GIBBS MEASURE FOR THE PERIODIC DERIVATIVE NONLINEAR SCHRÖDINGER EQUATION

by

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Abstract. — In this paper we construct a Gibbs measure for the derivative Schrödinger equation on the circle. The construction uses some renormalisations of Gaussian series and Wiener chaos estimates, ideas which have already been used by the second author in a work on the Benjamin-Ono equation.

1. Introduction

Denote by $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ the circle. The purpose of this work is to construct a Gibbs measure associated to the derivative nonlinear Schrödinger equation

$$\begin{cases} i\partial_t u + \partial_x^2 u = i\partial_x\big(|u|^2 u\big), & (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u(0,x) = u_0(x). \end{cases} \tag{1.1}$$

Many recent results (see the end of Section 1.2) show that a Gibbs measure is an efficient tool to construct global rough solutions of nonlinear dispersive equations. This is the main motivation of this paper: we hope that our result combined with a local existence theory for (1.1) (e.g. a result like Grünrock-Herr [**6**]) on the support of the measure will give a global existence result for irregular initial conditions. A second motivation is the fact that an invariant measure is an object which fits well in the study of recurrence properties given by the Poincaré theorem, of the flow of (1.1).

2000 ***Mathematics Subject Classification.*** — 35BXX ; 37K05 ; 37L50 ; 35Q55.
Key words and phrases. — Nonlinear Schrödinger equation, random data, Gibbs measure.

The authors were supported in part by the grant ANR-07-BLAN-0250.

For $f \in L^2(\mathbb{T})$, denote by $\int_{\mathbb{T}} f(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} f(x)\mathrm{d}x$. The following quantities are conserved (at least formally) by the flow of the equation

• The mass

$$M(u(t)) = \|u(t)\|_{L^2} = \|u_0\|_{L^2} = M(u_0).$$

• The energy

$$\begin{aligned} H(u(t)) &= \int_{\mathbb{T}} |\partial_x u|^2 \mathrm{d}x + \frac{3}{2}\mathrm{Im}\int_{\mathbb{T}} |u|^2 u\, \partial_x \overline{u} \mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}} |u|^6 \mathrm{d}x \\ &= \int_{\mathbb{T}} |\partial_x u|^2 \mathrm{d}x - \frac{3}{4}\mathrm{Im}\int_{\mathbb{T}} \overline{u}^2\, \partial_x(u^2) \mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}} |u|^6 \mathrm{d}x \\ &= \int_{\mathbb{T}} |\partial_x u|^2 \mathrm{d}x + \frac{3}{4}i\int_{\mathbb{T}} \overline{u}^2\, \partial_x(u^2) \mathrm{d}x + \frac{1}{2}\int_{\mathbb{T}} |u|^6 \mathrm{d}x \\ &= H(u_0). \end{aligned}$$

The conservation of the energy can be seen by a direct computation (see also the appendix of this paper.)
Notice that the momentum

$$\begin{aligned} P(u(t)) &= \frac{1}{2}\int_{\mathbb{T}} |u|^4 \mathrm{d}x + i\int_{\mathbb{T}} \overline{u}\, \partial_x u \mathrm{d}x \\ &= \frac{1}{2}\int_{\mathbb{T}} |u|^4 \mathrm{d}x - \mathrm{Im}\int_{\mathbb{T}} \overline{u}\, \partial_x u \mathrm{d}x = P(u_0), \end{aligned}$$

is also formally conserved by (1.1). Indeed it is the Hamiltonian of (1.1) associated to a symplectic structure involving ∂_x (see [**7**]). However, we won't use this fact here. Instead, our measure will be deduced from a Hamiltonian formulation based on H of a transformed form of (1.1).

Let us define the complex vector space $E_N = \mathrm{span}\Big((\mathrm{e}^{inx})_{-N\leq n\leq N}\Big)$. Then we introduce the spectral projector Π_N on E_N by

$$\Pi_N\Big(\sum_{n\in\mathbb{Z}} c_n \mathrm{e}^{inx}\Big) = \sum_{n=-N}^{N} c_n \mathrm{e}^{inx}\,. \tag{1.2}$$

Let $(\Omega, \mathcal{F}, \mathbf{p})$ be a probability space and $\big(g_n(\omega)\big)_{n\in\mathbb{Z}}$ a sequence of independent complex normalised gaussians, $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$. We can write

$$g_n(\omega) = \frac{1}{\sqrt{2}}\big(h_n(\omega) + i\ell_n(\omega)\big), \tag{1.3}$$

where $\big(h_n(\omega)\big)_{n\in\mathbb{Z}}$, $\big(\ell_n(\omega)\big)_{n\in\mathbb{Z}}$ are independent standard real Gaussians $\mathcal{N}_{\mathbb{R}}(0,1)$.

1.1. Definition of the measure for (1.1). —

In the sequel we will use the notation $\langle n\rangle = \sqrt{n^2+1}$.

Now write $c_n = a_n + ib_n$.
For $N \geq 1$, consider the probability measure on $\mathbb{R}^{2(2N+1)}$ defined by

$$\mathrm{d}\mu_N = d_N \prod_{n=-N}^{N} e^{-\langle n\rangle^2(a_n^2+b_n^2)}\mathrm{d}a_n\mathrm{d}b_n, \tag{1.4}$$

where d_N is such that

$$\begin{aligned}\frac{1}{d_N} &= \prod_{n=-N}^{N}\int_{\mathbb{R}^2} e^{-\langle n\rangle^2(a_n^2+b_n^2)}\mathrm{d}a_n\mathrm{d}b_n \\ &= \pi^{2N+1}\Big(\prod_{n=-N}^{N}\frac{1}{\langle n\rangle}\Big)^2 = \pi^{2N+1}\Big(\prod_{n=1}^{N}\frac{1}{\langle n\rangle}\Big)^4.\end{aligned} \tag{1.5}$$

The measure μ_N defines a measure on E_N via the map

$$(a_n,b_n)_{n=-N}^{N} \longmapsto \sum_{n=-N}^{N}(a_n+ib_n)e^{inx},$$

which will still be denoted by μ_N. Then μ_N may be seen as the distribution of the E_N valued random variable

$$\omega \longmapsto \sum_{|n|\leq N}\frac{g_n(\omega)}{\langle n\rangle}\mathrm{e}^{inx} \equiv \varphi_N(\omega,x), \tag{1.6}$$

where $(g_n)_{n=-N}^{N}$ are Gaussians as in (1.3).

Let $\sigma < \frac{1}{2}$. Then (φ_N) is a Cauchy sequence in $L^2\big(\Omega;\, H^\sigma(\mathbb{T})\big)$ which defines

$$\varphi(\omega,x) = \sum_{n\in\mathbb{Z}}\frac{g_n(\omega)}{\langle n\rangle}\mathrm{e}^{inx}, \tag{1.7}$$

as the limit of (φ_N). Indeed, the map

$$\omega \longmapsto \sum_{n\in\mathbb{Z}}\frac{g_n(\omega)}{\langle n\rangle}\mathrm{e}^{inx},$$

defines a (Gaussian) measure on $H^\sigma(\mathbb{T})$ which will be denoted by μ.

For $u \in L^2(\mathbb{T})$, we will write $u_N = \Pi_N u$. Now define

$$f_N(u) = \mathrm{Im}\int_{\mathbb{T}}\overline{u_N^2(x)}\,\partial_x(u_N^2(x))\mathrm{d}x.$$

Let $\kappa > 0$, and let $\chi : \mathbb{R} \longrightarrow \mathbb{R}$, $0 \leq \chi \leq 1$ be a continuous function with support $\text{supp}\,\chi \subset [-\kappa, \kappa]$ and so that $\chi = 1$ on $[-\frac{\kappa}{2}, \frac{\kappa}{2}]$. We define the density

$$G_N(u) = \chi\big(\|u_N\|_{L^2(\mathbb{T})}\big)\mathrm{e}^{\frac{3}{4}f_N(u) - \frac{1}{2}\int_{\mathbb{T}}|u_N(x)|^6\mathrm{d}x}, \tag{1.8}$$

and the measure ρ_N on $H^\sigma(\mathbb{T})$ by

$$\mathrm{d}\rho_N(u) = G_N(u)\mathrm{d}\mu(u). \tag{1.9}$$

1.2. Statement of the main result. —
Our main result which defines a formally invariant measure for (1.1) reads

Theorem 1.1. — *The sequence $G_N(u)$ defined in* (1.8) *converges in measure, as $N \to \infty$, with respect to the measure μ. Denote by $G(u)$ the limit of* (1.8) *as $N \to \infty$, and we define $d\rho(u) \equiv G(u)d\mu(u)$.*
Moreover, for every $p \in [1, \infty[$, there exists $\kappa_p > 0$ so that for all $0 < \kappa \leq \kappa_p$, $G(u) \in L^p(d\mu(u))$ and the sequence G_N converges to G in $L^p(d\mu(u))$, as N tends to infinity.

Remark 1.2. — In particular, for any Borel set $A \subset H^\sigma(\mathbb{T})$, $\lim_{N\to\infty} \rho_N(A) = \rho(A)$.

It is not clear to us how to prove the convergence property, if we define ρ_N as follows : For any Borel set $A \subset H^\sigma(\mathbb{T})$, $\rho_N(A) = \tilde{\rho}_N(A \cap E_N)$ where $\mathrm{d}\tilde{\rho}_N = G_N(u)\mathrm{d}\mu_N(u)$. In particular, the convergence stated in [**11**, Theorem 1] is not proven there. However, if we define in the context of [**11**] ρ_N as we did here, the convergence property holds true. In addition the measure ρ_N defined here (see also [**4**]) is more natural, since it is invariant by the truncated flow $\Phi_N(t)$ of equation (A.16).

One can show that by varying the cut-off χ, the support of ρ describes the support of μ (see Lemma 4.2 below).

The main ideas of this paper come from the work of the second author [**11**] where a similar construction is made for the Benjamin-Ono equation using the pioneering work of Bourgain [**3**]. In [**11**], one of the main difficulties is that on the support of the measure μ, the L^2 norm is a.s. infinite, which is not the case in our setting, since for any $\sigma < \frac{1}{2}$, $\varphi(\omega) \in H^\sigma(\mathbb{T})$, for almost all $\omega \in \Omega$. Here the difficulty is to treat the term $\displaystyle\int_{\mathbb{T}} \overline{u}^2\,\partial_x(u^2)\mathrm{d}x$ in the conserved quantity H. Roughly speaking, it should be controlled by the $H^{\frac{1}{2}}$ norm, but this is not enough, since $\|u\|_{H^{\frac{1}{2}}(\mathbb{T})} = \infty$ on the support of $\mathrm{d}\mu$. However, we will see in

Section 2, that we can handle this term thanks to an adapted decomposition and thanks to the integrability properties of the Gaussians. This is the main new idea in this paper.

The result of Theorem 1.1 may be the first step to obtain almost sure global well-posedness for (1.1), with initial conditions of the form (1.7). To reach such a result, we will also need a suitable local existence theory on the statistical set, and prove the invariance of the measure $\mathrm{d}\rho$ under this flow. For instance, this program was fruitful for Bourgain [**2, 3**] and Zhidkov [**14**] for NLS on the torus, Tzvetkov [**12, 13**] for NLS on the disc, Burq-Tzvetkov [**5**] for the wave equation, Oh [**8, 9**] for Schrödinger-Benjamin-Ono and KdV systems, and Burq-Thomann-Tzvetkov [**4**] for the one-dimensional Schrödinger equation. For the DNLS equation, we plan to pursue this issue in a subsequent work.

1.3. Notations and structure of the paper. —

Notations. — *In this paper c, C denote constants the value of which may change from line to line. These constants will always be universal, or uniformly bounded with respect to the other parameters.*
We denote by $\mathbb{Z}$ (resp. $\mathbb{N}$) the set of the integers (resp. non negative integers), and $\mathbb{N}^* = \mathbb{N}\backslash\{0\}$.
For $x \in \mathbb{R}$, we write $\langle x \rangle = \sqrt{x^2+1}$. For $u \in L^2(\mathbb{T})$, we usually write $u_N = \Pi_N u$, where Π_N is the projector defined in (1.2).
The notation L^q stands for $L^q(\mathbb{T})$ and $H^s = H^s(\mathbb{T})$.

The paper is organised as follows. In Section 2 we give some large deviation bounds and some results on the Wiener chaos at any order. In Section 3 we study the term of the Hamiltonian containing the derivative, and Section 4 is devoted to the proof of Theorem 1.1.
In the appendix, we give the Hamiltonian formulation of the transformed form of (1.1).

Acknowledgements. — *The authors warmly thank Justin Brereton for pointing out an error in Lemma 2.1 and in Proposition 3.1 in the previous version of the paper.*

2. Preliminaries : some stochastic estimates

2.1. Large deviation estimates. —

Lemma 2.1. — *Fix $\sigma < \frac{1}{2}$ and $p \in [2, \infty)$. Then*

$$\exists\, C > 0, \exists\, c > 0,\ \forall\, \lambda \geq 1, \forall\, N \geq 1,$$
$$\mu\big(\, u \in H^\sigma \ :\ \|\Pi_N u\|_{L^p(\mathbb{T})} > \lambda \,\big) \leq C e^{-c\lambda^2}\,.$$

Moreover there exists $\beta > 0$ such that

$$\exists\, C > 0, \exists\, c > 0,\ \forall\, \lambda \geq 1, \forall\, M \geq N \geq 1,$$
$$\mu\big(\, u \in H^\sigma \ :\ \|\Pi_M u - \Pi_N u\|_{L^p(\mathbb{T})} > \lambda \,\big) \leq C e^{-cN^\beta \lambda^2}\,.$$

Proof. — This result is consequence of the hypercontractivity of the Gaussian random variables : There exists $C > 0$ such that for all $r \geq 2$ and $(c_n) \in l^2(\mathbb{N})$

$$\|\sum_{n\geq 0} g_n(\omega)\, c_n\|_{L^r(\Omega)} \leq C\sqrt{r}\Big(\sum_{n\geq 0} |c_n|^2\Big)^{\frac{1}{2}}.$$

See e.g. [**4**, Lemma 3.3] for the details of the proof. □

2.2. Wiener chaos estimates. —

The aim of this subsection is to obtain $L^p(\Omega)$ bounds on Gaussian series. These are obtained thanks to the smoothing effects of the Ornstein-Uhlenbeck semi-group. The following considerations are inspired from [**11**]. See also [**1, 10**] for more details on this topic.

For $d \geq 1$, denote by L the operator

$$L = \Delta - x \cdot \nabla = \sum_{j=1}^{d} \big(\frac{\partial^2}{\partial x_j^2} - x_j \,\frac{\partial}{\partial x_j} \,\big).$$

This operator is self adjoint on $\mathcal{K} = L^2(\mathbb{R}^d, \mathrm{e}^{-|x|^2/2}\mathrm{d}x)$ with domain

$$\mathcal{D} = \Big\{ u \ :\ u(x) = \mathrm{e}^{|x|^2/4} v(x), \quad v \in \mathcal{H}^2 \Big\},$$

where $\mathcal{H}^2 = \big\{ u \in L^2(\mathbb{R}^d),\ x^\alpha \partial_x^\beta v(x) \in L^2(\mathbb{R}^d),\ \ \forall\, (\alpha, \beta) \in N^{2d},\ |\alpha| {+} |\beta| \leq 2 \big\}$. Denote by $\mathbf{k} = k_1 + \cdots + k_d$ and by $(P_n)_{n\geq 0}$ the Hermite polynomials defined by

$$P_n(x) = (-1)^n \mathrm{e}^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n}\big(\, \mathrm{e}^{-x^2}\,\big).$$

Then a Hilbertian basis of eigenfunctions of L on $\mathcal{K}$ is given by

$$P_{\mathbf{k}}(x_1,\dots,x_d) = P_{k_1}(x_1)\dots P_{k_d}(x_d),$$

with eigenvalue $-\mathbf{k} = -(k_1+\dots+k_d)$.
Finally define the measure γ_d on $\mathbb{R}^d$ by

$$\mathrm{d}\gamma_d(x) = (2\pi)^{-d/2}\mathrm{e}^{-|x|^2/2}\mathrm{d}x.$$

The next result is a direct consequence of [**11**, Proposition 3.1]. See [**1**] for the proof.

Lemma 2.2. — *Let $d \geq 1$ and $k \in \mathbb{N}$. Assume that $\widetilde{P}_k$ is an eigenfunction of L with eigenvalue $-k$. Then for all $p \geq 2$*

$$\|\widetilde{P}_k\|_{L^p(\mathbb{R}^d,d\gamma_d)} \leq (p-1)^{\frac{k}{2}}\|\widetilde{P}_k\|_{L^2(\mathbb{R}^d,d\gamma_d)}.$$

Thanks to Lemma 2.2, we will prove the following L^p smoothing effect for some stochastic series.

***Proposition 2.3* (Wiener chaos)**. —
Let $d \geq 1$ and $c(n_1,\dots,n_k) \in \mathbb{C}$. Let $(g_n)_{1\leq n\leq d} \in \mathcal{N}_{\mathbb{C}}(0,1)$ be complex L^2-normalised independent Gaussians.
For $k \geq 1$ denote by $A(k,d) = \big\{(n_1,\dots,n_k) \in \{1,\dots,d\}^k,\ n_1 \leq \dots \leq n_k\big\}$ and

$$S_k(\omega) = \sum_{A(k,d)} c(n_1,\dots,n_k)\, g_{n_1}(\omega)\cdots g_{n_k}(\omega). \tag{2.1}$$

Then for all $d \geq 1$ and $p \geq 2$

$$\|S_k\|_{L^p(\Omega)} \leq \sqrt{k+1}\,(p-1)^{\frac{k}{2}}\,\|S_k\|_{L^2(\Omega)}.$$

Proof. — Let $g_n \in \mathcal{N}_{\mathbb{C}}(0,1)$. Then we can write $g_n = \frac{1}{\sqrt{2}}(\gamma_n + i\,\widetilde{\gamma}_n)$ with $\gamma_n, \widetilde{\gamma}_n \in \mathcal{N}_{\mathbb{R}}(0,1)$ mutually independent Gaussians. Hence, up to a change of indexes (and with d replaced with $2d$) we can assume that the random variables in (2.1) are real valued. Thus in the following we assume that $g_n \in \mathcal{N}_{\mathbb{R}}(0,1)$ and are independent.
Denote by

$$\Sigma_k(x_1,\dots,x_d) = \sum_{A(k,d)} c(n_1,\dots,n_k)\, x_{n_1}\cdots x_{n_k}.$$

Then obviously for all $p \geq 1$,

$$\|S_k\|_{L^p(\Omega)} = \|\Sigma_k\|_{L^p(\mathbb{R}^d,\mathrm{d}\gamma_d)}. \tag{2.2}$$

Let $(n_1, \dots, n_k) \in A(k,d)$. Then we can write

$$x_{n_1} \cdots x_{n_k} = x_{m_1}^{p_1} \, \cdots \, x_{m_l}^{p_l},$$

where $l \leq k$, $p_1 + \cdots + p_l = k$ and $n_1 = m_1 < \cdots < m_l \leq n_k$. Now, each monomial $x_{m_j}^{p_j}$ can be expanded on the Hermite polynomials $(P_n)_{n\geq 0}$

$$x_{m_j}^{p_j} = \sum_{k_j=0}^{p_j} \alpha_{j,k_j} \, P_{k_j}(x_{m_j}).$$

Therefore there exists $\beta(k_1, \dots, k_l) \in \mathbb{C}$ so that

$$x_{n_1} \cdots x_{n_k} = \sum_{j=0}^{k} \sum_{\substack{k_1+\cdots+k_l=j \\ 0\leq k_i\leq p_i}} \beta(k_1, \dots, k_l) P_{k_1}(x_{m_1}) \cdots P_{k_l}(x_{m_l}),$$

and we have

$$\Sigma_k(x_1, \dots, x_d) = \sum_{j=0}^{k} \widetilde{P}_j(x_1, \dots, x_d), \tag{2.3}$$

where the polynomial $\widetilde{P}_j$ is given by

$$\begin{aligned} &\widetilde{P}_j(x_1, \dots, x_d) \\ &\qquad = \sum_{A(k,d)} \sum_{\substack{k_1+\cdots+k_l=j \\ 0\leq k_i\leq p_i}} c(n_1, \dots, n_k) \beta(k_1, \dots, k_l) P_{k_1}(x_{m_1}) \cdots P_{k_l}(x_{m_l}). \end{aligned}$$

For $0 \leq k_i \leq p_i$ so that $k_1 + \cdots + k_l = j$, the polynomial $\widetilde{P}_j$ is an eigenfunction of L with eigenvalue $-j$, hence by Lemma 2.2 we have that for all $p \geq 2$

$$\|\widetilde{P}_j\|_{L^p(\mathbb{R}^d, \mathrm{d}\gamma_d)} \leq (p-1)^{\frac{j}{2}} \, \|\widetilde{P}_j\|_{L^2(\mathbb{R}^d, \mathrm{d}\gamma_d)}.$$

Therefore, by (2.3) and by the Cauchy-Schwarz inequality,

$$\begin{aligned} \|\Sigma_k\|_{L^p(\mathbb{R}^d, \mathrm{d}\gamma_d)} &\leq (p-1)^{\frac{k}{2}} \sum_{j=0}^{k} \|\widetilde{P}_j\|_{L^2(\mathbb{R}^d, \mathrm{d}\gamma_d)} \\ &\leq \sqrt{k+1}\,(p-1)^{\frac{k}{2}} \Big(\sum_{j=0}^{k} \|\widetilde{P}_j\|^2_{L^2(\mathbb{R}^d, \mathrm{d}\gamma_d)} \Big)^{\frac{1}{2}} \\ &\leq \sqrt{k+1}\,(p-1)^{\frac{k}{2}} \, \|\Sigma_k\|_{L^2(\mathbb{R}^d, \mathrm{d}\gamma_d)}, \end{aligned}$$

where in the last line we used that the polynomials $\widetilde{P}_j$ are orthogonal. This concludes the proof by (2.2) $\square$

We will need the following lemma which is proved in [**11**, Lemma 4.5]

Lemma 2.4. — *Let $F : H^\sigma(\mathbb{T}) \longrightarrow \mathbb{R}$ be a measurable function. Assume that there exist $\alpha > 0, N > 0, k \geq 1$, and $C > 0$ so that for every $p \geq 2$*

$$\|F\|_{L^p(d\mu)} \leq CN^{-\alpha}\, p^{\frac{k}{2}}.$$

Then there exist $\delta > 0, C_1$ independent of N and α such that

$$\int_{H^\sigma(\mathbb{T})} e^{\delta N^{\frac{2\alpha}{k}} |F(u)|^{\frac{2}{k}}}\, d\mu(u) \leq C_1.$$

As a consequence, for all $\lambda > 0$,

$$\mu\big(\, u \in H^\sigma(\mathbb{T}) \,:\, |F(u)| > \lambda \,\big) \leq C_1 e^{-\delta N^{\frac{2\alpha}{k}} \lambda^{\frac{2}{k}}}.$$

3. Study of the sequence $\big(f_N(u)\big)_{N\geq 1}$

Recall that $f_N(u)$ is defined by $f_N(u) = \operatorname{Im} \displaystyle\int_{\mathbb{T}} \overline{u_N^2(x)}\, \partial_x(u_N^2(x))\mathrm{d}x$.
The main result of this section is the following

Proposition 3.1. — *The sequence $\big(f_N\big)_{N\geq 1}$ is a Cauchy sequence in $L^2\big(H^\sigma(\mathbb{T}), \mathcal{B}, d\mu\big)$. More precisely, there exists $C > 0$ so that for all $M > N \geq 1$*

$$\|f_M(u) - f_N(u)\|_{L^2\left(H^\sigma(\mathbb{T}),\mathcal{B},d\mu\right)} \leq \frac{C}{N^{\frac{1}{2}}}. \tag{3.1}$$

Moreover, for all $p \geq 2$ and $M > N \geq 1$

$$\|f_M(u) - f_N(u)\|_{L^p\left(H^\sigma(\mathbb{T}),\mathcal{B},d\mu\right)} \leq \frac{C\,(p-1)^2}{N^{\frac{1}{2}}}. \tag{3.2}$$

Then a combination of the estimate (3.2) and Lemma 2.4 yields the following large deviation estimate

Corollary 3.2. — *There exist $C, \delta > 0$ such that for all $M > N \geq 1$ and $\lambda > 0$*

$$\mu\big(\, u \in H^\sigma(\mathbb{T}) \,:\, |f_M(u) - f_N(u)| > \lambda \,\big) \leq Ce^{-\delta(N^{\frac{1}{2}}\lambda)^{\frac{1}{2}}}.$$

Thanks to Proposition 3.1, we are able to define the limit in $L^2(\Omega)$ of the sequence $\big(f_N\big)_{N\geq 1}$, which will be denoted by

$$f(u) = \operatorname{Im} \int_{\mathbb{T}} \overline{u^2(x)}\, \partial_x(u^2(x))\mathrm{d}x. \tag{3.3}$$

This gives a sense to the r.h.s. of (3.3) for u in the support of μ.

Notice that Corollary 3.2 implies in particular the convergence in measure

$$\forall\, \varepsilon > 0, \quad \lim_{N\to\infty} \mu\big(u \in \mathcal{H}^{-\sigma} \,:\, \big|f_N(u) - f(u)\big| > \varepsilon\big) = 0. \tag{3.4}$$

For the proof of Proposition 3.1, we have to put $\displaystyle\int_{\mathbb{T}} \overline{\varphi_N^2(\omega)}\, \partial_x(\varphi_N^2(\omega))\mathrm{d}x$ in a suitable form.
Recall the notation (1.6), then

$$\varphi_N^2(\omega) = \sum_{|n_1|,|n_2|\leq N} \frac{g_{n_1}(\omega)\, g_{n_2}(\omega)}{\langle n_1\rangle\langle n_2\rangle} \mathrm{e}^{i(n_1+n_2)x}. \tag{3.5}$$

Therefore we deduce that

$$\partial_x\big(\varphi_N^2(\omega)\big) = \sum_{|m_1|,|m_2|\leq N} i(m_1+m_2)\frac{g_{m_1}(\omega)\ g_{m_2}(\omega)}{\langle m_1\rangle\langle m_2\rangle} \mathrm{e}^{i(m_1+m_2)x}. \tag{3.6}$$

Now, by (3.5), (3.6) and the fact that $\big(\mathrm{e}^{inx}\big)_{n\in\mathbb{Z}}$ is an orthonormal family in $L^2(\mathbb{T})$ (endowed with the scalar product $\langle f, g\rangle = \displaystyle\int_{\mathbb{T}} f(x)\overline{g}(x)\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi} f(x)\overline{g}(x)\mathrm{d}x$), we obtain

$$\int_{\mathbb{T}} \overline{\varphi_N^2(\omega)}\, \partial_x(\varphi_N^2(\omega))\mathrm{d}x = \sum_{A_N} i(n_1+n_2)\, \frac{g_{m_1}(\omega)\ g_{m_2}(\omega)\ \overline{g_{n_1}(\omega)}\ \overline{g_{n_2}(\omega)}}{\langle m_1\rangle\ \langle m_2\rangle\ \langle n_1\rangle\ \langle n_2\rangle}, \tag{3.7}$$

where

$A_N = \{(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 \ \text{ s.t. } \ |m_1|, |m_2|, |n_1|, |n_2| \leq N \ \text{ and } \ m_1{+}m_2 = n_1{+}n_2\}.$

We now split the sum (3.7) in two parts, by distinguishing the cases $m_1 = n_1$ and $m_1 \neq n_1$ in A_N and write

$$\int_{\mathbb{T}} \overline{\varphi_N^2(\omega)}\, \partial_x(\varphi_N^2(\omega))\mathrm{d}x = S_N^1 + S_N^2,$$

with

$$S_N^1 = \sum_{B_N} i(n_1+n_2)\, \frac{g_{m_1}(\omega)\ g_{m_2}(\omega)\ \overline{g_{n_1}(\omega)}\ \overline{g_{n_2}(\omega)}}{\langle m_1\rangle\ \langle m_2\rangle\ \langle n_1\rangle\ \langle n_2\rangle}, \tag{3.8}$$

where $B_N = A_N \cap \{\, m_1 = n_1 \ \text{ or } \ m_1 = n_2 \,\}$, and

$$S_N^2 = \sum_{D_N} i(n_1+n_2)\, \frac{g_{m_1}(\omega)\ g_{m_2}(\omega)\ \overline{g_{n_1}(\omega)}\ \overline{g_{n_2}(\omega)}}{\langle m_1\rangle\ \langle m_2\rangle\ \langle n_1\rangle\ \langle n_2\rangle}, \tag{3.9}$$

where

$$D_N = \big\{(m_1, m_2, n_1, n_2) \in \mathbb{Z}^4 \ \text{ s.t. } \ |m_1|, |m_2|, |n_1|, |n_2| \leq N,$$
$$\text{and } \ m_1 + m_2 = n_1 + n_2, \ \ m_1 \neq n_1, \ \ m_1 \neq n_2\big\}.$$

3.1. Study of S^1_N. —

Lemma 3.3. — *Let S^1_N be defined by* (3.8)*. Then there exists $C > 0$ so that for all $M > N > 0$,*

$$\|S^1_M - S^1_N\|_{L^2(\Omega)} \leq \frac{C}{N^{\frac{1}{2}}}.$$

Proof. — Let $(m_1, m_2, n_1, n_2) \in B_N$. Then as $m_1 + m_2 = n_1 + n_2$, we have $(m_1, m_2) = (n_1, n_2)$ or $(m_1, m_2) = (n_2, n_1)$, and deduce that

$$S^1_N = \sum_{|n_1|,|n_2| \leq N} 2i(n_1 + n_2)\, \frac{|g_{n_1}(\omega)|^2\, |g_{n_2}(\omega)|^2}{\langle n_1\rangle^2\, \langle n_2\rangle^2} = X_N + Y_N,$$

where

$$X_N = \sum_{|n| \leq N} 4in\, \frac{|g_n(\omega)|^4}{\langle n\rangle^4},$$

and

$$Y_N = \sum_{\substack{|n_1|,|n_2| \leq N,\\ n_1 \neq n_2}} 2i(n_1 + n_2)\, \frac{|g_{n_1}(\omega)|^2\, |g_{n_2}(\omega)|^2}{\langle n_1\rangle^2\, \langle n_2\rangle^2}.$$

♠ First we will show that there exists $C > 0$ so that for all $M > N > 0$,

(3.10) $$\|X_M - X_N\|_{L^2(\Omega)} \leq \frac{C}{N^2}.$$

Let $M > N \geq 1$. Then

$$|X_M - X_N|^2 = \sum_{N < |n_1|,|n_2| \leq M} 16 n_1 n_2\, \frac{|g_{n_1}(\omega)|^4\, |g_{n_2}(\omega)|^4}{\langle n_1\rangle^4 \langle n_2\rangle^4}.$$

Thus

$$\|X_M - X_N\|^2_{L^2(\Omega)} \leq C \sum_{N < |n_1|,|n_2| \leq M} \frac{1}{\langle n_1\rangle^3 \langle n_2\rangle^3} \leq \frac{C}{N^4},$$

which proves (3.10).

♠ To complete the proof of Lemma 3.3, it remains to check that there exists $C > 0$ so that for all $M > N > 0$,

(3.11) $$\|Y_M - Y_N\|_{L^2(\Omega)} \leq \frac{C}{N^{\frac{1}{2}}}.$$

For $M \geq N \geq 1$ we write

$$\begin{aligned} Y_N &= \sum_{\substack{|n_1|,|n_2|\leq N,\\ n_1\neq n_2}} i(n_1+n_2)\frac{|g_{n_1}(\omega)|^2\,|g_{n_2}(\omega)|^2}{\langle n_1\rangle^2\langle n_2\rangle^2} \\ &= Y_N^1+Y_N^2+Y_N^3, \end{aligned}$$

with

$$Y_N^1 = \sum_{\substack{|n_1|,|n_2|\leq N,\\ n_1\neq n_2}} i(n_1+n_2)\frac{\big(|g_{n_1}(\omega)|^2-1\big)\,\big(|g_{n_2}(\omega)|^2-1\big)}{\langle n_1\rangle^2\langle n_2\rangle^2}, \tag{3.12}$$

$$Y_N^2 = \sum_{\substack{|n_1|,|n_2|\leq N,\\ n_1\neq n_2}} i(n_1+n_2)\frac{\big(|g_{n_1}(\omega)|^2-1\big)+\big(|g_{n_2}(\omega)|^2-1\big)}{\langle n_1\rangle^2\langle n_2\rangle^2}, \tag{3.13}$$

and

$$Y_N^3 = \sum_{\substack{|n_1|,|n_2|\leq N,\\ n_1\neq n_2}} i(n_1+n_2)\frac{1}{\langle n_1\rangle^2\langle n_2\rangle^2}.$$

By the symmetry $(n_1,n_2)\mapsto(-n_1,-n_2)$, we have that $Y_N^3=0$. For $n\in\mathbb{Z}$, denote by

$$G_n(\omega) = |g_n(\omega)|^2-1.$$

Let $n\neq m$. Then, since g_n and g_m are independent and since $\mathbb{E}\big[|g_n(\omega)|^2\big]=1$, we have

$$\mathbb{E}\big[G_n(\omega)\,G_m(\omega)\big] = \mathbb{E}\big[G_n(\omega)\big]\mathbb{E}\big[G_m(\omega)\big] = 0. \tag{3.14}$$

First we analyse (3.12). We have

$$\begin{aligned} Y_M^1-Y_N^1 &= \sum_{\substack{|n_2|\leq M,\\ N<|n_1|\leq M,\\ n_1\neq n_2}} i(n_1+n_2)\frac{G_{n_1}(\omega)\,G_{n_2}(\omega)}{\langle n_1\rangle^2\langle n_2\rangle^2} + \sum_{\substack{|n_1|\leq N,\\ N<|n_2|\leq M,\\ n_1\neq n_2}} i(n_1+n_2)\frac{G_{n_1}(\omega)\,G_{n_2}(\omega)}{\langle n_1\rangle^2\langle n_2\rangle^2} \\ &:= \Sigma^1_{M,N}+\Gamma^1_{M,N}. \end{aligned}$$

We estimate only the term $\Sigma^1_{M,N}$, since the term $\Gamma^1_{M,N}$ can be estimated similarly. We compute

$$|\Sigma^1_{M,N}|^2 = \sum_{(n,m)\in C_{M,N}\times C_{M,N}} (n_1+n_2)(m_1+m_2)\frac{G_{m_1}(\omega)\,G_{m_2}(\omega)\,G_{n_1}(\omega)\,G_{n_2}(\omega)}{\langle m_1\rangle^2\langle m_2\rangle^2\langle n_1\rangle^2\langle n_2\rangle^2},$$

where $n=(n_1,n_2)$, $m=(m_1,m_2)$ and

$$C_{M,N} = \big\{(n_1,n_2)\in\mathbb{Z}^2 \ \text{ s.t. }\ N<|n_1|\leq M,\ |n_2|\leq M,\ \text{ and }\ n_1\neq n_2\big\}.$$

We compute $\mathbb{E}\big[|\Sigma^1_{M,N}|^2\big]$, and thanks to (3.14) we see that only the terms ($n_1 = m_1$ and $n_2 = m_2$) or ($n_1 = m_2$ and $n_2 = m_1$) give some contribution, hence

$$\|\Sigma^1_{M,N}\|^2_{L^2(\Omega)} \leq C \sum_{\substack{|n_1|,|n_2|\leq M\\ |n_1|\geq N}} \frac{(n_1+n_2)^2}{\langle n_1\rangle^4\langle n_2\rangle^4} \leq \frac{C}{N},$$

and therefore

$$\|Y^1_M - Y^1_N\|^2_{L^2(\Omega)} \leq \frac{C}{N}. \tag{3.15}$$

We now turn to (3.13). Similarly, we write

$$\begin{aligned} Y^2_M - Y^2_N &= \sum_{\substack{|n_2|\leq M,\\ N<|n_1|\leq M,\\ n_1\neq n_2}} i(n_1+n_2)\frac{G_{n_1}(\omega)+G_{n_2}(\omega)}{\langle n_1\rangle^2\langle n_2\rangle^2} + \sum_{\substack{|n_1|\leq N,\\ N<|n_2|\leq M,\\ n_1\neq n_2}} i(n_1+n_2)\frac{G_{n_1}(\omega)+G_{n_2}(\omega)}{\langle n_1\rangle^2\langle n_2\rangle^2}\\ &:= \Sigma^2_{M,N} + \Gamma^2_{M,N}. \end{aligned}$$

As previously, it is enough to estimate the contribution of $\Sigma^2_{M,N}$. Then we decompose

$$\begin{aligned} \Sigma^2_{M,N} &= \sum_{\substack{|n_2|\leq M,\\ N<|n_1|\leq M,\\ n_1\neq n_2}} i(n_1+n_2)\frac{G_{n_1}(\omega)}{\langle n_1\rangle^2\langle n_2\rangle^2} + \sum_{\substack{|n_2|\leq M,\\ N<|n_1|\leq M,\\ n_1\neq n_2}} i(n_1+n_2)\frac{G_{n_2}(\omega)}{\langle n_1\rangle^2\langle n_2\rangle^2}\\ &:= \Sigma^{2,1}_{M,N} + \Sigma^{2,2}_{M,N}. \end{aligned}$$

We estimate the contribution of $\Sigma^{2,1}_{M,N}$. We have

$$|\Sigma^{2,1}_{M,N}|^2 = \sum_{(n,m)\in C_{M,N}\times C_{M,N}} (n_1+n_2)(m_1+m_2)\frac{G_{m_1}(\omega)\,G_{n_1}(\omega)}{\langle m_1\rangle^2\langle m_2\rangle^2\langle n_1\rangle^2\langle n_2\rangle^2},$$

and using (3.14), we obtain

$$\|\Sigma^{2,1}_{M,N}\|^2_{L^2(\Omega)} = C \sum_{\substack{|m_2|,|n_2|\leq M,\\ N<|n_1|\leq M\\ n_1\neq n_2,m_2}} \frac{(n_1+n_2)(n_1+m_2)}{\langle m_2\rangle^2\langle n_1\rangle^4\langle n_2\rangle^2}. \tag{3.16}$$

We expand the numerator of the previous sum and estimate each term.

- We have

$$\sum_{\substack{|m_2|,|n_2|\leq M,\\ N<|n_1|\leq M\\ n_1\neq n_2,m_2}} \frac{n_1^2}{\langle m_2\rangle^2\langle n_1\rangle^4\langle n_2\rangle^2} \leq \sum_{\substack{|m_2|,|n_2|\leq M,\\ |n_1|>N\\ n_1\neq n_2,m_2}} \frac{1}{\langle m_2\rangle^2\langle n_1\rangle^2\langle n_2\rangle^2} \leq \frac{C}{N},$$

• Fix $m_2 \in \mathbb{Z}$. Then thanks to the symmetry $n_2 \to -n_2$, we have

$$\Big|\sum_{\substack{|n_2|\leq M,\\ N<|n_1|\leq M\\ n_1\neq n_2}} \frac{n_1 n_2}{\langle m_2\rangle^2\langle n_1\rangle^4\langle n_2\rangle^2}\Big| \leq C\sum_{|n_1|>N}\frac{n_1^2}{\langle m_2\rangle^2\langle n_1\rangle^6}\leq \frac{C}{N\langle m_2\rangle^2}$$

Then summing up in m_2 we get

$$\Big|\sum_{\substack{|m_2|,|n_2|\leq M,\\ N<|n_1|\leq M\\ n_1\neq n_2, m_2}} \frac{n_1 n_2}{\langle m_2\rangle^2\langle n_1\rangle^4\langle n_2\rangle^2}\Big| \leq \frac{C}{N}.$$

Similarly, we have

$$\Big|\sum_{\substack{|m_2|,|n_2|\leq M,\\ N<|n_1|\leq M\\ n_1\neq n_2, m_2}} \frac{n_1 m_2}{\langle m_2\rangle^2\langle n_1\rangle^4\langle n_2\rangle^2}\Big| \leq \frac{C}{N}.$$

• Analogously, using the symmetries $n_2 \to -n_2$ and $m_2 \to -m_2$ we get

$$\sum_{\substack{|m_2|,|n_2|\leq M,\\ N<|n_1|\leq M\\ n_1\neq n_2, m_2}} \frac{n_2 m_2}{\langle m_2\rangle^2\langle n_1\rangle^4\langle n_2\rangle^2} \leq C\sum_{|n_1|>N}\frac{n_1^2}{\langle n_1\rangle^8}\leq \frac{C}{N}.$$

Therefore, from (3.16), $\|\Sigma^{2,1}_{M,N}\|^2_{L^2(\Omega)} \leq \frac{C}{N}$. The term $\Sigma^{2,2}_{M,N}$ can be handled similarly, therefore

$$\|Y^2_M - Y^2_N\|^2_{L^2(\Omega)} \leq \frac{C}{N}. \tag{3.17}$$

Finally, (3.15) and (3.17) yield the estimate (3.11). □

3.2. Study of S^2_N. — We are now able to prove

Lemma 3.4. — *Let S^2_N be defined by* (3.9). *The there exists $C>0$ so that for all $M>N>0$,*

$$\|S^2_M - S^2_N\|_{L^2(\Omega)} \leq \frac{C}{N^{\frac12}}. \tag{3.18}$$

Proof. — Set $S^2_M = \displaystyle\sum_{D_N} i(n_1+n_2)\,\frac{g_{m_1}(\omega)\,g_{m_2}(\omega)\,\overline{g_{n_1}(\omega)}\,\overline{g_{n_2}(\omega)}}{\langle m_1\rangle\,\langle m_2\rangle\,\langle n_1\rangle\,\langle n_2\rangle} := \sum_{D_M} a_{m,n}$, where

$$D_M = \big\{(m_1,m_2,n_1,n_2)\in\mathbb{Z}^4 \ \text{ s.t. } \ |m_1|,|m_2|,|n_1|,|n_2|\leq M,$$
$$\text{and } \ m_1+m_2 = n_1+n_2,\ \ m_1\neq n_1,\ \ m_1\neq n_2\big\}.$$

We make the decomposition
(3.19)

$$S_M^2-S_N^2=\sum_{\substack{D_M\\ N<|m_1|\le M}} a_{m,n}+\sum_{\substack{D_M\\ |m_1|\le N\\ N<|m_2|\le M}} a_{m,n}+\sum_{\substack{D_M\\ |m_1|,|m_2|\le N\\ N<|n_1|\le M}} a_{m,n}+\sum_{\substack{D_M\\ |m_1|,|m_2|,|n_1|\le N\\ N<|n_2|\le M}} a_{m,n}.$$

It is enough to study the contribution of the first sum, since the other terms are similar. We have

$$\begin{aligned}
&\Big|\sum_{\substack{D_M\\ N<|m_1|\le M}} a_{m,n}\Big|^2 =\\
&=\sum_{\substack{(m,n)\in D_M\\ N<|m_1|\le M}}\ \sum_{\substack{(q,p)\in D_M\\ N<|q_1|\le M}} (n_1+n_2)(p_1+p_2)\,\frac{g_{m_1}\,g_{m_2}\,\overline{g_{n_1}}\,\overline{g_{n_2}}\,g_{p_1}\,g_{p_2}\,\overline{g_{q_1}}\,\overline{g_{q_2}}}{\langle m_1\rangle\,\langle m_2\rangle\,\langle n_1\rangle\,\langle n_2\rangle\,\langle p_1\rangle\,\langle p_2\rangle\,\langle q_1\rangle\,\langle q_2\rangle}.
\end{aligned}$$

The expectation of each term of the previous sum vanishes, unless $\big((m_1,m_2)=(q_1,q_2)\ \text{ or }\ (q_2,q_1)\big)$ and $\big((n_1,n_2)=(p_1,p_2)\ \text{ or }\ (p_2,p_1)\big)$. Hence

$$\|\sum_{\substack{D_M\\ N<|m_1|\le M}} a_{m,n}\|^2_{L^2(\Omega)}\le C\sum_{\substack{D_M\\ N<|m_1|\le M}}\frac{(n_1+n_2)^2}{\langle n_1\rangle^2\,\langle n_2\rangle^2\,\langle m_1\rangle^2\,\langle m_2\rangle^2}. \tag{3.20}$$

Now observe that on D_M, $|m_1|=|n_1+n_2-m_2|\le 3\max(|n_1|,|n_2|,|m_2|)$. Thus $|m_1|>N$ implies $\max(|n_1|,|n_2|,|m_2|)>N/3$. We split the r.h.s of (3.21) into the corresponding three parts :

- Case $|n_1|\ge N/3$. Write $n=n_1+n_2=m_1+m_2$, therefore

$$\sum_{\substack{D_M\\ N<|m_1|\le M\\ |n_1|\ge N/3}}\frac{(n_1+n_2)^2}{\langle n_1\rangle^2\,\langle n_2\rangle^2\,\langle m_1\rangle^2\,\langle m_2\rangle^2}\le\sum_{n\in\mathbb{Z}}\ \sum_{\substack{|m_1|>N\\ |n_1|\ge N/3}}\frac{n^2}{\langle n_1\rangle^2\,\langle n-n_1\rangle^2\,\langle m_1\rangle^2\,\langle n-m_1\rangle^2}.$$

Next we have $n^2 \leq C(\langle n_1\rangle^2 + \langle n-n_1\rangle^2)$, thus

$$\sum_{\substack{D_M\\ N<|m_1|\leq M\\ |n_1|\geq N/3}} \frac{(n_1+n_2)^2}{\langle n_1\rangle^2 \langle n_2\rangle^2 \langle m_1\rangle^2 \langle m_2\rangle^2} \leq$$

$$\leq C\sum_{n\in\mathbb{Z}} \sum_{\substack{|m_1|>N\\ |n_1|\geq N/3}} \Big[\frac{1}{\langle n-n_1\rangle^2 \langle m_1\rangle^2 \langle n-m_1\rangle^2} + \frac{1}{\langle n_1\rangle^2 \langle m_1\rangle^2 \langle n-m_1\rangle^2}\Big]$$

$$\leq C\sum_{|m_1|>N} \sum_{n\in\mathbb{Z}} \frac{1}{\langle m_1\rangle^2 \langle n-m_1\rangle^2} + C\sum_{\substack{|m_1|>N\\ |n_1|\geq N/3}} \frac{1}{\langle n_1\rangle^2 \langle m_1\rangle^2}$$

$$\leq \frac{C}{N},$$

which is an admissible contribution. The case $|n_2| \geq N/3$ is similar.

- Case $|m_2| \geq N/3$. On D_M, we have $n_1+n_2 = m_1+m_2$. Then by symmetry, we can reduce to the case $|m_2| \leq |m_1|$, therefore

$$\sum_{\substack{D_M\\ N<|m_1|\leq M\\ |m_2|\geq N/3}} \frac{(n_1+n_2)^2}{\langle n_1\rangle^2 \langle n_2\rangle^2 \langle m_1\rangle^2 \langle m_2\rangle^2} \leq$$

$$\leq C \sum_{\substack{D_M\\ N<|m_1|\leq M\\ |m_1|\geq|m_2|\geq N/3}} \frac{(m_1+m_2)^2}{\langle n_1\rangle^2 \langle n_2\rangle^2 \langle m_1\rangle^2 \langle m_2\rangle^2}$$

$$\leq C \sum_{\substack{|n_1|\leq M\\ N<|m_1|\leq M\\ |m_1|\geq|m_2|\geq N/3}} \frac{1}{\langle n_1\rangle^2 \langle m_1+m_2-n_1\rangle^2 \langle m_2\rangle^2}.$$

In this last sum, we first sum in m_1, then in m_2, and finally in n_1. This gives the bound

$$\sum_{\substack{D_M\\ N<|m_1|\leq M\\ |m_2|\geq N/3}} \frac{(n_1+n_2)^2}{\langle n_1\rangle^2 \langle n_2\rangle^2 \langle m_1\rangle^2 \langle m_2\rangle^2} \leq \frac{C}{N},$$

which is admissible.

As a consequence, we have obtained that

$$\| \sum_{\substack{D_M \\ N<|m_1|\leq M}} a_{m,n} \|^2_{L^2(\Omega)} \leq \frac{C}{N}. \tag{3.21}$$

The other terms in (3.19) are treated similarly. This proves (3.18). □

The results of Lemma 3.3 and Lemma 3.4 imply (3.1).

To complete the proof of Proposition 3.1, it remains to show (3.2). But this is a direct consequence of (3.1) and Proposition 2.3

We are now able to define the density $G : H^\sigma(\mathbb{T}) \longrightarrow \mathbb{R}$ (with respect to the measure μ) of the measure ρ. By (3.4) and Proposition 3.1 and Lemma 2.1, we have the following convergences in the μ measure : $f_N(u)$ converges to $f(u)$ and $\|u_N\|_{L^6(\mathbb{T})}$ to $\|u\|_{L^6(\mathbb{T})}$. Then, by composition and multiplication of continuous functions, we obtain

$$\chi\big(\|u_N\|_{L^2(\mathbb{T})}\big)\mathrm{e}^{\frac{3}{4}f_N(u)-\frac{1}{2}\int_{\mathbb{T}}|u_N(x)|^6\mathrm{d}x} \to \chi\big(\|u\|_{L^2(\mathbb{T})}\big)\mathrm{e}^{\frac{3}{4}f(u)-\frac{1}{2}\int_{\mathbb{T}}|u(x)|^6\mathrm{d}x} \equiv G(u), \tag{3.22}$$

in measure, with respect to the measure μ. As a consequence, G is measurable from $\big(H^\sigma(\mathbb{T}), \mathcal{B}\big)$ to $\mathbb{R}$.

4. Integrability of the density of dρ

Proposition 4.1. — *For all $1 \leq p < \infty$, there exists $\kappa_p > 0$ so that for all $0 < \kappa \leq \kappa_p$ there exists $C > 0$ such that for every $N \geq 1$.*

$$\Big\|\chi\big(\|u_N\|_{L^2(\mathbb{T})}\big)\, e^{\frac{3}{4}f_N(u)-\frac{1}{2}\int_{\mathbb{T}}|u_N(x)|^6\,dx}\Big\|_{L^p(d\mu(u))} \leq C\,.$$

Proof. — Here we can follow the proof of [**11**, Proposition 4.9]. To prove the proposition, it is sufficient to show that the integral

$$\int_0^\infty \lambda^{p-1}\mu(A_{\lambda,N})d\lambda, \tag{4.1}$$

is convergent uniformly with respect to N for $\kappa > 0$ small enough and where

$$A_{\lambda,N} = \Big\{u \in H^\sigma \;:\; \chi\big(\|u_N\|_{L^2(\mathbb{T})}\big)\mathrm{e}^{\frac{3}{4}f_N(u)-\frac{1}{2}\int_{\mathbb{T}}|u_N(x)|^6\mathrm{d}x} > \lambda\Big\}.$$

We set

$$N_0 = \alpha^2(\ln\lambda)^{\frac{1}{2}} \quad \text{and} \quad N_1 = \beta^2(\ln\lambda)^2,$$

where $\alpha, \beta > 0$ are constants which will be fixed below.

• Assume that $N \leq N_0$.
Firstly, using a Littlewood-Paley decomposition one shows that

$$|f_N(u)| \lesssim \Big(\sum_{M \leq N} M^{\frac{1}{2}} \|P_M u\|_{L^2} \Big)^4, \tag{4.2}$$

where P_M are the usual Littlewood-Paley projectors and the summation is done over the dyadic integers.

Indeed, after a Littlewood-Paley decomposition, in the proof of (4.2) one needs to estimate

$$\int |P_{M_1}(\partial_x u) P_{M_2}(u) P_{M_3}(u) P_{M_4}(u)|,$$

for $M_1 \geq M_2 \geq M_3 \geq M_4$ with $M_1 \sim M_2$. It therefore suffices to write

$$\begin{aligned}
&\int |P_{M_1}(\partial_x u) P_{M_2}(u) P_{M_3}(u) P_{M_4}(u)| \leq \\
&\qquad \leq \|P_{M_1}(\partial_x u)\|_{L^2} \|P_{M_2}(u)\|_{L^2} \|P_{M_3}(u)\|_{L^\infty} \|P_{M_4}(u)\|_{L^\infty} \\
&\qquad \leq C M_1 \|P_{M_1}(u)\|_{L^2} \|P_{M_2}(u)\|_{L^2} \|P_{M_3}(u)\|_{L^\infty} \|P_{M_4}(u)\|_{L^\infty} \\
&\qquad \leq C (M_1 M_2)^{\frac{1}{2}} \|P_{M_1}(u)\|_{L^2} \|P_{M_2}(u)\|_{L^2} \|P_{M_3}(u)\|_{L^\infty} \|P_{M_4}(u)\|_{L^\infty}
\end{aligned}$$

which by Sobolev is bounded by

$$C(M_1 M_2 M_3 M_4)^{\frac{1}{2}} \|P_{M_1}(u)\|_{L^2} \|P_{M_2}(u)\|_{L^2} \|P_{M_3}(u)\|_{L^2} \|P_{M_4}(u)\|_{L^2}.$$

Therefore, we have (4.2).

Thanks to (4.2), we need to evaluate $\mu(B_{\lambda,N})$, where

$$B_{\lambda,N} \equiv \big(u \, : \, \sum_{M \leq N} M^{\frac{1}{2}} \|P_M u\|_{L^2} \geq (\ln \lambda)^{\frac{1}{4}}, \; \|u\|_{L^2} \leq \kappa \big) \, .$$

We can write

$$\sum_{M \leq N_0} M^{\frac{1}{2}} \|P_M u\|_{L^2} \leq N_0^{\frac{1}{2}} \|P_{\leq N_0} u\|_{L^2} \leq N_0^{\frac{1}{2}} \kappa \leq \frac{1}{2} (\ln \lambda)^{\frac{1}{4}}, \tag{4.3}$$

provided $\alpha > 0$ is small enough. This implies that $\mu(B_{\lambda,N}) = 0$.

• Assume that $N_0 \leq N \leq N_1$.
By (4.3) we have

$$\mu(B_{\lambda,N}) \leq \mu\big(\sum_{N_0 \leq M \leq N} M^{\frac{1}{2}} \|P_M u\|_{L^2} \geq \frac{1}{2} (\ln \lambda)^{\frac{1}{4}} \big).$$

If $(\sigma_M)_{M\geq N_0}$ is a sequence such that

$$\sum_{M\geq N_0}\sigma_M\leq\frac{1}{4}$$

then we can write

$$\mu(B_{\lambda,N})\leq\sum_{M\geq N_0}\mu\big(u: M^{\frac12}\|P_Mu\|_{L^2}\geq\sigma_M(\ln\lambda)^{\frac14}\big)\,.$$

We choose σ_M as

$$\sigma_M=c_0\Big(\frac{N_0}{M}\Big)^{\frac{1}{1000}},$$

for $c_0>0$ small enough. Now we can write

$$\mu\big(u: M^{\frac12}\|P_Mu\|_{L^2}\geq\sigma_M(\ln\lambda)^{\frac14}\big)\lesssim p\big(\omega\,:\,(\sum_{n\sim M}|g_n(\omega)|^2)^{\frac12}\gtrsim M^{\frac12}\sigma_M(\ln\lambda)^{\frac14}\big).$$

Using that $N^{\frac12}\leq\beta\ln\lambda$ and invoking [**12**, Lemma 2.2], we obtain that

$$p\Big(\omega\,:\,(\sum_{n\sim M}|g_n(\omega)|^2)^{\frac12}\gtrsim M^{\frac12}\sigma_M(\ln\lambda)^{\frac14}\Big)\lesssim e^{-cM\sigma_M^2(\ln\lambda)^{\frac12}}.$$

Consequently

$$\mu(B_{\lambda,N})\lesssim\sum_{M\geq N_0}e^{-cM\sigma_M^2(\ln\lambda)^{\frac12}}\lesssim e^{-cN_0(\ln\lambda)^{\frac12}}\lesssim e^{-c\kappa^{-2}\ln\lambda}\leq C_L\lambda^{-L},$$

provided $\kappa>0$ is sufficiently small (depending on L).

• Assume now $N_1\leq N$.
Thanks to the triangle inequality $A_{\lambda,N}\subset C_{\lambda,N}\cup D_{\lambda,N}$, where

$$C_{\lambda,N}\equiv\Big\{u\in H^\sigma\,:\,|f_{N_1}(u)|>\frac12\ln\lambda,\ \ \|u_N\|_{L^2(\mathbb{T})}\leq\kappa\Big\},$$

and

$$D_{\lambda,N}\equiv\Big\{u\in H^\sigma\,:\,|f_N(u)-f_{N_1}(u)|>\frac12\ln\lambda,\ \ \|u_N\|_{L^2(\mathbb{T})}\leq\kappa\Big\}.$$

The measure of $C_{\lambda,N}$ has been estimated in the previous point. Finally, by Corollary 3.2, as $N_1=\beta^2(\ln\lambda)^2$, we obtain that for all $L\geq1$

$$\mu(D_{\lambda,N})\leq Ce^{-\delta(N_1^{1/2}\ln\lambda)^{\frac12}}=Ce^{-\delta\beta\ln\lambda}\leq C_L\lambda^{-L}\,,$$

provided that $\beta>L/\delta$. This completes the proof of the proposition. □

Proof of Theorem 1.1. — Recall (3.22). Let $p \in [1, +\infty)$ and choose $\kappa_p > 0$ so that Proposition 4.1 holds. Then there exists a subsequence $G_{N_k}(u)$ so that $G_{N_k}(u) \longrightarrow G(u)$, μ a.s. Then by Fatou's lemma,

$$\int_{H^{\sigma}(\mathbb{T})} |G(u)|^p \mathrm{d}\mu(u) \leq \liminf_{k\to\infty} \int_{H^{-\sigma}(\mathbb{T})} |G_{N_k}(u)|^p \mathrm{d}\mu(u) \leq C,$$

thus $G(u) \in L^p(\mathrm{d}\mu(u))$.
Now it remains to check the convergence in $L^p(\mathrm{d}\mu(u))$ for $1 \leq p < \infty$. As in [**11**], for $N \geq 0$ and $\varepsilon > 0$, we introduce the set

$$A_{N,\varepsilon} = \big\{ u \in H^{\sigma}(\mathbb{T}) \;:\; |G_N(u) - G(u)| \leq \varepsilon \big\},$$

and denote by $\overline{A_{N,\varepsilon}}$ its complement.
Firstly, there exists $C > 0$ so that for all $N \geq 0$, $\varepsilon > 0$

$$\int_{A_{N,\varepsilon}} \big|G_N(u) - G(u)\big|^p \mathrm{d}\mu(u) \leq C\varepsilon^p.$$

Secondly, by Cauchy-Schwarz, Proposition 4.1 and as $G(u) \in L^{2p}(\mathrm{d}\mu(u))$, we obtain

$$\begin{aligned} \int_{\overline{A_{N,\varepsilon}}} \big|G_N(u) - G(u)\big|^p \mathrm{d}\mu(u) &\leq \|G_N - G\|^p_{L^{2p}(\mathrm{d}\mu)} \mu(\,\overline{A_{N,\varepsilon}}\,)^{\frac{1}{2}} \\ &\leq C\mu(\,\overline{A_{N,\varepsilon}}\,)^{\frac{1}{2}}. \end{aligned}$$

By (3.22), we deduce that for all $\varepsilon > 0$,

$$\mu(\,\overline{A_{N,\varepsilon}}\,) \longrightarrow 0, \quad N \longrightarrow +\infty,$$

which yields the result. This ends the proof of Theorem 1.1. □

Lemma 4.2. — *The measure ρ is not trivial*

Proof. — First observe that for all $\kappa > 0$

$$\mu\big(\, u \in H^{\sigma}(\mathbb{T}) \;:\; \|u\|_{L^2(\mathbb{T})} \leq \kappa \,\big) = \mathbf{p}\big(\, \omega \in \Omega \;:\; \sum_{n\in\mathbb{Z}} \frac{1}{\langle n \rangle^2} |g_n(\omega)|^2 \leq \kappa^2 \,\big) > 0.$$

Then, by Lemma 2.1 and Proposition 3.1, the quantities $\|u\|_{L^6(\mathbb{T})}$ and $f(u)$ are μ almost surely finite. Hence, the density of ρ does not vanish on a set of positive μ measure. In other words, ρ is not trivial. □

A
Appendix

A.1. Hamiltonian structure of the transformed form of DNLS. — In this section we give the Hamiltonian structure of the equation related to (1.1). First we define the projection Π on the 0-mean functions :

$$\Pi\big(f\big) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \alpha_n \mathrm{e}^{inx}, \quad \text{for} \quad f(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathrm{e}^{inx},$$

then we introduce the integral operator

$$\partial^{-1} \,:\, f(x) = \sum_{n\in\mathbb{Z}} \alpha_n \mathrm{e}^{inx} \longmapsto \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{\alpha_n}{in} \mathrm{e}^{inx}.$$

Notice that we have

$$\partial^{-1}\big(f'\big) = \Pi f = f - \int_{\mathbb{T}} f(x)\,\mathrm{d}x.$$

Next we define the operator

$$K(u,v) = \begin{pmatrix} -u\partial^{-1}u\cdot & -i + u\partial^{-1}v\cdot \\ i + v\partial^{-1}u\cdot & -v\partial^{-1}v\cdot \end{pmatrix}. \tag{A.1}$$

Lemma A.1. — *For u, v, the operator $K(u,v)$ is skew symmetric : $K(u,v)^* = -K(u,v)$.*

Proof. — This is a straightforward computation. We only have to use that $\big(\partial^{-1}\big)^* = -\partial^{-1}$. □

Define

$$H(u,v) = \int_{\mathbb{T}} \partial_x u\, \partial_x v + \frac{3}{4} i \int_{\mathbb{T}} v^2 \partial_x(u^2) + \frac{1}{2}\int_{\mathbb{T}} u^3 v^3.$$

Notice that we also have the expressions

$$\begin{aligned} H(u,v) &= -\int_{\mathbb{T}} \partial_x^2 u\, v + \frac{3}{4} i \int_{\mathbb{T}} v^2 \partial_x(u^2) + \frac{1}{2}\int_{\mathbb{T}} u^3 v^3 \\ &= -\int_{\mathbb{T}} u\, \partial_x^2 v - \frac{3}{4} i \int_{\mathbb{T}} u^2 \partial_x(v^2) + \frac{1}{2}\int_{\mathbb{T}} u^3 v^3, \end{aligned}$$

therefore, we can deduce the variational derivatives

$$\frac{\delta H}{\delta u}(u,v) = -\partial_x^2 v - \frac{3}{2}\, iu\, \partial_x(v^2) + \frac{3}{2} u^2 v^3 \tag{A.2}$$

$$\frac{\delta H}{\delta v}(u,v) = -\partial_x^2 u + \frac{3}{2}\, iv\, \partial_x(u^2) + \frac{3}{2} u^3 v^2. \tag{A.3}$$

We consider the Hamiltonian system

$$\begin{pmatrix} \partial_t u \\ \partial_t v \end{pmatrix} = K(u,v) \begin{pmatrix} \frac{\delta H}{\delta u}(u,v) \\ \frac{\delta H}{\delta v}(u,v) \end{pmatrix}. \tag{A.4}$$

Denote by

$$F_u(t) = 2\,\mathrm{Im} \int_{\mathbb{T}} u \partial_x \overline{u} + \frac{3}{2} \int_{\mathbb{T}} |u|^4,$$

and notice that for all $t \in \mathbb{R}$, $F_u(t) \in \mathbb{R}$.

Proposition A.2. — *The system* (A.4) *is a Hamiltonian formulation of the equation*

$$i\partial_t u + \partial_x^2 u = i\partial_x\big(|u|^2 u\big) + F_u(t) u, \tag{A.5}$$

in the coordinates $(u,v) = (u, \overline{u})$.

As a consequence, if we set

$$v(t,x) = \mathrm{e}^{i\int_0^t F_u(s)\mathrm{d}s} u(t,x), \tag{A.6}$$

then v is the solution of the equation

$$\begin{cases} i\partial_t v + \partial_x^2 v = i\partial_x\big(|v|^2 v\big), \quad (t,x) \in \mathbb{R}\times\mathbb{T}, \\ v(0,x) = u_0(x). \end{cases} \tag{A.7}$$

Moreover, if u and v are linked by (A.6), we have $F_u = F_v$.

Proof. — We have

$$u\,\partial_x^2 v = v\,\partial_x^2 u + (u\,\partial_x v)' - (v\,\partial_x u)',$$

therefore

$$\partial^{-1}\big(u\,\partial_x^2 v\big) = \partial^{-1}\big(v\,\partial_x^2 u\big) + u\,\partial_x v - v\,\partial_x u - \int_{\mathbb{T}} \big(u\,\partial_x v - v\,\partial_x u\big). \tag{A.8}$$

Similarly we obtain the relation

$$\partial^{-1}\big(u^2\,\partial_x(v^2)\big) = -\partial^{-1}\big(v^2\,\partial_x(u^2)\big) + u^2\,v^2 - \int_{\mathbb{T}} u^2\,v^2. \tag{A.9}$$

By (A.2), (A.3), using (A.8) and (A.9), a straightforward computation gives

$$\begin{aligned} \partial_t u &= -u\partial^{-1}\big(u\,\frac{\delta H}{\delta u}\big) - i\frac{\delta H}{\delta v} + u\partial^{-1}\big(v\,\frac{\delta H}{\delta v}\big) \\ &= i\partial_x^2 u + \partial_x\big(u^2\,v\big) - u\int_{\mathbb{T}}(u\,\partial_x v - v\,\partial_x u) - \frac{3}{2} i u \int_{\mathbb{T}} u^2 v^2, \end{aligned}$$

and

$$\begin{aligned}\partial_t v &= i\frac{\delta H}{\delta u} + v\partial^{-1}\big(u\,\frac{\delta H}{\delta u}\big) - v\partial^{-1}\big(v\,\frac{\delta H}{\delta v}\big)\\ &= -i\partial_x^2 v + \partial_x\big(u\,v^2\big) - v\int_{\mathbb{T}}(v\,\partial_x u - u\,\partial_x v) - \frac{3}{2}iu\int_{\mathbb{T}} u^2v^2.\end{aligned}$$

Now assume that $v=\overline{u}$. This yields the result, as

$$\int_{\mathbb{T}}(u\,\partial_x\overline{u} - \overline{u}\,\partial_x u) = 2i\,\mathrm{Im}\int_{\mathbb{T}} u\,\partial_x\overline{u}.$$

□

A.2. Invariance of the measure ρ_N under a truncated flow of (A.5)**.** — We present here a natural finite dimensional approximation of (A.5) for which ρ_N is an invariant measure.

Let $N\geq 1$. Recall that E_N is the the complex vector space $E_N=\mathrm{span}\Big((\mathrm{e}^{inx})_{-N\leq n\leq N}\Big)$, and that Π_N is the spectral projector from $L^2(\mathbb{T})$ to E_N.

Let K be given by (A.1), and consider the following system

$$\begin{pmatrix}\partial_t u\\ \partial_t v\end{pmatrix} = \Pi_N K(u_N,v_N)\Pi_N\begin{pmatrix}\frac{\delta H}{\delta u}(u_N,v_N)\\ \frac{\delta H}{\delta v}(u_N,v_N)\end{pmatrix}. \tag{A.10}$$

This an Hamiltonian system with Hamiltonian $H(\Pi_N u,\Pi_N v)$. Now we assume that $v=\overline{u}$ and we compute the equation satisfied by u_N : this will be a finite dimensional approximation of (A.5). Denote by $\Pi_N^{\perp}=1-\Pi_N$, then we have

Lemma A.3. — *In the coordinates $v_N=\overline{u_N}$, the system* (A.10) *reads*

$$i\partial_t u + \partial_x^2 u_N = i\Pi_N\Big(\partial_x(|u_N|^2u_N)\Big) + u_N F_{u_N}(t) + R_N(u_N), \tag{A.11}$$

where

$$\begin{aligned}R_N(u_N) &= \frac{3}{2}\Pi_N\Big(u_N\partial^{-1}\Big[u_N\Pi_N^{\perp}\big(u_N\partial_x(\overline{u_N}^2)\big) + \overline{u_N}\Pi_N^{\perp}\big(\overline{u_N}\partial_x(u_N{}^2)\big)\Big]\Big)\\ &\quad+\frac{3}{2}i\Pi_N\Big(u_N\partial^{-1}\Big[u_N\Pi_N^{\perp}\big(|u_N|^4\overline{u_N}\big) - \overline{u_N}\Pi_N^{\perp}\big(|u_N|^4u_N\big)\Big]\Big).\end{aligned}$$

Proof. — The proof is a direct computation. By (A.10), the equation on u_N reads

$$\partial_t u = \Pi_N\Big(-u_N\partial^{-1}(u_Nf_N) - i\overline{f_N} + u_N\partial^{-1}(\overline{u_Nf_N})\Big), \tag{A.12}$$

where

$$f_N = \Pi_N\Big(-\partial_x^2\overline{u_N} - \frac{3}{2}iu_N\partial_x(\overline{u_N}^2) + \frac{3}{2}|u_N|^4\overline{u_N}\Big).$$

Thanks to (A.8) we deduce from (A.12) that

$$\begin{aligned}(A.13)\quad \partial_t u = i\partial_x^2 u_N + \frac{3}{2}\Pi_N\Big(\overline{u_N}\partial_x(u_N^2)\Big) - \frac{3}{2}i\Pi_N\Big(|u_N|^4 u_N)\Big)+\\ + \Pi_N\Big(u_N^2\partial_x\overline{u_N} - |u_N|^2\partial_x u_N\Big) - u_N\int_{\mathbb{T}}(u_N\partial_x\overline{u_N} - \overline{u_N}\partial_x u_N)+\\ + \frac{3}{2}i\Pi_N\Big(u_N\partial^{-1}\Big[u_N\Pi_N\big(u_N\partial_x(\overline{u_N}^2)\big) + \overline{u_N}\Pi_N\big(\overline{u_N}\partial_x({u_N}^2)\big)\Big]\Big)+\\ + \frac{3}{2}\Pi_N\Big(u_N\partial^{-1}\Big[\overline{u_N}\Pi_N\big(|u_N|^4u_N\big) - u_N\Pi_N\big(|u_N|^4\overline{u_N})\big)\Big]\Big).\end{aligned}$$

Using (A.9) we obtain, with $\Pi_N^{\perp} = 1 - \Pi_N$

$$\begin{aligned}(A.14)\quad &\partial^{-1}\Big[u_N\Pi_N\big(u_N\partial_x(\overline{u_N}^2)\big) + \overline{u_N}\Pi_N\big(\overline{u_N}\partial_x({u_N}^2)\big)\Big] =\\ &= -\partial^{-1}\Big[u_N\Pi_N^{\perp}\big(u_N\partial_x(\overline{u_N}^2)\big) + \overline{u_N}\Pi_N^{\perp}\big(\overline{u_N}\partial_x({u_N}^2)\big)\Big] + \partial^{-1}\Big[u_N^2\partial_x(\overline{u_N}^2) + \overline{u_N}^2\partial_x({u_N}^2)\Big]\\ &= -\partial^{-1}\Big[u_N\Pi_N^{\perp}\big(u_N\partial_x(\overline{u_N}^2)\big) + \overline{u_N}\Pi_N^{\perp}\big(\overline{u_N}\partial_x({u_N}^2)\big)\Big] + |u_N|^4 - \int_{\mathbb{T}}|u_N|^4.\end{aligned}$$

We can also write

$$\begin{aligned}(A.15)\quad \overline{u_N}\Pi_N\big(|u_N|^4u_N\big) - u_N\Pi_N\big(|u_N|^4\overline{u_N})\big) =\\ - \overline{u_N}\Pi_N^{\perp}\big(|u_N|^4u_N\big) + u_N\Pi_N^{\perp}\big(|u_N|^4\overline{u_N})\big).\end{aligned}$$

Thus, by (A.14) and (A.15), equation (A.13) becomes

$$\begin{aligned}\partial_t u \;=\; & i\partial_x^2 u_N + \Pi_N\Big(\partial_x(|u_N|^2u_N)\Big) - iu_N F_{u_N}(t) +\\ & -\frac{3}{2}i\Pi_N\Big(u_N\partial^{-1}\Big[u_N\Pi_N^{\perp}\big(u_N\partial_x(\overline{u_N}^2)\big) + \overline{u_N}\Pi_N^{\perp}\big(\overline{u_N}\partial_x({u_N}^2)\big)\Big]\Big)\\ & +\frac{3}{2}\Pi_N\Big(u_N\partial^{-1}\Big[u_N\Pi_N^{\perp}\big(|u_N|^4\overline{u_N})\big) - \overline{u_N}\Pi_N^{\perp}\big(|u_N|^4u_N\big)\Big]\Big),\end{aligned}$$

which is the claim. □

In the sequel we fix $\sigma < \frac{1}{2}$, and we consider (A.17) as a Cauchy problem with initial condition in $H^\sigma(\mathbb{T})$

(A.16)

$$\begin{cases} i\partial_t u + \partial_x^2 u_N = i\Pi_N\Big(\partial_x(|u_N|^2u_N)\Big) + u_N F_{u_N}(t) + R_N(u_N), \quad (t,x)\in\mathbb{R}\times\mathbb{T},\\ u(0,x) = u_0(x) \in H^\sigma(\mathbb{T}).\end{cases}$$

We now state the main result of this section.

Proposition A.4. — *The equation* (A.16) *has a well-defined global flow* Φ_N*. Moreover, the measure* ρ_N *is invariant under* Φ_N *: For any Borel set* $A \subset H^\sigma(\mathbb{T})$ *and for all* $t \in \mathbb{R}$, $\rho_N\big(\Phi_N(t)(A)\big) = \rho_N\big(A\big)$.

For the proof of Proposition A.4, we first need the following result

Lemma A.5. — *The equation*

$$\begin{cases} i\partial_t u + \partial_x^2 u_N = i\Pi_N\Big(\partial_x(|u_N|^2 u_N)\Big) + u_N F_{u_N}(t) + R_N(u_N), & (t,x) \in \mathbb{R}\times\mathbb{T}, \\ u(0,x) = \Pi_N\big(u_0(x)\big) \in E_N. \end{cases} \tag{A.17}$$

is an Hamiltonian ODE. Moreover, the mass $\|u(t)\|_{L^2(\mathbb{T})}$ *is conserved under the flow of* (A.17)*. As a consequence,* (A.17) *has a well-defined global flow* $\widetilde{\Phi}_N$.

Proof. — The first statement is clear by the previous construction. We now check that the L^2-norm of u is conserved. Multiply (A.11) with $\overline{u}$, integrate over $x \in \mathbb{T}$ and take the imaginary part. In the sequel we use that $\Pi_N^2 = \Pi_N$ and $\Pi_N^* = \Pi_N$. Firstly by integration by parts,

$$\int_{\mathbb{T}} \overline{u}\,\partial_x^2 u_N = \int_{\mathbb{T}} \overline{u_N}\,\partial_x^2 u_N = -\int_{\mathbb{T}} |\partial_x u_N|^2 \in \mathbb{R}. \tag{A.18}$$

Then

$$\begin{aligned} \mathrm{Im}\int_{\mathbb{T}} i\overline{u}\,\Pi_N\Big(\partial_x(|u_N|^2u_N)\Big) &= \mathrm{Re}\int_{\mathbb{T}} \overline{u_N}\partial_x(|u_N|^2u_N) \\ &= -\mathrm{Re}\int_{\mathbb{T}} (\partial_x\overline{u_N})|u_N|^2u_N \\ &= -\frac{1}{4}\int_{\mathbb{T}} \partial_x(|u_N|^4) = 0. \end{aligned} \tag{A.19}$$

Now observe that if f is real-valued, then $\partial^{-1}f$ is also real valued. Then it is easy to see that

$$\int_{\mathbb{T}} \overline{u}\,R_N(u_N) = \int_{\mathbb{T}} \overline{u_N}\,R_N(u_N) \in \mathbb{R}. \tag{A.20}$$

Finally by (A.18), (A.19) and (A.20) we obtain that $\dfrac{d}{dt}\|u(t)\|^2_{L^2(\mathbb{T})} = 0$ which yields the result. □

Recall the definitions (1.4) of μ_N and (1.8) of G_N. Then we define the measure $\widetilde{\rho}_N$ on E_N by

$$\mathrm{d}\widetilde{\rho}_N(u) = G_N(u)\mathrm{d}\mu_N(u).$$

Then we have

Lemma A.6. — *The measure* $\widetilde{\rho}_N$ *is invariant under the flow* $\widetilde{\Phi}_N$ *of* (A.17).

Proof. — The proof is a direct application of the Liouville thereom. See e.g. [**4**, Section 8] for a similar argument. □

Proof of Proposition A.4. — We decompose the space $H^\sigma(\mathbb{T}) = E_N^\perp \oplus E_N$. From the previous analysis, we observe that the flow Φ_N of (A.16) is given by $\Phi_N = \big(Id, \widetilde{\Phi}_N\big)$. Finally, the invariance of ρ_N follows from Lemma A.6 and invariance of the Gaussian measure under the trivial flow on the high frequency part. □

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