Math. Anal. Appl. 326 (2007) 691–707.

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028

[27] J.E. Muñoz Rivera, M.G. Naso, Optimal energy decay rate for a class of weakly dissipative second-order systems with memory, Appl. Math. Lett. 23 (2010) 743–746.
[28] A. Pazy, Semigroups of Linear Operators and Applications to Partial Differential Equations, Springer-Verlag, New York, 1983.
[29] D.L. Russel, A general framework for the study of indirect damping mechanisms in elastic systems, J. Math. Anal. Appl. 173 (1993) 339–358.

Please cite this article in press as: M.M. Cavalcanti et al., Weak stability for coupled wave and/or Petrovsky systems with complementary frictional damping and infinite memory, J. Differential Equations (2015), http://dx.doi.org/10.1016/j.jde.2015.08.028