Invariant Gibbs measures for dispersive PDEs

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Chapter 3 : Almost sure global wellposedness of the LLL equation below $L^2(\mathbb{C})$

In this chapter, we show how we can use a Gibbs measure to prove almost sure global existence results. We will present the method on the Landau Lowest Level (LLL) equation. The results are taken from Germain-Hani-Thomann and some analysis from Germain-Hani-Thomann and Gérard-Germain-Thomann.

For $1 \leq p \leq +\infty$ we define the Bargmann-Fock spaces

$$F^p(\mathbb{C}) = \left\{ u(z) = e^{-\frac{|z|^2}{2}} f(z) \ , \ f \text{ entire holomorphic} \right\} \cap L^p(\mathbb{C}) \ .$$

In the sequel we consider the Lowest Landau Level equation which reads

$$\begin{cases} i\partial_t u = \Pi(|u|^2 u), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ u(0,z) = u_0(z), \end{cases} \tag{LLL}$$

where Π is the orthogonal projector on $F^2(\mathbb{C})$.

This equation is used in the description of fast rotating Bose-Einstein condensates, see *e.g.* the book of Aftalion and references therein.

The equation (LLL) can be obtained as the restriction of the continuous resonant equation (CR) which was introduced by Faou-Germain-Hani and further studied by Germain-Hani-Thomann.

Let $z = x + iy$. Denote by H the harmonic oscillator $H = -\partial_x^2 - \partial_y^2 + x^2 + y^2$. A Hilbertian basis of normalized eigenfunctions of H for $F^2(\mathbb{C})$ is given by the so-called special Hermite functions defined for $n \geq 0$ by

$$\varphi_n(z) = \frac{1}{\sqrt{\pi n!}} z^n e^{-|z|^2/2},$$

and which satisfy

$$H\varphi_n = 2(n+1)\varphi_n.$$

Therefore, every $u \in F^2(\mathbb{C})$ can be decomposed in a series

$$u = \sum_{n=0}^{+\infty} c_n \varphi_n. \tag{1}$$

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Therefore, every $u \in F^2(\mathbb{C})$ can be decomposed in a series

$$u = \sum_{n=0}^{+\infty} c_n \varphi_n. \tag{2}$$

We are able to explicitly compute the kernel of Π

$$\sum_{n=0}^{+\infty} \varphi_n(z)\overline{\varphi_n(w)} = \frac{1}{\pi}\Big(\sum_{n=0}^{+\infty} \frac{1}{n!}(z\overline{w})^n\Big)\mathrm{e}^{-|z|^2/2-|w|^2/2} = \frac{1}{\pi}\mathrm{e}^{z\overline{w}-|z|^2/2-|w|^2/2}.$$

As a consequence,

$$[\Pi u](z) = \frac{1}{\pi}\mathrm{e}^{-\frac{|z|^2}{2}} \int_{\mathbb{C}} \mathrm{e}^{z\overline{w}-\frac{|w|^2}{2}} u(w)\, dL(w),$$

where dL stands for the Lebesgue measure on $\mathbb{C}$.

We define the trilinear operator $\mathcal{T}$ by

$$\mathcal{T}(u_1, u_2, u_3) = \Pi(u_1 u_2 \overline{u_3}). \tag{3}$$

The equation (LLL) is Hamiltonian : indeed, introducing the functional

$$\begin{aligned}\mathcal{E}(u_1, u_2, u_3, u_4) &\overset{def}{=} \langle \mathcal{T}(u_1, u_2, u_3)\,,\ u_4 \rangle_{L^2(\mathbb{C})} \\ &= \int_{\mathbb{C}} (u_1 u_2 \overline{u_3 u_4})(z) dL(z)\end{aligned}$$

and setting

$$\mathcal{E}(u) := \mathcal{E}(u, u, u, u) = \int_{\mathbb{C}} |u(z)|^4 dL(z) = \|u\|^4_{L^4(\mathbb{C})},$$

then (LLL) derives from the Hamiltonian $\mathcal{E}$ given the symplectic form

$$\omega(f, g) = \mathfrak{Im} \int_{\mathbb{C}} f \overline{g}\, dL,$$

so that (LLL) is equivalent to

$$i\partial_t u = \frac{1}{2} \frac{\partial \mathcal{E}(u)}{\partial \bar{u}}.$$

The family $(\varphi_n)_{n\geq 0}$ is particularly well adapted in the study of the operator $\mathcal{T}$ since on has

$$\mathcal{T}(\varphi_{n_1}, \varphi_{n_2}, \varphi_{n_3}) = \alpha_{n_1,n_2,n_3,n_4}\varphi_{n_4}, \qquad n_4 = n_1 + n_2 - n_3, \tag{4}$$

with

$$\alpha_{n_1,n_2,n_3,n_4} = \mathcal{E}(\varphi_{n_1}, \varphi_{n_2}, \varphi_{n_3}, \varphi_{n_4}) = \frac{\pi}{2}\frac{(n_1+n_2)!}{2^{n_1+n_2}\sqrt{n_1!n_2!n_3!n_4!}}\mathbf{1}_{n_1+n_2=n_3+n_4}.$$

We can prove that $e^{itH}\mathcal{T}(u_1, u_2, u_3) = \mathcal{T}(e^{itH}u_1, e^{itH}u_2, e^{itH}u_3)$, and therefore with the change of unknowns $v = e^{itH}u$ we see that (LLL) is equivalent to the equation

$$i\partial_t v + Hv = \Pi(|v|^2 v), \quad (t, z) \in \mathbb{R} \times \mathbb{C}. \tag{5}$$

Some deterministic results

Well-posedness of the LLL equation
Define the harmonic Sobolev spaces for $s \in \mathbb{R}$, by

$$\mathcal{H}^s = \mathcal{H}^s(\mathbb{C}) = \big\{ u \in \mathcal{S}'(\mathbb{C}),\ H^{s/2} u \in L^2(\mathbb{C}) \big\}.$$

This is a weighted Sobolev norm. In the Bargmann-Fock space, it simply corresponds to a weighted L^2-norm. Set $\langle z \rangle = (1+|z|^2)^{1/2}$, then we have

Lemma

Let $s \in \mathbb{R}$. There exists $C > 0$ such that for all $u \in F^2(\mathbb{C}) \cap \mathcal{H}^s(\mathbb{C})$

$$\frac{1}{C} \| \langle z \rangle^s u \|_{L^2(\mathbb{C})} \leq \|u\|_{\mathcal{H}^s(\mathbb{C})} \leq C \| \langle z \rangle^s u \|_{L^2(\mathbb{C})}.$$

Exercise

Prove the previous lemma in the particular case where $s \in 2\mathbb{N}$. Hint : use the decomposition (2), and the relations $z\varphi_n = \sqrt{n+1}\varphi_{n+1}$ and $H\varphi_n = 2(n+1)\varphi_n$.

Proposition

The following quantities are conservation laws for (LLL) *:*

$$\mathcal{E}(u) = \int_{\mathbb{C}} |u(z)|^4 dL(z) \qquad \textit{(Hamiltonian)}$$

$$M(u) = \int_{\mathbb{C}} |u(z)|^2 dL(z) \qquad \textit{(Mass)}$$

$$P(u) = \int_{\mathbb{C}} (|z|^2 - 1)|u(z)|^2 \, dL(z) \qquad \textit{(Angular momentum)}$$

$$Q(u) = \int_{\mathbb{C}} z|u|^2(z) dL(z) \qquad \textit{(Magnetic momentum).}$$

Notice that the $\mathcal{H}^1$ norm is also preserved, since in coordinates we can check that

$$\int_{\mathbb{C}} |H^{1/2}u(z)|^2 dL(z) = 2\int_{\mathbb{C}} |z|^2 |u(z)|^2 \, dL(z) = 2(P(u) + M(u)).$$

An important tool in the study of the (LLL) equation are the hypercontractivity inequalities of Carlen.

Proposition

Assume that $1 \leq p \leq q \leq \infty$. *Then* $F^p(\mathbb{C}) \subset F^q(\mathbb{C})$ *and*

$$\left(\frac{q}{2\pi}\right)^{1/q} \|u\|_{L^q(\mathbb{C})} \leq \left(\frac{p}{2\pi}\right)^{1/p} \|u\|_{L^p(\mathbb{C})}, \tag{6}$$

with optimal constants.

This result can be understood as smoothing estimate in the L^p scale. Compare with the Khintchine Lemma.

Proof : We prove prove the result for $p = 1$ and $q = +\infty$. Write $u(z) = f(z)e^{-|z|^2/2}$ where f is entire. By the Cauchy formula, for all $r > 0$,

$$|f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})| d\theta.$$

Thus by integration in $r > 0$

$$|f(0)| \int_0^{+\infty} re^{-r^2/2} dr \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{+\infty} |f(re^{i\theta})| re^{-r^2/2} dr d\theta,$$

in other words

$$|u(0)| = |f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \int_0^{+\infty} |f(re^{i\theta})| re^{-r^2/2} dr d\theta =$$

$$= \frac{1}{2\pi} \int_{\mathbb{C}} |f(z)| e^{-|z|^2/2} dL(z) = \frac{1}{2\pi} \|u\|_{L^1(\mathbb{C})}.$$

More generally, for any $z \in \mathbb{C}$ and f we apply the previous inequality to the entire function

$$w \longmapsto f(z - w) e^{w\overline{z} - |z|^2/2},$$

and deduce the announced bound $\|u\|_{L^\infty(\mathbb{C})} \leq \dfrac{1}{2\pi} \|u\|_{L^1(\mathbb{C})}$. $\square$

As a consequence, we observe that for all $u \in F^2(\mathbb{C})$

$$\mathcal{E}(u) = \|u\|^4_{L^4(\mathbb{C})} \leq \frac{1}{2\pi} \|u\|^4_{L^2(\mathbb{C})}.$$

We refer to the book [Zhu] for more analysis on Bargmann-Fock spaces.

Exercise

1. *Show that with a slight modification in the previous proof one can also obtain the case $q = \infty$ and any $p \geq 1$.*
2. *Prove directly the inequality* (6) *for* $(q, p) = (\infty, 2)$. *Hint : use the identity*

$$\int_{\mathbb{C}} e^{-|w|^2 + aw + c\overline{w}} \, dL(w) = \pi e^{ac}.$$

We are now able to show that (LLL) is globally well-posed in $F^p(\mathbb{C})$ with $2 \leq p \leq 4$.

Proposition (Gérard-Germain-LT)

Assume that $2 \leq p \leq 4$. *The equation* (LLL) *is globally well-posed for data in* $F^p(\mathbb{C})$ *and such data lead to solutions in* $\mathcal{C}^\infty\big(\mathbb{R}, F^p(\mathbb{C})\big)$.
Moreover, there exists $C = C(\|u_0\|_{L^p(\mathbb{C})}) > 0$ *such that*

$$\|u(t) - u_0\|_{L^p(\mathbb{C})} \leq C|t|^{4/p-1}, \qquad \|u(t) - u_0\|_{L^2(\mathbb{C})} \leq C|t|, \qquad \forall t \in \mathbb{R}. \quad (7)$$

Proof : Local well-posedness is obtained by a fixed point argument from the following a priori estimate : using successively the boundedness of Π, Hölder's inequality, and (6),

$$\big\|\Pi(|u|^2u)\big\|_{L^p} \leq C_1\big\||u|^2u\big\|_{L^p} = C_1\|u\|^3_{L^{3p}} \leq C_2\|u\|^2_{L^4}\|u\|_{L^p}.$$

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Let us now prove the bound (7). We write $u = u_0 + v$, then for $t \geq 0$ we have

$$v(t) = -i\int_0^t \mathcal{T}(u_0+v)(s)ds.$$

We take the L^2-norm and get with the help of (6)

$$\|v(t)\|_{L^2(\mathbb{C})} \leq C_1 t\|u_0+v\|^3_{L^6(\mathbb{C})} \leq C_2 t(\|u_0\|^3_{L^6(\mathbb{C})}+\|v\|^3_{L^6(\mathbb{C})}) \leq C_3 t(\|u_0\|^3_{L^p(\mathbb{C})}+\|v\|^3_{L^4(\mathbb{C})}).$$

Therefore, by the conservation of the energy, we obtain $\|v(t)\|_{L^2(\mathbb{C})} \leq Ct$ which is the second bound. The first bound follows from interpolation with the energy. $\square$

KAM results for a perturbed equation

In the sequel, we consider the (non-local) perturbation of the (LLL) equation

$$i\partial_t u + \nu M u = \varepsilon \Pi(|u|^2 u), \quad (t,z) \in \mathbb{R} \times \mathbb{C}, \tag{8}$$

where $\nu, \varepsilon > 0$ are small and where M is the (Hermite) multiplier, defined by $M\varphi_j = \xi_j \varphi_j$ with $-1 \leq \xi_j \leq 1$.

Notice that M and H commute and that we have the following conservation laws :

$$\int_{\mathbb{C}} |u(z)|^2 dL(z), \qquad \int_{\mathbb{C}} \overline{u} Hu(z) dL(z), \qquad \nu \int_{\mathbb{C}} \overline{u} Mu(z) dL(z) + \varepsilon \int_{\mathbb{C}} |u(z)|^4 dL(z),$$

which are the L^2 and $\mathcal{H}^1$ norms as well the Hamiltonian (there are other conservation laws).

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which are the L^2 and $\mathcal{H}^1$ norms as well the Hamiltonian (there are other conservation laws).

Using the commutation of M and H, as well as the relation

$$e^{itH} \mathcal{T}(u_1, u_2, u_3) = \mathcal{T}(e^{itH} u_1, e^{itH} u_2, e^{itH} u_3),$$

we see that (10) is equivalent to the equation ($v = e^{itH} u$)

$$i\partial_t v + Hv + \nu M v = \Pi(|v|^2 v), \quad (t,z) \in \mathbb{R} \times \mathbb{C}. \tag{11}$$

The abstract KAM result of Grébert-Thomann can directly be applied to the equation (11) and hence (10)

Theorem

Let $n \geq 1$ be an integer and set $\mathcal{A} = [-1,1]^{n+1}$. There exist $\varepsilon_0 > 0$, $\nu_0 > 0$, $C_0 > 0$ and, for each $\varepsilon < \varepsilon_0$, a Cantor set $\mathcal{A}_\varepsilon \subset \mathcal{A}$ of asymptotic full measure when $\varepsilon \to 0$, such that for each $\xi \in \mathcal{A}_\varepsilon$ and for each $C_0\varepsilon \leq \nu < \nu_0$, the solution of

$$i\partial_t u + \nu M u = \varepsilon \Pi(|u|^2 u), \quad (t,z) \in \mathbb{R} \times \mathbb{C}, \tag{12}$$

with initial datum

$$u_0(z) = \sum_{j=0}^{n} I_j^{1/2} e^{i\theta_j} \varphi_j(z), \tag{13}$$

with $(I_0, \cdots, I_n) \subset (0,1]^{n+1}$ and $\theta \in \mathbb{T}^{n+1}$, is quasi periodic with a quasi period $\omega^\star$ close to $\omega_0 = (2j+2)_{j=0}^n$: $|\omega^\star - \omega_0| < C\nu$.

Control of Sobolev norms for a perturbed equation

We define the Hermite multiplier M by $M\varphi_j = m_j\varphi_j$, where $(m_j)_{j\in\mathbb{N}}$ is a bounded sequence of real numbers chosen in the following classes : for any $k \geq 1$, we define the class

$$\mathcal{W}_k = \Big\{(m_j)_{j\in\mathbb{N}} : \ m_j = \frac{\tilde{m}_j}{(j+1)^k} \ \text{ with } \ \tilde{m}_j \in [-1/2, 1/2]\Big\}$$

which is endowed with the product Lebesgue (probability) measure. Consider the problem

$$i\partial_t u + Mu = \Pi(|u|^2 u), \quad (t,z) \in \mathbb{R}\times\mathbb{C}. \tag{14}$$

The following almost global existence result is proved in [Grébert-Imekraz-Paturel].

Theorem

Let $k, r \in \mathbb{N}$. There exists a set $\mathcal{B}_k \subset \mathcal{W}_k$ of measure 1 such that if $(m_j)_{j\in\mathbb{N}} \in \mathcal{B}_k$ there exists $s_0 \in \mathbb{N}$ such that for any $s \geq s_0$, there are $\varepsilon_0 > 0$, $c > 0$, such that for any $\varepsilon \in (0, \varepsilon_0)$, for any $u_0 \in \mathcal{H}^s(\mathbb{C})$ with

$$\|u_0\|_{\mathcal{H}^s(\mathbb{C})} \leq \varepsilon,$$

the equation (14) with initial datum u_0 has a unique global solution $u \in \mathcal{C}^\infty\big(\mathbb{R}, \mathcal{H}^s(\mathbb{C})\big)$ and it satisfies

$$\|u(t)\|_{\mathcal{H}^s(\mathbb{C})} \leq 2\varepsilon, \quad |t| \leq c\varepsilon^{-r}.$$

To prove this result, we apply the result of Grébert-Imekraz-Paturel to the equation $i\partial_t v + Hv + Mv = \Pi(|v|^2 v)$, obtained with the change unknown $v = \mathrm{e}^{itH} u$.
This result shows that if the initial condition is strongly localised in space, then the corresponding solution remains localised for large times.

Statement of the probabilistic results

Set

$$X^0_{hol}(\mathbb{C}) := \big(\cap_{\sigma>0}\mathcal{H}^{-\sigma}(\mathbb{C})\big)\cap\big(\mathcal{O}(\mathbb{C})e^{-|z|^2/2}\big).$$

Define $\gamma \in L^2(\Omega; X^0(\mathbb{C}))$ by

$$\gamma(\omega, z) = \sum_{n=0}^{+\infty} \frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z),$$

and for $\beta > 0$ we define $\gamma_\beta = \gamma/\sqrt{\beta}$. Consider the Gaussian probability measure

$$\mu_\beta = (\gamma_\beta)_\# \mathbf{p} := \mathbf{p}\circ\gamma_\beta^{-1}.$$

We will check later that μ_β is a probability measure on $X^0_{hol}(\mathbb{C})$. Let $2 < p \leq +\infty$, then for almost all $\omega \in \Omega$,

$$\gamma(\omega, .) \in F^p(\mathbb{C}) \quad \text{but} \quad \gamma(\omega, .) \notin F^2(\mathbb{C}).$$

As a consequence $\mu_\beta(L^2(\mathbb{C})) = 0$.

Notice that since (LLL) conserves the $\mathcal{H}^1(\mathbb{C})$ norm, μ_β is formally invariant by its flow. More generally, we can define a family $(\rho_\beta)_{\beta>0}$ of probability measures on $X^0_{hol}(\mathbb{C})$ which are formally invariant by (LLL) in the following way : define for $\beta > 0$ the measure ρ_β by

$$d\rho_\beta(u) = C_\beta e^{-\beta\mathcal{E}(u)} d\mu_\beta(u), \tag{15}$$

where $C_\beta > 0$ is a normalising constant. By the Kakutani theorem and its corollary, the measures ρ_β are mutually singular. Actually, the $(\rho_\beta)_{\beta>0}$ are the Gibbs measures of the equation (5).

We are now able to state the following global existence result, which also gives some qualitative information on the long time dynamics.

Theorem (Germain-Hani-LT)

Let $\beta > 0$. There exists a set $\Sigma \subset X^0_{hol}(\mathbb{C})$ of full ρ_β measure so that for every $u_0 \in \Sigma$ the equation (LLL) with initial condition $u(0) = u_0$ has a unique global solution $u(t) = \Phi(t, u_0)$ such that for any $0 < s < 1/2$

$$u(t) - u_0 \in \mathcal{C}\big(\mathbb{R}; \mathcal{H}^s(\mathbb{C})\big).$$

Moreover, for all $\sigma > 0$ and $t \in \mathbb{R}$

$$\|u(t)\|_{L^3(\mathbb{C})} + \|u(t)\|_{\mathcal{H}^{-\sigma}(\mathbb{C})} \leq C\big(\Lambda(u_0, \sigma) + \ln^{1/2}(1 + |t|)\big), \tag{16}$$

where the constant $\Lambda(u_0, \sigma)$ satisfies the bound $\mu_\beta\big(u_0 : \Lambda(u_0, \sigma) > \lambda\big) \leq Ce^{-c\lambda^2}$. Furthermore, the measure ρ_β is invariant by Φ : for any ρ_β measurable set $A \subset \Sigma$ and for any $t \in \mathbb{R}$,

$$\rho_\beta(A) = \rho_\beta(\Phi(t, A)).$$

Finally, for all $t \in \mathbb{R}$

$$\|u(t)\|_{L^4(\mathbb{C})} = \|u_0\|_{L^4(\mathbb{C})}.$$

The same result (with the *ad hoc* measures μ and ρ) holds for the perturbed equations (11) and (14).

Remark

By the Birkhoff-Kintchine Theorem we have for all $k \geq 1$

$$\frac{1}{T}\int_0^T \|u(t)\|^k_{\mathcal{H}^{-\sigma}(\mathbb{C})} dt \longrightarrow G_k(u_0), \quad \textit{when} \quad T \longrightarrow +\infty, \tag{17}$$

and the fonction G_k *is a conservation law : for all* $t \in \mathbb{R}$, $G_k(u(t)) = G_k(u_0)$. *Moreover*

$$\int_{\mathcal{H}^{-\sigma}} G_k(u) d\mu(u) = \int_{\mathcal{H}^{-\sigma}} \|u\|^k_{\mathcal{H}^{-\sigma}(\mathbb{C})} d\mu(u).$$

One even has

$$\frac{1}{T}\int_0^T e^{\frac{1}{2}\|u(t)\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})}} dt \longrightarrow G_\infty(u_0), \quad \textit{when} \quad T \longrightarrow +\infty,$$

By the theorem, there may be initial conditions such that $\|u(t)\|_{\mathcal{H}^{-\sigma}(\mathbb{C})}$ *may grow like* $\ln^{1/2}(t)$, *but not many since in mean it stays bounded, by* (17). *Compare with the bound obtained from the deterministic result.*

Remark

Formally, the (LLL) *equation looks like the Szegö equation introduced and studied by Gérard and Grellier, but their properties are different. For instance, unlike* (16) *there is no nonlinear smoothing for the Szegö equation, as was shown by Oh, therefore it is not clear if an analogous result holds for the Szegö equation.*

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Remark

Let us compare the types of results given by the KAM method, the Birkhoff normal form method and the probabilistic methods :

- *Smooth vs rough solutions*
- *Randomness*
- *Resonances ?*

Let us conclude this section with a few reference concerning the use of Gibbs measure in the construction of global strong solutions to PDEs. In a compact setting : Lebowitz-Rose-Speer, Bourgain, Zhidkov, Tzvetkov, Burq-Tzvetkov, Oh, Burq-Thomann-Tzvetkov, Deng, Nahmod et al, Suzzoni, Deng-Tzvetkov-Visciglia, Bourgain-Bulut, Richards and others.

There are also other types of a.s. global wellposedness results, without the use of invariant measures, mainly for the wave equation, but we do not comment on them.

Sketch of the proof of the global wellposedness result

In the sequel we fix $\beta = 1$ (say) and write $\mu = \mu_\beta$, $\rho = \rho_\beta$. We only prove the result for $s = 0$.

Lemma

The measure μ is a probability measure on $X^0_{hol}(\mathbb{C})$.

Proof : It is enough to show that $\gamma \in X^0_{hol}(\mathbb{C})$, $\mathbf{p}$-a.s. First, for all $\sigma > 0$ we have

$$\int_\Omega \|\gamma\|^2_{\mathcal{H}^{-\sigma}(\mathbb{C})} d\mathbf{p}(\omega) = \int_\Omega \sum_{n=0}^{+\infty} \frac{|g_n|^2}{\big(2(n+1)\big)^{\sigma+1}} d\mathbf{p}(\omega) = C \sum_{n=0}^{+\infty} \frac{1}{(n+1)^{\sigma+1}} < +\infty,$$

therefore $\gamma \in \bigcap_{\sigma>0} L^2\big(\Omega\,;\,\mathcal{H}^{-\sigma}(\mathbb{C})\big)$. Next, for all $A \geq 1$ there exists a set $\Omega_A \subset \Omega$ such that $\mathbf{p}(\Omega_A^c) \leq \exp\left(-A^\delta\right)$ and for all $\omega \in \Omega_A$, $\varepsilon > 0$, $n \geq 0$

$$|g_n(\omega)| \leq CA(n+1)^\varepsilon.$$

Then for $\omega \in \bigcup_{A \geq 1} \Omega_A$, $\displaystyle\sum_{n=0}^{+\infty} \frac{z^n g_n(\omega)}{\sqrt{(n+1)!}} \in \mathcal{O}(\mathbb{C})$. $\square$

We first define a smooth version of the usual spectral projector. Let $\chi \in \mathcal{C}_0^\infty(-1,1)$, so that $0 \leq \chi \leq 1$, with $\chi = 1$ on $[-\frac{1}{2}, \frac{1}{2}]$. We define the operators $S_N = \chi\big(\frac{H}{N+1}\big)$ as

$$S_N\Big(\sum_{n=0}^{\infty} c_n \varphi_n\Big) = \sum_{n=0}^{\infty} \chi\Big(\frac{n+1}{N+1}\Big) c_n \varphi_n.$$

Then for all $1 < p < +\infty$, the operator S_N is bounded in $L^p(\mathbb{C})$. This result does not hold true if one replaces S_N with a crude frequency truncation.

Local existence

Recall the definition of $\mathcal{T}$ in (3). It will be useful to work with an approximation of (LLL). We consider the dynamical system given by the Hamiltonian $\mathcal{E}_N(u) := \mathcal{E}(S_N u)$. This system reads

$$\begin{cases} i\partial_t u_N = \mathcal{T}_N(u_N), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ u_N(0,z) = u_0(z), \end{cases} \tag{18}$$

and $\mathcal{T}_N(u_N) := S_N \mathcal{T}(S_N u, S_N u, S_N u)$.

Denote by E_k the space on $\mathbb{C}$ spanned by φ_k. Observe that (18) is a finite dimensional dynamical system on $\bigoplus_{k=0}^N E_k$ and that the projection of $u_N(t)$ on its complement is constant. For $N \geq 0$ we define the measures ρ_N by

$$d\rho_N(u) = C_N e^{-\mathcal{E}_N(u)} d\mu(u),$$

where $C_N > 0$ is a normalising constant. We have the following result

$$\begin{cases} i\partial_t u_N = \mathcal{T}_N(u_N), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ u_N(0,z) = u_0(z). \end{cases} \tag{15}$$

Lemma

The system (18) *is globally well-posed in* $L^2(\mathbb{C})$. *Moreover, the measures* ρ_N *are invariant by its flow denoted by* Φ_N.

Proof : The global existence follows from the conservation of $\|u_N\|_{L^2(\mathbb{C})}$. The invariance of the measures is a consequence of the Liouville theorem and the conservation of $\sum_{k=0}^{\infty} \lambda_k |c_k|^2$ by the flow of (LLL). $\square$

We now state a result concerning dispersive bounds of Hermite functions

Lemma

For all $2 \leq p \leq +\infty$,

$$\|\varphi_n\|_{L^p(\mathbb{C})} \leq C n^{\frac{1}{2p}-\frac{1}{4}}. \tag{19}$$

Proof : By Stirling, we easily get that $\|\varphi_n\|_{L^\infty(\mathbb{C})} \leq C n^{-\frac{1}{4}}$, which is (19) for $p = +\infty$; the estimate for $2 \leq p \leq \infty$ follows by interpolation. $\square$

Lemma

(*i*) *For all* $2 < p < +\infty$

$$\exists C > 0, \exists c > 0, \forall \lambda \geq 1, \forall N \geq 1,$$
$$\mu\big(u \in X^0_{hol}(\mathbb{C}) : \|S_N u\|_{L^p(\mathbb{C})} > \lambda \big) \leq Ce^{-c\lambda^2},$$
$$\mu\big(u \in X^0_{hol}(\mathbb{C}) : \|u\|_{L^p(\mathbb{C})} > \lambda \big) \leq Ce^{-c\lambda^2}. \tag{20}$$

(*ii*) *For all* $2 < p < +\infty$, *there exists* $\delta > 0$ *such that*

$$\exists\, C > 0, \exists\, c > 0,\ \forall \lambda \geq 1,\ \forall N \geq N_0 \geq 1,$$
$$\mu\big(u \in X^0_{hol}(\mathbb{C})\ :\ \|(S_N - S_{N_0})u\|_{L^p(\mathbb{C})} > \lambda \big) \leq Ce^{-cN_0^{\delta}\lambda^2}. \tag{21}$$

Proof : We have that

$$\mu\big(\, u\in X^0_{hol}(\mathbb{C})\ :\ \|u\|_{L^p(\mathbb{C})}>\lambda\,\big)=\mathbf{p}\Big(\Big\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\Big\|_{L^p(\mathbb{C})}>\lambda\Big).$$

Let $q\geq p\geq 2$. Recall here the Khintchine inequality : there exists $C>0$ such that for all real $k\geq 2$ and $(a_n)\in\ell^2(\mathbb{N})$

$$\Big\|\sum_{n\geq 0} g_n(\omega)\,a_n\Big\|_{L^k_{\mathbf{p}}}\leq C\sqrt{k}\Big(\sum_{n\geq 0}|a_n|^2\Big)^{\frac{1}{2}}, \tag{22}$$

if the g_n are iid normalized Gaussians.

Applying it to (22) we get

$$\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^q_\omega}\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\frac{|\varphi_n(z)|^2}{2(n+1)}\Big)^{1/2},$$

and using twice the Minkowski inequality for $q\geq p$ gives

$$\begin{aligned}\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^q_\omega L^p_z} &\leq \|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^p_z L^q_\omega}\\ &\leq C\sqrt{q}\Big(\sum_{n=0}^{\infty}\frac{\|\varphi_n(z)\|^2_{L^p(\mathbb{C})}}{\langle n\rangle}\Big)^{1/2}. \qquad (23)\end{aligned}$$

We are now ready to prove (20). Since we have $\|\varphi_n\|_{L^p(\mathbb{C})}\leq Cn^{\frac{1}{2p}-\frac{1}{4}}$, we get from (23)

$$\|\sum_{n=0}^{\infty}\frac{g_n(\omega)}{\sqrt{2(n+1)}}\varphi_n(z)\|_{L^q_\omega L^p_z}\leq C\sqrt{q}\,.$$

The Bienaymé-Tchebichev inequality gives then

$$\mathbf{p}\left(\| \sum_{n=0}^{\infty} \frac{g_n(\omega)}{\sqrt{2(n+1)}} \varphi_n(z) \|_{L^p(\mathbb{C})} > \lambda \right) \leq (\lambda^{-1} \| \sum_{n=0}^{\infty} \frac{g_n(\omega)}{\sqrt{2(n+1)}} \varphi_n(z) \|_{L^q_\omega L^p_z})^q$$
$$\leq (C\lambda^{-1}\sqrt{q})^q \,.$$

Thus by choosing $q = \delta\lambda^2 \geq 4$, for δ small enough, we get the bound

$$\mathbf{p}\big(\| \sum_{n=0}^{\infty} \frac{g_n(\omega)}{\sqrt{2(n+1)}} \varphi_n(z) \|_{L^p(\mathbb{C})} > \lambda \big) \leq C e^{-c\lambda^2} \,,$$

which was the claim. □

Remark

From the previous result we deduce that on the support of μ (resp. ρ) we have $u \in L^4(\mathbb{C})$, thus we get a global existence result. However the invariance of the measures is not directly implied.

Lemma

Let $p \in [1, \infty[$, then when $N \longrightarrow +\infty$.

$$C_N e^{-\mathcal{E}_N(u)} \longrightarrow C e^{-\mathcal{E}(u)} \quad in \quad L^p(d\mu(u)).$$

In particular, for all measurable sets $A \subset X^0_{hol}(\mathbb{C})$,

$$\rho_N(A) \longrightarrow \rho(A).$$

Proof : Denote by $G_N(u) = \mathrm{e}^{-\mathcal{E}_N(u)}$ and $G(u) = \mathrm{e}^{-\mathcal{E}(u)}$. By (21), we deduce that $\mathcal{E}_N(u) \longrightarrow \mathcal{E}(u)$ in measure, w.r.t. μ. In other words, for $\varepsilon > 0$ and $N \geq 1$ we denote by

$$A_{N,\varepsilon} = \big\{ u \in X^0_{hol}(\mathbb{C}) \;:\; |G_N(u) - G(u)| \leq \varepsilon \big\},$$

then $\mu(A^c_{N,\varepsilon}) \longrightarrow 0$, when $N \longrightarrow +\infty$. Since $0 \leq G, G_N \leq 1$,

$$\begin{aligned} \|G - G_N\|_{L^p_\mu} &\leq \|(G - G_N)\mathbf{1}_{A_{N,\varepsilon}}\|_{L^p_\mu} + \|(G - G_N)\mathbf{1}_{A^c_{N,\varepsilon}}\|_{L^p_\mu} \\ &\leq \varepsilon\big(\mu(A_{N,\varepsilon})\big)^{1/p} + 2\big(\mu(A^c_{N,\varepsilon})\big)^{1/p} \leq C\varepsilon, \end{aligned}$$

for N large enough. Finally, we have when $N \longrightarrow +\infty$

$$C_N = \Big(\int \mathrm{e}^{-\mathcal{E}_N(u)} d\mu(u)\Big)^{-1} \longrightarrow \Big(\int \mathrm{e}^{-\mathcal{E}(u)} d\mu(u)\Big)^{-1} = C,$$

and this ends the proof. $\square$

We look for a solution to (LLL) of the form $u = u_0 + v$, thus v has to satisfy

$$\begin{cases} i\partial_t v = \mathcal{T}(u_0 + v), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ v(0,z) = 0, \end{cases} \tag{24}$$

with $\mathcal{T}(u) = \mathcal{T}(u,u,u)$. Similarly, we introduce

$$\begin{cases} i\partial_t v_N = \mathcal{T}_N(u_0 + v_N), & (t,z) \in \mathbb{R} \times \mathbb{C}, \\ v(0,z) = 0. \end{cases} \tag{25}$$

Recall that equation (25) is globally well posed in $L^2(\mathbb{C})$, and its flowmap is denoted by Φ_N.
Let $\sigma > 0$ and let us define

$$A(R) = \big\{ u_0 \in X^0_{hol}(\mathbb{C}) \ : \ \ \|u_0\|_{\mathcal{H}^{-\sigma}(\mathbb{C})} + \|u_0\|_{L^3(\mathbb{C})} \leq R^{1/2} \big\}.$$

Then we have the following result

Lemma

There exist $c, C > 0$ *so that for all* $N \geq 0$

$$\rho_N\big(A(R)^c\big) \leq Ce^{-cR}, \quad \rho\big(A(R)^c\big) \leq Ce^{-cR}, \quad \mu\big(A(R)^c\big) \leq Ce^{-cR}. \qquad (26)$$

Proof : Observe that we have $\rho_N\big(A(R)^c\big), \rho\big(A(R)^c\big) \leq C\mu\big(A(R)^c\big)$. The result is therefore given by (20). $\square$

Proposition

There exists $c > 0$ such that, for any $R > 1$ $c_0 > 0$, setting $\tau(R) = cR^{-2}$, for any $u_0 \in A(R)$ there exists a unique solution $v \in L^\infty([-\tau,\tau]; L^2(\mathbb{C}))$ to the equation (24) and a unique solution $v_N \in L^\infty([-\tau,\tau]; L^2(\mathbb{C}))$ to the equation (25) which furthermore satisfy

$$\|v\|_{L^\infty([-\tau,\tau];L^2(\mathbb{C}))} \leq c_0 R^{-1/2}, \quad \|v_N\|_{L^\infty([-\tau,\tau];L^2(\mathbb{C}))} \leq c_0 R^{-1/2}.$$

As a consequence, for all $|t| \leq cR^{-2}$, if $c_0 \ll 1$

$$\Phi(t, u_0) \in A(R+1), \qquad \Phi_N(t, u_0) \in A(R+1). \tag{27}$$

Proof : We only consider the equation (24), the other case being similar by the boundedness of S_N on $L^p(\mathbb{C})$. We define the space

$$Z(\tau) = \big\{ v \in \mathcal{C}\big([-\tau,\tau]; L^2(\mathbb{C})\big) \text{ s.t. } v(0) = 0 \text{ and } \|v\|_{Z(\tau)} \leq c_0 R^{-1/2} \big\},$$

with $\|v\|_{Z(\tau)} = \|v\|_{L^\infty_{[-\tau,\tau]} L^2(\mathbb{C})}$, and for $u_0 \in A(R)$ we define the operator

$$K(v) = -i \int_0^t \mathcal{T}(u_0 + v) ds.$$

We will show that K has a unique fixed point $v \in Z(\tau)$.

We have

$$\begin{aligned} \|K(v)\|_{Z(\tau)} &\leq \tau \big\| \mathcal{T}(u_0 + v) \big\|_{Z(\tau)} \\ &\leq C\tau \big(\|\mathcal{T}(u_0, u_0, u_0)\|_Z + \|\mathcal{T}(u_0, u_0, v)\|_Z + \|\mathcal{T}(u_0, v, v)\|_Z \\ &\qquad + \|\mathcal{T}(v, v, v)\|_Z \big). \end{aligned}$$

We estimate each term. The conjugation plays no role, so we forget it. We only detail the first and the last term.

• Estimate of the trilinear term in v : by the hypercontractivity estimates

$$\|\mathcal{T}(v,v,v)\|_{L^2(\mathbb{C})} \leq C\|v\|^3_{L^6(\mathbb{C})} \leq C\|v\|^3_{L^2(\mathbb{C})}.$$

• Estimate of the constant term in v : for u_0 in $A(R)$

$$\|\mathcal{T}(u_0,u_0,u_0)\|_{L^2(\mathbb{C})} \leq C\|u_0\|^3_{L^6(\mathbb{C})} \leq C\|u_0\|^3_{L^3(\mathbb{C})} \leq CR^{3/2},$$

(recall here that the bound $\|u_0\|_{L^2(\mathbb{C})}$ is forbidden since $\|u_0\|_{L^2(\mathbb{C})} = +\infty$ on the support of μ.)

With these estimates at hand, the result follows by the Picard fixed point theorem. $\square$

Approximation and invariance of the measure

Lemma

Fix $R \geq 0$. Then for all $\varepsilon > 0$, there exists $N_0 \geq 0$ such that for all $u_0 \in A(R)$ and $N \geq N_0$

$$\left\| \Phi(t, u_0) - \Phi_N(t, u_0) \right\|_{L^\infty([-\tau_1, \tau_1]; L^2(\mathbb{C}))} \leq \varepsilon,$$

where $\tau_1 = cR^{-2}$ for some $c > 0$.

Proof : We have

$$v - v_N = -i \int_0^t \left[S_N\big(\mathcal{T}(u_0 + v) - \mathcal{T}(u_0 + v_N)\big) + (1 - S_N)\mathcal{T}(u_0 + v) \right] ds.$$

Then we get

$$\|v - v_N\|_{Z(\tau)} \leq C\tau R^2 \|v - v_N\|_{Z(\tau)} + \int_{-\tau}^{\tau} \|(1 - S_N)\mathcal{T}(u_0 + v)\|_{L^2(\mathbb{C})} ds,$$

which in turn implies when $C\tau R^2 \leq 1/2$

$$\|v - v_N\|_{Z(\tau)} \leq 2 \int_{-\tau}^{\tau} \|(1 - S_N)\mathcal{T}(u_0 + v)\|_{L^2(\mathbb{C})} ds.$$

Here we need a bit a compactness to conclude. We refer to [Germain-Hani-Thomann] for the details. □

Let $D_{i,j} = (i + j^{1/2})^{1/2}$, with $i, j \in \mathbb{N}$ and set $T_{i,j} = \sum_{\ell=1}^{j} \tau_1(D_{i,\ell})$. Let

$$\Sigma_{N,i} := \big\{ u_0 \; : \forall j \in \mathbb{N}, \;\; \Phi_N(\pm T_{i,j}, u_0) \in A(D_{i,j+1}) \big\},$$

and

$$\Sigma_i := \limsup_{N \to +\infty} \Sigma_{N,i}, \qquad \Sigma := \bigcup_{i \in \mathbb{N}} \Sigma_i.$$

Proposition

The following holds true :

- (*i*) *The set Σ is of full ρ measure.*
- (*ii*) *For all $u_0 \in \Sigma$, there exists a unique global solution $u = u_0 + v$ to* (LLL)*. This defines a global flow Φ on Σ.*
- (*iii*) *For all measurable set $A \subset \Sigma$, and all $t \in \mathbb{R}$,*

$$\rho(A) = \rho\big(\Phi(t, A)\big).$$

The proof of (*ii*) relies on the invariance of the measure ρ_N under the flow Φ_N. A repeated use of the approximation result will be crucial to prove (*iii*).

Let us show how one uses the Gibbs measure to define a global flow and to get the quantitative bound in $\ln^{1/2}(t)$ in the main theorem.

Let $c > 0$ be given by (26). For $T \leq e^{cR/2}$ we define the set of the good data

$$\Sigma_R = \bigcap_{k=-[T/\tau]}^{[T/\tau]} \Phi_N\big(-k\tau, B_R\big). \tag{28}$$

Now we crucially use the invariance of the measure and get

$$\begin{aligned} \rho_N(X^0_{hol}(\mathbb{R})\backslash\Sigma_R) &\leq (2[T/\tau]+1)\rho_N(X^0(\mathbb{R})\backslash B_R) \\ &\leq CR^2 e^{cR/2} e^{-cR} \leq Ce^{-cR/4}, \end{aligned}$$

which shows that Σ_R is a big subset of $X^0_{hol}(\mathbb{R})$ when $R \longrightarrow +\infty$.

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which shows that Σ_R is a big subset of $X^0_{hol}(\mathbb{R})$ when $R \longrightarrow +\infty$. Now, by the definition (29) of Σ_R and (27), we deduce that for all $|t| \leq T$ and $u_0 \in \Sigma_R$

$$\|\Phi_N(t,u_0)\|_{L^3(\mathbb{C})} + \|\Phi_N(t,u_0)\|_{\mathcal{H}^{-\sigma}(\mathbb{C})} \leq (R+1)^{1/2}.$$

In particular, for $|t| = T \sim e^{cR/2}$

$$\|\Phi_N(t,u_0)\|_{L^3(\mathbb{C})} + \|\Phi_N(t,u_0)\|_{\mathcal{H}^{-\sigma}(\mathbb{C})} \leq C(\ln|t|+1)^{1/2},$$

and this bound is uniform in $N \geq 1$. The term $\ln^{1/2}(t)$ is reminiscent from the large deviation estimates involving Gaussian random variables.