GLOBAL INFINITE ENERGY SOLUTIONS FOR THE CUBIC WAVE EQUATION

by

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Abstract. — We prove the existence of infinite energy global solutions of the cubic wave equation in dimension greater than 3. The data is a typical element on the support of suitable probability measures.

Résumé. — On considère l'équation des ondes cubique sur un tore de dimension supérieure à 3, et on montre l'existence de solutions globales d'énergie infinie. La condition initiale de l'équation est un élément typique du support d'une mesure de probabilité.

1. Introduction

This paper is a higher dimensional sequel of the recent article [8] by the first and the third authors (and also of [9, 10, 4]). As such it aims to construct global in time solutions of the cubic wave equation with low regularity (infinite energy) random initial data. To the best of our knowledge such a regularity is out of reach of the present deterministic methods. The major difference between the present paper and [8] is that here we only establish existence results and in particular no uniqueness statement is proven. Let us recall that in [8] a suitable uniqueness and a probabilistic continuity of the flow were proven. This result was followed by more recent results by Nahmod-Pavlovic-Staffilani [15] on the 2 and 3-dimensional homogeneous Navier-Stokes equation, where the authors obtain strong (in 2-d) and weak (in 3-d) results, and in turn, here we are inspired by this latter 3-d weak-existence result. Related weak-existence results had been already used in the context of the randomly forced Navier-Stokes equation by Da Prato-Debussche [12] and the Euler equation by Albeverio-Cruzeiro [1],

using more sophisticated probabilistic tools (Prokhorov and Skorohod Theorems). This approach may be seen as the analogue in the random setting of the Leray compactness method for constructing solutions of nonlinear evolution equations. It has the advantage to require less regularity on the initial data, one allows infinite energy while the Leray method requires finite energy of the data. It should however be emphasised that as in the Leray method our approach still makes a crucial use of the energy functional. In this paper we will only need an invariance property for the linear evolution combined with large deviation estimates on the nonlinear part which are much easier to achieve than the invariance properties as in [**12, 1**]. Let us now describe our model. Let $d \geq 4$ and consider the cubic wave equation on the torus $\mathbb{T}^d = (\mathbb{R}/2\pi\mathbb{Z})^d$

$$(1.1) \qquad \begin{cases} \partial_t^2 u - \mathbf{\Delta} u + u^3 = 0, & (t, x) \in \mathbb{R} \times \mathbb{T}^d, \\ (u, \partial_t u)(0, \cdot) = (u_0, u_1) \in \mathcal{H}^s, \end{cases}$$

where $\mathbf{\Delta} := \mathbf{\Delta}_{\mathbb{T}^d}$ is the Laplace operator and

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{T}^d) := H^s(\mathbb{T}^d) \times H^{s-1}(\mathbb{T}^d).$$

Denote by $s_c = (d-2)/2$ the critical (scaling) Sobolev index for (1.1). Then one can show that (1.1) is well-posed in $\mathcal{H}^s$ for $s > s_c$ ([**13**]) and ill-posed when $s < s_c$ ([**13, 11, 14**]). See the introduction of [**8**] for more details. The energy of (1.1) reads

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{T}^d} \left(|\nabla u|^2 + (\partial_t u)^2 \right) + \frac{1}{4} \int_{\mathbb{T}^d} u^4,$$

thus with deterministic compactness methods due to Leray (see *e.g.* Lebeau [**14**, Section 6] for the application of the method in the context of (1.1)), we can construct global weak solutions to (1.1) so that

$$(u, \partial_t u) \in \mathcal{C}_w\left(\mathbb{R}; H^1(\mathbb{T}^d) \cap L^4(\mathbb{T}^d)\right) \times \mathcal{C}_w\left(\mathbb{R}; L^2(\mathbb{T}^d)\right),$$

(here $\mathcal{C}_w$ means weak continuity in time) and $\mathcal{E}(u)(t) \leq \mathcal{E}(u)(0)$ for all $t \in \mathbb{R}$. Observe that for $d > 4$ one has $1 < s_c$ and thus the construction of weak solutions works for data of supercritical regularity with respect to the scaling of the equation. However it requires finite energy of the initial data. The main goal of this paper is to show that weak solutions still exist for infinite energy, almost surely with respect to a large class of probability measures.

Let us now describe precisely the initial data sets (statistical ensembles) that we shall consider in this article. Here we follow [**8**]. Let $0 < s < 1$ and let $(v_0, v_1) \in \mathcal{H}^s$ with Fourier series

$$v_j(x) = a_j + \sum_{n \in \mathbb{Z}_\star^d} \left(b_{n,j} \cos(n \cdot x) + c_{n,j} \sin(n \cdot x) \right), \quad j = 0, 1,$$

where $\mathbb{Z}_\star^d = \mathbb{Z}^d \backslash \{0\}$. Then let $\big(\alpha_j(\omega), \beta_{n,j}(\omega), \gamma_{n,j}(\omega)\big)$, $n \in \mathbb{Z}_\star^d$, $j = 0, 1$ be a sequence of independent real random variables given on a probability space $(\Omega, \mathcal{F}, \mathbf{p})$ with a joint distribution θ satisfying

$$\exists\, c > 0, \quad \forall\, \gamma \in \mathbb{R}, \quad \int_{-\infty}^{\infty} e^{\gamma x} d\theta(x) \leq e^{c\gamma^2}.$$

We then define the random variables v_j^ω by

$$v_j^\omega(x) = \alpha_j(\omega) a_j + \sum_{n \in \mathbb{Z}_\star^d} \big(\beta_{n,j}(\omega) b_{n,j} \cos(n \cdot x) + \gamma_{n,j}(\omega) c_{n,j} \sin(n \cdot x)\big),$$

and we define the measure $\mu_{(v_0, v_1)}$ on $\mathcal{H}^s$ as the image of $\mathbf{p}$ under the map

$$\omega \longmapsto (v_0^\omega, v_1^\omega) \in \mathcal{H}^s.$$

We then define $\mathcal{M}^s$ by

$$\mathcal{M}^s = \bigcup_{(v_0, v_1) \in \mathcal{H}^s} \big\{\mu_{(v_0, v_1)}\big\}.$$

For $(u_0, u_1) \in \mathcal{H}^s$, denote by

$$(1.2) \qquad S(t)(u_0, u_1) = \cos\big(t\sqrt{-\mathbf{\Delta}}\,\big)(u_0) + \frac{\sin\big(t\sqrt{-\mathbf{\Delta}}\,\big)}{\sqrt{-\mathbf{\Delta}}}(u_1),$$

the free wave evolution. Then our result reads

Theorem 1.1. — *Let $d \geq 4$, $0 < s < 1$ and $\mu = \mu_{(v_0, v_1)} \in \mathcal{M}^s$. Then there exists a set Σ of full μ measure so that for every $(u_0, u_1) \in \Sigma \subset \mathcal{H}^s$ the equation (1.1) with initial condition $(u(0), \partial_t u(0)) = (u_0, u_1)$ has a solution*

$$u(t) = S(t)(u_0, u_1) + w(t),$$

where for any $\varepsilon > 0$

$$\big(w, \partial_t w\big) \in \mathcal{C}\big(\mathbb{R}; H^{1-\varepsilon}(\mathbb{T}^d) \times H^{-\varepsilon}(\mathbb{T}^d)\big).$$

Moreover, for all $t \in \mathbb{R}$

$$\|(w(t), \partial_t w(t))\|_{\mathcal{H}^1(\mathbb{T}^d)} \leq C(M + |t|)^{\frac{1-s}{s} + \varepsilon},$$

$$\|w(t)\|_{L^4(\mathbb{T}^d)} \leq C(M + |t|)^{\frac{1-s}{2s} + \varepsilon},$$

with $\mu(M > \lambda) \leq Ce^{-\lambda^\delta}$ for some $\delta > 0$.

Remark 1.2. — Let us recall (see **[8]**) that if the measure $\mu \in \mathcal{M}^s$ is constructed using data $(v_0, v_1) \in \mathcal{H}^s(\mathbb{T}^d)$, then $\mu(\mathcal{H}^s) = 1$, while if for some $s < \sigma$, we have $(v_0, v_1) \notin \mathcal{H}^\sigma(\mathbb{T}^d)$, then as soon as the random variables $(\alpha_j, \beta_{n,j}, \gamma_{n,j})$ do not accumulate at 0 (for example, in the case where they are non trivial and identically distributed, then

$\mu(\mathcal{H}^\sigma) = 0$. On the other hand, under rather weak assumptions, $\mu(B^s) > 0$ for any non empty open ball $B^s \subset \mathcal{H}^s$ (see [8, Proposition 1.2]).

Let us now mention two possible extensions of our result. In the case $d = 4$ one may expect to get uniqueness by combining the analysis of [8] with the critical H^1 theory for (1.1). One may also expect to include the case $s = 0$ by elaborating on the arguments developed in [8] to treat this case. It is not clear to us what happens for $s < 0$ (and in [8] as well). In particular we do not know whether $s = 0$ is the optimal regularity one may achieve by our approach. Invariant Gibbs measures for dispersive equations were extensively studied (see *e.g.* [20, 3, 2, 19, 18, 16, 17, 6]). In these papers the Gibbs measure is combined with a suitable local in time result (which can sometimes be quite involved) to get global existence and uniqueness on the support of the measure. By an extension of the method (using in particular Skorohod and Prokhorov theorems) we use in this paper one may construct a dynamics (without any uniqueness) on the support of a Gibbs measure and prove its invariance. We plan to give several relevant examples of this observation in [7]. We however do not see how to make work such an approach in the context of (1.1). Indeed, the present methods of renormalization of Gibbs measures are restricted to dimensions ≤ 2 (see [3]). Let us also recall that as mentioned above a global existence based on Gibbs measures only works for a very specific choice of the initial distribution. On the other hand, it has of course the advantage to give a quite remarkable dynamical property of the flow.

The rest of the paper is organised as follows. In Section 2 we recall stochastic properties of the linear flow which were proven in [8]. In Section 3 we study the dynamics of an approximation of (1.1). Section 4 is devoted to the proof of Theorem 1.1.

2. Stochastic estimates on the linear flow

Once for all we fix $0 < s < 1$ and $\mu = \mu_{(v_0,v_1)} \in \mathcal{M}^s$. Recall the definition (1.2) of the linear wave propagator $S(t)$. In this section we prove estimates which reflect the invariance of μ under $S(t)$. This is the only measure invariance aspect used in this paper.

2.1. The projectors. — Denote by $\mathbb{Z}_\star^d = \mathbb{Z}^d \backslash \{0\}$. For a Fourier series u

$$u(x) = a + \sum_{n \in \mathbb{Z}_\star^d} \big(b_n \cos(n \cdot x) + c_n \sin(n \cdot x)\big),$$

we denote by $\Pi_0(u) = a$ and for $N \geq 1$

$$\Pi_N(u) = a + \sum_{1 \leq |n| \leq N} \left(b_n \cos(n \cdot x) + c_n \sin(n \cdot x)\right) \quad \text{and} \quad \Pi^N = 1 - \Pi_N.$$

Let $\chi \in \mathcal{C}_0^\infty(-1, 1)$, so that $\chi \equiv 1$ on $(-1/2, 1/2)$. Let us also introduce the smooth spectral projector

$$S_N(u) \equiv \chi(-N^{-2}\boldsymbol{\Delta}) = a + \sum_{n \in \mathbb{Z}_\star^d} \chi\left(\frac{|n|^2}{N^2}\right) \left(b_n \cos(n \cdot x) + c_n \sin(n \cdot x)\right),$$

which will be needed in the next section. This operator has the following property (see e.g. [5] for a proof).

Lemma 2.1. — *Let M be a compact Riemannian manifold. Let $\boldsymbol{\Delta}$ be the Laplace-Beltrami operator on M. Let $1 \leq p \leq \infty$ and denote by $L^p = L^p(M)$. Then $S_N = \chi(-N^{-2}\boldsymbol{\Delta}) : L^p \longrightarrow L^p$ is continuous and there exists $C > 0$ so that for every $N \geq 1$,*

$$\|S_N\|_{L^p \to L^p} \leq C.$$

Moreover, for all $f \in L^p$, $S_N f \longrightarrow f$ in L^p, when $N \longrightarrow +\infty$.

2.2. The estimates. — Following [8], we introduce the following sets for

$$\delta > 1/2, \; \widetilde{\delta} > 1/3, \; \check{\delta} > 0, \; \varepsilon > 0$$

$$
\begin{aligned}
F_M &= \left\{(u_0, u_1) \; : \; \|\Pi_M(u_0, u_1)\|_{\mathcal{H}^1(\mathbb{T}^d)} \leq M^{1-s+\varepsilon}\right\}, \\
G_M &= \left\{(u_0, u_1) : \; \|\Pi_M(u_0)\|_{L^4(\mathbb{T}^d)} \leq M^\varepsilon\right\}, \\
H_M &= \left\{(u_0, u_1) : \; \|\langle t\rangle^{-\delta} S(t)(\Pi^M(u_0, u_1))\|_{L^2(\mathbb{R}_t; L^\infty(\mathbb{T}^d))} \leq M^{\varepsilon-s}\right\} \\
K_M &= \left\{(u_0, u_1) : \; \|\langle t\rangle^{-\widetilde{\delta}} S(t)(\Pi^M(u_0, u_1))\|_{L^3(\mathbb{R}_t; L^6(\mathbb{T}^d))} \leq M^{\varepsilon-s}\right\} \\
R_M &= \left\{(u_0, u_1) \; : \; \|\langle t\rangle^{-\check{\delta}} S(t)\Pi^M(u_0, u_1)\|_{L^\infty(\mathbb{R}; L^4(\mathbb{T}^d))} \leq M^{\varepsilon-s}\right\},
\end{aligned}
$$

and $E_M = F_M \cap G_M \cap H_M \cap K_M \cap R_M$. Then the following result holds true.

Lemma 2.2. — *For any $\varepsilon > 0$, there exists $\varepsilon_0 > 0$ such that there exist $C, c > 0$ such that for every $M \geq 1$*

$$\mu(F_M^c) \leq Ce^{-cM^{2\varepsilon_0}}, \quad \mu(G_M^c) \leq Ce^{-cM^{2\varepsilon_0}},$$

$$\mu(H_M^c) \leq Ce^{-cM^{2\varepsilon_0}}, \quad \mu(K_M^c) \leq Ce^{-cM^{2\varepsilon_0}}, \quad \mu(R_M^c) \leq Ce^{-cM^{2\varepsilon_0}}.$$

Proof. — This result is very close to [**8**, Lemma 4.2]. Indeed, the only new point is the bound on the measure of R_M, whose proof follows the same lines as the proof of the bound on K_M, once we notice that by (1-d) Sobolev injection, with p sufficiently large and such that $\check{\delta} > \frac{1}{p}$, $\sigma > \frac{1}{p}$, $\sigma < s$,

$$\text{(2.1)} \quad \|\langle t \rangle^{-\check{\delta}} S(t) \Pi^M(u_0, u_1)\|_{L^\infty(\mathbb{R};L^4(\mathbb{T}^d))}$$
$$\leq C \|(1 + |D_t|)^\sigma \langle t \rangle^{-\check{\delta}} S(t) \Pi^M(u_0, u_1)\|_{L^p(\mathbb{R};L^4(\mathbb{T}^d))}$$
$$\leq C' \|\langle t \rangle^{-\check{\delta}} (1 + |D_t|)^\sigma S(t) \Pi^M(u_0, u_1)\|_{L^p(\mathbb{R};L^4(\mathbb{T}^d))}$$
$$\leq C' \|\langle t \rangle^{-\check{\delta}} (1 + |D_x|)^\sigma S(t) \Pi^M(u_0, u_1)\|_{L^p(\mathbb{R};L^4(\mathbb{T}^d))}.$$

$\square$

3. Uniform bounds on the Sobolev norms, $s > 0$

For $N \gg 1$ we consider the following truncation of (1.1)

$$\text{(3.1)} \quad \begin{cases} \partial_t^2 u_N - \boldsymbol{\Delta} u_N + S_N\big((S_N u_N)^3\big) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{T}^d, \\ (u_N, \partial_t u_N)(0, \cdot) = (u_0, u_1) \in \mathcal{H}^s. \end{cases}$$

In fact, equation (3.1) is an ODE in low frequencies, and is the linear wave equation in high frequencies. Indeed, if K is large enough so that $\Pi_K S_N = S_N$, then the equation (3.1) is equivalent to the uncoupled system

$$\begin{cases} \partial_t^2 \Pi_K u_N - \boldsymbol{\Delta} \Pi_K u_N + S_N\big((S_N u_N)^3\big) = 0, \quad (t, x) \in \mathbb{R} \times \mathbb{T}^d, \\ (\Pi_K u_N, \partial_t \Pi_K u_N)(0, \cdot) = (\Pi_K u_0, \Pi_K u_1), \\ (\mathrm{Id} - \Pi_K)(u_N) = S(t)\big((\mathrm{Id} - \Pi_K)u_0, (\mathrm{Id} - \Pi_K)u_1\big). \end{cases}$$

Then from the conservation of the energy

$$\mathcal{E}_N(\Pi_K(u_N))(t) = \frac{1}{2} \int_{\mathbb{T}^d} \Big((\partial_t \Pi_K u_N)^2 + |\nabla_x \Pi_K u_N|^2 + \frac{1}{2}(S_N u_N)^4\Big) \mathrm{d}x,$$

we deduce that, for all $N \geq 1$, (3.1) admits a global flow $\Phi_N(t)$. The goal of this section is to prove the following statement.

Proposition 3.1. — *Let $0 < s < 1$ and $\mu \in \mathcal{M}^s$. Then for any $\varepsilon > 0$ there exist $C, \delta > 0$ such that for every $(u_0, u_1) \in \Sigma$, there exists $M > 0$ such that the family of*

global solution $(u_N)_{N\in\mathbb{N}}$ *to* (3.1) *satisfies*

$$u_N(t) = S(t)\Pi^0(u_0, u_1) + w_N(t),$$

$$\|(w_N(t), \partial_t w_N(t))\|_{\mathcal{H}^1} \leq C(M^s + |t|)^{\frac{1-s}{s}+\varepsilon},$$

$$\|S_N(u_N)\|_{L^4(\mathbb{T}^d)} \leq C(M^s + |t|)^{\frac{1-s}{2s}+\varepsilon},$$

with $\mu(M > \lambda) \leq Ce^{-\lambda^\delta}$.

Proof. — We only give the proof for positive times, the analysis for negative times being analogous. Fix $\varepsilon > 0$ and $\varepsilon_1 > 0$ such that

(3.2) $$\varepsilon < \frac{s}{2}, \qquad \frac{1-s+\varepsilon}{s-2\varepsilon} \leq \frac{1-s}{s} + \varepsilon_1,$$

and fix $\delta > 1/2, \widetilde{\delta} > 1/3$ such that

(3.3) $$(\delta - \frac{1}{2})s < 2\delta\varepsilon, \qquad \widetilde{\delta} < 1.$$

We have the following statement.

Lemma 3.2. — *For every* $c > 0$ *there exists* $C > 0$ *such that for every* $t \geq 1$, *every integer* $M \geq 1$ *such that* $t \leq cM^{s-2\varepsilon}$, *every* $(u_0, u_1) \in E_M$ *the solution of* (3.1) *with data* (u_0, u_1) *satisfies*

$$\|u_N(t) - S(t)\Pi^0(u_0, u_1)\|_{\mathcal{H}^1(\mathbb{T}^d)} \leq CM^{1-s+\varepsilon}.$$

In particular, thanks to (3.2), *if* $t \approx M^{s-2\varepsilon}$ *then*

$$\|u_N(t) - S(t)\Pi^0(u_0, u_1)\|_{\mathcal{H}^1(\mathbb{T}^d)} \lesssim t^{\frac{1-s}{s}+\varepsilon_1}.$$

Proof. — For $(u_0, u_1) \in E_M$ we decompose the solution of (3.1) with data (u_0, u_1) as

$$u_N(t) = S(t)\Pi^M(u_0, u_1) + w_{N,M},$$

where $w_{N,M}$ solves the problem

$$\begin{cases} (\partial_t^2 - \Delta_{\mathbb{T}^d})w_{N,M} + S_N\big((S_N w_{N,M} + S_N S(t)\Pi^M(u_0, u_1))^3\big) = 0, \\ (w_{N,M}(0), \partial_t w_{N,M}(0)) = \Pi_M(u_0, u_1). \end{cases}$$

Then thanks to an integration by parts and the fact that S_N is self adjoint, we get

(3.4) $\dfrac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(w_{N,M}) =$

$$= \int_{\mathbb{T}^d}\Big(\partial_t^2 w_{N,M}\partial_t w_{N,M} + \nabla_x w_{N,M}\cdot\partial_t\nabla_x w_{N,M} + (S_N w_{N,M})^3\partial_t S_N w_{N,M}\Big)\mathrm{d}x$$

$$= \int_{\mathbb{T}^d}\partial_t w_{N,M}\Big(\partial_t^2 w_{N,M} - \mathbf{\Delta}w_{N,M} + S_N\big((S_N w_{N,M})^3\big)\Big)\mathrm{d}x$$

$$= \int_{\mathbb{T}^d}\partial_t w_{N,M}\Big(S_N\big((S_N w_{N,M})^3\big) - S_N\big((S_N S(t)\Pi^M(u_0,u_1) + S_N w_{N,M})^3\big)\Big)\mathrm{d}x.$$

Denote by

$$g_M(t) = \|S(t)\Pi^M(u_0,u_1)\|^3_{L^6(\mathbb{T}^d)} \quad\text{and}\quad f_M(t) = \|S(t)\Pi^M(u_0,u_1)\|_{L^\infty(\mathbb{T}^d)}.$$

Therefore from (3.4) and the Cauchy-Schwarz inequality, we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{E}_N(w_{N,M})$$

$$\leq C\mathcal{E}_N^{1/2}(w_{N,M})\|(S_N w_{N,M})^3 - \big(S_N S(t)\Pi^M(u_0,u_1) + S_N w_{N,M}\big)^3\|_{L^2(\mathbb{T}^d)}$$

$$\leq C\mathcal{E}_N^{1/2}(w_{N,M})\big(\|S(t)\Pi^M(u_0,u_1)\|^3_{L^6(\mathbb{T}^d)} + \|S(t)\Pi^M(u_0,u_1)\|_{L^\infty(\mathbb{T}^d)}\|S_N w_{N,M}\|^2_{L^4(\mathbb{T}^d)}\big)$$

$$\leq C\mathcal{E}_N^{1/2}(w_{N,M})\Big(g_M(t) + f_M(t)\mathcal{E}_N^{1/2}(w_{N,M})\Big),$$

and with the Gronwall lemma, we obtain

$$\mathcal{E}_N^{1/2}(w_{N,M})(t) \;\leq\; C\mathrm{e}^{C\int_0^t f_M(\tau)\mathrm{d}\tau}\Big(\mathcal{E}_N^{1/2}(w_{N,M})(0) + \int_0^t g_M(\tau)\mathrm{d}\tau\Big)$$

(3.5) $$\leq\; C\mathrm{e}^{C\int_0^T f_M(\tau)\mathrm{d}\tau}\Big(\mathcal{E}_N^{1/2}(w_{N,M})(0) + \int_0^T g_M(\tau)\mathrm{d}\tau\Big) := \mathcal{G}_M(T)$$

(notice that since $w_{N,M}(0)$ does not depend on N, the right-hand side in the last inequality is also independent on N). We now observe that for $(u_0,u_1)\in E_M$

$$\Big|\int_0^t g_M(\tau)d\tau\Big| \leq CM^{3(-s+\varepsilon)}\langle t\rangle^{3\widetilde{\delta}} \leq CM^{3(-s+\varepsilon)+3\widetilde{\delta}(s-2\varepsilon)} \leq C,$$

provided

$$-s + \varepsilon + \widetilde{\delta}(s - 2\varepsilon) \leq 0.$$

The last condition can be readily satisfied according to (3.3).

Next, we have (using Cauchy-Schwarz inequality in time) that for $(u_0,u_1)\in E_M$,

$$\Big|\int_0^t f_M(\tau)d\tau\Big| \leq \|\langle\tau\rangle^{-\delta}f_M\|_{L^2(\mathbb{R})}\langle t\rangle^{\delta+\frac{1}{2}} \leq CM^{-s+\varepsilon}\langle t\rangle^{\delta+\frac{1}{2}} \leq CM^{-s+\varepsilon+(\delta+\frac{1}{2})(s-2\varepsilon)} \leq C,$$

provided $-s + \varepsilon + (\delta + \frac{1}{2})(s - 2\varepsilon) \leq 0$, a condition which is satisfied thanks to (3.3).

For $(u_0, u_1) \in E_M$, we have

$$\mathcal{E}^{1/2}(w_{N,M}(0)) \leq C(\|\Pi_M(u_0, u_1)\|_{\mathcal{H}^1} + \|\Pi_M(u_0)\|_{L^4}^2) \leq CM^{1-s+\varepsilon},$$

and coming back to (3.5), we get

(3.6) $$\mathcal{E}^{1/2}(w_{N,M}(t)) \leq CM^{1-s+\varepsilon}.$$

Recall that

$$u_N(t) = w_{N,M}(t) + S(t)\Pi^M(u_0, u_1) = S(t)\Pi^0(u_0, u_1) + w_{N,M}(t) - S(t)\Pi_M\Pi^0(u_0, u_1).$$

We have that for a solution to the linear wave equation the linear energy

$$\|\nabla_x u\|_{L^2(\mathbb{T}^d)}^2 + \|\partial_t u\|_{L^2(\mathbb{T}^d)}^2$$

is independent of time and that if $(u, \partial_t u)$ is orthogonal to constants $((u, \partial_t u) = \Pi^0(u, \partial_t u))$, then this energy controls the $\mathcal{H}^1(\mathbb{T}^d)$-norm, we deduce for $(u_0, u_1) \in E_M \subset F_M$ that

$$\|S(t)\Pi_M\Pi^0(u_0, u_1)\|_{\mathcal{H}^1(\mathbb{T}^d)} \leq CM^{1-s+\varepsilon}$$

and therefore

$$\|u_N(t) - S(t)\Pi^0(u_0, u_1)\|_{\mathcal{H}^1(\mathbb{T}^d)} \leq CM^{1-s+\varepsilon}.$$

This completes the proof of Lemma 3.2. □

Next we set

$$E^M = \bigcap_{K \geq M} E_K,$$

where the intersection is taken over the dyadic values of K, *i.e.* $K = 2^j$ with j an integer. Thus $\mu(E^M)$ tends to 1 as M tends to infinity. Using Lemma 3.2, we obtain that there exists $C > 0$ such that for every $t \geq 1$, every M, every $(u_0, u_1) \in E^M$, and every $N \in \mathbb{N}$,

$$\|u_N(t) - S(t)\Pi^0(u_0, u_1)\|_{\mathcal{H}^1(\mathbb{T}^d)} \leq C\left(M^{1-s+\varepsilon} + t^{\frac{1-s}{s}+\varepsilon_1}\right).$$

Furthermore, by (3.6) and the definition of R_M, we get that for $(u_0, u_1) \in E_M$, and $t \leq cM^{s-2\varepsilon}$

$$\begin{aligned}
\|S_N(u_N)\|_{L^4(\mathbb{T}^d)}(t) &\leq \|S_N(w_{N,M})\|_{L^4(\mathbb{T}^d)}(t) + \|S_N(S(t)\Pi^M(u_0, u_1))\|_{L^4(\mathbb{T}^d)}(t) \\
&\leq \mathcal{E}^{1/4}(w_{N,M})(t) + M^{-s+2\epsilon} \leq CM^{\frac{1-s+\epsilon}{2}}.
\end{aligned}$$

Finally, we set

$$E = \bigcup_{M=1}^{\infty} E^M.$$

We have thus shown the μ almost sure bounds on the possible growths of the Sobolev norms of the solutions established in the previous section for data in E which is of full μ measure. This completes the proof of Proposition 3.1. $\square$

4. Passing to the limit

4.1. Some deterministic estimates. — We now need an interpolation result. Define the space $W_T^{1,\infty}$ by the norm $\|u\|_{W_T^{1,\infty}} = \|u\|_{L_T^\infty} + \|\partial_t u\|_{L_T^\infty}$, and denote by $H^\sigma = H^\sigma(\mathbb{T}^d)$.

Lemma 4.1. — *Let $T > 0$, $-\infty < \sigma_2 \leq \sigma_1 < +\infty$ and assume that*

$$u \in L^\infty\big([-T,T]; H^{\sigma_1}\big), \qquad \partial_t u \in L^\infty\big([-T,T]; H^{\sigma_2}\big).$$

Then for all $\theta \in (0,1)$, and all $t_1, t_2 \in [-T,T]$

$$\|u(t_1) - u(t_2)\|_{H^{\theta\sigma_1 + (1-\theta)\sigma_2}} \leq C|t_1 - t_2|^{1-\theta} \|u\|_{L_T^\infty H^{\sigma_1}}^\theta \|u\|_{W_T^{1,\infty} H^{\sigma_2}}^{1-\theta}.$$

Proof. — By Hölder we get

$$\|u(t_1) - u(t_2)\|_{H^{\sigma_2}} = \|\int_{t_1}^{t_2} \partial_\tau u(\tau)\mathrm{d}\tau\|_{H^{\sigma_2}} \leq |t_1 - t_2| \|\partial_t u\|_{L_T^\infty H^{\sigma_2}}.$$

Next we clearly have

$$\|u(t_1) - u(t_2)\|_{H^{\sigma_1}} \leq 2\|u\|_{L_T^\infty H^{\sigma_1}},$$

and we conclude using that

$$\|u\|_{H^{\theta\sigma_1 + (1-\theta)\sigma_2}} \leq \|u\|_{H^{\sigma_1}}^\theta \|u\|_{H^{\sigma_2}}^{1-\theta}.$$

$\square$

Now for $\sigma \in \mathbb{R}$ and $\alpha \in (0,1)$, let us define the space $\mathcal{C}_T^\alpha H^\sigma = \mathcal{C}^\alpha\big([-T,T]; H^\sigma(\mathbb{T}^d)\big)$ by the norm

$$\|u\|_{\mathcal{C}_T^\alpha H^\sigma} = \sup_{t_1, t_2 \in [-T,T], t_1 \neq t_2} \frac{\|u(t_1) - u(t_2)\|_{H_x^\sigma}}{|t_1 - t_2|^\alpha} + \|u\|_{L_T^\infty H_x^\sigma}.$$

According to Ascoli theorem, we obtain

Lemma 4.2. — *For any $T > 0$, any $\alpha > 0$ and any $\epsilon > 0$, the embedding*

$$\mathcal{C}_T^\alpha H^\sigma \mapsto C((0,T); H^{\sigma-\epsilon})$$

is compact.

4.2. The compactness argument. — According to Proposition 3.1, we know that almost surely, there exists $M \geq 1$ such that the family of solutions to (3.1)

$$u_N(t) = S(t)\Pi^0(u_0, u_1) + w_N(t),$$

is such that

$$\|(w_N(t), \partial_t w_N(t))\|_{\mathcal{H}^1(\mathbb{T}^d)} \leq C(M^s + |t|)^{\frac{1-s}{s}+\varepsilon}$$

$$\|S_N(u_N)\|_{L^4((0,t)\times\mathbb{T}^d)} \leq C(M^s + |t|)^{\frac{1-s}{2s}+\varepsilon}|t|^{1/4}.$$

We apply Lemma 4.1 with $\sigma_1 = 1$ and $\sigma_2 = 0$ and we deduce that the sequence w_N is for any $\epsilon > 0$ bounded in $\mathcal{C}_T^{\epsilon/2} H^{1-\epsilon/2}$. According to Lemma 4.2 we can almost surely extract a sequence converging for any T in $\mathcal{C}\big((0, T); H^{1-\epsilon}\big)$, to a limit that we denote by w. On the other hand, the sequence $S_N(u_N)$ is, for any T bounded in $L^4_{t,x}$ and we can consequently extract a sequence converging weakly in $L^4_{loc,t,x}$ to a limit that we denote by u. But for any $K \in \mathbb{N}$, if $K \leq N - 2$, we have

$$S_K(S_N(u_N)) = S_K(u_N) = S_K(S(t)\Pi^0(u_0, u_1) + w_N(t)),$$

and we deduce that (in distribution sense), $S_K(S_N(u_N))$ is converging to $S_K(u)$ on the one hand and to $S_K\big(S(t)(u_0, u_1) + w\big)$ on the other hand. Hence

$$\forall K \in \mathbb{N}, \quad S_K(u) = S_K\big(S(t)(u_0, u_1) + w\big).$$

We deduce that (in distribution sense) $u = S(t)(u_0, u_1) + w$. Now we deduce that $S_N(u_N)$ is converging weakly in $L^4_{loc,t,x}$ and strongly in $L^2_{loc,t,x}$ to u (here by strong convergence in $L^p_{loc,t,x}$ we mean that the convergence is strong on any compact set). By interpolation, we deduce that $S_N(u_N)$ is converging strongly to u in $L^p_{loc,t,x}$ for $2 \leq p < 4$. In particular using this property for $p = 3$, we can pass to the limit in (3.1) (here we use Lemma 2.1 to pass to the limit in the nonlinear term) and obtain that u satisfies (1.1). To prove the convergence of $\partial_t w_N$ in $\mathcal{C}\big((0, T); H^{-\varepsilon}(\mathbb{T}^d)\big)$, we estimate

$$\partial_t^2 w_N = \Delta w_N - S_N\big((S_N w_N + S_N S(t)\Pi^0(u_0, u_1))^3\big),$$

in $L^\infty\big((0, T); H^{-\tau}(\mathbb{T}^d)\big)$ with $\tau = \max(d/4, 1)$ (here we use $L^{4/3}(\mathbb{T}^d) \subset H^{-d/4}(\mathbb{T}^d)$), and we can conclude thanks to Lemma 4.1 with $\sigma_1 = 0$ and $\sigma_2 = -\tau$.

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