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ON THE BOUNDARY STABILIZATION OF A COMPACTLY COUPLED SYSTEM OF NONLINEAR WAVE EQUATIONS

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ABSTRACT. We consider a compactly coupled system of nonlinear wave equations with nonlinear feedbacks localized on a part of the boundary. We first linearize the problem and use a fixed point argument to establish a local existence result, for arbitrary initial data. We then show that, under some conditions on the form of the nonlinearity and on the form of the feedbacks, this unique solution is global and with an exponentially decaying energy.

Keywords: exponential decay, fixed point, global existence, local existence, multiplier method, wave equation

AMS Subject Classification: 35 L 45 - 35 B 40 - 35 L 05 - 35 L 55

1. **Introduction.** In [1], Aassila studied the following problem

$$\begin{cases} u_1'' - \Delta u_1 + \alpha(x)(u_1 - u_2) = 0, & (x,t) \in \Omega \times \mathbb{R}^+ \\ u_2'' - \Delta u_2 + \alpha(x)(u_2 - u_1) = 0, & (x,t) \in \Omega \times \mathbb{R}^+ \\ u_1 = u_2 = 0, & (x,t) \in \Gamma_0 \times \mathbb{R}^+ \\ \frac{\partial u_i}{\partial \nu} + a_i u_i + g_i(u_i') = 0, \ i = 1,2, & (x,t) \in \Gamma_1 \times \mathbb{R}^+ \\ u_i(0) = u_i^0, \quad u_i'(0) = u_i^1, \ i = 1,2, & x \in \Omega \end{cases}$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \in \mathbb{N}^*$) with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ and $\{\Gamma_0, \Gamma_1\}$ is a partition of the boundary, α: $\Omega \to \mathbb{R}^+$, a_1, a_2: $\Gamma_1 \to \mathbb{R}^+$ and g_1, g_2: $\mathbb{R} \to \mathbb{R}$ are given functions. They proved an energy decay result of the strong solution with weak dissipation. In his proof he used the multipliers method introduced by Komornik in [14] and Lions in [20].

This type of problems is motivated by similar problems in ordinary differential equations for coupled oscillators. It has an application in engineering such as in the case of isolation of objects from outside disturbances. Also, modeling structures like beams or plates sandwiched with rubber or similar materials lead to equations similar to the above system (See [2], [15] and the references therein).

Several authors have considered the issue of the energy decay for solutions of systems with boundary dissipation. In this regard we mention the work of Conrad and Rao [8], Komornik and Zuazua [12], Komornik [13], Lagnese [16], Lasiecka [17], Lasiecka and Tataru [18], Lions [20], Zuazua [21, 22] and Zuazua and Liu [23]. In particular, we shall stress on the work of Bardos, Lebeau and Rauch [5], where the authors analyzed the observability, controllability and stabilization of solutions of second-order hyperbolic partial differential equations. They obtained results for multidimensional problems that are as precise as those in the one-dimensional cases. Though they treated linear equations, as they pointed out, their results extend to nonlinear equations by linearization use of the microlocal analysis.

In fact the result in [1] improves an earlier one by Komornik and Rao [15] in the sense that the author in [1] allows the dissipative effect, caused by g_i, to be weaker at the origin and at infinity. At the best of our knowledge, the first works, where the possibility of splitting the behavior of the nonlinearity at the origin and at infinity were pointed out, are [11] and [22].

Recently there has been a lot of work on the stabilization of coupled systems. For instance, the works by Alabau-Boussouira are worth mentioning. This issue is related with the theory of polarization of waves for systems of wave equations developed by Burq [7] and Asch and Lebeau [4]. Alabau-Boussouira [3] considered a coupled system of two linear wave equations with linear weak coupling and only one damping placed in first equation, and proved some polynomial decay estimates. The coupling under consideration in the present paper is nonlinear and so is more challenging.

In this article, we deal with the global existence and energy decay of solutions for the initial boundary value problem

$$\begin{cases} u_1'' - \Delta u_1 + b_1\alpha(x) f(b_1u_1 + b_2u_2) = 0, & (x,t) \in \Omega\times\mathbb{R}^+ \\ u_2'' - \Delta u_2 + b_2\alpha(x) f(b_1u_1 + b_2u_2) = 0, & (x,t) \in \Omega\times\mathbb{R}^+ \\ u_1 = u_2 = 0, & (x,t) \in \Gamma_0\times\mathbb{R}^+ \\ \frac{\partial u_i}{\partial \nu} + a_i u_i + g_i(u_i') = 0, \qquad i = 1,2, & (x,t) \in \Gamma_1\times\mathbb{R}^+ \\ u_i(x,0) = u_i^0(x) \qquad u_i'(x,0) = u_i^1(x), \qquad i = 1,2, & x \in \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^n$ ($n \in \mathbb{N}^*$) with a smooth boundary $\partial\Omega = \Gamma_0 \cup \Gamma_1$ where $\{\Gamma_0, \Gamma_1\}$ is a partition of the boundary, b_1 and b_2 are constants, $\alpha : \Omega \to \mathbb{R}^+$, $a_1, a_2 : \Gamma_1 \to \mathbb{R}^+$ and g_1, g_2, $f : \mathbb{R} \to \mathbb{R}$ are given functions.

In system (1.1) we consider more general coupling (represented by the function f) than the ones studied in [1], [3] and [15]. After proving the well-posedness of (1.1) by linearizing the problem and using a fixed point argument, we prove, under some suitable conditions on the coupling function f and the feedbacks g_1 and g_2, that the multiplier method introduced by Komornik [14] and Lions [20] is applicable and (1.1) is exponentially stable.

This work is divided into five parts. In part two we establish a local existence theorem. In part three we show that this local solution is, in fact, global. In part four we prove some decay estimates and in part five, some applications and open questions are presented and discussed.

2. **Local Existence.** In order to state and prove our local existence result we make the following assumptions:

(H1) Ω is smooth and the partition of $\partial\Omega$ satisfies $\Gamma_0 \cap \Gamma_1 = \emptyset$, such that there exists x_0 in $\mathbb{R}^n$ and $\delta > 0$ satisfying $(x - x_0).\nu \leq 0$ on Γ_0 and $(x - x_0).\nu \geq \delta > 0$ on Γ_1.

(H2) $a_i \in C^1(\Gamma_1)$ such that $a_i \geq 0$, $i = 1, 2$, on Γ_1.

(H3) The function $\alpha \in C^1(\Omega) \cap L^\infty(\Omega)$.

(H4) The functions g_1 and g_2 are nondecreasing, continuous and

$$g_i(s) = 0 \Leftrightarrow s = 0, \; i = 1, 2.$$

Furthermore, there exists a constant $d_1 > 0$ such that

$$|g_i(s)| \leq 1 + d_1|s|, \; i = 1, 2, \quad \forall s \in \mathbb{R}.$$

(H5) The function f is of class $C^1(\mathbb{R})$ and satisfying

$$|f(s)| \leq d_2|s|^p, \qquad \forall s \in \mathbb{R}$$

for some constant $d_2 > 0$ and $p \geq 1$ with $(n - 2)p \leq n$.

Remark 2.1. Assumption (H1) implies that the domain Ω is not simply connected. This drawback was overcome in [12] using the analysis by Grisvard regarding singularities on interfaces. The assumption (H2) was also weakened in [12]. Recently, an important progress in this direction has been done by Bey *et al* [6]. So the general case where the interface is not ruled out by artificial geometry assumption can be considered. Because the study of the geometry of the domain is not our main objective, we shall consider in this paper hypothesis (H1).

Remark 2.2. For alternative conditions on g_1 and g_2 see [1].

Remark 2.3. Thanks to (H3) and (H5), the coupling in (1.1) is compact. In fact if we write (1.1) in the form

$$V' + AV + BV = 0,$$

where $V = (u_1, u_2, u_1', u_2')$, $AV = (-u_1', -u_2', -\Delta u_1, -\Delta u_2)$ and

$$BV = (0, 0, b_1\alpha f(b_1u_1 + b_2u_2), b_2\alpha f(b_1u_1 + b_2u_2)),$$

we easily see that B is a compact operator (See [15], equation (3.2)).

First let us consider the linear problem

$$\begin{cases} u_1'' - \Delta u_1 = f_1(x,t), & (x,t) \in \Omega\times\mathbb{R}^+ \\ u_2'' - \Delta u_2 = f_2(x,t), & (x,t) \in \Omega\times\mathbb{R}^+ \\ u_1 = u_2 = 0, & (x,t) \in \Gamma_0\times\mathbb{R}^+ \\ \frac{\partial u_i}{\partial \nu} + a_iu_i + g_i(u_i') = 0, \quad i = 1, 2, & (x,t) \in \Gamma_1\times\mathbb{R}^+ \\ u_i(x,0) = u_i^0(x) \quad u_i'(x,0) = u_i^1(x), \quad i = 1, 2, & x \in \Omega. \end{cases} \tag{2.1}$$

Lemma 2.1. *Assume that (H1), (H2), (H4) hold and $f_1, f_2 \in L^2(\Omega \times (0,T))$. Then for any $(u_i^0, u_i^1), i = 1, 2$, in $H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, problem (2.1) has a unique solution*

$$u_i \in C\left([0,\; T];\; H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left([0,\; T];\; L^2(\Omega)\right), \quad i = 1, 2, \tag{2.2}$$

where

$$H^1_{\Gamma_0}(\Omega) := \{v \in H^1(\Omega) \ / \ v = 0 \ on \ \Gamma_0\}.$$

This lemma can be proved by repeating the argument of Theorem 1 in [2] (see also Theorem 1 in [15]).

Theorem 2.2. *Assume that (H1) - (H5) hold. Then for any* $(u_i^0, u_i^1), i = 1, 2,$ *in* $H^1_{\Gamma_0}(\Omega) \times L^2(\Omega)$, *problem (1.1) has a unique solution*

$$u_i \in C\left([0,\ T];\ H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left([0,\ T];\ L^2(\Omega)\right), \quad i = 1, 2, \tag{2.3}$$

where T *is small enough. Furthermore, if* g *is globally Lipschitz continuous and*

$$(u_i^0, u_i^1), \in \left(H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \times H^1_{\Gamma_0}(\Omega), \quad i = 1, 2$$

and satisfying the compatibility conditions

$$\frac{\partial u_i^0}{\partial \nu} + a_i u_i^0 + g_i(u_i^1) = 0, \qquad i = 1, 2, \tag{2.4}$$

then the solution of (1.1) satisfies

$$u_i \in L^\infty\left([0,\ T];\ H^2(\Omega) \cap H^1_{\Gamma_0}(\Omega)\right) \cap W^{1,\infty}\left([0,\ T];\ H^1_{\Gamma_0}(\Omega)\right)$$
$$\cap W^{2,\infty}\left([0,\ T];\ L^2(\Omega)\right), \qquad i = 1, 2. \tag{2.5}$$

Proof. For $M > 0$ large and $T > 0$, we define a class of functions $Z(M, T)$ which consists of all functions $w = (w_1, w_2)$ in

$$\mathbf{W} := \left[C\left([0,\ T];\ H^1_{\Gamma_0}(\Omega)\right) \cap C^1\left([0,\ T];\ L^2(\Omega)\right)\right]^2 \tag{2.6}$$

satisfying the initial conditions of (2.1) and

$$||w||^2_{\mathbf{W}} := \max_{0 \le t \le T} \{ \int_\Omega (w_1'^2 + w_2'^2 + |\nabla w_1|^2 + |\nabla w_2|^2)(x,t)dx$$
$$+ \int_{\Gamma_1} (a_1 w_1^2 + a_2 w_2^2)(x,t) d\Gamma\} \le M^2. \tag{2.7}$$

The set $Z(M, T)$ is nonempty if M is large enough. This follows from the trace theorem [19]. We also define the map h from $Z(M, T)$ into $\mathbf{W}$ by $u =: h(\phi)$, where u is the unique solution of the linear problem

$$\begin{cases} u_1'' - \Delta u_1 + b_1 \alpha(x) f(b_1 \phi_1 + b_2 \phi_2) = 0, & (x,t) \in \Omega \times \mathbb{R}^+ \\ u_2'' - \Delta u_2 + b_2 \alpha(x) f(b_1 \phi_1 + b_2 \phi_2) = 0, & (x,t) \in \Omega \times \mathbb{R}^+ \\ u_1 = u_2 = 0, & (x,t) \in \Gamma_0 \times \mathbb{R}^+ \\ \frac{\partial u_i}{\partial \nu} + a_i u_i + g_i(u_i') = 0, \quad i = 1, 2, & (x,t) \in \Gamma_1 \times \mathbb{R}^+ \\ u_i(x,0) = u_i^0(x) \quad u_i'(x,0) = u_i^1(x), \quad i = 1, 2, & x \in \Omega. \end{cases} \tag{2.8}$$

By virtue of (H3) and (H5) and the embedding of $H^1_{\Gamma_0}(\Omega) \subset L^q(\Omega)$, $1 \le q \le p$, $(n-2)p \le n$, we have $b_i \alpha(x) f(b_1 \phi_1 + b_2 \phi_2) \in L^2(\Omega \times (0, T))$, $i = 1, 2$, so Lemma 2.1 guarantees the existence of a unique solution $u = (u_1, u_2) \in \mathbf{W}$. We would like to show, for M sufficiently large and T sufficiently small, that h is a contraction from

$Z(M,T)$ into itself. For this purpose, we multiply the first and the second equations in (2.8) by u_1' and u_2', respectively and integrate over $\Omega \times (0,T)$ to get

$$
\begin{aligned}
0 \leq & \frac{1}{2}\int_\Omega (u_1'^2 + u_2'^2 + |\nabla u_1|^2 + |\nabla u_2|^2)dx + \frac{1}{2}\int_{\Gamma_1} (a_1 u_1^2 + a_2 u_2^2)(x,t)d\Gamma \\
& + \int_0^t \int_{\Gamma_1} (g_1(u_1')u_1' + g_1(u_2')u_2')(x,t)d\Gamma ds \\
\leq & \frac{1}{2}\int_\Omega [(u_1^1)^2 + (u_2^1)^2 + |\nabla u_1^0|^2 + |\nabla u_2^0|^2]dx + \frac{1}{2}\int_{\Gamma_1} [a_1(u_1^0)^2 + a_2(u_2^0)^2]d\Gamma \\
& + C\int_0^t \int_\Omega (|\phi_1|^p + |\phi_2|^p][|u_1'| + |u_2'|)dxds \quad \forall\, t \in [0,\ T], \qquad (2.9)
\end{aligned}
$$

where C is a generic positive constant depending only on $p, b_1, b_2, d_1, d_2, d_3$, and the L^∞ norm of α. To estimate the last term of (2.9), we note that

$$
\begin{aligned}
\int_\Omega |\phi_1|^p |u_1'| dx &\leq \left(\int_\Omega |\phi_1|^{2p} dx\right)^{1/2} \left(\int_\Omega |u_1'|^2 dx\right)^{1/2} \\
&\leq C||\nabla \phi_1||^p ||u_1'||_{L^2} \leq CM^p ||u_1'||_{L^2} \qquad (2.10)
\end{aligned}
$$

by virtue of (2.6) and (2.7). Therefore (2.9), (2.10), and (H4) yield

$$
\begin{aligned}
||u||_{\mathbf{W}}^2 \leq & CM^pT||u||_{\mathbf{W}} + \frac{1}{2}\int_{\Gamma_1} [a_1(u_1^0)^2 + a_2(u_2^0)^2]d\Gamma \\
& + \frac{1}{2}\int_\Omega [(u_1^1)^2 + (u_2^1)^2 + |\nabla u_1^0|^2 + |\nabla u_2^0|^2]dx.
\end{aligned}
$$

By choosing M large enough and T sufficiently small, we get

$$||u||_{\mathbf{W}}^2 \leq M^2;$$

hence $u \in Z(M,T)$. Next we verify that h is a contraction. To this end we set $U = u - v$ and $\Phi = \phi - \psi$, where $u = h(\phi)$ and $v = h(\psi)$. It is straightforward to verify that U satisfies

$$
\begin{aligned}
U_1'' - \Delta U_1 + b_1\alpha(x)\,[f(b_1\phi_1 + b_2\phi_2) - f(b_1\psi_1 + b_2\psi_2)] &= 0, \quad (x,t) \in \Omega \times \mathbb{R}^+ \\
U_2'' - \Delta U_2 + b_2\alpha(x)\,[f(b_1\phi_1 + b_2\phi_2) - f(b_1\psi_1 + b_2\psi_2)] &= 0, \quad (x,t) \in \Omega \times \mathbb{R}^+ \\
U_1 = U_2 = 0, \qquad (x,t) \in \Gamma_0 \times \mathbb{R}^+ & \qquad (2.11) \\
\frac{\partial U_i}{\partial \nu} + a_i U_i + [g_i(u_i') - g_i(v_i')] = 0,\ i = 1,2, & \qquad (x,t) \in \Gamma_1 \times \mathbb{R}^+ \\
U_i(x,0) = U_i'(x,0) = 0,\ i = 1,2, \quad x \in \Omega. &
\end{aligned}
$$

By multiplying the first and the second equations in (2.11) by U_1' and U_2', respectively, integrating over $\Omega \times (0,\ t)$, adding the resulting equalities, and using the fact that g_1 and g_2 are nondecreasing we arrive at

$$
\begin{aligned}
||U||_{\mathbf{W}}^2 &\leq C\int_0^t \int_\Omega |f(b_1\phi_1 + b_2\phi_2) - f(b_1\psi_1 + b_2\psi_2)|\,|U_1' - U_2'|\,dxds \\
&\leq C\int_0^t (||U_1'||_2 + ||U_2'||_2)||\Phi||_{2n/(n-2)} \qquad (2.12)
\end{aligned}
$$

$$\times\{||\phi_1||^{p-1}_{n(p-1)} + ||\phi_2||^{p-1}_{n(p-1)} + ||\psi_1||^{p-1}_{n(p-1)} + ||\psi_2||^{p-1}_{n(p-1)}\}ds.$$

The Sobolev embedding ($H^1_{\Gamma_0}(\Omega) \subset L^{2n/(n-2)}$, $n > 2$) and condition on p in (H5) give

$$||\Phi||_{2n/(n-2)} \le C||\nabla\Phi||_2,\ ||\phi_i||^{p-1}_{n(p-1)} \le C||\nabla\phi_i||^{p-1}_2, ||\psi_i||^{p-1}_{n(p-1)} \le C||\nabla\psi_i||^{p-1}_2.$$

Thus we have

$$||U||^2_{\mathbf{W}} \le CTM^{p-1}||U||_{\mathbf{W}}||\Phi||_{\mathbf{W}};$$

hence

$$||U||_{\mathbf{W}} \le CTM^{p-1}||\Phi||_{\mathbf{W}}. \tag{2.13}$$

By choosing T so small that $CTM^{p-1} < 1$, estimate (2.13) shows that h is a contraction. The contraction mapping theorem guarantees the existence of a unique u satisfying $u = h(u)$. Obviously it is a solution of (1.1). The uniqueness of this solution follows from the inequality (2.12). The extra regularity (2.5) of the solution can be established in a standard way (see [9] for instance). The proof is completed.

3. Global existence.

In this section, we establish a global existence result. For this aim we set $F(s) = \int_0^s f(\tau)d\tau$

Theorem 3.1. *Assume that (H1) - (H5) and $F(s) \ge 0$ hold. Then, for any initial data $(u_i^0, u_i^1) \in H^1_{\Gamma_0}(\Omega) \times L^2(\Omega), i = 1, 2$, the solution (2.3) is global; i.e*

$$u := (u_1, u_2) \in C\left([0,\ \infty);\ H^1_{\Gamma_0}(\Omega)\right)^2 \cap C^1\left([0,\ \infty);\ L^2(\Omega)\right)^2. \tag{3.1}$$

Proof. To establish (3.1), it suffices to show that the solution (2.3) remains bounded, independently of T, in its space. So we have to prove that there exists a constant K independent of T such that

$$\int_\Omega (u_1'^2 + u_2'^2 + |\nabla u_1|^2 + |\nabla u_2|^2)dx + \int_{\Gamma_1} (a_1 u_1^2 + a_2 u_2^2)d\Gamma \le K, \quad \forall\, t \ge 0. \tag{3.2}$$

To achieve this we only have to multiply the first and the second equations in (1.1) by u_1' and u_2' respectively and integrate over $\Omega \times [0, t]$ to get

$$\frac{1}{2}\int_\Omega (u_1'^2 + u_2'^2 + |\nabla u_1|^2 + |\nabla u_2|^2)dx + \frac{1}{2}\int_{\Gamma_1} (a_1 u_1^2 + a_2 u_2^2)d\Gamma$$

$$+ \int_\Omega \alpha(x)F(b_1u_1 + b_2u_2)dx + \int_0^t \int_{\Gamma_1} (g_1(u_1')u_1' + g_2(u_2')u_2')d\Gamma ds$$

$$\le \frac{1}{2}\int_\Omega \{(u_1^1)^2 + (u_2^1)^2 + |\nabla u_1^0|^2 + |\nabla u_2^0|^2\}dx$$

$$+\frac{1}{2}\int_{\Gamma_1} \{a_1(u_1^0)^2 + a_2(u_2^0)^2\}d\Gamma = K, \quad \forall t \in [0, T]$$

which yields (3.1). The theorem is proved.

4. **Decay of energy.** We define the energy of (1.1) by

$$E(t) = \frac{1}{2}\int_\Omega \{u_1'^2 + u_2'^2 + |\nabla u_1|^2 + |\nabla u_2|^2 + 2\alpha F(b_1u_1 + b_2u_2)\}dx$$

$$+\frac{1}{2}\int_{\Gamma_1}(a_1u_1^2 + a_2u_2^2)d\Gamma \tag{4.1}$$

where F is the function defined in Section 3 satisfying

$$0 \le F(s) \le \frac{1}{2b}sf(s), \quad b > 1 \tag{4.2}$$

Remark 4.1. As an example of such function we can take $f(s) = a|s|^{p-1}s$ with $a \ge 0$ and $p > 1$. Then (4.2) and (H5) are satisfied with $b = (p+1)/2$.

Concerning the functions g_1 and g_2, we assume that there exist two positive constants c_1 and c_2 such that

$$c_1|s| \le |g_i(s)| \le c_2|s|, \qquad i = 1, 2. \tag{4.3}$$

We have the following stabilization result for system (1.1)
Theorem 4.1. *Assume that* $\max\{\|a_1\|_{L^\infty(\Gamma_1)}, \|a_2\|_{L^\infty(\Gamma_1)}\}$ *is small enough and*

$$\sup_\Omega\,(\alpha(x)(b-1)n - (x - x_0).\nabla\alpha(x)) \ge \beta > 0. \tag{4.4}$$

Then there exist two positive constants c and ω independent of t and the initial data, such that the energy (4.1) *satisfies the following decay estimate*

$$E(t) \le ce^{-\omega t}, \qquad \forall t > 0. \tag{4.5}$$

Remark 4.2. This result can be generalized to the internal feedback case.
Remark 4.3. Using the techniques in [8,10,13], we may consider more general non degenerate nonlinearities at the origin and obtain similar exponential and polynomial decay results. To keep this paper short, we restrict ourselves to the case (4.3).
Remark 4.4. If α is constant then (4.4) is always satisfied.
Proof (of Theorem 4.1). We are going to show that, for any $0 \le S < \infty$,

$$\int_S^\infty E(t) \le cE(S) \tag{4.6}$$

Here and in what follows, c denotes a generic positive constant and ε denotes diverse positive "small enough" constants. The inequality (4.6) gives (4.5) thanks to the following
Lemma 4.2: ([14], Theorem 9.1). *Let E: $\mathbb{R}^+ \longrightarrow \mathbb{R}^+$ be a nonincreasing absolutely continuous function satisfying* (4.5). *Then E satisfies* (4.4)

Thanks to the fact that g_1, g_2 are nondecreasing, a simple computation yields

$$E'(t) = -\int_{\Gamma_1}(u_1'g_1(u_1') + u_2'g_2(u_2'))d\Gamma \le 0; \tag{4.7}$$

hence E is nonincreasing.

To prove (4.6), we denote by $m = x - x_0$ and multiply the first two equations in (1.1) by

$$2m \cdot \nabla u_i + (n - \varepsilon_0)u_i, \ i = 1, 2,$$

respectively, where $\varepsilon_0 \in]0, 1]$ will be chosen later. By integrating over $\Omega \times [S, T]$, we obtain

$$I_1 + I_2 + J_1 + J_2 + L = 0, \tag{4.8}$$

where

$$I_i := \int_S^T \int_\Omega u_i''(2m \cdot \nabla u_i + (n - \varepsilon_0)u_i)dxdt, \quad i = 1, 2,$$

$$J_i := \int_S^T \int_\Omega u(-\Delta u_i)(2m \cdot \nabla u_i + (n - \varepsilon_0)u_i)dxdt, \quad i = 1, 2$$

and

$$L := \int_S^T \int_\Omega \alpha f(b_1u_1+b_2u_2)[b_1(2m\cdot\nabla u_1+(n-\varepsilon_0)u_1)+b_2(2m\cdot\nabla u_2+(n-\varepsilon_0)u_2)]dxdt.$$

Using the boundary conditions, we estimate the above integrals as follows

$$I_i := \int_S^T \int_\Omega u_i''(2m \cdot \nabla u_i + (n - \varepsilon_0)u_i)dxdt$$

$$= \left[\int_\Omega u_i'(2m \cdot \nabla u_i + (n - \varepsilon_0)u_i)dx\right]_{t=S}^{t=T} - \int_S^T \int_\Omega (m \cdot \nabla(u_i')^2 + (n - \varepsilon_0) \left|u_i'\right|^2)dxdt$$

$$= \varepsilon_0 \int_S^T \int_\Omega \left|u_i'\right|^2 dxdt - \int_S^T \int_{\Gamma_1} (m \cdot \nu) \left|u_i'\right|^2 d\Gamma dt$$

$$+ \left[\int_\Omega u_i'(2m \cdot \nabla u_i + (n - \varepsilon_0)u_i)dx\right]_{t=S}^{t=T}.$$

We estimate the last term in this inequality in the following manner

$$\int_\Omega (2m \cdot \nabla u_i + (n - \varepsilon_0)u_i)^2 dx - \int_\Omega (2m \cdot \nabla u_i)^2 dx$$

$$= \int_\Omega \left((n - \varepsilon_0)^2 |u_i|^2 + 2(n - \varepsilon_0)m \cdot \nabla(u_i)^2\right) dx$$

$$= \int_\Omega \left((n - \varepsilon_0)^2 |u_i|^2 - 2(n - \varepsilon_0)n |u_i|^2\right) dx + 2(n - \varepsilon_0) \int_{\Gamma_1} (m \cdot \nu) |u_i|^2 d\Gamma$$

$$= (\varepsilon_0 + n)(\varepsilon_0 - n) \int_\Omega |u_i|^2 dx + 2(n - \varepsilon_0) \int_{\Gamma_1} (m \cdot \nu) |u_i|^2 d\Gamma$$

$$\leq 2(n - \varepsilon_0)R \int_{\Gamma_1} |u_i|^2 d\Gamma$$

where $R = \|m\|_{L^\infty(\Omega)}$. Hence

$$\int_\Omega (2m \cdot \nabla u_i + (n - \varepsilon_0)u_i)^2 dx \leq \int_\Omega (2m \cdot \nabla u_i)^2 dx + 2(n - \varepsilon_0)R \int_{\Gamma_1} |u_i|^2 d\Gamma. \tag{4.9}$$

We then use Young's inequality and (4.10) to get, for any $\varepsilon > 0$,

$$\begin{aligned}
&\left|\int_\Omega u_i'(2m\cdot\nabla u_i + (n-\varepsilon_0)u_i)dx\right| \\
\leq\ & \frac{\varepsilon}{2}\int_\Omega |u_i'|^2\,dx + \frac{1}{2\varepsilon}\left(\int_\Omega (2m.\nabla u_i)^2 dx + 2(n-\varepsilon_0)R\int_{\Gamma_1}|u_i|^2\,d\Gamma\right) \\
\leq\ & \int_\Omega\left(\frac{\varepsilon}{2}|u_i'|^2 + \frac{2R^2}{\varepsilon}|\nabla u_i|^2\right)dx + \frac{R}{\varepsilon}(n-\varepsilon_0)\,\bar{c}\int_\Omega |\nabla u_i|^2 dx
\end{aligned}$$

where $\bar{c}$ is the smallest positive constant satisfying

$$\int_{\Gamma_1}|v|^2\,d\Gamma \leq \bar{c}\int_\Omega |\nabla v|^2 dx, \quad \forall v\in H^1_{\Gamma_0}(\Omega).$$

By choosing

$$\varepsilon = 2\sqrt{R\left[R+\frac{\bar{c}}{2}(n-\varepsilon_0)\right]}$$

we obtain

$$\left|\int_\Omega u_i'(2m\cdot\nabla u_i + (n-\varepsilon_0)u_i)dx\right| \leq 2\sqrt{R\left[R+\frac{\bar{c}}{2}(n-\varepsilon_0)\right]}E(t) = cE(t).$$

Therefore we arrive, using the fact that E is nonincreasing, at

$$I_i \geq -cE(S) - R\int_S^T\int_{\Gamma_1}\left|u_i'\right|^2 d\Gamma dt + \varepsilon_0\int_S^T\int_\Omega\left|u_i'\right|^2 dxdt. \tag{4.10}$$

In the other hand, by the generalized Green formula and the identities

$$\begin{aligned}
2\nabla u_i.\nabla\,(m\cdot\nabla u_i) &= 2|\nabla u_i|^2 + m.\nabla\left(|\nabla u_i|^2\right) \\
\nabla u_i &= (\partial_\nu u_i)\,\nu, \quad on\ \Gamma_0,
\end{aligned}$$

we infer

$$\begin{aligned}
J_i &= -\int_S^T\int_\Omega(\Delta u_i)\,(2m\cdot\nabla u_i + (n-\varepsilon_0)u_i)\,dxdt \\
&= (2-\varepsilon_0)\int_S^T\int_\Omega|\nabla u_i|^2 dxdt - 2\int_S^T\int_{\Gamma_0}(m.\nu)|\nabla u_i|^2 d\Gamma dt \\
&\quad + \int_S^T\int_{\Gamma_1}\left((m.\nu)|\nabla u_i|^2 - (n-\varepsilon_0)u_i\partial_\nu u_i - 2(m.\nabla u_i)\partial_\nu u_i\right)d\Gamma dt.
\end{aligned}$$

Thanks to (H1), we have $m.\nu \geq \delta > 0$ on Γ_1, then, using the definitions of Γ_0 and Γ_1, we deduce that

$$J_i \geq (2-\varepsilon_0)\int_S^T\int_\Omega|\nabla u_i|^2 dxdt$$

$$+\int_S^T\int_{\Gamma_1}\left(\delta|\nabla u_i|^2 - (n-\varepsilon_0)u_i\partial_\nu u_i - \delta|\nabla u_i|^2 - \frac{R^2}{\delta}\left(\partial_\nu u_i\right)^2\right)d\Gamma dt$$

so

$$J_i \geq (2-\varepsilon_0)\int_S^T\int_\Omega |\nabla u_i|^2 dxdt - \int_S^T\int_{\Gamma_1}\left((n-\varepsilon_0)u_i\partial_\nu u_i + \frac{R^2}{\delta}(\partial_\nu u_i)^2\right) d\Gamma dt. \tag{4.11}$$

Next we exploit (4.2) and the fact that $F(0)=0$ to estimate

$$L =$$

$$= \int_S^T\int_\Omega \alpha f(b_1u_1+b_2u_2)\left[b_1\{2m.\nabla u_1 + (n-\varepsilon_0)u_1 + b_2(2m.\nabla u_2 + (n-\varepsilon_0)u_2\}\right] dxdt$$

$$\geq (n-\varepsilon_0)b\int_S^T\int_\Omega 2\alpha F(b_1u_1+b_2u_2)dxdt + \int_S^T\int_\Omega 2\alpha m.\nabla\left(F(b_1u_1+b_2u_2)\right)dxdt$$

$$\geq [(n-\varepsilon_0)b-n]\int_S^T\int_\Omega 2\alpha F(b_1u_1+b_2u_2)dxdt - \int_S^T\int_\Omega (2m.\nabla\alpha)F(b_1u_1+b_2u_2)dxdt$$

$$+\int_S^T\int_{\Gamma_1} 2\alpha(x)(m.\nu)F(b_1u_1+b_2u_2)d\Gamma dt$$

then, since $m.\nu \geq 0$ on Γ_1, we conclude that

$$L \geq 2\int_S^T\int_\Omega (\beta - b\varepsilon_0\alpha)F(b_1u_1+b_2u_2)dxdt, \tag{4.12}$$

where β is given in (4.4). Combining (4.10) - (4.12) and taking in account the fact that $I_1+I_2+J_1+J_2+L=0$, we arrive at

$$\int_S^T\int_\Omega \Big[\varepsilon_0(|u_1'|^2+|u_2'|^2) + (2-\varepsilon_0)(|\nabla u_1|^2+|\nabla u_2|^2)$$

$$+2(\beta-b\varepsilon_0\alpha)F(b_1u_1+b_2u_2)\Big]dxdt \leq cE(S) \tag{4.13}$$

$$+\int_S^T\int_{\Gamma_1}\left[R(|u_1'|^2+|u_2'|^2) + (n-\varepsilon_0)(u_1\partial_\nu u_1 + u_2\partial_\nu u_2) + \frac{R^2}{\delta}[(\partial_\nu u_1)^2 + (\partial_\nu u_2)^2]\right]$$

$$d\Gamma dt.$$

By recalling the boundary conditions and using Young's inequality, we have

$$(n-\varepsilon_0)u_i\partial_\nu u_i + \frac{R^2}{\delta}(\partial_\nu u_i)^2 = -(n-\varepsilon_0)u_i\left(a_iu_i+g_i(u_i')\right) + \frac{R^2}{\delta}\left(a_iu_i+g_i(u_i')\right)^2$$

$$= (\frac{R^2}{\delta}a_i + \varepsilon_0 - n)a_i|u_i|^2 + (\frac{2R^2}{\delta}a_i+\varepsilon_0-n)u_ig_i(u_i') + \frac{R^2}{\delta}\left(g_i(u_i')\right)^2 \tag{4.14}$$

$$\leq (\frac{R^2}{\delta}a+\varepsilon_0-n)a_i|u_i|^2 + \varepsilon_0a_i|u_i|^2 + c\left(g_i(u_i')\right)^2 = (\frac{R^2}{\delta}a+2\varepsilon_0-n)a_i|u_i|^2 + c\left(g_i(u_i')\right)^2$$

where

$$a = \max\{||a_1||_{L^\infty(\Gamma_1)}, ||a_2||_{L^\infty(\Gamma_1)}\}.$$

By inserting (4.14) in (4.13), we obtain

$$\int_S^T\int_\Omega \Big[\varepsilon_0(|u_1'|^2+|u_2'|^2) + (2-\varepsilon_0)(|\nabla u_1|^2+|\nabla u_2|^2)$$

$$+2(\beta-b\varepsilon_0\alpha)F(b_1u_1+b_2u_2)\Big]dxdt \tag{4.15}$$

$$+\int_S^T\int_{\Gamma_1}[n-(\frac{R^2}{\delta}a+2\varepsilon_0)](a_1|u_1|^2+a_2|u_2|^2)d\Gamma dt \le cE(S)$$

$$+\int_S^T\int_{\Gamma_1}\left[R(|u_1'|^2+|u_2'|^2)+c\left(g_1(u_1')\right)^2+c\left(g_2(u_2')\right)^2\right]d\Gamma dt.$$

If a is small enough so that $a<\delta n/R^2$ we then choose $\varepsilon_0>0$ so small that

$$2\varepsilon_0<n-\frac{R^2}{\delta}a, \qquad \beta-b\varepsilon_0\alpha>0.$$

Consequently (4.15) becomes

$$\int_S^T E(t)dt \le cE(S)+c\int_S^T\int_{\Gamma_1}\left(|u_1'|^2+|u_2'|^2+g_1^2(u_1')+g_2^2(u_2')\right)d\Gamma dt. \tag{4.16}$$

Finally, using (4.3) and (4.7), we arrive at

$$\int_S^T\int_{\Gamma_1}\left(|u_i'|^2+g_i^2(u_i')\right)d\Gamma dt \le c\int_S^T\int_{\Gamma_1}u_i'g_i(u_i')d\Gamma dt \le -c\int_S^T E'(t)dt \le cE(S).$$

Hence (4.15) yields $\int_S^T E(t)dt \le cE(S)$. By letting T go to ∞, we arrive at (4.6). This completes the proof of Theorem 4.1.

Remark 4.5. Following the proof carefully, it is easy to notice, from (4.15), that $\sup_\Omega\left(\alpha(x)(b-1)n-(x-x_0).\nabla\alpha(x)\right)$ and $n-\frac{R^2}{\delta}a$ can be taken negative and bounded below by constants depending on the initial data and Ω. In this case the third and the fourth terms in the LHS of (4.15) will be majorized by the second one, using the embedding and Poincaré's inequality.

5. **Applications.** In [10], the first author considered the following coupled (Wave-Petrovsky) system:

$$\begin{cases} u_1''+\Delta^2u_1+\alpha(x)u_2+g_1(u_1')=0, & in\ \Omega\times\mathbb{R}^+\\ u_2''-\Delta u_2+\alpha(x)u_1+g_1(u_2')=0, & in\ \Omega\times\mathbb{R}^+\\ \frac{\partial u_1}{\partial\nu}=u_1=u_2=0, & on\ \Gamma\times\mathbb{R}^+\\ u_i(x,0)=u_i^0(x),\quad u_i'(x,0)=u_i^1(x), \quad i=1,2, & in\ \ \Omega \end{cases} \tag{5.1}$$

where Ω is a bounded domain in $\mathbb{R}^n$, with a smooth boundary Γ and ν is the outward unit normal vector to Γ. For g_1 and g_2 continuous, increasing, satisfying $g_1(0)=g_2(0)=0$, and $\alpha:\Omega\to\mathbb{R}$ a bounded function, he proved a global existence and a regularity result. He also established, under suitable growth conditions on g_1 and g_2, decay results for weak, as well as strong, solutions. Precisely, he showed that the solution decays exponentially if g_1 and g_2 behaves like a linear function, whereas the decay is of a polynomial order otherwise. Using the method developed in this paper, we can extend these results to the more general coupled system:

$$\begin{cases} u_1'' + \Delta^2 u_1 + b_1\alpha(x) f(b_1u_1 + b_2u_2) + g_1(u_1') = 0, & in \quad \Omega \times \mathbb{R}^+ \\ u_2'' - \Delta u_2 + b_2\alpha(x) f(b_1u_1 + b_2u_2) = 0, & in \quad \Omega \times \mathbb{R}^+ \\ \frac{\partial u_1}{\partial \nu} = u_1 = 0, & on \quad \Gamma \times \mathbb{R}^+ \\ u_2 = 0, & on \quad \Gamma_0 \times \mathbb{R}^+ \\ \frac{\partial u_2}{\partial \nu} + a_2u_2 + g_1(u_2') = 0, & on \quad \Gamma_1 \times \mathbb{R}^+ \\ u_i(x,0) = u_i^0(x), \quad u_i'(x,0) = u_i^1(x), \quad i = 1, 2, & in \quad \Omega \end{cases} \tag{5.2}$$

with the same hypotheses (H1) - (H5) above. In this case we define the energy of (5.2) by

$$E(t) := \frac{1}{2}\int_\Omega \left(u_1'^2 + u_2'^2 + (\Delta u_1)^2 + |\nabla u_2|^2 + 2\alpha F(b_1u_1 + b_2u_2)\right) dx + \frac{1}{2}\int_{\Gamma_1} a_2 u_2^2 d\Gamma$$

and we obtain similar results to the one in Theorem 4.1. The method developed in this paper is direct and very flexible; it can be applied to various coupled systems of PDE's (Wave-Wave, Petrovsky-Petrovsky, Wave-Petrovsky) with internal or boundary feedbacks.

Open questions. The main restrictive assumption under which the stability result is valid is the particular coupling function represented by $f(b_1u_1 + b_2u_2)$, this choice is in order to define an appropriate energy to our systems. It would be very interesting (in particular from the point of view of applications) to explore more general coupling functions $f(u_1, u_2)$, where f is a two-variable function and to know if our system can be polynomially stable by only one control; that is $g_1 \equiv 0$ (or $g_2 \equiv 0$). In the negative-answer case, one might be interested in knowing if other weaker stability estimates can be proved using more sophisticated tools as general multipliers. In the case where f is linear, F. Alabau-Boussouira [3] proved that this system can not be exponentially stable by only one control, and proved some polynomial decay estimates in the case of linear feedback. Another good open question concerns the stability of our system in the situation, where the two controls are not defined on the same boundary region. This is a challenging question and, at the best of our knowledge, no result exists concerning stability of coupled hyperbolic equations defined in the same domain with different regions of controls.

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6. References.

1. M. Aassila, A note on the boundary stabilization of a compactly coupled system of wave equations, *Appl. Math. Letters* **12** (1999), 19-24.
2. M. Aassila, Strong asymptotic stability of a compactly coupled system of wave equations, *Appl. Math. Letters* **12** (2001), 285-290.
3. F. Alabau-Boussouira, Stabilisation frontière indirecte de systèmes linéaires hyperboliques faiblement couplés, *SICON,* **41** no **2** (2002), 511-541.

4. M. Asch, G. Lebeau, Geometrical aspects of exact boundary controllability for the wave equation—a numerical study, *ESAIM Control Optim. Calc. Var.* **3** (1998), 163–212.
5. C. Bardos, G. Lebeau and J. Rauch, Sharp sufficient conditions for the observation, control, and stabilization of waves from the boundary, *SIAM J. control Optim.*, **30** no. **5** (1992), 1024-1065.
6. R. Bey, A. Heminna and J. P. Lohéac, Stabilisation frontière du système de l'élasticité. Nouvelle approche, *C. R. Acad. Sci. Paris Sér. I Math.* **330** no. **7** (2000), 563–566.
7. N. Burq, Contrôle de l'équation des ondes dans des ouverts comportant des coins, *Bull. Soc. Math. France* **126** no. **4** (1998), 601–637.
8. F. Conrad and B. Rao, Decay of solutions of wave equations in a star-shaped domain with nonlinear boundary feedback, *Asymptotic Anal.* **7** (1993), 159-177.
9. C. M. Dafermos and W. J. Hrusa, Energy methods for quasilinear hyperbolic initial-boundary value problems. Applications to elastodynamics. *Arch. Rational Mech. Anal.* **87** no. **3** (1985), 267–292.
10. A. Guesmia, Energy decay for a damped nonlinear coupled system, *J. Math. Anal. Appl.* **239** (1999), 38-48.
11. A. Haraux and E. Zuazua, Decay estimates for some damped hyperbolic equations, *Arch. Rat. Mech. Anal.* **100** no. **2** (1988), 191-206.
12. V. Komornik and E. Zuazua, A direct method for the boundary stabilization of the wave equation, *J. Math. Pure Appl.* **69** (1990), 33-54.
13. V. Komornik, On the nonlinear boundary stabilization of the wave equation, *Chen. Ann. of Math.* **14B** no. **2** (1993), 153-164.
14. V. Komornik, Exact Controllability and Stabilization: The multiplier method, *Masson, Paris* (1994).
15. V. Komornik and B. Rao, Boundary stabilization of compactly coupled wave eqautions, *Asympt. Anal.* **14** (1997), 339-359.
16. J. Lagnese, Boundary stabilization of thin plates, *SIAM Studies in Applied Math.*, Vol. **10** *SIAM* 1989.
17. J. Lasiecka, Stabilization of wave and plate-like equations with nonlinear dissipation on the boundary, *J. Differential Equations,* Vol. **79** (1989), 340-381.
18. I. Lasiecka and D. Tataru, Uniform boundary stabilization of semilinear wave equation with nonlinear boundary damping, *Differential & Integral Equations* **6** no. **3** (1993), 507-533.
19. J. L. Lions, Quelques méthodes de résolution des problèmes aux limites nonlinéaires, *Dunod Gautier-Villars, Paris,* 1969.
20. J. L. Lions, Exact contollability, stabilizations and perturbations of distributed systems, *SIAM Rev.* **30** (1988), 1-68.
21. E. Zuazua, Uniform stabilization of the wave equation by nonlinear boundary feedback, *SIAM J. Control and Opt.*, **28** (1990), 466-477.
22. E. Zuazua, Stability and decay for a class of nonlinear hyperbolic problems, *Asympt. Anal.* **1** no. **2** (1988), 1-28.

23. E. Zuazua and W. J. Liu, Decay rates for dissipative wave equations, *Ricerche di Matematica* **XLVIII** (1999), 61-75.