Invariant Gibbs measures for dispersive PDEs

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Chapter 4 : Global weak probabilistic solutions of the LLL equation below $\mathcal{H}^{-1}(\mathbb{C})$

White noise measure and global weak solutions for LLL

Up to now, we have considered strong probabilistic solutions. We show here how we can construct global probabilistic solutions to PDEs thanks to compactness methods in the space of measures. As an application, we will construct a global dynamics on the support of the white noise measure of the LLL equation which lives at the very low regularity $\mathcal{H}^{-1}(\mathbb{C})$.

Our aim is now to construct weak solutions to the Lowest Landau Level equation

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ u(0,z) = u_0(z), \end{cases} \tag{LLL}$$

on the support of the white noise measure.

Denote by $(e_n)_{n\geq 0}$ a Hilbertian basis of $L^2(0,1)$ and consider independent standard Gaussians $(g_n)_{n\geq 0}$ on a probability space $(\Omega, \mathcal{F}, \mathbf{p})$. Then it is well-known that the random series

$$B_t = \sum_{n=0}^{+\infty} g_n \int_0^t e_n(s)ds$$

converges in $L^2(\Omega, \mathcal{F}, \mathbf{p})$ and defines a Brownian motion. The white noise measure is then defined by the map

$$\omega \longmapsto W(t,\omega) = \frac{dB_t}{dt}(\omega) = \sum_{n=0}^{+\infty} g_n(\omega) e_n(t). \tag{1}$$

Now consider a Hilbert space $\mathcal{K}$ which is a space of functions on a manifold M and consider a Hilbertian basis $(e_n)_{n\geq 0}$ of $\mathcal{K}$.

We define the mean-zero Gaussian white noise (measure) on $\mathcal{K}$ as $\mu = \mathbf{p} \circ W^{-1}$, where

$$W(x,\omega) = \sum_{n=0}^{+\infty} g_n(\omega) e_n(x).$$

Notice that this measure is independent of the choice of the Hilbertian basis of $\mathcal{K}$.

For more details on Gaussian measures on Hilbert spaces, we refer to [Janson].

Recall the definition of the harmonic Sobolev spaces : for $s \in \mathbb{R}$ we define

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{C}) = \big\{u \in \mathcal{S}'(\mathbb{C}),\ H^{s/2}u \in L^2(\mathbb{C})\big\},$$

and the norm on $F^2(\mathbb{C}) \cap \mathcal{H}^s(\mathbb{C})$ is a weighted L^2-norm.

Consider the Gaussian random variable

$$\eta(\omega, z) = \sum_{n=0}^{+\infty} g_n(\omega)\varphi_n(z) = \frac{1}{\sqrt{\pi}}\Big(\sum_{n=0}^{+\infty} \frac{z^n g_n(\omega)}{\sqrt{n!}}\Big)e^{-|z|^2/2},$$

and the measure $\mu = \mathbf{p} \circ \eta^{-1}$. We can show that the measure μ is a probability measure on

$$X^{-1}_{hol}(\mathbb{C}) := \big(\cap_{\sigma>1} \mathcal{H}^{-\sigma}(\mathbb{C})\big) \cap \big(\mathcal{O}(\mathbb{C})e^{-|z|^2/2}\big).$$

We have already seen that, for any $2 \leq p < +\infty$, $\eta(\omega, .) \notin F^p(\mathbb{C})$ for a.a. $\omega \in \Omega$.

Since $\|u\|_{L^2(\mathbb{C})}$ is preserved by (LLL), μ is formally invariant under (LLL). We are not able to define a flow at this level of regularity, however using compactness arguments combined with probabilistic methods, we will construct weak solutions.

Theorem (**Germain-Hani-LT**)

There exists a set $\Sigma \subset X^{-1}_{hol}(\mathbb{C})$ *of full* μ *measure so that for every* $u_0 \in \Sigma$ *the equation* (LLL) *with initial condition* $u(0) = u_0$ *has a solution*

$$u \in \bigcap_{\sigma > 1} \mathcal{C}\big(\mathbb{R}\,;\mathcal{H}^{-\sigma}(\mathbb{C})\big).$$

The distribution of the random variable $u(t)$ *is equal to* μ *(and thus independent of* $t \in \mathbb{R}$*) :*

$$\mathscr{L}_{X^{-1}_{hol}}\big(u(t)\big) = \mathscr{L}_{X^{-1}_{hol}}\big(u(0)\big) = \mu, \quad \forall\, t \in \mathbb{R}.$$

The proof is based on a compactness argument in the space of measures (the Prokhorov theorem) combined with a representation theorem of random variables (the Skorohod theorem).

This approach has been first applied to the Navier-Stokes and Euler equations in Albeverio-Cruzeiro and Da Prato-Debussche and extended to dispersive equations by Burq-Thomann-Tzvetkov. See also Germain-Hani-Thomann, Oh-Thomann and Oh-Richards-Thomann. For results in a non compact setting, see Suzzoni.

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Remark

For the Szegö equation, using that the $H^{1/2}(\mathbb{T})$ norm is preserved by the flow, the method used in the proof of the main theorem allows to construct a global dynamics in $\bigcap_{\sigma>0}\mathcal{C}\big(\mathbb{R}\,;H_+^{-\sigma}(\mathbb{T})\big)$. See Burq-Thomann-Tzvetkov for details.

The Prokhorov and Skorokhod theorems

We state two basic results, concerning the convergence of random variables. To begin with, recall the following definition

Definition

Let S be a metric space and $(\rho_N)_{N\geq 1}$ a family of probability measures on the Borel $\sigma-$algebra $\mathcal{B}(S)$. The family (ρ_N) on $(S, \mathcal{B}(S))$ is said to be tight if for any $\varepsilon > 0$ one can find a compact set $K_\varepsilon \subset S$ such that $\rho_N(K_\varepsilon) \geq 1 - \varepsilon$ for all $N \geq 1$.

Then, we have the following compactness criterion

Theorem (**Prokhorov**)

Assume that the family $(\rho_N)_{N\geq 1}$ of probability measures on the metric space S is tight. Then it is weakly compact, i.e. there is a subsequence $(N_k)_{k\geq 1}$ and a limit measure ρ_∞ such that for every bounded continuous function $F : S \to \mathbb{R}$,

$$\lim_{k\to\infty} \int_S F(x) d\rho_{N_k}(x) = \int_S F(x) d\rho_\infty(x).$$

Remark

Let us make a remark on the case $S = \mathbb{R}^d$. The measure given by the theorem allows mass concentration in a point and the tightness condition forbids the escape of mass to infinity.
The Prokhorov theorem is of different nature compared to the compactness theorems giving the deterministic weak solutions : In the latter case there can be a loss of energy. A weak limit of L^2 functions may lose some mass whereas in the Prokhorov theorem a limit measure is a probability measure.

We now state the Skorokhod theorem

Theorem (Skorokhod)

Assume that S is a separable metric space. Let $(\rho_N)_{N\geq 1}$ and ρ_∞ be probability measures on S. Assume that $\rho_N \longrightarrow \rho_\infty$ weakly. Then there exists a probability space on which there are $S-$valued random variables $(Y_N)_{N\geq 1}$, Y_∞ such that $\mathcal{L}(Y_N) = \rho_N$ for all $N \geq 1$, $\mathcal{L}(Y_\infty) = \rho_\infty$ and $Y_N \longrightarrow Y_\infty$ a.s.

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We illustrate this result with two elementary but significant examples :

- Assume that $S = \mathbb{R}$. Let $(Y_N)_{1\leq N \leq \infty}$ be standard Gaussians, *i.e.* $\mathcal{L}(Y_N) = \mathcal{L}(Y_\infty) = \mathcal{N}_{\mathbb{R}}(0,1)$. Then the convergence in law obviously holds, but in general we can not expect the almost sure convergence of the Y_N to Y_∞ (define for example $Y_N = (-1)^N Y_\infty$).

- Assume that $S = \mathbb{R}$. Let $(Y_N)_{1\leq N\leq\infty}$ be random variables. For any random variable Y on $\mathbb{R}$ we denote by $F_Y(t) = P(Y \leq t)$ its cumulative distribution function. Here we assume that for all $1 \leq N \leq \infty$, F_{Y_N} is bijective and continuous, and we prove the Skorokhod theorem in this case. Let U be a r.v. so that $\mathcal{L}(U)$ is the uniform distribution on $[0,1]$ and define the r.v. $\widetilde{Y}_N = F_{Y_N}^{-1}(U)$. We now check that the $\widetilde{Y}_N$ satisfy the conclusion of the theorem. To begin with,

$$F_{\widetilde{Y}_N}(t) = P(\widetilde{Y}_N \leq t) = P(U \leq F_{Y_N}(t)) = F_{Y_N}(t),$$

therefore we have for $1 \leq N \leq \infty$, $\mathcal{L}(Y_N) = \mathcal{L}(\widetilde{Y}_N)$. Now if we assume that $Y_N \longrightarrow Y_\infty$ in law, we have for all $t \in \mathbb{R}$, $F_{Y_N}(t) \longrightarrow F_{Y_\infty}(t)$ and in particular $\widetilde{Y}_N \longrightarrow \widetilde{Y}_\infty$ almost surely.

General strategy of the proof

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\geq 1}$ a sequence of independent complex normalised Gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$. Let $\mathcal{M}$ be a Riemanian compact manifold and let $(e_n)_{n\geq 1}$ be an Hilbertian basis of $L^2(\mathcal{M})$ (with obvious changes, we can allow $n \in \mathbb{Z}$). Consider one of the equations mentioned in the introduction. Denote by

$$X^\sigma = X^\sigma(\mathcal{M}) = \bigcap_{\tau<\sigma} H^\tau(\mathcal{M}).$$

The general strategy for proving a global existence result is the following :

Step 1 : The Gaussian measure μ : We define a measure μ on $X^{\sigma}(\mathcal{M})$ which is invariant by the flow of the linear part of the equation. The index $\sigma_c \in \mathbb{R}$ is determined by the equation and the manifold $\mathcal{M}$. Indeed this measure can be defined as $\mu = \mathbf{p} \circ \gamma^{-1}$, where $\gamma \in L^2\big(\Omega;\, H^{\sigma}(\mathcal{M})\big)$ for all $\sigma < \sigma_c$ is a Gaussian random variable which takes the form

$$\gamma(\omega, x) = \sum_{n \geq 1} \frac{g_n(\omega)}{\lambda_n} e_n(x).$$

Here the (λ_n) satisfy $\lambda_n \sim cn^{\alpha}$, $\alpha > 0$ and are given by the linear part and the Hamiltonian structure of the equation. Notice in particular that for all measurable $F : X^{\sigma_c}(\mathcal{M}) \longrightarrow \mathbb{R}$

$$\int_{X^{\sigma_c}(\mathcal{M})} F(u) \mathrm{d}\mu(u) = \int_{\Omega} F\big(\gamma(\omega, \cdot)\big) \mathrm{d}\mathbf{p}(\omega). \tag{2}$$

Step 2 : The invariant measure ρ_N : By working on the Hamiltonian formulation of the equation, we introduce an approximation of the initial problem which has a global flow Φ_N, and for which we can construct a measure ρ_N on $X^{\sigma_c}(\mathcal{M})$ which has the following properties

(*i*) The measure ρ_N is a probability measure which is absolutly continuous with respect to μ

$$d\rho_N(u) = \Psi_N(u)d\mu(u).$$

(*ii*) The measure ρ_N is invariant by the flow Φ_N by the Liouville theorem.

(*iii*) There exists $\Psi \not\equiv 0$ such that for all $p \geq 1$, $\Psi(u) \in L^p(d\mu)$ and

$$\Psi_N(u) \longrightarrow \Psi(u), \quad \text{in} \quad L^p(d\mu).$$

(In particular $\|\Psi_N(u)\|_{L^p_\mu} \leq C$ uniformly in $N \geq 1$.) This enables to define a probability measure on $X^{\sigma_c}(\mathcal{M})$ by

$$d\rho(u) = \Psi(u)d\mu(u),$$

which is formally invariant by the equation.

Step 3 : The measure ν_N : We abuse notation and write

$$\mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big) = \bigcap_{\sigma<\sigma_c} \mathcal{C}\big([-T,T];H^{\sigma}(\mathcal{M})\big).$$

We denote by $\nu_N = \rho_N \circ \Phi_N^{-1}$ the measure on $\mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big)$, defined as the image measure of ρ_N by the map

$$\begin{array}{ccc} X^{\sigma_c}(\mathcal{M}) & \longrightarrow & \mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big) \\ v & \longmapsto & \Phi_N(t,v). \end{array}$$

In particular, for any measurable $F : \mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big) \longrightarrow \mathbb{R}$

$$\int_{\mathcal{C}\left([-T,T];X^{\sigma_c}\right)} F(u)\mathrm{d}\nu_N(u) = \int_{X^{\sigma_c}} F\big(\Phi_N(t,v)\big)\mathrm{d}\rho_N(v). \tag{3}$$

Assume that the corresponding sequence of measures (ν_N) is tight in $\mathcal{C}\big([-T,T];H^\sigma(\mathcal{M})\big)$ for all $\sigma<\sigma_c$ (this has to be shown for the considered equation). Therefore, for all $\sigma<\sigma_c$, by the Prokhorov theorem, there exists a measure $\nu_\sigma=\nu$ on $\mathcal{C}\big([-T,T];H^\sigma(\mathcal{M})\big)$ so that the weak convergence holds (up to a sub-sequence) : For all $\sigma<\sigma_c$ and all bounded continuous $F:\mathcal{C}\big([-T,T];H^\sigma(\mathcal{M})\big)\longrightarrow\mathbb{R}$

$$\lim_{N\to\infty}\int_{\mathcal{C}\big([-T,T];H^\sigma\big)}F(u)\mathrm{d}\nu_N(u)=\int_{\mathcal{C}\big([-T,T];H^\sigma\big)}F(u)\mathrm{d}\nu(u).$$

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$$\lim_{N\to\infty}\int_{\mathcal{C}\big([-T,T];H^\sigma\big)}F(u)\mathrm{d}\nu_N(u)=\int_{\mathcal{C}\big([-T,T];H^\sigma\big)}F(u)\mathrm{d}\nu(u).$$

At this point, observe that if $\sigma_1<\sigma_2$, then $\nu_{\sigma_1}\equiv\nu_{\sigma_2}$ on $\mathcal{C}\big([-T,T];H^{\sigma_1}(\mathcal{M})\big)$. Moreover, by the standard diagonal argument, we can ensure that ν is a measure on $\mathcal{C}\big([-T,T];X^{\sigma_c}(\mathcal{M})\big)$.
Finally, with the Skorokhod theorem, we can construct a sequence of random variables which converges to a solution of the initial problem.

We now state a result which will be useful in the sequel. Assume that ρ_N satisfies the properties mentioned in Step 2.

Proposition

Let $\sigma < \sigma_c$. Let $p \geq 2$ and $r > p$. Then for all $N \geq 1$

$$\big\| \|u\|_{L^p_T H^\sigma_x} \big\|_{L^p_{\nu_N}} \leq C T^{1/p} \big\| \|v\|_{H^\sigma_x} \big\|_{L^r_\mu}.$$

Let $q \geq 1$, $p \geq 2$ and $r > p$. Then for all $N \geq 1$

$$\big\| \|u\|_{L^p_T L^q_x} \big\|_{L^p_{\nu_N}} \leq C T^{1/p} \big\| \|v\|_{L^q_x} \big\|_{L^r_\mu}.$$

In case $\Psi_N \leq C$, one can take $r = p$ in the previous inequalities.

Proof : We apply (3) with the function $u \longmapsto F(u) = \|u\|^p_{L^p_T H^\sigma_x}$. Here and after, we make the abuse of notation

$$\big\| \|u\|_{L^p_T H^\sigma_x} \big\|_{L^p_{\nu_N}} = \|u\|_{L^p_{\nu_N} L^p_T H^\sigma_x}.$$

Then

$$\begin{aligned} \|u\|^p_{L^p_{\nu_N} L^p_T H^\sigma_x} &= \int_{\mathcal{C}\left([-T,T];X^{\sigma_c}\right)} \|u\|^p_{L^p_T H^\sigma_x} \mathrm{d}\nu_N(u) \\ &= \int_{X^{\sigma_c}} \|\Phi_N(t,v)\|^p_{L^p_T H^\sigma_x} \mathrm{d}\rho_N(v) \\ &= \int_{X^{\sigma_c}} \Big[\int_{-T}^{T} \|\Phi_N(t,v)\|^p_{H^\sigma_x} \mathrm{d}t \Big] \mathrm{d}\rho_N(v) \\ &= \int_{-T}^{T} \Big[\int_{X^{\sigma_c}} \|\Phi_N(t,v)\|^p_{H^\sigma_x} \mathrm{d}\rho_N(v) \Big] \mathrm{d}t, \end{aligned} \tag{4}$$

where in the last line we used Fubini.

Now we use the invariance of ρ_N under Φ_N, and we deduce that for all $t \in [-T, T]$

$$\int_{X^{\sigma_c}} \|\Phi_N(t,v)\|^p_{H^\sigma_x} \mathrm{d}\rho_N(v) = \int_{X^{\sigma_c}} \|v\|^p_{H^\sigma_x} \mathrm{d}\rho_N(v).$$

Therefore, from (4) and Hölder we obtain with $1/r_1 + 1/r_2 = 1$

$$\begin{aligned} \|u\|^p_{L^p_{\nu_N} L^p_T H^\sigma_x} &= 2T \int_{X^{\sigma_c}} \|v\|^p_{H^\sigma_x} \mathrm{d}\rho_N(v) \\ &= 2T \int_{X^{\sigma_c}} \|v\|^p_{H^\sigma_x} \Psi_N(v) \mathrm{d}\mu(v) \\ &\leq 2T \|v\|^p_{L^{pr_1}_\mu H^\sigma_x} \|\Psi_N(v)\|_{L^{r_2}_\mu}. \end{aligned}$$

Now, let $r > p$, take $r_1 = r/p$ and we can conclude since $\Psi_N(v) \in L^{r_2}(\mathrm{d}\mu)$. For the proof of the second estimate, we proceed similarly. We take $F(u) = \|u\|^p_{L^p_T L^q_x}$ in (3), and use the same arguments as previously. □

The probabilistic argument of convergence

Definition of $\mathcal{T}(u,u,u)$ on the support of μ Denote by E_k the space on $\mathbb{C}$ spanned by φ_k. For $N \geq 0$, denote by Π_N the orthogonal projector on the space $\bigoplus_{k=0}^{N} E_k$ (in this section, we do not need the smooth cut-offs S_N). In the sequel, we denote by $\mathcal{T}(u) = \mathcal{T}(u,u,u)$ and $\mathcal{T}_N(u) = \Pi_N \mathcal{T}(\Pi_N u, \Pi_N u, \Pi_N u)$

Proposition

For all $p \geq 2$ and $\sigma > 1$, the sequence $\big(\mathcal{T}_N(u)\big)_{N \geq 1}$ is a Cauchy sequence in $L^p\big(X^{-1}_{hol}, \mathcal{B}, d\mu; \mathcal{H}^{-\sigma}(\mathbb{C})\big)$. Namely, for all $p \geq 2$, there exist $\delta > 0$ and $C > 0$ so that for all $1 \leq M < N$,

$$\int_{X^{-1}_{hol}} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|^p_{\mathcal{H}^{-\sigma}(\mathbb{C})} d\mu(u) \leq C M^{-\delta}.$$

We denote by $\mathcal{T}(u) = \mathcal{T}(u,u,u)$ the limit of this sequence and we have for all $p \geq 2$

$$\|\mathcal{T}(u)\|_{L^p_\mu \mathcal{H}^{-\sigma}(\mathbb{C})} \leq C_p. \tag{5}$$

Proof : By the Proposition on the Wiener chaos, we only have to prove the statement for $p = 2$.
Firstly, by definition of the measure μ

$$\int_{X^{-1}_{hol}} \|\mathcal{T}_N(u) - \mathcal{T}_M(u)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})} \mathrm{d}\mu(u) = \int_\Omega \|\mathcal{T}_N\big(\eta(\omega)\big) - \mathcal{T}_M\big(\eta(\omega)\big)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})} \mathrm{d}\mathbf{p}(\omega).$$

Therefore, it is enough to prove that $\big(\mathcal{T}_N(\eta)\big)_{N\geq 1}$ is a Cauchy sequence in $L^2\big(\Omega; \mathcal{H}^{-\sigma}(\mathbb{C})\big)$.
Let $1 \leq M < N$ and fix $\sigma > 1$. Then an explicit computation gives

$$\|\mathcal{T}_N(\eta) - \mathcal{T}_M(\eta)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})} =$$

$$= \frac{\pi^2}{64 \cdot 2^\sigma} \sum_{p=0}^{N} \frac{1}{(p+1)^\sigma} \sum_{(n,m)\in A^{(p)}_{M,N} \times A^{(p)}_{M,N}} \frac{(n_1+n_2)!\,(m_1+m_2)!\, g_{n_1} g_{n_2} \overline{g_{n_3} g_{m_1} g_{m_2}} g_{m_3}}{2^{n_1+n_2} 2^{m_1+m_2} p! \sqrt{n_1!\,n_2!\,n_3!}\sqrt{m_1!\,m_2!\,m_3!}}$$

where $A^{(p)}_{M,N}$ is the set defined by

$$A^{(p)}_{M,N} = \Big\{ n \in \mathbb{N}^3 \ \text{ s.t. } \ 0 \leq n_j \leq N, \ \ n_1 + n_2 - n_3 = p \in \{0 \dots N\}, \\ \big(n_1 > M \ \text{ or } \ n_2 > M \ \text{ or } \ n_3 > M \ \text{ or } \ p > M\big) \Big\}.$$

Now we take the integral over Ω. Here, the key fact is to use that the $(g_n)_{n\geq 0}$ are independent and centred Gaussians : we deduce that each term in the r.h.s. vanishes, unless

Case 1 : $(n_1, n_2, n_3) = (m_1, m_2, m_3)$ or $(n_1, n_2, n_3) = (m_2, m_1, m_3)$

or

Case 2 : $(n_1, n_2, m_1) = (n_3, m_2, m_3)$ or $(n_1, n_2, m_2) = (n_3, m_1, m_3)$ or $(n_1, n_2, m_3) = (m_1, n_3, m_2)$ or $(n_1, n_2, m_3) = (m_2, n_3, m_1)$.

With a careful inspection of each contribution, we are able to bound the different sums. □

Study of the measure ν_N
Let $N \geq 1$. We then consider the following approximation of (LLL)

$$\begin{cases} i\partial_t u = \mathcal{T}_N(u), \quad (t,z) \in \mathbb{R} \times \mathbb{C}, \\ u(0,z) = u_0(z) \in X_{hol}^{-1}. \end{cases} \tag{6}$$

The equation (6) is an ODE in the frequencies less than N, whereas for the large frequencies, the solution is constant in time : $(1-\Pi_N)u(t) = (1-\Pi_N)u_0$ and for all $t \in \mathbb{R}$.

The main motivation to introduce this system is the following proposition

Proposition

The equation (6) has a global flow Φ_N. Moreover, the measure μ is invariant under Φ_N : For any Borel set $A \subset X_{hol}^{-1}$ and for all $t \in \mathbb{R}$, $\mu\big(\Phi_N(t,A)\big) = \mu\big(A\big)$. In particular if $\mathscr{L}_{X^{-1}}(v) = \mu$ then for all $t \in \mathbb{R}$, $\mathscr{L}_{X^{-1}}(\Phi_N(t,v)) = \mu$.

Proof : The proof is a direct application of the Liouville theorem. □

We denote by ν_N the measure on $\mathcal{C}\big([-T,T];X^{-1}_{hol}\big)$, defined as the image measure of μ by the map

$$\begin{array}{ccc} X^{-1}_{hol} & \longrightarrow & \mathcal{C}\big([-T,T];X^{-1}_{hol}\big) \\ v & \longmapsto & \Phi_N(t,v). \end{array}$$

Lemma

Let $\sigma > 1$ and $p \geq 2$. Then there exists $C > 0$ so that for all $N \geq 1$

$$\Big\| \|u\|_{W^{1,p}_T \mathcal{H}^{-\sigma}_z} \Big\|_{L^p_{\nu_N}} \leq C. \tag{7}$$

Proof : Firstly, we have that for $\sigma > 1$, $p \geq 2$ and $N \geq 1$

$$\Big\| \|u\|_{L^p_T \mathcal{H}_z^{-\sigma}} \Big\|_{L^p_{\nu_N}} \leq C.$$

Indeed, by the definition of ν_N and the invariance of μ by Φ_N we have

$$\|u\|_{L^p_{\nu_N} L^p_T \mathcal{H}_z^{-\sigma}} = (2T)^{1/p} \|v\|_{L^p_\mu \mathcal{H}_z^{-\sigma}} = (2T)^{1/p} \|\eta\|_{L^p_{\mathbf{p}} \mathcal{H}_z^{-\sigma}}.$$

Then, by the Khintchine inequality, for all $p \geq 2$

$$\|\eta\|_{L^p_{\mathbf{p}} \mathcal{H}_z^{-\sigma}} \leq C\sqrt{p} \|\eta\|_{L^2_{\mathbf{p}} \mathcal{H}_z^{-\sigma}} \leq C.$$

Next, we show that $\Big\| \|\partial_t u\|_{L^p_T \mathcal{H}_z^{-\sigma}} \Big\|_{L^p_{\nu_N}} \leq C$. By definition of ν_N

$$\begin{aligned} \|\partial_t u\|^p_{L^p_{\nu_N} L^p_T \mathcal{H}_z^{-\sigma}} &= \int_{\mathcal{C}\left([-T,T]; X^{-\mathbf{1}}_{hol}\right)} \|\partial_t u\|^p_{L^p_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\nu_N(u) \\ &= \int_{X^{-\mathbf{1}}_{hol}} \|\partial_t \Phi_N(t,v)\|^p_{L^p_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(v). \end{aligned}$$

Now, since $\Phi_N(t,v)$ satisfies (6) and by the invariance of μ, we have

$$\begin{aligned}\|\partial_t u\|^p_{L^p_{\nu_N} L^p_T \mathcal{H}_z^{-\sigma}} &= \int_{X^{-1}_{hol}} \|\mathcal{T}_N(\Phi_N(t,v))\|^p_{L^p_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(v) \\ &= 2T \int_{X^{-1}_{hol}} \|\mathcal{T}_N(v)\|^p_{\mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(v),\end{aligned}$$

and conclude with (5) and proposition defining $\mathcal{T}(u)$. □

The convergence argument

We are able to establish the following tightness result for the measures ν_N.

Proposition

Let $T > 0$ and $\sigma > 1$. Then the family of measures

$$(\nu_N)_{N\geq 1} \quad \textit{with} \quad \nu_N = \mathscr{L}_{\mathcal{C}_T\mathcal{H}^{-\sigma}}\big(u_N(t); t \in [-T, T]\big)$$

is tight in $\mathcal{C}\big([-T, T]; \mathcal{H}^{-\sigma}(\mathbb{C})\big)$.

Proof : Let $\sigma > 1$. Fix $\sigma > s' > s'' > 1$ and $\alpha > 0$.
We define the space $\mathcal{C}^\alpha_T\mathcal{H}^{-s'} = \mathcal{C}^\alpha\big([-T, T]; \mathcal{H}^{-s'}(\mathbb{C})\big)$ by the norm

$$\|u\|_{\mathcal{C}^\alpha_T\mathcal{H}^{-s'}} = \sup_{t_1, t_2 \in [-T,T],\, t_1 \neq t_2} \frac{\|u(t_1) - u(t_2)\|_{\mathcal{H}_z^{-s'}}}{|t_1 - t_2|^\alpha} + \|u\|_{L^\infty_T\mathcal{H}_z^{-s'}},$$

and it is classical that the embedding $\mathcal{C}^\alpha_T\mathcal{H}^{-s'} \subset \mathcal{C}\big([-T, T]; \mathcal{H}^{-\sigma}(\mathbb{C})\big)$ is compact.

We now claim that there exists $0 < \alpha \ll 1$ so that for all $p \geq 1$ we have the bound

$$\|u\|_{L^p_{\nu_N}\mathcal{C}^\alpha_T\mathcal{H}^{-s'}} \leq C. \tag{8}$$

With an interpolation argument we obtain that for some $p \gg 1$

$$\|u\|_{\mathcal{C}^\alpha_T\mathcal{H}^{-s'}} \leq C\|u\|^{1-\theta}_{L^p_T\mathcal{H}^{-s''}}\|u\|^{\theta}_{W^{1,p}_T\mathcal{H}^{-\sigma}} \leq C\|u\|_{L^p_T\mathcal{H}^{-s''}} + C\|u\|_{W^{1,p}_T\mathcal{H}^{-\sigma}},$$

for some small $\alpha > 0$. By (7) we then deduce (9).

We now claim that there exists $0 < \alpha \ll 1$ so that for all $p \geq 1$ we have the bound

$$\|u\|_{L^p_{\nu_N} \mathcal{C}^\alpha_T \mathcal{H}^{-s'}} \leq C. \tag{9}$$

With an interpolation argument we obtain that for some $p \gg 1$

$$\|u\|_{\mathcal{C}^\alpha_T \mathcal{H}^{-s'}} \leq C \|u\|^{1-\theta}_{L^p_T \mathcal{H}^{-s''}} \|u\|^{\theta}_{W^{1,p}_T \mathcal{H}^{-\sigma}} \leq C \|u\|_{L^p_T \mathcal{H}^{-s''}} + C \|u\|_{W^{1,p}_T \mathcal{H}^{-\sigma}},$$

for some small $\alpha > 0$. By (7) we then deduce (9).

Next, let $\delta > 0$ and define the subset of $\mathcal{C}_T \mathcal{H}^{-\sigma}$

$$K_\delta = \big\{ u \in \mathcal{C}_T \mathcal{H}^{-\sigma} \;\; s.t. \;\; \|u\|_{\mathcal{C}^\alpha_T \mathcal{H}^{-s'}} \leq \delta^{-1} \big\},$$

endowed with the natural topology of $\mathcal{C}_T \mathcal{H}^{-\sigma}$. Thanks to the previous considerations, the set K_δ is compact. Finally, by Markov and (9) we get that

$$\nu_N(K^c_\delta) \leq \delta \, \|u\|_{L^1_{\nu_N} \mathcal{C}^\alpha_T \mathcal{H}^{-s'}} \leq \delta C,$$

which shows the tightness of (ν_N). □

The result of the previous proposition enables us to use the Prokhorov theorem : For each $T>0$ there exists a sub-sequence ν_{N_k} and a measure ν on the space $\mathcal{C}\big([-T,T];X^{-1}_{hol}\big)$ so that for all $\tau>1$ and all bounded continuous function $F:\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{C})\big)\longrightarrow\mathbb{R}$

$$\int_{\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{C})\big)} F(u)\mathrm{d}\nu_{N_k}(u)\longrightarrow\int_{\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{C})\big)} F(u)\mathrm{d}\nu(u).$$

The result of the previous proposition enables us to use the Prokhorov theorem : For each $T > 0$ there exists a sub-sequence ν_{N_k} and a measure ν on the space $\mathcal{C}\big([-T,T]; X^{-1}_{hol}\big)$ so that for all $\tau > 1$ and all bounded continuous function $F : \mathcal{C}\big([-T,T]; \mathcal{H}^{-\tau}(\mathbb{C})\big) \longrightarrow \mathbb{R}$

$$\int_{\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{C})\big)} F(u)\mathrm{d}\nu_{N_k}(u) \longrightarrow \int_{\mathcal{C}\big([-T,T];\mathcal{H}^{-\tau}(\mathbb{C})\big)} F(u)\mathrm{d}\nu(u).$$

By the Skohorod theorem, there exists a probability space $(\widetilde{\Omega}, \widetilde{\mathcal{F}}, \widetilde{\mathbf{p}})$, a sequence of random variables $(\widetilde{u}_{N_k})$ and a random variable $\widetilde{u}$ with values in $\mathcal{C}\big([-T,T]; X^{-1}_{hol}\big)$ so that

$$\mathscr{L}\big(\widetilde{u}_{N_k}; t \in [-T,T]\big) = \mathscr{L}\big(u_{N_k}; t \in [-T,T]\big) = \nu_{N_k}, \quad \mathscr{L}\big(\widetilde{u}; t \in [-T,T]\big) = \nu, \tag{12}$$

and for all $\tau > 1$

$$\widetilde{u}_{N_k} \longrightarrow \widetilde{u}, \qquad \widetilde{\mathbf{p}} - \text{a.s. in } \mathcal{C}\big([-T,T]; \mathcal{H}^{-\tau}(\mathbb{C})\big). \tag{13}$$

We now claim that $\mathscr{L}_{X^{-1}}(u_{N_k}(t)) = \mathscr{L}_{X^{-1}}(\widetilde{u}_{N_k}(t)) = \mu$, for all $t \in [-T, T]$ and $k \geq 1$. Indeed, for all $t \in [-T, T]$, the evaluation map

$$\begin{array}{rcl} R_t \, : \, \mathcal{C}\big([-T,T]; X^{-1}_{hol}\big) & \longrightarrow & X^{-1}_{hol} \\ u & \longmapsto & u(t,.), \end{array}$$

is well defined and continuous.
Thus, for all $t \in [-T, T]$, $u_{N_k}(t)$ and $\widetilde{u}_{N_k}(t)$ have same distribution $(R_t)_{\#}\nu_{N_k}$.
We then obtain that this distribution is μ.
Thus from (13) we deduce that

$$\mathscr{L}_{X^{-1}}(\widetilde{u}(t)) = \mu, \quad \forall\, t \in [-T, T]. \tag{14}$$

Let $k \geq 1$ and $t \in \mathbb{R}$ and consider the r.v. X_k given by

$$X_k = u_{N_k}(t) - R_0(u_{N_k}(t)) + i \int_0^t \mathcal{T}_{N_k}(u_{N_k}) ds.$$

Define $\widetilde{X}_k$ similarly to X_k with u_{N_k} replaced with $\widetilde{u}_{N_k}$. Then by (12),

$$\mathscr{L}_{\mathcal{C}_T X^{-1}}(\widetilde{X}_{N_k}) = \mathscr{L}_{\mathcal{C}_T X^{-1}}(X_{N_k}) = \delta_0.$$

In other words, $\widetilde{X}_k = 0$ $\widetilde{\mathbf{p}}-$a.s. and $\widetilde{u}_{N_k}$ satisfies the following equation $\widetilde{\mathbf{p}}-$a.s.

$$\widetilde{u}_{N_k}(t) = R_0(\widetilde{u}_{N_k}(t)) - i \int_0^t \mathcal{T}_{N_k}(\widetilde{u}_{N_k}) ds. \tag{15}$$

We now show that we can pass to the limit $k \longrightarrow +\infty$ in (15) in order to show that $\widetilde{u}$ is $\widetilde{\mathbf{p}}-$a.s. a solution to (LLL) written in integral form as :

$$\widetilde{u}(t) = R_0(\widetilde{u}(t)) - i\int_0^t \mathcal{T}(\widetilde{u})ds. \tag{16}$$

Firstly, from (13) we deduce the convergence of the linear terms in equation (15) to those in (16).

The following lemma gives the convergence of the nonlinear term.

Lemma

Up to a sub-sequence, the following convergence holds true

$$\mathcal{T}_{N_k}(\widetilde{u}_{N_k}) \longrightarrow \mathcal{T}(\widetilde{u}), \qquad \widetilde{\mathbf{p}} - a.s.\ in\ \ L^2([-T,T];\mathcal{H}^{-\sigma}(\mathbb{C})).$$

Proof : In order to simplify the notations, in this proof we drop the tildes and write $N_k = k$. Let $M \geq 1$ and write

$$\begin{aligned}\mathcal{T}_k(u_k) - \mathcal{T}(u) = \big(\mathcal{T}_k(u_k) - \mathcal{T}(u_k)\big) + \big(\mathcal{T}(u_k) - \mathcal{T}_M(u_k)\big) + \\ + \big(\mathcal{T}_M(u_k) - \mathcal{T}_M(u)\big) + \big(\mathcal{T}_M(u) - \mathcal{T}(u)\big).\end{aligned}$$

To begin with, by continuity of the product in finite dimension, when $k \longrightarrow +\infty$

$$\mathcal{T}_M(u_k) \longrightarrow \mathcal{T}_M(u), \qquad \widetilde{\mathbf{p}} - \text{a.s. in } L^2\big([-T, T]; \mathcal{H}^{-\sigma}(\mathbb{C})\big).$$

We now deal with the other terms. It is sufficient to show the convergence in the space $X := L^2\big(\Omega \times [-T, T]; \mathcal{H}^{-\sigma}(\mathbb{C})\big)$, since the almost sure convergence follows after exaction of a sub-sequence.

By definition and the invariance of μ we obtain

$$\begin{aligned}
\| \mathcal{T}_M(u_k) - \mathcal{T}(u_k) \|_X^2 &= \int_{\mathcal{C}([-T,T];X^{-1})} \| \mathcal{T}_M(v) - \mathcal{T}(v) \|^2_{L^2_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\nu_k(v) \\
&= \int_{X^{-1}(\mathbb{C})} \Big\| \mathcal{T}_M\big(\Phi_k(t,g)\big) - \mathcal{T}\big(\Phi_k(t,g)\big) \Big\|^2_{L^2_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(g) \\
&= \int_{X^{-1}(\mathbb{C})} \| \mathcal{T}_M(g) - \mathcal{T}(g)) \|^2_{L^2_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(g) \\
&= 2T \int_{X^{-1}(\mathbb{C})} \| \mathcal{T}_M(g) - \mathcal{T}(g) \|^2_{\mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(g),
\end{aligned}$$

which tends to 0 uniformly in $k \geq 1$ when $M \longrightarrow +\infty$, according to proposition on the Cauchy sequence.

By definition and the invariance of μ we obtain

$$\begin{aligned}
\big\| \mathcal{T}_M(u_k) - \mathcal{T}(u_k) \big\|_X^2 &= \int_{\mathcal{C}([-T,T];X^{-1})} \big\| \mathcal{T}_M(v) - \mathcal{T}(v) \big\|^2_{L^2_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\nu_k(v) \\
&= \int_{X^{-1}(\mathbb{C})} \Big\| \mathcal{T}_M\big(\Phi_k(t,g)\big) - \mathcal{T}\big(\Phi_k(t,g)\big) \Big\|^2_{L^2_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(g) \\
&= \int_{X^{-1}(\mathbb{C})} \big\| \mathcal{T}_M(g) - \mathcal{T}(g)\big) \big\|^2_{L^2_T \mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(g) \\
&= 2T \int_{X^{-1}(\mathbb{C})} \big\| \mathcal{T}_M(g) - \mathcal{T}(g) \big\|^2_{\mathcal{H}_z^{-\sigma}} \mathrm{d}\mu(g),
\end{aligned}$$

which tends to 0 uniformly in $k \geq 1$ when $M \longrightarrow +\infty$, according to proposition on the Cauchy sequence.
The term $\big\| \mathcal{T}_M(u) - \mathcal{T}(u) \big\|_X$ is treated similarly. Finally, with the same argument we show

$$\big\| \mathcal{T}_k(u_k) - \mathcal{T}(u_k) \big\|_X \leq C \big\| \mathcal{T}_k(g) - \mathcal{T}(g) \big\|_{L^2_\mu \mathcal{H}_z^{-\sigma}},$$

which tends to 0 when $k \longrightarrow +\infty$. This completes the proof. □

Conclusion of the proof of the main theorem

Define $\widetilde{u}_0 = \widetilde{u}(0) := R_0(\tilde{u})$. Then by (14), $\mathscr{L}_{X^{-1}}(\,\widetilde{u}_0\,) = \mu$ and by the previous arguments, there exists $\widetilde{\Omega'} \subset \widetilde{\Omega}$ such that $\widetilde{\mathbf{p}}(\widetilde{\Omega'}) = 1$ and for each $\omega' \in \widetilde{\Omega'}$, the random variable $\widetilde{u}$ satisfies the equation

$$\widetilde{u} = \widetilde{u}_0 - i\int_0^t \mathcal{T}(\widetilde{u})dt, \quad (t,z) \in \mathbb{R} \times \mathbb{C}. \tag{17}$$

Set $\Sigma = \widetilde{u}_0(\Omega')$, then $\mu(\Sigma) = \widetilde{\mathbf{p}}(\widetilde{\Omega'}) = 1$.

Conclusion of the proof of the main theorem

Define $\widetilde{u}_0 = \widetilde{u}(0) := R_0(\tilde{u})$. Then by (14), $\mathscr{L}_{X^{-1}}(\,\widetilde{u}_0\,) = \mu$ and by the previous arguments, there exists $\widetilde{\Omega'} \subset \widetilde{\Omega}$ such that $\widetilde{\mathbf{p}}(\widetilde{\Omega'}) = 1$ and for each $\omega' \in \widetilde{\Omega'}$, the random variable $\widetilde{u}$ satisfies the equation

$$\widetilde{u} = \widetilde{u}_0 - i\int_0^t \mathcal{T}(\widetilde{u})dt, \quad (t,z) \in \mathbb{R}\times\mathbb{C}. \tag{18}$$

Set $\Sigma = \widetilde{u}_0(\Omega')$, then $\mu(\Sigma) = \widetilde{\mathbf{p}}(\widetilde{\Omega'}) = 1$. It remains to check that we can construct a global dynamics. Take a sequence $T_N \to +\infty$, and perform the previous argument for $T = T_N$. For all $N \geq 1$, let Σ_N be the corresponding set of initial conditions and set $\Sigma = \cap_{N\in\mathbb{N}}\Sigma_N$. Then $\mu(\Sigma) = 1$ and for all $\widetilde{u}_0 \in \Sigma$, there exists

$$\widetilde{u} \in \mathcal{C}\big(\mathbb{R}\,;X^{-1}_{hol}\big),$$

which solves (18). This completes the proof. □