ON INVARIANT GIBBS MEASURES FOR THE GENERALIZED KDV EQUATIONS

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ABSTRACT. We consider the defocusing generalized KdV equations on the circle. In particular, we construct global-in-time solutions with initial data distributed according to the Gibbs measure and show that the law of the random solutions, at any time, is again given by the Gibbs measure. In handling a nonlinearity of an arbitrary high degree, we make use of the Hermite polynomials and the white noise functional.

1. INTRODUCTION

1.1. Generalized KdV equations.
We consider the generalized KdV equation (gKdV) on the circle:

$$\begin{cases} \partial_t u + \partial_x^3 u = \pm \partial_x(u^k) \\ u|_{t=0} = u_0, \end{cases} \qquad (t, x) \in \mathbb{R} \times \mathbb{T}, \qquad (1.1)$$

where $k \geq 2$ is an integer and u is a real-valued function on $\mathbb{R} \times \mathbb{T}$ with $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$. When $k = 2$ and 3, the equation (1.1) corresponds to the famous Korteweg-de Vries equation (KdV) and the modified KdV equation (mKdV), respectively, and has been studied extensively from both theoretical and applied points of view.

The gKdV equation (1.1) is known to possess the following Hamiltonian structure:

$$\partial_t u = \partial_x \frac{d\mathcal{E}(u)}{du}$$

where $\mathcal{E}(u)$ is the Hamiltonian given by

$$\mathcal{E}(u) = \frac{1}{2} \int_{\mathbb{T}} (\partial_x u)^2 dx \pm \frac{1}{k+1} \int_{\mathbb{T}} u^{k+1} dx.$$

In particular, $\mathcal{E}(u)$ is conserved under the dynamics of (1.1). Moreover, the spatial mean $\int_{\mathbb{T}} u \, dx$ and the mass $M(u) = \int_{\mathbb{T}} u^2 \, dx$ are also conserved. While KdV ($k = 2$) and mKdV ($k = 3$) are known to be completely integrable and thus possess infinitely many conservation laws, there are no other known conservation laws for higher values of $k \geq 4$.

In view of the conservation of the spatial mean, we assume that both the initial condition u_0 and the solution u are of spatial mean 0 in the following. In other words, defining the Fourier coefficient $\widehat{f}(n)$ by

$$\widehat{f}(n) = \mathcal{F}(f)(n) = \int_{\mathbb{T}} f(x) e^{-inx} dx,$$

we will work on real-valued functions of the form:[1]

$$f(x) = \frac{1}{2\pi} \sum_{n \in \mathbb{Z}^*} \widehat{f}(n) e^{inx}$$

where $\mathbb{Z}^* = \mathbb{Z} \setminus \{0\}$.

1.2. **Gibbs measures.** Consider the following Hamiltonian dynamics on $\mathbb{R}^{2n}$:

$$\dot{p}_j = \frac{\partial \mathcal{E}}{\partial q_j} \qquad \text{and} \qquad \dot{q}_j = -\frac{\partial \mathcal{E}}{\partial p_j} \tag{1.2}$$

with Hamiltonian $\mathcal{E}(p,q) = \mathcal{E}(p_1, \cdots, p_n, q_1, \cdots, q_n)$. Liouville's theorem states that the Lebesgue measure $\prod_{j=1}^n dp_j dq_j$ on $\mathbb{R}^{2n}$ is invariant under the dynamics. Then, it follows from the conservation of the Hamiltonian $\mathcal{E}$ that the Gibbs measure $e^{-\mathcal{E}(p,q)} \prod_{j=1}^n dp_j dq_j$ is invariant under the dynamics of (1.2).

In view of the Hamiltonian structure of gKdV (1.1), we expect the Gibbs measure of the form:[2]

$$d\mu_k = Z^{-1} \exp(-\mathcal{E}(u)) du = Z^{-1} e^{\mp \frac{1}{k+1} \int_{\mathbb{T}} u^{k+1} dx} e^{-\frac{1}{2} \int_{\mathbb{T}} (\partial_x u)^2 dx} du \tag{1.3}$$

to be invariant under the dynamics of (1.1). As it stands, (1.3) is merely a formal expression. It turns out, however, that the Gibbs measure μ_k in (1.3) can be defined as a probability measure absolutely continuous with respect to the following Gaussian measure:

$$d\rho = Z_0^{-1} e^{-\frac{1}{2} \int_{\mathbb{T}} (\partial_x u)^2 dx} du. \tag{1.4}$$

Note that ρ in (1.4) is the induced probability measure under the map:

$$\omega \in \Omega \longmapsto u(x) = u(x; \omega) = \sum_{n \in \mathbb{Z}^*} \frac{g_n(\omega)}{|n|} e^{inx}, \tag{1.5}$$

where $\{g_n\}_{n \in \mathbb{N}}$ is a sequence of independent standard[3] complex-valued Gaussian random variables on a probability space $(\Omega, \mathcal{F}, P)$ conditioned that $g_{-n} = \overline{g_n}$, $n \in \mathbb{N}$. The random function u under ρ, represented by the Fourier-Wiener series in (1.5), corresponds to a mean-zero Brownian loop (periodic Wiener process) on $\mathbb{T}$. See [3]. Therefore, we refer to ρ as the (mean-zero periodic) Wiener measure in the following. Lastly, note that u in (1.5) lies in $H^s(\mathbb{T}) \setminus H^{\frac{1}{2}}(\mathbb{T})$ for any $s < \frac{1}{2}$, almost surely.

From Sobolev's inequality, we see that $\int_{\mathbb{T}} (u(x; \omega))^{k+1} dx$ is finite almost surely. Hence, in the defocusing case, i.e. with the $+$ sign in (1.1) and an odd integer $k \geq 3$, the Gibbs measure μ_k is a well-defined probability measure on $H^s(\mathbb{T})$, $s < \frac{1}{2}$, absolutely continuous with respect to ρ.

If (1.1) is not defocusing, then one encounters a normalization issue in (1.3) since $\int_{\mathbb{T}} u^{k+1} dx$ is unbounded. In [24], Lebowitz-Rose-Speer showed that by inserting a mass cutoff $\mathbf{1}_{\{\int u^2 dx \leq R\}}$, one can realize the Gibbs measure μ_k in (1.3) as a probability measure on $H^s(\mathbb{T})$, $s < \frac{1}{2}$, provided that $1 < k \leq 5$ and that the mass cutoff size $R > 0$ is sufficiently small when $k = 5$.

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[1]Hereafter, we drop the harmless 2π.

[2]In the following, Z, Z_N, and etc. denote various normalizing constants so that the corresponding measures are probability measures when appropriate.

[3]Namely, g_n has mean 0 and variance 1.

In the seminal work [5], Bourgain proved the invariance of the Gibbs measures for KdV ($k = 2$) and mKdV ($k = 3$). Here, by invariance, we mean that

$$\mu_k\big(\Phi(-t)A\big) = \mu_k(A) \tag{1.6}$$

for any measurable set $A \in \mathcal{B}_{H^s(\mathbb{T})}$, $s < \frac{1}{2}$, and any $t \in \mathbb{R}$, where $\Phi(t) : u_0 \in H^s(\mathbb{T}) \mapsto u(t) = \Phi(t)u_0 \in H^s(\mathbb{T})$ is a well-defined solution map to (1.1), at least almost surely with respect to μ_k. While KdV was known to be globally well-posed in $L^2(\mathbb{T}) \supset \operatorname{supp} \mu_2$, as shown in [4], there was no well-posedness result for mKdV in the support of the Gibbs measure. In [5], Bourgain first established local well-posedness of mKdV in $H^s(\mathbb{T}) \cap \mathcal{F}L^{s_1,\infty}(\mathbb{T}) \supset \operatorname{supp} \mu_3$ for some $s < \frac{1}{2} < s_1 < 1$, where $\|f\|_{\mathcal{F}L^{s_1,\infty}(\mathbb{T})} = \|\langle n \rangle^{s_1} \widehat{f}(n)\|_{\ell_n^\infty}$. He then used a probabilistic argument to construct almost sure global-in-time dynamics. In fact, the main novelty of the paper [5] is this globalization argument, exploiting the invariance of the finite dimensional Gibbs measures for the finite dimensional approximations to a given PDE. There have been many results on the construction of invariant Gibbs measures for Hamiltonian PDEs that followed and further developed the approach in [5]. See for example [6, 8, 35, 36, 13, 14, 28, 29, 37, 34, 25, 21, 10, 9, 20, 32]. We also refer to the book by Zhidkov [38, Chapter 4] for the construction of infinitely many invariant measures for KdV associated to the conservation laws of the equation at different levels of Sobolev regularity.

In [32], the second author considered the same problem for the quartic gKdV ($k = 4$). As in the case of mKdV, the main challenge in [32] was the construction of the local-in-time dynamics in the support of the Gibbs measure. Following the approach in [30], the second author constructed almost sure local-in-time dynamics by establishing a probabilistic a priori estimate and proved the invariance of the Gibbs measure.[4]

1.3. **Main result.** We now state our main result. In particular, for large values of $k \geq 5$, this theorem addresses the invariance of the Gibbs measures for gKdV (1.1) for the first time.

Theorem 1.1. *Assume one of the following conditions:*

   (i) *defocusing gKdV: with the $+$ sign in (1.1) and odd $k \geq 3$, or*

   (ii) *non-defocusing gKdV: $2 \leq k \leq 5$. When $k = 5$, the mass threshold $R > 0$ is sufficiently small.*

Then, there exists a set Σ of full measure with respect to μ_k such that for every $\phi \in \Sigma$, the generalized KdV equation (1.1) with mean-zero initial condition $u(0) = \phi$ has a global-in-time solution $u \in C(\mathbb{R}; H^s(\mathbb{T}))$ for any $s < \frac{1}{2}$. Moreover, for all $t \in \mathbb{R}$, the law of the random function $u(t)$ is given by μ_k.

Theorem 1.1 asserts two statements: global existence of solutions (without uniqueness) and invariance of μ_k. Regarding the invariance part, Theorem 1.1 only claims that, given any $t \in \mathbb{R}$, the law $\mathcal{L}(u(t))$ of the H^s-valued random variable $u(t)$ is given by the Gibbs measure μ_k. This implies the invariance property of the Gibbs measure μ_k in some mild sense, but it is weaker than the actual invariance described in (1.6). While the well-posedness results for gKdV in low regularity setting [33, 15, 32] are obtained via gauge transforms, we

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[4]Strictly speaking, the invariance of the Gibbs measure in [32] was shown only for the gauged quartic gKdV. See Remark 1.3 below.

work directly on the equation (1.1) in the following. This is crucial to study the invariance property of the Gibbs measure μ_k. See Remark 1.3 for more details.

A precursor to the existence part of Theorem 1.1 appears in the work by the third author with Burq and Tzvetkov [11], where they used the energy conservation and a regularization property under randomization to construct global-in-time solutions to the cubic NLW on $\mathbb{T}^d$ for $d \geq 3$. The main ingredient in [11] is the compactness of the solutions to the approximating PDEs.

We prove Theorem 1.1 by following the approach presented in the work by the third author with Burq and Tzvetkov [12], which was in turn motivated by the works of Albeverio-Cruzeiro [1] and Da Prato-Debussche [17] in the study of fluids. This method allows us to construct global dynamics, even for very rough initial conditions [1, 22]. The main idea is to exploit the invariance of the truncated Gibbs measures $\mu_{k,N}$ (see (2.5) below) and construct a tight (= compact) sequence of measures ν_N on space-time functions. We then apply Skorokhod's theorem (see Lemma 4.6 below) to construct global-in-time *weak* solutions for gKdV (1.1).

In order to prove Theorem 1.1 following the approach in [12], one needs a uniform bound on the nonlinearity of gKdV and its truncated version (see (2.2) below). This is quite easy, since we have $u \in L^p$ for all $2 \leq p < \infty$, almost surely with respect to the Gibbs measure μ_k. In the following, we prove a stronger regularity result on the nonlinearity. Recalling that u defined in (1.5) lies in $H^s(\mathbb{T})$ for any $s < \frac{1}{2}$ almost surely, we show that u^k also lies in $H^s(\mathbb{T})$ for any $s < \frac{1}{2}$, almost surely. See Proposition 3.1. We wanted to include this optimal bound in this paper, since we believe that this could be a first step towards a probabilistic strong well-posedness result for gKdV on the support of the Gibbs measure. The main source of difficulty in proving Proposition 3.1 comes from the more complicated combinatorics[5] for larger values of k. In order to overcome this combinatorial difficulty, we make use of the *white noise functional* (see Definition 3.3 below) and the orthogonality property of the Hermite polynomials (Lemma 3.4) and entirely avoid combinatorial arguments of increasing complexity in k. This allows us to prove Proposition 3.1 in a concise and uniform manner. See Da Prato-Debussche [18] and Da Prato-Tubaro [19] for a presentation of this method in the context of the stochastic quantization equation on $\mathbb{T}^2$. See also a related recent work by the first and third authors [31] on the invariant Gibbs measures for the nonlinear Schrödinger equations on $\mathbb{T}^2$.

For simplicity of the presentation, we only treat (1.1) with the $+$ sign and drop the mass cutoff $\mathbf{1}_{\{\int u^2 dx \leq R\}}$ required for normalization of the Gibbs measures in the non-defocusing case. Note that the restriction on the values of k in the non-defocusing case in Theorem 1.1 simply comes from the normalization of the Gibbs measures as probability measures [24, 5] and that it does not appear in the proof of Theorem 1.1 in an explicit manner.

Remark 1.2. The mean-zero assumption in the construction of the Gibbs measure (and hence in Theorem 1.1) is not essential. In view of the mass conservation, one can consider

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[5]See [12] for examples of probabilistic estimates on various nonlinearities of low degrees, where the required combinatorics is relatively simple thanks to the low degree of the nonlinearities. See also [31, Appendix A] for a concrete combinatorial computation for the (Wick ordered) quintic nonlinearity.

the Gibbs measure of the form:

$$d\widehat{\mu}_k = Z^{-1} \exp\big(-\mathcal{E}(u) - \tfrac{1}{2}M(u)\big)du = Z^{-1}e^{\mp\frac{1}{k+1}\int_{\mathbb{T}} u^{k+1}dx}d\widehat{\rho}, \qquad (1.7)$$

where $\widehat{\rho}$ denotes the Gaussian measure given by

$$d\widehat{\rho} = Z_0^{-1}e^{-\frac{1}{2}\int_{\mathbb{T}}(\partial_x u)^2 dx - \frac{1}{2}\int_{\mathbb{T}} u^2 dx}du.$$

Under $\widehat{\rho}$, a typical element u is represented by

$$u(x) = u(x;\omega) = \sum_{n\in\mathbb{Z}^*} \frac{g_n(\omega)}{\sqrt{1+n^2}}e^{inx}, \qquad (1.8)$$

where $\{g_n\}_{n\in\mathbb{Z}_{\geq 0}}$ is a sequence of independent standard complex-valued Gaussian random variables conditioned that $g_{-n} = \overline{g_n}$, $n \in \mathbb{Z}_{\geq 0}$. Unlike (1.5), the random function u in (1.8) has a spatial mean $g_0(\omega)$. We point out that Theorem 1.1 with the exact same proof holds for the Gibbs measure $\widehat{\mu}_k$ in (1.7).

Remark 1.3. (i) Let $k \geq 4$. On the one hand, gKdV (1.1) is known to be locally well-posed in $H^s(\mathbb{T})$, $s \geq \frac{1}{2}$; see [33, 15]. On the other hand, it is known to be mildly ill-posed for $s < \frac{1}{2}$ in the sense that the solution map fails to be smooth in this range of regularity. See [7, 15]. Therefore, it is non-trivial to construct (local-in-time) solutions in the support of the Gibbs measure.

When $k = 4$, local-in-time dynamics was constructed in a probabilistic manner [32]. As in the deterministic case [33, 15], this probabilistic construction of solutions was carried out through a gauge transform and thus the uniqueness statement in [32] was very mild. While one can apply a similar probabilistic construction of solutions for $k \geq 5$, such construction requires case-by-case consideration (see [6, 16, 32]) and thus combinatorics gets out of control for large values of k. At this point, there seems to be no uniform way to perform this probabilistic construction for all values of k, rather than working out case-by-case analysis for a fixed value of k.

(ii) For the quartic gKdV ($k = 4$), the second author [32] proved the invariance of the Gibbs measure μ_4 for the following *gauged* gKdV:

$$\partial_t u + \partial_x^3 u = \pm 4\big(\mathbf{P}_{\neq 0}[u^3]\big)\partial_x u,$$

where $\mathbf{P}_{\neq 0}$ denotes the orthogonal projection onto the mean-zero functions. Through the inverse gauge transform, this result yields almost sure global well-posedness of the ungauged gKdV (1.1). The invariance of μ_4 under the dynamics of (1.1), however, is unknown. Moreover, we do not know if the Gibbs measure μ_k is absolutely continuous with respect to the pushforward of μ_k under the inverse gauge transform. This is a sharp contrast to the situation for the derivative cubic nonlinear Schrödinger equation. See [25, 26].

While Theorem 1.1 asserts the existence of global-in-time dynamics under which μ_4 is invariant, we do not know if these solutions coincide with the almost sure global solutions constructed in [32] due to the mild uniqueness statement under the inverse gauge transform.

2. ON THE TRUNCATED GKDV EQUATION

In this section, we introduce the truncated gKdV equation and the truncated Gibbs measure $\mu_{k,N}$ and discuss their basic properties. Given $N \in \mathbb{N}$, define E_N and $E_N^\perp$ by

$$E_N = \text{span}\{e^{in\cdot x}\}_{|n|\leq N} \qquad \text{and} \qquad E_N^\perp = \text{span}\{e^{in\cdot x}\}_{|n|>N}.$$

Consider the following truncated gKdV on $\mathbb{T}$:

$$\partial_t u^N + \partial_x^3 u^N = \partial_x \mathbf{P}_N[(\mathbf{P}_N u^N)^k], \tag{2.1}$$

where $\mathbf{P}_N$ denotes the Dirichlet projection onto the frequencies $\{|n| \leq N\}$. Letting $v^N = \mathbf{P}_N u^N$, we can decouple (2.1) into the following finite dimensional system of ODEs on E_N:

$$\partial_t v^N + \partial_x^3 v^N = \partial_x \mathbf{P}_N[(v^N)^k] \tag{2.2}$$

and the linear flow for high frequencies $\{|n| > N\}$:

$$\partial_t \mathbf{P}_N^\perp u^N + \partial_x^3 \mathbf{P}_N^\perp u^N = 0. \tag{2.3}$$

Here, $\mathbf{P}_N^\perp$ is the Dirichlet projection onto the high frequencies $\{|n| > N\}$. Note that the truncated gKdV (2.1) is a Hamiltonian PDE with

$$\mathcal{E}_N(u^N) = \frac{1}{2} \int_\mathbb{T} (\partial_x u^N)^2 dx + \frac{1}{k+1} \int_\mathbb{T} (\mathbf{P}_N u^N)^{k+1} dx. \tag{2.4}$$

Associated to the truncated gKdV (2.1), let us define the truncated Gibbs measure $\mu_{k,N}$ by[6]

$$d\mu_{k,N} = Z_N^{-1} \exp(-\mathcal{E}_N(u^N)) du^N = Z_N^{-1} R_N(u) d\rho(u), \tag{2.5}$$

where ρ is the Wiener measure defined in (1.4) and $R_N(u)$ is defined by

$$R_N(u) := e^{-\frac{1}{k+1} \int_\mathbb{T} (\mathbf{P}_N u)^{k+1} dx}.$$

It basically follows from the argument in [24, 5] that $R_N(u)$ converges to $R_\infty(u)$ in $L^p(\rho)$, $1 \leq p < \infty$, as $N \to \infty$. Consequently, for any $1 \leq p < \infty$, we have

$$\|R_N(u)\|_{L^p(\rho)} \leq C_p < \infty, \tag{2.6}$$

uniformly in $N \in \mathbb{N}$, and

$$\lim_{N \to \infty} \mu_{k,N}(A) = \mu_k(A) \tag{2.7}$$

for any measurable set $A \in \mathcal{B}_{H^s(\mathbb{T})}$, $s < \frac{1}{2}$. See also [12, 31].

We now decompose the Wiener measure ρ as

$$\rho = \rho_N \otimes \rho_N^\perp,$$

where ρ_N and $\rho_N^\perp$ are the marginals of ρ on E_N and $E_N^\perp$, respectively. Then, we can write the truncated Gibbs measure $\mu_{k,N}$ in (2.5) as

$$\mu_{k,N} = \widehat{\mu}_{k,N} \otimes \rho_N^\perp, \tag{2.8}$$

where $\widehat{\mu}_{k,N}$ is the finite dimensional Gibbs measure defined by

$$d\widehat{\mu}_{k,N} = Z_N^{-1} e^{-\frac{1}{k+1} \int_\mathbb{T} (\mathbf{P}_N u^N)^{k+1} dx} d\rho_N.$$

We have the following lemma on global well-posedness of the truncated gKdV (2.1) and the invariance of the truncated Gibbs measure $\mu_{k,N}$ under the dynamics of (2.1).

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[6]In the non-defocusing case, there is a mass cutoff $\mathbf{1}_{\{\int u^2 dx \leq R\}}$ which we omit for simplicity of the presentation.

Lemma 2.1. *Let $N \in \mathbb{N}$ and $s < \frac{1}{2}$. Then, the truncated gKdV (2.1) is globally well-posed in $H^s(\mathbb{T})$. Moreover, the truncated Gibbs measure $\mu_{k,N}$ is invariant under the dynamics of (2.1).*

In particular, Lemma 2.1 states that if the law of $u^N(0)$ is given by $\mu_{k,N}$, then the law of the corresponding solution $u^N(t)$ is again given by $\mu_{k,N}$ for any $t \in \mathbb{R}$.

Proof. We first prove global well-posedness of the truncated gKdV (2.1). We use the decomposition of (2.1) by the low frequency part (2.2) and the high frequency part (2.3). As a linear equation, the high frequency part (2.3) is globally well-posed. By viewing (2.2) on the Fourier side, we see that (2.2) is a finite dimensional system of ODEs of dimension $2N$. Hence, by the Cauchy-Lipschitz theorem, (2.2) is locally well-posed.

By a direct computation, it is easy to see from (2.2) that $\int_{\mathbb{T}} |v^N|^2 dx$ is conserved for (2.2). In particular, this shows that the Euclidean norm on the phase space $\mathbb{C}^{2N}$

$$\left\| \{\widehat{v^N}(n)\}_{|n| \leq N} \right\|_{\mathbb{C}^{2N}} = \left(\sum_{|n| \leq N} |\widehat{v^N}(n)|^2 \right)^{\frac{1}{2}} = \left(\int_{\mathbb{T}} |v^N|^2 dx \right)^{\frac{1}{2}}$$

is conserved under (2.2). This proves global existence for (2.2) and hence for the truncated gKdV (2.1).

On the one hand, the linear flow (2.3) leaves the Gaussian measure $\rho_N^\perp$ on $E_N^\perp$ invariant under the dynamics. On the other hand, noting that (2.2) is the finite dimensional Hamiltonian dynamics corresponding to $\mathcal{E}_N(v^N)$ defined in (2.4), we see that $\widehat{\mu}_{k,N}$ is invariant under (2.2). Therefore, in view of (2.8), the truncated Gibbs measure $\mu_{k,N}$ is invariant under the dynamics of (2.1). $\square$

3. HERMITE FUNCTIONS AND WHITE NOISE FUNCTIONAL

Let u be the random function defined in (1.5) distributed according to the Wiener measure ρ. Then, the nonlinearity $\partial_x(u^k)$ makes sense as a (spatial) distribution almost surely, since $u \in H^{\frac{1}{2}-\varepsilon}(\mathbb{T})$ for any $\varepsilon > 0$ and hence $u \in L^p(\mathbb{T})$ for any $p < \infty$ almost surely. Given $N \in \mathbb{N}$, define $F_N(u)$ and $F(u)$ by

$$F_N(u) := \mathbf{P}_N[(\mathbf{P}_N u)^k] \quad \text{and} \quad F(u) := F_\infty(u) = u^k. \tag{3.1}$$

The main goal of this section is to establish the following convergence property of $F_N(u)$ to $F(u)$.

Proposition 3.1. *Let $k \geq 2$ be an integer and $s < \frac{1}{2}$. Then, there exists $C_{k,s} > 0$ such that*

$$\|F_N(u)\|_{L^p(\rho;H^s)}, \ \|F(u)\|_{L^p(\rho;H^s)} \leq C_{k,s}(p-1)^{\frac{k}{2}} \tag{3.2}$$

for any $p \geq 1$ and any $N \in \mathbb{N}$. Moreover, given $\varepsilon > 0$ with $s + \varepsilon < \frac{1}{2}$, there exists $C_{k,s,\varepsilon} > 0$ such that

$$\|F_M(u) - F_N(u)\|_{L^p(\rho;H^s)} \leq C_{k,s,\varepsilon}(p-1)^{\frac{k}{2}} \frac{1}{N^\varepsilon} \tag{3.3}$$

for any $p \geq 1$ and any $1 \leq N \leq M \leq \infty$. In particular, $F_N(u)$ converges to $F(u)$ in $L^p(\rho;H^s(\mathbb{T}))$ as $N \to \infty$.

Remark 3.2. In order to construct global-in-time weak solutions claimed in Theorem 1.1, one only needs to prove in (3.2) and (3.3) with $s = 0$. This easily follows from the fact that $u \in L^q(\mathbb{T})$ for any $q < \infty$ almost surely. Then, one can proceed as in [12, Lemma 5.6]. On the other hand, Proposition 3.1 is optimal in the range of $s < \frac{1}{2}$ and shows the stability of $u \mapsto F_N(u)$ in the H^s-norm.

3.1. **Hermite polynomials and white noise functional.** First, recall from [23] the Hermite polynomials $H_n(x; \sigma)$ defined through the generating function:

$$G(t, x; \sigma) := e^{tx - \frac{1}{2}\sigma t^2} = \sum_{k=0}^{\infty} \frac{t^k}{k!} H_k(x; \sigma). \tag{3.4}$$

For simplicity, we set $G(t, x) := G(t, x; 1)$ and $H_k(x) := H_k(x; 1)$ in the following. For readers' convenience, we write out the first few Hermite polynomials:

$$H_0(x; \sigma) = 1, \qquad H_1(x; \sigma) = x, \qquad H_2(x; \sigma) = x^2 - \sigma,$$
$$H_3(x; \sigma) = x^3 - 3\sigma x, \qquad H_4(x; \sigma) = x^4 - 6\sigma x^2 + 3\sigma^2.$$

Then, the monomial x^n can be expressed in term of the Hermite polynomials:

$$x^k = \sum_{m=0}^{\left[\frac{k}{2}\right]} \binom{k}{2m} (2m - 1)!! \, \sigma^m H_{k-2m}(x; \sigma), \tag{3.5}$$

where $(2m - 1)!! = (2m - 1)(2m - 3) \cdots 3 \cdot 1 = \frac{(2m)!}{2^m m!}$ and $(-1)!! := 1$ by convention.

Next, we define the white noise functional. Let $w(x; \omega)$ be the (real-valued) mean-zero Gaussian white noise on $\mathbb{T}$ defined by

$$w(x; \omega) = \sum_{n \in \mathbb{Z}^*} g_n(\omega) e^{inx}.$$

Definition 3.3. The *white noise functional* $W_{(\cdot)} : L^2(\mathbb{T}) \to L^2(\Omega)$ is defined by

$$W_f(\omega) = \langle f, w(\omega) \rangle_{L_x^2} = \sum_{n \in \mathbb{Z}^*} \widehat{f}(n) \overline{g_n}(\omega) \tag{3.6}$$

for a real-valued function $f \in L^2(\mathbb{T})$. Here, $\{g_n\}_{n \in \mathbb{N}}$ is a sequence of independent standard complex-valued Gaussian random variables conditioned that $g_{-n} = \overline{g_n}$, $n \in \mathbb{N}$, as in (1.5).

For real-valued $f \in L^2(\mathbb{T})$, W_f is a real-valued Gaussian random variable with mean 0 and variance $\|f\|_{L^2}^2$. Moreover, we have

$$E[W_f W_h] = \langle f, h \rangle_{L_x^2}$$

for $f, h \in L^2(\mathbb{T})$. In particular, the white noise functional $W_{(\cdot)}$ is an isometry from $L^2(\mathbb{T})$ onto $L^2(\Omega)$.

The following orthogonality lemma on the white noise functional and Hermite polynomials is well known [19] and will play an essential role in the subsequent analysis. We present the proof for readers' convenience.

Lemma 3.4. *Let $f, h \in L^2(\mathbb{T})$ such that $\|f\|_{L^2} = \|h\|_{L^2} = 1$. Then, for $k, m \in \mathbb{Z}_{\geq 0}$, we have*

$$\mathbb{E}[H_k(W_f) H_m(W_h)] = \delta_{km} k! [\langle f, h \rangle_{L_x^2}]^k. \tag{3.7}$$

Here, δ_{km} denotes the Kronecker delta function.

Proof. First recall the following identity:

$$\int_\Omega e^{W_f(\omega)}dP = \prod_{n\in\mathbb{N}} \frac{1}{\pi}\int_\mathbb{C} e^{2\,\mathrm{Re}(\widehat{f}(n)\bar{g}_n)}e^{-|g_n|^2}dg_n$$

$$= \prod_{n\in\mathbb{N}} \frac{1}{\pi}\int_\mathbb{R} e^{2\,\mathrm{Re}\,\widehat{f}(n)\,\mathrm{Re}\,g_n}e^{-(\mathrm{Re}\,g_n)^2}d\,\mathrm{Re}\,g_n\int_\mathbb{R} e^{2\,\mathrm{Im}\,\widehat{f}(n)\,\mathrm{Im}\,g_n}e^{-(\mathrm{Im}\,g_n)^2}d\,\mathrm{Im}\,g_n$$

$$= e^{\sum_{n\in\mathbb{N}}|\widehat{f}(n)|^2} = e^{\frac{1}{2}\|f\|_{L^2}^2}.$$

Let G be as in (3.4). Then, for any $t,s\in\mathbb{R}$ and $f,h\in L^2(\mathbb{T})$ with $\|f\|_{L^2} = \|h\|_{L^2} = 1$, we have

$$\int_\Omega G(t,W_f(\omega))G(s,W_h(\omega))dP(\omega) = e^{-\frac{t^2+s^2}{2}}\int_\Omega e^{W_{tf+sh}(\omega)}dP(\omega)$$

$$= e^{-\frac{t^2+s^2}{2}}e^{\frac{1}{2}\|tf+sh\|_{L^2}^2} = e^{ts\langle f,h\rangle_{L_x^2}}. \qquad (3.8)$$

Thus, it follows from (3.4) and (3.8) that

$$e^{ts\langle f,h\rangle_{L^2}} = \sum_{k,m=0}^\infty \frac{t^k s^m}{k!m!}\int_\Omega H_k(W_f(\omega))H_m(W_h(\omega))dP(\omega).$$

By comparing the coefficients of $t^k s^m$, we obtain (3.7). $\qquad\square$

Given $N\in\mathbb{N}\cup\{\infty\}$, define

$$\sigma_N := \mathbb{E}\big[\|\mathbf{P}_N u\|_{L^2}^2\big] = \sum_{1\le|n|\le N}\frac{1}{n^2}$$

with the understanding that $\mathbf{P}_\infty = \mathrm{Id}$. For *fixed* $x\in\mathbb{T}$ and $N\in\mathbb{N}\cup\{\infty\}$, we also define

$$\eta_N(x)(\cdot) := \frac{1}{\sigma_N^{\frac{1}{2}}}\sum_{1\le|n|\le N}\frac{\overline{e_n(x)}}{|n|}e_n(\cdot), \qquad (3.9)$$

$$\gamma_N(\cdot) := \sum_{1\le|n|\le N}\frac{1}{n^2}e_n(\cdot),$$

where $e_n(y) = e^{iny}$. Note that

$$\|\eta_N(x)\|_{L^2(\mathbb{T})} = 1 \qquad (3.10)$$

for all fixed $x\in\mathbb{T}$ and all $N\in\mathbb{N}\cup\{\infty\}$. Moreover, we have

$$\langle\eta_M(x),\eta_N(y)\rangle_{L_x^2} = \frac{1}{\sigma_M^{\frac{1}{2}}\sigma_N^{\frac{1}{2}}}\gamma_N(y-x), \qquad (3.11)$$

for fixed $x,y\in\mathbb{T}$ and $N,M\in\mathbb{N}\cup\{\infty\}$ with $M\ge N$. Note that $\sigma_N \le \sigma_\infty = \frac{\pi^2}{3}$ for all $N\in\mathbb{N}$.

We now establish a second moment bound on the Fourier coefficients of the (truncated) nonlinearity $F_N(u)$ and $F(u)$ defined in (3.1).

Lemma 3.5. *Let $k \geq 2$ be an integer. Then, there exists $C_k > 0$ such that*

$$\|\langle F_N(u), e_n\rangle_{L_x^2}\|_{L^2(\rho)}, \ \|\langle F(u), e_n\rangle_{L_x^2}\|_{L^2(\rho)} \leq C_k \frac{1}{|n|} \tag{3.12}$$

for any $n \in \mathbb{Z}^*$ and any $N \in \mathbb{N}$. Moreover, given positive $\varepsilon < \frac{1}{2}$, there exists $C_{k,\varepsilon} > 0$ such that

$$\|\langle F_M(u) - F_N(u), e_n\rangle_{L_x^2}\|_{L^2(\rho)} \leq C_{k,\varepsilon} \frac{1}{N^\varepsilon |n|^{1-\varepsilon}} \tag{3.13}$$

for any $n \in \mathbb{Z}^*$ and any $1 \leq N \leq M \leq \infty$.

Proof. We first prove (3.12). Let $N \in \mathbb{N} \cup \{\infty\}$. Given $x \in \mathbb{T}$, it follows from (1.5), (3.6), and (3.9) that

$$\mathbf{P}_N u(x) = \sigma_N^{\frac{1}{2}} \frac{u_N(x)}{\sigma_N^{\frac{1}{2}}} = \sigma_N^{\frac{1}{2}} \overline{W_{\eta_N(x)}} = \sigma_N^{\frac{1}{2}} W_{\eta_N(x)}. \tag{3.14}$$

Then, from (3.5) and (3.14), we have

$$[\mathbf{P}_N u(x)]^k = \sigma_N^{\frac{k}{2}} \sum_{m=0}^{[\frac{k}{2}]} \binom{k}{2m} (2m-1)!! \, H_{k-2m}(W_{\eta_N(x)}). \tag{3.15}$$

Clearly, $\langle F_N(u), e_n\rangle_{L_x^2} = 0$ when $|n| > N$. Thus, we only need to consider the case $|n| \leq N$. From Lemma 3.4 with (3.15), (3.10), and (3.11), we have

$$\|\langle F_N(u), e_n\rangle_{L_x^2}\|_{L^2(\rho)}^2 = \sigma_N^k \int_{\mathbb{T}_x \times \mathbb{T}_y} e_n(x)\overline{e_n(y)}$$

$$\times \sum_{m,\widetilde{m}=0}^{[\frac{k}{2}]} \binom{k}{2m} (2m-1)!! \binom{k}{2\widetilde{m}} (2\widetilde{m}-1)!!$$

$$\times \int_\Omega H_{k-2m}(W_{\eta_N(x)}) H_{k-2\widetilde{m}}(W_{\eta_N(y)}) dP dx dy$$

$$= \sum_{m=0}^{[\frac{k}{2}]} \binom{k}{2m}^2 \left[(2m-1)!!\right]^2 (k-2m)! \, \sigma_N^{2m} \int_{\mathbb{T}_x \times \mathbb{T}_y} \left[\gamma_N(y-x)\right]^{k-2m} \overline{e_n(y-x)} dx dy$$

$$= \sum_{m=0}^{[\frac{k}{2}]} c_{k,m} \sigma_N^{2m} \mathcal{F}\left[\gamma_N^{k-2m}\right](n). \tag{3.16}$$

Given $n = n_1 + \cdots + n_{k-2m}$, we have $\max_j |n_j| \gtrsim |n|$ and thus

$$\mathcal{F}\left[\gamma_N^{k-2m}\right](n) = \sum_{\substack{n=n_1+\cdots+n_{k-2m} \\ 1 \leq |n_j| \leq N}} \prod_{j=1}^{k-2m} \frac{1}{n_j^2} \leq d_{k,m} \frac{1}{n^2}. \tag{3.17}$$

Hence, (3.12) follows from (3.16) and (3.17).

Next, we prove (3.13). Proceeding as before with (3.15), Lemma 3.4 for $1 \leq N \leq M$, and (3.11), we have

$$\|\langle F_M(u) - F_N(u), e_n \rangle_{L_x^2}\|_{L^2(\rho)}^2 = \int_{\mathbb{T}_x \times \mathbb{T}_y} e_n(x)\overline{e_n(y)}$$

$$\times \sum_{m,\widetilde{m}=0}^{[\frac{k}{2}]} \binom{k}{2m} (2m-1)!! \binom{k}{2\widetilde{m}} (2\widetilde{m}-1)!!$$

$$\times \int_\Omega \Big[\mathbf{1}_{[1,M]}(|n|)\sigma_M^k H_{k-2m}(W_{\eta_M(x)})H_{k-2\widetilde{m}}(W_{\eta_M(y)})$$

$$- \mathbf{1}_{[1,N]}(|n|)\sigma_M^{\frac{k}{2}}\sigma_N^{\frac{k}{2}} H_{k-2m}(W_{\eta_M(x)})H_{k-2\widetilde{m}}(W_{\eta_N(y)})$$

$$- \mathbf{1}_{[1,N]}(|n|)\sigma_M^{\frac{k}{2}}\sigma_N^{\frac{k}{2}} H_{k-2m}(W_{\eta_N(x)})H_{k-2\widetilde{m}}(W_{\eta_M(y)})$$

$$+ \mathbf{1}_{[1,N]}(|n|)\sigma_N^k H_{k-2m}(W_{\eta_N(x)})H_{k-2\widetilde{m}}(W_{\eta_N(y)})\Big] dP dx dy$$

$$= \sum_{m=0}^{[\frac{k}{2}]} c_{k,m}\mathbf{1}_{[1,N]}(|n|)\Big\{ \sigma_M^{2m}\mathcal{F}\big[\gamma_M^{k-2m}\big](n) - 2\sigma_M^m \sigma_N^m \mathcal{F}\big[\gamma_N^{k-2m}\big](n) + \sigma_N^{2m}\mathcal{F}\big[\gamma_N^{k-2m}\big](n) \Big\}$$

$$+ \sum_{m=0}^{[\frac{k}{2}]} c_{k,m}\mathbf{1}_{(N,M]}(|n|)\sigma_M^{2m}\mathcal{F}\big[\gamma_M^{k-2m}\big](n). \tag{3.18}$$

On the one hand, noting that $|n| > N$, we can use (3.17) to estimate the second sum on the right-hand side of (3.18), yielding (3.13). On the other hand, noting that

$$\Big|\mathcal{F}\big[\gamma_M^{k-2m}\big](n) - \mathcal{F}\big[\gamma_N^{k-2m}\big](n)\Big| \leq \sum_{\substack{n=n_1+\cdots+n_{k-2m} \\ 1 \leq |n_j| \leq M \\ \max_j |n_j| \geq N}} \prod_{j=1}^{k-2m} \frac{1}{n_j^2} \leq d_{k,m}\frac{1}{\max(N,|n|)^2} \tag{3.19}$$

and

$$|\sigma_M^m - \sigma_N^m| \leq C_m |\sigma_M - \sigma_N| \lesssim \frac{1}{N}, \tag{3.20}$$

we can use (3.17), (3.19), and (3.20) to estimate the first sum on the right-hand side of (3.18), yielding (3.13). $\square$

As an immediate corollary to Lemma 3.5, we obtain the following estimate on the H^s-norm of $F_N(u)$, establishing Proposition 3.1 for $p = 2$.

Corollary 3.6. *Let $k \geq 2$ be an integer. Let $s < \frac{1}{2}$. Then, there exists $C_{k,s} > 0$ such that*

$$\|F_N(u)\|_{L^2(\rho;H^s)}, \ \|F(u)\|_{L^2(\rho;H^s)} \leq C_{k,s} \tag{3.21}$$

for any $N \in \mathbb{N}$. Moreover, given $\varepsilon > 0$ with $s + \varepsilon < \frac{1}{2}$, there exists $C_{k,s,\varepsilon} > 0$ such that

$$\|F_M(u) - F_N(u)\|_{L^2(\rho;H^s)} \leq \frac{C_{k,s,\varepsilon}}{N^\varepsilon} \tag{3.22}$$

for any $1 \leq N \leq M \leq \infty$.

3.2. **Wiener chaos estimates.** In this subsection, we prove Proposition 3.1 by extending (3.21) and (3.22) in Corollary 3.6 to any finite $p \geq 2$. This is achieved by an application of the Wiener chaos estimate (Lemma 3.9).

Fix $d \in \mathbb{N}$.[7] Consider the Hilbert space $H = L^2(\mathbb{R}^d, \mu_d)$ endowed with the Gaussian measure $d\mu_d = (2\pi)^{-\frac{d}{2}} \exp(-|x|^2/2)dx$, $x = (x_1, \ldots, x_d) \in \mathbb{R}^d$. Let $L := \Delta - x \cdot \nabla$ be the Ornstein-Uhlenbeck operator. Then, we have the following hypercontractivity of the Ornstein-Uhlenbeck semigroup $S(t) := e^{tL}$ due to Nelson [27].

Lemma 3.7. *Let $p \geq 2$. Then, for every $u \in L^p(\mathbb{R}^d, \mu_d)$ and $t \geq \frac{1}{2} \log(p-1)$, we have*

$$\|S(t)u\|_{L^p(\mathbb{R}^d, \mu_d)} \leq \|u\|_{L^2(\mathbb{R}^d, \mu_d)}. \tag{3.23}$$

We stress that the estimate (3.23) is independent of the dimension d.

Next, we define a *homogeneous Wiener chaos of order k* to be an element of the form $\prod_{j=1}^d H_{k_j}(x_j)$, where $k = k_1 + \cdots + k_d$ and H_{k_j} is the Hermite polynomial of degree k_j defined in (3.4). Then, we have the following Ito-Wiener decomposition:

$$L^2(\mathbb{R}^d, \mu_d) = \bigoplus_{k=0}^\infty \mathcal{H}_k,$$

where $\mathcal{H}_k$ is the closure of homogeneous Wiener chaoses of order k under $L^2(\mathbb{R}^d, \mu_d)$. We obtain the following corollary to Lemma 3.7 for elements in $\mathcal{H}_k$.

Lemma 3.8. *Let $F \in \mathcal{H}_k$. Then, for $p \geq 2$, we have*

$$\|F\|_{L^p(\mathbb{R}^d, \mu_d)} \leq (p-1)^{\frac{k}{2}} \|F\|_{L^2(\mathbb{R}^d, \mu_d)}. \tag{3.24}$$

It is known that any element in $\mathcal{H}_k$ is an eigenfunction of L with eigenvalue $-k$. Then, the estimate (3.24) follows immediately from noting that F is an eigenfunction of $S(t) = e^{tL}$ with eigenvalue e^{-tk} and setting $t = \frac{1}{2} \log(p-1)$ in (3.23).

As a further consequence to Lemma 3.8, we obtain the following Wiener chaos estimate.

Lemma 3.9. *Fix $k \in \mathbb{N}$ and $c(n_1, \ldots, n_k) \in \mathbb{C}$. Given $d \in \mathbb{N}$, let $\{g_n\}_{n=1}^d$ be a sequence of independent standard complex-valued Gaussian random variables and set $g_{-n} = \overline{g_n}$. Define $S_k(\omega)$ by*

$$S_k(\omega) = \sum_{\Gamma(k,d)} c(n_1, \ldots, n_k) g_{n_1}(\omega) \cdots g_{n_k}(\omega),$$

where $\Gamma(k, d)$ is defined by

$$\Gamma(k, d) = \left\{ (n_1, \ldots, n_k) \in \{\pm 1, \ldots, \pm d\}^k \right\}.$$

Then, for $p \geq 2$, we have

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k+1}(p-1)^{\frac{k}{2}} \|S_k\|_{L^2(\Omega)}. \tag{3.25}$$

Note that the estimate (3.25) is independent of $d \in \mathbb{N}$. Lemma 3.9 follows from (3.5) and Lemma 3.8. See Proposition 2.4 in [34] for details. Lemmas 3.8 and 3.9 have been very effective in the probabilistic study of dispersive PDEs and related areas. [37, 34, 3, 16, 25, 32, 12].

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[7]Indeed, the discussion presented here also holds for $d = \infty$ in the context of abstract Wiener spaces. For simplicity, however, we only consider finite values for d.

We are now ready to present the proof of Proposition 3.1.

Proof of Proposition 3.1. We only prove (3.2) for $N = \infty$. The proofs of (3.2) for $N \in \mathbb{N}$ and (3.3) are analogous in view of Lemma 3.5.

Let $p \geq 2$. By Minkowski's integral inequality with (1.5) and (3.1) followed by Lemma 3.9 and Lemma 3.5, we have

$$
\begin{aligned}
\|F(u)\|_{L^p(\rho;H^s)} &= \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \left\| \sum_{\substack{n=n_1+\cdots n_k \\ n_j \in \mathbb{Z}^*}} \prod_{j=1}^{k} \frac{g_{n_j}(\omega)}{|n_j|} \right\|^2_{L^p(\Omega)} \right)^{\frac{1}{2}} \\
&\leq \sqrt{k+1}(p-1)^{\frac{k}{2}} \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \left\| \sum_{\substack{n=n_1+\cdots n_k \\ n_j \in \mathbb{Z}^*}} \prod_{j=1}^{k} \frac{g_{n_j}(\omega)}{|n_j|} \right\|^2_{L^2(\Omega)} \right)^{\frac{1}{2}} \\
&= \sqrt{k+1}(p-1)^{\frac{k}{2}} \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s} \|\langle F(u), e_n \rangle_{L^2_x}\|^2_{L^2(\rho)} \right)^{\frac{1}{2}} \\
&\leq C_k(p-1)^{\frac{k}{2}} \left(\sum_{n \in \mathbb{Z}^*} |n|^{2s-2} \right)^{\frac{1}{2}} \leq C_{k,s}(p-1)^{\frac{k}{2}}
\end{aligned}
$$

as long as $s < \frac{1}{2}$. This proves (3.2) for $N = \infty$.   □

4. Proof of Theorem 1.1

In this section, we present the proof of Theorem 1.1. Fix an integer $k \geq 2$ and $s < \frac{1}{2}$ in the following. The basic structure of the argument follows that in [12, 31]. We point out a (minor) difference in the presentations in [12] and [31]. On the one hand, the argument in [12] was first carried out on a finite time interval $[-T, T]$ for $T > 0$. Namely, given $T > 0$, we construct a set Σ_T of full probability, guaranteeing the existence of solutions on $[-T, T]$, such that the law of the random function $u(t)$, $t \in [-T, T]$, is given by μ_k. Then, the desired set Σ of full probability of global existence was constructed as $\Sigma = \bigcap_{N \in \mathbb{N}} \Sigma_N$. On the other hand, the desired set Σ of full probability of global existence was directly constructed in [31] without restricting the argument onto finite time intervals. In the following, we follow the approach presented in [31].

Given $N \in \mathbb{N}$, let $\mu_{k,N}$ be the invariant truncated Gibbs measure for the truncated gKdV (2.1) constructed in Section 2. We first extend $\mu_{k,N}$ to a measure on space-time functions. Let $\Phi_N : H^s(\mathbb{T}) \to C(\mathbb{R}; H^s(\mathbb{T}))$ be the solution map to (2.1) constructed in Lemma 2.1. By endowing $C(\mathbb{R}; H^s(\mathbb{T}))$ with the compact-open topology,[8] it follows from the local Lipschitz continuity of $\Phi_N(\cdot)$ that Φ_N is continuous from $H^s(\mathbb{T})$ into $C(\mathbb{R}; H^s(\mathbb{T}))$. We now define a probability measure ν_N on $C(\mathbb{R}; H^s(\mathbb{T}))$ by setting

$$
\nu_N = \mu_{k,N} \circ \Phi_N^{-1}.
$$

---

[8]Under the compact-open topology, a sequence $\{u_n\}_{n \in \mathbb{N}} \subset C(\mathbb{R}; H^s(\mathbb{T}))$ converges if and only if it converges uniformly on any compact time interval.

Namely, we define ν_N as the induced probability measure of $\mu_{k,N}$ under the map Φ_N. In particular, we have

$$\int_{C(\mathbb{R};H^s)} F(u)d\nu_N(u) = \int_{H^s} F(\Phi_N(\phi))d\mu_{k,N}(\phi)$$

for any measurable function $F : C(\mathbb{R};H^s(\mathbb{T})) \to \mathbb{R}$.

Our first goal is to show that $\{\nu_N\}_{N\in\mathbb{N}}$ converges to some probability measure ν on $C(\mathbb{R};H^s(\mathbb{T}))$. For this purpose, recall the following definition of tightness for a sequence of probability measures.

Definition 4.1. A sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of probability measures on a metric space $\mathcal{S}$ is said to be *tight* if, for every $\varepsilon > 0$, there exists a compact set K_ε such that $\rho_n(K_\varepsilon^c) \leq \varepsilon$ for all $n \in \mathbb{N}$.

Recall the following Prokhorov's theorem on a tight sequence of probability measures. See [2].

Lemma 4.2 (Prokhorov's theorem)**.** *If a sequence of probability measures on a metric space $\mathcal{S}$ is tight, then there is a subsequence that converges weakly to a probability measure on $\mathcal{S}$.*

The following proposition shows that the family $\{\nu_N\}_{N\in\mathbb{N}}$ is tight and hence converges weakly to some probability measure ν on $C(\mathbb{R};H^s(\mathbb{T}))$.

Proposition 4.3. *The family $\{\nu_N\}_{N\in\mathbb{N}}$ of the probability measures on $C(\mathbb{R};H^s(\mathbb{T}))$ is tight.*

Similar tightness results were proven in [12, 31] in the context of the Gibbs measures for other evolution equations. Before proceeding to the proof of Proposition 4.3, we first state several lemmas. We use the following notations. Given $T > 0$, we write $L_T^p H^s$ for $L^p([-T,T];H^s(\mathbb{T}))$. We use a similar abbreviation for other function spaces in time.

The first lemma provides a uniform control on the size of random space-time functions under ν_N. It follows as a consequence of the invariance of $\mu_{k,N}$ under the dynamics of the truncated gKdV (2.1) (Lemma 2.1). See [12, 31] for the proof.

Lemma 4.4. *Let $s < \frac{1}{2}$ and $p \geq 1$. Then, there exists $C_p > 0$ such that*

$$\big\|\|u\|_{L_T^p H^s}\big\|_{L^p(\nu_N)} \leq C_p T^{\frac{1}{p}},$$

$$\big\|\|u\|_{W_T^{1,p} H^{s-3}}\big\|_{L^p(\nu_N)} \leq C_p T^{\frac{1}{p}},$$

uniformly in $N \in \mathbb{N}$.

Recall also the following lemma on deterministic functions from [12].

Lemma 4.5 ([12, Lemma 3.3])**.** *Let $T > 0$ and $1 \leq p \leq \infty$. Suppose that $u \in L_T^p H^{s_1}$ and $\partial_t u \in L_T^p H^{s_2}$ for some $s_2 \leq s_1$. Then, for $\delta > p^{-1}(s_1 - s_2)$, we have*

$$\|u\|_{L_T^\infty H^{s_1-\delta}} \lesssim \|u\|_{L_T^p H^{s_1}}^{1-\frac{1}{p}} \|u\|_{W_T^{1,p} H^{s_2}}^{\frac{1}{p}}.$$

Moreover, there exist $\alpha > 0$ and $\theta \in [0,1]$ such that for all $t_1, t_2 \in [-T,T]$, we have

$$\|u(t_2) - u(t_1)\|_{H^{s_1-2\delta}} \lesssim |t_2 - t_1|^\alpha \|u\|_{L_T^p H^{s_1}}^{1-\theta} \|u\|_{W_T^{1,p} H^{s_2}}^\theta.$$

We now present the proof of Proposition 4.3.

Proof of Proposition 4.3. Let $s < s_1 < s_2 < \frac{1}{2}$. For $\alpha \in (0,1)$ and $T > 0$, we define the Lipschitz space $C_T^\alpha H^{s_1} = C^\alpha([-T, T]; H^{s_1}(\mathbb{T}))$ by the norm

$$\|u\|_{C_T^\alpha H^{s_1}} = \sup_{\substack{t_1, t_2 \in [-T,T] \\ t_1 \neq t_2}} \frac{\|u(t_1) - u(t_2)\|_{H^{s_1}}}{|t_1 - t_2|^\alpha} + \|u\|_{L_T^\infty H^{s_1}}.$$

Note that the embedding $C_T^\alpha H^{s_1} \subset C_T H^s$ is compact for each $T > 0$. This follows from the compact embedding of $H^{s_1}(\mathbb{T})$ into $H^s(\mathbb{T})$ and the Hölder regularity in time of functions in $C_T^\alpha H^{s_1}$, allowing us to apply Arzelà-Ascoli's theorem.

For $j \in \mathbb{N}$, let $T_j = 2^j$. Given $\varepsilon > 0$, define K_ε by

$$K_\varepsilon = \left\{ u \in C(\mathbb{R}; H^s) : \|u\|_{C_{T_j}^\alpha H^{s_1}} \leq c_0 \varepsilon^{-1} T_j^{1+\frac{1}{p}} \text{ for all } j \in \mathbb{N} \right\}$$

for some $p \geq 1$ (to be chosen later). Let $\{u_n\}_{n \in \mathbb{N}} \subset K_\varepsilon$. By the definition of K_ε, $\{u_n\}_{n \in \mathbb{N}}$ is bounded in $C_{T_j}^\alpha H^{s_1}$ for each $j \in \mathbb{N}$. Then, in view of the compact embedding $C_T^\alpha H^{s_1} \subset C_T H^s$, we can apply the diagonal argument to extract a subsequence $\{u_{n_\ell}\}_{\ell \in \mathbb{N}}$ convergent in $C_{T_j}^\alpha H^s$ for each $j \in \mathbb{N}$. In particular, $\{u_{n_\ell}\}_{\ell \in \mathbb{N}}$ converges uniformly in H^s on any compact time interval. Hence, $\{u_{n_\ell}\}_{\ell \in \mathbb{N}}$ converges in $C(\mathbb{R}; H^s)$ endowed with the compact-open topology. This proves that K_ε is compact in $C(\mathbb{R}; H^s)$.

By Lemma 4.5 with large $p \gg 1$ and Young's inequality followed by Lemma 4.4, we have

$$\left\| \|u\|_{C_T^\alpha H^{s_1}} \right\|_{L^p(\nu_N)} \lesssim \left\| \|u\|_{L_T^p H^{s_2}}^{1-\theta} \|u\|_{W_T^{1,p} H^{s_2-3}}^\theta \right\|_{L^p(\nu_N)}$$

$$\lesssim \left\| \|u\|_{L_T^p H^{s_2}} \right\|_{L^p(\nu_N)} + \left\| \|u\|_{W_T^{1,p} H^{s_2-3}} \right\|_{L^p(\nu_N)} \leq C_p T^{\frac{1}{p}}. \qquad (4.1)$$

for some $\alpha \in (0,1)$ and $\theta \in [0,1]$, uniformly in $N \in \mathbb{N}$. Then, by Markov's inequality with (4.1) and choosing $c_0 > 0$ sufficiently large, we have

$$\nu_N(K_\varepsilon^c) \leq c_0^{-1} \varepsilon T_j^{-1-\frac{1}{p}} \left\| \|u\|_{C_{T_j}^\alpha H^{s_1}} \right\|_{L^1(\nu_N)} \leq c_0^{-1} C_p \varepsilon \sum_{j=1}^\infty T_j^{-1} = c_0^{-1} C_p \varepsilon < \varepsilon.$$

This completes the proof of Proposition 4.3. □

As a consequence of Proposition 4.3 and Lemma 4.2, we conclude that, passing to a subsequence, ν_{N_j} converges weakly to some probability measure ν on $C(\mathbb{R}; H^s(\mathbb{T}))$.

Next, recall the following Skorokhod's theorem. See [2] for the proof.

Lemma 4.6 (Skorokhod's theorem). *Let $\mathcal{S}$ be a complete separable metric space. Suppose that ρ_n are probability measures on $\mathcal{S}$ converging weakly to a probability measure ρ. Then, there exist random variables $X_n : \widetilde{\Omega} \to \mathcal{S}$ with laws ρ_n and a random variable $X : \widetilde{\Omega} \to \mathcal{S}$ with law ρ such that $X_n \to X$ almost surely.*

It follows from the weak convergence of ν_{N_j} to ν and Lemma 4.6 that there exist another probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{P})$, a sequence $\{\widetilde{u^{N_j}}\}_{j \in \mathbb{N}}$ of $C(\mathbb{R}; H^s)$-valued random variables, and a $C(\mathbb{R}; H^s)$-valued random variable u such that

$$\mathcal{L}(\widetilde{u^{N_j}}) = \mathcal{L}(u^{N_j}) = \nu_N, \qquad \mathcal{L}(u) = \nu, \qquad (4.2)$$

and $\widetilde{u^{N_j}}$ converges to u in $C(\mathbb{R}; H^s(\mathbb{T}))$ almost surely with respect to $\widetilde{P}$. Then, Theorem 1.1 follows from the following proposition.

Proposition 4.7. *Let $\widetilde{u}_{N_j}$, $j \in \mathbb{N}$, and u be as above. Then, $\widetilde{u^{N_j}}$ and u are global-in-time distributional solutions to the truncated gKdV (2.1) and to gKdV (1.1), respectively. Moreover, we have*

$$\mathcal{L}\big(\widetilde{u^{N_j}}(t)\big) = \mu_{k,N_j} \quad and \quad \mathcal{L}(u(t)) = \mu_k \tag{4.3}$$

for any $t \in \mathbb{R}$.

Proof. We first prove (4.3). Fix $t \in \mathbb{R}$. Let $R_t : C(\mathbb{R}; H^s) \to H^s$ be the evaluation map defined by $R_t(v) = v(t)$. Then, from Lemma 2.1, we have

$$\mu_{k,N_j} = \nu_{N_j} \circ R_t^{-1}. \tag{4.4}$$

Denoting by $\nu_{N_j}^t$ the distribution of $\widetilde{u^{N_j}}(t)$, it follows from (4.2) and (4.4) that

$$\nu_{N_j}^t = \nu_{N_j} \circ R_t^{-1} = \mu_{k,N_j}. \tag{4.5}$$

In view of the almost sure convergence of $\widetilde{u^{N_j}}$ to u in $C(\mathbb{R}; H^s)$, $\widetilde{u^{N_j}}(t)$ converges to $u(t)$ in H^s almost surely for any $t \in \mathbb{R}$. Then, denoting by ν^t the distribution of $u(t)$, it follows from the dominated convergence theorem with (4.5) and (2.7) that

$$\nu^t(A) = \int \mathbf{1}_{\{u(t)(\omega) \in A\}} d\widetilde{P} = \lim_{j \to \infty} \int \mathbf{1}_{\left\{\widetilde{u^{N_j}}(t)(\omega) \in A\right\}} d\widetilde{P} = \lim_{j \to \infty} \mu_{k,N_j}(A) = \mu_k(A)$$

for any $A \in \mathcal{B}_{H^s(\mathbb{T})}$, $s < \frac{1}{2}$. This proves that (4.3).

Hence, it remains to show that $\widetilde{u^{N_j}}$ and u are global-in-time distributional solutions to the truncated gKdV (2.1) and to gKdV (1.1), respectively. For $j \in \mathbb{N}$, define the $\mathcal{D}'_{t,x}$-valued random variable X_j by

$$X_j = \partial_t u^{N_j} + \partial_x^3 u^{N_j} - \partial_x \mathbf{P}_{N_j}\big[(\mathbf{P}_{N_j} u^{N_j})^k\big].$$

Here, $\mathcal{D}'_{t,x} = \mathcal{D}'(\mathbb{R} \times \mathbb{T})$ denotes the space of space-time distributions on $\mathbb{R} \times \mathbb{T}$. We define $\widetilde{X}_j$ for $\widetilde{u^{N_j}}$ in an analogous manner. Noting that u^{N_j} is a global solution to (2.1), we see that $\mathcal{L}_{\mathcal{D}'_{t,x}}(X_j) = \delta_0$, where δ_0 denotes the Dirac delta measure. By (4.2), we also have

$$\mathcal{L}_{\mathcal{D}'_{t,x}}(\widetilde{X}_j) = \delta_0,$$

for each $j \in \mathbb{N}$. In particular, $\widetilde{u^{N_j}}$ is a global solution to the truncated gKdV (2.1) in the distributional sense, almost surely with respect to $\widetilde{P}$.

Recall that $\widetilde{u^{N_j}}$ converges to u in $C(\mathbb{R}; H^s)$ almost surely with respect to $\widetilde{P}$. Hence, we have the almost sure convergence of the linear part:

$$\partial_t \widetilde{u^{N_j}} + \partial_x^3 \widetilde{u^{N_j}} \longrightarrow \partial_t u + \partial_x^3 u$$

in $\mathcal{D}'(\mathbb{R} \times \mathbb{T})$ as $j \to \infty$.

Next, we discuss the almost sure convergence of the truncated nonlinearity in the distributional sense. It suffices to show that $F_{N_j}\big(\widetilde{u^{N_j}}\big) = \mathbf{P}_{N_j}\big[(\mathbf{P}_{N_j} u^{N_j})^k\big]$ converges to $F(u) = u^k$ in the distributional sense, almost surely. For simplicity of notation, let $F_j = F_{N_j}$ and $u_j = \widetilde{u^{N_j}}$.

Fix $T > 0$ and let $s < \frac{1}{2}$. By Lemma 2.1 and Proposition 3.1 with (2.5) and (2.6), we have

$$\left\| \|F_j(u_j) - F(u_j)\|_{L_T^2 H^s} \right\|_{L^2(\nu_{N_j})} = \left\| \|F_j(\Phi_{N_j}(t)\phi) - F(\Phi_{N_j}(t)\phi)\|_{L^2(\mu_{k,N_j}) H^s} \right\|_{L_T^2}$$

$$= (2T)^{\frac{1}{2}} \|F_j(\phi) - F(\phi)\|_{L^2(\mu_{k,N_j}) H^s}$$

$$\lesssim T^{\frac{1}{2}} \|R_{N_j}\|_{L^4(\rho)} \|F_j(\phi) - F(\phi)\|_{L^4(\rho) H^s}$$

$$\lesssim T^{\frac{1}{2}} N_j^{-\varepsilon}, \tag{4.6}$$

for some small $\varepsilon > 0$. In the third step, we used the fact that $Z_N \gtrsim 1$ in view of $Z_N = \|R_N(u)\|_{L^1(\rho)} \to \|R_\infty(u)\|_{L^1(\rho)} > 0$ as $N \to \infty$. Fix $M \in \mathbb{N}$. Then, proceeding as in (4.6), we have

$$\left\| \|F(u_j) - F_M(u_j)\|_{L_T^2 H^s} \right\|_{L^2(\nu_{N_j})} \lesssim T^{\frac{1}{2}} M^{-\varepsilon},$$

uniformly in $j \in \mathbb{N} \cup \{\infty\}$. Lastly, note that it follows from the almost sure convergence of $\widetilde{u^{N_j}}$ to u in $C(\mathbb{R}; H^s)$ and the continuity of F_M that $F_M(u_j)$ converges to $F_M(u)$ in $C(\mathbb{R}; H^s)$ as $j \to \infty$, almost surely with respect to $\widetilde{P}$. Hence, by writing $F_j(u_j) - F(u)$ as

$$F_j(u_j) - F(u) = \big(F_j(u_j) - F(u_j)\big) + \big(F(u_j) - F_M(u_j)\big)$$
$$+ \big(F_M(u_j) - F_M(u)\big) + \big(F_M(u) - F(u)\big),$$

we see that, after passing to a subsequence, $F_j(u_j)$ converges to $F(u)$ in $L^2([-T, T]; H^s(\mathbb{T}))$ almost surely with respect to $\widetilde{P}$.

By iteratively applying the above argument on time intervals $[-2^\ell, 2^\ell]$, $\ell \in \mathbb{N}$, we construct a sequence $\{\Omega_\ell\}_{\ell \in \mathbb{N}}$ of sets of full probability with $\Omega_{\ell+1} \subset \Omega_\ell$ such that a subsequence $F_{j(\ell+1)}(u_{j(\ell+1)})(\omega)$ of $F_{j(\ell)}(u_{j(\ell)})(\omega)$ from the previous step converges to $F(u)(\omega)$ in $L^2([-2^\ell, 2^\ell]; H^s(\mathbb{T}))$ for all $\omega \in \Omega_{\ell+1}$. Then, by a diagonal argument, passing to a subsequence, the term $F_j(u_j)$ converges to $F(u)$ in $L^2_{\text{loc}} H^s$ almost surely with respect to $\widetilde{P}$. In particular, up to a subsequence, $F_j(u_j)$ converges to $F(u)$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{T})$ almost surely with respect to $\widetilde{P}$. Therefore, u is a global-in-time distributional solution to (1.1). $\square$

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