On very ample vector bundles on curves

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Abstract: We study very ample vector bundles on curves. We first give numerical conditions for the existence of non-special such bundles. Then we prove the inequality

$$h^0(\det E) \geq h^0(E) + rank(E) - 2$$

over curves of genus at least two. We apply this to prove some special cases of a conjecture on scrolls of small codimension.

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0 Introduction

Algebraic vector bundles over projective curves have been intensively studied over the last decades. Special classes such as stable and ample ones were the subject of many contributions by several authors. On the other hand, many classical and modern studies were devoted to understanding embeddings of curves, i.e. to very ample line bundles. Long before the modern concept of vector bundle was introduced, classical algebraic geometers had dealt with embedded projective bundles (over curves) with degree -1 fibers, known as "scrolls". In modern terms this amounts to studying very ample vector bundles of arbitrary rank over curves.

In the first section of the paper we give numerical necessary and sufficient conditions for the existence of non-special very ample vector bundles over a given curve, generalizing a well-known result of Halphen for the rank -1 case (Proposition 1). Next we remark that on hyperelliptic curves very ample vector bundles are always non-special so the existence problem is completely solved in this case. We also give a numerical criterion ensuring the ampleness of a vector bundle, which generalizes a result of Hartshorne's (Proposition 2).

The second section contains the main result of the paper which is the following inequality:

$$h^0(\det E) \geq h^0(E) + rank(E) - 2,$$

where E is a very ample vector bundle over a curve of genus at least two. Moreover, we completely classify the cases where equality holds. We end the paper by proposing a conjecture on scrolls embedded with small codimension, which is supported by facts proved in both sections. More precisely, the conjecture holds if either the base curve is hyperelliptic or $rank(E) \leq 4$. See our paper [7] for a discussion of this conjecture in the context of manifolds embedded with small codimension.

1

Let C be a smooth projective curve of genus g over the complex field. Throughout the paper we denote by E a vector bundle on C, by r its rank and by d its degree.

Definition A vector bundle E over C is called **(very) ample** if the tautological line bundle $\mathcal{O}_{\mathbb{P}(E)}(1)$ is (very) ample on $\mathbb{P}(E)$.

When E is very ample, $\mathbb{P}(E)$ under the corresponding embedding is classically known as a **scroll**.

As in the case of line bundles one easily proves the following:

Lemma 1 *A rank r vector bundle E on C is very ample if and only if for* *any two (possibly equal) points P, Q on C one has*

$$h^0(E(-P-Q)) = h^0(E) - 2r.$$

$\square$

As usual we call a vector bundle E special if $H^1(C, E) \neq 0$.

Already in the rank 1 case the numerical classification of special very ample vector bundles is a very subtle problem, so we first restrict our attention to the non-special case. The following Proposition generalizes a well-known result of Halphen (see e.g.[5]) to arbitrary rank.

Proposition 1

(i) If $g \leq 1$, any very ample vector bundle on C is non-special and such a vector bundle having rank r and degree d exists if and only if $d \geq r$ in case $g = 0$ and $d \geq 2r + 1$ in case $g = 1$.

(ii) If $g \geq 2$, there exists a non-special very ample vector bundle of rank r and degree d on C if and only if $d \geq r(g + 1) + 2$.

Proof: The genus-zero case is classical and easy. For $g = 1$ see e.g. [6], p.151.

Let now $g \geq 2$. Take E very ample and non-special and consider the scroll $\mathbb{P}(E)$ in $\mathbb{P}^N$ where $N := h^0(E) - 1$. Lemma 1 implies $N \geq 2r - 1$. If $N = 2r - 1$ if follows $g = 0$ (e.g. by Barth-Lefschetz; in fact, $\mathbb{P}(E)$ is the Segre embedding of $\mathbb{P}^1 \times \mathbb{P}^{r-1}$ in this case). If $N = 2r$ we may apply the double-point formula

$$c_r(N_{\mathbb{P}(E)/\mathbb{P}^{2r}}) = (deg(\mathbb{P}(E)))^2$$

[9], which in this case takes the form

$$r(r + 1)g = (d - r)(d - (r + 1))$$

(remark that $deg(\mathbb{P}(E)) = d$ and compute the Chern classes of the normal bundle out of its associated exact sequence and the Euler exact sequences for $\mathbb{P}(E)$ and for $\mathbb{P}^N$). Using the Riemann-Roch theorem and the non-speciality of E it follows $g \leq 1$.

So $h^0(E) - 1 = N \geq 2r + 1$. Applying once again Riemann-Roch we get $d \geq r(g + 1) + 2$.

Now we turn to the converse implication. Let d and r be such that $d \geq r(g+1)+2$. We look for a non-special very ample E with these invariants. Using Lemma 1, Riemann-Roch and Serre duality we see that a vector bundle E is very ample and non-special if and only if

$$H^0(C, E^\vee \otimes \omega_C(P+Q)) = 0$$

for any points P, Q of C (where ω_C stands for the canonical line bundle of C). Remark that the degree of $E^\vee \otimes \omega_C(P+Q)$ is

$$d' := -d + 2r(g-1) + 2r \leq r(g-1) - 2.$$

Let $U_{r,d'}$ be the moduli space of stable vector bundles of rank r and degree d' on C. Let $W^0 \subset U_{r,d'}$ be the locus of vector bundles having nonzero global sections. Consider the natural map

$$\varphi : U_{r,d'} \times C^{(2)} \to U_{r,d'-2r},$$

which sends $(F, P+Q)$ to $F(-P-Q)$, $C^{(2)}$ being the second symmetric product of C. It is enough to prove that the restriction of φ to $W^0 \times C^{(2)}$ is not surjective. Since $\dim U_{r,d'-2r} = \dim U_{r,d'} = r^2(g-1)+1$ and $d' \leq r(g-1)-2$, this will be a consequence of the fact that $\dim W^0 \leq r(r-1)(g-1)+d'$. The estimate for $\dim W^0$ is the easy part of Sundaram's work on Brill-Noether loci in arbitrary rank, [13]. We remark that a slight refinement of his argument makes it work for $g = 2$ too. $\square$

Lemma 2 *For a vector bundle E on C one has: E is special if and only if E possesses a rank -1 special locally free quotient.*

Proof: One way is clear from the associated cohomology sequence.

Conversely, assume E special and take a nonzero global section of $E^\vee \otimes \omega_C$. Dualizing we get a quotient of $E \otimes \omega_C^{-1}$ of the form $\mathcal{O}_C(-D)$ for some effective divisor D. So E has the quotient $\omega_C(-D)$ which is special. $\square$

Corollary 1 *Let C be hyperelliptic and $g \geq 2$. Then any very ample vector bundle on C is non-special. In particular, there exists a very ample vector bundle of degree d and rank r on C if and only if $d \geq r(g+1)+2$.*

Proof: Use Lemma 2 and the facts that a quotient of a very ample vector bundle remains very ample and any very ample line bundle on a hyperelliptic curve is non-special. $\square$

Remarks:

1. Since any genus 2 curve is hyperelliptic the existence problem for very ample vector bundles is settled in this case too.
2. For $g = 3$ the only non-hyperelliptic curves are the plane quartics. On a plane quartic C, we may take $E = \omega_C^{\oplus r}$ which is very ample of degree $4r < r(g+1)+2$. This example will play a special role later on.

Now we look for more specific criteria ensuring the (very) ampleness of a given vector bundle.

Definition For a vector bundle E on C we set $d_1(E) := \min\{deg(L) \mid L$ a quotient line bundle of $E\}$.

One sees that $d_1(E)$ is well defined and the following easy properties hold:

(i) $d_1(E) \leq \deg E$ if E is globally generated,

(ii) $d_1(E \otimes L) = d_1(E) + \deg L$ for any line bundle L on C,

(iii) d_1 is related to the invariant s_1 of $\mathbb{P}(E)$ (cf. [10],[5; V, 2.8]) by $s_1(\mathbb{P}(E)) = \deg E + r d_1(E^\vee)$.

Proposition 2
Let E be a rank r vector bundle on C.

(i) If $d_1(E) > \dfrac{r-1}{r} g$ then E is ample.

(ii) If $d_1(E) \geq 2g-1$ then E is non-special.

(iii) If $d_1(E) \geq 2g$ then E is globally generated.

(iv) If $d_1(E) \geq 2g+1$ then E is very ample.

For $g = 1$, (i) is an old result due to Hartshorne, [4].

Proof:

(i) Let L be a line bundle quotient of E of degree $d_1(E)$ and

$$0 \to F \to E \to L \to 0$$

be the corresponding exact sequence.

Using the theorem of Mukai and Sakai [10], one has

$$\frac{\deg L}{rankL} - \frac{\deg F}{rankF} \leq g.$$

In our case this gives

$$\deg E \geq rd_1(E) - (r-1)g > 0.$$

If E is semi-stable, the ampleness follows from Hartshorne's criterion [3]. Otherwise, take E' a maximal destabilizing subbundle of E of rank r' and degree d' and argue by induction on $r = rankE$. Since $d_1(E/E') \geq d_1(E) > (r-1)g/r > (r'-1)g/r'$ and $\frac{d'}{r'} > \frac{d}{r} > 0$, E/E' and E' are ample by the induction hypothesis and Hartshorne's criterion. Being an extension of ample vector bundles, E is ample too, [3].

(ii), (iii) and (iv) are easily derived from suitable versions of Lemma 1 and the following remark:

$d_1(E) \geq a$ if and only if
$H^0(C, E^\vee \otimes L) = 0$ for all $L \in Pic^{a-1}(C)$.

$\square$

Corollary 2 *A semi-stable rank r bundle over C of degree $d \geq 2rg+1$ is very ample.*

2

The main result of this section is the following:

Theorem *If E is a very ample vector bundle of rank r on C and $g \geq 2$ then*

$$h^0(\det E) \geq h^0(E) + r - 2.$$

Moreover, equality holds if and only if C admits an embedding in $\mathbb{P}^2$ by means of a line bundle L and $E = L^{\oplus r}$ for $r \geq 2$ when $g = 3$ and for $r = 2$ or $r = 3$ when $g > 3$.

For a very ample vector bundle E on C let $X := \mathbb{P}(E)$ be the associated scroll embedded in $\mathbb{P}^N$ by means of the complete linear system $|\mathcal{O}_{\mathbb{P}(E)}(1)|$ and $\pi : X \to C$ the projection.

We shall prove the theorem by successively cutting X with hyperplanes containing two fibers of π and comparing the invariants of X to those of the residual scroll in such hyperplane sections.

For any effective divisor D on C we denote by $< \pi^*(D) >$ the linear span of $\pi^*(D)$ in $\mathbb{P}^N$, that is the base locus of the linear subsystem of $|\mathcal{O}_{\mathbb{P}^N}(1)|$ given by sections of $H^0(\mathbb{P}^N, \mathcal{I}_{\pi^*(D)}(1)) \cong H^0(X, \mathcal{O}_X(1)(-\pi^* D)) \cong H^0(C, E(-D))$.

Suppose from now on that $g \geq 1$ and let $A = P_1 + P_2$ be a general effective degree 2 divisor on C. Recall that since $g \geq 1$ we have $N \geq 2r$ (see proof of Proposition 1) and $\dim(< \pi^*(A) >) = 2r - 1$ (Lemma 1). Suppose that the linear span of $\pi^*(A)$ contains some other fibers of π over an effective divisor B on C of degree b. We first remark that b is independent of the choice of A, provided A is general. Consider indeed the subvariety T of $C \times C \times C$ parametrizing effective divisors D of degree 3 such that $\dim(< \pi^*(D) >) \leq 2r - 1$. This is a proper subvariety whose projection on any factor $C \times C$ is generically finite of degree b. Moreover, no fiber of the projection $C \times C \times C \to C \times C$ is contained in T. Otherwise there would exist 2 fibers of π such that any other fiber of π lies in their span.

Lemma 3 *If $b > 0$, then for various choices of general effective divisors A of degree 2 the associated divisors $A + B$ are linearly equivalent on C and $|A + B|$ is very ample.*

Proof: $b > 0$ means that in the above construction $\dim T = 2$. We consider a general fiber $C \times C$ of the projection $C \times C \times C \to C$. (i.e. we fix P_1 in A and vary P_2).

Then $T \cap (C \times C)$ is one-dimensional and has no vertical, and by symmetry also no horizontal components with respect to the projection $C \times C \xrightarrow{p} C$. Let Z be the pure 1-dimensional part of $T \cap (C \times C)$. The symmetry of Z in $C \times C$ allows us to introduce an equivalence relation on C:

$$P \sim P' \text{ if } P = P' \text{ or } (P, P') \in Z$$

(for the transitivity, remark that if (P, Q) and (Q, R) are in Z then $\dim(< \pi^*(P_1 + P + Q + R) >) = 2r - 1$ and thus (P, R) is in Z). There exist $b + 1$ distinct points all equivalent modulo this relation. Otherwise the diagonal of $C \times C$ must lie in Z so the divisor B associated to $A = P_1 + P_2$ always contains P_2. This cannot happen for all P_1, for then B would also contain P_1 by symmetry and thus Z would have horizontal components.

By factorizing through $\sim$ we get a (possibly ramified) covering of degree $b+1$ of Riemann surfaces $C \xrightarrow{f} C/_{\sim} =: C_0$.

If now P_1 moves the corresponding fibers $p^{-1}(P_2) \cap Z$ must also vary, otherwise we get horizontal components when we fix P_2.

This produces a nontrivial family of coverings (not coming from automorphisms of C_0) parametrized by a neighborhood S of P_1 :

$$\begin{tikzcd} C \times S \arrow[rr] \arrow[dr] & & \mathcal{C} \arrow[dl] \\ & S & \end{tikzcd}$$

There exists a maximal family of holomorphic maps of this type parameterized by $\{(\zeta, s) \in U \times S; \alpha(\zeta, s) = 0\}$ (for a smaller S if necessary) where U is an open neighborhood of 0 in $H^0(C, f^*T_{C_0})$, $\alpha : U \times S \to H^1(C, f^*T_{C_0})$ is holomorphic with $d\alpha_{(0,P_1)} = (0, f^* \circ \rho)$,

$$f^* : H^1(C_0, T_{C_0}) \to H^1(C, f^*T_{C_0})$$

and $\rho : T_0S \to H^1(C_0, T_{C_0})$ the Kodaira-Spencer map associated to the family $\mathcal{C} \to S$ (cf. [11], 3.2.3.).

Since f^* is injective it follows $H^0(C, f^*T_{C_0}) \neq 0$ so $g(C_0) \leq 1$. Moreover, for $g(C_0) = 1$, $H^0(C, f^*T_{C_0}) \cong H^0(C_0, T_{C_0}) \cong \mathbb{C}$ and one can see that all the coverings of the maximal family come from automorphisms of C_0 composed by f, and this is also excluded in our case. Thus $g(C_0) = 0$ and the fibers of f are linearly equivalent divisors on C. (Remark that, at least for $g(C_0) \geq 2$, the above deformation theory argument may be replaced by a well known finiteness result due to Severi; see e.g. [12]).

So, moving any point of A in an open set of C we get by adding the corresponding B linearly equivalent divisors $A+B$. We may give up the genericity assumption on A by looking at the pure 2-dimensional part T' of T and taking the fibers of the projection $T' \to C \times C$. It is then easy to see that the divisors $A+B$ remain linearly equivalent. Since T' has no "horizontal fibers" $|A+B|$ has no base points, and for two given fibers F_1, F_2 of π there always exist a third one F such that $F_1 \not\subset < F_2 + F >$ by Lemma 1. This proves that $|A+B|$ is very ample. $\square$

Lemma 4 *If $r \geq 2$ then for a general divisor A of degree 2 on C the general hyperplane of $\mathbb{P}^N$ containing $\pi^*(A)$ does not contain other fibers of π excepting those which are already in $< \pi^*(A) >$.*

Proof: We consider the linear system $\mathcal{H}$ of hyperplanes of $\mathbb{P}^N$ containing $\pi^*(A)$ and the incidence variety $I \subset \mathcal{H} \times C$ given by pairs $(H, P) \in \mathcal{H} \times C$ such that $H \supset < \pi^*(A + B + P) >$.

Suppose that I projects onto $\mathcal{H}$. Then $\dim I = \dim \mathcal{H} = N - 2r$. Consider the part I' of I consisting of components covering $\mathcal{H}$. If $< \pi^*(A) >$ cuts the general fiber of π along an $(r-1-t)$-dimensional subspace, then $\dim I' = \dim \mathcal{H} - t + 1$, hence $t = 1$. But then $< \pi^*(A) >$ cuts X along an $(r-1)$-dimensional scroll X' plus the fibers of π over $A + B$, and an element of $\mathcal{H}$ contains extra fibers over some divisor, B' say, of C. Hence an exact sequence

$$0 \to \mathcal{O}_X(\pi^*(A + B + B')) \to \mathcal{O}_X(1) \to \mathcal{O}_{X'}(1) \to 0$$

and by applying π_* also

$$0 \to \mathcal{O}_C(A + B + B') \to E \to E' \to 0$$

where $E' := \pi_* \mathcal{O}_{X'}(1)$.

So $\mathcal{O}_C(A+B+B') = \det E \otimes (\det E')^{-1}$ does not depend on the choice of the hyperplane. Thus B' moves in a linear system (whose degree is the degree of the projection $I' \to \mathcal{H}$ and) whose dimension is $\dim \mathcal{H}$. Moreover, one sees as in Lemma 3 that the divisors $A + B + B'$ are linearly equivalent to each other and that $|A + B + B'|$ is very ample.

In particular $\dim |A + B + B'| \geq \dim |B + B'| + 2 \geq N - 2r + 2$ and the restriction morphism

$$H^0(\mathbb{P}^N, \mathcal{O}_{\mathbb{P}^N}(1)) \cong H^0(X, \mathcal{O}_X(1)) \to H^0(X', \mathcal{O}_{X'}(1))$$

has a $(2r-2)$-dimensional image at most, showing that X' embeds in $\mathbb{P}^{2r-3}$, a contradiction since $g \geq 1$ (see proof of Proposition 1). $\square$

Corollary 3 *Keeping the previous notations, if* $b > 0$ *then* $\dim |A + B| = 2$.

Proof: By Lemma 4, $\mathcal{O}_C(A + B)$ is a subbundle of E. Hence any global section s of $\mathcal{O}_C(A + B)$ extends to a section of E vanishing on the same divisor D as s. There exists then a hyperplane in $\mathbb{P}^N$ containing $\pi^* D$ and no other fibers of π. Thus D is uniquely determined by two of its points. $\square$

Lemma 5 *Let* $g \geq 2$ *and* $|D_1|$, $|D_2|$ *two very ample linear systems on* C *with* $\dim |D_1| = 2$, $\dim |D_2| > 2$. *Then*

$$\dim |D_1 + D_2| \geq 4 + \dim |D_2|.$$

This is a special case of the Clifford type results contained in ([1], III B) which is sufficient for our purposes. For the reader's convenience we sketch a proof.

Proof: Let $\delta = \dim|D_2| + 2$. Then $\delta \leq \deg D_2$, since $g \geq 2$.

Let $\Gamma \in |D_2|$ be general with respect to $|D_1|$ and $\Gamma' \subset \Gamma$ consist of δ distinct points $P_1, \ldots, P_\delta$. By the Uniform Position Theorem [1], there exists divisors D_1 in the first linear system through P_2, P_3 and D_2 in the second system through $P_4, \ldots, P_\delta$, neither of which containing P_1. It follows that Γ' imposes independent conditions on $|D_1 + D_2|$. Thus

$$\dim|D_1 + D_2| = \delta + \dim|D_1 + D_2 - \Gamma'| \geq \delta + \dim|D_1| = 4 + \dim|D_2|.$$

□

Proof of the theorem Let $g \geq 2$ and consider the fibers of $\pi : X \to C$ over a general effective degree 2 divisor A, B the associated divisor corresponding to supplementary fibers in $< \pi^*(A) >$ and $D = A + B$. By Lemma 4, there exists a hyperplane section of X containing no other fibers of π excepting those over D. Let X' be the residual $(r-1)$-dimensional scroll in this hyperplane section and $E' = \pi_*\mathcal{O}_{X'}(1)$. We have the same exact sequences as before

$$0 \to \mathcal{O}_X(\pi^*(D)) \to \mathcal{O}_X(1) \to \mathcal{O}_{X'}(1) \to 0,$$
$$0 \to \mathcal{O}_C(D) \to E \to E' \to 0.$$

Suppose first that $\deg D = 2$ (i.e. $b = 0$).

Then

$$h^0(E) \leq h^0(E') + h^0(\mathcal{O}_C(D)) = h^0(E') + 1.$$

Since $|\det(E)|$ is very ample we also have

$$h^0(\det E') = h^0(\det(E)(-D)) = h^0(\det E) - 2$$

so $h^0(\det E) - h^0(E) \geq h^0(\det E') - h^0(E') + 1$.

Let now $\deg D > 2$ (i.e. $b > 0$). By the Corollary above,

$$h^0(E) \leq h^0(E') + 3.$$

When $h^0(\det E') > 3$ then Lemma 5 gives

$$h^0(\det E) \geq 4 + h^0(\det E').$$

So in this case

$$h^0(\det E) - h^0(E) \geq h^0(\det E') - h^0(E') + 1$$

too.

In the remaining case $h^0(\det E') = 3$ so $r = 2$, $\det E' = \mathcal{O}_C(D)$ and $h^0(\det E) - h^0(E) = h^0(\det E') - h^0(E')$. (Notice that $h^0(\det E) = 6$ and that $h^0(E) < 6$ would contradict the double-point formula; see proof of Proposition 1).

Applying further this cutting procedure to the residual scrolls we see that the difference $h^0(\det E) - h^0(E)$ decreases by at least one at each step excepting possibly at the last one. Summing up it follows that $h^0(\det E) \geq h^0(E) + r - 2$.

Now we show by induction on r that $h^0(\det E) = h^0(E) + (r-2)$ implies $E = L^{\oplus r}$, where L is a very ample line bundle on C with $h^0(L) = 3$.

For $r = 2$, we must have

$$0 \to L \to E \to L \to 0$$

exact and $h^0(E) = 2h^0(L) = 6$.

Two fibers of π span now a 2-codimensional subspace of $\mathbb{P}^5$ so $h^0(E \otimes L^{-1}) = 2$ and the extension

$$0 \to \mathcal{O}_C \to E \otimes L^{-1} \to \mathcal{O}_C \to 0$$

splits since the surjection $E \otimes L^{-1} \to \mathcal{O}_C$ admits a section. (This splitting can also be deduced directly from $h^0(E) = 2h^0(L)$ with no very ampleness assumption on E). Suppose now that our claim is true for $r - 1 \geq 2$, and let E be of rank r such that

$$h^0(\det E) = h^0(E) + r - 2.$$

Cutting X as usual we find

$$0 \to \mathcal{O}_C(D) \to E \to E' \to 0,$$

$$\begin{aligned}
&h^0(\det E) - h^0(E) = h^0(\det E') - h^0(E') + 1 \\
&h^0(E) = h^0(E') + h^0(\mathcal{O}_C(D)), \\
&h^0(\det E') = h^0(E') + r - 3, \\
&E' = L^{\oplus(r-1)}
\end{aligned}$$

and $\mathcal{O}_C(D) = (\det E) \otimes L^{-(r-1)}$, implying that $\mathcal{O}_C(D)$ doesn't depend on the choice of A. Thus $\mathcal{O}_C(D) = L$ (Lemma 3) and one proves that the exact sequence

$$0 \to L \to E \to L^{\oplus(r-1)} \to 0.$$

splits in the same way as before.

Finally, it follows by direct computation that $h^0(L^{\oplus r}) = h^0(L^r) - r + 2$ exactly in the cases appearing in the theorem's statement. $\square$

As a first consequence we get a Clifford type result for very ample vector bundles.

Corollary 4 *Let E be a very ample vector bundle on C.*

i) If $\det E$ *is special then*

$$h^0(E) \leq \frac{d}{2} + 3 - r.$$

ii) If $\det E$ *is non-special then*

$$h^0(E) \leq d - g + 3 - r.$$

$\square$

Example: If $g = 3$ then for any very ample vector bundle E one has

$$h^1(E) \leq r$$

and equality holds if and only if C is a plane curve and $E = \omega_C^{\oplus r}$.

We have seen that r-dimensional scrolls over C embedded in $\mathbb{P}^N$ exist only when $N \geq 2r - 1$, equality occuring only for the Segre scroll $\mathbb{P}^1 \times \mathbb{P}^{r-1}$. Assume now $N = 2r$ and recall that the double-point formula reads:

$$r(r+1)g = (d-r)(d-(r+1)).$$

We have also seen that for $g = 0$ and $g = 1$ such scrolls do exist for all $r \geq 1$. We propose the following:

Conjecture If X is an r-dimensional scroll over C embedded in $\mathbb{P}^{2r}$ $(r \geq 2)$, then $g(C) \leq 1$.

Supporting evidence for the conjecture is given by the hyperelliptic case (see. Corollary 1 and Proof of Proposition 1) the case $r = 2$ ([8], [2]) and the case $r = 3$ ([14]).

As a final application of our theorem we prove the Conjecture for $r \leq 4$ by an uniform method.

Corollary 5 *For $2 \leq r \leq 4$ an r-dimensional scroll, over C, embedded in $\mathbb{P}^{2r}$ satisfies*

$$g(C) \leq 1.$$

Proof: Let $X = \mathbb{P}(E) \to C$ be our scroll and assume $g \geq 2$. We want to apply Castelnuovo's inequality (cf. [1]) to C embedded by means of $|\det E|$. The double-point formula excludes the equality case in the theorem so we have

$$h^0(\det E) > h^0(E) + r - 2 \geq 3r - 1.$$

Letting $M = [(d-1)/(3r-2)]$ and $\varepsilon = d - 1 - (3r-2)M$ we get the following expression of Castelnuovo's inequality

$$g \leq \frac{M(M-1)}{2}(3r-2) + M \cdot \varepsilon$$

which combined with the double-point formula gives our assertion.

$\square$

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References

[1] E Arbarello, M Cornalba, P A Griffiths, J Harris, *Geometry of Algebraic Curves*, Springer, 1985.

[2] A Aure, On Surfaces in Projective 4-Space, Thesis, Oslo, 1987.

[3] R Hartshorne, *Ample vector bundles*, Publ. Math. IHES **29** (1966), 63-94.

[4] R Hartshorne, *Ample vector bundles on curves*, Nagoya Math. J. **43** (1971), 73-89.

[5] R Hartshorne, *Algebraic Geometry*, Springer, 1977.

[6] P Ionescu, *Embedded projective varieties of small invariants* III, in Algebraic Geometry, L'Aquila 1988, Springer LNM 1417, 138-154.

[7] P Ionescu, M Toma, Projective manifolds embedded with small codimension, in preparation.

[8] A Lanteri, *On the Existence of Scrolls in* $\mathbb{P}^4$, Lincei-Rend. Sc. fis. mat. e nat. **59** (1980), 223-227.

[9] A T Lascu, D Mumford, D B Scott, *The self-intersection formula and the "formule-clef"*, Math. Proc. Camb. Phil. Soc. **78** (1975), 117-123.

[10] S Mukai, F Sakai, *Maximal subbundles of vector bundles on a curve*, Manuscr. Math. **52** (1985), 251-256.

[11] M Namba, *Families of Meromorphic Functions on Compact Riemann Surfaces*, Springer LNM 767, 1979.

[12] P Samuel, *Lectures on Old and New Results on Algebraic Curves*, Tata Institute of Fundamental Research, Bombay, 1966.

[13] N Sundaram, *Special divisors and vector bundles*, Tohoku. Math. J. **39** (1987), 175-213.

[14] M Toma, *Three-dimensional scrolls in* $\mathbb{P}^6$, Arch. Math. (Basel) **65** (1995), 444-448.