International Mathematics Research Notices Advance Access published October 27, 2016
A. Hébert (2016) “Gindikin–Karpelevich Finiteness for Kac–Moody Groups Over Local Fields,”
International Mathematics Research Notices, Vol. 2016, No. 00, pp. 1–22
doi: 10.1093/imrn/rnw224

Gindikin–Karpelevich Finiteness for Kac–Moody Groups Over Local Fields

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In this article, we prove some finiteness results about split Kac–Moody groups over local non-archimedean fields. Our results generalize those of "An affine Gindikin–Karpelevich formula" by Alexander Braverman, Howard Garland, David Kazhdan, and Manish Patnaik and give a positive answer to a conjecture of Alexander Braverman and David Kazdhan formulated in their article in sixth European Congress of Mathematicians 2012. We do not require our groups to be affine. We use the hovel associated to this situation, which is the analogue of the Bruhat-Tits building for a reductive group.

1 Introduction

The classical Gindikin–Karpelevich formula was introduced in 1962 by Simon Gindikin and Fridrikh Karpelevich. It applies to real semi-simple Lie groups and it enables to compute certain Plancherel densities for semi-simple Lie groups. This formula was established in the non-archimedean case by Robert Langlands in [9]. In 2014, in [1], Alexander Braverman, Howard Garland, David Kazhdan, and Manish Patnaik obtained a generalization of this formula in the affine Kac–Moody case.

Let $\mathcal{F}$ be a ultrametric field. Let $\mathcal{O}$ be its ring of integers, π be a generator of the maximal ideal of $\mathcal{O}$ and q denote the cardinal of the residue field $\mathcal{O}/\pi\mathcal{O}$. Let **G** be a split Kac–Moody group over $\mathcal{F}$. Let **T** be a maximal split torus of **G**. Choose a pair

Received March 10, 2016; Revised September 21, 2016; Accepted September 22, 2016
Communicated by Prof. Michael Finkelberg

$\mathbf{B}$, $\mathbf{B}^-$ of opposite Borel subgroups such that $\mathbf{B} \cap \mathbf{B}^- = \mathbf{T}$ and $\mathbf{U}$, $\mathbf{U}^-$ be their unipotent radicals. We use boldface letters to denote schemes : $\mathbf{G}, \mathbf{T}, \ldots$ and their sets of $\mathcal{F}$-points are denoted by G,T,... Let Λ and $\Lambda^\vee$ be the root lattice and the coroot lattice of $\mathbf{T}$. Let R be the set of roots of $\mathbf{G}$ and R^+ be its set of positive roots. Let $K = \mathbf{G}(\mathcal{O})$. Let $S^\vee$ be the set of simple coroots ($S^\vee$ depends on the choice of $\mathbf{B}$). For a coroot $\nu^\vee = \sum_{s^\vee \in S^\vee} n_{s^\vee} s^\vee \in \Lambda^\vee$, one sets $|\nu^\vee| = \sum n_{s^\vee}$. Let $\Lambda^\vee_+ = \bigoplus_{s^\vee \in S^\vee} \mathbb{N} s^\vee$ and $\Lambda^\vee_- = -\Lambda^\vee_+$. Let $\mathbb{C}[[\Lambda^\vee]]$ be the Looijenga's coweight algebra of $\mathbf{G}$, with generators $e^{\lambda^\vee}$, for $\lambda^\vee \in \Lambda^\vee$. Alexander Braverman, Howard Garland, David Kazhdan, and Manish Patnaik proved that when $\mathbf{G}$ is the affine Kac–Moody group associated to a simply connected semi-simple split group, we have the following formula:

$$\sum_{\mu^\vee \in \Lambda^\vee} |K \backslash K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee - \mu^\vee} U| e^{\lambda^\vee - \mu^\vee} q^{|\lambda^\vee - \mu^\vee|} = \frac{1}{H_0} \prod_{\alpha \in R_+} \left(\frac{1 - q^{-1} e^{-\alpha^\vee}}{1 - e^{-\alpha^\vee}}\right)^{m_\alpha}, \tag{1}$$

where $\alpha \in R_+$, m_α denotes the multiplicity of the coroot $\alpha^\vee$ in the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$, and H_0 is some term (there is a formula describing H_0 in [1]) depending on $\mathfrak{g}$. When $\mathbf{G}$ is a reductive group, $H_0 = 1$, each m_α is equal to 1 and this formula is equivalent to Gindikin–Karpelevich formula.

The aim of this article is to show the following four theorems:

Theorem 3.1: Let $\mu^\vee \in \Lambda^\vee$. Then if $\mu^\vee \notin \Lambda^\vee_-$, $K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee + \mu^\vee} U$ is empty for all $\lambda^\vee \in \Lambda^\vee$. If $\mu^\vee \in \Lambda^\vee_+$, then for $\lambda^\vee \in \Lambda^\vee$ sufficiently dominant, $K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee + \mu^\vee} U \subset K\pi^{\lambda^\vee} K \cap K\pi^{\lambda^\vee + \mu^\vee} U$.

Corollary 5.3: Let $\mu^\vee \in \Lambda^\vee$. For all $\lambda \in \Lambda^\vee$, $K \backslash K\pi^{\lambda^\vee} K \cap K\pi^{\lambda^\vee + \mu^\vee} U$ is finite and is empty if $\mu^\vee \notin \Lambda^\vee_-$.

Theorem 5.6: Let $\mu^\vee \in \Lambda^\vee_-$ and $\lambda^\vee \in \Lambda^\vee$. Then $|K \backslash K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee + \mu^\vee} U| = |K \backslash K\pi^0 U^- \cap K\pi^{\mu^\vee} U|$ and these sets are finite.

Theorem 6.1: Let $\mu^\vee \in \Lambda^\vee$. Then for $\lambda^\vee \in \Lambda^\vee$ sufficiently dominant, $K\pi^{\lambda^\vee} U^- \cap K\pi^{\lambda^\vee + \mu^\vee} U = K\pi^{\lambda^\vee} K \cap K\pi^{\lambda^\vee + \mu^\vee} U$.

These theorems correspond to Theorem 1.9 of [1] and this is a positive answer to Conjecture 4.4 of [2].

Theorem 5.6 is named "Gindikin–Karpelevich finiteness" in [3] and it enables to make sense to the left side of formula 1. Thus this could be a first step to a generalization of Gindikin–Karpelevich formula to general split Kac–Moody groups. However, we do not know yet how to express the term H_0 when $\mathbf{G}$ is not affine.

The main tool that we use to prove these theorems is the hovel $\mathcal{I}$ associated to **G**, which was defined by Stéphane Gaussent and Guy Rousseau in [6] and generalized by Guy Rousseau in [12]. This hovel is a geometric object similar to a Bruhat-Tits building. It is a set covered by apartments isomorphic to a standard apartment $\mathbb{A}$ on which **G** acts. The standard apartment $\mathbb{A}$ is an affine space with a structure obtained from a lattice Y containing $\Lambda^\vee$ (which can be thought of as $\Lambda^\vee$ in a first approximation) and from a set of hyperplanes called walls, defined by the roots of **G**. In general, $\mathcal{I}$ is not a building: there may be two points such that no apartment of $\mathcal{I}$ contains these two points simultaneously. However some properties of euclidean buildings remain and, for example one can still define the retraction onto an apartment centered at a sector-germ. We can always define a fundamental vectorial chamber C_f^v in $\mathbb{A}$ and its opposite, which gives rise to two retractions: $\rho_{+\infty}$ and $\rho_{-\infty}$. We also have a "vectorial distance" d^v defined on a subset of $\mathcal{I}^2$. These maps enable to give other expressions to the sets defined above: for all $\lambda^\vee \in \Lambda^\vee$, $K\backslash K\pi^{\lambda^\vee}U^-$ corresponds to $\rho_{-\infty}^{-1}(\{\lambda^\vee\})$, $K\backslash K\pi^{\lambda^\vee}U$ corresponds to $\rho_{+\infty}^{-1}(\{\lambda^\vee\})$ and $K\backslash K\pi^{\lambda^\vee}K$ corresponds to $S^v(0,\lambda^\vee) = \{x \in \mathcal{I}|\ d^v(0,x) \text{ is defined and } d^v(0,x) = \lambda^\vee\}$. This enables to use properties of "Hecke paths", which are more or less the images of segments in the hovel by retractions, to prove these theorems. These paths were first defined by Misha Kapovich and John J.Millson in [8].

Actually, when we will write Theorem 3.1 and Theorem 6.1 using the hovel, we will show that these inclusion or equality are true modulo K. But as if $X \subset G$ is invariant by left multiplication by K, $X = \bigcup_{x\in K\backslash X} Kx$, this will be sufficient.

The framework of hovels is a bit more general than that of split Kac–Moody groups over local non-archimedean field. For example, we might also apply this to almost split Kac–Moody groups over local non-archimedean field (see [12]).

In the sequel, the results are not stated as in this introduction. Their statements use retractions and vectorial distance. They are also a bit more general: they take into account the inessential part of the standard apartment $\mathbb{A}$, which will be defined later. Corollary 5.3, as it is stated in this introduction was proved in Section 5 of [6] and it is slightly generalized in the following. The lattice $\Lambda^\vee$ will be denoted by $Q^\vee$.

In Section 2, we set the general frameworks, we set the notation, we define hovels and we prove that $Y^{++} = Y \cap \overline{C_f^v}$ is a finitely generated monoid, which will be useful to prove Theorem 6.1.

In Section 3, we first define two applications: $T_\nu : \mathcal{I} \to \mathbb{R}_+$ and $y_\nu : \mathcal{I} \to \mathbb{A}$, for a fixed $\nu \in C_f^v$. For $x \in \mathcal{I}$, $T_\nu(x)$ measures the distance between the point x and the apartment $\mathbb{A}$ along $\mathbb{R}_+\nu$ and $y_\nu(x)$ defines the projection of x on $\mathbb{A}$ along $\mathbb{R}_+\nu$. We also determine the antecedents of some kinds of paths by $\rho_{-\infty}$.

In Section 4, we show that T_ν is bounded by some function of $\rho_{+\infty} - \rho_{-\infty}$ and we deduce Theorem 3.1.

In Section 5, we study some kinds of translations of $\mathcal{I}$, which enables us to show Corollary 5.3 and Theorem 5.6.

In Section 6, we use the tools of the preceding sections to show Theorem 6.1.

2 General Frameworks

2.1 Root generating system

A Kac–Moody matrix (or generalized Cartan matrix) is a square matrix $C = (c_{i,j})_{i,j \in I}$ with integers coefficients, indexed by a finite set I and satisfying:

1. $\forall i \in I,\ c_{i,i} = 2$
2. $\forall (i,j) \in I^2 | i \neq j,\ c_{i,j} \leq 0$
3. $\forall (i,j) \in I^2,\ c_{i,j} = 0 \Leftrightarrow c_{j,i} = 0$.

A root generating system is a 5-tuple $\mathcal{S} = (C, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac–Moody matrix C indexed by I, of two dual free $\mathbb{Z}$-modules X (of characters) and Y (of cocharacters) of finite rank $\mathrm{rk}(X)$, a family $(\alpha_i)_{i\in I}$ (of simple roots) in X and a family $(\alpha_i^\vee)_{i\in I}$ (of simple coroots) in Y. They have to satisfy the following compatibility condition: $c_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j \in I$. We also suppose that the family $(\alpha_i)_{i\in I}$ is free in X and that the family $(\alpha_i^\vee)_{i\in I}$ is free in Y.

We now fix a Kac–Moody matrix C and a root generating system with matrix C.

Let $V = Y \otimes \mathbb{R}$. Every element of X induces a linear form on V. We will consider X as a subset of the dual V^* of V: the α_i, $i \in I$ are viewed as linear form on V. For $i \in I$, we define an involution r_i of V by $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ for all $v \in V$. Its space of fixed points is $\ker \alpha_i$. The subgroup of $\mathrm{GL}(V)$ generated by the α_i for $i \in I$ is denoted by W^v and is called the Weyl group of $\mathcal{S}$.

For $x \in V$, we let $\underline{\alpha}(x) = (\alpha_i(x))_{i\in I} \in \mathbb{R}^I$.

Let $Q^\vee = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$ and $P^\vee = \{v \in V | \underline{\alpha}(v) \in \mathbb{Z}^I\}$. We call $Q^\vee$ the *coroot-lattice* and $P^\vee$ the *coweight-lattice* (but if $\bigcap_{i\in I} \ker \alpha_i \neq \{0\}$, this is not a lattice). Let $Q_+^\vee = \bigoplus_{i\in I} \mathbb{N}\alpha_i^\vee$, $Q_-^\vee = -Q_+^\vee$ and $Q_\mathbb{R}^\vee = \bigoplus_{i\in I} \mathbb{R}\alpha_i^\vee$. This enables us to define a preorder $\leq_{Q^\vee}$ on V by the following way: for all $x, y \in V$, one writes $x \leq_{Q^\vee} y$ if $y - x \in Q_+^\vee$.

One defines an action of the group W^v on V^* by the following way: if $x \in V$, $w \in W^v$ and $\alpha \in V^*$ then $(w.\alpha)(x) = \alpha(w^{-1}.x)$. Let $\Phi = \{w.\alpha_i | (w, i) \in W^v \times I\}$, Φ is the set of *real roots*. Then $\Phi \subset Q$, where $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$. Let $W^a = Q^\vee \rtimes W^v \subset \mathrm{GA}(V)$ the *affine Weyl group* of $\mathcal{S}$, where $\mathrm{GA}(V)$ is the group of affine isomorphisms of V.

Let $V_{in} = \bigcap_{i\in I} \ker \alpha_i$. Then one has $Y + V_{in} \subset P^\vee$.

Remark 2.1. Let $C = (c_{i,j})_{i,j\in I}$ a Kac–Moody matrix. Then there exists a generating root system $(C, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ such that $Y + V_{in} = P^\vee$. For this, suppose $I = [\![1, k]\!]$ for some $k \in \mathbb{N}$. Let $n = 2k$ and let $(\alpha_i)_{i\in[\![1,n]\!]}$ be the canonical basis of the dual of $\mathbb{Z}^n$. Let $(e_i)_{i\in[\![1,n]\!]}$ be the canonical basis of $\mathbb{Z}^n$. For $i \in [\![1, k]\!]$, let $\alpha_i^\vee = \sum_{j=1}^k c_{i,j}e_j + e_{i+k}$. Let $X = \bigoplus_{i\in[\![1,n]\!]} \mathbb{Z}\alpha_i = (\mathbb{Z}^n)^*$ and $Y = \mathbb{Z}^n$. Then $(C, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ is a generating root system which is more or less $\mathcal{D}^C_{un}$ of 7.1.2 of [10].

Let $u \in P^\vee$. Then there exists $y \in Y$ such that $(\alpha_i(y))_{i\in I} = (\alpha_i(u))_{i\in I}$ and thus $u \in Y + V_{in}$. Therefore, $P^\vee = Y + V_{in}$. □

2.2 Description of Y^{++}

In this subsection, we show that $Y^{++} = Y \cap \overline{C^v_f}$ is a finitely generated monoid, which will be useful to prove Theorem 6.1.

Let $l \in \mathbb{N}^*$. Let us define a binary relation $\prec$ on $\mathbb{N}^l$. Let $x, y \in \mathbb{N}^l$, $x = (x_1, \dots, x_l)$, $y = (y_1, \dots, y_l)$, then one says $x \prec y$ if $x \neq y$ and for all $i \in [\![1, l]\!]$, $x_i \leq y_i$.

Lemma 2.2. Let $l \in \mathbb{N}^*$ and F be a subset of $\mathbb{N}^l$ satisfying property (INC(l)):

for all $x, y \in F$, x and y are not comparable for $\prec$.

Then F is finite. □

Proof. This is clear for $l = 1$ because a set F satisfying INC(1) is a singleton or $\emptyset$.

Suppose that $l > 1$ and that we have proved that all set satisfying INC($l - 1$) is finite.

Let F be a set satisfying INC(l) and suppose F infinite. Let $(\lambda_n)_{n\in\mathbb{N}}$ be an injective sequence of F. One writes $(\lambda_n) = (\lambda_n^1, \dots, \lambda_n^l)$. Let $M = \max \lambda_0$ and, for any $n \in \mathbb{N}$, $m_n = \min(\lambda_n)$. Then for all $n \in \mathbb{N}$, $m_n \leq M$ (if $m_n > M$, we would have $\lambda_0 \prec \lambda_n$). Maybe extracting a sequence of λ, one can suppose that $(m_n) = (\lambda_n^i)$ for some $i \in [\![1, l]\!]$ and that (m_n) is constant and equal to $m_0 \in [\![0, M]\!]$. For $x \in \mathbb{N}^l$, we define $\tilde{x} = (x_j)_{j\in[\![1,l]\!]\setminus\{i\}} \in \mathbb{N}^{l-1}$.

Set $\tilde{F} = \{\tilde{\lambda_n} | n \in \mathbb{N}\}$. The set $\tilde{F}$ satisfies INC($l - 1$). By the induction hypothesis, $\tilde{F}$ is finite and thus F is finite, which is absurd. Hence F is finite and the lemma is proved. ■

Lemma 2.3. There exists a finite set $E \subset Y$ such that $Y^{++} = \sum_{e\in E} \mathbb{N}e$. □

Proof. The set $Y_{in} = Y \cap V_{in}$ is a lattice in the vectorial space it spans. Consequently, it is a finitely generated $\mathbb{Z}$-module and thus a finitely generated monoid. Let E_1 be a finite set generating Y_{in} as a monoid.

Let $Y_{\succ 0} = Y^{++} \backslash Y_{in}$. Let $\mathcal{P} = \{a \in Y_{\succ 0} | a \neq b + c \ \forall b, c \in Y_{\succ 0}\}$. Let $\alpha : Y^{++} \to \mathbb{N}^I$ such that $\alpha(x) = (\alpha_i(x))_{i \in I}$ for all $x \in Y^{++}$. Let $a, b \in \mathcal{P}$. If $\alpha(a) \prec \alpha(b)$, then $b = b - a + a$, with $a, b - a \in Y_{\succ 0}$, which is absurd and by symmetry we deduce that $\alpha(a)$ and $\alpha(b)$ are not comparable for $\prec$. Therefore, by Lemme 2.2, $\alpha(\mathcal{P})$ is finite. Let E_2 be a finite set of $Y_{\succ 0}$ such that $\alpha(\mathcal{P}) = \{\alpha(x) | x \in E_2\}$. Then $Y^{++} = \sum_{e \in E_2} \mathbb{N}e + Y_{in} = \sum_{e \in E} \mathbb{N}e$, where $E = E_1 \cup E_2$. ■

2.3 Vectorial faces

Define $C_f^v = \{v \in V | \ \alpha_i(v) > 0, \ \forall i \in I\}$. We call it the *fundamental chamber*. For $J \subset I$, one sets $F^v(J) = \{v \in V | \ \alpha_i(v) = 0 \ \forall i \in J, \alpha_i(v) > 0 \ \forall i \in J \backslash I\}$. Then the closure $\overline{C_f^v}$ of C_f^v is the union of the $F^v(J)$ for $J \subset I$. The *positive* (resp. *negative*) *vectorial faces* are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. A *vectorial face* is either a positive vectorial face or a negative vectorial face. We call *positive* (resp. *negative*) *chamber* every cone of the shape $w.C_f^v$ for some $w \in W^v$ (resp. $-w.C_f^v$). For all $x \in C_f^v$ and for all $w \in W^v$, $w.x = x$ implies that $w = 1$. In particular, the action of W^v on the positive chambers is simply transitive. The *Tits cone* $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$. We also consider the negative cone $-\mathcal{T}$. We define a W^v invariant preorder $\leq$ on V by: $\forall (x, y) \in V^2, x \leq y \ \Leftrightarrow \ y - x \in \mathcal{T}$.

2.4 Filters

Definition 2.4. A filter in a set E is a nonempty set F of nonempty subsets of E such that, for all subsets S, S' of E, if $S, S' \in F$ then $S \cap S' \in F$ and, if $S' \subset S$, with $S' \in F$ then $S \in F$. □

If F is a filter in a set E, and E' is a subset of E, one says that F contains E' if every element of F contains E'. If E' is nonempty, the set $F_{E'}$ of subsets of E containing E' is a filter. By abuse of language, we will sometimes say that E' is a filter by identifying $F_{E'}$ and E'. If F is a filter in E, its closure $\overline{F}$ (resp. its convex envelope) is the filter of subsets of E containing the closure (resp. the convex envelope) of some element of F. A filter F is said to be contained in an other filter F': $F \subset F'$ (resp. in a subset Z in E: $F \subset Z$) if and only if any set in F' (resp. if Z) is in F.

If $x \in V$ and Ω is a subset of V containing x in its closure, then the *germ* of Ω in x is the filter $germ_x(\Omega)$ of subsets of V containing a neighborhood in Ω of x.

A *sector* in V is a set of the shape $\mathfrak{s} = x + C^v$ with $C^v = \pm w.C^v_f$ for some $x \in \mathbb{A}$ and $w \in W^v$. The point x is its *base point* and C^v is its *direction*. The intersection of two sectors of the same direction contains a sector of the same direction.

A *sector-germ* of a sector $\mathfrak{s} = x + C^v$ is the filter $\mathfrak{S}$ of subsets of V containing a V-translate of $\mathfrak{s}$. It only depends on the direction C^v. We denote by $+\infty$ (resp. $-\infty$) the sector-germ of C^v_f (resp. of $-C^v_f$).

A ray δ with base point x and containing $y \neq x$ (or the interval $]x, y] = [x, y] \backslash \{x\}$ or $[x, y]$) is called *preordered* if $x \leq y$ or $y \leq x$ and *generic* if $y - x \in \pm\mathring{\mathcal{T}}$, the interior of $\pm\mathcal{T}$.

In the next subsection, we define the notions of faces, enclosures and chimneys defined in [11] 1.7 and 1.10 and in [6] 1.4. For a first reading, one can just know the following facts about these objects and skip this subsection:

1. To any filter F of V is associated its enclosure $\mathrm{cl}_{\mathbb{A}}(F)$ which is a filter in $\mathbb{A}$ containing the convex envelope of the closure of F.
2. A face or a chimney is a filter in V.
3. A sector is a chimney which is solid and splayed.
4. If a chimney is a sector, its germ as a chimney coincides with its germ as a sector.
5. Every $x \in V$ is in some face of V.
6. The group W^a permutes the sectors, the enclosures, the faces and the chimneys of V.

2.5 Definitions of enclosures, faces, chimneys, and related notions

Let $\Delta = \Phi \cup \Delta^+_{im} \cup \Delta^-_{im} \subset Q$ be the set of all roots (recall that $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i$) defined in [7]. The group W^v stabilizes Δ. For $\alpha \in \Delta$, and $k \in \mathbb{Z} \cup +\infty$, let $D(\alpha, k) = \{v \in V | \alpha(v) + k \geq 0\}$ (and $D(\alpha, +\infty) = V$ for all $\alpha \in \Delta$) and $D^\circ(\alpha, k) = \{v \in V | \alpha(v) + k > 0\}$ (for $\alpha \in \Delta$ and $k \in \mathbb{Z} \cup \{+\infty\}$).

Given a filter F of subsets of V, its *enclosure* $\mathrm{cl}_V(F)$ is the filter made of the subsets of V containing an element of F of the shape $\bigcap_{\alpha \in \Delta} D(\alpha, k_\alpha)$ where $k_\alpha \in \mathbb{Z} \cup \{+\infty\}$ for all $\alpha \in \Delta$.

A *face* F in V is a filter associated to a point $x \in V$ and a vectorial face $F^v \subset V$. More precisely, a subset S of V is an element of the face $F = F(x, F^v)$ if and only if, it contains an intersection of half-spaces $D(\alpha, k_\alpha)$ or open half-spaces $D^\circ(\alpha, k_\alpha)$, with $k_\alpha \in \mathbb{Z}$ for all $\alpha \in \Delta$, that contains $\Omega \cap (x + F^v)$, where Ω is an open neighborhood of x in V.

There is an order on the faces: if $F \subset \overline{F'}$ we say that "F is a face of F'" or "F' dominates F". The dimension of a face F is the smallest dimension of an affine space generated by some $S \in F$. Such an affine space is unique and is called its support. A face is said to be *spherical* if the direction of its support meets the open Tits cone $\mathring{\mathcal{T}}$ or its opposite $-\mathring{\mathcal{T}}$; then its pointwise stabilizer W_F in W^v is finite.

We have $W^a \subset P^\vee \rtimes W^v$. As $\alpha(P^\vee) \subset \mathbb{Z}$ for all α in $\bigoplus_{i\in I} \mathbb{Z}\alpha_i$, if τ is a translation of V of a vector $p \in P^\vee$, then for all $\alpha \in Q$, τ permutes the sets of the shape $D(\alpha, k)$ where k runs over $\mathbb{Z}$. As W^v stabilizes Δ, any element of W^v permutes the sets of the shape $D(\alpha, k)$ where α runs over Δ. Therefore, W^a permutes the sets $D(\alpha, k)$, where (α, k) runs over $\Delta \times \mathbb{Z}$ and thus W^a permutes the enclosures, faces, chimneys, ... of V.

A *chamber* (or alcove) is a maximal face, or equivalently, a face such that all its elements contains a nonempty open subset of V.

A *panel* is a spherical face maximal among faces that are not chambers or, equivalently, a spherical face of dimension $n-1$.

A *chimney* in V is associated to a face $F = F(x, F_0^v)$ and to a vectorial face F^v; it is the filter $\mathfrak{r}(F, F^v) = \mathrm{cl}_{\mathbb{A}}(F + F^v)$. The face F is the basis of the chimney and the vectorial face F^v its direction. A chimney is *splayed* if F^v is spherical, and is *solid* if its support (as a filter, i.e., the smallest affine subspace of V containing $\mathfrak{r}$) has a finite pointwise stabilizer in W^v. A splayed chimney is therefore solid.

A *shortening* of a chimney $\mathfrak{r}(F, F^v)$, with $F = F(x, F_0^v)$ is a chimney of the shape $\mathfrak{r}(F(x+\xi, F_0^v), F^v)$ for some $\xi \in \overline{F^v}$. The *germ* of a chimney $\mathfrak{r}$ is the filter of subsets of V containing a shortening of $\mathfrak{r}$ (this definition of shortening is slightly different from the one of [11] 1.12 but follows [12] 3.6) and we obtain the same germs with these two definitions).

2.6 Hovel

We now denote by $\mathbb{A}$, the affine space V equipped with its faces, chimneys, ...

An apartment of type $\mathbb{A}$ is a set A with a nonempty set $\mathrm{Isom}(\mathbb{A}, A)$ of bijections (called isomorphisms) such that if $f_0 \in \mathrm{Isom}(\mathbb{A}, A)$ then $f \in \mathrm{Isom}(\mathbb{A}, A)$ if and only if, there exists $w \in W^a$ satisfying $f = f_0 \circ w$. An isomorphism between two apartments $\phi : A \to A'$ is a bijection such that ($f \in \mathrm{Isom}(\mathbb{A}, A)$ if, and only if, $\phi \circ f \in \mathrm{Isom}(\mathbb{A}, A')$). We extend all the notions that are preserved by W^a to each apartment. By the fact 2.4 of the above subsection, sectors, enclosures, faces and chimneys are well defined in any apartment of type $\mathbb{A}$.

Definition 2.5. An ordered affine hovel of type $\mathbb{A}$ (also called a masure of type $\mathbb{A}$) is a set $\mathcal{I}$ endowed with a covering $\mathcal{A}$ of subsets called apartments such that:

(MA1) Any $A \in \mathcal{A}$ admits a structure of an apartment of type $\mathbb{A}$.

(MA2) If F is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment A and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists an isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$.

(MA3) If $\mathfrak{R}$ is the germ of a splayed chimney and if F is a face or a germ of a solid chimney, then there exists an apartment that contains $\mathfrak{R}$ and F.

(MA4) If two apartments A, A' contain $\mathfrak{R}$ and F as in (MA3), then there exists an isomorphism from A to A' fixing $\mathrm{cl}_A(\mathfrak{R} \cup F)$.

(MAO) If x, y are two points contained in two apartments A and A', and if $x \leq_A y$ then the two segments $[x, y]_A$ and $[x, y]_{A'}$ are equal. □

In this definition, we say that an apartment contains a germ of a filter, if it contains at least one element of this germ. We say that an application fixes a germ if it fixes at least one element of this germ.

Remark 2.6. (consequence 2.2 3) of [11]) By (MA2), the axioms (MA3) and (MA4) also apply in a hovel when F is a point, a germ of a preodered segment and when $\mathfrak{R}$ or F is a germ of a generic ray or a germ of a spherical sector face. □

Until the end of this article, $\mathcal{I}$ will be an affine ordered hovel. We suppose that $\mathcal{I}$ is thick of *finite thickness*: the number of chambers (=alcoves) containing a given panel has to be finite, greater or equal to 3. This assumption will be crucial to use some theorems of [6] but we will not use it directly.

We assume that $\mathcal{I}$ has a strongly transitive group of automorphisms G, which means that all isomorphisms involved in the above axioms are induced by elements of G. We choose in $\mathcal{I}$ a fundamental apartment, that we identify with $\mathbb{A}$. As G is strongly transitive, the apartments of $\mathcal{I}$ are the sets $g.\mathbb{A}$ for $g \in G$. The stabilizer N of $\mathbb{A}$ induces a group $\nu(N)$ of affine automorphisms of $\mathbb{A}$ and we suppose that $\nu(N) = W^v \ltimes Y$.

An example of such a hovel $\mathcal{I}$ is the hovel associated to a split Kac–Moody group over a ultrametric field constructed in [5] and in [12]. We will precise it in Subsection 2.8

2.6.1 Preorder and vectorial distance

As the preorder $\leq$ on $\mathbb{A}$ (induced by the Tits cone) is invariant under the action of W^v, we can equip each apartment A with a preorder $\leq_A$. Let A be an apartment of

$\mathcal{I}$ and $x, y \in A$ such that $x \leq_A y$. Then by Proposition 5.4 of [11], if B is an apartment containing x and y, $x \leq_B y$. As a consequence, one can define a relation $\leq$ on $\mathcal{I}$ as follow: let $x, y \in \mathcal{I}$, one says that $x \leq y$ if there exists an apartment A of $\mathcal{I}$ containing x and y and such that $x \leq_A y$. By Théorèmes 5.9 of [11], this defines a G-invariant preorder on $\mathcal{I}$.

For $x \in \mathcal{T}$, we denote by x^{++} the unique element in $\overline{C_f^v}$ conjugated by W^v to x.

Let $\mathcal{I} \times_{\leq} \mathcal{I} = \{(x, y) \in \mathcal{I}^2 | x \leq y\}$ be the set of increasing pairs in $\mathcal{I}$. Let (x, y) be such a pair and $g \in G$ such that $g.x, g.y \in \mathbb{A}$. Then $g.x \leq g.y$ and we define the *vectorial distance* $d^v(x, y) \in \overline{C_f^v}$ by $d^v(x, y) = (g.y - g.x)^{++}$. It does not depend on the choices we made. This "distance" is G-invariant.

For $x \in \mathcal{I}$ and $\lambda \in \overline{C_f^v}$, one defines $S^v(x, \lambda) = \{y \in \mathcal{I} | x \leq y \text{ and } d^v(x, y) = \lambda\}$.

Remark 2.7. a) If $a \in Y$ and $\lambda \in \overline{C_f^v}$, then $S^v(a, \lambda) = \{x \in \mathcal{I} | \exists g \in G | g.a = a \text{ and } g.x = a + \lambda\}$.

b) Let $x, y \in \mathcal{I}$ and suppose that for some $g \in G$, $g.y - g.x \in \overline{C_f^v}$. Then $x \leq y$ and $d^v(x, y) = g.y - g.x$. □

2.7 Retractions and Hecke paths

Let $\mathfrak{R}$ be the germ of a splayed chimney of an apartment A. Let $x \in \mathcal{I}$. By (MA3), for all $x \in \mathcal{I}$, there exists an apartment A_x of $\mathcal{I}$ containing x and $\mathfrak{R}$. By (MA4), there exists an isomorphism of apartments $\phi : A_x \to A$ fixing $\mathfrak{R}$. By [11] 2.6, $\phi(x)$ does not depend on the choices we made and thus we can set $\rho_{A,\mathfrak{R}}(x) = \phi(x)$.

The map $\rho_{A,\mathfrak{R}}$ is a retraction from $\mathcal{I}$ onto A. It only depends on $\mathfrak{R}$ and A and we call it the *retraction onto A centered at* $\mathfrak{R}$.

We denote by $\rho_{+\infty}$ (resp. $\rho_{-\infty}$), the retraction onto $\mathbb{A}$ centered at $+\infty$ (resp. $-\infty$).

We now define Hecke paths. They are more or less the images by $\rho_{-\infty}$ of preordered segments $[x, y]$ in $\mathcal{I}$. The definition is a bit technical but it expresses the fact that the image of such a path "goes nearer to $+\infty$" when it crosses a wall. A consequence of that is Remark 2.9 and we will not use directly this definition in the following.

We consider piecewise linear continuous paths $\pi : [0, 1] \to \mathbb{A}$ such that the values of π' belong to some orbit $W^v.\lambda$ for some $\lambda \in \overline{C_f^v}$. Such a path is called a λ-*path*. It is increasing with respect to the preorder relation $\leq$ on $\mathbb{A}$. For any $t \neq 0$ (resp. $t \neq 1$), we let $\pi'_-(t)$ (resp. $\pi'_+(t)$) denote the derivative of π at t from the left (resp. from the right).

Definition 2.8. A Hecke path of shape λ with respect to $-C_f^v$ is a λ-path such that $\pi'_+(t) \leq_{W^v_{\pi(t)}} \pi'_-(t)$ for all $t \in [0, 1]\backslash\{0, 1\}$, which means that there exists a $W^v_{\pi(t)}$-chain from

$\pi'_-(t)$ to $\pi'_+(t)$, i.e., a finite sequence $(\xi_0 = \pi'_-(t), \xi_1, \dots, \xi_s = \pi'_+(t))$ of vectors in V and $(\beta_1, \dots, \beta_s) \in \Phi^s$ such that, for all $i \in [\![1, s]\!]$,

1. $r_{\beta_i}(\xi_{i-1}) = \xi_i$.
2. $\beta_i(\xi_{i-1}) < 0$.
3. $r_{\beta_i} \in W^v_{\pi(t)}$; i.e., $\beta_i(\pi(t)) \in \mathbb{Z}$: $\pi(t)$ is in a wall of direction $\ker(\beta_i)$.
4. Each β_i is positive with respect to $-C^v_f$; i.e., $\beta_i(C^v_f) > 0$. □

Remark 2.9. Let $\pi : [0,1] \to \mathbb{A}$ be a Hecke path of shape $\lambda \in \overline{C^v_f}$ with respect to $-C^v_f$. Then if $t \in [0,1]$ such that π is differentiable in t and $\pi'(t) \in \overline{C^v_f}$, then for all $s \geq t$, π is differentiable in s and $\pi'(s) = \lambda$. □

2.8 Hovel associated to a split Kac–Moody group

Let $\mathbf{G}$ be a split Kac–Moody group over a ultrametric field $\mathcal{F}$ with ring of integer $\mathcal{O}$ and $G = \mathbf{G}(\mathcal{F})$. We use notation of the introduction. We associate a hovel to G as in [5] and [12]. Let us explain the dictionary, given in the introduction, between objects in G and objects in $\mathcal{I}$.

The group G acts strongly transitively on $\mathcal{I}$. We have $Q = \Lambda$ and $Q^\vee = \Lambda^\vee$. The group K is the fixator of 0 in G and U (resp. U^-) is the fixator of $+\infty$ (resp. $-\infty$) in G. The action of T on $\mathbb{A}$ is as follows: if $t = \pi^{\lambda^\vee} \in T$ for some $\lambda^\vee \in \Lambda^\vee$, t acts on $\mathbb{A}$ by the translation of vector $-\lambda^\vee$ and thus $\lambda^\vee = \pi^{-\lambda^\vee}.0$ for all $\lambda^\vee \in \Lambda^\vee$.

Let $\mathcal{I}_0 = G.0$ be the set of *vertices of type* 0. The map $G \to \mathcal{I}_0$ sending $g \in G$ to $g.0$ induces a bijection $\phi : G/K \to \mathcal{I}_0$. If $g \in G$, then $\phi^{-1}(g.0) = gK$. If $x \in \mathbb{A}$, then $\rho^{-1}_{+\infty}(\{x\}) = U.x$ and $\rho^{-1}_{-\infty}(\{x\}) = U^-.x$. By Remark 2.7, if $\lambda^\vee \in \Lambda^\vee$, then $S^v(0, \lambda^\vee) = K.\lambda^\vee$.

Therefore, for all $\lambda^\vee, \mu^\vee \in \Lambda^\vee$, $\phi^{-1}\big(S^v(0,\lambda^\vee) \cap \rho^{-1}_{+\infty}(\{\mu^\vee\})\big) = (K\pi^{-\lambda^\vee}K \cap U\pi^{-\mu^\vee}K)/K$ and $\phi^{-1}\big(\rho^{-1}_{-\infty}(\{\lambda^\vee\}) \cap \rho^{-1}_{+\infty}(\{\mu^\vee\})\big) = (U^-\pi^{-\lambda^\vee}K \cap U\pi^{-\mu^\vee}K)/K$.

We then use the bijection $\psi : K\backslash G \to G/K$ defined by $\psi(Kg) = g^{-1}K$ to obtain the sets considered in the Section 1.

3 Segments and Rays in $\mathcal{I}$

In this section, we begin by defining for all $\nu \in C^v_f$ two maps $y_\nu : \mathcal{I} \to \mathbb{A}$ and $T_\nu : \mathcal{I} \to \mathbb{R}_+$, where for all $x \in \mathcal{I}$, $T_\nu(x)$ and $y_\nu(x)$ can be considered as the distance between x and $\mathbb{A}$ along $\mathbb{R}_+\nu$ and the projection of x on $\mathbb{A}$ along $\mathbb{R}_+\nu$.

We also show that the only antecedent of some paths for $\rho_{-\infty}$ are themselves (this is Lemma 3.6).

Let $Q^\vee_{\mathbb{R}_+} = \bigoplus_{i\in I} \mathbb{R}_+\alpha_i^\vee$ and $Q^\vee_{\mathbb{R}_-} = -Q^\vee_{\mathbb{R}_+}$. Recall that we want to prove the following theorem:

Theorem 3.1. Let $\mu \in \mathbb{A}$. Then if $\mu \notin Q^\vee_{\mathbb{R}_-}$, $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ is empty for all $\lambda \in \mathbb{A}$. If $\mu \in Q^\vee_{\mathbb{R}_-}$, then for $\lambda \in \mathbb{A}$ sufficiently dominant, $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) \subset S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})$. □

This theorem is a bit more general than its statement in the introduction because $\lambda, \mu \in \mathbb{A}$ are arbitrary and not only in $Q^\vee$. It will be proved in Section 4.

Let us recall briefly the notion of parallelism in $\mathcal{I}$. This is done more completely in [11] Section 3. Let δ and δ' be two generic rays in $\mathcal{I}$. Then there exists a splayed chimney R containing δ and a solid chimney F containing δ'. By (MA3), there exists an apartment A containing the germ $\mathfrak{R}$ of R and F. Therefore A contains translates of δ and δ' and we say that δ and δ' are *parallel*, if these translates are parallel in A. Parallelism is an equivalence relation and its equivalence classes are called *directions*.

Lemma 3.2. Let $x \in \mathcal{I}$ and δ be a generic ray. Then there exists a unique ray $x+\delta$ in $\mathcal{I}$ with base point x and direction δ. In any apartment A containing x and a ray δ' parallel to δ, this ray is the translate in A of δ' having x as a base point. □

This lemma is analogous to Proposition 4.7.1) of [11]. The difficult part of this lemma is the uniqueness of such a ray because second part of the lemma yields a way to construct a ray having direction δ and x as a base point. This uniqueness can be shown exactly in the same manner as the proof of Proposition 4.7.1) by replacing "spherical sector face" by "generic ray". This is possible by NB.a) of Proposition 2.7 and by 2.2 3) (or by Remark 2.6 of this article) of [11].

Definition of y_ν and T_ν (resp. y_ν^- and T_ν^-)

Let $x \in \mathcal{I}$. Let $\nu \in C_f^v$ and $\delta = \mathbb{R}_+\nu$, which is a generic ray. According to axiom (MA3) applied to a face containing x and the splayed chimney C_f^v, there exists an apartment A containing x and $+\infty$. Then A contains $x+\delta$. The set $x+\delta\cap\mathbb{A}$ is nonempty. Let $z \in x+\delta\cap\mathbb{A}$. Then $A\cap\mathbb{A}$ contains z, $+\infty$ and by (MA4), $A\cap\mathbb{A}$ contains $\mathrm{cl}(z,+\infty)$. As $\mathrm{cl}(z,+\infty) \supset z+\overline{C_f^v}$, $A\cap\mathbb{A} \supset z+\delta$ and thus $x+\delta\cap\mathbb{A} = y+\delta$ or $x+\delta\cap\mathbb{A} = y+\mathring{\delta}$ for some $y \in x+\delta$, where $\mathring{\delta} = \mathbb{R}_+^*\nu$.

Suppose $x+\delta\cap\mathbb{A} = y+\mathring{\delta}$. Let $z \in y+\mathring{\delta}$. Then by (MA2) applied to $germ_y([y,z]\backslash\{y\})$, $\mathbb{A}\cap A \supset \mathrm{cl}(germ_y([y,z]\backslash\{y\})) \ni y$ because $\mathrm{cl}(germ_y([y,z]\backslash\{y\}))$ contains the closure of

$germ_y([y,z]\backslash\{y\})$. This is absurd and thus $\mathbb{A} \cap x + \delta = y + \delta$, with $y \in \mathbb{A}$. One sets $y_\nu(x) = y \in \mathbb{A}$ (actually, y_ν only depends on δ).

One has $\rho_{+\infty}(x+\delta) = \rho_{+\infty}(x)+\delta$ and $y \in \rho_{+\infty}(x)+\delta$. We define $T_\nu(x)$ as the unique element T of $\mathbb{R}_+$ such that $y = \rho_{+\infty}(x) + T\nu$.

Let $\delta^- = -\mathbb{R}_+\nu$ and $x \in \mathcal{I}$. Similarly, one defines y_ν^- as the first point of $x + \delta^-$ meeting $\mathbb{A}$ and $T_\nu^-(x)$ as the element T of $\mathbb{R}_+$ such that $\rho_{-\infty}(x) = y + T\nu$.

The proof of Theorem 3.1 will rely on the fact that T_ν^- is bounded by some function of $\rho_{+\infty} - \rho_{-\infty}$, which is Corollary 4.2. This bounding will be obtained by studying Hecke paths.

Remark 3.3. In the following, the choice of ν will not be very important. We will often need to choose $\nu \in Y \cap C_f^v$. □

Example 3.4. Let us describe the action of y_ν on a simple example. Let $\mathbf{G}$ be as in Subsection 2.8 (we keep notation as in the introduction). For $\alpha \in \Phi$, let U_α be the root subgoup associated to α and $x_\alpha : (\mathcal{F},+) \to U_\alpha$ be an isomorphism of algebraic group. Let $\omega : \mathcal{F} \to \mathbb{Z} \cup \{+\infty\}$ be a surjective valuation inducing the structure of ultrametric field of $\mathcal{F}$. Define $\phi_\alpha : U_\alpha \to \mathbb{R}$ by $\phi_\alpha(u) = \omega(x_\alpha^{-1}(u))$ for all $u \in U_\alpha$. For $k \in \mathbb{R} \cap \{+\infty\}$, one sets $U_{\alpha,k} = \phi_\alpha^{-1}([k,+\infty])$ and $D(\alpha,k) = \{x \in \mathbb{A} | \alpha(x) + k \geq 0\}$. Let $H = \mathbf{T}(\mathcal{O})$. Then by 4.2 5) and 4.2 7) of [5], H is the fixator of $\mathbb{A}$ in $\mathcal{I}$ and for all $k \in \mathbb{R}$, $HU_{\alpha,k}$ is the fixator of $D(\alpha,k)$ in $\mathcal{I}$.

Let $k \in \mathbb{Z}$, $u \in HU_{\alpha,k} \backslash HU_{\alpha,k+1}$, and $A = u.\mathbb{A}$. One has $A \cap \mathbb{A} = \{x \in \mathbb{A} | u.x = x\} = \{u \in \mathbb{A} | u.x \in \mathbb{A}\}$. In order to simplify, suppose that $\mathbf{G}$ is a reductive group (and thus $\mathcal{I}$ is a usual Bruhat-Tits building). Let us show that $D(\alpha,k) = A \cap \mathbb{A}$ (actually this remains true if G is not a reductive group but the proof is a bit more difficult and we do not want to develop it here).

By 2.5.7 of [4], $A \cap \mathbb{A}$ is enclosed (which means that the enclosure of $A \cap \mathbb{A}$ is $A \cap \mathbb{A}$) and thus $A \cap \mathbb{A} = D(\alpha, l)$ for some $l \in [k,+\infty) \cap \mathbb{Z}$. As $u \in HU_{\alpha,l}$, we deduce that $k = l$.

One writes $\mathbb{A} = M \oplus \mathbb{R}\nu$, where $M = \{x \in \mathbb{A} | \alpha(x) = -k\}$. Then if $x = m + \lambda\nu$, with $m \in M$ and $\lambda \in \mathbb{R}$, $y_\nu(u.x) = m + \mu(\lambda)\nu$, with $\mu(\lambda) = \min(\lambda, 0)$. □

Lemma 3.5. Let $x \in \mathcal{I}$ and $\nu \in C_f^v$. Let $y = y_\nu(x)$ and $T = T_\nu(x)$.

a) Then $x \leq y$ and $d^v(x,y) = T\nu$.
b) One has $\rho_{+\infty}(x) \in Y$ if and only if $\rho_{-\infty}(x) \in Y$ if and only if $x \in \mathcal{I}_0$. In this case, $\rho_{+\infty}(x) \leq_{Q^\vee} \rho_{-\infty}(x)$. □

Proof. Let A be an apartment containing x and $+\infty$ and $g \in G$ fixing $+\infty$ such that $A = g^{-1}.\mathbb{A}$. Then $x+\delta$ is the translate of a shortening $\delta' \subset A$ of δ (which means $\delta' = z+\delta$, with $z \in \delta$). As for all $z' \in z+\delta$, $z \leq z'$, one has $x \leq y$. As $d^v(x,y) = d^v(g.x, g.y)$ and $g_{|A} = \rho_{+\infty}$ one gets a).

For $x \in \mathcal{I}$, there exists $g_-, g_+ \in G$ such that $\rho_{-\infty}(x) = g_-.x$ and $\rho_{+\infty}(x) = g_+.x$, which shows the claimed equivalence because $Y = G.0 \cap \mathbb{A}$.

Suppose $x \in \mathcal{I}_0$. One chooses $\nu \in Y \cap C_f^v$. Let $S = \lfloor T \rfloor + 1$, where $\lfloor . \rfloor$ is the floor function, and $z = \rho_{+\infty}(x) + S\nu \in x+\delta$. Then $d^v(x,z) = d^v(g_+.x, g_+.z) = d^v(\rho_{+\infty}(x), z) = S\nu \in Y \cap C_f^v$.

According to paragraph 2.3 of [6], the image π of $[x,z]$ by $\rho_{-\infty}$ is a Hecke path of shape $z - \rho_{+\infty}(x) = S\nu$ with respect to $-C_f^v$ (unless the contrary is specified, "Hecke path" will mean with respect to $-C_f^v$). By applying Lemma 2.4b) of [6] to π, one gets that $z - \rho_{-\infty}(x) \leq_{Q^\vee} d^v(x,z) = z - \rho_{+\infty}(x)$ and thus $\rho_{+\infty}(x) \leq_{Q^\vee} \rho_{-\infty}(x)$ and one has b). ∎

Lemma 3.6. Let $\tau : [0,1] \to \mathcal{I}$ be a segment of $\mathcal{I}$ such that $\tau(1) \in \mathbb{A}$ and $\rho_{-\infty} \circ \tau$ is a segment of $\mathbb{A}$ satisfying $(\rho_{-\infty} \circ \tau)' = \nu \in \overline{C_f^v}$. Then $\tau([0,1]) \subset \mathbb{A}$ and thus $\rho_{-\infty} \circ \tau = \tau$. □

Proof. Suppose $\tau([0,1]) \not\subset \mathbb{A}$. Let $u = \sup\{t \in [0,1] | \tau(u) \notin \mathbb{A}\}$. Then by the same reasoning as in the proof that "$x+\delta \cap \mathbb{A} = y+\delta$" in the paragraph "Definition of y_ν and T_ν", $x = \tau(u) \in \mathbb{A}$. One has $\tau(0) \leq x \leq \tau(1)$ (by the same reasoning as in the proof of Lemma 3.5 a)).

By Remark 2.6, there exists an apartment $A = g^{-1}.\mathbb{A}$ with $g \in G$ containing $\mathfrak{R} = -\infty$ and $germ_x([x, \tau(0)])$. By axiom (MA4) (and Remark 2.6) applied to $\mathfrak{R} = -\infty$ and to x, we can suppose that g fixes $\mathrm{cl}(x, -\infty) \supset x - C_f^v$. Let $x' \in [x, \tau(0)] \backslash \{x\}$ such that $[x, x'] \subset A$ and $x' \notin \mathbb{A}$. Then $g.x' = \rho_{-\infty}(x') \in x - \mathbb{R}_+\nu \subset x - C_f^v$. Therefore, $x' = g.x' \in \mathbb{A}$, which is absurd. Hence $\tau([0,1]) \subset \mathbb{A}$. ∎

4 Bounding of T_ν and Proof of Theorem 3.1

One defines $h : Q_{\mathbb{R}}^\vee \to \mathbb{R}$ by $h(x) = \sum_{i\in I} x_i$, for all $x = \sum_{i \in I} x_i \alpha_i^\vee \in Q_{\mathbb{R}}^\vee$.

Lemma 4.1. Let $T \in \mathbb{R}_+$, $\mu \in \mathbb{A}$, $a \in \mathbb{A}$, $\nu \in Y^{++} = Y \cap \overline{C_f^v}$ and suppose there exists a Hecke path π from a to $a + T\nu - \mu$ of shape $T\nu$. Then

a) $\mu \in Q_{\mathbb{R}_+}^\vee$. Consequently $h(\mu)$ is well defined.

b) if $T > h(\mu)$, there exists t such that π is differentiable on $(t, 1]$ and $\pi'_{|(t,1]} = T\nu$. Furthermore, let t^* be the smallest $t \in [0, 1]$ having this property, then $t^* \leq \frac{h(\mu)}{T}$. □

Proof. The main idea of b) is to use the fact that during the time when $\pi'(t) \neq T\nu$, $\pi'(t) = T\nu - T\lambda(t)$ with $\lambda(t) \in Q_+^\vee \backslash \{0\}$. Hence for T large, π decreases quickly for the $Q^\vee$ order, but it cannot decrease too much because μ is fixed.

Let $t_0 = 0, t_1, \ldots, t_n = 1$ be a subdivision of $[0, 1]$ such that for all $i \in [\![0, n-1]\!]$, $\pi_{|(t_i,t_{i+1})}$ is differentiable and let $w_i \in W^v$ be such that $\pi'_{|(t_i,t_{i+1})} = w_i.T\nu$. If $w_i.\nu = \nu$, one chooses $w_i = 1$.

For $i \in [\![0, n-1]\!]$, according to Lemma 2.4 a) of [6], $w_i.\nu = \nu - \lambda_i$, with $\lambda_i \in Q_+^\vee$ and if $w_i \neq 1$, $\lambda_i \neq 0$. One has

$$\pi(1) - \pi(0) = T\nu - \sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i) T\lambda_i = T\nu - \mu$$

and one deduces a).

Suppose now $T > h(\mu)$. Let us show that there exists $i \in [\![0, n-1]\!]$ such that $w_i = 1$.

For all i such that $w_i \neq 1$, one has $h(\lambda_i) \geq 1$. Hence $T \sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i) \leq h(\mu)$, and $\sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i) < 1 = \sum_{i=0}^{n-1} (t_{i+1} - t_i)$. Thus there exists $i \in [\![0, n-1]\!]$ such that $w_i = 1$.

By Remark 2.9, if $w_i = 1$ for some i, then $w_j = 1$ for all $j \geq i$. This shows the existence of t^*. We also have $t^* \leq \sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i)$ and hence the claimed inequality follows. ■

From now on and until the end of this subsection, ν will be a fixed element of $C_f^v \cap Y$. We define $T_\nu : \mathcal{I} \to \mathbb{R}_+$ and $y_\nu : \mathcal{I} \to \mathbb{A}$ as in the paragraph "Definition of y_ν and T_ν".

Corollary 4.2. Let $x \in \mathcal{I}$ and $\mu = \rho_{-\infty}(x) - \rho_{+\infty}(x)$. Then $\mu \in Q_{\mathbb{R}_+}^\vee$ and $T_\nu(x) \leq h(\mu)$. □

Proof. Let $y = y_\nu(x)$ and $T = T_\nu(x)$. Let π be the image by $\rho_{-\infty}$ of $[x, y]$. This is a Hecke path from $\rho_{-\infty}(x)$ to $y = \rho_{+\infty}(x) + T\nu$, of shape $T\nu$. The minimality of T and Lemma 3.6 imply that $\pi'(t) \neq \nu$ for all $t \in [0, 1]$ where π is differentiable. By applying Lemma 4.1b), we deduce that $T \leq h(\mu)$. Lemma 4.1a) applied to $a = \rho_{-\infty}(x)$ and $\mu = \rho_{-\infty}(x) - \rho_{+\infty}(x)$ completes the proof. ■

Remark 4.3. With the same reasoning but by considering Hecke with respect to C_f^v we obtain an analogous bounding for T_ν^-: for all $x \in \mathcal{I}$, $T_\nu^-(x) \leq h(\rho_{-\infty}(x) - \rho_{+\infty}(x))$. □

Corollary 4.4. Let $x \in \mathcal{I}$ such that $\rho_{+\infty}(x) = \rho_{-\infty}(x)$. Then $x \in \mathbb{A}$. Therefore, $\forall z \in \mathbb{A}$, $\rho_{+\infty}^{-1}(\{z\}) \cap \rho_{-\infty}^{-1}(\{z\}) = \{z\}$. □

Proof. Let $x \in \mathcal{I}$ such that $\rho_{+\infty}(x) = \rho_{-\infty}(x)$. Then by Corollary 4.2, $T_\nu(x) = 0$ and thus $x \in \mathbb{A}$. ■

Lemma 4.5. Let $x \in \mathcal{I}$ such that $y_\nu^-(x) \in C_f^v$. Then $0 \leq x$ and $\rho_{-\infty}(x) = d^v(0, x)$. □

Proof. Let $y^- = y_\nu^-(x)$. Let A be an apartment containing x and $-\infty$. By (MA4), there exists $g \in G$ such that $A = g^{-1}.\mathbb{A}$ and g fixes $\mathrm{cl}(y^-, -\infty) \supset y^- - \overline{C_f^v} \ni 0$. Then $g.x - g.y^- = \rho_{-\infty}(x) - y^- = T^-\nu \in C_f^v$ and $g.y^- - g.0 = y^-$. Thus $g.x - g.0 = \rho_{-\infty}(x) \in C_f^v$ and we can conclude by Remark 2.7. ■

We can now prove Theorem 3.1:

Let $\mu \in \mathbb{A}$. Then if $\mu \notin Q^\vee_{\mathbb{R}_-}$, $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ is empty for all $\lambda \in \mathbb{A}$. If $\mu \in Q^\vee_{\mathbb{R}_-}$, then for $\lambda \in \mathbb{A}$ sufficiently dominant, $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) \subset S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})$.

Proof. The condition on μ comes from Corollary 4.2.

Suppose $\mu \in Q^\vee_{\mathbb{R}_-}$. Let $\lambda \in \mathbb{A}$. Let $y^- = y_\nu^-$ and $T^- = T_\nu^-$. By Corollary 4.2 and Remark 4.3, if $x \in \rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$, then $y^-(x) \in [\lambda - T^-(x)\nu, \lambda] \subset \lambda - [0, h(-\mu)]\nu$. For all $i \in I$, $\alpha_i([0, h(-\mu)]\nu)$ is bounded. Consequently for λ sufficiently dominant, $\alpha_i(\lambda - [0, h(-\mu)]\nu) \subset \mathbb{R}_+^*$ for all $i \in I$. For such a λ, $y^-\left(\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})\right) \subset C_f^v$. We conclude the proof with Lemma 4.5. ■

5 Study of "Translations" of $\mathcal{I}$ and Proof of Theorem 5.6

Let $\mathbb{A}_{in} = \bigcap_{i\in I} \ker \alpha_i$.

In this subsection, we introduce some kind of "translations" of $\mathcal{I}$ of an inessential vector and show that they have very nice properties. It will be useful to "generalize theorems from Y to $Y + \mathbb{A}_{in}$" by getting rid of the inessential part. First example of this technique will be Corollary 5.3 which generalizes a theorem of [6]. Then we study elements of G inducing a translation on $\mathbb{A}$. We show that they commute with $\rho_{+\infty}$ and $\rho_{-\infty}$. We then can see that for a fixed $\mu \in Q^\vee$, the $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$, for $\lambda \in Y + \mathbb{A}_{in}$, are some translates of $\rho_{+\infty}^{-1}(\{\mu\}) \cap \rho_{-\infty}^{-1}(\{0\})$, which enables to show Theorem 5.6 by using Theorem 3.1 and Corollary 5.3.

Lemma 5.1. Let $\nu \in \mathbb{A}_{in}$ and $a \in \mathcal{I}$. Then $|S^v(a,\nu)| = 1$. Moreover, if $a \in \mathbb{A}$, $S^v(a,\nu) = \{a+\nu\}$. □

Proof. Let $h \in G$ such that $h.a \in \mathbb{A}$. Then $S^v(a,\nu) = h^{-1}.S^v(h.a,\nu)$. Therefore one can assume $a \in \mathbb{A}$.

Let $x \in S^v(a,\nu)$. Let $g \in G$ such that $x, a \in g^{-1}.\mathbb{A}$ and $g.x - g.a = \nu$. Let $\tau : [0,1] \to \mathcal{I}$ defined by $\tau(t) = g^{-1}.(g.a + (1-t)\nu)$. Then $\pi = \rho_{-\infty} \circ \tau$ is a Hecke path of shape $-\nu$ and in particular, it is a $-\nu$-path. For all t where π is differentiable, there exists $w(t) \in W^v$ such that $\pi'(t) = -w_i(t).\nu = -\nu$. As W^v acts trivially on $\mathbb{A}_{in}$. Thus π is differentiable on $[0,1]$ and $\pi' = -\nu$. As $\tau(1) = a \in \mathbb{A}$, one can apply Lemma 3.6 and we get that $\tau([0,1]) \subset \mathbb{A}$. Therefore, $x \in \mathbb{A}$ and there exists $w \in W^v$ such that $x - a = w.\nu = \nu$. ■

This lemma enables us to define a kind of translation of an inessential vector. Let $\nu \in \mathbb{A}_{in}$. Let $\tau_\nu : \mathcal{I} \to \mathcal{I}$ which associates to $x \in \mathcal{I}$ the unique element of $S^v(x,\nu)$. Then we have the following lemma:

Lemma 5.2. Let $\nu \in \mathbb{A}_{in}$ and $\tau = \tau_\nu$. Then:

1. For all $x \in \mathbb{A}$, $\tau(x) = x + \nu$.
2. For all $g \in G$, $g \circ \tau = \tau \circ g$. In particular, for all $x \in \mathcal{I}$, if x is in an apartment A, then $\tau(x) \in A$, and if $x = g.a$ with $g \in G$ and $a \in \mathbb{A}$, then $\tau(x) = g.(x+\nu)$.
3. The map τ is a bijection, its inverse being $\tau_{-\nu}$.
4. The map τ commutes with $\rho_{+\infty}$ and $\rho_{-\infty}$.
5. Let $x \in \mathcal{I}$ and $\lambda \in \overline{C^v_f}$, then $\tau(S^v(x,\lambda)) = S^v(\tau(x),\lambda) = S^v(x,\lambda+\nu)$. □

Proof. Part 1 is a part of Lemma 5.1.

Choose $x \in \mathcal{I}$ and $g \in G$. Then $d^v(x,\tau(x)) = \nu$ and thus $d^v(g.x, g.\tau(x)) = \nu$. Consequently $g.\tau(x) = \tau(g.x)$ by definition of τ. Let A be an apartment containing x, $A = h.\mathbb{A}$, with $h \in G$. Then $\tau(x) = \tau(h.a)$ with $a \in \mathbb{A}$, hence $\tau(x) = h.(x+\nu) \in A = h.\mathbb{A}$ and we have 2.

Let $x \in \mathcal{I}$, $x = g.a$ with $a \in \mathbb{A}$. By part 2 applied to τ and $\tau_{-\nu}$, one has $\tau_{-\nu}(\tau(g.a)) = g.\tau_{-\nu}(\tau(a)) = g.a$ and thus $\tau_{-\nu} \circ \tau = \mathrm{Id}$. This is enough to show 3.

Let $x \in \mathcal{I}$ and $g \in G$ fixing $+\infty$ such that $g.x = \rho_{+\infty}(x)$. Then $\tau(x) \in g^{-1}.\mathbb{A}$, thus $g.\tau(x) = \rho_{+\infty}(\tau(x))$ and by part 2, $g.\tau(x) = \tau(g.x)$. Hence, τ and $\rho_{+\infty}$ commute and by the same reasoning, this is also true for τ and $\rho_{-\infty}$.

Let $x \in \mathcal{I}$ and $\lambda \in \overline{C^v_f}$. Let $u \in S^v(x,\lambda)$. There exists $g' \in G$ such that $x, u \in g'^{-1}.\mathbb{A}$ and $g'.u - g'.x = w.\lambda$, for some $w \in W^v$. Let $n \in N$ inducing w on $\mathbb{A}$ and $g = n^{-1}g'$. Then

$x, u \in g^{-1}.\mathbb{A}$ and $g.u - g.x = \lambda$. We have $g.\tau(u) - g.x = \tau(g.u) - g.x = \lambda + \nu$. Therefore, $\tau(S^v(x,\lambda)) \subset S^v(x,\lambda+\nu)$. Applying this result with $\tau_{-\nu}$ yields $\tau_{-\nu}(S^v(x,\lambda+\nu)) \subset S^v(x,\lambda)$ and thus $\tau(S^v(x,\lambda)) = S^v(x,\lambda+\nu)$.

One has $g.\tau(u) - g.\tau(x) = (g.u+\nu) - (g.x+\nu) = \lambda$ and thus $\tau(S^v(x,\lambda)) \subset S^v(\tau(x),\lambda)$. Again, by considering $\tau_{-\nu}$, we have that $\tau(S^v(x,\lambda)) = S^v(\tau(x),\lambda)$. ■

Corollary 5.3. Let $\lambda \in \mathbb{A}$ and $\mu \in Y + \mathbb{A}_{in}$. Then for all $\lambda_{in} \in \mathbb{A}_{in}$, $\tau_{\lambda_{in}}\big(S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\mu\})\big) = S^v(0,\lambda+\lambda_{in}) \cap \rho_{+\infty}^{-1}(\{\mu+\lambda_{in}\})$. In particular, for all $\lambda \in Y + \mathbb{A}_{in}$, $S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\mu+\lambda\})$ is finite and is empty if $\mu \notin Q_-^\vee$. □

Proof. The first assertion is a consequence of Lemma 5.2 part 3, 4, and 5.

Let $\lambda \in Y + \mathbb{A}_{in}$ and $\lambda_{in} \in \mathbb{A}_{in}$ such that $\tau(\lambda) \in Y$, with $\tau = \tau_{\lambda_{in}}$. Then $\tau\big(S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})\big) = S^v(0,\tau(\lambda)) \cap \rho_{+\infty}^{-1}(\{\tau(\lambda+\mu)\})$. Consequently, one can assume $\lambda \in Y$.

Suppose $S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\mu+\lambda\})$ is nonempty. Let x be in this set. Then there exists $g, h \in G$ such that $g.x = \lambda$ and $h.x = \mu + \lambda$. Thus $\lambda + \mu = h.g^{-1}.\lambda \in \mathcal{I}_0 \cap \mathbb{A} = Y$ and therefore, $\mu \in Y$. We can now conclude because the finiteness and the condition on μ are shown in [6], Section 5: if $\lambda, \mu \in Y$ then $S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\mu\})$ is finite and it is empty if $\mu \notin Q_-^\vee$ (the cardinals of these sets correspond to the $n_\lambda(\nu)$ of loc. cit.). ■

We now show a lemma similar to Lemma 5.2 part 4 for translations of G:

Lemma 5.4. Let $n \in G$ inducing a translation on $\mathbb{A}$. Then $n \circ \rho_{+\infty} = \rho_{+\infty} \circ n$ and $n \circ \rho_{-\infty} = \rho_{-\infty} \circ n$. □

Proof. Let $x \in \mathcal{I}$ and A be an apartment containing x and $+\infty$. Then $n.A$ is an apartment containing $+\infty$. Let $\phi : A \to \mathbb{A}$ an isomorphism fixing $+\infty$. We have $n.x \in n.A$, and $n \circ \phi \circ n^{-1} : n.A \to \mathbb{A}$ fixes $+\infty$. Hence $\rho_{+\infty}(n.x) = n \circ \phi \circ n^{-1}(n.x) = n \circ \phi(x) = n \circ \rho_{+\infty}(x)$ and thus $n \circ \rho_{+\infty} = \rho_{+\infty} \circ n$. By the same reasoning applied to $\rho_{-\infty}$, we get the lemma. ■

Lemma 5.5. Let $n \in G$ inducing a translation on $\mathbb{A}$. Let $\lambda_{in} \in \mathbb{A}_{in}$. Set $\tau = \tau_{\lambda_{in}} \circ n$. Let $\nu \in C_f^v$ and $y^- = y_\nu^-$. Then $\tau \circ y^- = y^- \circ \tau$. □

Proof. If $x \in \mathbb{A}$, then $y^-(x) = x$, $y^-(\tau(x)) = \tau(x)$ and there is nothing to prove.

Suppose $x \notin \mathbb{A}$. Then $[x, y^-(x)] \backslash \{y^-(x)\} \subset (x - \mathbb{R}_+\nu) \backslash \mathbb{A}$, thus $\tau([x, y^-(x)] \backslash \{y^-(x)\}) \subset (\tau(x) - \mathbb{R}_+\nu) \backslash \mathbb{A}$ and $\tau(y^-(x)) \in \mathbb{A}$. ■

Theorem 5.6. Let $\mu \in \mathbb{A}$ and $\lambda \in Y + \mathbb{A}_{in}$. One writes $\lambda = \lambda_{in} + \Lambda$, with $\lambda_{in} \in \mathbb{A}_{in}$ and $\Lambda \in Y$. Let $n \in G$ inducing the translation of vector Λ and $\tau = \tau_{\lambda_{in}} \circ n$. Then

$\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) = n\big(\rho_{+\infty}^{-1}(\{\mu\}) \cap \rho_{-\infty}^{-1}(\{0\})\big)$. Therefore these sets are finite and if $\mu \notin Q_-^\vee$ these sets are empty. □

Proof. First assertion is a consequence of Lemma 5.4 and Lemma 5.2 part 4. Then Lemma 3.5 b) shows that these sets are empty unless $\mu \in Q_-^\vee$.

By Theorem 3.1, for $\lambda' \in Y$ sufficiently dominant, $\rho_{+\infty}^{-1}(\{\lambda'+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda'\}) \subset S^v(0,\lambda') \cap \rho_{+\infty}^{-1}(\{\lambda'+\mu\})$, which is a finite set by [6] Section 5 (or by Corollary 5.3). ■

6 Proof of Theorem 6.1

Recall that we want to prove the following theorem:

Theorem 6.1. Let $\mu \in Q^\vee$. Then for $\lambda \in Y^{++} + \mathbb{A}_{in}$ sufficiently dominant, $S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\}) = \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})$. □

The proof is postponed to the end of this seciton. The basic idea of this proof is that if $\mu \in Q^\vee$ there exists a finite set $F \subset Y^{++}$ such that for all $\lambda \in Y^{++} + \mathbb{A}_{in}$, $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap S^v(0,\lambda) \subset \bigcup_{f \in F | \lambda - f \in \overline{C_f^v}} \rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap S^v(\lambda-f,f)$ (this is Corollary 6.4, which generalizes Lemma 6.2 and uses Section 2.2). Then we use Subsection 5 to show that $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap S^v(\lambda-f,f)$ is the image of $\rho_{+\infty}^{-1}(\mu+f) \cap S^v(0,f)$ by a "translation" $\tau_{\lambda-f}$ of G of vector $\lambda-f$ (which means that $\tau_{\lambda-f}$ induces the translation of vector $\lambda-f$ on $\mathbb{A}$). We fix $\nu \in C_f^v$ and set $y^- = y_\nu^-$. By Lemma 5.5,

$$y^-\big(\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap S^v(0,\lambda)\big) \subset \bigcup_{f \in F} \tau_{\lambda-f} \circ y^-\big(S^v(0,f) \cap \rho_{+\infty}^{-1}(\{\mu+f\})\big).$$

According to Section 5 of [6], for all $f, \mu \in Y$, $S^v(0,f) \cap \rho_{+\infty}^{-1}(\{\mu+f\})$ is finite. Consequently, for λ sufficiently dominant, $\bigcup_{f \in F} \tau_{\lambda-f}\big(y^-(S^v(0,f) \cap \rho_{+\infty}^{-1}(\{\mu+f\})\big) \subset C_f^v$ and one concludes with Lemma 4.5.

Lemma 6.2. Let $\mu \in Q_-^\vee$ and $H = -h(\mu)+1 \in \mathbb{N}$. Let $a \in Y$, $T \in [H,+\infty)$, $\nu \in Y^{++}$ and $x \in S^v(a,T\nu) \cap \rho_{+\infty}^{-1}(\{a+T\nu+\mu\})$. Let $g \in G$ such that $g.a = a$ and $g.x = T\nu + a$. Then g fixes $[a, a+(T-H)\nu]$ and in particular $x \in S^v(a+(T-H)\nu, H\nu)$. □

Proof. Let $\tau : [0,1] \to \mathbb{A}$ defined by $\tau(t) = a + (1-t)T\nu$. The main idea is to apply Lemma 4.1 to $\rho_{+\infty} \circ (g^{-1}.\tau)$ but we cannot do it directly because $\rho_{+\infty} \circ \tau$ is not a Hecke path with respect to $-C_f^v$. Let $\mathbb{A}'$ be the vectorial space $\mathbb{A}$ equipped with a structure of apartment of type $-\mathbb{A}$: the fundamental chamber of $\mathbb{A}'$ is $C_f'^v = -C_f^v$ etc ... Let $\mathcal{I}'$ be the set $\mathcal{I}$, whose apartments are the $-A$ where A runs over the apartment of $\mathcal{I}$. Then $\mathcal{I}'$ is a hovel of standard apartment $\mathbb{A}'$. We have $a \leq x$ in $\mathcal{I}$ and so $x \leq' a$ in $\mathcal{I}'$.

Then the image π of $g^{-1}.\tau$ by $\rho_{+\infty}$ is a Hecke path of shape $-T\nu$ from $\rho_{+\infty}(x) = a + T\nu + \mu$ to a. By Lemma 4.1 (for $\mathcal{I}'$), for $t > -h(\mu)/T$, $\pi'(t) = -T\nu$, and thus Lemma 3.6 (for $\mathcal{I}'$ and $\mathbb{A}'$) implies $\rho_{+\infty}(g^{-1}.\tau(t)) = g^{-1}.\tau(t)$ for all $t > -h(\mu)/T$. Therefore, $g^{-1}.\tau_{|(-h(\mu)/T,1]}$ is a segment in $\mathbb{A}$ ending in a, with derivative $-T\nu$ and thus, for all $t \in (\frac{-h(\mu)}{T}, 1]$, $g^{-1}.\tau(t) = \tau(t)$. In particular, g fixes $[a, \tau(\frac{H}{T})] = [a, a+(T-H)\nu]$, and $d^v(a+(T-H)\nu, x) = d^v(g^{-1}.(a+(T-H)\nu), g^{-1}.x) = d^v(a+(T-H)\nu, a+T\nu) = H\nu$. ∎

Let E be as in Lemma 2.3.

Lemma 6.3. Let $\mu \in Q^\vee_-$, $H = -h(\mu) + 1 \in \mathbb{N}$ and $a \in Y$. Let $\lambda \in Y^{++}$. One writes $\lambda = \sum_{e\in E} \lambda_e e$ with $\lambda_e \in \mathbb{N}$ for all $e \in E$. Let $e \in E$. Then if $\lambda_e \geq H$, $S^v(a, \lambda) \cap \rho_{+\infty}^{-1}(\{a+\lambda+\mu\}) \subset S^v(a + (\lambda_e - H)e, \lambda - (\lambda_e - H)e)$. □

Proof. Let $x \in S^v(a, \lambda) \cap \rho_{+\infty}^{-1}(\{a + \lambda + \mu\})$ and $g \in G$ fixing a such that $g.x = a + \lambda$. Let $z = g^{-1}(a + \lambda_e e)$.

Then one has $d^v(a, z) = \lambda_e e$ and $d^v(z, x) = \lambda - \lambda_e e$.

According to Lemma 2.4.b) of [6] (adapted because one considers Hecke paths with respect to C^v_f), one has:

$\rho_{+\infty}(z) - a \leq_{Q^\vee} d^v(a, z) = \lambda_e e$ and $\rho_{+\infty}(x) - \rho_{+\infty}(z) \leq_{Q^\vee} d^v(z, x) = \lambda - \lambda_e e$.

Therefore,

$$a + \lambda + \mu = \rho_{+\infty}(x) \leq_{Q^\vee} \rho_{+\infty}(z) + \lambda - \lambda_e e \leq_{Q^\vee} a + \lambda.$$

Hence, $\rho_{+\infty}(z) = a + \lambda_e e + \mu'$, with $\mu \leq_{Q^\vee} \mu' \leq_{Q^\vee} 0$. One has $-h(\mu') + 1 \leq H$. By Lemma 6.2, g fixes $[a, a + (\lambda_e - H)e]$ and thus g fixes $a + (\lambda_e - H)e$.

As $d^v(g^{-1}.(a+(\lambda_e-H)e), x) = \lambda-(\lambda_e-H)e$, $x \in S^v(a+(\lambda_e-H)e, \lambda-(\lambda_e-H)e)$. ∎

Corollary 6.4. Let $\mu \in Q^\vee_-$. Let $H = -h(\mu)+1$. Let $\lambda \in Y^{++}$. We fix a writing $\lambda = \sum_{e\in E} \lambda_e e$, with $\lambda_e \in \mathbb{N}$ for all $e \in E$. Let $J = \{e \in E | \lambda_e \geq H\}$. Then $S^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \subset S^v(\lambda - H\sum_{e\in J} e - \sum_{e\notin J} \lambda_e e, H\sum_{e\in J} e + \sum_{e\notin J} \lambda_e)$. □

Proof. This is a generalization by induction of Lemma 6.3. ∎

We now prove Theorem 6.1:

Let $\mu \in Q^\vee$. Then for $\lambda \in Y^{++} + \mathbb{A}_{in}$ sufficiently dominant, $S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\}) = \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})$.

Proof. Theorem 3.1 yields one inclusion. It remains to show that $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap S^v(0,\lambda) \subset \rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ for λ sufficiently dominant.

Let $H = -h(\mu) + 1$ and $F = \{\sum_{e \in E} \nu_e e | (\nu_e) \in [\![0, H]\!]^E\}$. This set is finite. Let $\lambda \in Y^{++} + \mathbb{A}_{in}$, $\lambda = \lambda_{in} + \Lambda$, with $\lambda_{in} \in \mathbb{A}_{in}$ and $\Lambda \in Y^{++}$.

Let $x \in S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})$. Then by Corollary 6.4, there exists $f \in F$ such that $\lambda - f \in \overline{C_f^v}$ and $x \in S^v(\lambda - f, f)$ (one can take $f = H \sum_{e \in J} e + \sum_{e \notin J} \lambda_e$ where J is as in Corollary 6.4). Let n be an element of G inducing the translation of vector $\Lambda - f = \lambda - \lambda_{in} - f$ on $\mathbb{A}$ and $\tau_{\lambda,f} = \tau_{\lambda_{in}} \circ n$. Then $x \in \tau_{\lambda,f}(B_f)$ where $B_f = S^v(0,f) \cap \rho_{+\infty}^{-1}(\{\mu+f\})$.

Let $B = \bigcup_{f \in F} B_f$. Then one has proved that

$$S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\}) \subset \bigcup_{f \in F} \tau_{\lambda,f}(B).$$

By Section 5 of [6] (or Corollary 5.3), B_f is finite for all $f \in F$ and thus $B = \bigcup_{f \in F} B_f$ is finite. Let $\nu \in C_f^v$ and $y^- = y_\nu^-$. Then $y^-(B)$ is finite and for λ sufficiently dominant, $\bigcup_{f \in F} \tau_{\lambda,f} \circ y^-(B) \subset C_f^v$. Moreover, according to Lemma 5.5, $\bigcup_{f \in F} \tau_{\lambda,f} \circ y^-(B) = \bigcup_{f \in F} y^- \circ \tau_{\lambda,f}(B)$. Hence

$$y^-\left(S^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})\right) \subset C_f^v$$

for λ sufficiently dominant. Eventually one concludes with Lemma 4.5. ■

Funding

This work was supported by the ANR grants ANR-15-CE40-0012 to A.H.

Acknowledgments

I thank Stéphane Gaussent for suggesting me this problem and for helping me improving the manuscript. I also thank Guy Rousseau for his remarks on a previous version of this article.

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