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Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay

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In this paper, we consider a second-order abstract linear equation with infinite memory and time delay terms. Under appropriate assumptions on the convolution kernel and on the weight of the delay, we prove the well-posedness and the exponential stability of the system. Our stability estimate proves that the unique dissipation given by the memory term is strong enough to stabilize exponentially the system in presence of delay. Some applications are also given.

Keywords: well-posedness; asymptotic behaviour; memory; delay; semigroups.

1. Introduction

Let H be a real Hilbert space with inner product and related norm denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let $A : D(A) \to H$ be a self-adjoint linear positive definite operator with domain $D(A) \subset H$ such that the embedding is dense and compact. Let $\tau \in]0,+\infty[$, $\mu \in \mathbb{R}^*$ and $g : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function. We consider the following class of second-order linear integro-differential equations:

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Au(t-s)\,ds + \mu u_t(t-\tau) = 0, \quad \forall\, t > 0 \tag{1.1}$$

with initial conditions

$$\begin{cases} u(-t) = u_0(t) & \forall\, t \in \mathbb{R}_+, \\ u_t(0) = u_1, \quad u_t(t-\tau) = f_0(t-\tau) & \forall\, t \in]0,\tau[, \end{cases} \tag{1.2}$$

where (u_0, u_1, f_0) are given initial data belonging to a suitable space (see Section 2) and $u : \mathbb{R}_+ \to H$ is the state (unknown) of the system (1.1 and 1.2). The infinite integral and the constant τ represent, respectively, the memory term and time delay. For a generic function f, the notation f_y means the derivative of f with respect to y. When f has only one variable, the derivative of f is noted by f'.

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Equation (1.1) can describe the dynamics of linear viscoelastic solids, a generalized Kirchhoff viscoelastic beam with memory and systems governing the longitudinal motion of a viscoelastic configuration obeying a nonlinear Boltzmann's model; see, for example, Fabrizio *et al.* (2010), Giorgi *et al.* (2001), Muñoz Revira & Grazia Naso (2010) and Pata (2006) for more details concerning the physical phenomena which are modelled by differential equations with memory.

The subject of our paper is the well-posedness and asymptotic behaviour as time goes to infinity of solutions of (1.1 and 1.2) under appropriate assumptions on the operator A, the convolution kernel g and the constant μ.

The questions related to well-posedness and stability/instability of evolution equations with delay or memory have attracted considerable attention in recent years and many authors have shown that delays can destabilize a system that is asymptotically stable in the absence of delays and presence of memory. Before we state and prove our main results, let us first recall some works related to the problem we address.

In the absence of the delay term in (1.1) (i.e. $\mu = 0$):

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Au(t-s)\,\mathrm{d}s = 0 \quad \forall\, t > 0, \tag{1.3}$$

a large amount of literature is available on this model, addressing problems of the existence, uniqueness and asymptotic behaviour in time (see Dafermos, 1970; Fabrizio & Lazzari, 1991; Liu & Zheng, 1996; Giorgi *et al.*, 2001; Chepyzhov & Pata, 2006; Muñoz Revira & Grazia Naso, 2007; Pata, 2010; Guesmia, 2011 and the references cited therein). The nonlinear one-dimensional viscoelastic wave equation has been investigated by Dafermos (1970). He showed that the energy of the problem tends to zero asymptotically under the Dirichlet boundary conditions, but no decay rate was given in Dafermos (1970). Under the condition

$$\exists \delta > 0: \quad g'(s) \leqslant -\delta g(s) \quad \forall\, s \in \mathbb{R}_+, \tag{1.4}$$

the exponential decay of solutions of (1.3) was obtained in Fabrizio & Lazzari (1991), Giorgi *et al.* (2001), Liu & Zheng (1996) and Muñoz Revira & Grazia Naso (2007) (in different contexts and using different approaches). In Chepyzhov & Pata (2006), it was proved that the weaker condition

$$\exists \delta_1 \geqslant 1, \exists \delta_2 > 0: \quad g(t+s) \leqslant \delta_1\, e^{-\delta_2 t} g(s) \quad \forall\, t \in \mathbb{R}_+ \text{ for a.e. } s \in \mathbb{R}_+ \tag{1.5}$$

is necessary for (1.3) to be exponentially stable. In the particular case of the wave equation, it was proved in Pata (2010) that the exponential stability holds if and only if g satisfies (1.5) and the set $\{s \in \mathbb{R}_+ : g'(s) < 0\}$ has positive Lebesgue measure. When g has a general growth at infinity, a general decay estimate of the solutions of (1.3) was established in Guesmia (2011).

When $\int_0^{+\infty}$ is replaced by $\int_0^t$ in (1.3):

$$u_{tt}(t) + Au(t) - \int_0^{t} g(s)Au(t-s)\,\mathrm{d}s = 0 \quad \forall\, t > 0, \tag{1.6}$$

the stability of (1.6) has received considerable attention and there is now a large literature on this subject, where different decay estimates were obtained depending on the growth of g at infinity, see in this regard

Berrimi & Messaoudi (2006), Cavalcanti *et al.* (2002), Cavalcanti & Oquendo (2003), Guesmia *et al.* (2011), Messaoudi (2008a,b), Messaoudi & Tatar (2003, 2007) and the references cited therein. For the particular case of the wave equation with (internal or boundary) finite memory, see Aassila *et al.* (2000, 2002), Cavalcanti *et al.* (2008), Said-Houari & Falcão Nascimento (2013) and Vicente (2009). See also Guesmia & Messaoudi (2012) for the wave equation with complementary finite and infinite memories.

When the memory term is replaced by Bu_t in (1.1):

$$u_{tt}(t) + Au(t) + Bu_t(t) + \mu u_t(t-\tau) = 0 \quad \forall\, t > 0, \tag{1.7}$$

where B is a given operator, there exist in the literature different stability/instability results of (1.7) depending, in particular, on the connection between B and μ; see Datko *et al.* (1986), Datko (1991) and Nicaise *et al.* (2009) for the one-dimensional wave equation with internal and/or boundary feedback and constant delay, Ammari *et al.* (2010), Nicaise & Pignotti (2006, 2008, 2011) and Nicaise *et al.* (2011) for the N-dimensional case, and Fridman *et al.* (2010) and Nicaise & Valein (2010) for the general system (1.7) with constant or variable delay. These results show that the damping Bu_t is strong enough to stabilize (1.7) provided that $|\mu|$ is small enough with respect to B (in some sense).

Recently, the wave equation in N-dimensional bounded domain with constant delay, finite memory and linear frictional damping was considered in Kirane & Said-Houari (2011); that is,

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^t g(s)\Delta u(x,t-s)\,\mathrm{d}s + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0 \quad \forall\, t > 0, \tag{1.8}$$

and the exponential stability of (1.8) was proved under the assumption $0 \leqslant \mu_2 \leqslant \mu_1$. The same result was obtained in Said-Houari (2011) in the case of Timoshenko systems. As is indicated in Kirane & Said-Houari (2011), the case $\mu_1 = 0$ is an open problem. We recall that (1.8) is instable if $0 \leqslant \mu_1 \leqslant \mu_2$ and $g = 0$ (see Nicaise & Pignotti, 2006).

As a consequence of the results of the papers cited above, a small delay time is a source of instability. Consequently, to stabilize a hyperbolic system involving input delay terms, control terms (such as memory or frictional damping) will be necessary. According to this observation, two main questions naturally arise:

- Is it possible for the memory term, which plays solely the role of dissipation for (1.1 and 1.2), to build the robustness of controllers against delay and stabilize (1.1 and 1.2) exponentially? As far as we know, this situation has never been considered before in the literature.
- Is it possible to get an exponential decay rate of solutions explicitly in terms of, in particular, the connection between the delay and the memory terms?

One of the main goal of this paper is to give satisfactory answers to the above two questions. In addition, the method presented in the proof is considerably simple and allows one to consider various practical applications.

The plan of the paper is as follows. In Section 2, we give appropriate assumptions on A and g, and state and prove the well-posedness of (1.1 and 1.2). While Section 3 is devoted to the proof of the exponential stability of (1.1 and 1.2) under an additional smallness condition on $|\mu|$. In Section 4, we give some applications of (1.1 and 1.2). Finally, in Section 5, we discuss some general issues and indicate some open questions.

2. Well-posedness

In this section, we state some assumptions on A and g, and prove the global existence, uniqueness and smoothness of the solution of (1.1 and 1.2). We assume that

(A1) There exists a positive constant a satisfying

$$a\|w\|^2 \leqslant \|A^{1/2}w\|^2 \quad \forall\, w \in D(A^{1/2}). \tag{2.1}$$

(A2) g is of class $C^1(\mathbb{R}_+)$ and satisfies, for a positive constant δ,

$$g'(s) \leqslant -\delta g(s) \quad \forall\, s \in \mathbb{R}_+. \tag{2.2}$$

(A3) The function g is integrable on $\mathbb{R}_+$ and is such that

$$g_0 := \int_0^{+\infty} g(s)\, \mathrm{d}s \in]0,1[. \tag{2.3}$$

Following a method devised in the pioneering paper of Dafermos (1970) (see also Pata, 2006; Muñoz Revira & Grazia Naso, 2007, 2010; Fabrizio *et al.*, 2010) and the idea of Nicaise & Pignotti (2006) (see also Nicaise & Pignotti, 2008, 2011; Nicaise *et al.*, 2009, 2011; Nicaise & Valein, 2010) to treat the memory and delay terms by considering two new auxiliary variables η and z, we will formulate the system (1.1–1.2) in the following abstract linear first-order system:

$$\begin{cases} \mathscr{U}_t(t) = (\mathscr{A} + \mathscr{B})\mathscr{U}(t) \quad \forall\, t > 0, \\ \mathscr{U}(0) = \mathscr{U}_0, \end{cases} \tag{2.4}$$

where $\mathscr{U} = (u, u_t, \eta, z)^\top$, $\mathscr{U}_0 = (u_0(0), u_1, \eta_0, z_0)^\top \in \mathscr{H}$,

$$\mathscr{H} = D(A^{1/2}) \times H \times L^2_g(\mathbb{R}_+, D(A^{1/2})) \times L^2(]0,1[, H)$$

and

$$\begin{cases} \eta(t,s) = u(t) - u(t-s) & \forall\, t, s \in \mathbb{R}_+, \\ \eta_0(s) = \eta(0,s) = u_0(0) - u_0(s) & \forall\, s \in \mathbb{R}_+, \\ z(t,p) = u_t(t - \tau p) & \forall\, t \in \mathbb{R}_+, \forall\, p \in]0,1[, \\ z_0(p) = z(0,p) = f_0(-\tau p) & \forall\, p \in]0,1[. \end{cases} \tag{2.5}$$

The sets $L^2_g(\mathbb{R}_+, D(A^{1/2}))$ and $L^2(]0,1[, H)$ are the weighted spaces with respect to the measures $g(s)\,\mathrm{d}s$ and $\mathrm{d}p$, respectively, defined by

$$L^2_g(\mathbb{R}_+, D(A^{1/2})) = \left\{ w : \mathbb{R}_+ \to D(A^{1/2}),\ \int_0^{+\infty} g(s) \|A^{1/2}w(s)\|^2\, \mathrm{d}s < +\infty \right\}$$

and

$$L^2(]0,1[, H) = \left\{ w :]0,1[\to H,\ \int_0^1 \|w(p)\|^2\, \mathrm{d}p < +\infty \right\},$$

endowed with the inner products

$$\langle w_1, w_2\rangle_{L^2_g(\mathbb{R}_+,D(A^{1/2}))} = \int_0^{+\infty} g(s)\langle A^{1/2}w_1(s), A^{1/2}w_2(s)\rangle\, \mathrm{d}s$$

and

$$\langle w_1, w_2\rangle_{L^2(]0,1[,H)} = \int_0^1 \langle w_1(p), w_2(p)\rangle\, \mathrm{d}p.$$

The operators $\mathscr{A}$ and $\mathscr{B}$ are linear and given by

$$\mathscr{A}\begin{pmatrix} w_1\\ w_2\\ w_3\\ w_4\end{pmatrix} = \begin{pmatrix} w_2\\ -(1-g_0)Aw_1 - \displaystyle\int_0^{+\infty} g(s)Aw_3(s)\,\mathrm{d}s - |\mu|w_2 - \mu w_4(1)\\ -\dfrac{\partial w_3}{\partial s} + w_2\\ -\dfrac{1}{\tau}\dfrac{\partial w_4}{\partial p}\end{pmatrix} \tag{2.6}$$

and

$$\mathscr{B}(w_1,w_2,w_3,w_4)^\top = |\mu|(0,w_2,0,0)^\top. \tag{2.7}$$

The domains $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}(\mathscr{B})$ of $\mathscr{A}$ and $\mathscr{B}$, respectively, are given by

$$\mathscr{D}(\mathscr{A}) = \left\{\begin{array}{c}(w_1,w_2,w_3,w_4)^\top \in \mathscr{H}, \dfrac{\partial w_4}{\partial p} \in L^2(]0,1[,H), \dfrac{\partial w_3}{\partial s} \in L^2_g(\mathbb{R}_+, D(A^{1/2})),\\ w_2 \in D(A^{1/2}), (1-g_0)w_1 + \displaystyle\int_0^{+\infty} g(s)w_3(s)\,\mathrm{d}s \in D(A), w_3(0)=0, w_4(0)=w_2\end{array}\right\} \tag{2.8}$$

and $\mathscr{D}(\mathscr{B}) = \mathscr{H}$. Bearing in mind the definition (2.5) of η and z, we have

$$\begin{cases}\eta_t(t,s) + \eta_s(t,s) = u_t(t) & \forall\, t,s \in \mathbb{R}_+,\\ \eta(t,0) = 0 & \forall\, t\in\mathbb{R}_+\end{cases} \tag{2.9}$$

and

$$\begin{cases}\tau z_t(t,p) + z_p(t,p) = 0 & \forall\, t\in\mathbb{R}_+,\ \forall\, p \in]0,1[,\\ z(t,0) = u_t(t) & \forall\, t\in\mathbb{R}_+.\end{cases} \tag{2.10}$$

Therefore, we conclude from (2.9 and 2.10) that the systems (1.1 and 1.2) and (2.4) are equivalent.

Clearly, thanks to (2.3), $\mathscr{H}$ endowed with the inner product

$$\begin{aligned}\langle (w_1,w_2,w_3,w_4)^\top, (\tilde{w}_1,\tilde{w}_2,\tilde{w}_3,\tilde{w}_4)^\top\rangle_{\mathscr{H}} &= (1-g_0)\langle A^{1/2}w_1, A^{1/2}\tilde{w}_1\rangle + \langle w_2,\tilde{w}_2\rangle\\ &\quad + \langle w_3,\tilde{w}_3\rangle_{L^2_g(\mathbb{R}_+,D(A^{1/2}))} + \tau|\mu|\langle w_4,\tilde{w}_4\rangle_{L^2(]0,1[,H)}\end{aligned}$$

is a Hilbert space and $\mathscr{D}(\mathscr{A}) \subset \mathscr{H}$ with dense embedding. The well-posedness of problem (2.4) is ensured by the following theorem.

THEOREM 2.1 Assume that (A1)–(A3) hold. Then, for any $\mathscr{U}_0 \in \mathscr{H}$, the system (2.4) has a unique weak solution

$$\mathscr{U} \in C(\mathbb{R}_+, \mathscr{H}).$$

Moreover, if $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, then the solution of (2.4) satisfies (classical solution)

$$\mathscr{U} \in C^1(\mathbb{R}_+, \mathscr{H}) \cap C(\mathbb{R}_+, \mathscr{D}(\mathscr{A})).$$

Proof. To prove Theorem 2.1, we use the semigroup approach. So, first, we show that the linear operator $\mathscr{A}$ is dissipative. Indeed, let $W = (w_1, w_2, w_3, w_4)^\top \in \mathscr{D}(\mathscr{A})$, then

$$\begin{aligned}\langle \mathscr{A}W, W\rangle_{\mathscr{H}} = &+ \left\langle -(1-g_0)Aw_1 - \int_0^{+\infty} g(s)Aw_3(s)\,\mathrm{d}s - |\mu|w_2 - \mu w_4(1), w_2 \right\rangle \\ &+ (1-g_0)\langle A^{1/2}w_2, A^{1/2}w_1\rangle + \left\langle -\frac{\partial w_3}{\partial s} + w_2, w_3 \right\rangle_{L^2_g(\mathbb{R}_+, D(A^{1/2}))} \\ &+ \tau|\mu| \left\langle -\frac{1}{\tau}\frac{\partial w_4}{\partial p}, w_4 \right\rangle_{L^2(]0,1[,H)}. \end{aligned} \tag{2.11}$$

It is clear that, thanks to the definition of $A^{1/2}$ and the fact that H is a real Hilbert space,

$$\langle -(1-g_0)Aw_1, w_2\rangle = -(1-g_0)\langle A^{1/2}w_2, A^{1/2}w_1\rangle,$$
$$\left\langle -\int_0^{+\infty} g(s)Aw_3(s)\,\mathrm{d}s, w_2 \right\rangle = -\langle w_2, w_3\rangle_{L^2_g(\mathbb{R}_+, D(A^{1/2}))}$$

and

$$\langle -|\mu|w_2, w_2\rangle = -|\mu|\|w_2\|^2.$$

On the other hand, the Cauchy–Schwarz and Young's inequalities imply that

$$\langle -\mu w_4(1), w_2\rangle \leqslant \frac{|\mu|}{2}(\|w_4(1)\|^2 + \|w_2\|^2).$$

Integrating by parts and using the fact that $w_3(0) = 0$ (definition of $\mathscr{D}(\mathscr{A})$) give

$$\left\langle -\frac{\partial w_3}{\partial s}, w_3 \right\rangle_{L^2_g(\mathbb{R}_+, D(A^{1/2}))} \leqslant \frac{1}{2}\int_0^{+\infty} g'(s)\|A^{1/2}w_3(s)\|^2\,\mathrm{d}s.$$

Also recalling that $w_4(0) = w_2$ (definition of $\mathscr{D}(\mathscr{A})$), we may write

$$\tau|\mu| \left\langle -\frac{1}{\tau}\frac{\partial w_4}{\partial p}, w_4 \right\rangle_{L^2(]0,1[,H)} = \frac{|\mu|}{2}(-\|w_4(1)\|^2 + \|w_4(0)\|^2) = \frac{|\mu|}{2}(-\|w_4(1)\|^2 + \|w_2\|^2).$$

Consequently, inserting these six formulas in the previous identity (2.11) and using the fact that g is non-increasing (according to (2.2)), we have

$$\langle \mathscr{A} W, W\rangle_{\mathscr{H}} \leqslant \frac{1}{2}\int_0^{+\infty} g'(s)\|A^{1/2}w_3(s)\|^2\,\mathrm{d}s \leqslant 0, \tag{2.12}$$

which means that $\mathscr{A}$ is dissipative.

Next, we shall prove that $Id - \mathscr{A}$ is surjective. Indeed, let $F = (f_1, f_2, f_3, f_4)^\top \in \mathscr{H}$, we show that there exists $W = (w_1, w_2, w_3, w_4)^\top \in \mathscr{D}(\mathscr{A})$ satisfying

$$(\mathrm{Id} - \mathscr{A})W = F, \tag{2.13}$$

which is equivalent to

$$\begin{cases} w_2 = w_1 - f_1, \\ w_3 + \dfrac{\partial w_3}{\partial s} = f_3 + w_1 - f_1, \\ w_4 + \dfrac{1}{\tau}\dfrac{\partial w_4}{\partial p} = f_4, \\ ((1-g_0)A + (1+|\mu|)\,\mathrm{Id})w_1 + \displaystyle\int_0^{+\infty} g(s)Aw_3(s)\,\mathrm{d}s = (1+|\mu|)f_1 + f_2 - \mu w_4(1). \end{cases} \tag{2.14}$$

We note that the second equation in (2.14) with $w_3(0) = 0$ has a unique solution

$$w_3 = \left(\int_0^s e^y (f_3(y) + w_1 - f_1)\,\mathrm{d}y\right) e^{-s}. \tag{2.15}$$

On the other hand, the third equation in (2.14) with $w_4(0) = w_2 = w_1 - f_1$ has a unique solution

$$w_4 = \left(w_1 - f_1 + \tau \int_0^p f_4(y)\, e^{\tau y} \mathrm{d}y\right) e^{-\tau p}. \tag{2.16}$$

Next, plugging (2.15) and (2.16) into the fourth equation in (2.14), we get

$$(lA + (|\mu| + e^{-\tau}\mu + 1)\,\mathrm{Id})w_1 = \tilde{f}, \tag{2.17}$$

where

$$l = 1 - g_0 + \int_0^{+\infty} g(s)\, e^{-s}\left(\int_0^s e^y\,\mathrm{d}y\right)\mathrm{d}s = 1 - \int_0^{+\infty} g(s)\, e^{-s}\,\mathrm{d}s$$

((2.3) implies that $l > 0$) and

$$\tilde{f} = f_2 + (|\mu| + e^{-\tau}\mu + 1)f_1 - \int_0^{+\infty} g(s)\, e^{-s}\left(\int_0^s e^y A(f_3(y) - f_1)\,\mathrm{d}y\right)\mathrm{d}s - \tau\mu\, e^{-\tau}\int_0^1 f_4(y)\, e^{\tau y}\,\mathrm{d}y.$$

We have just to prove that (2.17) has a solution $w_1 \in D(A^{1/2})$ and replace in (2.15), (2.16) and the first equation in (2.14) to obtain $W \in \mathscr{D}(\mathscr{A})$ satisfying (2.13). Since, applying the Lax–Milgram theorem and classical regularity arguments, we conclude that (2.17) has a unique solution $w_1 \in D(A^{1/2})$ satisfying, using (2.15),

$$(1-g_0)w_1 + \int_0^{+\infty} g(s)w_3(s)\,\mathrm{d}s \in D(A).$$

This proves that $\mathrm{Id} - \mathscr{A}$ is surjective. Finally, (2.12) and (2.13) mean that $-\mathscr{A}$ is maximal monotone operator. Then, using Lummer–Phillips theorem (see Pazy, 1983), we deduce that A is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$.

On the other hand, it is clear that the linear operator $\mathscr{B}$ is Lipschitz continuous. Finally, also $\mathscr{A} + \mathscr{B}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$ (see Pazy, 1983, Chapter 3, Theorem 1.1). Consequently, (2.4) is well-posed in the sense of Theorem 2.1 (see Pazy, 1983; see also Komornik, 1994). □

3. Exponential stability

In this section, we investigate the asymptotic behaviour of the solution of problem (2.4). In fact, using the energy method to produce a suitable Lyapunov functional, we will prove that, under a smallness condition on $|\mu|$, the solution of (2.4) decays to zero as t tends to infinity; that is,

$$\lim_{t\to+\infty} \|\mathscr{U}(t)\|^2_{\mathscr{H}} = 0, \tag{3.1}$$

and the decay of $\|\mathscr{U}\|^2_{\mathscr{H}}$ is at least exponential. Our result reads as follows.

THEOREM 3.1 Assume that (A1)–(A3) hold. Then there exists a positive constant δ_0 independent of μ such that, if

$$|\mu| < \delta_0, \tag{3.2}$$

then, for any $\mathscr{U}_0 \in \mathscr{H}$, there exist positive constants δ_1 and δ_2 (depending on $\|\mathscr{U}_0\|_{\mathscr{H}}$, a, g_0, $g(0)$, δ, τ and μ) such that the solution of (2.4) satisfies

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} \leqslant \delta_2\, e^{-\delta_1 t} \quad \forall\, t \in \mathbb{R}_+. \tag{3.3}$$

Proof. Assume that (A1)–(A3) are satisfied and let $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, so that all the calculations below are justified. By a simple density argument, (3.3) remains valid for any weak solution ($\mathscr{U}_0 \in \mathscr{H}$). We start our proof by providing a bound on the derivative of the energy functional E associated with the solution of (2.4) corresponding to $\mathscr{U}_0$

$$E(t) = \frac{1}{2}\|\mathscr{U}(t)\|^2_{\mathscr{H}} = \frac{1}{2}(\|u_t(t)\|^2 + (1-g_0)\|A^{1/2}u(t)\|^2) + \frac{\tau|\mu|}{2}\int_0^1 \|z(t,p)\|^2\,\mathrm{d}p$$
$$+ \frac{1}{2}\int_0^{+\infty} g(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{d}s \quad \forall\, t \in \mathbb{R}_+. \tag{3.4}$$

Using (2.4), (2.7) and (2.12), we obtain

$$E'(t) \leqslant \frac{1}{2}\int_0^{+\infty} g'(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{d}s + |\mu|\|u_t(t)\|^2 \quad \forall\, t \in \mathbb{R}_+. \tag{3.5}$$

Note that, in contrast to the situation where we have a frictional damping in (1.7 and 1.8) and no delay in (1.3) and (1.6), the inequality (3.5) shows that E' is not negative in general, and therefore the system (2.4) is, in general, not necessarily dissipative with respect to E. In order to continue the proof of Theorem 3.1, we need the following four lemmas. □

LEMMA 3.2 Let us define the functional

$$I_1(t) = -\left\langle u_t(t), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s \right\rangle \quad \forall\, t \in \mathbb{R}_+.$$

Then

$$I_1'(t) \leqslant -(g_0 - \epsilon)\|u_t(t)\|^2 + \epsilon\|A^{1/2}u(t)\|^2 + c_1 \int_0^{+\infty} g(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{d}s$$
$$- c_2 \int_0^{+\infty} g'(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{d}s + \mu \left\langle z(t,1), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s \right\rangle \quad \forall\, t \in \mathbb{R}_+, \tag{3.6}$$

where (the constants a and g_0 are defined in (2.1) and (2.3), respectively)

$$\begin{cases} \epsilon = \dfrac{g_0(1-g_0)}{2(2+g_0)}, \\ c_1 = g_0 + \dfrac{1}{2}(1-g_0)(2+g_0), \\ c_2 = \dfrac{g(0)(2+g_0)}{2ag_0(1-g_0)}. \end{cases} \tag{3.7}$$

Proof. As in Messaoudi (2008a,b) and Muñoz Revira & Grazia Naso (2007), multiplying (1.1) by $\int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s$, we obtain

$$0 = \left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s \right\rangle + (1-g_0)\left\langle Au(t), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s \right\rangle$$
$$+ \left\langle \int_0^{+\infty} g(s)A\eta(t,s)\,\mathrm{d}s, \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s \right\rangle + \left\langle \mu u_t(t-\tau), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s \right\rangle.$$

Using the definition of $A^{1/2}$, we obtain

$$0 = \left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s \right\rangle + (1-g_0)\left\langle A^{1/2}u(t), \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{d}s \right\rangle$$
$$+ \left\langle \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{d}s, \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{d}s \right\rangle$$
$$+ \left\langle \mu u_t(t-\tau), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s \right\rangle. \tag{3.8}$$

On the other hand, by using the fact that $\eta_t(t,s) = -\eta_s(t,s) + u_t(t)$, we find

$$\left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{ds}\right\rangle = \frac{\partial}{\partial t}\left\langle u_t(t), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{ds}\right\rangle - \left\langle u_t(t), \int_0^{+\infty} g(s)\eta_t(t,s)\,\mathrm{ds}\right\rangle$$
$$= -I_1'(t) - g_0\|u_t(t)\|^2 + \left\langle u_t(t), \int_0^{+\infty} g(s)\eta_s(t,s)\,\mathrm{ds}\right\rangle.$$

Integrating by parts with respect to s in the infinite memory integral, and using the fact that $\lim_{s\to+\infty} g(s) = 0$ and $\eta(t,0) = 0$, we obtain

$$\left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{ds}\right\rangle = -I_1'(t) - g_0\|u_t(t)\|^2 - \left\langle u_t(t), \int_0^{+\infty} g'(s)\eta(t,s)\,\mathrm{ds}\right\rangle. \tag{3.9}$$

Exploiting (3.8–3.9) and the fact that $u_t(t-\tau) = z(t,1)$, we deduce that

$$I_1'(t) = -g_0\|u_t(t)\|^2 + \mu\left\langle z(t,1), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{ds}\right\rangle - \left\langle u_t(t), \int_0^{+\infty} g'(s)\eta(t,s)\,\mathrm{ds}\right\rangle$$
$$+ (1-g_0)\left\langle A^{1/2}u(t), \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{ds}\right\rangle$$
$$+ \left\langle \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{ds}, \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{ds}\right\rangle. \tag{3.10}$$

Using Cauchy–Schwarz and Young's inequalities for the last three terms of (3.10), and (2.1) to estimate $\|\eta(t,s)\|^2$ by $(1/a)\|A^{1/2}\eta(t,s)\|^2$, we obtain, for ϵ defined in (3.7) (ϵ is a positive constant according to (2.3)),

$$-\left\langle u_t(t), \int_0^{+\infty} g'(s)\eta(t,s)\,\mathrm{ds}\right\rangle \leqslant \epsilon\|u_t(t)\|^2 + \frac{1}{4\epsilon}\left(\int_0^{+\infty}\sqrt{-g'(s)}\sqrt{-g'(s)}\|\eta(t,s)\|\,\mathrm{ds}\right)^2$$
$$\leqslant \epsilon\|u_t(t)\|^2 - \frac{g(0)}{4a\epsilon}\int_0^{+\infty} g'(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{ds},$$

$$(1-g_0)\left\langle A^{1/2}u(t), \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{ds}\right\rangle$$
$$\leqslant \epsilon\|A^{1/2}u(t)\|^2 + \frac{(1-g_0)^2}{4\epsilon}\left(\int_0^{+\infty}\sqrt{g(s)}\sqrt{g(s)}\|A^{1/2}\eta(t,s)\|\,\mathrm{ds}\right)^2$$
$$\leqslant \epsilon\|A^{1/2}u(t)\|^2 + \frac{g_0(1-g_0)^2}{4\epsilon}\int_0^{+\infty} g(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{ds}$$

and

$$\left\langle \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{ds}, \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{ds}\right\rangle \leqslant \left(\int_0^{+\infty}\sqrt{g(s)}\sqrt{g(s)}\|A^{1/2}\eta(t,s)\|\,\mathrm{ds}\right)^2$$
$$\leqslant g_0\int_0^{+\infty} g(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{ds}.$$

Inserting these three inequalities into (3.10), we obtain (3.6) with c_1 and c_2 defined in (3.7) (c_1 and c_2 are positive constants thanks to (A1)–(A3)). □

LEMMA 3.3 Define the functional

$$I_2(t) = \langle u_t(t), u(t)\rangle \quad \forall\, t \in \mathbb{R}_+.$$

Then

$$I_2'(t) \leqslant \|u_t(t)\|^2 - (1-g_0-\epsilon)\|A^{1/2}u(t)\|^2 - \mu\langle z(t,1), u(t)\rangle + c_3\int_0^{+\infty} g(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{d}s \quad \forall\, t\in\mathbb{R}_+, \tag{3.11}$$

where ϵ is defined in (3.7) and

$$c_3 = \frac{2+g_0}{2(1-g_0)}. \tag{3.12}$$

Proof. Multiplying (1.1) by u, we find

$$0 = \langle u_{tt}(t), u(t)\rangle + (1-g_0)\langle Au(t), u(t)\rangle + \left\langle \int_0^{+\infty} g(s)A\eta(t,s)\,\mathrm{d}s, u(t)\right\rangle + \langle \mu u_t(t-\tau), u(t)\rangle.$$

Consequently, using the definition of $A^{1/2}$ and the fact that $u_t(t-\tau) = z(t,1)$, we have

$$0 = \frac{\partial}{\partial t}\langle u_t(t), u(t)\rangle - \|u_t(t)\|^2 + (1-g_0)\|A^{1/2}u(t)\|^2 + \left\langle \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{d}s, A^{1/2}u(t)\right\rangle + \langle \mu z(t,1), u(t)\rangle,$$

which implies that

$$I_2'(t) = \|u_t(t)\|^2 - (1-g_0)\|A^{1/2}u(t)\|^2 - \mu\langle z(t,1), u(t)\rangle - \left\langle \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{d}s, A^{1/2}u(t)\right\rangle. \tag{3.13}$$

By using Cauchy–Schwarz and Young's inequalities for the last term of (3.13), we obtain (as in the proof of (3.6))

$$-\left\langle \int_0^{+\infty} g(s)A^{1/2}\eta(t,s)\,\mathrm{d}s, A^{1/2}u(t)\right\rangle \leqslant \epsilon\|A^{1/2}u(t)\|^2 + \frac{g_0}{4\epsilon}\int_0^{+\infty} g(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{d}s.$$

Reporting this inequality in (3.13) and using the definition of ϵ in (3.7), (3.11) holds. □

Now, as in Fridman *et al.* (2010) and Nicaise *et al.* (2011), we prove the following estimate.

LEMMA 3.4 The functional

$$I_3(t) = \int_0^1 e^{-2\tau p} \|z(t,p)\|^2 \, dp \quad \forall\, t \in \mathbb{R}_+,$$

satisfies

$$I_3'(t) \leqslant -2\, e^{-2\tau} \int_0^1 \|z(t,p)\|^2 \, dp + \frac{1}{\tau} \|u_t(t)\|^2 - \frac{e^{-2\tau}}{\tau} \|z(t,1)\|^2 \quad \forall\, t \in \mathbb{R}_+. \tag{3.14}$$

Proof. Using (2.10), the derivative of I_3 entails

$$\begin{aligned} I_3'(t) &= 2 \int_0^1 e^{-2\tau p} \langle z_t(t,p), z(t,p) \rangle \, dp \\ &= -\frac{2}{\tau} \int_0^1 e^{-2\tau p} \langle z_p(t,p), z(t,p) \rangle \, dp \\ &= -\frac{1}{\tau} \int_0^1 e^{-2\tau p} \frac{\partial}{\partial p} (\|z(t,p)\|^2) \, dp. \end{aligned}$$

Then, by integrating by parts and $z(t,0) = u_t(t)$, the above formula can be rewritten as

$$I_3'(t) = -2 \int_0^1 e^{-2\tau p} \|z(t,p)\|^2 \, dp + \frac{1}{\tau} \|u_t(t)\|^2 - \frac{e^{-2\tau}}{\tau} \|z(t,1)\|^2,$$

which gives (3.14), since $e^{-2\tau p} \geqslant e^{-2\tau}$, for any $p \in]0,1[$. □

LEMMA 3.5 There exists a positive constant δ_0 independent of μ such that, if (3.2) holds, then there exist positive constants ϵ_1 and δ_1 such that the functional

$$F = E + \epsilon_1 \left(I_1 + \frac{\epsilon^2 + g_0(1-g_0)}{2(1-g_0-\epsilon)} I_2 + \frac{\tau(\epsilon^2 - 2\epsilon + g_0(1-g_0))}{8(1-g_0-\epsilon)} I_3 \right) \tag{3.15}$$

satisfies $F \sim E$ and

$$F'(t) \leqslant -\delta_1 F(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.16}$$

Proof. Let $\epsilon_1 > 0$, which will be fixed later carefully. The constants $(\epsilon^2 + g_0(1-g_0))/2(1-g_0-\epsilon)$ and $\tau(\epsilon^2 - 2\epsilon + g_0(1-g_0))/8(1-g_0-\epsilon)$ are positive according to (2.3) and the choice of ϵ in (3.7). Then, combining (3.5), (3.6), (3.11) and (3.14), we obtain

$$\begin{aligned} F'(t) \leqslant & -\epsilon_1 \left(\left(c_4 - \frac{|\mu|}{\epsilon_1} \right) \|u_t(t)\|^2 + c_5 \|A^{1/2} u(t)\|^2 + c_6 \int_0^1 \|z(t,p)\|^2 \, dp \right. \\ & \left. + c_5 \int_0^{+\infty} g(s) \|A^{1/2} \eta(t,s)\|^2 \, ds + c_7 \|z(t,1)\|^2 \right) \\ & + \left(\frac{1}{2} - \epsilon_1 c_2 \right) \int_0^{+\infty} g'(s) \|A^{1/2} \eta(t,s)\|^2 \, ds + \epsilon_1 c_8 \int_0^{+\infty} g(s) \|A^{1/2} \eta(t,s)\|^2 \, ds \\ & + \mu \epsilon_1 \left\langle z(t,1), -c_9 u(t) + \int_0^{+\infty} g(s) \eta(t,s) \, ds \right\rangle \quad \forall\, t \in \mathbb{R}_+, \end{aligned} \tag{3.17}$$

where

$$\begin{cases} c_4 = \dfrac{3(\epsilon^2 - 2\epsilon + g_0(1-g_0))}{8(1-g_0-\epsilon)}, \\ c_5 = \dfrac{\epsilon^2 - 2\epsilon + g_0(1-g_0)}{2}, \\ c_6 = \dfrac{\tau(\epsilon^2 - 2\epsilon + g_0(1-g_0))}{4(1-g_0-\epsilon)}\, e^{-2\tau}, \\ c_7 = \dfrac{\epsilon^2 - 2\epsilon + g_0(1-g_0)}{8(1-g_0-\epsilon)}\, e^{-2\tau}, \\ c_8 = c_1 + c_5 + \dfrac{\epsilon^2 + g_0(1-g_0)}{2(1-g_0-\epsilon)} c_3, \\ c_9 = \dfrac{\epsilon^2 + g_0(1-g_0)}{2(1-g_0-\epsilon)}. \end{cases} \tag{3.18}$$

Bearing in mind (2.3), (3.7) and (3.12), the constants c_i $(i = 4, \ldots, 9)$ are positive and independent of μ. On the other hand, (2.2) implies that

$$\epsilon_1 c_8 \int_0^{+\infty} g(s) \|A^{1/2}\eta(t,s)\|^2 \, ds + \left(\frac{1}{2} - \epsilon_1 c_2\right) \int_0^{+\infty} g'(s) \|A^{1/2}\eta(t,s)\|^2 \, ds$$
$$\leqslant \left(\frac{1}{2} - \epsilon_1 c_{10}\right) \int_0^{+\infty} g'(s) \|A^{1/2}\eta(t,s)\|^2 \, ds,$$

where (noting that c_{10} is also independent of μ)

$$c_{10} = c_2 + \frac{c_8}{\delta}. \tag{3.19}$$

Next, the use of Cauchy–Schwarz and Young's inequalities and (2.1) to estimate $\|\eta(t,s)\|^2$ and $\|u(t)\|^2$ by $(1/a)\|A^{1/2}\eta(t,s)\|^2$ and $(1/a)\|A^{1/2}u(t)\|^2$, respectively, gives (as in the proof of (3.6))

$$\mu\epsilon_1 \left\langle z(t,1), -c_9 u(t) + \int_0^{+\infty} g(s)\eta(t,s)\, ds \right\rangle$$
$$\leqslant \epsilon_1 c_7 \|z(t,1)\|^2 + \frac{\epsilon_1 \mu^2}{4c_7} \left(c_9 \|u(t)\| + \int_0^{+\infty} g(s) \|\eta(t,s)\| \, ds \right)^2$$
$$\leqslant \epsilon_1 c_7 \|z(t,1)\|^2 + \frac{\epsilon_1 \mu^2}{2ac_7} \left(c_9^2 \|A^{1/2}u(t)\|^2 + g_0 \int_0^{+\infty} g(s) \|A^{1/2}\eta(t,s)\|^2 \, ds \right)$$
$$\leqslant \epsilon_1 c_7 \|z(t,1)\|^2 + \epsilon_1 \mu^2 c_{11} \left(\|A^{1/2}u(t)\|^2 + \int_0^{+\infty} g(s) \|A^{1/2}\eta(t,s)\|^2 \, ds \right),$$

where (noting that c_{11} is also independent of μ)

$$c_{11} = \frac{1}{2ac_7} \max\{c_9^2, g_0\}. \tag{3.20}$$

Therefore, inserting the last two inequalities in (3.17) and using the definition (3.4) of E, we obtain

$$F'(t) \leqslant -\epsilon_1 c_{12} E(t) + \left(\frac{1}{2} - \epsilon_1 c_{10}\right) \int_0^{+\infty} g'(s) \|A^{1/2}\eta(t,s)\|^2 \mathrm{d}s \quad \forall\, t \in \mathbb{R}_+, \tag{3.21}$$

where

$$c_{12} = 2 \min\left\{c_4 - \frac{|\mu|}{\epsilon_1}, \frac{c_6}{\tau|\mu|}, c_5 - \mu^2 c_{11}\right\}. \tag{3.22}$$

Finally, by definition of E, I_1, I_2 and I_3, we have, using again Cauchy–Schwarz and Young's inequalities, (2.1) and (2.3),

$$\begin{aligned}
|I_1(t)| &= \left|\left\langle u_t(t), \int_0^{+\infty} g(s)\eta(t,s)\,\mathrm{d}s\right\rangle\right| \\
&\leqslant \frac{1}{2}\left(\|u_t(t)\|^2 + \frac{g_0}{a}\int_0^{+\infty} g(s)\|A^{1/2}\eta(t,s)\|^2\,\mathrm{d}s\right) \\
&\leqslant \max\left\{1, \frac{g_0}{a}\right\} E(t),
\end{aligned} \tag{3.23}$$

$$\begin{aligned}
|I_2(t)| &= |\langle u_t(t), u(t)\rangle| \\
&\leqslant \frac{1}{2}\left(\|u_t(t)\|^2 + \frac{1}{a}\|A^{1/2}u(t)\|^2\right) \\
&\leqslant \max\left\{1, \frac{1}{a(1-g_0)}\right\} E(t)
\end{aligned} \tag{3.24}$$

and

$$|I_3(t)| = \int_0^1 e^{-2\tau p}\|z(t,p)\|^2\,\mathrm{d}p \leqslant \int_0^1 \|z(t,p)\|^2\,\mathrm{d}p \leqslant \frac{2}{\tau|\mu|}E(t). \tag{3.25}$$

Therefore, (3.23–3.25) imply that

$$|I_1(t) + \frac{\epsilon^2 + g_0(1-g_0)}{2(1-g_0-\epsilon)} I_2 + \frac{\tau(\epsilon^2 - 2\epsilon + g_0(1-g_0))}{8(1-g_0-\epsilon)} I_3(t)| \leqslant c_{13}E(t),$$

where

$$c_{13} = \max\left\{1, \frac{g_0}{a}\right\} + \frac{\epsilon^2 + g_0(1-g_0)}{2(1-g_0-\epsilon)} \max\left\{1, \frac{1}{a(1-g_0)}\right\} + \frac{\epsilon^2 - 2\epsilon + g_0(1-g_0)}{4|\mu|(1-g_0-\epsilon)}, \tag{3.26}$$

which, using (3.15), gives

$$(1 - \epsilon_1 c_{13})E(t) \leqslant F(t) \leqslant (1 + \epsilon_1 c_{13})E(t) \quad \forall\, t \in \mathbb{R}_+. \tag{3.27}$$

Now, we assume that $|\mu|$ satisfies (3.2) with

$$\delta_0 = \min\left\{\sqrt{\frac{c_5}{c_{11}}}, \frac{c_4}{2c_{10}}, \frac{c_4 - ((\epsilon^2 - 2\epsilon + g_0(1-g_0))/4(1-g_0-\epsilon))}{\max\{1, g_0/a\} + ((\epsilon^2 + g_0(1-g_0))/2(1-g_0-\epsilon))\max\{1, 1/a(1-g_0)\}}\right\} \tag{3.28}$$

and we fix ϵ_1 such that

$$\frac{|\mu|}{c_4} < \epsilon_1 < \min\left\{\frac{1}{2c_{10}}, \frac{1}{c_{13}}\right\}. \tag{3.29}$$

First, from (3.7) and (3.18), we conclude that

$$c_4 - \frac{\epsilon^2 - 2\epsilon + g_0(1-g_0)}{4(1-g_0-\epsilon)} = \frac{\epsilon^2 - 2\epsilon + g_0(1-g_0)}{8(1-g_0-\epsilon)} > 0,$$

and then δ_0 is a positive constant and independent of μ. Second, (3.2), (3.28) and (3.29) imply:

(i) ϵ_1 exists,

(ii) $c_4 - |\mu|/\epsilon_1 > 0$ and $c_5 - \mu^2 c_{11} > 0$, which gives $c_{12} > 0$, in view of (3.22),

(iii) $1 - \epsilon_1 c_{13} > 0$, which gives $F \sim E$ thanks to (3.27),

(iv) $\frac{1}{2} - \epsilon_1 c_{10} > 0$, which implies that the last term of (3.21) is non-positive (note that g in non-increasing), and then, using (3.21) and (3.27), (3.16) holds with $\delta_1 = \epsilon_1 c_{12}/(1+\epsilon_1 c_{13})$.

This finishes the proof of Lemma 3.5. □

Now, going back to the proof of Theorem 3.1, we have to just integrate the differential inequality (3.16) over $[0,t]$ to obtain

$$F(t) \leqslant F(0)\, e^{-\delta_1 t} \quad \forall\, t \in \mathbb{R}_+.$$

Consequently, using (3.4) and (3.27), we find

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} = 2E(t) \leqslant \frac{2}{1-\epsilon_1 c_{13}} F(t) \leqslant \frac{2}{1-\epsilon_1 c_{13}} F(0)\, e^{-\delta_1 t} \quad \forall\, t \in \mathbb{R}_+,$$

which gives (3.3) with $\delta_2 = (2/(1-\epsilon_1 c_{13}))F(0)$. Thus the proof of Theorem 3.1 is completed.

4. Applications

We present in this section some extensions and particular applications included in our abstract equation (1.1).

4.1 *More general model*

Our results hold for the more general form

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)\,\mathrm{d}s + Cu_t(t-\tau) = 0 \quad \forall\, t > 0, \tag{4.1}$$

where $B: D(B) \to H$ is a self-adjoint linear positive definite operator having domain $D(A) \subset D(B) \subset H$ with dense and compact embeddings and $C: H \to H$ is a self-adjoint linear operator such that, for some positive constants a_0, a_1, a_2 and a_3,

$$\|v\|^2 \leqslant a_0\|B^{1/2}v\|^2 \leqslant a_1\|A^{1/2}v\|^2 \leqslant a_2\|B^{1/2}v\|^2 \quad \forall\, v \in D(A^{1/2}) \tag{4.2}$$

and

$$\|Cv\|^2 \leqslant a_3\|v\|^2 \quad \forall\, v \in H$$

with $g_0 \in]0, 1/a_1[$ and a_3 (which plays the role of $|\mu|$) is small enough so that (3.2) holds for $|\mu| = a_3$.

4.2 *Finite memory*

Our model (1.1) includes the case of finite memory

$$u_{tt}(t) + Au(t) - \int_0^t g(s)Au(t-s)\,\mathrm{d}s + \mu u_t(t-\tau) = 0 \quad \forall\, t > 0.$$

This equation corresponds to (1.1) with a null past history; that is $u_0 \equiv 0$.

4.3 *Wave equation*

In this application, as well as in the next two ones, let α be a positive constant and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary Γ, where $n \in \mathbb{N}^*$.

Our results hold for the following wave equation with Dirichlet boundary condition:

$$\begin{cases} u_{tt}(x,t) - \alpha\Delta u(x,t) + \alpha\displaystyle\int_0^{+\infty} g(s)\Delta u(x,t-s)\,\mathrm{d}s + \mu u_t(x,t-\tau) = 0, & \Omega \times \mathbb{R}_+, \\ u(x,t) = 0, & \Gamma \times \mathbb{R}_+, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), & \Omega \times \mathbb{R}_+, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \Omega \times]0,\tau[, \end{cases} \tag{4.3}$$

which is equivalent to (1.1 and 1.2) with $A = -\alpha\Delta$, $D(A) = H^2(\Omega) \cap H_0^1(\Omega)$, $H = L^2(\Omega)$ and $\langle w_1, w_2\rangle = \int_\Omega w_1 w_2\,\mathrm{d}x$.

We can also consider the general wave equation $A = -\sum_{i,j=1}^n (\partial/\partial x_i)(a_{ij}(\partial/\partial x_j))$ with variable coefficients a_{ij} depending only on the space variable and satisfying classical smoothness, symmetry and coercivity conditions.

4.4 *Petrovsky-type system*

Our results also hold for the following Petrovsky system with Dirichlet and Neumann boundary conditions:

$$\begin{cases} u_{tt}(x,t) + \alpha\Delta^2 u(x,t) - \alpha\displaystyle\int_0^{+\infty} g(s)\Delta^2 u(x,t-s)\,\mathrm{d}s + \mu u_t(x,t-\tau) = 0, & \Omega \times \mathbb{R}_+, \\ u(x,t) = \dfrac{\partial u}{\partial \nu}(x,t) = 0, & \Gamma \times \mathbb{R}_+, \\ u(x,-t) = u_0(x,t), \quad u_t(x,0) = u_1(x), & \Omega \times \mathbb{R}_+, \\ u_t(x,t-\tau) = f_0(x,t-\tau), & \Omega \times]0,\tau[, \end{cases} \tag{4.4}$$

which is equivalent to (1.1 and 1.2) with $A = \alpha\Delta^2$, $D(A) = H^4(\Omega) \cap H_0^2(\Omega)$ and $H = L^2(\Omega)$.

4.5 *Coupled systems*

We can also consider the following coupled wave–wave, Petrovsky–Petrovsky and wave–Petrovsky systems with Dirichlet and Neumann boundary conditions:

$$\begin{cases} w_{tt}(x,t) - \alpha\Delta w(x,t) + \alpha\displaystyle\int_0^{+\infty} g(s)\Delta w(x,t-s)\,\mathrm{d}s + \mu w_t(x,t-\tau) + dv(x,t) = 0, & \Omega\times\mathbb{R}_+, \\ v_{tt}(x,t) - \beta\Delta v(x,t) + \beta\displaystyle\int_0^{+\infty} g(s)\Delta v(x,t-s)\,\mathrm{d}s + \mu v_t(x,t-\tau) + dw(x,t) = 0, & \Omega\times\mathbb{R}_+, \\ w(x,t) = v(x,t) = 0, & \Gamma\times\mathbb{R}_+, \\ (w(x,-t), v(x,-t)) = (w_0(x,t), v_0(x,t)), \quad (w_t(x,0), v_t(x,0)) = (w_1(x), v_1(x)), & \Omega\times\mathbb{R}_+, \\ (w_t(x,t-\tau), v_t(x,t-\tau)) = (k_0(x,t-\tau), h_0(x,t-\tau)), & \Omega\times]0,\tau[, \end{cases} \tag{4.5}$$

$$\begin{cases} w_{tt}(x,t) + \alpha\Delta^2 w(x,t) - \alpha\displaystyle\int_0^{+\infty} g(s)\Delta^2 w(x,t-s)\,\mathrm{d}s + \mu w_t(x,t-\tau) + dv(x,t) = 0, & \Omega\times\mathbb{R}_+, \\ v_{tt}(x,t) + \beta\Delta^2 v(x,t) - \beta\displaystyle\int_0^{+\infty} g(s)\Delta^2 v(x,t-s)\,\mathrm{d}s + \mu v_t(x,t-\tau) + dw(x,t) = 0, & \Omega\times\mathbb{R}_+, \\ w(x,t) = v(x,t) = \dfrac{\partial w}{\partial \nu}(x,t) = \dfrac{\partial v}{\partial \nu}(x,t) = 0, & \Gamma\times\mathbb{R}_+, \\ (w(x,-t), v(x,-t)) = (w_0(x,t), v_0(x,t)), \quad (w_t(x,0), v_t(x,0)) = (w_1(x), v_1(x)), & \Omega\times\mathbb{R}_+, \\ (w_t(x,t-\tau), v_t(x,t-\tau)) = (k_0(x,t-\tau), h_0(x,t-\tau)) & \Omega\times]0,\tau[\end{cases} \tag{4.6}$$

and

$$\begin{cases} w_{tt}(x,t) - \alpha\Delta w(x,t) + \alpha\displaystyle\int_0^{+\infty} g(s)\Delta w(x,t-s)\,\mathrm{d}s + \mu w_t(x,t-\tau) + dv(x,t) = 0, & \Omega\times\mathbb{R}_+, \\ v_{tt}(x,t) + \beta\Delta^2 v(x,t) - \beta\displaystyle\int_0^{+\infty} g(s)\Delta^2 v(x,t-s)\,\mathrm{d}s + \mu v_t(x,t-\tau) + dw(x,t) = 0, & \Omega\times\mathbb{R}_+, \\ w(x,t) = v(x,t) = \dfrac{\partial v}{\partial \nu}(x,t) = 0, & \Gamma\times\mathbb{R}_+, \\ (w(x,-t), v(x,-t)) = (w_0(x,t), v_0(x,t)), \quad (w_t(x,0), v_t(x,0)) = (w_1(x), v_1(x)), & \Omega\times\mathbb{R}_+, \\ (w_t(x,t-\tau), v_t(x,t-\tau)) = (k_0(x,t-\tau), h_0(x,t-\tau)), & \Omega\times]0,\tau[, \end{cases} \tag{4.7}$$

where α and β are positive constants, and d is a constant with $|d|$ small enough such that (2.1) holds. Systems (4.5–4.7) are equivalent to (1.1 and 1.2) with $u = (w, v)$, $f_0 = (k_0, h_0)$, $H = (L^2(\Omega))^2$, $\langle (w_1, \tilde{w}_1), (w_2, \tilde{w}_2)\rangle = \int_\Omega (w_1 w_2 + \tilde{w}_1\tilde{w}_2)\,\mathrm{d}x$,

$$Au = \begin{cases} -(\alpha\Delta w, \beta\Delta v) + d(v, w) & \text{in the case of (4.5)}, \\ (\alpha\Delta^2 w, \beta\Delta^2 v) + d(v, w) & \text{in the case of (4.6)}, \\ (-\alpha\Delta w, \beta\Delta^2 v) + d(v, w) & \text{in the case of (4.7)} \end{cases}$$

and

$$D(A) = \begin{cases} (H^2(\Omega) \cap H_0^1(\Omega))^2 & \text{in the case of (4.5)}, \\ (H^4(\Omega) \cap H_0^2(\Omega))^2 & \text{in the case of (4.6)}, \\ (H^2(\Omega) \cap H_0^1(\Omega)) \times (H^4(\Omega) \cap H_0^2(\Omega)) & \text{in the case of (4.7)}. \end{cases}$$

5. General comments and open problems

We give in this last section some general comments and open problems.

(1) It is interesting to determine the biggest value of δ_0 in (3.2) for which the exponential stability (3.3) of (1.1 and 1.2) holds. On the other hand, is the system (1.1 and 1.2) instable when $|\mu|$ is not small enough? Some instabilities hold for (1.7) if $|\mu|$ is not small enough (see Nicaise & Pignotti, 2006).

(2) Assumption (4.2) implies that A and B are equivalent. When $C = 0$, some (non-exponential) decay estimates of (4.1) have been proved in Guesmia (2011) and Muñoz Revira & Grazia Naso (2007, 2010) even if the last inequality of (4.2) does not hold. We do not know if such decay estimates can be proved in case of presence of delay in (4.1); that is, $C \neq 0$.

(3) Condition (2.2) implies that g converges to zero at infinity at least exponentially. Does the strong stability (3.1) of system (1.1 and 1.2) still hold for g converging to zero at infinity less faster than exponentially, and what is the decay estimate satisfied by $\|\mathscr{U}\|_{\mathscr{H}}^2$ in this case?

(4) Our results do not include the case of wave equation and Petrovsky systems (4.3) and (4.4), for example, with boundary memory and delay, or internal memory and boundary delay, or conversely. It is interesting to study these situations.

(5) Another interesting question concerns the stability of coupled systems like (4.5–4.7) with two delays and one memory considered only on one equation of the system.

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