Study of masures and of their applications in arithmetic

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Contents

Chapter 1

Introduction

1.1 Kac-Moody groups and Masures

In this thesis, we study masures and some of their applications to the theory of Kac-Moody groups over local fields. Masures are some generalizations of Bruhat-Tits building. They were introduced in order to study Kac-Moody groups - which are generalizations of reductive groups- over local fields. Let us introduce those objects.

Kac-Moody algebras Since the works of Cartan and Killing at the end of the nineteenth century, the classification of semi-simple Lie algebras over $\mathbb{C}$ is known. One way to enumerate this classification is to use Cartan matrices. To each such Lie algebra, one can associate a Cartan matrix, which is a matrix composed with integers satisfying some conditions. In 1966, Serre gave a presentation of each semi-simple Lie algebra, by generators and relations (see chapitre 6 of [Ser66]). This presentation involves the coefficients of the associated Cartan matrix. In 1967, Kac and Moody independently introduce a new class of Lie algebras, the Kac-Moody algebras. A Kac-Moody algebra is a Lie algebra defined by a presentation, Serre's presentation, but for a matrix more general than a Cartan matrix, a Kac-Moody matrix (also known as generalized Cartan matrix). Let $\mathfrak{g}$ be a Kac-Moody algebra. Unless $\mathfrak{g}$ is associated to a Cartan matrix, $\mathfrak{g}$ is infinite dimensional. Despite this difference, Kac-Moody algebras share many properties with semi-simple Lie algebras.

Kac-Moody groups Reductive groups integrate in some sense finite dimensional complex reductive Lie algebras, which are by definition direct sums of an abelian Lie algebra and of a semi-simple Lie algebra. An important example of such a group is $\mathrm{SL}_n(\mathbb{C})$. The question of integrating Kac-Moody algebras to obtain groups arises naturally. Many authors studied this question and proposed "solutions" to this "problem" and each of this solution could be called a Kac-Moody group. There are (at least) two classes of Kac-Moody groups: minimal Kac-Moody groups and maximal or completed ones. The difference between these two notions corresponds to the difference between $\mathrm{SL}_n\big(\mathbb{C}[t,t^{-1}]\big)$ and $\mathrm{SL}_n\big(\mathbb{C}(\!(t)\!)\big)$, where t is an indeterminate (or more generally between the functor sending a field $\mathcal{K}$ to $\mathrm{SL}_n\big(\mathcal{K}[t,t^{-1}]\big)$ and the one sending $\mathcal{K}$ to $\mathrm{SL}_n\big(\mathcal{K}(\!(t)\!)\big)$). Modulo some central extensions, the first one is a minimal (affine) Kac-Moody group over $\mathbb{C}$ and the second is a maximal (affine) Kac-Moody group over $\mathbb{C}$. In this thesis, the Kac-Moody groups we consider are the minimal ones. More precisely Kac-Moody group functors defined by Tits in [Tit87]. In this definition, a split Kac-Moody group is a group functor from the category of rings to the category of groups, which satisfies the nine axioms proposed by Tits. They are inspired by the properties of the group schemes over $\mathbb{Z}$ (hence of the group functors on the category of rings) associated by Chevalley and Demazure to reductive groups over $\mathbb{C}$. In particular if $\mathbf{G}$ is such a functor, one of the axioms requires the existence of a morphism $\mathrm{Ad} : \mathbf{G}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g})$, where $\mathfrak{g}$ is the Kac-Moody algebra "integrated" by $\mathbf{G}$.

Affine Kac-Moody groups and loop groups There are three types of Kac-Moody matrices: the Cartan matrices, the affine matrices and the indefinite matrices. The affine Kac-Moody matrices are in bijective (explicit) correspondence with the Cartan matrices. If A is an affine (indecomposable) Kac-Moody matrix and $\mathfrak{g}$ is its Kac-Moody algebra, then $\mathfrak{g}$ is isomorphic to some extension of $\mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$, where $\mathring{\mathfrak{g}}$ is the simple finite dimensional Lie algebra associated the Cartan matrix $\mathring{A}$ obtained from A, where the Lie Bracket on $\mathring{\mathfrak{g}} \otimes_{\mathbb{C}} \mathbb{C}[t, t^{-1}]$ can be described explicitly. This isomorphism integrates and Kac-Moody group functors associated to an affine Kac-Moody matrix A resembles the loop group functor $\mathcal{K} \mapsto \mathring{\mathbf{G}}\big(\mathcal{K}[t, t^{-1}]\big)$ or $\mathcal{K} \mapsto \mathring{\mathbf{G}}\big(\mathcal{K}(\!(t)\!)\big)$, where $\mathring{\mathbf{G}}$ is the group functor associated to $\mathring{A}$. So far, the affine Kac-Moody groups are the best understood and the most studied groups among Kac-Moody groups which are not reductive. When the masure is indefinite, Kac-Moody groups are obtained as an amalgamated product and it seems that there is no "explicit" description of them as in the reductive or in the affine case.

Bruhat-Tits buildings The Bruhat-Tits buildings, introduced in [BT72] and [BT84] are a powerful tool to study reductive groups over non archimedean local fields. Let G be a split reductive group over a local field. Then a building $\mathcal{I}$ is associated to G. The group G acts on it, which enables to obtain informations on G from the study of $\mathcal{I}$. The building $\mathcal{I}$ is an object of geometric and combinatoric nature. It is a gluing of affine spaces called apartments, all isomorphic to a standard one $\mathbb{A}$. Let us describe $\mathbb{A}$. Let $T \subset G$ be a maximal split torus. The root system Φ of (G, T) can be viewed as a set of linear forms on some vector space $\mathbb{A}$ (which is a real form of the Cartan algebra of the Lie algebra associated to G). The hyperplanes of the shape $\alpha^{-1}(\{k\})$, for some $\alpha \in \Phi$ and $k \in \mathbb{Z}$ are called walls. The standard apartment $\mathbb{A}$ is the vector space $\mathbb{A}$ equipped with this arrangement of walls. As Φ is finite, the connected components of $\mathbb{A} \setminus \bigcup_{\alpha \in \Phi, k \in \mathbb{Z}} \alpha^{-1}(\{k\})$ are open and when G is simple, their closures are actually simplices of $\mathbb{A}$. Then $\mathcal{I}$ is a simplicial complex satisfying some axioms. One of them is that every pair of faces is included in an apartment. For example, when $G = \mathrm{SL}_2\big(\mathbb{F}_q(\!(t)\!)\big)$ for some prime power q, $\mathbb{A}$ is isomorphic to $(\mathbb{R}, \mathbb{Z})$ (the walls of $\mathbb{A}$ are the points of $\mathbb{Z}$) and $\mathcal{I}$ is a homogeneous tree with valency $q + 1$.

Masures In order to use certain techniques of building theoretic nature in the Kac-Moody frameworks, Gaussent and Rousseau associated an object called masure (or hovel) to some Kac-Moody groups over local fields in [GR08]. It generalizes the Bruhat-Tits buildings. Charignon and Rousseau developed this theory in [Cha10], [Rou11], [Rou16] and [Rou17] and a masure is now associated to every almost-split Kac-Moody group over a field endowed with a non trivial valuation. Let G be a split Kac-Moody group over a local field and $\mathcal{I}$ be its masure. Similarly to buildings, G acts on $\mathcal{I}$ and $\mathcal{I}$ is a union of affine spaces called apartments, which are all isomorphic to a standard one $\mathbb{A}$. Let $T \subset G$ be a maximal torus. The root system Φ_{all} of (G, T) can be viewed as a set of linear forms on some vector space $\mathbb{A}$. The hyperplanes of the shape $\alpha^{-1}(\{k\})$, for some $\alpha \in \Phi_{all}$ and $k \in \mathbb{Z}$ are called walls. The standard apartment $\mathbb{A}$ is the vector space $\mathbb{A}$ equipped with this arrangement of walls. Contrary to the reductive case, Φ_{all} is infinite and the connected components of $\mathbb{A} \setminus \bigcup_{\alpha \in \Phi_{all}, k \in \mathbb{Z}} \alpha^{-1}(\{k\})$ have empty interior, unless G is reductive. Thus $\mathbb{A}$ is a priori not a simplicial complex. The faces are defined using the notion of filter (this is already the case of buildings when the valuation of the field $\mathcal{K}$ is not discrete). An other important difference is that there can exist a pair of points which is not included in any apartment of $\mathcal{I}$.

Recently, Freyn, Hartnick, Horn and Köhl made an analog construction in the archimedean case (see [FHHK17]): to each split real Kac-Moody group, they associate a "symmetric" space on which the group acts, generalizing the notion of riemannian symmetric space.

1.2 Kac-Moody groups over local fields

We now survey some results obtained recently for Kac-Moody groups over local fields.

Hecke algebras of a reductive group Let $G = \mathbf{G}(\mathcal{K})$ be a split reductive group over a non archimedean local field, where $\mathbf{G}$ is a reductive group functor and $\mathcal{K}$ is a local field. The Hecke algebras are central objects in the theory of smooth representations of G over $\mathbb{C}$. Such an algebra is associated to each open compact subgroup of G. Two choices of open compact subgroups are of particular interest: the spherical subgroup $K_s = \mathbf{G}(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of $\mathcal{K}$, and the Iwahori subgroup K_I, which is roughly speaking the inverse image of $\mathbf{B}(k)$ in K_s, where $\mathbf{B}$ is a Borel sub-functor and $k = \mathcal{O}/m$, where m is the maximal ideal of $\mathcal{O}$. The corresponding Hecke algebras are the spherical Hecke algebra $\mathcal{H}_s$ and the Iwahori-Hecke algebra $\mathcal{H}$. The spherical Hecke algebra can be explicitly described by the Satake isomorphism. Let $Q^\vee_\mathbb{Z}$ be the coweight lattice and W^v be the Weyl group of (G,T). The group W^v acts on $Q^\vee_\mathbb{Z}$ and thus on the algebra $\mathbb{C}[Q^\vee_\mathbb{Z}]$. The Satake isomorphism S is an isomorphism between $\mathcal{H}_s$ and the algebra of W^v-invariant polynomials $\mathbb{C}[Q^\vee_\mathbb{Z}]^{W^v}$. In particular $\mathcal{H}_s$ is commutative.

For what we study, two presentations of the Iwahori-Hecke algebra are particularly interesting: the Iwahori-Matsumoto and the Bernstein-Lusztig presentations. Let us describe them. Let $W = W^v \ltimes Q^\vee_\mathbb{Z}$ be the affine Weyl group of G. Then if m denotes the rank of $Q^\vee_\mathbb{Z}$, there exists a set $(r_i)_{i\in[\![0,m]\!]}$ of simple reflections, such that $(W,(r_i)_{i\in[\![0,m]\!]})$ and $(W^v,(r_i)_{i\in[\![1,m]\!]})$ are Coxeter systems. Let ℓ be the corresponding length. Let $q = |k|$ be the residue cardinal of $\mathcal{K}$. Then $\mathcal{H}$ is generated by a basis $(T_w)_{w\in W}$, subject to the following relations (this is Iwahori-Matsumoto presentation):

(IM1): $T_w * T_{w'} = T_{ww'}$ if $\ell(w)+\ell(w') = \ell(ww')$
(IM2): $T_{r_i}^2 = qT_1 + (q-1)T_{r_i}$ for all $i \in [\![0,m]\!]$.

We denote the simple coroots of $Q^\vee_\mathbb{Z}$ by $\alpha_i^\vee$, $i\in[\![1,m]\!]$. The algebra $\mathcal{H}$ also has the following presentation (Bernstein-Lusztig presentation): it is generated by elements Z^λ, $\lambda\in Q^\vee_\mathbb{Z}$ and H_w, $w \in W^v$ (where $H_w = q^{-\frac{1}{2}\ell(w)}T_w$), subject to the following relations:

(BL1): for all $(w,i)\in W^v\times[\![1,m]\!]$,

$$H_{r_i} * H_w = \begin{cases} H_{r_iw} & \text{if } \ell(r_iw)=\ell(w)+1 \\ (\sqrt{q}+\sqrt{q}^{-1})H_w + H_{r_iw} & \text{if } \ell(r_iw)=\ell(w)-1. \end{cases}$$

(BL2): for all $\lambda,\mu\in Q^\vee_\mathbb{Z}$, $Z^\lambda * Z^\mu = Z^{\lambda+\mu}$. Therefore we can identify $\mathbb{C}[Q^\vee_\mathbb{Z}]$ and the algebra spanned by the Z^λ's, $\lambda\in Q^\vee_\mathbb{Z}$.

(BL3): for all $(\lambda,i)\in Q^\vee_\mathbb{Z}\times[\![1,m]\!]$,

$$H_{r_i} * Z^\lambda - Z^{r_i(\lambda)} * H_{r_i} = (\sqrt{q}+\sqrt{q}^{-1})\frac{(Z^\lambda - Z^{-\lambda})}{1-Z^{-\alpha_i^\vee}}.$$

The Bernstein-Luzstig presentation enables to see that $\mathcal{H}$ contains a large commutative subalgebra. It also permits to determine the center of $\mathcal{H}$: $\mathcal{Z}(\mathcal{H}) = \mathbb{C}[Q^\vee_\mathbb{Z}]^{W^\vee} \simeq \mathcal{H}_s$.

Kac-Moody groups over local fields This thesis has two components: one is purely building theoretic, the other focuses on the applications of masures in arithmetic. We now survey recent arithmetic results which are closely related to the second component. We will detail the results of [Héb17b] and [AH17] in Section 1.3 and in the introductions of the corresponding chapters.

Its seems that the study of Kac-Moody groups over local fields begins with Garland in the affine case in [Gar95]. In 2011, Braverman and Kazhdan extended the definition of the spherical Hecke algebra to the affine Kac-Moody frameworks ([BK11]). They obtain a Satake isomorphism which describes $\mathcal{H}_s$ as an algebra of formal series. Using masures, Gaussent and Rousseau generalized their construction to the general Kac-Moody setting ([GR14]). Let G be a split Kac-Moody group over a local field and $\mathcal{I}$ be its masure. Then K_s is the fixer of some vertex of $\mathcal{I}$. The spherical Hecke algebra is then obtained as an algebra of functions from a set of pairs of vertices of $\mathcal{I}$ into $\mathbb{C}$. Note that construction of spherical Hecke algebras from Bruhat-Tits buildings had already be done in [Car73], [Car01] and [Par06].

In [BKP16], Braverman, Kazhdan and Patnaik defined the Iwahori-Hecke algebra for affine Kac-Moody groups. They obtain Iwahori-Matsumoto and Bernstein-Lusztig presentations. They also obtain Macdonald's formula: they compute "explicitly" the image by the Satake isomorphism of some basis of $\mathcal{H}_s$. Using again masures, Bardy-Panse, Gaussent and Rousseau obtained these results in the general case ([BPGR16] and [BPGR17]). The Iwahori subgroup is then the fixer of some chamber C_0^+ and $\mathcal{H}$ is obtained as an algebra of functions from a set of pairs of chambers of $\mathcal{I}$ into $\mathbb{C}$.

In [BGKP14], using Macdonald's formula of [BKP16], Braverman, Garland, Kazhdan and Patnaik obtain a Gindikin-Karpelevich formula. Let G be an affine Kac-Moody group over a local field. This formula relates a formal sum involving cardinals of quotient of subgroups of G to a formal infinite product of elements of $\mathbb{C}(Q^\vee_\mathbb{Z})$ (see the introduction of Chapter 6). The fact that these cardinals are finite - the Gindikin-Karpelevich finiteness - is an important step of their proof. In [Héb17b], we prove the Gindikin-Karpelevich finiteness for general G (not necessarily affine) using masures. Patnaik and Puskás use this finiteness to define analogs of Whitakker functions in the Kac-Moody frameworks: [Pat17] for the affine case and [PP17] for the general (and more generally, metaplectic) case. They obtain a Casselman-Shalika formula. This formula is more explicit in the affine case thanks to the Gindikin-Karpelevich formula.

In [AH17], together with Abdellatif, we prove that the center of the Iwahori-Hecke algebra $\mathcal{H}$ is more or less trivial and define a completion of $\mathcal{H}$ to obtain a bigger center. We associate Hecke algebras to subgroups of G more general than the Iwahori subgroup, the analogs of the parahoric subgroups. These algebras are defined as algebras from a set of pairs of faces of $\mathcal{I}$ into $\mathbb{C}$.

A motivation (or a dream) for Braverman and Kazhdan would be to develop a theory of smooth representations and of automorphic representations of (affine) Kac-Moody groups and a Langlands correspondence in the (affine) Kac-Moody frameworks, see [BK14]. They hope it would lead to powerful results for automorphic forms of reductive groups.

1.3 Main results

We now describe the main results of these thesis. Most of them come from [Héb17b], [Héb16], [AH17] and [Héb17a]. We do not respect the chronological order of publication. For more motivations and details on these results, we refer to the introduction of the corresponding chapters. As mentioned above a part of the thesis is dedicated to the study of the properties of masures and the other is dedicated to the applications of masures in arithmetic.

1.3.1 Axiomatic of masures

In [Rou11] and [Rou17], Rousseau gives an axiomatic of masures (we write it in 3.1.5) which contains five axioms. We propose an equivalent axiomatic, which is shorter and closer to the axiomatic of Bruhat-Tits buildings.

Let G be a Kac-Moody group over a local field and T be a maximal split torus. Let Φ_{re} be the set of real roots of (G,T). One considers Φ_{re} as a set of linear forms of some vector space $\mathbb{A}$. The hyperplanes of the shape $\alpha^{-1}(\{k\})$ for some $k \in \mathbb{Z}$ and $\alpha \in \Phi_{re}$ are called walls of $\mathbb{A}$ and a half-space delimited by a wall is called a half-apartment of $\mathbb{A}$. An apartment of type $\mathbb{A}$ is more or less an affine space equipped with a set of walls which is isomorphic to $(\mathbb{A}, \mathcal{M})$ where $\mathcal{M}$ is the set of walls of $\mathbb{A}$.

Let us begin with affine Kac-Moody groups. In the statement of the next theorem, we use the notion of chimney. They are some kind of thickened sector faces. In the affine case, a chimney is splayed if the associated sector face is not trivial.

Theorem 1. (see Theorem 4.4.33) Suppose that G is an affine split Kac-Moody group over a local field. Let $\mathbb{A}$ be the apartment associated to the root system of (G,T). Let $(\mathcal{I}, \mathcal{A})$ be a couple such that $\mathcal{I}$ is a set and $\mathcal{A}$ is a set of subsets of $\mathcal{I}$ called apartments. Then $(\mathcal{I}, \mathcal{A})$ is a masure of type $\mathbb{A}$ in the sense of [Rou11] if and only if it satisfies the following axioms:

(MA af i) Each apartment is an apartment of type $\mathbb{A}$.

(MA af ii) If A and A' are two apartments, then $A \cap A'$ is a finite intersection of half-apartments and there exists an isomorphism $\phi : A \to A'$ fixing $A \cap A'$.

(MA af iii) If $\mathfrak{R}$ is the germ of a splayed chimney and F is a face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

Similarly to buildings, we can define a fundamental chamber C_f^v in the standard apartment $\mathbb{A}$. This enables to define the Tits cone $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$, where W^v is the vectorial Weyl group of G. An important difference between buildings and masures is that when G is reductive, $\mathcal{T} = \mathbb{A}$ and when G is not reductive, $\mathcal{T}$ is only a proper convex cone in $\mathbb{A}$. This defines a preorder on $\mathbb{A}$ by saying that $x, y \in \mathbb{A}$ satisfy $x \leq y$ if $y \in x + \mathcal{T}$. This preorder extends to a preorder on $\mathcal{I}$ - the Tits preorder - by using isomorphisms of apartments.

A ray (half-line) of $\mathbb{A}$ is said to be generic if its direction meets the interior $\mathring{\mathcal{T}}$ of $\mathcal{T}$. A splayed chimney is a chimney containing a generic ray. The main result of this chapter is the following theorem:

Theorem 2. (see Theorem 4.4.1) Let $\mathbb{A}$ be the apartment associated to the root system of (G,T). Let $(\mathcal{I}, \mathcal{A})$ be a couple such that $\mathcal{I}$ is a set and $\mathcal{A}$ is a set of subsets of $\mathcal{I}$ called apartments. Then $(\mathcal{I}, \mathcal{A})$ is a masure of type $\mathbb{A}$ in the sense of [Rou11] if and only if it satisfies the following axioms:

(MA i) Each apartment is an apartment of type $\mathbb{A}$.

(MA ii) If two apartments A and A' are such that $A \cap A'$ contains a generic ray, then $A \cap A'$ is a finite intersection of half-apartments and there exists an isomorphism $\phi : A \to A'$ fixing $A \cap A'$.

(MA iii) If $\mathfrak{R}$ is the germ of a splayed chimney and F is a face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

Actually we do not limit our study to masures associated to Kac-Moody groups: for us a masure is a set satisfying the axioms of [Rou11] and whose apartments are associated to a root generating system (and thus to a Kac-Moody matrix). We do not assume that there exists a group acting strongly transitively on it. We do not either make any discreteness hypothesis for the standard apartment: if M is a wall, the set of walls parallel to it is not necessarily discrete; this enables to handle masures associated to almost-split Kac-Moody groups over any ultrametric field.

1.3.2 Hecke algebras

The main results of this chapter come from a joint work with Abdellatif, see [AH17]. Let G be a split Kac-Moody group over a local field, T be a maximal split torus and $\mathcal{I}$ be the masure of G. In [BK11] (when G is affine), [GR14], [BKP16] (when G is affine) and [BPGR16], the authors define and study the spherical Hecke algebra and the Iwahori-Hecke algebra of G. When G is reductive, these are particular cases (for the definition) of Hecke algebras associated to compact open subgroups of G. We no more suppose G to be reductive. Two questions naturally arise:

1. Is there a way to turn G into a topological group in such a way that the spherical subgroup K_s or the Iwahori subgroup K_I is open and compact?
2. Can we associate a Hecke algebra to subgroups K more general than K_s or K_I?

We first prove that the answer to the first question is no. Let $\mathcal{I}$ be the masure of G. Then K_s is the fixer of some vertex 0 in G and K_I is the fixer of some chamber C_0^+ based at 0. By an argument of incommensurability using $\mathcal{I}$, we give a more general answer:

Proposition 3. (see Proposition 5.2.8) Let $K \subset G$ be the fixer of some face between $\{0\}$ and C_0^+. Then there is no way to turn G into a topological group in such a way that K is open and compact, unless G is reductive.

For the second question, our idea is to use the same technique as Bardy-Panse, Gaussent and Rousseau in [BPGR16]. Let F be a face between 0 and C_0^+. We associate to F a space ${}^F\mathcal{H}$ of functions from some pairs of faces of $\mathcal{I}$ into $\mathbb{C}$. We equip it with the convolution product if $f, g \in {}^F\mathcal{H}$ and $F_1, F_2 \in G.F$,

$$f * g(F_1, F_2) = \sum_{F_3 \in G.F} f(F_1, F_3) g(F_3, F_2).$$

Our definition is very close to the definition of [BPGR16] and when $F = C_0^+$ we get their algebra. There are some definition issues, we have to prove that this definition does not lead to infinite coefficients. We prove that when F is spherical, which mean that its fixer in the Weyl group of G is finite, then the convolution product is well-defined and $({}^F\mathcal{H}, *)$ is an associative algebra (see Theorem 5.3.14). We prove that when F is non-spherical and different from $\{0\}$ however, this definition leads to infinite structure coefficients (see Proposition 5.3.22).

Suppose that G is reductive. Let W^v be the Weyl group of (G, T) and $Q_{\mathbb{Z}}^{\vee}$ be the coweight lattice. Then by Satake isomorphism and Bernstein-Lusztig relations, we have the following diagram:

$$\mathcal{H}_s \xrightarrow[S]{\backsimeq} \mathbb{C}[Q_\mathbb{Z}^\vee]^{W^v} \underset{g}{\hookrightarrow} \mathcal{H}, \text{ and } \mathrm{Im}(g) = \mathcal{Z}(\mathcal{H})$$

where S is the Satake isomorphism and g comes from the Bernstein-Lusztig basis.

We no more suppose G to be reductive. Let W^v be the Weyl group of G. The Satake isomorphism is an isomorphism between $\mathcal{H}_s$ and $\mathbb{C}[[Y]]^{W^v}$, where Y is a lattice which can be thought of as the coroot lattice in a first approximation (but it can be different, notably when G is affine) and $\mathbb{C}[[Y]]$ is the Looijenga's algebra of Y, which is some completion of the group algebra $\mathbb{C}[Y]$ of Y. As we shall see (Theorem 5.5.19), the center of $\mathcal{H}$ is more or less trivial. Moreover, $\mathbb{C}[[Y]]^{W^v}$ is a set of infinite formal series and there is no obvious injection from $\mathbb{C}[[Y]]$ to $\mathcal{H}$. For these reasons, we define a "completion" $\widehat{\mathcal{H}}$ of $\mathcal{H}$. More precisely, let $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$, where Y^+ is a sub-monoid of Y, be the Bernstein-Lusztig basis of $\mathcal{H}$. Then $\widehat{\mathcal{H}}$ is the set of formal series $\sum_{w\in W^v,\lambda\in Y^+} c_{w,\lambda} Z^\lambda H_w$ whose support satisfies some conditions similar to what appears in the definition of $\mathbb{C}[[Y]]$. We equip it with a convolution compatible with the inclusion $\mathcal{H}\subset\widehat{\mathcal{H}}$. The fact that this product is well defined is not obvious and this is our main result: Theorem 5.5.10. We then determine the center of $\widehat{\mathcal{H}}$ and we show that it is isomorphic to $\mathbb{C}[[Y]]^{W^v}$ (Theorem 5.5.19), which is similar to the classical case. We thus get the following diagram:

$$\mathcal{H}_s \xrightarrow[S]{\backsimeq} \mathbb{C}[[Y]]^{W^v} \underset{g}{\hookrightarrow} \widehat{\mathcal{H}}, \text{ and } \mathrm{Im}(g) = \mathcal{Z}(\widehat{\mathcal{H}}),$$

where S is the Satake isomorphism (see Section 8 of [BK11] or Theorem 5.4 of [GR14]), and g comes from the Bernstein-Lusztig basis.

1.3.3 Gindikin-Karpelevich finiteness

Let $\mathbf{G}$ be a split Kac-Moody functor, $\mathcal{K}$ be a local field, $G=\mathbf{G}(\mathcal{K})$ and T be a maximal torus of G. Let $\mathcal{O}$ be the ring of integers of $\mathcal{K}$, π be a generator of the maximal ideal of $\mathcal{O}$ and q denote the cardinal of the residue field $\mathcal{O}/\pi\mathcal{O}$. Choose a pair B, B^- of opposite Borel subgroups such that $B\cap B^-=T$ and let U, U^- be their "unipotent radicals". Let $K_s=\mathbf{G}(\mathcal{O})$. Let $(\alpha_i^\vee)_{i\in I}$ denote the simple coroots of (G,T), $Q_\mathbb{N}^\vee=\bigoplus_{i\in I}\mathbb{N}\alpha_i^\vee$ and $Q_\mathbb{Z}^\vee=\bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee$.

We prove the following theorem (see Theorem 6.5.1, Corollary 6.6.2, Theorem 6.6.7 and Theorem 6.7.1):

Theorem 4. Let $\mu\in Q_\mathbb{Z}^\vee$. Then for all $\lambda\in Q_\mathbb{Z}^\vee$, $K_s\backslash K_s\pi^\lambda U^-\cap K_s\pi^{\lambda+\mu}U$ is finite and is empty if $\mu\notin -Q_\mathbb{N}^\vee$. Moreover $|K_s\backslash K_s\pi^\lambda U^-\cap K_s\pi^{\lambda+\mu}U|$ does not depend on $\lambda\in Q_\mathbb{Z}^\vee$ and for λ sufficiently dominant, $K_s\pi^\lambda U^-\cap K_s\pi^{\lambda+\mu}U = K_s\pi^\lambda K_s\cap K_s\pi^{\lambda+\mu}U$.

This theorem corresponds to Theorem 1.9 of [BGKP14] and this is a positive answer to Conjecture 4.4 of [BK14]. It proves that

$$\sum_{\mu\in Q_\mathbb{Z}^\vee} |K_s\backslash K_s\pi^\lambda U^-\cap K_s\pi^{\lambda-\mu}U| q^{|\lambda-\mu|} e^{\lambda-\mu}$$

has a meaning in the algebra $\mathbb{C}[[Q_\mathbb{Z}^\vee]]$ of $Q_\mathbb{Z}^\vee$ (loosely speaking, $\mathbb{C}[[Q_\mathbb{Z}^\vee]]$ is a completion of the group algebra $\mathbb{C}[Q_\mathbb{Z}^\vee]$ with generators e^ν for $\nu\in Q_\mathbb{Z}^\vee$) for a general split Kac-Moody group over a local field. This is one side of the Gindikin-Karpelevich formula of [BGKP14].

1.3.4 Distances on a masure

Let $\mathcal{I}$ be a Bruhat-Tits building associated to a split reductive group G over a local field. Then $\mathcal{I}$ is equipped with a distance d such that G acts isometrically on $\mathcal{I}$ and such that the restriction of d to each apartment is a euclidean distance. This distances are important tools in the study of buildings. Let now G be a split Kac-Moody group over a local field. We show that unless G is reductive, we cannot equip $\mathcal{I}$ with a distance having these properties. It seems natural to ask whether we can define distances on a masure having "good" properties. We limit our study to distances inducing the affine topology on each apartment. We show that under assumptions of continuity for retractions, the metric space we have is not complete nor locally compact (see Section 7.2). We show that there is no distance on $\mathcal{I}$ such that the restriction to each apartment is a norm. However, we prove the following theorem (Corollary 7.4.8, Lemma 7.3.9, Corollary 7.4.9 and Theorem 7.4.14):

Theorem 5. Let $\mathfrak{q}$ be a sector germ of $\mathcal{I}$, then there exists a distance d on $\mathcal{I}$ having the following properties:

- the topology induced on each apartment is the affine topology;
- each retraction with center $\mathfrak{q}$ is 1-Lipschitz continuous;
- each retraction with center a sector-germ of the same sign as $\mathfrak{q}$ is Lipschitz continuous;
- each $g \in G$ is Lipschitz continuous when we see it as an automorphism of $\mathcal{I}$.

We call the distances constructed in the proof of this theorem distances of **positive** or of **negative type**, depending on the sign of $\mathfrak{q}$. A distance of positive or negative type is called a signed distance. We prove that all distances of positive type on a masure (resp. of negative type) are equivalent, where we say that two distances d_1 and d_2 are equivalent if there exist $k, \ell \in \mathbb{R}_+^*$ such that $kd_1 \leq d_2 \leq ld_1$ (this is Theorem 7.4.7). We thus get a **positive topology** $\mathscr{T}_+$ and a **negative topology** $\mathscr{T}_-$. We prove (Corollary 7.5.4) that these topologies are different when $\mathcal{I}$ is not a building. When $\mathcal{I}$ is a building these topologies are the usual topology on a building (Proposition 7.4.15).

Let $\mathcal{I}_0$ be the orbit of some special vertex under the action of G. If $\mathcal{I}$ is not a building, $\mathcal{I}_0$ is not discrete for $\mathscr{T}_-$ and $\mathscr{T}_+$. We also prove that if ρ is a retraction centered at a positive (resp. negative) sector-germ, ρ is not continuous for $\mathscr{T}_-$ (resp. $\mathscr{T}_+$), see Proposition 7.5.3. For these reasons we introduce **mixed distances**, which are the sum of a distance of positive type and of a distance of negative type. We then have the following theorem (Theorem 7.5.7):

Theorem 6. All the mixed distances on $\mathcal{I}$ are equivalent; moreover, if d is a mixed distance and $\mathcal{I}$ is equipped with d we have:

- each $g : \mathcal{I} \to \mathcal{I} \in G$ is Lipschitz continuous;
- each retraction centered at a sector-germ is Lipschitz continuous;
- the topology induced on each apartment is the affine topology;
- the set $\mathcal{I}_0$ is discrete.

The topology $\mathscr{T}_c$ associated to mixed distances is the initial topology with respect to the retractions of $\mathcal{I}$ (see Corollary 7.5.10).

1.3.5 Tits preorder on a masure of affine type

Suppose that G is an affine Kac-Moody group over a local field. In this situation, $\mathcal{T}$ is well understood: this is more or less a half-space of $\mathbb{A}$ defined by some linear form $\delta_{\mathbb{A}} : \mathbb{A} \to \mathbb{R}$ (the smallest imaginary root of G). Using the action of G, one naturally extends $\delta_{\mathbb{A}}$ to a map $\delta : \mathcal{I} \to \mathbb{A}$. The aim of this short chapter is to prove the following theorem:

Theorem 7. Let $x, y \in \mathcal{I}$ such that $\delta(x) < \delta(y)$. Then $x \leq y$.

As in restriction to each apartments, δ is an affine map, this proves that "almost all" pair of points is included in an apartment. This answers the question of the last paragraph of Section 5 of [Rou11].

1.4 Frameworks

In this thesis, we consider abstract masures in the definition of Rousseau, not necessarily associated to a group (depending on the chapters). Our framework is adapted to the study of almost-split Kac-Moody groups over valued fields or local fields, depending on the chapter. However, we only define split Kac-Moody groups and refer to [Rém02] and [Rou17] for the definition of almost split Kac-Moody groups.

1.5 Organization of the memoir

The memoir is organized as follows. Chapters 2 and 3 detail the framework of this thesis. Chapters 4 to 8 are based on the works of the author. They contain an introduction, more detailed than above in 1.3. There is an index of definitions at the end of the memoir and a section "notation" at the end of Chapter 3.

In Chapter 2, we define Kac-Moody algebras, recall some basic properties and define their vectorial apartments.

In Chapter 3, we define masures, we briefly define Kac-Moody groups according to [Tit87], sketch the construction of masures and describe the action of a Kac-Moody group on its masure.

In Chapter 4, we simplify the axiomatic of masures.

In Chapter 5, we study the Hecke algebras associated to Kac-Moody groups over local fields.

In Chapter 6, we prove the Gindikin-Karpelevich finiteness.

In Chapter 7, we study distances on a masure.

In Chapter 8, we give a simple criterion for the Tits preorder on a masure associated to an affine Kac-Moody group.

The most important definitions are in Chapters 2 and 3. Chapters 4 to 8 can more or less be read independently, punctually admitting results of previous chapters. Chapter 5 to Chapter 8 are written using the axiomatic definition of 4 (see Theorem 4.4.1). This induces minor changes in comparison with [Héb17b], [Héb16], [AH17] where Rousseau's axiomatic is used.

In Appendix A, we give a short definition of masures, using the axiomatic of 4 and limiting the framework to masures associated to split Kac-Moody groups over local fields. The reader only interested in arithmetic results (Chapter 5 and Chapter 6) can replace Section 3.1 by this appendix (or replace Chapters 2 and 3 by this appendix).

Chapter 2

Kac-Moody algebras and Vectorial apartments

2.1 Introduction

In this chapter we define Kac-Moody algebras, recall some of their basic properties and define their vectorial apartments.

The masure is a union of apartments, all isomorphic to a standard one $\mathbb{A}$. In order to define $\mathbb{A}$, we first introduce vectorial apartments, which are some $\mathbb{R}$-forms of Cartan subalgebras of Kac-Moody algebras over $\mathbb{C}$. Most of the results of this chapter are known.

Let us describe briefly the construction of the vectorial apartment of a Kac-Moody algebra. Similarly to semi-simple Lie algebras, a Kac-Moody algebra $\mathfrak{g}$ admits a root decomposition. Let us be more precise. The definition of $\mathfrak{g}$ involves a Cartan subalgebra $\mathfrak{h}$ which is abelian and finite dimensional. If $\alpha \in \mathfrak{h}^*$, one defines $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h,x] = \alpha(h)x \ \forall h \in \mathfrak{h}\}$. It appears that for a subset Φ_{all}^+ of $\mathfrak{h}^*$, $\mathfrak{g} = \bigoplus_{\alpha \in -\Phi_{all}^+} \mathfrak{g}_\alpha \oplus \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{all}^+} \mathfrak{g}_\alpha$.

When $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$, $\mathfrak{h}$ is the vector space of diagonal matrices and the $\mathfrak{g}_\alpha$'s are the spaces $\mathbb{C}E_{i,j}$, where the $E_{i,j}$'s are the elementary matrices (if $\alpha \in \Phi_{all}^+$, $i < j$ and if $\alpha \in -\Phi_{all}^+$, $i > j$), see Example 2.2.5.

If $\mathfrak{h}_\mathbb{R}$ is some $\mathbb{R}$-form of $\mathfrak{h}$ (that is $\mathfrak{h}_\mathbb{R} \otimes_\mathbb{R} \mathbb{C} = \mathfrak{h}$), the elements of Φ_{all} define hyperplanes of $\mathfrak{h}_\mathbb{R}$, that we call vectorial walls. This arrangement of walls defines a structure of simplicial complex on some cone $\mathcal{T}$ of $\mathfrak{h}_\mathbb{R}$ (the Tits cone) and $\mathfrak{h}_\mathbb{R}$ equipped with this arrangement is called the vectorial apartment. If $C_f^v = \{x \in \mathfrak{h}_\mathbb{R} | \ \alpha(x) > 0 \ \forall \alpha \in \Phi_{all}^+\}$, then C_f^v is a cone called the fundamental chamber. It is delimited by a finite number of walls and the subgroup of $\mathrm{GL}(\mathfrak{h}_\mathbb{R})$ spanned by certain reflexions with respect to these walls is a Coxeter group called the Weyl group W^v of $\mathfrak{g}$. Then one has $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$. The faces of $\mathcal{T}$ are the $w.F^v$, where $w \in W^v$ and F^v is a face of C_f^v. We also consider the opposite of the Tits cone $-\mathcal{T}$.

Organization of the chapter In Section 2.2, we recall the definition of Kac-Moody algebras, of vectorial apartments and their basic properties.

In Section 2.3, we describe the vectorial apartment in the affine case and in the indefinite case.

2.2 Kac-Moody algebras and vectorial apartment of a masure

2.2.1 Kac-Moody algebras

In this subsection, we define Kac-Moody algebras by generators and relations. We first define the realization of a Kac-Moody matrix, which corresponds more or less to the Cartan subalgebra of the associated Kac-Moody algebra. We then recall the root decomposition of these Lie algebras and define their set of roots. Our references are mainly [Kac94], Chapter I of [Kum02] and Section 1 of [Rou11].

2.2.1.1 Realization of a Kac-Moody matrix

Let A be a **Kac-Moody matrix** (also known as generalized Cartan matrix) i.e a square matrix $A = (a_{i,j})_{i,j\in I}$ with integers coefficients, indexed by a finite set I and satisfying:

1. $\forall i \in I,\ a_{i,i} = 2$
2. $\forall (i,j) \in I^2 | i \neq j,\ a_{i,j} \leq 0$
3. $\forall (i,j) \in I^2,\ a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0$.

The matrix A is said to be **decomposable** if for some reordering of the indices, A, one can write as a non trivial block diagonal matrix $\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$. One says that A is **indecomposable** if it is not decomposable.

A **root generating system** of type A is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac-Moody matrix A indexed by I, of two dual free $\mathbb{Z}$-modules X (of **characters**) and Y (of **cocharacters**) of finite rank $\mathrm{rk}(X)$, a family $(\alpha_i)_{i\in I}$ (of **simple roots**) in X and a family $(\alpha_i^\vee)_{i\in I}$ (of **simple coroots**) in Y. They have to satisfy the following compatibility condition: $a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j \in I$. We also suppose that the family $(\alpha_i)_{i\in I}$ is free in X and that the family $(\alpha_i^\vee)_{i\in I}$ is free in Y.

If $\mathcal{K}$ is a subfield of $\mathbb{C}$ (one will consider the case $\mathcal{K} = \mathbb{R}$ and $\mathcal{K} = \mathbb{C}$), a **realization of** A **over** $\mathcal{K}$ is a triple $(\mathfrak{h}, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ such that $\mathfrak{h}$ is a finite dimensional vector space over $\mathcal{K}$, $(\alpha_i^\vee)_{i\in I}$ is a free family of $\mathfrak{h}$, $(\alpha_i)_{i\in I}$ is a free family of the dual $\mathfrak{h}^*$ of $\mathfrak{h}$ and for all $i, j \in I$, $a_{i,j} = \alpha_j(\alpha_i^\vee)$. The realization is called **minimal** if moreover $\dim \mathfrak{h} = |I| + \mathrm{corank}(A)$.

Two realizations $(\mathfrak{h}, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$, $(\mathfrak{h}', (\alpha_i')_{i\in I}, (\alpha_i'^\vee)_{i\in I})$ of A are called **isomorphic** if there exists a vector space isomorphism $\phi : \mathfrak{h} \to \mathfrak{h}'$ such that $\phi((\alpha_i^\vee)_{i\in I}) = (\alpha_i'^\vee)_{i\in I}$ and $\phi^*((\alpha_i)_{i\in I}) = (\alpha_i')_{i\in I}$, where $\phi^* : \mathfrak{h}^* \to \mathfrak{h}^*$ maps each $f \in \mathfrak{h}^*$ on $f \circ \phi$. By Proposition 1.1 of [Kac94], for all subfield of $\mathbb{C}$, there exists up to isomorphism a unique minimal realization of A over $\mathcal{K}$.

Let $\mathcal{K}$ be a subfield of $\mathbb{C}$. If $(A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ is a root generating system then $(Y \otimes \mathcal{K}, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ is a realization of A over $\mathcal{K}$. We view the elements of X as linear forms on $Y \otimes \mathcal{K}$. If $(\mathfrak{h}_\mathcal{K}, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ is a realization of A over $\mathcal{K}$ then $(\mathfrak{h}_\mathcal{K} \otimes \mathbb{C}, (\alpha)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ is a complex realization of A.

The following lemma is easy to prove.

Lemma 2.2.1. *Let A be a Kac-Moody matrix, $\mathcal{K}$ be a subfield of $\mathbb{C}$ and $\underline{\mathcal{S}}_\mathcal{K} = (\underline{\mathfrak{h}}, (\underline{\alpha}_i)_{i\in I}, (\underline{\alpha}_i^\vee)_{i\in I}))$ be the minimal realization of A over $\mathcal{K}$. Let $\mathcal{S}_\mathcal{K} = (\mathfrak{h}, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ be a realization of A over $\mathcal{K}$. Then there exist subspaces $\mathfrak{h}_0 \subset \mathfrak{h}$ and $\mathfrak{h}' \subset \mathfrak{h}$ such that*

- $\mathfrak{h}_0 \supset (\alpha_i^\vee)_{i\in I}$

- $\mathfrak{h}' \subset \bigcap_{i\in I} \ker \alpha_i$
- $\mathfrak{h} = \mathfrak{h}_0 \oplus \mathfrak{h}'$
- *there exists an isomorphism of realizations* $\Upsilon : \mathcal{S}_{0,\mathcal{K}} = (\mathfrak{h}_0, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I}) \to \underline{\mathcal{S}}_{\mathcal{K}}$.

2.2.1.2 Kac-Moody algebras

Lie algebra defined by a presentation If V is a vector space, one denotes by $T(V)$ its tensor algebra and one equips it with the bracket $[a,b] = a\otimes b - b\otimes a$. Let X be a set of symbols and $\mathbb{C}^{(X)}$ be the set of maps from X to $\mathbb{C}$ having finite support. The **free Lie algebra generated by** X is the Lie subalgebra of $T(\mathbb{C}^{(X)})$ spanned by $\mathbb{C}^{(X)} = T^1(\mathbb{C}^{(X)})$. We denote it $F(X)$. If $(R_k)_{k\in K}$, $(S_k)_{k\in K}$ are elements of $F(X)$, the **Lie algebra generated by** X **and subject to the relations** $R_k = S_k$, $k\in K$, is the quotient of $F(X)$ by the ideal spanned by $\{R_k - S_k |\ k \in K\}$.

Remark 2.2.2. *If* $\mathfrak{g}$ *is a Lie algebra,* $k\in\mathbb{N}$ *and* $e_1,\ldots,e_k\in\mathfrak{g}$*, one defines* $[e_1,\ldots,e_k]$ *by induction on* k *as follows. One sets* $[e_k] = e_k$ *and if* $\ell \in [\![1, k-1]\!]$, $[e_\ell,\ldots,e_k] = [e_\ell,[e_{\ell+1},\ldots,e_k]]$. *Then using Jacobi identity, one can prove that if* $\{e_k,\ k\in K\}$ *is a set of symbols,* $F(\{e_k|\ k\in K\})$ *is spanned by* $\{[e_{k_1},\ldots,e_{k_\ell}] | \ell\in\mathbb{N}$ and $(k_1,\ldots,k_\ell)\in K^\ell\}$.

Definition of Kac-Moody algebras Let $A = (a_{i,j})_{i,j\in I}$ be a Kac-Moody matrix and $(\mathfrak{h}, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ be a complex realization of A. The Kac-Moody algebra $\mathfrak{g} := \mathfrak{g}(A)$ is the Lie algebra over $\mathbb{C}$ generated by $\mathfrak{h}$ and the symbols e_i, f_i for $i\in I$ subject to the following relations for all $i,j\in I$ and $h\in\mathfrak{h}$:
(R_1) $[\mathfrak{h},\mathfrak{h}] = 0$
(R_2) $[h,e_i] = \alpha_i(h)e_i$; $[h,f_i] = -\alpha_i(h)f_i$,
(R_3) $[e_i,f_j] = \delta_{i,j}\alpha_i^\vee$
(R_4) $(\mathrm{ad}\ e_i)^{1-a_{i,j}}(e_j) = 0$, $i\neq j$
(R_5) $(\mathrm{ad}\ f_i)^{1-a_{i,j}}(f_j) = 0$, $i\neq j$.
The e_i, f_i, $i\in I$ are called the **Chevalley generators** of $\mathfrak{g}$.

Kac-Moody algebras associated to free realizations of A Let $\mathcal{S}_\mathbb{C} = (\mathfrak{h}, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I}))$ be a complex realization of A. Let $\mathfrak{g}$ be the algebra generated by $\mathfrak{h}$ and symbols e_i, f_i, $i\in I$ subject to the relations R_1 to R_5. The Lie algebra $\mathfrak{g}$ is called the **Kac-Moody algebra of** $\mathcal{S}_\mathbb{C}$.

Let $\underline{\mathcal{S}}_\mathbb{C} = (\underline{\mathfrak{h}}, (\underline{\alpha}_i)_{i\in I}, (\underline{\alpha}_i^\vee)_{i\in I}))$ be the minimal realization of A. Let $\underline{\mathfrak{g}}$ be the Kac-Moody algebra of $\underline{\mathcal{S}}_\mathbb{C}$ (generated by $\underline{\mathfrak{h}}$ and symbols $\underline{e}_i$, $\underline{f}_i$, $i\in I$). One writes $\mathfrak{h} = \mathfrak{h}_0\oplus\mathfrak{h}'$ as in Lemma 2.2.1. Let $\underline{\mathfrak{g}}\times\mathfrak{h}'$ be the Lie algebra with underlying vector space $\underline{\mathfrak{g}}\times\mathfrak{h}'$ and equipped with the bracket $[(g,h),(g'+h')] = [g,g]$ for all $(g,h),(g',h')\in\underline{\mathfrak{g}}\times\mathfrak{h}'$. Although we do not know yet that the map $\mathfrak{h}'\to\mathfrak{g}$ is injective, we denote by $\mathfrak{h}'$ the image of $\mathfrak{h}'$ in $\mathfrak{g}$

Proposition 2.2.3. *There exists a unique Lie algebra isomorphism* $\Psi : \mathfrak{g}\to\underline{\mathfrak{g}}\times\mathfrak{h}'$ *such that* $\Psi_{|\mathfrak{h}'} = \mathrm{Id}_{|\mathfrak{h}'}$, $\Psi_{|\mathfrak{h}_0} = \Upsilon^{-1}_{|\mathfrak{h}_0}$ *(where* $\Upsilon : \underline{\mathcal{S}}\to\mathcal{S}_0$ *is an isomorphism of realizations),* $\Psi(e_i) = \underline{e}_i$ *and* $\Psi(f_i) = \underline{f}_i$ *for all* $i\in I$.

Proof. Let $\widetilde{\mathfrak{g}}$ be the free Lie algebra generated by $\mathfrak{h}$ and symbols e_i, f_i, $i\in I$. Let $\widetilde{\Psi} : \widetilde{\mathfrak{g}}\to\underline{\mathfrak{g}}\times\mathfrak{h}'$ defined by $\widetilde{\Psi}(e_i) = \underline{e}_i$, $\widetilde{\Psi}(f_i) = \underline{f}_i$, $\widetilde{\Psi}_{|\mathfrak{h}'} = \mathrm{Id}_{\mathfrak{h}'}$ and $\widetilde{\Psi}_{|\mathfrak{h}_0} = \Upsilon^{-1}_{|\mathfrak{h}_0}$. Then $\widetilde{\Psi}$ satisfies relations R_1 to R_5 and thus it induces a morphism of Lie algebras $\Psi : \mathfrak{g}\to\underline{\mathfrak{g}}\times\mathfrak{h}'$.

Let $\widetilde{\underline{\mathfrak{g}}}$ be the free Lie algebra generated by $\mathfrak{h}$ and symbols $\underline{e}_i$, $\underline{f}_i$, $i \in I$. Let $\widetilde{\Omega} : \widetilde{\underline{\mathfrak{g}}} \to \mathfrak{g}$ be the Lie algebra morphism defined by $\widetilde{\Omega}_{|\underline{\mathfrak{h}}} = \Upsilon_{|\underline{\mathfrak{h}}}$, $\widetilde{\Omega}(\underline{e}_i) = e_i$ and $\widetilde{\Omega}(\underline{f}_i) = f_i$, for all $i \in I$. Then $\widetilde{\Omega}$ satisfies relations R_1 to R_5 and thus it induces a Lie algebra morphism $\Omega : \underline{\mathfrak{g}} \to \mathfrak{g}$. Let $\Omega' : \underline{\mathfrak{g}} \times \mathfrak{h}' \to \mathfrak{g}$ defined by $\Omega'_{|\underline{\mathfrak{g}}} = \Omega$ and $\Omega'_{|\mathfrak{h}'}$ is the natural map $\mathfrak{h}' \to \mathfrak{g}$. Then Ω' is a Lie algebra morphism and $\Omega' \circ \Psi$ (resp. $\Psi \circ \Omega'$) is a Lie algebra morphisms whose restriction to a set of generators of $\mathfrak{g}$ (resp. $\underline{\mathfrak{g}} \times \mathfrak{h}'$) is the the identity: Ω' and Ψ are inverse of each others and in particular, they are isomorphisms. □

Our framework is slightly more general than the one of [Kac94] and [Kum02] because we consider non-necessarily minimal realizations. We use Proposition 2.2.3 throughout this section to use results of [Kac94] and [Kum02] in our frameworks.

Cartan involution Let $\omega : \mathfrak{h} \cup \{e_i |\ i \in I\} \cup \{f_i -\ i \in I\}$ defined by $\omega(h) = -h$ if $h \in \mathfrak{h}$, $\omega(e_i) = -f_i$ and $\omega(f_i) = -e_i$ if $i \in I$. Then ω extends uniquely to a Lie algebra morphism (still denoted ω) from $\mathfrak{g}$ to $\mathfrak{g}$. Moreover ω is an involution and it is called the **Cartan involution of** $\mathfrak{g}$.

Let $Q = \bigoplus_{i \in I} \mathbb{Z}\alpha_i \subset \mathfrak{h}^*$ be the **root lattice** of $\mathfrak{g}$. Let $Q_{\mathbb{N}} = \bigoplus_{i \in I} \mathbb{N}\alpha_i \subset Q$. For any $\alpha \in \mathfrak{h}^*$, let $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} | [h, x] = \alpha(h)x,\ \forall h \in \mathfrak{h}\}$. Let $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) be the sub-Lie algebra of $\mathfrak{g}$ generated by $\{e_i |\ i \in I\}$ (resp. $\{f_i |\ i \in I\}$). Then we have the following Theorem (Theorem 1.2.1 of [Kum02]):

Theorem 2.2.4. *The canonical map $\mathfrak{h} \to \mathfrak{g}$ is an embedding of Lie algebras. One considers $\mathfrak{h}$ as a sub-Lie algebra of $\mathfrak{g}$. One has the following properties.*

1. *The Lie algebra $\mathfrak{g}$ decomposes as $\mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$.*

2. *The following root space decomposition holds: $\mathfrak{n}^\pm = \bigoplus_{\alpha \in Q_{\mathbb{N}} \setminus \{0\}} \mathfrak{g}_{\pm\alpha}$. An for all $\alpha \in Q$, $\mathfrak{g}_\alpha$ is finite dimensional.*

3. *The Lie algebra $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) is generated by $\{e_i |\ i \in I\}$ (resp. $\{f_i |\ i \in I\}$) subject to the only relations R_4 (resp. R_5).*

Let $\Phi_{all} = \{\alpha \in Q \setminus \{0\} |\ \mathfrak{g}_\alpha \neq 0\}$. Then Φ_{all} is called the **set of roots** and the elements of Φ_{all} are called roots. Let $\Phi_{all}^+ = \Phi_{all} \cap Q_{\mathbb{N}}$ and $\Phi_{all}^- = -\Phi_{all}^+$. Then $\Phi_{all} = \Phi_{all}^+ \sqcup \Phi_{all}^-$. If $\alpha \in \Phi_{all}$, the dimension of $\mathfrak{g}_\alpha$ is called the **multiplicity of** α.

Example 2.2.5. *A simple example of a Kac-Moody algebra is $\mathfrak{g} = \mathfrak{sl}_n(\mathbb{C})$ for $n \in \mathbb{N}_{\geq 2}$. As a vector space, $\mathfrak{g}$ is the set of $n \times n$ matrices whose trace is 0. If $a, b \in \mathfrak{sl}_n(\mathbb{C})$, $[a, b] = ab - ba$. The standard Cartan subalgebra $\mathfrak{h}$ is the set of diagonal matrices whose trace is 0. For $i, j \in [\![1, n-1]\!]$, one denotes by $E_{i,j}$ the matrix having 1 in i, j and 0 elsewhere. If $i \in [\![1, n-1]\!]$, one sets $e_i = E_{i,i+1}$ and $f_i = E_{i+1,i}$. One sets $\alpha_i^\vee = E_{i,i} - E_{i+1,i+1}$ for all $i \in [\![1, n-1]\!]$.*

The associated Kac-Moody matrix is the matrix having 2 on the diagonal, -1 on the strictly upper and lower diagonal and 0 elsewhere.

If $i \in [\![1, n-1]\!]$, then $\alpha_i(\sum_{j=1}^n h_j e_j) = h_i - h_{i+1}$ for all $(h_j) \in \mathbb{C}^n$ such that $\sum_{j=1}^n h_j = 0$. If $(i, j) \in [\![1, n]\!]^2$ such that $i \neq j$, one defines $\alpha_{i,j} : \mathfrak{h} \to \mathbb{C}$ by $\alpha_{i,j}(\sum_{k=1}^n h_k E_{k,k}) = h_i - h_j$ (one has $\alpha_{i,j} = \sum_{k=i}^{j-1} \alpha_k$ if $i < j$). Then $\Phi_{all}^+ = \{\alpha_{i,j} | (i, j) \in [\![1, n-1]\!]^2 | i < j\}$ and $\Phi_{all} = \Phi_{all}^+ \cup -\Phi_{all}^+$. If $\alpha \in \Phi$, $\alpha = \alpha_{i,j}$, then $\mathfrak{g}_\alpha = \mathbb{C}E_{i,j}$ with the notation of Theorem 2.2.4. Thus $\mathfrak{n}^+$ (resp. $\mathfrak{n}^-$) is the set of strictly upper (resp. lower) triangular matrix.

We will write explicitly the root decomposition of an affine Kac-Moody algebra in Subsection 2.3.1 (see Lemma 2.3.2 and Corollary 2.3.4).

Remark 2.2.6. *1. Using 3, one can prove that for all $i \in I$ that $\mathrm{ad}(e_i)^{-a_{i,j}}(e_j) \neq 0$ and $\mathrm{ad}(f_i)^{-a_{i,j}}(f_j) \neq 0$ (in $\mathfrak{g}$). In particular, $e_i \neq 0$ and $f_i \neq 0$.*

2. If $k \in \mathbb{N}$ and $i_1, \ldots, i_k \in I$, $[e_{i_1}, \ldots, e_{i_k}] \in \mathfrak{g}_{\alpha_{i_1}+\ldots+\alpha_{i_k}}$ and $[f_{i_1}, \ldots, f_{i_k}] \in \mathfrak{g}_{-\alpha_{i_1}-\ldots-\alpha_{i_k}}$. Thus for all $\alpha \in Q_{\mathbb{N}}$ (resp. $-Q_{\mathbb{N}}$), $\mathfrak{g}_\alpha$ is the vector space spanned by the $[e_{i_1}, \ldots, e_{i_k}]$ (resp. $[f_{i_1}, \ldots, f_{i_k}]$) such that $i_1, \ldots, i_k \in I$ and $\alpha_{i_1}+\ldots+\alpha_{i_k} = \alpha$ (resp. $\alpha_{i_1}+\ldots+\alpha_{i_k} = -\alpha$).

2.2.2 Vectorial apartment of a Kac-Moody algebra

We now define the vectorial apartment of a root generating system $\mathcal{S}$. If $\mathcal{S}$ is associated to a matrix A, this apartment is a vector space $\mathbb{A}$(a realization of A over $\mathbb{R}$) equipped with a cone (the Tits cone), which is a geometric realization of the vectorial Weyl group of $\mathbb{A}$.

2.2.2.1 Weyl group, real and imaginary roots

Let $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee))$ be a root generating system and $\mathfrak{g}$ be the Kac-Moody algebra of $\mathcal{S}_\mathbb{C} = (Y \otimes \mathbb{C}, (\alpha_i)_{i\in I}, (\alpha_i^\vee))$. Let $\mathbb{A} = Y \otimes \mathbb{R}$. If $i \in I$, one defines the **fundamental reflection** r_i of the space $\mathbb{A}$ by $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$. Then $r_i(\alpha_i^\vee) = -\alpha_i^\vee$ and r_i fixes $\{x \in \mathbb{A} |\ \alpha_i(x) = 0\}$. The **Weyl group of $\mathfrak{g}$** is the subgroup of $\mathrm{GL}(\mathbb{A})$ spanned by the fundamental reflections. We denote it by W^v. By Proposition 1.3.21 of [Kum02], $(W^v, \{r_i|\ i \in I\})$ is a Coxeter system. The action of W^v on $\mathbb{A}$ induces an action of W^v on $\mathbb{A}^*$ by the formula $w.f(x) := f(w^{-1}.x)$ for all $f \in \mathbb{A}^*$, $x \in \mathfrak{h}$ and $w \in W^v$.

By Corollary 1.3.6 of [Kum02], the set of roots $\Phi_{all} \subset \bigoplus_{i\in I} \mathbb{Z}\alpha_i \subset \mathbb{A}^*$ is stable under the action of W^v. Moreover for all $\alpha \in \Phi_{all}$ and $w \in W^v$, $\dim \mathfrak{g}_\alpha = \dim \mathfrak{g}_{w.\alpha}$. A root is said to be **real** if it is of the form $w.\alpha_i$, for some $i \in I$ and $w \in W^v$. One denotes by Φ_{re} the set of **real roots**. Let $\Phi_{im} = \Phi_{all} \backslash \Phi_{re}$. An element of Φ_{im} is called an **imaginary root**. By Remark 2.2.6, $\mathfrak{g}_{\alpha_i} = \mathbb{C}e_i \neq \{0\}$ and thus $\dim \mathfrak{g}_\alpha = 1$ for all $\alpha \in \Phi_{re}$. By Remark 2.2.6, if $i \in I$ and $n \in \mathbb{C}$, $n\alpha_i \in \Phi_{all}$ if and only if $n \in \{-1, 1\}$. Thus we get that if $\alpha \in \Phi_{re}$ and $n \in \mathbb{C}$ such that $n\alpha \in \Phi_{all}$, then $n \in \{-1, 1\}$. On the contrary, by Proposition 5.4 of [Kac94], if $\alpha \in \Phi_{im}$ and $r \in \mathbb{Q}^*$ such that $r\alpha \in Q$, then $r\alpha \in \Phi_{im}$.

2.2.2.2 Vectorial apartment

Let $C_f^v := \{x \in \mathbb{A} | \alpha_i(x) > 0\ \forall i \in I\}$ be the **vectorial fundamental chamber**. For $J \subset I$, one sets $F^v(J) = \{v \in \mathbb{A} |\ \alpha_i(v) = 0\ \forall i \in J, \alpha_i(v) > 0\ \forall i \in J \backslash I\}$. Then the closure $\overline{C_f^v}$ of C_f^v is the union of the $F^v(J)$ for $J \subset I$. The **positive** (resp. **negative**) **vectorial faces** are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. A **vectorial face** is either a positive vectorial face or a negative vectorial face. We call **positive chamber** (resp. **negative**) every cone of the shape $w.C_f^v$ for some $w \in W^v$ (resp. $-w.C_f^v$). A vectorial face of codimension 1 is called a **vectorial panel**. The **Tits cone** $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{w\in W^v} w.\overline{C_f^v}$.

Proposition 2.2.7. *(Proposition 3.12 b,c of [Kac94])*

1. *The fundamental chamber $\overline{C_f^v}$ is a fundamental domain for the action of W^v on $\mathcal{T}$, i.e., any orbit $W^v.h$ of $h \in \mathcal{T}$ intersects $\overline{C_f^v}$ in exactly one point. In particular, the positive faces form a partition of $\mathcal{T}$.*

2. *One has $\mathcal{T} = \{h \in \mathbb{A} |\ \{\alpha \in \Phi_{all}^+ | \alpha(h) < 0\}$ is finite$\}$. In particular, $\mathcal{T}$ is a convex cone of $\mathbb{A}$.*

Tits preorder on $\mathbb{A}$ One defines the **Tits preorder** on $\mathbb{A}$ as follows. If x, y in $\mathcal{T}$, then one writes $x \leq y$ if $y - x \in \mathcal{T}$. This is indeed a preorder by Proposition 2.2.7. By definition, it is W^v invariant.

A vectorial face F^v is said to be **spherical** if its fixer in W^v is finite. Vectorial chambers and panels are examples of spherical vectorial faces.

Proposition 2.2.8. *(Théorème 5.2.3 (ii) of [Rém02]) Let $\mathring{\mathcal{T}}$ be the interior of $\mathcal{T}$. Then $\mathring{\mathcal{T}}$ is the union of the positive spherical vectorial faces of $\mathbb{A}$.*

A hyperplane of the form $\alpha^{-1}(\{0\})$ for some $\alpha \in \Phi_{re}$ is called a **vectorial wall** of $\mathbb{A}$.

The $Q^\vee$-orders in $\mathbb{A}$ If $E \subset \mathbb{R}$, one sets $Q^\vee_E = \bigoplus_{i\in I} E\alpha_i^\vee$. If $x, y \in \mathbb{A}$, one denotes $x \leq_{Q^\vee_{\mathbb{R}}} y$ (resp. $x \leq_{Q^\vee_{\mathbb{Z}}} y$) if $y - x \in Q^\vee_{\mathbb{R}_+}$ (resp. $y - x \in Q^\vee_{\mathbb{N}}$).

The following lemma is the writing of Proposition 3.12 d) of [Kac94] and Lemma 2.4 a of [GR14] in our context.

Lemma 2.2.9. *Let $\lambda \in \overline{C^v_f}$ and $w \in W^v$. Then $w.\lambda \leq_{Q^\vee_{\mathbb{R}}} \lambda$. If moreover $\lambda \in Y$, then $w.\lambda \leq_{Q^\vee_{\mathbb{Z}}} \lambda$.*

2.2.2.3 Essential apartment

Let $\mathbb{A}_{in} = \bigcap_{i\in I} \ker(\alpha_i)$. Let $\mathbb{A}_{es} = \mathbb{A}/\mathbb{A}_{in}$ be the **essentialization of** $\mathbb{A}$.

If X is an affine space, one denotes by $\mathrm{Aut}(X)$ its group of affine automorphisms.

Lemma 2.2.10. *Let $w \in W^v$. Then the map $\overline{w} : \mathbb{A}_{es} \to \mathbb{A}_{es}$ defined by $\overline{w}(x + \mathbb{A}_{in}) = w(x) + \mathbb{A}_{in}$ is well defined. Moreover, the morphism $\Gamma : W^v \to \mathrm{Aut}(\mathbb{A}_{es})$ sending each $v \in W^v$ on $\overline{v}$ is injective.*

Proof. Let $x \in \mathbb{A}$ and $x' \in \mathbb{A}_{in}$. Then $w(x + x') = w(x) + x'$, thus $w(x + \mathbb{A}_{in}) = w(x) + \mathbb{A}_{in}$ and hence $\overline{w}$ is well defined.

Let $v \in W^v \backslash \{1\}$. Let $x \in C^v_f$. As the action of W^v on the positive chambers is simply transitive, $w.x \notin C^v_f$. Therefore, there exists $i \in I$ such that $\alpha_i(w.x) < 0$. In particular $\alpha_i(v.x - x) < 0$ and thus $v.x - x \notin \mathbb{A}_{in}$. As a consequence, $\overline{v} \neq 1$ and Γ is injective, which is our assertion. □

2.2.3 Classification of indecomposable Kac-Moody matrices

We now recall the classification of indecomposable Kac-Moody matrices in three types: Cartan matrices (also called Kac-Moody matrix of finite type), affine Kac-Moody matrices and indefinite matrices. If $u \in \mathbb{R}^I$, one writes $u \geq 0$ (resp. $u > 0$, ...) if $u \in (\mathbb{R}_+)^I$ (resp $u \in (\mathbb{R}_+^*)^I$, ...). Suppose A indecomposable. Then by Theorem 4.3 of [Kac94] one of the following three mutually exclusive possibilities holds for A and ${}^tA$:

(Fin) $\det(A) \neq 0$; there exists $u > 0$ such that $Au > 0$ and $Av \geq 0$ implies $v > 0$ or $v = 0$.

(Aff) $\mathrm{corank}(A) = 1$; there exists $u > 0$ such that $Au = 0$ and $Av \geq 0$ implies $Av = 0$.

(Ind) there exists $u > 0$ such that $Au < 0$; $Av \geq 0$, $v \geq 0$ implies $v = 0$.

One says that A if of **finite type** or is a **Cartan matrix** if A satisfies (Fin). One says that A is of **affine type** resp. **indefinite type**) if A satisfies (Aff) (resp. (Ind)). As we shall see (Subsection 2.3.1), there is a correspondence between Cartan matrices and Kac-Moody matrices of affine type.

Proposition 2.2.11. *Suppose that A is indecomposable. The following conditions are equivalent:*

1. *The Kac-Moody matrix A is a Cartan matrix.*
2. *The vectorial Weyl group W^v is finite.*
3. *One has $\mathbb{A} = \mathcal{T}$.*
4. *The set of roots Φ_{all} is finite.*
5. *The set of real roots Φ_{re} is finite.*
6. *The set Φ_{im} is empty.*
7. *$\mathfrak{g}(A)$ is a finite dimensional Lie algebra.*

Proof. By Proposition 4.9 of [Kac94], 1, 4 and 2 are equivalent. By Proposition 5.8 of [Kac94], 3 is equivalent to 1. If Φ_{re} is infinite then $\Phi_{all} \supset \Phi_{re}$ is infinite. Suppose W^v infinite. Then by Lemma 1.3.14 of [Kum02], Φ_{re} is infinite. By Theorem 2.2.4 1 and 2 and Remark 2.2.6, 7 and 4 are equivalent. By Theorem 5.6 of [Kac94], 1 is equivalent to 6 and we get the proposition. □

Proposition 2.2.12. *Let $A = \begin{pmatrix} 2 & -a \\ -b & 2 \end{pmatrix}$, with $a, b \in \mathbb{N}^*$ be an indecomposable Kac-Moody matrix of size 2. Then:*

- *if $ab \leq 3$, then A is of finite type,*
- *if $ab = 4$, then A is of affine type,*
- *otherwise A is of indefinite type.*

Proof. The vectorial Weyl group W^v of A is spanned by the two simple reflexions r_1 and r_2. Therefore W^v is finite if and only if $r_1 r_2$ has finite order. By Proposition 1.3.21 of [Kum02], this happens if and only if $ab \leq 3$. By Proposition 2.2.11, we deduce that A is of finite type if and only if $ab \leq 3$.

Suppose that A is of affine type. Then $\det(A) = 0$, thus $ab = 4$. Reciprocally, if $a = b = 2$, $A \begin{pmatrix} 1 \\ 1 \end{pmatrix} = 0$, if $a = 1$ and $b = 4$, $A \begin{pmatrix} 1 \\ 2 \end{pmatrix} = 0$ and if $a = 4$, $b = 1$, $A \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 0$ and by Theorem 4.3 of [Kac94], A is of affine type. □

2.3 Vectorial apartments in the non-reductive case

In this section, we describe the vectorial apartment in the affine case and the indefinite case.

In 2.3.1, we recall the link between Kac-Moody algebras associated to affine Kac-Moody algebras and loop groups. We then describe the vectorial apartment of such algebras by using the affine apartment of the underlying semi-simple finite dimensional matrix.

In 2.3.2, we describe the vectorial apartment in the indefinite case. We give a more precise description in the two dimensional case.

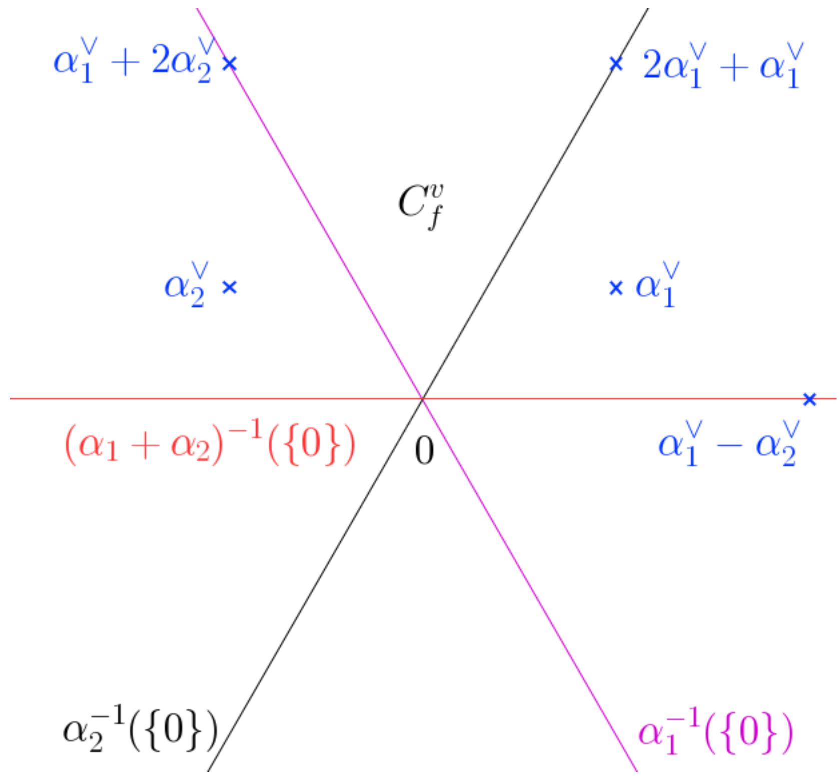

Figure 2.2.1 – Vectorial apartment of $\mathfrak{sl}_3$ (see Example 2.2.5)
I used Geogebra to build all the pictures of the thesis, except figure 3.1.2.

2.3.1 Affine case

2.3.1.1 Realizations of affine Kac-Moody algebras as loop algebras

Classification of finite dimensional semi-simple Lie algebras using Cartan matrices Let $\mathring{A} = (a_{i,j})_{1\leq i,j\leq l}$ be an indecomposable Cartan matrix. By Proposition 8 of V.11 of [Ser01], this determines a reduced root system uniquely up to isomorphism. By Theorem 8 and Theorem of the appendix of VI of [Ser01], there exists a unique semi-simple Lie algebra $\mathring{\mathfrak{g}}(\mathring{A})$ having $\mathring{A}$ as a Cartan matrix and this algebra is the Kac-Moody algebra $\mathring{\mathfrak{g}}(\mathring{A})$.

For this section, we mainly use Chapter XIII of [Kum02].

Underlying finite dimensional simple Lie algebra Let $\mathring{\mathcal{S}} = (\mathring{\mathfrak{h}}, (\alpha_i)_{i\in[\![1,\ell]\!]}, (\alpha_i^\vee)_{i\in[\![1,\ell]\!]})$ be the complex minimal free realization of $\mathring{A}$. Let $\mathring{\mathfrak{g}} = \mathring{\mathfrak{g}}(\mathring{\mathcal{S}})$. We denote by $\mathring{e}_i, \mathring{f}_i$, $i \in [\![1,\ell]\!]$ the Chevalley generators of $\mathring{\mathfrak{g}}$. Let $\mathring{\Phi} \subset \mathring{\mathfrak{h}}^*$ be the root system of $(\mathring{\mathfrak{g}}, \mathring{\mathcal{S}})$.

Let $\mathring{\mathfrak{h}}_\mathbb{Z} = \bigoplus_{i=1}^{\ell} \mathbb{Z}\alpha_i^\vee$ and $\mathring{\mathfrak{h}}_\mathbb{Z}^* = \mathrm{Hom}_\mathbb{Z}(\mathring{\mathfrak{h}}_\mathbb{Z}, \mathbb{Z})$. Let c, d be symbols and $\mathfrak{h}_\mathbb{Z} = \mathring{\mathfrak{h}}_\mathbb{Z} \oplus \mathbb{Z}c \oplus \mathbb{Z}d$. Let $\mathfrak{h} = \mathfrak{h}_\mathbb{Z} \otimes_\mathbb{Z} \mathbb{C}$ and $\mathfrak{h}^* = \mathfrak{h}_\mathbb{Z}^* \otimes_\mathbb{Z} \mathbb{C}$. One embeds $\mathring{\mathfrak{h}}_\mathbb{Z}^*$ in $\mathfrak{h}_\mathbb{Z}^*$ by requiring $\lambda(\mathbb{Z}c + \mathbb{Z}d) = \{0\}$ for all $\lambda \in \mathring{\mathfrak{h}}_\mathbb{Z}^*$. Let $\delta \in \mathfrak{h}_\mathbb{Z}$ defined by $\delta(d) = 1$ and $\delta(\mathring{\mathfrak{h}}_\mathbb{Z} \oplus \mathbb{Z}c) = 0$.

The system $\mathring{\mathcal{S}}^\vee := ({}^t\mathring{A}, \mathring{\mathfrak{h}}_\mathbb{Z}^*, \mathring{\mathfrak{h}}_\mathbb{Z}, (\alpha_i^\vee)_{i\in[\![1,\ell]\!]}, (\alpha_i)_{i\in[\![1,\ell]\!]})$ is a root generating system. By Proposition 25 (i) of VI of [Bou68], $\mathring{\Phi}^+$ admits a unique highest roots: this is a root $\theta = \sum_{i=1}^{\ell} a_i\alpha_i \in \mathring{\Phi}^+$ such that for all $\alpha = \sum_{i=1}^{\ell} n_i\alpha_i \in \mathring{\Phi}^+$, $n_i \leq a_i$. In particular $a_i \in \mathbb{N}^*$ for all $i \in [\![1,\ell]\!]$.

Let $\theta^\vee \in \mathring{\mathfrak{h}}_\mathbb{Z}^*$ be the highest root of $\mathring{\Phi}^{\vee+}$ (where $\mathring{\Phi}^\vee$ is the root space of $\mathring{\mathcal{S}}^\vee$). One sets $\alpha_0 = \delta - \theta$ and $\alpha_0^\vee = c - \theta^\vee$.

One sets $a_{0,0} = 2 = \theta(\theta^\vee)$ and for $j \in [\![1,\ell]\!]$, $a_{0,j} = -\alpha_j(\theta^\vee)$ and $a_{j,0} = -\theta(\alpha_j^\vee)$. Let $A = (a_{i,j})_{0\leq i,j\leq l}$.

Let $V = \mathbb{R} \otimes X$. By VI 1.1 of [Bou68], there exists a scalar product on V such that if one identifies V and V^* through this scalar product, $\alpha^\vee \in \mathbb{R}^*\alpha$ for all $\alpha \in \mathring{\Phi}$.

Lemma 2.3.1. *The matrix A is a Kac-Moody matrix of affine type.*

Let $I = [\![0, \ell]\!]$. The system $\mathcal{S} = (A, \mathfrak{h}_\mathbb{Z}^*, \mathfrak{h}_\mathbb{Z}, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ is a root generating system. The triple $\mathcal{S}_\mathbb{C} = (\mathfrak{h}, (\alpha_i)_{i \in [\![0,\ell]\!]}, (\alpha_i^\vee)_{i \in [\![0,\ell]\!]})$ is a realization of A.

Let $\mathfrak{g} = (\mathbb{C}[t, t^{-1}] \otimes_\mathbb{C} \mathring{\mathfrak{g}}) \oplus \mathbb{C}c \oplus \mathbb{C}d$, where t is an indeterminate. Let $\langle , \rangle$ be the invariant (symmetric nondegenerate) bilinear form on $\mathring{\mathfrak{g}}$ defined in Theorem 1.5.4 of [Kum02] (the properties of $\langle , \rangle$ are important to prove that $\mathfrak{g}$ is isomorphic to the Kac-Moody algebra of $\mathcal{S}$ but as we admit this result, we will not explicitly use them). One equips $\mathfrak{g}$ with the bracket:

$$[t^m \otimes x + \lambda c + \mu d, t^{m'} \otimes x' + \lambda' c + \mu' d] = t^{m+m'} \otimes [x, x'] + \mu m' t^{m'} \otimes x' - \mu' m t^m \otimes x + m\delta_{m,-m'}\langle x, x'\rangle c,$$

for $\lambda, \mu, \lambda', \mu' \in \mathbb{C}$, $m, m' \in \mathbb{Z}$, $x, x' \in \mathring{\mathfrak{g}}$.

Let $\Phi_{all} = \{j\delta | j \in \mathbb{Z}\backslash\{0\}\} \cup \{j\delta + \beta |\ j \in \mathbb{Z},\ \beta \in \mathring{\Phi}\}$. If $\alpha \in \mathfrak{h}^*$, one sets $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} |\ [h, x] = \alpha(h)x\ \forall h \in \mathfrak{h}\}$.

Lemma 2.3.2. *One has $\mathfrak{g} = \mathfrak{h} \oplus \bigoplus_{\alpha \in \Phi_{all}} \mathfrak{g}_\alpha$. Moreover if $j \in \mathbb{Z}\backslash\{0\}$, $\mathfrak{g}_{j\delta} = t^j \otimes \mathring{\mathfrak{h}}$ and if $j \in \mathbb{Z}$ and $\beta \in \mathring{\Phi}$, $\mathfrak{g}_{\beta + j\delta} = t^j \otimes \mathring{\mathfrak{g}}_\beta$, where $\mathring{\mathfrak{g}}_\beta = \{x \in \mathring{\mathfrak{g}} |\ [h, x] = \beta(h)x\ \forall h \in \mathring{\mathfrak{h}}\}$.*

Let $\mathring{\omega}$ be the Cartan involution of $\mathring{\mathfrak{g}}$. Let $x_0 \in \mathring{\mathfrak{g}}_\Theta$ such that $\langle x_0, \mathring{\omega}(x_0)\rangle = -1$. Let $E_0 = -t \otimes \mathring{\omega}(x_0)$ and $F_0 = -t \otimes x_0$.

Let $\mathfrak{g}(\mathcal{S}_\mathbb{C})$ be the Kac-Moody algebra with generators $(\mathfrak{h}, (e_i)_{i \in I}, (f_i)_{i \in I})$ associated to $\mathcal{S}_\mathbb{C} = (\mathfrak{h}, (\alpha_i)_{i \in I}, (\alpha_i^\vee)_{i \in I})$ in 1.1 of [Kum02].

Theorem 2.3.3. *(Theorem 1.3.1.3 of [Kum02]) There exists a unique Lie algebra isomorphism $\Upsilon : \mathfrak{g}(\mathcal{S}_\mathbb{C}) \xrightarrow{\sim} \mathfrak{g}$ fixing $\mathfrak{h}$ and mapping e_0 to E_0, e_i to $\mathring{e}_i$ and f_i to $\mathring{f}_i$ for all $i \in [\![1, \ell]\!]$.*

Corollary 2.3.4. *The set Φ_{all} is the set of roots of $\mathfrak{g}$. One has $\Phi_{all}^+ = \{j\delta + \beta | j \in \mathbb{N}^*, \beta \in \mathring{\Phi} \cup \{0\}\} \cup \mathring{\Phi}^+$ and $\Phi_{im} = (\mathbb{Z}\backslash\{0\})\delta$. Moreover, if $j \in \mathbb{Z}\backslash\{0\}$, $\dim \mathfrak{g}_{j\delta} = \ell$.*

2.3.1.2 Apartment of $\mathring{A}$ viewed as a subspace of $\mathbb{A}_{es}$

In this subsection, we show that we can see the affine apartment of $\mathring{A}$ as an affine hyperplane of the essentialization $\mathbb{A}_{es}$ of $\mathbb{A}$. This is inspired by Section 6 of [Kac94].

Let $\mathbb{A} = \mathfrak{h}_\mathbb{Z} \otimes \mathbb{R}$. Let $\mathbb{A}^1 = \{x \in \mathbb{A} |\ \delta(x) = 1\}$. Then $\mathbb{A}^1$ is an affine subspace of $\mathbb{A}$ whose direction contains $\mathbb{R}c = \mathbb{A}_{in}$.

One sets $\widetilde{\mathbb{A}} = \mathbb{A}^1/\mathbb{R}c \subset \mathbb{A}_{es}$. One can consider each element of Φ_{re} as a linear form on $\mathbb{A}_{es}$ and one considers W^v as a subgroup of $\mathrm{Aut}(\mathbb{A}_{es})$, which is possible by Lemma 2.2.10. Let $\mathring{\mathbb{A}} = \bigoplus_{i=1}^\ell \mathbb{R}\alpha_i^\vee \subset \mathbb{A}$. The map $\iota : \mathring{\mathbb{A}} \to \mathbb{A}_{es}$ defined by $\iota(\sum_{i=1}^\ell x_i\alpha_i^\vee) = \sum_{i=1}^\ell x_i\alpha_i^\vee + \mathbb{R}c$ is injective and we consider $\mathring{\mathbb{A}}$ as a subspace of $\mathbb{A}_{es}$ through this embedding.

Let $\psi : \mathring{\mathbb{A}} \to \widetilde{\mathbb{A}}$ defined by $\psi(x) = x + d$. Then ψ is an isomorphism of affine spaces.

One equips $\mathring{\mathbb{A}}$ with its structure of affine apartment. The walls of $\mathring{\mathbb{A}}$ are the $\mathring{\alpha}^{-1}(\{k\})$ such that $\mathring{\alpha} \in \mathring{\Phi}$ and $k \in \mathbb{Z}$. Let $\mathring{Q}_\mathbb{Z}^\vee = \bigoplus_{i=1}^\ell \mathbb{Z}\alpha_i^\vee$ and $\mathring{W} = \mathring{W}^v \ltimes \mathring{Q}_\mathbb{Z}^\vee$ be the affine Weyl group of $\mathring{\mathbb{A}}$. Let $\mathcal{M}_0 = \{\alpha^{-1}(\{0\}) |\ \alpha \in \Phi_{re}\}$ be the set of vectorial walls of $\mathbb{A}_{es}$ and $\widetilde{\mathcal{M}} = \{M \cap \widetilde{\mathbb{A}} |\ M \in \mathcal{M}\}$. Then by Corollary 2.3.4, if $\mathring{\mathcal{M}} := \{\mathring{\alpha}^{-1}(\{k\}) | (\mathring{\alpha}, k) \in \mathring{\Phi} \times \mathbb{Z}\}$ is the set of walls of $\mathring{\mathbb{A}}$, then $\mathring{\mathcal{M}} = \psi^{-1}(\widetilde{\mathcal{M}})$.

Let $\mathring{C}_f^v = \{x \in \mathring{\mathbb{A}} | \alpha_i(x) > 0,\ \forall i \in [\![1, \ell]\!]\}$ and $\mathring{C}_0^+ = \mathring{C}_f^v \cap \theta^{-1}(]0, 1[)$. By Proposition 5 of VI of [Bou68], $\overline{\mathring{C}_0^+}$ is a fundamental domain for the action of $\mathring{W}$ on $\mathring{\mathbb{A}}$ and is a connected component of $\mathring{\mathbb{A}} \backslash \bigcup_{M \in \mathcal{M}_0} M$.

Let $C^v_{f,e} \subset \mathbb{A}_{es}$ be the image of C^v_f in $\mathbb{A}_{es}$. Let $\mathcal{F}(C^v_{f,e})$ (resp. $\mathcal{F}(\mathring{C}^+_0)$) be the poset of faces of $C^v_{f,e}$ (resp. $\mathring{C}^+_0$).

Lemma 2.3.5. *The map $\Lambda : \mathcal{F}(C^v_{f,e}) \to \mathcal{F}(\mathring{C}^+_0)$ defined by $\Lambda(F^v) = \psi^{-1}\big(F^v \cap \widetilde{\mathbb{A}}\big)$ is well defined and is an isomorphism of poset.*

Proof. Let $F \in \mathcal{F}(C^v_{f,e})$. Let $J \subset [\![0,\ell]\!]$ such that $F = F^v_e(J)$, where $F^v_e(J)$ is the image of F^v in $\mathbb{A}_{es}$. Let $(R_i) \in \{=,>\}^{[\![0,l]\!]}$ sending each $i \in J$ on "=" and each $i \in [\![0,l]\!]\backslash J$ on ">". Then $F^v_e(J) = \{x \in \mathbb{A}_{es} | \big(\alpha_i(x)\ R_i\ 0\big),\ \forall i \in [\![0,\ell]\!]\}$.

Let $x \in \widetilde{\mathbb{A}}$. One writes $x = y+d = \psi(y)$, with $y \in \mathring{\mathbb{A}}$. Then for all $i \in [\![1,\ell]\!]$, $\alpha_i(x) = \alpha_i(y)$ and $\alpha_0(x) = \delta(y+d) - \theta(y+d) = 1 - \theta(y)$.

Therefore $x \in F^v_e(J)$ if and only if $\big(1-\theta(y) R_0 0\big)$ and $\big(\alpha_i(y) R_i 0\big)$ for all $i \in [\![0,\ell]\!]$ and we get the lemma because the faces of $\mathring{C}^+_0$ are the

$$\{y \in \mathring{\mathbb{A}} | \big(1-\theta(y)\ T_0\ 0\big) \text{ and } \big(\alpha_i(y)\ T_i\ 0\big),\ \forall i \in [\![0,\ell]\!]\},$$

such that $(T_i) \in \{>,=\}^{[\![0,\ell]\!]}$. □

As $\delta(Q^\vee_\mathbb{R}) = \{0\}$, $\delta \circ w = \delta$ for all $w \in W^v$. Therefore, W^v acts on $\widetilde{\mathbb{A}}$. We deduce an action of W^v on $\mathring{\mathbb{A}}$ through ψ. If $w \in W^v$, we denote by $\mathring{w} = \psi^{-1} \circ w \circ \psi : \mathring{\mathbb{A}} \to \mathring{\mathbb{A}}$ the corresponding affine automorphism.

Lemma 2.3.6. *1. Let $i \in [\![1,\ell]\!]$. Then for all $x \in \mathring{\mathbb{A}}$, $\mathring{r}_i(x) = x - \alpha_i(x)\alpha_i^\vee$.*

2. For all $x \in \mathring{\mathbb{A}}$, $\mathring{r}_0(x) = x - (\theta(x)-1)\theta^\vee$.

Proof. Let $i \in [\![1,\ell]\!]$ and $x \in \mathring{\mathbb{A}}$. Then

$$\begin{aligned} r_i(\psi(x)) &= r_i(x+d) \\ &= x + d - \alpha_i(x+d)\alpha_i^\vee \\ &= x - \alpha_i(x)\alpha_i^\vee + d \\ &= \psi(x - \alpha_i(x)\alpha_i^\vee), \end{aligned}$$

and thus $\mathring{r}_i(x) = x - \alpha_i(x)\alpha_i^\vee$.

One has

$$\begin{aligned} r_0(\psi(x)) &= r_0(x+d) = x + d - \alpha_0(x+d)\alpha_0^\vee \\ &= x + d - (\delta-\theta)(x+d)(c-\theta^\vee) \\ &= x + d - (1-\theta(x))(c-\theta^\vee) \\ &= \psi(x - (\theta(x)-1)\theta^\vee), \end{aligned}$$

because $c = 0$ in $\mathbb{A}_{es}$ and thus $\mathring{r}_0(x) = x - (\theta(x)-1)\theta^\vee$. □

Let $\mathcal{F}^+(\mathbb{A}_{es})$ (resp. $\mathcal{F}(\mathring{\mathbb{A}})$) be the simplicial complex of positive vectorial faces of $\mathbb{A}_{es}$ (resp. of faces of $\mathring{\mathbb{A}}$).

Proposition 2.3.7. *1. The morphism $\Xi : W^v \to \mathrm{Aut}(\mathring{\mathbb{A}})$ mapping each $w \in W^v$ to $\mathring{w}$ is injective and its image is $\mathring{W} = \mathring{W}^v \ltimes \mathring{Q}^\vee_\mathbb{Z}$.*

2. The map $\Lambda : \mathcal{F}^+(\mathbb{A}_{es}) \to \mathcal{F}(\mathring{\mathbb{A}})$ sending each F^v on $\psi^{-1}(F^v \cap \widetilde{\mathbb{A}})$ is well defined and is an isomorphism compatible with the actions of W^v and $\mathring{W}$: for all $w \in W^v$ and $F^v \in \mathcal{F}^+(\mathbb{A})$ and $w \in W^v$, $\Lambda(w.F^v) = \mathring{w}.\Lambda(F^v)$.

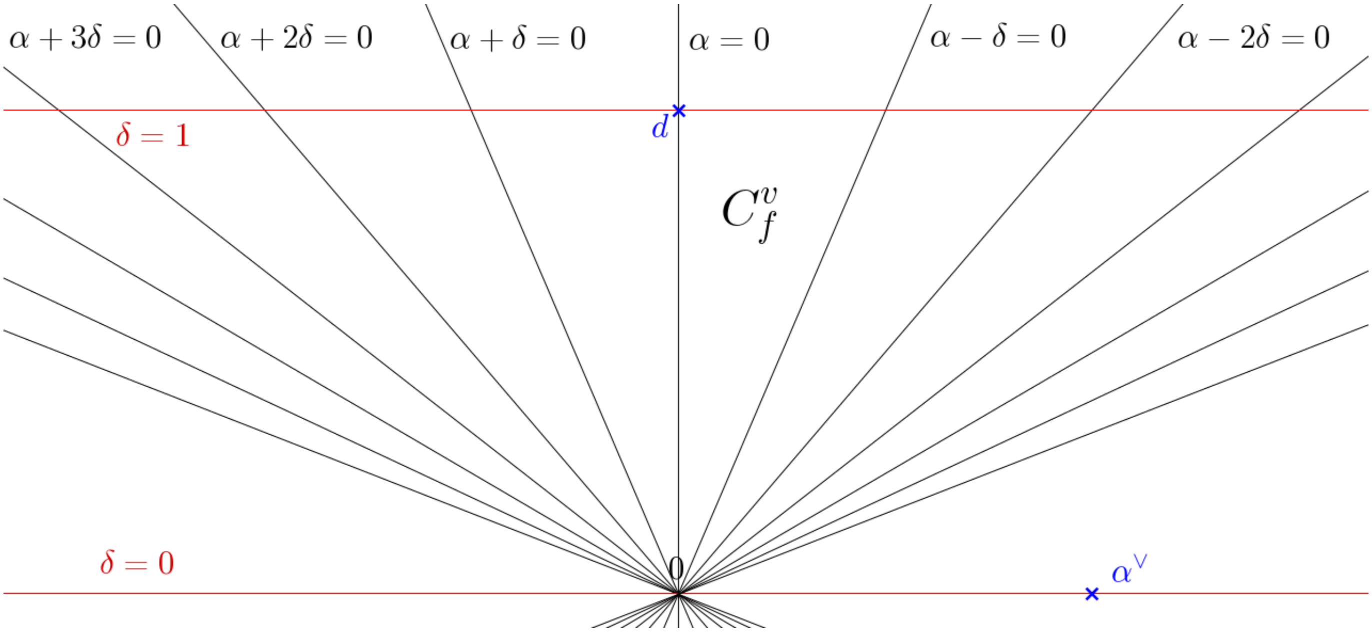

Figure 2.3.1 – Vectorial apartment of $\widetilde{\mathrm{SL}_2}$ with thirteen walls

Proof. Let us first prove that Ξ is injective. Let $w \in W^v \setminus \{1\}$. Let $x \in \mathbb{A}_{es}$ such that $w(x) \neq x$. By continuity of w, one can suppose that $\delta(x) \neq 0$. Then $x' = \frac{1}{\delta(x)} x \in \widetilde{\mathbb{A}}$ and $w(x') \neq x'$. Therefore $w_{|\widetilde{\mathbb{A}}} \neq \mathrm{Id}$ and thus $\mathring{w} \neq \mathrm{Id}$. Consequently, Ξ is injective. As $W^v = \langle r_i | i \in I \rangle$, $\Xi(W^v) = \langle \mathring{r}_i |\ i \in I \rangle$. By Lemma 2.3.6, Theorem 1 (i) of V.3.2 of [Bou68] and VI.2.1 of [Bou68], $\Xi(W^v) = \mathring{W}$.

Let $w \in W^v$ and $x \in \widetilde{\mathbb{A}}$. Then by definition, $\psi^{-1}(w.x) = \mathring{w}.\psi^{-1}(x)$. Therefore if $F^v \in \mathcal{F}^+(\mathbb{A}_{es})$, $\psi^{-1}(w.F^v \cap \widetilde{\mathbb{A}}) = \mathring{w}.\psi^{-1}(F^v \cap \widetilde{\mathbb{A}})$. Using Lemma 2.3.5 and 1, we deduce that Λ is well defined and is an isomorphism (because $\overline{\mathring{C}_0^+}$ is a fundamental domain for the action of $\mathring{W}$ on $\mathring{\mathbb{A}}$). The proposition follows. □

Corollary 2.3.8. *The Tits cone $\mathcal{T}$ of $\mathbb{A}$ is $\delta^{-1}(\mathbb{R}_+^*) \cup \mathbb{A}_{in}$. In particular, $\overline{\mathcal{T}} = \delta^{-1}(\mathbb{R}_+)$.*

Proof. Let $\mathcal{T}_e$ be the image of $\mathcal{T}$ in $\mathbb{A}_{es}$. Let $x \in \mathcal{T}_e$. Then there exists $w \in W^v$ such that $w.x \in \overline{C_{f,e}^v}$. One has $\delta(x) = \delta(w.x) = \theta(w.x) + \alpha_0(w.x) \geq 0$. As $\theta \in \bigoplus_{i=1}^{\ell} \mathbb{N}^* \alpha_i$, if $\delta(w.x) = 0$, then $\alpha_i(w.x) = 0$ for all $i \in [\![0, \ell]\!]$ and thus $w.x = 0$. Therefore $\mathcal{T}_e \subset \{0\} \cup \{x \in \mathbb{A}_{es} |\ \delta(x) > 0\}$.

One has $0 \in \mathcal{T}_e$. By Proposition 2.3.7 2, $\widetilde{\mathbb{A}} \subset \mathcal{T}_e$. Moreover $\mathcal{T}_e$ is a convex cone and thus $\mathcal{T}_e \supset \{x \in \mathbb{A}_{es} | \delta(x) > 0\} \cup \{0\}$. Consequently, $\mathcal{T}_e = \{x \in \mathbb{A}_{es} |\ \delta(x) > 0\} \cup \{0\}$ and hence $\mathcal{T} = \{x \in \mathbb{A} |\ \delta(x) > 0\} \cup \mathbb{A}_{in}$, which is our assertion.

□

2.3.2 Vectorial apartment in the indefinite case

We now study the vectorial apartment in the indefinite case. Let $\mathcal{S}$ be a root generating system associated to an indefinite Kac-Moody matrix. We keep the same notation as above in 2.2.

Proposition 2.3.9. *Suppose that A is an indefinite Kac-Moody matrix. Then:*

1. $\overline{\mathcal{T}} = \{x \in \mathbb{A} |\ \alpha(x) \geq 0,\ \forall \alpha \in \Phi_{im}^+\}$*, and in particular,* $\mathring{\mathcal{T}} \subset \{x \in \mathbb{A} |\ \alpha(x) > 0,\ \forall \alpha \in \Phi_{im}^+\}$*,*

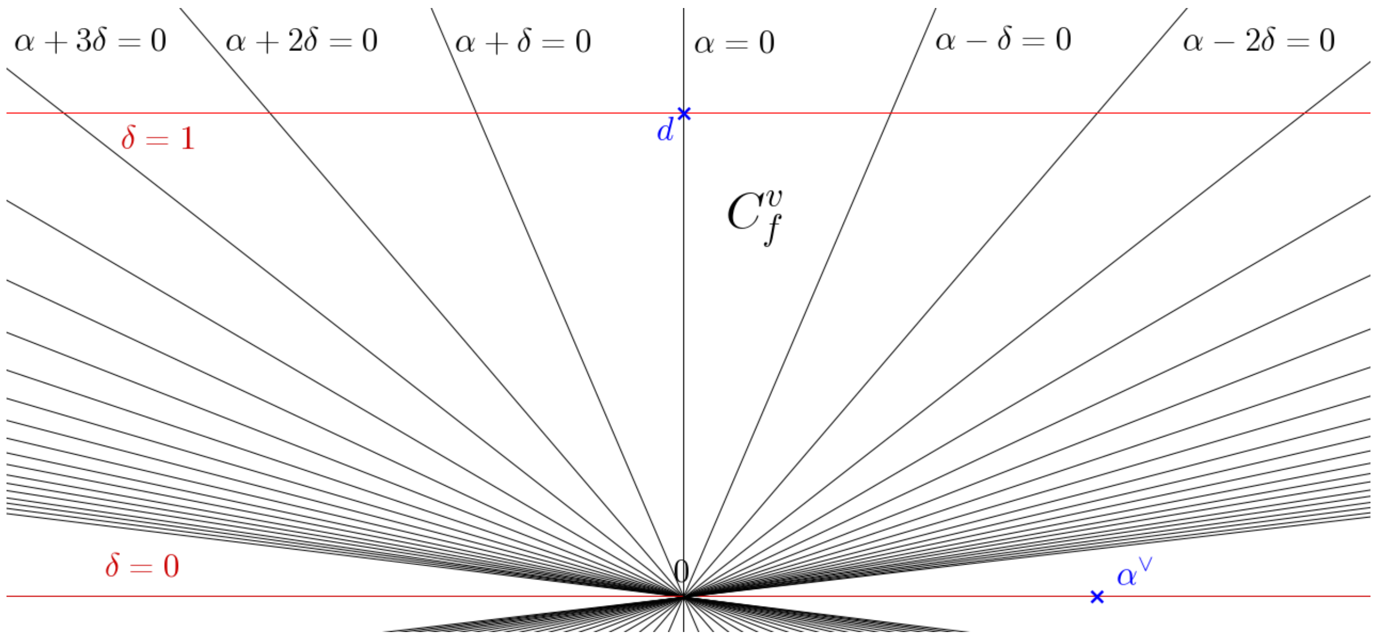

Figure 2.3.2 – Vectorial apartment of $\widetilde{\mathrm{SL}_2}$ with forty-one walls

2. *for all $i\in I$, $\alpha_i^\vee\in\mathbb{A}\backslash\overline{T}\cup\overline{-\mathcal{T}}$,*

3. *$\overline{\mathcal{T}}$ is not a half-space of $\mathbb{A}$,*

4. *for all $\alpha\in\Phi^+_{im}$, $(\mathbb{Z}\backslash\{0\})\alpha\subsetneq\Phi^+_{im}$.*

Proof. Point 1 is Proposition 5.8 c) of [Kac94]. By Theorem 5.6 c) of [Kac94], there exists $\delta\in\Phi^+_{im}$ such that $\delta(\alpha_i^\vee)<0$ for all $i\in I$. Let $i\in I$. By 1, $\alpha_i^\vee\in\mathbb{A}\backslash\overline{\mathcal{T}}$. As $\overline{\mathcal{T}}$ is W^v-invariant, $-\alpha_i^\vee=r_i(\alpha_i^\vee)\in\mathbb{A}\backslash\overline{\mathcal{T}}$ and thus $\alpha_i^\vee\in\mathbb{A}\backslash\overline{\mathcal{T}}\cup\overline{-\mathcal{T}}$.

Suppose that $\overline{\mathcal{T}}$ is a half-space of $\mathbb{A}$. Then by 1, $\overline{\mathcal{T}}=\alpha^{-1}(\mathbb{R}_+)$ for all $\alpha\in\Phi^+_{im}$. In particular, $\overline{\mathcal{T}}=\delta^{-1}(\mathbb{R}_+)$, where δ is as above. But then one would have $-\alpha_i^\vee\in\overline{\mathcal{T}}$, a contradiction with 2, and 3 follows. Point 4 is a consequence of 1 and 3. □

We now study the case where A is of size 2. One writes $A=A(a,b)=\begin{pmatrix}2&-a\\-b&2\end{pmatrix}$ with $a,b\in\mathbb{N}^*$. One has $ab\geq 5$ by Proposition 2.2.12. In particular, A is invertible and the minimal free realization $\mathbb{A}$ of A over $\mathbb{R}$ is of dimension 2. The vectorial faces of $\mathbb{A}$ are $\{0\}$, the vectorial panels and the vectorial chambers. Except $\{0\}$ these faces are spherical and by Proposition 2.2.8, $\mathring{\mathcal{T}}=\mathcal{T}\backslash\{0\}$.

Proposition 2.3.10. *(see Figure 2.3.3 and Figure 2.3.4) The set $\mathbb{A}\backslash(\mathcal{T}\cup-\mathcal{T})$ has two connected components. We denote them by Γ_1 and Γ_2. One has $\Gamma_1=-\Gamma_2$. If $i\in\{1,2\}$, then Γ_i is a convex cone, has nonempty interior, does not meet any vectorial wall of $\mathbb{A}$ and $\Gamma_i\cup\{0\}$ is closed.*

Proof. One identifies $\mathbb{A}$ and $\mathbb{C}$ as vector spaces over $\mathbb{R}$. Let $\mathbb{U}=\{x\in\mathbb{A}|\ |x|=1\}$. As $\mathbb{A}\backslash(\mathcal{T}\cup-\mathcal{T})$ is a cone, it suffices to study $\mathbb{U}\cap(\mathbb{A}\backslash(\mathcal{T}\cup-\mathcal{T}))$. Let $\chi:\mathbb{A}\backslash\{0\}\to\mathbb{U}$ defined by $\chi(x)=\frac{x}{|x|}$ for all $x\in\mathbb{A}\backslash\{0\}$. Let $x_0\in\mathcal{T}\cap\mathbb{U}$. One writes $x_0=e^{i\theta_0}$, with $\theta_0\in\mathbb{R}$. Let $f:[0,2\pi[\to\mathbb{U}$ defined by $f(\theta)=e^{i(\theta+\theta_0)}$. Then $\widetilde{f}=f^{|\mathbb{U}\backslash x_0}_{|]0,2\pi[}$ is a homeomorphism.

One has $f^{-1}(\mathring{\mathcal{T}})=f^{-1}(\mathring{\mathcal{T}}\backslash\mathbb{R}^*_+x_0)\cup\{0\}=\widetilde{f}^{-1}(\mathring{\mathcal{T}}\backslash\mathbb{R}^*_+x_0)\cup\{0\}$. As $\mathring{\mathcal{T}}$ is a convex cone which does not contain 0, $\mathring{\mathcal{T}}\backslash\mathbb{R}^*_+x_0$ has two connected components and thus there exists

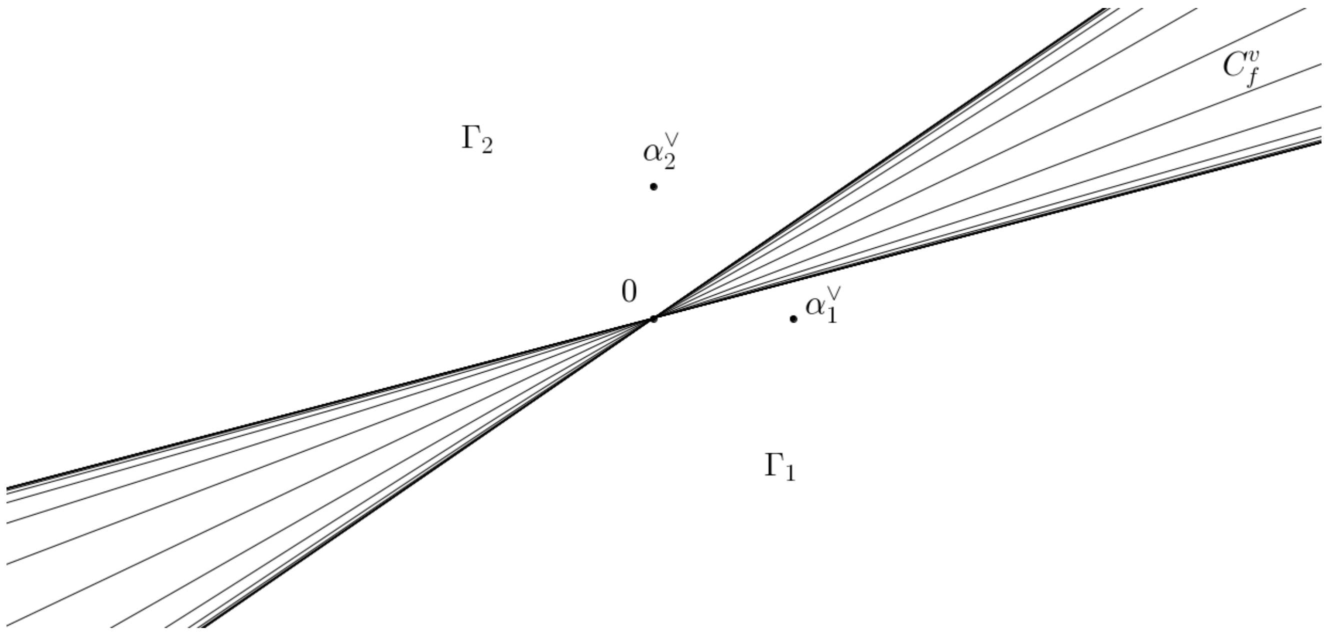

Figure 2.3.3 – Vectorial apartment of A(1,5)

$0 < t_1 < t_2 < 2\pi$ such that $f^{-1}(\mathring{\mathcal{T}}) = [0, t_1[\cup]t_2, 2\pi[$. By Proposition 2.3.9 1, $\mathring{\mathcal{T}} \cap -\mathring{\mathcal{T}} = \emptyset$. Therefore $f^{-1}(-\mathring{\mathcal{T}}) = \widetilde{f}^{-1}(-\mathring{\mathcal{T}})$ and there exist $t_3, t_4 \in [0, 2\pi]$ satisfying $t_1 < t_3 < t_4 < t_2$ and such that $\widetilde{f}^{-1}(-\mathring{\mathcal{T}}) =]t_3, t_4[$. Therefore $\widetilde{f}^{-1}(\mathbb{A}\backslash\mathring{\mathcal{T}} \cup -\mathring{\mathcal{T}}) = \widetilde{f}^{-1}(\mathbb{A}\backslash(\mathcal{T} \cup -\mathcal{T}) =]t_1, t_3[\cup]t_4, t_2[$ and thus $\mathbb{A}\backslash(\mathring{\mathcal{T}} \cup -\mathring{\mathcal{T}} \cup \{0\}) = \mathbb{R}_+^* \widetilde{f}([t_1, t_3] \cup [t_4, t_2])$. As a consequence, $\mathbb{A}\backslash(\mathcal{T} \cup -\mathcal{T})$ has two connected components Γ_1 and Γ_2, they have nonempty interior and they are convex cones.

Let $i \in \{1, 2\}$. Then $\Gamma_i \cup \{0\}$ is either $\mathbb{R}_+ \widetilde{f}([t_1, t_3])$ or $\mathbb{R}_+ \widetilde{f}([t_4, t_2])$ and thus it is closed.

As $\mathbb{A}\backslash(\mathcal{T} \cup -\mathcal{T}) = -(\mathbb{A}\backslash(\mathcal{T} \cup -\mathcal{T}))$, $-\Gamma_1$ is a connected component of $\mathbb{A}\backslash(\mathcal{T} \cup -\mathcal{T})$. Therefore $-\Gamma_1 \in \{\Gamma_1, \Gamma_2\}$. The fact that Γ_1 has nonempty interior implies that $\text{conv}(\Gamma_1, -\Gamma_1)$ contains 0 in its interior and thus $\text{conv}(\Gamma_1, -\Gamma_1) = \mathbb{A}$. Therefore $-\Gamma_1 = \Gamma_2$.

Let M be a vectorial wall. There exists $w \in W^v$ such that $M' = w.M$ is a wall of $\overline{C}_f^v$, thus $M' \cap \mathring{\mathcal{T}}$ is nonempty and hence $M \cap \mathring{\mathcal{T}}$ is nonempty. By symmetry, $M \cap -\mathring{\mathcal{T}}$ is nonempty and consequently, $M = (M \cap \mathring{\mathcal{T}}) \cup \{0\} \cup (M \cap -\mathring{\mathcal{T}}) \subset \mathcal{T} \cup -\mathcal{T}$, which completes the proof of the proposition. $\square$

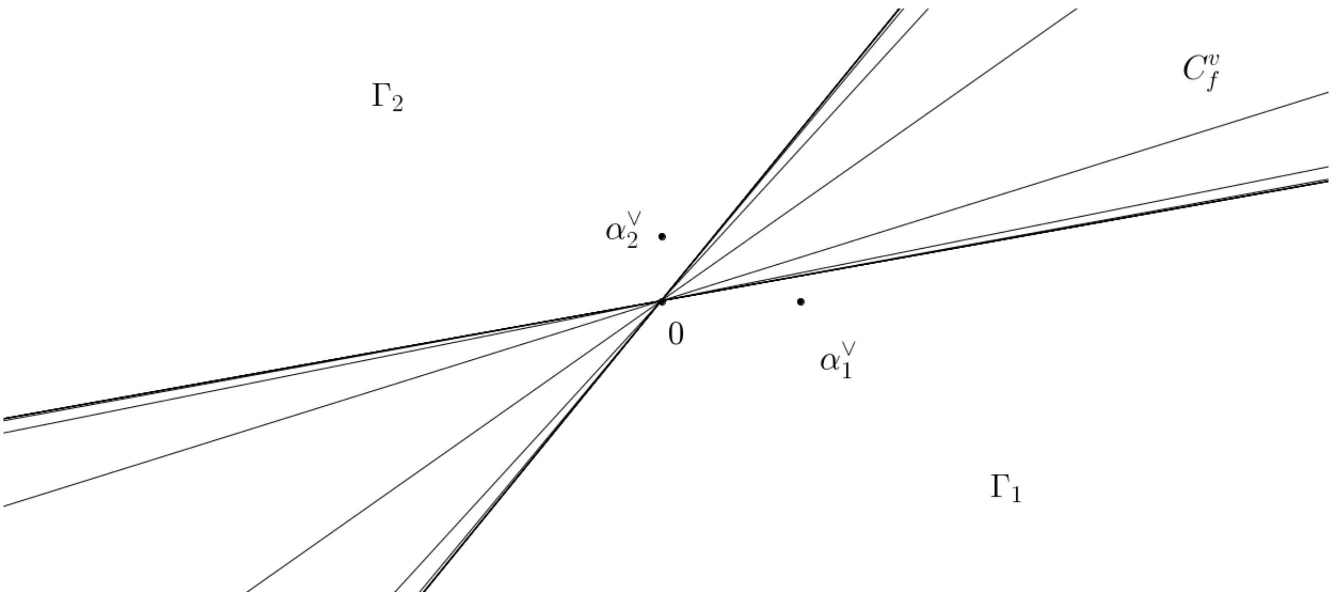

Figure 2.3.4 – Vectorial apartment of A(3,3)

Chapter 3

Masures and split Kac-Moody groups

Organization of the chapter In this chapter, we define masures and split Kac-Moody groups in the definition of [Tit87]. We begin by defining masures as abstract objects. They are some sets covered with subsets called apartments and satisfying some series of axioms and are not necessarily associated to a group. We then briefly define Kac-Moody groups, outline the construction of the associated masure and study the dictionary between notions in groups and notions in the associated masure.

In Subsection 3.1, we define affine apartments and give the axiomatic definition of masures of Rousseau.

In Subsection 3.2, we recall some notions on masures that will be useful for the sequel.

In Subsection 3.3, we briefly define minimal split Kac-Moody groups "à la Tits" and sketch the construction of the associated masure.

In Subsection 3.4, we study the action of a Kac-Moody group on its masure.

3.1 Affine apartment and abstract masure

Our main references for this subsection are [Rou11] and [Rou17]. Let A be a Kac-Moody matrix and $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ be a root generating system. Let $\mathbb{A} = Y \otimes \mathbb{R}$.

3.1.1 Affine Weyl group of $\mathbb{A}$

We now define the Weyl group W of $\mathbb{A}$. If X is an affine subspace of $\mathbb{A}$, one denotes by $\vec{X}$ its direction. We give the definition in the split case and in the general (almost-split) case. The reader only interested in the split case can skip the paragraph "general case".

Case of a masure associated to a split Kac-Moody group over a valued field Let $\mathcal{K}$ be a field with a non-trivial valuation $\omega : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$. Let $\Lambda = \omega(\mathcal{K}^*)$ (one can assume $\Lambda = \mathbb{Z}$ when $\mathcal{K}$ is local). Let $\mathcal{M} = \{\alpha^{-1}(\{k\}) | (\alpha, k) \in \Phi_{re} \times \Lambda\}$. The elements of $\mathcal{M}$ are called **walls** of $\mathbb{A}$. Let $Q^\vee = \bigoplus_{i\in I} \Lambda\alpha_i^\vee$. When $\Lambda = \mathbb{Z}$, $Q^\vee$ is the coroot lattice of $\mathbb{A}$. Let $W := W^v \ltimes Q^\vee$ be the **affine Weyl group** of $\mathbb{A}$. One sets $\Lambda_\alpha = \Lambda$ for all $\alpha \in \Phi_{re}$.

General case If X is an affine subspace of $\mathbb{A}$, one denotes by $\vec{X}$ its direction. One equips $\mathbb{A}$ with a family $\mathcal{M}$ of affine hyperplanes called **walls** such that:

1. For all $M \in \mathcal{M}$, there exists $\alpha_M \in \Phi_{re}$ such that $\vec{M} = \ker(\alpha_M)$.

2. For all $\alpha \in \Phi_{re}$, there exists an infinite number of hyperplanes $M \in \mathcal{M}$ such that $\alpha = \alpha_M$.

3. If $M \in \mathcal{M}$, we denote by r_M the reflexion of hyperplane M whose associated linear map is r_{α_M}. We assume that the group W generated by the r_M's for $M \in \mathcal{M}$ stabilizes $\mathcal{M}$.

The group W is the **affine Weyl group** of $\mathbb{A}$. A point x is said to be **special** if every real wall is parallel to a real wall containing x. We suppose that 0 is special and thus $W \supset W^v$.

If $\alpha \in \mathbb{A}^*$ and $k \in \mathbb{R}$, one sets $M(\alpha, k) = \{v \in \mathbb{A} | \alpha(v) + k = 0\}$. Then for all $M \in \mathcal{M}$, there exists $\alpha \in \Phi_{re}$ and $k_M \in \mathbb{R}$ such that $M = M(\alpha, k_M)$. If $\alpha \in \Phi_{re}$, one sets $\Lambda_\alpha = \{k_M | \ M \in \mathcal{M} \text{ and } \vec{M} = \ker(\alpha)\}$. Then $\Lambda_{w.\alpha} = \Lambda_\alpha$ for all $w \in W^v$ and $\alpha \in \Phi_{re}$.

If $\alpha \in \Phi_{re}$, one denotes by $\widetilde{\Lambda}_\alpha$ the subgroup of $\mathbb{R}$ generated by Λ_α. By 3, $\Lambda_\alpha = \Lambda_\alpha + 2\widetilde{\Lambda}_\alpha$ for all $\alpha \in \Phi_{re}$. In particular, $\Lambda_\alpha = -\Lambda_\alpha$ and when Λ_α is discrete, $\widetilde{\Lambda}_\alpha = \Lambda_\alpha$ is isomorphic to $\mathbb{Z}$.

One sets $Q^\vee = \bigoplus_{\alpha \in \Phi} \widetilde{\Lambda}_\alpha \alpha^\vee$. This is a subgroup of $\mathbb{A}$ stable under the action of W^v. Then one has $W = W^v \ltimes Q^\vee$.

3.1.2 Filters

For the definition of faces of a masure, one uses the notion of filters. Let us motivate this utilization. Suppose that $\mathbb{A}$ is associated to a split reductive group over a local field. Then one has $\mathcal{M} = \{\alpha^{-1}(\{k\}) | (\alpha, k) \in \Phi_{re} \times \mathbb{Z}\}$. This arrangement of hyperplanes is locally finite: for all bounded set E of $\mathbb{A}$, E meets a finite number of walls. The alcoves of $\mathbb{A}$ are then the connected components of $\mathbb{A} \backslash \bigcup_{M \in \mathcal{M}} M$. The following proposition shows that the situation is completely different when Φ_{re} is infinite, even when $\Lambda_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi_{re}$ and motivates the use of filters in the definitions of faces, enclosure maps, ... They are already used in [BT72] in the case when Λ_α is not discrete for some $\alpha \in \Phi_{re}$.

Proposition 3.1.1. *(see Figure 3.1.2) Suppose that Φ_{re} is infinite. Let $x \in \mathbb{A}$ and $y \in x + C^v_f$. Then there exists $(\beta_n) \in \Phi^{\mathbb{N}}$ such that $\beta_n(y) - \beta_n(x) \to +\infty$. As a consequence, all set having nonempty interior meets an infinite number of walls and the connected components of $\mathbb{A} \backslash \bigcup_{\alpha \in \Phi_{re}, k \in \mathbb{Z}} \alpha^{-1}(\{k\})$ have empty interior.*

Proof. Let $(\beta_n) \in (\Phi^+)^{\mathbb{N}}$ be an injective sequence. For all $n \in \mathbb{N}$, one writes $\beta_n = \sum_{i \in I} \lambda_{i,n} \alpha_i$ with $(\lambda_{i,n}) \in \mathbb{N}^I$. By injectivity of (β_n), one can suppose, extracting a sequence if necessary, that for some $i \in I$, $\lambda_{i,n} \to +\infty$. If $n \in \mathbb{N}$, $\beta_n(y) - \beta_n(x) \geq \lambda_{i,n} \alpha_i(y - x) \to +\infty$, which proves the proposition. □

A **filter** in a set E is a nonempty set F of nonempty subsets of E such that, for all subsets S, S' of E, if S, $S' \in F$ then $S \cap S' \in F$ and, if $S' \subset S$, with $S' \in F$ then $S \in F$.

If F is a filter in a set E, and E' is a subset of E, one says that F contains E' if every element of F contains E'. If E' is nonempty, the set $F_{E'}$ of subsets of E containing E' is a filter. By abuse of language, we will sometimes say that E' is a filter by identifying $F_{E'}$ and E'. If F is a filter in E, its closure $\overline{F}$ (resp. its convex envelope) is the filter of subsets of E containing the closure (resp. the convex envelope) of some element of F. A filter F is said to be contained in an other filter F': $F \subset F'$ (resp. in a subset Z in E: $F \subset Z$) if and only if any set in F' (resp. if Z) is in F.

If $x \in \mathbb{A}$ and Ω is a subset of $\mathbb{A}$ containing x in its closure, then the **germ** of Ω in x is the filter $germ_x(\Omega)$ of subsets of $\mathbb{A}$ containing a neighborhood of x in Ω.

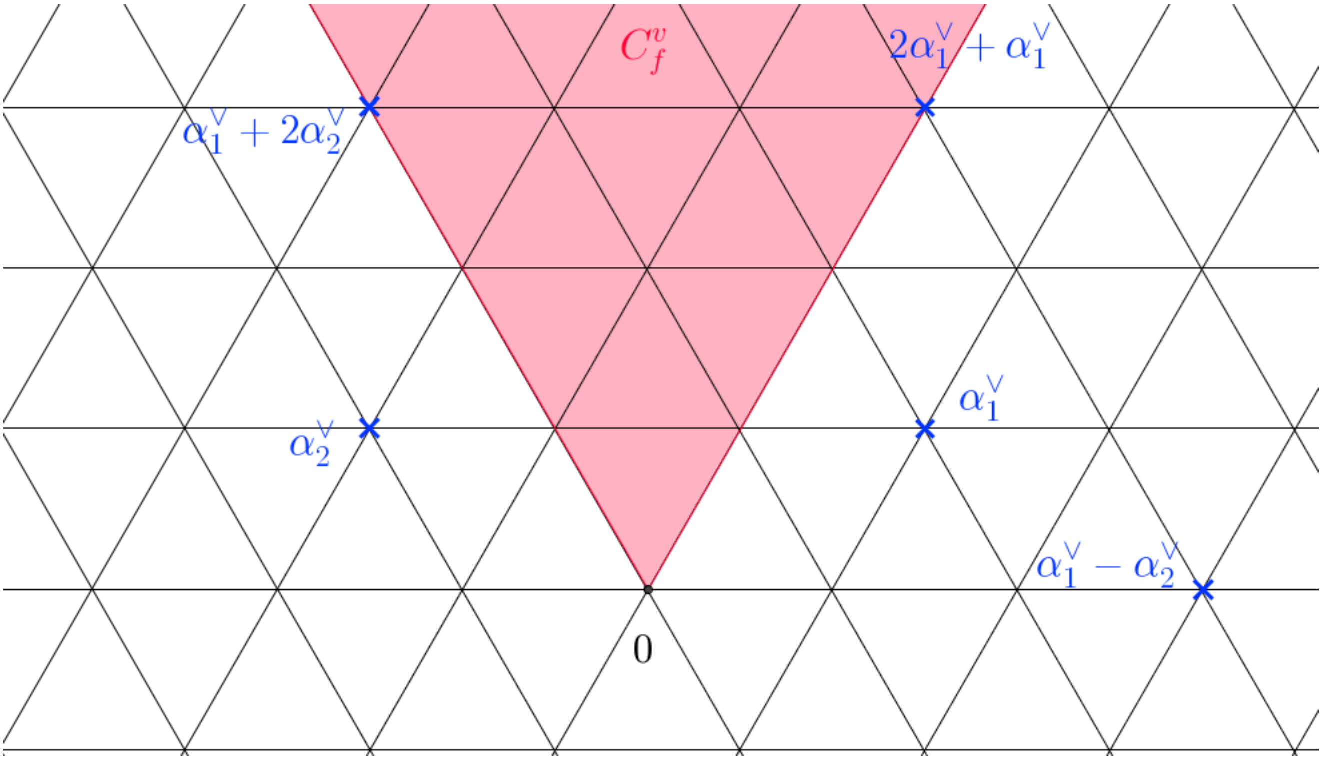

Figure 3.1.1 – Affine apartment of $\mathfrak{sl}_3$.

A **sector** in $\mathbb{A}$ is a set of the shape $\mathfrak{s} = x + C^v$ with $C^v = \pm w.C_f^v$ for some $x \in \mathbb{A}$ and $w \in W^v$. A point u such that $\mathfrak{s} = u + C^v$ is called a **base point of $\mathfrak{s}$** and C^v is its **direction**. The intersection of two sectors of the same direction is a sector of the same direction.

The **sector-germ** of a sector $\mathfrak{s} = x + C^v$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ containing an $\mathbb{A}$-translate of $\mathfrak{s}$. It only depends on the direction C^v. We denote by $+\infty$ (resp. $-\infty$) the sector-germ of C_f^v (resp. of $-C_f^v$).

A ray δ with base point x and containing $y \neq x$ (or the interval $]x, y] = [x, y]\backslash\{x\}$ or $[x, y]$ or the line containing x and y) is called **preordered** if $x \leq y$ or $y \leq x$ (see 2.2.2.2 for the definition of the Tits preorder $\leq$) and **generic** if $y - x \in \pm\mathring{\mathcal{T}}$, the interior of $\pm\mathcal{T}$.

3.1.3 Enclosure maps and affine apartment

In this subsection, we introduce the enclosure maps. This will enable us to define the faces, chimneys, ... When $\mathbb{A}$ is associated to a reductive group over a local field, the enclosure of a set E is $\text{cl}(E) = \bigcap_{\alpha\in\Phi} D(\alpha, k_\alpha)$, with $k_\alpha = \min\{k \in \mathbb{Z} \cup +\infty | D(\alpha, k) \supset E\}$. When $\mathbb{A}$ is associated to a split Kac-Moody group over a valued field, the enclosure map is defined using filters because the arrangement of walls can be dense in $\mathbb{A}$ (see Proposition 3.1.1).

There are many natural definitions of enclosures: one can consider an enclosure involving only real roots or involving every roots, one can consider only finite intersections or authorize infinite intersections. This explains that we define many enclosure maps below. Eventually (see Theorem 4.4.1), we will see that one of these enclosures is more adapted to the study of masures. In the split case, one defines only two enclosure maps.

Let $\Phi_{all} = \Phi_{re} \cup \Phi_{im}^+ \cup \Phi_{im}^-$ be the set of all roots. For $\alpha \in \Phi_{all}$, and $k \in \mathbb{R} \cup \{+\infty\}$, let $D(\alpha, k) = \{v \in \mathbb{A} | \alpha(v) + k \geq 0\}$ (and $D(\alpha, +\infty) = \mathbb{A}$ for all $\alpha \in \Phi_{all}$) and $D^\circ(\alpha, k) = \{v \in \mathbb{A} |\ \alpha(v) + k > 0\}$ (for $\alpha \in \Phi_{all}$ and $k \in \mathbb{R} \cup \{+\infty\}$).

If X is a set, one denotes by $\mathscr{P}(X)$ the set of subsets of X. Let $\mathscr{F}(\mathbb{A})$ be the set of filters

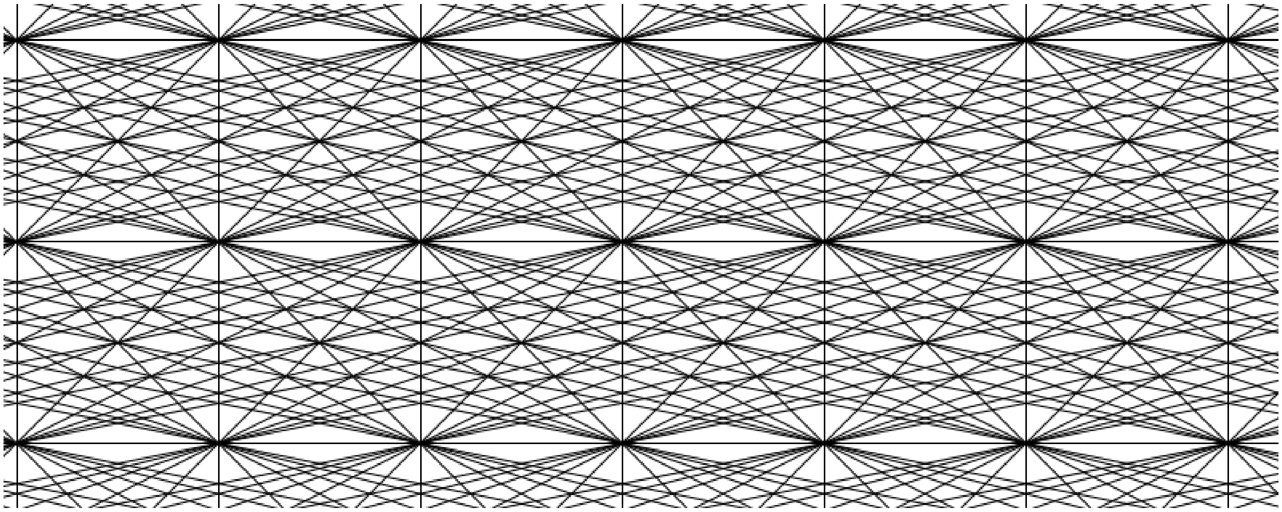

Figure 3.1.2 – Affine apartment of affine SL_2 ($\widetilde{\mathrm{SL}_2}$). Not all the walls are represented.

of $\mathbb{A}$.

We give the definitions in the split case and in the general (almost-split) case. The reader only interested in the split case can skip the paragraph "general case".

Case of a masure associated to a split Kac-Moody group over a valued field Recall that $\Lambda = \omega(\mathcal{K}^*)$. One sets $\Lambda_\alpha = \Lambda = \Lambda'_\alpha$ for all $\alpha \in \Phi_{re}$ and $\Lambda_\alpha = \mathbb{R} = \Lambda'_\alpha$ for all $\alpha \in \Phi_{im}$.

Let $\mathrm{cl}^{\Phi_{all}} : \mathscr{F}(\mathbb{A}) \to \mathscr{F}(\mathbb{A})$ be as follows. If $U \in \mathscr{F}(\mathbb{A})$,

$$\mathrm{cl}^{\Phi_{all}}(U) = \{V \in U |\ \exists (k_\alpha) \in (\Lambda \cup \{+\infty\})^{\Phi_{all}} |\ V \supset \bigcap_{\alpha \in \Phi_{all}} D(\alpha, k_\alpha) \supset U\}.$$

Let $\mathrm{cl}^\# : \mathscr{F}(\mathbb{A}) \to \mathscr{F}(\mathbb{A})$ be defined as follows. If $U \in \mathscr{F}(\mathbb{A})$,

$$\mathrm{cl}^\#(U) = \{V \in U |\ \exists n \in \mathbb{N}, (\beta_i) \in \Phi_{re}^n,\ (k_i) \in \Lambda^n |\ V \supset \bigcap_{i=1}^{n} D(\beta_i, k_i) \supset U\}.$$

Let $\mathcal{CL} = \mathcal{CL}_{\Lambda'} = \{\mathrm{cl}^{\Phi_{all}}, \mathrm{cl}^\#\}$. One sometimes writes $\mathrm{cl}_{\Lambda'}^{\Phi_{all}} = \mathrm{cl}^{\Phi_{all}} = \mathrm{cl}_{\Lambda}^{\Phi_{all}}$ and $\mathrm{cl}_{\Lambda'}^\# = \mathrm{cl}^\# = \mathrm{cl}_\Lambda^\#$. The enclosure map $\mathrm{cl}^{\Phi_{all}}$ is said to be **infinite** and $\mathrm{cl}^\#$ is said to be **finite**.

Definitions 3.1.2. *Let $(\mathcal{K}, \omega)$ be a valued field. An **affine apartment** is a root generating system $\mathcal{S}$ equipped with the affine Weyl group $W = W^v \ltimes Q^\vee$ and with the group $\Lambda' = \omega(K^*)$. We say apartment when there is no danger of confusion. By abuse of notation, if $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ is an apartment, we will often use $\mathbb{A}$ to denote the underlying vector space ($\mathbb{A} = Y \otimes \mathbb{R}$ if $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i \in I})$).*

*Let $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ be an apartment. A set of the shape $M(\alpha, k)$, with $\alpha \in \Phi_{re}$ and $k \in \omega(\mathcal{K}^*)$ is called a **wall** of $\mathbb{A}$ and a set of the shape $D(\alpha, k)$, with $\alpha \in \Phi_{re}$ and $k \in \omega(\mathcal{K}^*)$ is called a **half-apartment** of $\mathbb{A}$. A subset X of $\mathbb{A}$ is said to be enclosed if there exist $k \in \mathbb{N}$, $\beta_1, \ldots, \beta_k \in \Phi_{re}$ and $(\lambda_1, \ldots, \lambda_k) \in \omega(\mathcal{K}^*)^k$ such that $X = \bigcap_{i=1}^k D(\beta_i, \lambda_i)$ (i.e $X = \mathrm{cl}_{\Lambda'}^\#(X)$).*

We give an example of a realization of an affine apartment in figure 3.1.2. I thank Pierre Baumann for allowing me to use its picture.

General case Let $[\Phi_{re}, \Phi_{all}]$ be the set of sets $\mathcal{P}$ satisfying $\Phi_{re} \subset \mathcal{P} \subset \Phi_{all}$.

If $\alpha \in \Phi_{im}$, one sets $\Lambda_\alpha = \mathbb{R}$. Let $\mathcal{L}$ be the set of all families $(\Lambda'_\alpha) \in \mathscr{P}(\mathbb{R})^{\Phi_{all}}$ such that for all $\alpha \in \Phi_{all}$, $\Lambda_\alpha \subset \Lambda'_\alpha$ and $\Lambda'_\alpha = -\Lambda'_{-\alpha}$.

If $\mathcal{P} \in [\Phi_{re}, \Phi_{all}]$ and $\Lambda' \in \mathcal{L}$, one defines the map $\mathrm{cl}^{\mathcal{P}}_{\Lambda'} : \mathscr{F}(\mathbb{A}) \to \mathscr{F}(\mathbb{A})$ as follows. If $U \in \mathscr{F}(\mathbb{A})$,

$$\mathrm{cl}^{\mathcal{P}}_{\Lambda'}(U) = \{V \in U |\ \exists (k_\alpha) \in \prod_{\alpha \in \mathcal{P}} (\Lambda'_\alpha \cup \{+\infty\})|\ V \supset \bigcap_{\alpha \in \mathcal{P}} D(\alpha, k_\alpha) \supset U\}.$$

If $\Lambda' \in \mathcal{L}$, let $\mathrm{cl}^{\#}_{\Lambda'} : \mathscr{F}(\mathbb{A}) \to \mathscr{F}(\mathbb{A})$ defined as follows. If $U \in \mathscr{F}(\mathbb{A})$,

$$\mathrm{cl}^{\#}_{\Lambda'}(U) = \{V \in U |\ \exists n \in \mathbb{N}, (\beta_i) \in \Phi^n_{re},\ (k_i) \in \prod_{i=1}^{n} \Lambda'_{\beta_i} |\ V \supset \bigcap_{i=1}^{n} D(\beta_i, k_i) \supset U\}.$$

Let $\mathcal{CL}^\infty = \{\mathrm{cl}^{\mathcal{P}}_{\Lambda'} | \mathcal{P} \in [\Phi_{re}, \Phi_{all}] \text{ and } \Lambda' \in \mathcal{L}\}$. An element of $\mathcal{CL}^\infty$ is called an **infinite enclosure map**. Let $\mathcal{CL}^\# = \{\mathrm{cl}^{\#}_{\Lambda'} |\ \Lambda' \in \mathcal{L}\}$. An element of $\mathcal{CL}^\#$ is called a **finite** enclosure map. Although $\mathcal{CL}^\infty$ and $\mathcal{CL}^\#$ might not be disjoint (for example if $\mathbb{A}$ is associated to a reductive group over a local field), we define the set of **enclosure maps** $\mathcal{CL}$ by $\mathcal{CL}^\infty \sqcup \mathcal{CL}^\#$: in 3.1.4, the definition of the set of faces associated to an enclosure map cl depends on whether cl is finite or not.

If $\mathrm{cl} \in \mathcal{CL}$, $\mathrm{cl} = \mathrm{cl}^{\mathcal{P}}_{\Lambda'}$ with $\mathcal{P} \in [\Phi_{re}, \Phi_{all}] \cup \{\#\}$ and $\Lambda' \in \mathcal{L}$, then for all $\alpha \in \Phi_{all}$, $\Lambda'_\alpha = \{k \in \mathbb{R} |\ \mathrm{cl}(D(\alpha, k)) = D(\alpha, k)\}$. Therefore $\mathrm{cl}^\# := \mathrm{cl}^{\#}_{\Lambda'}$ is well defined. We do not use exactly the same notation as in [Rou17] in which $\mathrm{cl}^\#$ means $\mathrm{cl}^{\#}_{\Lambda}$.

If $\Lambda' \in \mathcal{L}$, one sets $\mathcal{CL}_{\Lambda'} = \{\mathrm{cl}^{\mathcal{P}}_{\Lambda'} |\ \mathcal{P} \in [\Phi_{re}, \Phi_{all}]\} \sqcup \{\mathrm{cl}^{\#}_{\Lambda'}\}$.

Definitions 3.1.3. *An **affine apartment** is a root generating system $\mathcal{S}$ equipped with an affine Weyl group W (i.e with a set $\mathcal{M}$ of real walls, see 3.1.1) and a family $\Lambda' \in \mathcal{L}$, where $\mathcal{L}$ is the set of families $(\Lambda'_\alpha) \in \mathscr{P}(\mathbb{R})^{\Phi_{all}}$ such that for all $\alpha \in \Phi_{all}$, $\Lambda_\alpha \subset \Lambda'_\alpha$ and $\Lambda'_\alpha = -\Lambda'_{-\alpha}$. We say apartment when there is no danger of confusion.*

*Let $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ be an apartment. A set of the shape $M(\alpha, k)$, with $\alpha \in \Phi_{re}$ and $k \in \Lambda'_\alpha$ is called a **wall** of $\mathbb{A}$ and a set of the shape $D(\alpha, k)$, with $\alpha \in \Phi_{re}$ and $k \in \Lambda'_\alpha$ is called a **half-apartment** of $\mathbb{A}$. A subset X of $\mathbb{A}$ is said to be enclosed if there exist $k \in \mathbb{N}$, $\beta_1, \ldots, \beta_k \in \Phi_{re}$ and $(\lambda_1, \ldots, \lambda_k) \in \prod_{i=1}^k \Lambda'_{\beta_i}$ such that $X = \bigcap_{i=1}^k D(\beta_i, \lambda_i)$ (i.e $X = \mathrm{cl}^{\#}_{\Lambda'}(X)$).*

As we shall see, if $\Lambda' \in \mathcal{L}$ is fixed, the definition of masures does not depend on the choice of an enclosure map in $\mathcal{CL}_{\Lambda'}$ and thus it will be more convenient to choose $\mathrm{cl}^{\#}_{\Lambda'}$, see Theorem 4.4.1 and Theorem 4.4.2.

Remark 3.1.4. *Here and in the following, we may replace Φ^+_{im} by any W^v-stable subset of $\bigoplus_{i \in I} \mathbb{R}_+ \alpha_i$ such that $\Phi^+_{im} \cap \bigcup_{\alpha \in \Phi} \mathbb{R}\alpha$ is empty. We then set $\Phi^-_{im} = -\Phi^+_{im}$. We could also assume Φ_{re} non reduced, with slight changes, see Remark 4.2.2. This is useful to include the case of almost split Kac-Moody groups, see 6.11.3 of [Rou17].*

3.1.4 Definitions of faces, chimneys and related notions

We now define the faces, chimneys, ... Let $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ be an apartment. We choose an enclosure map $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$.

A **local-face** is associated to a point x and a vectorial face F^v in $\mathbb{A}$; it is the filter $F^\ell(x, F^v) = germ_x(x + F^v)$ intersection of $x + F^v$ and the filter of neighborhoods of x in $\mathbb{A}$. A **face** F in $\mathbb{A}$ is a filter associated to a point $x \in \mathbb{A}$ and a vectorial face $F^v \subset \mathbb{A}$. More precisely, if cl is infinite (resp. cl is finite), $\mathrm{cl} = \mathrm{cl}_{\Lambda'}^{\mathcal{P}}$ with $\mathcal{P} \in [\Phi_{re}, \Phi_{all}]$ (resp. $\mathrm{cl} = \mathrm{cl}_{\Lambda'}^{\#}$), $F(x, F^v)$ is the filter made of the subsets containing an intersection (resp. a finite intersection) of half-spaces $D(\alpha, \lambda_\alpha)$ or $D^\circ(\alpha, \lambda_\alpha)$, with $\lambda_\alpha \in \Lambda'_\alpha \cup \{+\infty\}$ for all $\alpha \in \mathcal{P}$ (at most one $\lambda_\alpha \in \Lambda_\alpha$ for each $\alpha \in \mathcal{P}$) (resp. Φ_{re}). A **type 0 local face** is a local face based at an element of Y.

There is an order on the faces: if $F \subset \overline{F'}$ one says that"F is a face of F'" or "F' contains F". The dimension of a face F is the smallest dimension of an affine space generated by some $S \in F$. Such an affine space is unique and is called the **support** of F. A face is said to be **spherical** if the direction of its support meets the open Tits cone $\mathring{\mathcal{T}}$ or its opposite $-\mathring{\mathcal{T}}$; then its pointwise stabilizer W_F in W^v is finite.

A **chamber** (or alcove) is a face of the form $F(x, C^v)$ where $x \in \mathbb{A}$ and C^v is a vectorial chamber of $\mathbb{A}$.

A **panel** is a face of the form $F(x, F^v)$, where $x \in \mathbb{A}$ and F^v is a vectorial face of $\mathbb{A}$ spanning a wall.

A **chimney** in $\mathbb{A}$ is associated to a face $F = F(x, F_0^v)$ and to a vectorial face F^v; it is the filter $\mathfrak{r}(F, F^v) = \mathrm{cl}(F + F^v)$. The face F is the basis of the chimney and the vectorial face F^v its direction. A chimney is **splayed** if F^v is spherical, and is **solid** if its support (as a filter, i.e., the smallest affine subspace of $\mathbb{A}$ containing $\mathfrak{r}$) has a finite pointwise stabilizer in W^v. A splayed chimney is therefore solid.

A **shortening** of a chimney $\mathfrak{r}(F, F^v)$, with $F = F(x, F_0^v)$ is a chimney of the shape $\mathfrak{r}(F(x + \xi, F_0^v), F^v)$ for some $\xi \in \overline{F^v}$. The **germ** of a chimney $\mathfrak{r}$ is the filter of subsets of $\mathbb{A}$ containing a shortening of $\mathfrak{r}$ (this definition of shortening is slightly different from the one of [Rou11] 1.12 but follows [Rou17] 3.6) and we obtain the same germs with these two definitions).

3.1.5 Masure

Let $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ be an apartment. An **automorphism of** $\mathbb{A}$ is an affine bijection $\phi : \mathbb{A} \to \mathbb{A}$ stabilizing the family $\mathcal{M}$ of walls, and conjugating the corresponding reflections. We also ask that its linear map $\vec{\phi}$ stabilizes Φ_{re} (this is automatic when $\Lambda_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi_{re}$) and the union $\mathcal{T} \cup -\mathcal{T}$ of the Tits cones (this is automatic in the classical case). An **apartment of type** $\mathbb{A}$ is a set A with a nonempty set $\mathrm{Isom}_W(\mathbb{A}, A)$ of bijections called **Weyl-isomorphisms** such that if $f_0 \in \mathrm{Isom}_W(\mathbb{A}, A)$ then $f \in \mathrm{Isom}_W(\mathbb{A}, A)$ if and only if there exists $w \in W$ satisfying $f = f_0 \circ w$. An **isomorphism** (resp. **Weyl isomorphism**, **vectorially Weyl isomorphism**) between two apartments is a bijection $\phi : A \to A'$ such that for some $f_0 \in \mathrm{Isom}_W(\mathbb{A}, A)$ and $f'_0 \in \mathrm{Isom}_W(\mathbb{A}, A)$ (the choices have no importance), the map $(f'_0)^{-1} \circ \phi \circ f_0$ is an automorphism of $\mathbb{A}$ (resp. an element of W, the vectorial part of $(f'_0)^{-1} \circ \phi \circ f_0$ is an element of W^v). We extend all the notions that are preserved by W to each apartment. Thus sectors, enclosures, faces and chimneys are well defined in any apartment of type $\mathbb{A}$.

Definition 3.1.5. *Let* $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. *A masure of type* $(\mathbb{A}, \mathrm{cl})$ *is a set* $\mathcal{I}$ *endowed with a covering* $\mathcal{A}$ *of subsets called* ***apartments*** *such that:*

(MA1) Any $A \in \mathcal{A}$ *admits a structure of apartment of type* $\mathbb{A}$.

(MA2, cl*) If* F *is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment* A *and if* A' *is another apartment containing* F*, then* $A \cap A'$ *contains the enclosure* $\mathrm{cl}_A(F)$ *of* F *and there exists a Weyl isomorphism from* A *onto* A' *fixing* $\mathrm{cl}_A(F)$.

(MA3, cl*) If* $\mathfrak{R}$ *is the germ of a splayed chimney and if* F *is a face or a germ of a solid chimney, then there exists an apartment containing* $\mathfrak{R}$ *and* F.

(MA4, cl) If two apartments A, A' contain $\mathfrak{R}$ and F as in (MA3), then there exists a Weyl isomorphism from A to A' fixing $\mathrm{cl}_A(\mathfrak{R} \cup F)$.

(MAO) If x, y are two points contained in two apartments A and A', and if $x \leq_A y$ then the two segments $[x, y]_A$ and $[x, y]_{A'}$ are equal.

In this definition, one says that an apartment contains a germ of a filter if it contains at least one element of this germ. One says that a map fixes a germ if it fixes at least one element of this germ.

From now on, unless otherwise stated, the isomorphism of apartments will be vectorially Weyl isomorphisms. Actually in the split case, one could consider only Weyl isomorphisms.

The masure is said to be **semi-discrete** if the set of walls of $\mathbb{A}$ is $\{\alpha^{-1}(\{k\})|(\alpha, k) \in \Phi_{re} \times \mathbb{Z}\}$.

The main example of masure is the masure associated to an almost-split Kac-Moody group over a ultrametric field, see [Rou17].

Thickness of $\mathcal{I}$ One says that $\mathcal{I}$ is **thick** if for all panel of $\mathcal{I}$, there exist at least three chambers containing it. One says that $\mathcal{I}$ is of **finite thickness** if for all panel of $\mathcal{I}$, the number of chambers of $\mathcal{I}$ containing it is finite.

Group acting strongly transitively on a masure Let $\mathcal{I}$ be a masure. An **endomorphism of $\mathcal{I}$** is a map $\phi : \mathcal{I} \to \mathcal{I}$ such that for each apartment A, $\phi(A)$ is an apartment and $\phi_{|A}^{|\phi(A)}$ is an isomorphism of apartments. An **automorphism of $\mathcal{I}$** is a bijective endomorphism ϕ of $\mathcal{I}$ such that ϕ^{-1} is an endomorphism of $\mathcal{I}$. Let $\mathrm{Aut}(\mathcal{I})$ be the group of automorphisms of $\mathcal{I}$. A group $G \subset \mathrm{Aut}(\mathcal{I})$ is said to act **strongly transitively** if all the isomorphisms of apartment of (MA2), (MA4) and (MAO) are induced by elements of G. We give equivalent definitions in 4.4.5 and in Proposition 8.4.3.

3.2 Prerequisites on masures

In this section we recall some notions and results on masures that will be useful for us.

If $\mathcal{X}$ is a filter, one denotes by $\mathcal{A}(\mathcal{X})$ the set of apartments containing $\mathcal{X}$.

3.2.1 Tits preorder and Tits open preorder on $\mathcal{I}$

As the Tits preorder $\leq$ and the Tits open preorder $\mathring{\leq}$ on $\mathbb{A}$ are invariant under the action of W, one can equip each apartment A with preorders $\leq_A$ and $\mathring{\leq}_A$. Let A be an apartment of $\mathcal{I}$ and $x, y \in A$ such that $x \leq_A y$ (resp. $x \mathring{\leq}_A y$). Then by Proposition 5.4 of [Rou11], if B is an apartment containing x and y, $x \leq_B y$ (resp. $x \mathring{\leq}_B y$). This defines a relation $\leq$ (resp $\mathring{\leq}$) on $\mathcal{I}$. By Théorème 5.9 of [Rou11], this defines a preorder $\leq$ (resp. $\mathring{\leq}$) on $\mathcal{I}$. It is invariant by isomorphisms of apartments: if A, B are apartments, $\phi : A \to B$ is an isomorphism of apartments and $x, y \in A$ are such that $x \leq y$ (resp. $x \mathring{\leq} y$), then $\phi(x) \leq \phi(y)$ (resp. $\phi(x) \mathring{\leq} \phi(y)$). We call it the **Tits preorder on $\mathcal{I}$** (resp. the **Tits open preorder on $\mathcal{I}$**).

If $x, y \in \mathcal{I}$ are such that $x \leq y$, then the segment $[x, y]$ does not depend on the apartment containing x, y (conséquence 2) of Proposition 5.4 of [Rou11]). This property will be called **order-convexity**.

3.2.1.1 Vectorial distance on $\mathcal{I}$

For $x \in \mathcal{T}$, we denote by x^{++} the unique element in $\overline{C_f^v}$ conjugated by W^v to x.

Let $\mathcal{I} \times_{\leq} \mathcal{I} = \{(x,y) \in \mathcal{I}^2 | x \leq y\}$ be the set of increasing pairs in $\mathcal{I}$.

Definition/Proposition 3.2.1. *Let $(x,y) \in \mathcal{I} \times_{\leq} \mathcal{I}$. Let $\phi : A \to \mathbb{A}$ be an isomorphism of apartments. Then $\phi(x) \leq \phi(y)$ and we define the **vectorial distance** $d^v(x,y) \in \overline{C_f^v}$ by $d^v(x,y) = \big(\phi(y) - \phi(x)\big)^{++}$. It does not depend on the choices we made.*

Proof. Let A' be an apartment containing x, y and $\phi' : A' \to \mathbb{A}$ be an isomorphism of apartments. Let $\psi : A \to A'$ be an isomorphism fixing $[x,y]$, which exists by Proposition 5.4 of [Rou11]. Let $w \in W$ making the following diagram commute:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & \mathbb{A} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle w} \\ A' & \xrightarrow{\phi'} & \mathbb{A} \end{array}$$

Let $\vec{w} \in W^v$ be the vectorial part of w. Then:

$$\begin{aligned} \phi(y) - \phi(x) =& w^{-1} \circ \phi' \circ \psi(y) - w^{-1} \circ \phi' \circ \psi(y) \\ =& w^{-1} \circ \phi'(y) - w^{-1} \circ \phi'(x) \\ =& \vec{w}(\phi'(y) - \phi'(x)). \end{aligned}$$

Therefore $(\phi(y) - \phi(x))^{++} = (\phi'(y) - \phi'(x))^{++}$, which is our assertion. □

This "distance" is invariant by isomorphisms of apartments: if $x, y \in \mathcal{I}$ are such that $x \leq y$, A is an apartment containing $\{x,y\}$, B is an apartment of $\mathcal{I}$ and $\phi : A \to B$ is an isomorphism of apartments, $d^v(x,y) = d^v(\phi(x), \phi(y))$.

For $x \in \mathcal{I}$ and $\lambda \in \overline{C_f^v}$, one defines $\mathcal{S}^v(x,\lambda) = \{y \in \mathcal{I} | x \leq y \text{ and } d^v(x,y) = \lambda\}$.

Remark 3.2.2. *Let $a \in Y$ and $\lambda \in \overline{C_f^v}$. Then $\mathcal{S}^v(a,\lambda)$ is the set of points $x \in \mathcal{I}$ such that there exists $A \in \mathcal{A}(\{a,x\})$ and an isomorphism of apartments $\phi : A \to \mathbb{A}$ such that $\phi(x) = \phi(a) + \lambda$.*

3.2.2 Retractions centered at sector-germs

The reference for this subsection is 2.6 of [Rou11].

Definition/Proposition 3.2.3. *(see Figure 3.2.1) Let $\mathfrak{s}$ be a sector-germ of $\mathcal{I}$ and A be an apartment containing it. Let $x \in \mathcal{I}$. By (MA3), there exists an apartment A_x of $\mathcal{I}$ containing x and $\mathfrak{s}$. By (MA4), there exists an isomorphism of apartments $\phi : A_x \to A$ fixing $\mathfrak{s}$. Then $\phi(x)$ does not depend on the choices we made and thus we can set $\rho_{A,\mathfrak{s}}(x) = \phi(x)$. The map $\rho_{A,\mathfrak{s}}$ is a retraction from $\mathcal{I}$ onto A. It only depends on $\mathfrak{s}$ and A and one calls it the **retraction onto** A **centered at** $\mathfrak{s}$.*

Proof. Let $A'_x \in \mathcal{A}(x,\mathfrak{s})$ and $\phi' : A'_x \overset{\mathfrak{s}}{\to} A$. By (MA4), there exists an isomorphism $f : A_x \to A'_x$ fixing $\mathrm{conv}_A(x,\mathfrak{s})$. Let $\psi : \mathbb{A} \to \mathbb{A}$ making the following diagram commute:

$$\begin{array}{ccc} A_x & \xrightarrow{f} & A'_x \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi'} \\ \mathbb{A} & \xrightarrow{\psi} & \mathbb{A}. \end{array}$$

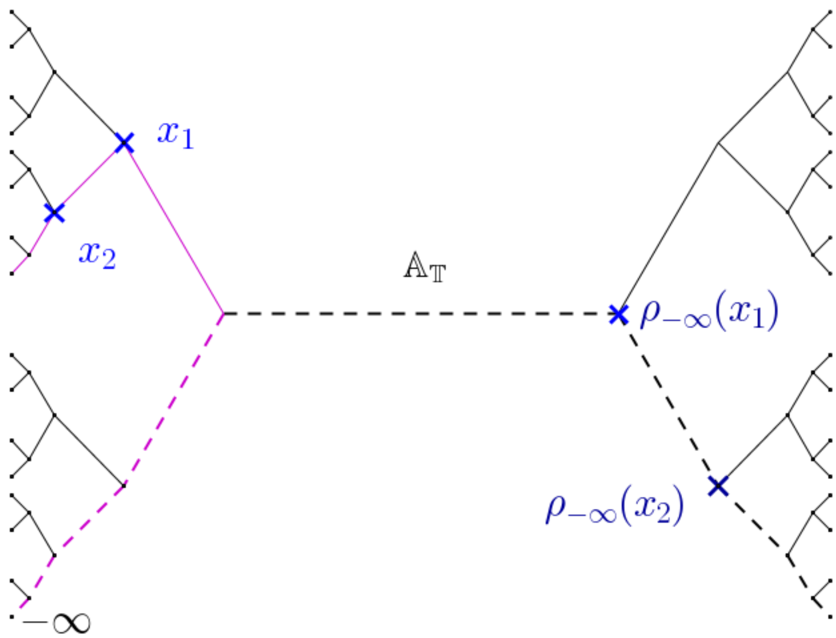

Figure 3.2.1 – Retraction centered at $-\infty$ on a tree, see 3.4.2. The standard apartment is drawn as dotted line. An apartment containing x_1, x_2 and $-\infty$ is colored in violet.

Then ψ is an affine automorphism of affine space fixing $\mathfrak{s}$ and thus $\psi = \mathrm{Id}_{\mathbb{A}}$. Therefore $\phi'(x) = \phi(x)$.

The fact that $\rho_{A,\mathfrak{s}}$ is a retraction is obtained by setting $A_x = A$ for all $x \in \mathbb{A}$. □

Remark 3.2.4. *Let $\mathfrak{s}$ be a sector-germ of $\mathbb{A}$ and A be an apartment containing $\mathfrak{s}$ and $\rho := \rho_{\mathfrak{s},\mathbb{A}}$. Let $\phi = \rho_{|A} : A \to \mathbb{A}$. Then ϕ is the unique isomorphism $A \overset{A\cap\mathbb{A}}{\to} \mathbb{A}$. Indeed, by (MA4), there exists an isomorphism of apartments $\psi : A \to \mathbb{A}$ fixing $\mathfrak{s}$. By definition $\phi = \psi$ and thus ϕ is an isomorphism of apartments. If $x \in \mathbb{A}$, $\rho(x) = x$ by definition of ρ and thus ϕ fixes $A \cap \mathbb{A}$. Let $\psi' : A \to \mathbb{A}$ be an isomorphism fixing $A \cap \mathbb{A}$. Then $\psi' \circ \phi^{-1} : \mathbb{A} \to \mathbb{A}$ is an isomorphism of affine spaces fixing $\mathfrak{s}$ and thus $\psi' = \phi$.*

If A and B are two apartments, and $\phi : A \to B$ is an isomorphism of apartments fixing some set X, one writes $\phi : A \overset{X}{\to} B$. If A and B share a sector-germ $\mathfrak{q}$, one denotes by $A \overset{A\cap B}{\to} B$ or by $A \overset{\mathfrak{q}}{\to} B$ the unique isomorphism of apartments from A to B fixing $\mathfrak{q}$ (and also $A \cap B$). We denote by $\mathcal{I} \overset{\mathfrak{q}}{\to} A$ the retraction onto A fixing $\mathfrak{q}$. One denotes by $\rho_{+\infty}$ the retraction $\mathcal{I} \overset{+\infty}{\to} \mathbb{A}$ and by $\rho_{-\infty}$ the retraction $\mathcal{I} \overset{-\infty}{\to} \mathbb{A}$.

3.2.3 Action of $\mathbb{A}_{in}$ on $\mathcal{I}$

We now define an action of $\mathbb{A}_{in}$ on $\mathcal{I}$ by translation in a more direct way than in Section 5 of [Héb17b].

Definition/Proposition 3.2.5. *Let $x \in \mathcal{I}$ and $\nu \in \mathbb{A}_{in}$. Let A be an apartment containing x and $\phi : \mathbb{A} \to A$ be an isomorphism of apartments. One sets $x + \nu = \phi(\phi^{-1}(x) + \nu)$. Then $x + \nu$ does not depend on the choice of A and ϕ.*

Proof. Let B be an apartment containing x and $\phi_1, \phi_2 : \mathbb{A} \to B$ be isomorphisms of apartments and $w = \phi_2^{-1} \circ \phi_1 : \mathbb{A} \to \mathbb{A}$. By definition, $w \in W$. Let $\vec{w}$ be the vectorial part of w.

One has

$$w(\phi_1^{-1}(x)+\nu) = w(\phi_1^{-1}(x)) + \vec{w}(\nu) = w(\phi_1^{-1}(x)) + \nu = \phi_2^{-1}(x) + \nu.$$

Therefore $\phi_1(\phi_1^{-1}(x)+\nu) = \phi_2(\phi_2^{-1}(x)+\nu)$.

Let now A' be an apartment containing x and $\phi' : \mathbb{A} \to A'$ be an isomorphism of apartments. By (MA2), there exists an isomorphism $f : A \to A'$ fixing the enclosure of x. By what we proved above, one can suppose that $\phi' = f \circ \phi$. One has

$$\phi'(\phi'^{-1}(x)+\nu) = (f\circ\phi)(\phi^{-1}\circ f^{-1}(x)+\nu) = (f\circ\phi)(\phi^{-1}(x)+\nu).$$

Moreover $\phi^{-1}(x)+\nu$ is in the enclosure (for any choice of enclosure map) of $\phi^{-1}(x)$ and thus $\phi(\phi^{-1}(x)+\nu)$ is in the enclosure of x. Hence $(f\circ\phi)(\phi^{-1}(x)+\nu) = \phi(\phi^{-1}(x)+\nu)$ and the proposition is proved. □

Remark 3.2.6. *Let $x \in \mathcal{I}$, $\nu \in \mathbb{A}_{in}$ and $\mathcal{S}^v(x,\lambda) := \{y \in \mathcal{I} |\ y \geq x \text{ and } d^v(x,y) = \nu\}$. Then $\mathcal{S}^v(x,\nu) = \{x+\nu\}$. Indeed, by Remark 3.2.2 if $y \in \mathcal{I}$, then $y \in \mathcal{S}^v(x,\nu)$ if and only if there exists an apartment A containing x, y and an isomorphism $\phi : \mathbb{A} \to A$ such that $\phi^{-1}(y) = \phi^{-1}(x)+\nu$.*

3.2.4 Parallelism in $\mathcal{I}$ and building at infinity

Let us explain briefly the notion of parallelism in $\mathcal{I}$. This is done more completely in [Rou11] Section 3.

Vectorial faces Let us begin with vectorial faces.

A **sector-face** F of $\mathbb{A}$, is a set of the shape $x + F^v$ for some vectorial face F^v and some $x \in \mathbb{A}$. The germ $\mathfrak{F} = germ_\infty(F)$ of this sector face is the filter containing an element of the shape $q + F^v$, for some $q \in \overline{x + F^v}$. The sector-face F is said to be **spherical** if $F^v \cap \mathring{\mathcal{T}}$, is nonempty that is if F^v is spherical. A **sector-panel** is a sector-face included in a wall and spanning it as an affine space. A sector-panel is spherical (see [Rou11] 1). We extend these notions to $\mathcal{I}$ thanks to the isomorphisms of apartments. Let us make a summary of the notion of parallelism for sector-faces. This is also more complete in [Rou11], 3.3.4)).

If A is an apartment and $F, F' \subset A$ are spherical sector-faces of A, we say that F and F' are parallel if for some isomorphism of apartment $\phi : A \to \mathbb{A}$, $\phi(F)$ and $\phi(F')$ are parallel (i.e there exists a translation τ of $\mathbb{A}$ such that $\tau(\phi(F)) = \phi(F')$). The **vectorial direction in** A of a spherical sector-face F is the set F^v_A of sector-faces of A that are parallel to F. We denote it F^v_A. The vectorial directions of sector-faces of A are called **vectorial faces** of A. When $A = \mathbb{A}$, this coincide with the previous notion of vectorial faces. If $y \in A$, then $y + F^v_A$ denote the translate in A of F based at y.

Let F and F' are two spherical sector-faces of $\mathcal{I}$. By (MA3), there exists an apartment B containing their germs $\mathfrak{F}$ and $\mathfrak{F}'$. One says that F and F' are **parallel** if there exists a vectorial face F^v_B of B such that $\mathfrak{F} = germ_\infty(x + F^v_B)$ and $\mathfrak{F}' = germ_\infty(y + F^v_B)$ for some $x, y \in B$. Parallelism is an equivalence relation. The parallelism class of a sector-face germ $\mathfrak{F}$ is denoted $\mathfrak{F}^\infty$ and is called the **direction of** $\mathfrak{F}$. We denote by $\mathcal{I}^\infty$ the set of directions of spherical faces of $\mathcal{I}$.

If M is a wall of $\mathcal{I}$, its direction M^∞ is the set of $\mathfrak{F}_\infty$ such that $\mathfrak{F} = germ_\infty(F)$, with F a spherical sector-face included in M.

By Proposition 4.7.1) of [Rou11], for all $x \in \mathcal{I}$ and all $\mathfrak{F}^\infty \in \mathcal{I}^\infty$, there exists a unique sector-face $x + \mathfrak{F}^\infty$ of direction $\mathfrak{F}^\infty$ and with base point x. Let us explain how to obtain such a sector-face. Let $x \in \mathcal{I}$, $\mathfrak{F}^\infty \in \mathcal{I}^\infty$, $\mathfrak{F}$ be an element of $\mathfrak{F}^\infty$ and A be an apartment containing $\mathfrak{F}$. One writes $\mathfrak{F} = germ_\infty(y + F_A^v)$, with $y \in A$ and F_A^v a vectorial face of A. Then $y + F_A^v$ is a splayed chimney and by (MA3), there exists an apartment B containing x and $\mathfrak{F}$. Maybe changing y, one can suppose that $y + F_A^v \subset B$ and by (MA4) (changing y if necessary), there exists $\phi : A \overset{y+F_A^v}{\to} B$. Let $F_B^v = \phi(F_A^v)$. Then $x + F_B^v \subset B$ is a sector-face of $\mathcal{I}$ based at x and having $\mathfrak{F}^\infty$ as a direction.

Generic rays We now recall the notion of parallelism for generic rays of $\mathcal{I}$. Let δ and δ' be two generic rays in $\mathcal{I}$. By (MA3) and [Rou11] 2.2 3) there exists an apartment A containing sub-rays of δ and δ' and we say that δ and δ' are **parallel**, if these sub-rays are parallel in A. Parallelism is an equivalence relation and its equivalence classes are called **directions**.

Lemma 3.2.7. *Let $x \in \mathcal{I}$ and δ be a generic ray. Then there exists a unique ray $x + \delta$ in $\mathcal{I}$ with base point x and direction δ. In any apartment A containing x and a ray δ' parallel to δ, this ray is the translate in A of δ' having x as a base point.*

This lemma is analogous to Proposition 4.7 1) of [Rou11]. The difficult part of this lemma is the uniqueness of such a ray because second part of the lemma yields a way to construct a ray having direction δ and x as a base point. This uniqueness can be shown exactly in the same manner as the proof of Proposition 4.7.1) by replacing "spherical sector face" by "generic ray". This is possible by NB.a) of Proposition 2.7 and by 2.2 3) of [Rou11].

Building at infinity The simplicial complex $\mathcal{I}^\infty$ is called the **building at infinity**. Let $\epsilon \in \{-, +\}$ and $\mathcal{I}^{\epsilon\infty}$ be the set of directions of spherical faces of sign ϵ. Then if $\mathcal{I}$ is not a building, $\mathcal{I}^{+\infty}$ and $\mathcal{I}^{-\infty}$ are disjoint and they are buildings. Moreover $(\mathcal{I}^{+\infty}, \mathcal{I}^{-\infty})$ is a twin building (see 3 of [Rou11] for more details). We will not use these properties directly.

3.3 Split Kac-Moody groups “à la Tits” and associated masures

In this section, we define minimal Kac-Moody groups as defined by Tits in [Tit87] and then briefly recall some of their properties. We then outline the construction of the masures associated to these groups over valued field. We will not really use this section in the sequel: we mainly consider abstract masures, not necessarily associated to Kac-Moody groups. However, the study of Kac-Moody groups is up to now the main motivation for the study of masures.

We do not recall all the definitions and properties in particular those concerning group schemes. The main properties of Kac-Moody groups for our purpose are summarized in Proposition 3.3.9.

In 3.3.1, we define nilpotent sets and prenilpotent pairs of roots. These notions are important to define Kac-Moody groups.

In 3.3.2, we give the Tits's axiomatic definition of Kac-Moody groups.

In 3.3.3, we define the constructive Tits functor of a generating root system. This is a group functor satisfying Tits axioms (at least on the fields).

In 3.3.4, we describe the root data of a Kac-Moody group. This prepares the construction of masures. The reading of 3.3.2 is not necessary for this part and the sequel.

In 3.3.5, we deal with the link between affine Kac-Moody groups and loop groups.

In 3.3.6, we outline the definition of parahoric subgroups and then define the masure of a split Kac-Moody group over a valued field.

In 3.3.7, we briefly introduce almost-split Kac-Moody groups.

3.3.1 Nilpotent sets and prenilpotent pairs of roots

In this subsection, we define the notion of nilpotency for sets of roots. This notion is important to define the constructive Tits functor and to study the Kac-Moody groups and particularly their root data.

Let Ψ be a subset of Φ_{re} (or Φ_{all}). Then Ψ is said to be **closed** if for all $\alpha, \beta \in \Psi$, $\alpha+\beta \in \Phi_{re}$ (or Φ_{all}) implies $\alpha+\beta \in \Psi$. The set Ψ is said to be **prenilpotent** if there exists $w, w' \in W^v$ such that $w.\Psi \subset \Phi_{all}^+$ and $w'.\Psi \subset \Phi_{all}^-$. In this case, Ψ is finite and contained in $w^{-1}.\Phi_{re}^+ \cap (w')^{-1}.\Phi_{re}^-$, which is **nilpotent** (prenilpotent and closed). A set Ψ is prenilpotent if and only if $-\Psi$ is prenilpotent.

If $\{\alpha,\beta\} \subset \Phi_{re}$ is a prenilpotent pair, one denotes by $]\alpha,\beta[$ the finite set $(\mathbb{N}^*\alpha+\mathbb{N}^*\beta)\cap\Phi_{re}$ and one sets $[\alpha,\beta] =]\alpha,\beta[\cup\{\alpha,\beta\}$. For all $\alpha \in \Phi_{re}$, the pair $\{\alpha,-\alpha\}$ is not prenilpotent.

In the remaining part of this subsection, we give criteria for pairs of roots to be prenilpotent in order to illustrate this notion. We will not use them in the sequel.

Lemma 3.3.1. *(see Lemme 5.4.2 of [Rém02])*

1. *Let $\Psi \subset \Phi_{re}$. Then Ψ is prenilpotent if and only if $\mathcal{T} \cap \bigcap_{\alpha\in\Psi} \alpha^{-1}(\mathbb{R}_+)$ and $-\mathcal{T} \cap \bigcap_{\alpha\in\Psi} \alpha^{-1}(\mathbb{R}_+)$ has nonempty interior.*

2. *Let $\alpha, \beta \in \Phi_{re}$ such that $\alpha \neq -\beta$. Then at least one pair of $\{\{\alpha,\beta\},\{\alpha,-\beta\}\}$ is prenilpotent.*

Proposition 3.3.2. *Suppose that A is of finite type. Then a pair $\{\alpha,\beta\} \subset \Phi_{re}$ is prenilpotent if and only if $\alpha \neq -\beta$.*

Proof. If $\alpha = -\beta$, $\{\alpha,\beta\}$ is not prenilpotent.

Suppose $\alpha \neq -\beta$. Then if $\alpha = \beta$, $\{\alpha,\beta\}$ is prenilpotent. Suppose $\alpha \neq \beta$. Then $\alpha^{-1}(\{0\})$ and $\beta^{-1}(\{0\})$ are not parallel and hence $\alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+)$ has nonempty interior. As $\mathcal{T} = \mathbb{A}$ (Proposition 2.2.11), Lemma 3.3.1 enables to conclude. ☐

Proposition 3.3.3. *Suppose that A is affine. Let $\alpha, \beta \in \Phi_{re}$ such that $\alpha \neq \beta$. One writes $\alpha = \mathring{\alpha} + k\delta$, $\beta = \mathring{\beta} + \ell\delta$, with $\mathring{\alpha}, \mathring{\beta} \in \mathring{\Phi}$, $k, \ell \in \mathbb{Z}$, which is possible by Corollary 2.3.4. Then $\{\alpha,\beta\}$ is prenilpotent if and only if $\mathring{\alpha} \neq \mathring{\beta}$.*

Proof. Suppose that $\mathring{\alpha} = -\mathring{\beta}$. If $\alpha = -\beta$, then $\{\alpha,\beta\}$ is not prenilpotent by definition. Suppose $\alpha \neq -\beta$. Then $\{(p+1)\alpha + p\beta | p \in \mathbb{N}^*\} = \{\mathring{\alpha} + ((p+1)k + pl)\delta | p \in \mathbb{N}^*\} \subset \Phi_{re}$. As this set is infinite, $\{\alpha,\beta\}$ is not prenilpotent.

Suppose that $\mathring{\alpha} = -\mathring{\beta}$. Let us prove that $\{\alpha,\beta\}$ is prenilpotent. By Lemma 3.3.1, it suffices to prove that $\alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+) \cap \mathcal{T}$ and $\alpha^{-1}(\mathbb{R}_-) \cap \beta^{-1}(\mathbb{R}_-) \cap \mathcal{T}$ have nonempty interior.

Let $E = \{x \in \mathbb{A} | \ \delta(x) = 1\} \subset \mathcal{T}$. Then

$$E \cap \alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+) = \{x \in E | \mathring{\alpha}(x) \geq -k\} \cap \{x \in E | \mathring{\beta}(x) \geq -l\}.$$

As $\mathring{\alpha} \neq -\mathring{\beta}$, $E \cap \alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+)$ has nonempty interior in E. Consequently, $\alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+) \cap \mathcal{T} \supset \mathbb{R}_+^* E$ has nonempty interior. By the same reasoning, $\alpha^{-1}(\mathbb{R}_-) \cap \beta^{-1}(\mathbb{R}_-) \cap \mathcal{T}$ has nonempty interior and thus $\{\alpha,\beta\}$ is prenilpotent, which proves the lemma. ☐

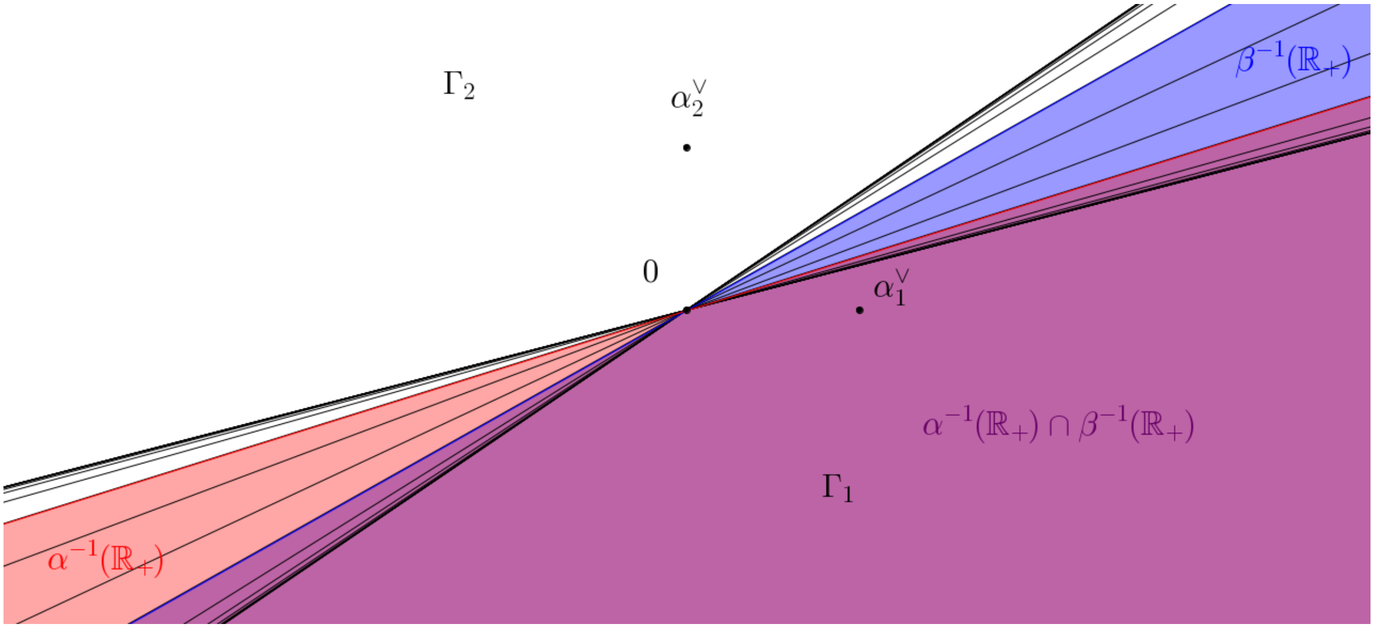

Figure 3.3.1 – Exemple of a prenilpotent pair $\{\alpha, \beta\}$

We now suppose that the matrix A is indefinite of size 2. Let Γ_1, Γ_2 be the connected components of $\mathbb{A}\backslash(\mathcal{T} \cup -\mathcal{T})$ (see Proposition 2.3.10).

Lemma 3.3.4. *Let $\gamma \in \Phi_{re}$ and $i \in \{1, 2\}$. Then $\gamma^{-1}(\mathbb{R}_+)$ contains Γ_i if and only if $\gamma^{-1}(\mathbb{R}_+) \cap \Gamma_i$ is nonempty.*

Proof. One implication is obvious. Suppose that $\gamma^{-1}(\mathbb{R}_+) \cap \Gamma_i$ contains a point x. Let $y \in \Gamma_i$. Then by Proposition 2.3.10, no vectorial wall separates x from y. Therefore $\gamma^{-1}(\mathbb{R}_+) \supset y$ and the lemma follows. □

Proposition 3.3.5. *Suppose that A is indefinite of size 2. Let $\alpha, \beta \in \Phi_{re}$. Let Γ_1, Γ_2 be as in Proposition 2.3.10. Then:*

1. *The pair $\{\alpha, \beta\}$ is prenilpotent if and only if there exists $i \in \{1, 2\}$ such that $\alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+) \supset \Gamma_i$.*
2. *Exactly one pair of $\{\{\alpha, \beta\}, \{\alpha, -\beta\}\}$ is prenilpotent.*

Proof. We can suppose that $\alpha \neq \pm\beta$ because it is clear in this case. Then $\alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+)$ has nonempty interior. Suppose $\{\alpha, \beta\}$ is prenilpotent. Then by Lemma 3.3.1, there exists $x \in \mathring{\mathcal{T}}$, $y \in -\mathring{\mathcal{T}}$ such that $\mathbb{R}x \neq \mathbb{R}y$ and $\{x, y\} \subset \alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+)$. Let $\delta \in \Phi_{im}^+$. By Proposition 2.3.9, $\delta(x) > 0$ and $\delta(y) < 0$. Therefore there exists $t \in]0, 1[$ such that $\delta(z) = 0$, where $z = (1 - t)x + ty$. By hypothesis on $\{x, y\}$, $z \neq 0$. By Proposition 2.3.9, $z \in \mathbb{A}\backslash\mathring{\mathcal{T}} \cup -\mathring{\mathcal{T}}$. As $\mathring{\mathcal{T}} = \mathcal{T}\backslash\{0\}$, we deduce that $z \in \mathbb{A}\backslash(\mathcal{T} \cup -\mathcal{T})$. Let $i \in \{1, 2\}$ such that $z \in \Gamma_i$. By Lemma 3.3.4, $\alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+) \supset \Gamma_i$, which proves one implication of 1.

Suppose that there exists $i \in \{1, 2\}$ such that $\alpha^{-1}(\mathbb{R}_+) \cap \beta^{-1}(\mathbb{R}_+) \supset \Gamma_i$. Let $j = 3 - i$. Suppose that $\alpha^{-1}(\mathbb{R}_+) \supset \Gamma_j$. Then $\alpha^{-1}(\mathbb{R}_+) \supset \mathrm{conv}(\Gamma_1, \Gamma_2) = \mathrm{conv}(\Gamma_1, -\Gamma_1) = \mathbb{A}$: this is absurd. Similarly, $-\beta(\mathbb{R}_+)$ does not contain Γ_i. Therefore $\alpha^{-1}(\mathbb{R}_+) \cap -\beta(\mathbb{R}_+)$ does not contain Γ_k for all $k \in \{1, 2\}$. Consequently $\{\alpha, -\beta\}$ is not prenilpotent. By Lemma 3.3.1, $\{\alpha, \beta\}$ is necessarily prenilpotent and we get the proposition. □

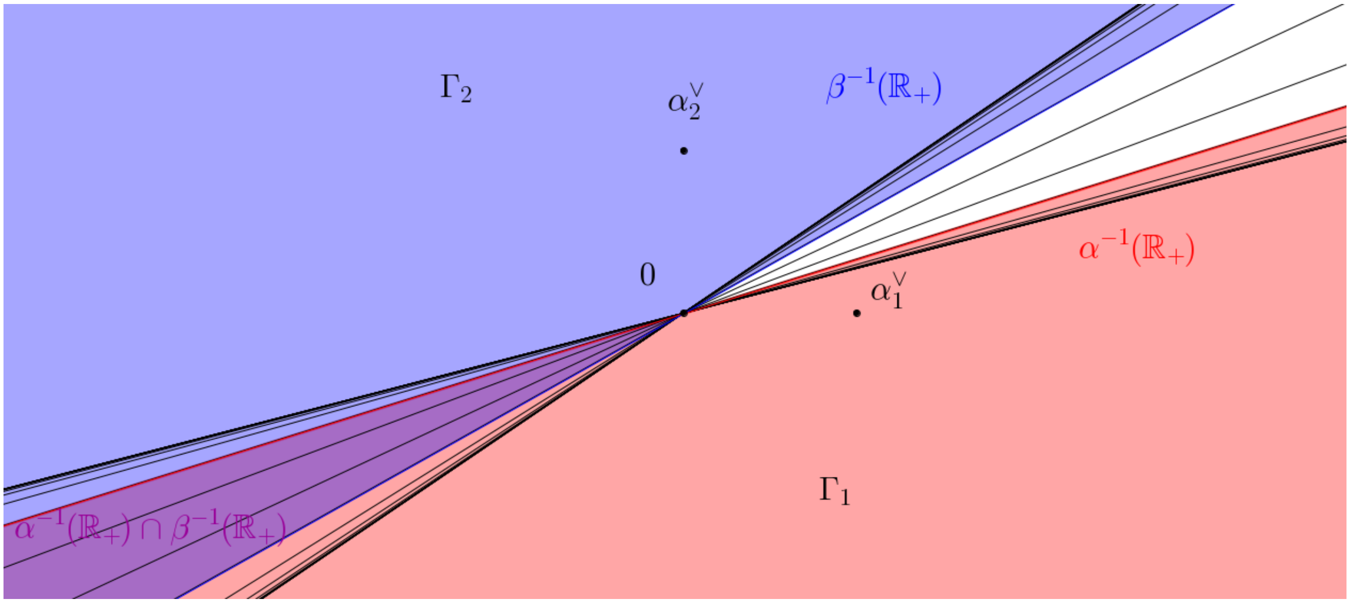

Figure 3.3.2 – Exemple of a non prenilpotent pair $\{\alpha, \beta\}$

3.3.2 Tits's axiomatic

Let $\mathcal{S}$ be a root generating system. We now define the minimal Kac-Moody group "à la Tits" associated to $\mathcal{S}$. This is the value on a field of the Tits-Kac-Moody functor associated to $\mathcal{S}$. This functor is a functor from the category of rings to the category of groups satisfying the axioms (KMG1) to (KMG 9) below. Our main references are Chapitre 8 of [Rém02] and [Rou16].

Let A be a Kac-Moody matrix and $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee))$ be a root generating system. Let $\mathfrak{g}$ be the Kac-Moody algebra of $(A, Y \otimes \mathbb{C}, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$. We write $\mathfrak{g} = \mathfrak{n}^- \oplus \mathfrak{h} \oplus \mathfrak{n}$ as is Theorem 2.2.4. By Corollary 1.3.4 of [Kum02], if $x \in \{e_i | i \in I\} \cup \{f_i | i \in I\}$, $\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ is **locally nilpotent**, which means that for all $v \in \mathfrak{g}$, there exists $n(v) \in \mathbb{N}$ such that $(\mathrm{ad}(x))^{n(v)}(v) = 0$. This enables to define $\exp \mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}$ by the usual formula.

Let $\mathbf{T} = \mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[X], \mathbf{G}_m)$, where $\mathbf{G}_m(R) = R^\times$ for all ring R. Let
$\mathbf{F} = (\mathbf{G}, \mathbf{U}, \mathbf{U}^-, (\varphi_i)_{i\in I}, \eta)$ be a 5-tuple consisting of a functor $\mathbf{G}$ from the category of rings to the category of groups, of functorial subgroups $\mathbf{U}$, $\mathbf{U}^-$, of functorial group morphisms $\varphi_i : \mathbf{SL}_2 \to \mathbf{G}$ for all $i \in I$ and of a functorial group morphism $\eta : \mathbf{T} \to \mathbf{G}$. The 5-tuple $\mathbf{F}$ is a **Tits functor** if it satisfies the following axioms:

(KMG1) If $\mathcal{K}$ is a field, $\mathbf{G}(\mathcal{K})$ is generated by the images of $\varphi_i(\mathcal{K})$, $i \in I$ and of $\eta(\mathcal{K})$.

(KMG2) For all ring R, the morphism $\eta(R) : \eta(R) \to \mathbf{G}(R)$ is injective.

(KMG3) For all ring R, $i \in I$ and $r \in R^\times$, one has $\varphi_i(\begin{pmatrix} r & 0 \\ 0 & r^{-1} \end{pmatrix}) = \eta(r^{\alpha_i^\vee})$, where $r^{\alpha_i^\vee}$ denotes the element $\lambda \in X \mapsto r^{\alpha_i^\vee(\lambda)}$ of $\mathbf{T}(R)$.

(KMG4) If ι is an injection of a ring R in a field $\mathcal{K}$, then $\mathbf{G}(\iota) : \mathbf{G}(R) \to \mathbf{G}(\mathcal{K})$ is injective.

(KMG5) There exists a morphism $\mathrm{Ad} : \mathbf{G}(\mathbb{C}) \to \mathrm{Aut}(\mathfrak{g})$ whose kernel is contained in $\eta(\mathbf{T}(\mathbb{C}))$, such that for all $c \in \mathbb{C}$, $t \in \mathbf{T}(\mathbb{C})$ and $i \in I$,

$$\mathrm{Ad}[\varphi_i(\begin{pmatrix} 1 & c \\ 0 & 1 \end{pmatrix})] = \exp \mathrm{ad}(ce_i), \quad \mathrm{Ad}[\varphi_i(\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix})] = \exp \mathrm{ad}(-cf_i),$$

$$\mathrm{Ad}(\eta(t))(e_i) = t(\alpha_i)(e_i), \quad \mathrm{Ad}(\eta(t))(f_i) = -t(\alpha_i)f_i.$$

(KMG6) The group $\mathbf{U}(\mathbb{C})$ (resp. $\mathbf{U}^-(\mathbb{C})$) is the derived group of the stabilizer in $\mathbf{G}(\mathbb{C})$ of $\mathfrak{n}$ (resp. $\mathfrak{n}^-$) for the adjoint action of the axiom (KM5).

(KMG7) If $\rho : R \to \mathcal{K}$ is the injection of a ring in a field and if ϵ is a sign, $\mathbf{U}^\epsilon(R)$ is the preimage by $\mathbf{G}(\rho)$ of $\mathbf{U}^\epsilon(\mathcal{K})$ (with $\mathbf{U}^+ = \mathbf{U}$).

(KMG8) The group $\mathbf{U}^\pm(\mathcal{K})$ is pronilpotent for all field $\mathcal{K}$

(KMG9) For all field $\mathcal{K}$ and all $i \in I$ the kernel of $\varphi_s(\mathcal{K}) : \mathbf{SL}_2(\mathcal{K}) \to \mathbf{G}(\mathcal{K})$ is included in the center of $\mathbf{SL}_2(\mathcal{K})$.

The axiom (KMG5) corresponds to the fact that $\mathbf{G}$ "integrates" $\mathfrak{g}$.

By Point b) of page 554 of [Tit87]), there exists a group functor satisfying these axioms. For each field $\mathcal{K}$, the value of $\mathbf{G}(\mathcal{K})$ is independent of the choice of the Tits functor $\mathbf{G}$ (once $\mathcal{S}$ is fixed). To prove this, Tits constructs an "explicit" functor $\widetilde{\mathbf{G}}_\mathcal{S}$ by generators and relations (see 3.3.3 for a sketch of its construction) and he proves that $\mathbf{G}(\mathcal{K}) = \widetilde{\mathbf{G}}_\mathcal{S}(\mathcal{K})$. More precisely, he proves the following theorem:

Theorem 3.3.6. *(Theorem 1' of [Tit87], see also 8.4.2 of [Rém02])*

Let $\mathbf{F} = (\mathbf{G}, \mathbf{U}, \mathbf{U}^-, (\varphi_i)_{i\in I}, \eta)$ be a system satisfying the axioms (KMG1) to (KMG9) for $\mathcal{S}$. Then there exists a functorial group morphism $\pi : \widetilde{\mathbf{G}}_\mathcal{S} \to \mathbf{G}$ satisfying the following properties.

1. *For all ring R, $\pi(\widetilde{\mathbf{U}}(R)) \subset \mathbf{U}(R)$ and $\pi(\widetilde{\mathbf{U}}^-(R)) \subset \mathbf{U}^-(R)$.*

2. *For all field $\mathcal{K}$, $\pi(\mathcal{K})$ is an isomorphism.*

This also proves that $\widetilde{\mathbf{G}}_\mathcal{S}$ satisfies (KM1) to (KM9) at least on the fields. Actually, he also proves that if $\mathbf{F} = (\mathbf{G}, (\varphi_i)_{i\in I}, \eta)$ is a triple satisfying the axioms (KMG1) to (KMG5), there exists a functorial morphism $\pi : \widetilde{\mathbf{G}}_\mathcal{S} \to \mathbf{G}$ which is an isomorphism on the fields except in some degenarate cases, see Theorem 1 of [Tit87] or 8.4.2 of [Rém02] for a precise statement. The construction of Tits relies on the existence of some $\mathbb{Z}$-form of the enveloping algebra of $\mathfrak{g}$ introduced in [Tit87], see 7.4 and 9 of [Rém02]. We do not introduce this algebra, but it is important for the proofs of many results of this subsection.

Definition 3.3.7. *If $\mathcal{K}$ is a field, a group of the form $\mathbf{G}(\mathcal{K})$ for some Tits functor $\mathbf{G}$ is called a **split Kac-Moody group over** $\mathcal{K}$.*

3.3.3 Constructive Tits functor

We now make the construction of the constructive Tits functor $\widetilde{\mathbf{G}}_\mathcal{S}$ of a root generating system $\mathcal{S}$. Let $\mathfrak{g}$ be the Kac-Moody algebra of $\mathcal{S}$. We use the same notation as in Section 2.2. Our reference is 8.3.2 and 8.3.3 of [Rém02].

If $\alpha \in \Phi_{re}$. Let $E_\alpha = \{e_\alpha, e_{-\alpha}\}$ be the double basis of Définition (ii) of 7.4.1 of [Rém02] (e_α is a particular choice of an element of $\mathfrak{g}_\alpha$). Let $\mathbf{U}_\alpha$ be the group scheme over $\mathbb{Z}$ isomorphic to $\mathbf{G}_a$ ($\mathbf{G}_a(R) = (R, +)$ for all ring R) whose Lie algebra is the $\mathbb{Z}$-Lie subalgebra of $\mathfrak{g}$ generated by E_α.

Let $\Psi \subset \Phi_{re}$ be a nilpotent set of roots. The sum $\mathfrak{g}_\Psi = \bigoplus_{\alpha\in\Psi} \mathfrak{g}_\alpha$ is a nilpotent Lie algebra. Let U_Ψ denote the unipotent complex algebraic group whose Lie algebra is $\mathfrak{g}_\Psi$.

Proposition 3.3.8. *(Proposition 1 of [Tit87])*

There exists a uniquely defined group scheme $\mathbf{U}_\Psi$ containing $\mathbf{U}_\alpha$ for $\alpha \in \Psi$, whose "generic fiber" $\mathbf{U}_\psi(\mathbb{C})$ is the group U_Ψ and such that, for any order put on Ψ, the product morphism $\prod_{\alpha\in\Psi} \mathbf{U}_\alpha \to \mathbf{U}_\Psi$ is an isomorphism of the underlying schemes.

The **Steinberg functor** of A is the functor $\mathbf{St}$ that associates to each ring R the amalgamated product of the $\mathbf{U}_\alpha(R)$, $\alpha \in \Phi_{re}$ and $\mathbf{U}_{[\alpha,\beta]}(R)$, for $\{\alpha, \beta\} \subset \Phi_{re}$ prenilpotent, ordered by the inclusion relations.

Let $i \in I$. Let $\mathbf{U}_i$ (resp. $\mathbf{U}_{-i}$) denote $\mathbf{U}_{\alpha_i}$ (resp. $\mathbf{U}_{-\alpha_i}$) and x_i (resp. x_{-i}) denote the isomorphism $\mathbf{G}_a \overset{\sim}{\to} \mathbf{U}_i$ induced by the choice of e_{α_i}. For $r \in R^\times$, one sets $\widetilde{s}_i(r) := x_i(r)x_{-i}(r^{-1})x_i(r)$. If $i \in I$, one sets $s_i^* := \exp\operatorname{ad}(e_i).\exp\operatorname{ad}(f_i).\exp\operatorname{ad}(e_i) = \exp\operatorname{ad}(f_i).\exp\operatorname{ad}(e_i).\exp\operatorname{ad}(f_i)$. We denote by s_i (instead of r_i) the fundamental reflexion of Section 2.2 to avoid confusion with the elements of r.

The **constructive Tits functor** of $\mathcal{S}$ is the functor $\widetilde{\mathbf{G}}_\mathcal{S}$ associating to each ring R the free product $\mathbf{St}(R) * \mathbf{T}(R)$ subject to the following relations: for all $t \in \mathbf{T}(R)$, $r \in R$, $r' \in R^\times$, $i \in I$, $\alpha \in \Phi_{re}$, $u \in \mathbf{U}_\alpha(R)$,

$$t.x_i(r).t^{-1} = x_i(\alpha_i(t)r), \quad \widetilde{s}_i(r').t.\widetilde{s}(r')^{-1} = s_i(t), \quad \widetilde{s}_i(r'^{-1}) = \widetilde{s}_i.r'^{\alpha_i^\vee}, \quad \widetilde{s}_i.u.\widetilde{s}_i = s^*(u),$$

where $r'^{\alpha_i^\vee}$ is defined in (KMG3).

3.3.4 Root datum for a split Kac-Moody group "à la Tits"

Let $\mathcal{K}$ be a field and $G = \widetilde{\mathbf{G}}_\mathcal{S}(\mathcal{K})$. We use boldfaces letters to denote the functors $\mathbf{T}$, $\mathbf{U}_\alpha$, ... and roman letters to denote their evaluation in $\mathcal{K}$: $T = \mathbf{T}(\mathcal{K})$, $U_\alpha = \mathbf{U}_\alpha(\mathcal{K})$, ...

The following proposition asserts in particular that $(G, (U_\alpha)_{\alpha\in\Phi}, T)$ is a "donnée radicielle jumelée entière" for the definition of 6.2.5 of [Rém02].

Proposition 3.3.9.

1. *The group T is isomorphic to $(\mathcal{K}^*)^{\mathrm{rk}(X)}$.*
2. *For all $\alpha \in \Phi_{re}$, U_α is isomorphic to $(\mathcal{K}, +)$.*
3. *The group G is generated by T and the U_α, $\alpha \in \Phi_{re}$.*
4. *For all $\alpha \in \Phi_{re}$, U_α is normalized by T. More precisely, $tx_\alpha(u)t^{-1} = x_\alpha(\alpha(t)u)$ for all $\alpha \in \Phi_{re}$, $t \in T$ and $u \in \mathcal{K}$.*
5. *Let $\{\alpha, \beta\}$, be a prenilpotent pair of roots. We fix an order on $[\alpha, \beta]$. Then the product map $\prod_{\gamma\in[\alpha,\beta]} U_\gamma \to \langle U_\gamma |\ \gamma \in [\alpha, \beta]\rangle$ is a bijection. Moreover, there exist integers $C_{p,q}$ such that for all $u, v \in \mathcal{K}$, $[x_\alpha(u), x_\beta(v)] = \prod_{\gamma\in]\alpha,\beta[,\gamma=p\alpha+q\beta} x_\gamma(C_{p,q}u^pv^q)$. These integers a priori depend on the order choosen on $[\alpha, \beta]$ but not on the field $\mathcal{K}$.*
6. *Let $\{\alpha, \beta\}$ be a non-prenilpotent pair, the canonical morphism $U_\alpha * U_\beta \to \langle U_\alpha, U_\beta\rangle$ (where $U_\alpha * U_\beta$ is the free product of U_α and U_β) is an isomorphism.*
7. *For all $\alpha \in \Phi_{re}$ and all $u \in U_\alpha\backslash\{1\}$, there exist $u', u'' \in U_{-\alpha}$ such that $m(u) := u'uu''$ conjugates U_β in $U_{r_\alpha(\beta)}$ for all $\beta \in \Phi_{re}$. Moreover, one can require that for all $u, v \in U_\alpha\backslash\{1\}$, $m(u)T = m(v)T$.*
8. *If U^+ (resp. U^-) is the subgroup of G spanned by the U_α's for $\alpha \in \Phi^+$ (resp. $\alpha \in \Phi^-$), one has $TU^+ \cap U^- = \{1\}$.*

Proof. Point 1 follows from the definition of T. Point 2 follows from the definition of $\mathbf{U}_\alpha$. Point 6 is Proposition 5 of Cours 1989-1990 of [Tit13]. The other assertions are a combination of 8.4.1 of [Rém02] or Proposition 1.5 of [Rou06] and of results of 1.6 of [Rou06]. □

Example 3.3.10. *If $n \in \mathbb{N}_{\geq 2}$, then the functor $k \mapsto \mathrm{SL}_n(k)$ is an example of Tits functor, where $\mathbf{T}(k)$ is the group of diagonal matrices, $\mathbf{U}(k)$ (resp. $\mathbf{U}^-(k)$) is the group of upper (resp. lower) triangular matrices having 1 on the diagonal. It is associated to the Lie algebra $\mathfrak{sl}_n(\mathbb{C})$. We keep the same notation as in Example 2.2.5. If $i, j \in [\![1, n-1]\!]$ such that $i < j$ and $\alpha = \alpha_{i,j}$, one can choose $\mathbf{U}_\alpha(k) = I + kE_{i,j}$ and $\mathbf{U}_{-\alpha}(k) = I + kE_{j,i}$.*

3.3.5 Affine Kac-Moody groups and loop groups

Let $\mathring{A}$ be an indecomposable Cartan matrix and A be the associated affine Kac-Moody matrix (see 2.3.1.1). Let $\mathfrak{g}$ be the Kac-Moody algebra of A and $\mathring{\mathfrak{g}}$ be the (finite dimensional) Kac-Moody algebra of $\mathfrak{g}$. We saw in Subsection 2.3.1 an explicit description of $\mathfrak{g}$ using the loop algebra $\mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]$ (with t an indeterminate). Moreover, by 13.1 of [Kum02] (page 483), $\mathfrak{g}$ can be obtained by taking two central extensions of $\mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}]$. It is natural to ask if one can have an "explicit" description of affine Kac-Moody groups using loop groups.

Let $\mathcal{S} = (A, X, Y, (\alpha_i^\vee)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ be a root generating data such that $Y/\sum_{i\in I} \mathbb{Z}\alpha_i^\vee$ is torsion-free (this is condition of 6.1.16 of [Kum02]). Let $\mathring{\mathbf{G}}$ be the connected, simply-connected algebraic group such that $\mathring{G} := \mathring{\mathbf{G}}(\mathbb{C})$ admits $\mathring{\mathfrak{g}}$ as a Lie algebra. One defines an action of $\mathbb{C}^*$ on $\mathbb{C}[t, t^{-1}]$ by setting $a.P(t) = P(at)$ for all $a \in \mathbb{C}^*$ and $P(t) \in \mathbb{C}[t, t^{-1}]$. This action defines an action of $\mathbb{C}^*$ on $\mathring{\mathbf{G}}(\mathbb{C}[t, t^{-1}])$. Let $C \subset \mathring{G} \subset \mathbb{C}^* \ltimes \mathring{\mathbf{G}}(\mathbb{C}[t, t^{-1}])$ be the center of $\mathring{G}$.

Let $G = \widetilde{\mathbf{G}}_{\mathcal{S}}(\mathbb{C})$. By 3.20 of [Rou16], G is the minimal Kac-Moody group of 7.4 of [Kum02] (the Kac-Peterson group). By Corollary 13.2.9 of [Kum02], there exists an isomorphism $\psi : G/\mathcal{Z}(G) \xrightarrow{\sim} (\mathbb{C}^* \ltimes \mathring{G}(\mathbb{C}[t, t^{-1}])/C$, where $\mathcal{Z}(G)$ is the center of G. Moreover, by Theorem 13.2.8 c) of [Kum02], if the $\mathbf{U}_\alpha$, $\alpha \in \Phi_{re}$ and $\mathring{\mathbf{U}}_{\mathring{\alpha}}$, $\mathring{\alpha} \in \mathring{\Phi}$ (with the same notation as in Subsection 2.3.1) are the root subgroups of $\widetilde{\mathbf{G}}_{\mathcal{S}}$ and $\mathring{\mathbf{G}}$, one has $\mathbf{U}_{\mathring{\alpha}+k\delta}(\mathbb{C}) = \{1\} \ltimes \mathring{\mathbf{U}}_{\mathring{\alpha}}(t^k\mathbb{C}) \subset \mathbb{C}^* \ltimes \mathring{\mathbf{G}}(\mathbb{C}[t, t^{-1}])$. Let $\mathring{\mathbf{T}} := \mathrm{Hom}_{\mathbb{Z}-\mathrm{alg}}(\mathbb{Z}[X], \mathbf{G}_m)$ be the standard maximal torus of $\mathring{\mathbf{G}}$ and $T = \mathbb{C}^* \ltimes \mathring{T} \subset \mathbb{C}^* \ltimes \mathring{\mathbf{G}}(\mathbb{C}[t, t^{-1}])$ (here the semi-direct product is the direct product because $\mathbb{C}^*$ acts trivially on $\mathring{G}$). Then by Theorem 13.2.8 b) of [Kum02], T/C is the image by ψ of the standard maximal torus of G. The adjoint action of $\mathbb{C}^* \ltimes \mathring{\mathbf{G}}(\mathbb{C}[t, t^{-1}])$ on $\mathring{\mathfrak{g}} \otimes \mathbb{C}[t, t^{-1}] \oplus \mathbb{C}c \oplus \mathbb{C}d$ can also be described explicitly, see page 493 of [Kum02].

Although it seems likely that loop groups or some central extensions of them satisfy the axioms (KMG1) to (KMG9) at least on the fields, I did not find a detailed proof of this fact in the literature. In [Tit89], Tits claims that the functor $R \mapsto \mathring{\mathbf{G}}(R[t, t^{-1}])$ coincide with $\widetilde{\mathbf{G}}_{\mathcal{S}}$ on the fields, for some root generating system $\mathcal{S}$ of A. In [Mas95], Masson proves that for some torus $\mathbf{T}_c$, the functor $R \mapsto \mathring{\mathbf{G}}(R[t, t^{-1}]) \rtimes \mathbf{T}_c(R)$ satisfies (KMG1) to (KMG5).

3.3.6 Parahoric subgroups and masure associated to a split Kac-Moody group over a valued field

Definition of the masure of G Suppose that $\mathcal{K}$ is a valued field. Let $G = \mathbf{G}(\mathcal{F})$. Let $N = N_G(T) = \{g \in G |\ gTg^{-1} = T\}$ be the normalizer of T in G. In order to associate a masure to G, Gaussent and Rousseau follow the same strategy as Bruhat and Tits to define their buildings: first they define an action of N on $\mathbb{A}$ and then they define the fixer $\widehat{P}_x$ of a point $x \in \mathbb{A}$. The second step is the hardest. To cope with the lack of commutation relations in G they use some completions of G defined by Kumar and Mathieu. We will not recall the details of the construction of the $\widehat{P}_x$'s because we will not use them once the masure is built. We refer to Section 2,3 and 4 of [Rou16] for this construction. However, we try to give an informal outline of the construction of these groups.

In [GR08], Gaussent and Rousseau considers valued fields whose residue fields contains $\mathbb{C}$. This enables them to use the "completed" Kac-Moody group defined by Kumar in [Kum02]. They also assume that their group is symmatrizable (which is a condition on the Kac-Moody matrix, see 1.5 of [Kum02] for a definition). Rousseau generalized this construction in [Rou16]: he drops the assumptions on the residue field and on symmatrizability. He replaces the completed group of Kumar by the Kac-Moody group defined by Mathieu in [Mat89]. Over $\mathbb{C}$, these groups are isomorphic (by 3.20 of [Rou16]).

Kac-Moody group "à la Mathieu" There is (up to now?) no unified definition of completed or maximal Kac-Moody groups. Several authors define such groups by different approaches. We focus here on the Kac-Moody group "à la Mathieu" and "à la Kumar" and refer to Chapter 6 of [Mar13] or 6 of [Rou16] for a comparison of maximal Kac-Moody groups. The positive (resp. negative) Kac-Moody groups of Mathieu is a functor $\mathbf{G}^{pma}$ (resp. $\mathbf{G}^{nma}$) from the category of commutative ring to the category of groups. These two functors are obtained from each other by permuting Φ_{all}^+ and Φ_{all}^- and thus we only describe $\mathbf{G}^{pma}$. It contains a Borel subgroup $\mathbf{B}^{pma}$, which is a semi-direct product $\mathbf{T} \ltimes \mathbf{U}^{pma}$ where $\mathbf{T}$ is a torus and $\mathbf{U}^{ma+}$ is a pro-unipotent group, which is some kind of completion of $\mathbf{U}$. If R is a field, the elements of $\mathbf{U}^{ma+}(R)$ can be written in some sense as a product (infinite if Φ_{all} is infinite) of elements of $\mathbf{U}_\alpha^{ma}(R)$ for $\alpha \in \Phi_{all}^+$, see Proposition 3.2 of [Rou16] for a precise statement, where if $\alpha \in \Phi_{all}^+$, $\mathbf{U}_\alpha^{ma}(R) = \mathbf{U}_\alpha(R) \simeq (R,+)$ and if $\alpha \in \Phi_{all}^+$, $\mathbf{U}_\alpha^{ma}(R)$ is some group. The group functor $\mathbf{G}^{pma}$ also contains group functors $\mathbf{U}_\alpha^{ma} = \mathbf{U}_\alpha$ for all $\alpha \in \Phi^-$.

There exists a injective morphism i_R which is the identity on $\mathbf{T}(R)$ and on $\mathbf{U}_\alpha(R)$ for all $\alpha \in \Phi_{re}$ which enables to consider $\mathbf{G}(R)$ as a subgroup of $\mathbf{G}^{ma}(R)$.

The functor $\mathbf{G}^{pma}$ By 3.20 of [Rou16], $\mathbf{G}^{ma}(\mathbb{C})$ is isomorphic to the Kac-Moody group of Kumar.

Action of N on $\mathbb{A}$ Let $(\mathcal{K},\omega)$ be a valued field, with ω non-trivial. Let us define an action ν of N on $\mathbb{A}$. The group T acts by translation on $\mathbb{A}$: if $t \in T$, $\nu(t)$ is the translation by the unique vector v of $\mathbb{A}$ satisfying $\chi(v) = -\omega(\chi(t))$ for all $\chi \in X$.

If A and B are affine spaces and $f : A \to B$ is an affine map, we denote by $\vec{f} : \vec{A} \to \vec{B}$ the linear part of f. Recall the definition of $m(u)$ for $u \in U_\alpha(\mathcal{K})$, $\alpha \in \Phi_{re}$ from Proposition 3.3.9.

By Lemma 2.1 of [GR08] or 4.2 of [Rou16], there exists an action ν of N on $\mathbb{A}$ by affine automorphisms such that $\vec{\nu}(m(u)) = r_i$ for all $i \in I$, $u \in U_{\pm\alpha_i}(\mathcal{K})$ and such that $\ker(\vec{\nu}) = T$.

As the r_i's, $i \in I$ span W^v, we deduce that $W^v \simeq N/T$. Let $\Lambda = \omega(\mathcal{K}^*) \subset \mathbb{R}$. The image $\nu(N)$ is $W_{Y\Lambda} = W^v \ltimes (Y \otimes \Lambda)$. The kernel of ν is $H = \mathcal{O}^* \otimes Y = \mathbf{T}(\mathcal{O})$, where $\mathcal{O}$ is the ring of integers of $\mathcal{K}$.

Unipotent groups associated to filters of $\mathbb{A}$ Suppose that $\mathbf{G}$ is a reductive group and $\mathcal{K}$ is a local field. Up to renormalization, $\Lambda = \omega(\mathcal{K}^*) = \mathbb{Z}$. Let $\mathcal{I}$ be the Bruhat-Tits building of $\mathbf{G}(\mathcal{K})$ and π be a uniformizer of $\mathcal{O}$. Then if $\alpha \in \Phi_{re}$, $\mathbf{U}_\alpha(\mathcal{K}).\mathbb{A} \subset \mathcal{I}$ is an extended homogeneous tree and for all $k \in \mathbb{Z}$, the fixer of the half-apartment $D(\alpha,k) \subset \mathbb{A}$ in G is $x_\alpha(\pi^{-k}\mathcal{O})$. One can expect similar properties for the masure and this motivates the following definitions.

Let Ω be a filter. If $\alpha \in X$, one sets

$$f_\Omega^\Lambda(\alpha) = \inf\{\lambda \in \Lambda |\ \Omega \subset D(\alpha,\lambda)\} = \inf\{\lambda \in \Lambda | \alpha(\Omega) + \lambda \subset [0,+\infty]\}.$$

Let U_Ω be the subgroup of U generated by the groups $U_{\alpha,\Omega} = U_{\alpha,f_\Omega(\alpha)}$ for $\alpha \in \Phi_{re}$.

In 4.5 of [Rou16], Rousseau associates subgroups $U_\Omega^{pma} \subset U^{pma}$ and $U_\Omega^{nma} \subset U^{nma}$. He then sets $U_\Omega^{pm+} = U_\Omega^{pma} \cap U$ and $U_\Omega^{nm-} = U_\Omega^{nma} \cap U^-$. One has $U_\Omega^{pm+} \supset U_\Omega^+ := U_\Omega \cap U^+$ and $U_\Omega^{nm-} \supset U_\Omega^- := U_\Omega \cap U^-$. By 4.5.4) f) of [Rou16], U_Ω^{pm+} depends only on $\mathrm{cl}(\Omega)$, which is a priori not true for $\mathrm{cl}^\#(\Omega)$.

In Definition 4.9 of [Rou16], Rousseau defines a group $\widetilde{P}_\Omega^{\mathcal{K}}$ by considering actions on the enveloping algebra and on highest weight modules. If Ω is a set, he defines $\widehat{P}_\Omega$ as the intersection of the $\widetilde{P}_{\Omega'}^{\mathcal{K}'}$ for $\mathcal{K}'/\mathcal{K}$ valued field extension and $\emptyset \subsetneq \Omega' \subset \overline{\Omega}$. For the definition of masures, we will only use the groups $\widehat{P}_x$ for $x \in \mathbb{A}$. One has $\widehat{P}_x \supset U_x$ for all $x \in \mathbb{A}$.

Proposition 3.3.11. *(see Proposition 4.14 of [Rou16])*
The groups $\widehat{P}_x$, $x \in \mathbb{A}$ have the following properties. Let $x \in \mathbb{A}$, then:

1. $\widehat{P}_x \cap N = \widehat{N}_x := \mathrm{Fix}_N(x)$,
2. *for all* $n \in N$, $n.\widehat{P}_x.n^{-1} = \widehat{P}_{\nu(n).x}$,
3. $\widehat{P}_x = U_x^{pm+}.U_x^{nm-}.\widehat{N_x}$.

Definition of the masure One defines an equivalence relation $\sim$ on $G \times \mathbb{A}$ as follows: $(g,x) \sim (h,y)$ if and only if there exists $n \in N$ such that $y = \nu(n).x$ and $g^{-1}.h.n \in \widehat{P}_x$. The **masure** $\mathcal{I}(G)$ of G (or of $\mathbf{G}$ over $\mathcal{K}$) is the set $G \times \mathbb{A}/\sim$. One defines an action of G on $\mathcal{I}$ by $g.(g',x) = (g.g',x)$ if $g, g' \in G$ and $x \in \mathbb{A}$. By Proposition 3.3.11, the map $\iota : \mathbb{A} \to \mathcal{I}$ defined by $\iota(x) = (1,x)$ for all $x \in \mathbb{A}$ is injective and enables to consider $\mathbb{A}$ as a subset of $\mathcal{I}$. An apartment of $\mathcal{I}$ is then a set of the form $g.\mathbb{A}$, for some $g \in G$.

Theorem 3.3.12. *Let $\Lambda = \omega(\mathcal{K}^*)$ and $\mathrm{cl} = \mathrm{cl}_\Lambda^{\Phi_{all}}$. Then $\mathcal{I}(G)$ is a masure of type $(\mathbb{A}, \mathrm{cl})$. The masure is thick, G acts strongly transitively on $\mathcal{I}$ and for all $g \in G$, if A is an apartment of $\mathcal{I}$. Each panel of $\mathcal{I}$ is included in $1 + |\mathcal{O}/\pi\mathcal{O}|$ chambers. If $\mathcal{K}$ is local, one can assume that $\Lambda = \mathbb{Z}$.*

Comments on the proof The theorem above is the Théorème 5.16 of [Rou16] (the fact that the isomorphisms of apartments induced by element of G are vectorially Weyl isomorphism is a part of Theorem 4.11 of [Rou17]). In order to prove (MA3), Rousseau uses decompositions of the group G. For the case where $\mathfrak{R}$ is the germ of a chimney and F is a face, he uses an analog of Iwasawa decomposition, see Proposition 3.6 of [GR08] or Proposition 4.7 of [Rou16]. For the case where $\mathfrak{R}$ and F are germs of chimneys, he generalizes Bruhat and Birkhoff decompositions for G, see Proposition 6.7 of [Rou11]; the case where $\mathfrak{R}$ and F have the same sign correspond to Bruhat decomposition and the case where they have opposite signs correspond to Birkhoff decomposition.

In order to study the properties relative to the enclosure map, for example (MA2) and (MA4), Gaussent and Rousseau introduce the notion of "good fixers" (see 4.1 of [GR08]). A set or a filter of $\mathbb{A}$ has a good fixer if its fixer G_Ω in G admits decompositions involving U_Ω^{pm+}, U_Ω^{nm-} and the fixer $\widehat{N}_\Omega$ of Ω in N and have some properties of transitivity on the apartments containing Ω. They prove that many usual filters (sector-germs, preordered segment-germs, ...) have good filters. The fact that U_Ω^{pm+} might depend on $\mathrm{cl}_\Lambda^\Delta(\Omega)$ and not only on $\mathrm{cl}_\Lambda^\#(\Omega)$ explains that we need a priori to use $\mathrm{cl}_\Lambda^{\Phi_{all}}$ instead of $\mathrm{cl}_\Lambda^\#$ in the definition of masures and in the theorem above. As we shall see, we actually only need $\mathrm{cl}_\Lambda^\#$ (see Theorem 4.4.1).

Case of a reductive group Suppose that A is of finite type. By Proposition 3.9 of [Rou16], $\mathbf{G}$ is isomorphic to $\mathbf{G}^{pma} = \mathbf{G}^{nma}$. One does not need to use completions of G and one has $\widehat{P}_\Omega = \widehat{N}_\Omega.U_\Omega$ for all subset Ω of $\mathbb{A}$ (see 7.1.8 of [BT72]). By Examples 3.14 of [GR08], $\mathcal{I}$ is then the usual Bruhat-Tits building of G.

3.3.7 The almost split case

Let $\mathcal{K}$ be a field and $\overline{\mathcal{K}}$ be its algebraic closure. As for reductive groups, it seems natural to study Kac-Moody groups that are not split on $\mathcal{K}$ but which splits over $\overline{\mathcal{K}}$. Rémy developed a theory of "almost-split Kac-Moody groups" in [Rém02]. Charignon associated masures to almost-split Kac-Moody groups over local fields in [Cha10] and then Rousseau extended this construction to valued fields in [Rou17]. We explain here briefly some differences between associated to split Kac-Moody groups and masures associated to almost split Kac-Moody groups.

If $\mathcal{K}$ is a field, one denotes by Sep($\mathcal{K}$) the category of separable field extensions of $\mathcal{K}$. A **Kac-Moody group** over the field $\mathcal{K}$ is a functor $\mathbf{G}$ from the category Sep($\mathcal{K}$) to the category of groups such that there exists a root generating system $\mathcal{S}$, a field $E \in \text{Sep}(\mathcal{K})$ and a functorial isomorphism between the restrictions $\mathbf{G}_E$ and $\widetilde{\mathbf{G}}_{\mathcal{S},E}$ of $\mathbf{G}$ and $\widetilde{\mathbf{G}}_{\mathcal{S}}$ to $\text{Sep}(E) = \{F \in \text{Sep}(K)|\ E \subset F\}$. One says that $\mathbf{G}$ is split over E if it satisfies moreover conditions (PREALG 1,2, SGR, ALG1,2, DCS2) of 11 and 12 of [Rém02]. These conditions involves in particular the action of the Galois group $\Gamma = \text{Gal}(\mathcal{K}_s/\mathcal{K})$, where $\mathcal{K}_s$ is a separable closure of $\mathcal{K}$. The functor $\mathbf{G}$ is said to be **almost-split** if the action of Γ preserves the conjugacy classes of Borel subgroups, see (PRD) of 11.3.1 of [Rém02].

There is still a notion of root system for the group $\mathbf{G}$ in the sense of 2.2 of [Bar96]. Let ${}_{\mathcal{K}}\Phi_{all}$ (resp. ${}_{\mathcal{K}}\Phi_{re}$) be the set of roots (resp. of real roots) of $\mathbf{G}$. Let $A = (a_{i,j})_{i,j\in I}$ be the Kac-Moody matrix of $\mathcal{S}$. By 2.5 of [Rou17], there exists sets ${}_{\mathcal{K}}I$, ${}_{\mathcal{K}}I_{\text{re}}$ such that ${}_{\mathcal{K}}I_{\text{re}} \subset {}_{\mathcal{K}}I$ and that we can consider as subset of I such that $\Phi_{all} \subset \bigoplus_{i\in{}_{\mathcal{K}}I} \mathbb{Z}_{\mathcal{K}}\alpha_i$ and ${}_{\mathcal{K}}\Phi_{re} \subset \bigoplus_{i\in{}_{\mathcal{K}}I_{\text{re}}} \mathbb{Z}_{\mathcal{K}}\alpha_i$. One has ${}_{\mathcal{K}}\Phi_{all} = {}_{\mathcal{K}}{\Phi_{all}}^+ \cup {}_{\mathcal{K}}{\Phi_{all}}^-$ and ${}_{\mathcal{K}}\Phi_{re} = {}_{\mathcal{K}}\Phi_{re}^+ \cup {}_{\mathcal{K}}\Phi_{re}^-$ with obvious notation. The root system is not necessarily reduced: there can exist $\alpha \in {}_{\mathcal{K}}\Phi_{re}$ such that $2\alpha \in {}_{\mathcal{K}}\Phi_{re}$. There still exists an affine space ${}_{\mathcal{K}}\mathbb{A}$ such that $\Phi_{all} \subset {}_{\mathcal{K}}\mathbb{A}^*$ and a Weyl group ${}_{\mathcal{K}}W^v \subset {}_{\mathcal{K}}\mathbb{A}$ such that $({}_{\mathcal{K}}W^v, ({}_{\mathcal{K}}s_i)_{i\in{}_{\mathcal{K}}I_{\text{re}}})$ is a Coxeter system. The set ${}_{\mathcal{K}}\Phi_{re}$ is included in $\{\frac{1}{2}, 1, 2\}.\{w.\mathcal{K}_{\alpha_i} | w \in {}_{\mathcal{K}}W^v, i \in {}_{\mathcal{K}}I_{\text{re}}\}$. Let ${}_{\mathcal{K}}\Delta^r$ be the set of roots defined in 2.9 3) of [Rou17]. Then ${}_{\mathcal{K}}\Delta^r \backslash {}_{\mathcal{K}}\Phi_{re}$ satisfies the condition of Remark 3.1.4. By [Rou17] and in particular its Proposition 6.11 3), if $(\mathcal{K}, \omega)$ is a valued field with non trivial valuation, $\mathbf{G}(\mathcal{K})$ acts on a masure $\mathcal{I}$ of type $\text{cl}_{\Lambda'}^{{}_{\mathcal{K}}\Delta^r}$, for some $\Lambda' = (\Lambda'_\alpha)_{\alpha\in{}_{\mathcal{K}}\Delta^r} \in \mathscr{P}(\mathbb{R})^{{}_{\mathcal{K}}\Delta^r}$ defined in 6.11 2) of [Rou17]. For all apartment A of $\mathcal{I}$ and all $g \in G$, $g_{|A}^{|g.A}$ is a vectorially Weyl isomorphism of apartments.

Suppose that $\mathcal{K}$ is local and let q be the (finite) residue cardinal of $\mathcal{K}$. By Lemma 1.3 of [GR14], one can suppose that $\Lambda'_\alpha = \mathbb{Z}$ for all $\alpha \in {}_{\mathcal{K}}\Phi$. One has $Q^\vee = \bigoplus_{i\in{}_{\mathcal{K}}I_{\text{re}}\mathbb{Z}_{\mathcal{K}}\alpha_i} \mathbb{Z}\alpha_i^\vee$. Let $i \in {}_{\mathcal{K}}I_{\text{re}}$. Let P (resp. P') be a panel of $\alpha_i^{-1}(\{0\})$ (resp. $\alpha_i^{-1}(\{1\})$). Then the number $1 + q_i$ (resp. q'_i) of chambers containing P does not depend on the choice of the panel $P \subset \alpha_i^{-1}(\{0\})$ (resp. $P' \subset \alpha_i^{-1}(\{1\})$). Moreover q_i and q'_i are powers of q. If $k \in \mathbb{Z}$ and P_k is a panel of $\alpha_i^{-1}(\{k\})$, then the number of chambers of $\mathcal{I}$ containing P_k is $1 + q_i$ if k is even and $1 + q'_i$ otherwise (this will be explained in 5.2.1). However contrary to the split case, one can have $q_i \neq q'_i$ and (q_i), $(q'_i)_{i\in I}$ are not necessarily constant.

3.4 Action of a Kac-Moody group on its masure

In this section, we recall some properties of the action of a Kac-Moody group on its masure.

In 3.4.1, we summarize the fixers and stabilizers of important filters of the masure.
In 3.4.2, we detail the example of a tree.
In 3.4.3, we prove Cartan and Iwasawa decompositions, using masures.

3.4.1 Fixers and stabilizers of usual filters of $\mathbb{A}$

In this subsection, we list the fixers and stabilizer in G of usual filters of the standard apartment.

Let A be a Kac-Moody matrix, $\mathcal{S}$ be a root generating system of type A and $\mathbf{G}$ be a split Kac-Moody groups associated to $\mathcal{S}$. Let $\mathcal{K}$ be a field equipped with a non-trivial valuation $\omega : \mathcal{K} \to \mathbb{R} \cup \{+\infty\}$ and $\Lambda = \omega(\mathcal{K}^*)$. We use the same notation as above for the subgroups of $\mathbf{G}$ and we still denote by boldface letters the functors and by roman letters their evaluation in $\mathcal{K}$.

One sets $B^+ = \langle T.U \rangle$ and $B^- = \langle T, U^- \rangle$ and one call B^+ and B^- the **positive** and **negative Borel subgroups** of G.

Let $\mathcal{I}$ be the masure associated to G constructed in [Rou16] and $\mathrm{cl} = \mathrm{cl}_\Lambda^{\Phi_{all}}$. If moreover $\mathcal{K}$ is local, one has (up to renormalization, see Lemma 1.3 of [GR14]) $\Lambda_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi_{re}$. Moreover, we have:

- the fixer of $\mathbb{A}$ in G is $H = \mathbf{T}(\mathcal{O})$ (by remark 3.2 of [GR08]),
- the stabilizer of $\mathbb{A}$ is N (by 5.7 5) of [Rou16]),
- the fixer of a half-apartment $D(\alpha, k)$ is $H.U_{\alpha,k}$ (by 5.7 7) of [Rou16]),
- the fixer of $\{0\}$ in G is $K_s = \mathbf{G}(\mathcal{O})$ by example 3.14 of [GR08],
- for all $\alpha \in \Phi_{re}$ and $k \in \mathbb{Z}$, the fixer of $D(\alpha, k)$ in G is $H.U_{\alpha,k}$ (by 4.2 7) of [GR08]),
- for all $\epsilon \in \{-, +\}$, $H.U^\epsilon$ is the fixer of $\epsilon\infty$ (by 4.2 4) of [GR08])
- the stabilizer of $\epsilon\infty$ is B^ϵ (see Lemma 3.4.1 below).

In particular, the map $G/K_s \to \mathcal{I}_0 := G.0$ is well defined and is a bijection.
If moreover, $\mathcal{K}$ is local, with residue cardinal q, each panel is contained in $1+q$ chambers.

Lemma 3.4.1. *Let $\epsilon \in \{-, +\}$. The stabilizer of $\epsilon\infty$ is B^ϵ.*

Proof. The groups T and U^ϵ stabilize $\epsilon\infty$ and thus B^ϵ stabilizes $\epsilon\infty$. Let $g \in G$ such that g stabilizes $\epsilon\infty$. Then for all $V \in +\infty$, $g.V \in +\infty$. The apartment $g.\mathbb{A}$ contains $\epsilon\infty$. By (MA4), there exists $h \in G$ such that $h.(g.\mathbb{A}) = \mathbb{A}$ and h fixes $+\infty$. One has $h.g.\mathbb{A} = \mathbb{A}$. Thus $h.g \in N$ and by paragraph "Action of N on $\mathbb{A}$", $w := h.g_{|\mathbb{A}}^{|h.g.\mathbb{A}} \in W^v \ltimes (Y \otimes \Lambda)$. One writes $w = \tau_y.w'$, where τ_y is the translation of $\mathbb{A}$ of vector $y \in Y \otimes \Lambda$ and $w' \in W^v$. Then $h.g(\epsilon C_f^v) \supset \epsilon\infty$ and hence $w' = 1$. Let $g' \in T$ inducing the translation of vector $-y \in Y \otimes \Lambda$ on $\mathbb{A}$. Then $g'.h.g_{|\mathbb{A}} = \mathrm{Id}_\mathbb{A}$ and thus $g'.h.g \in H$. Moreover, $g' \in T$ and $h \in H.U^\epsilon \subset T.U^\epsilon$ and thus $g \in B^\epsilon$. $\square$

3.4.2 Example of a discrete tree

In this subsection, we study discrete trees. There are important for us for two reasons. First because they are the "simplest" masures and give some intuitions on the problems. Secondly because a masure contains many trees as we shall see in paragraph "Extended tree associated to a wall of a masure".

Let us describe the apartment of a tree. An apartment is a simplicial complex isomorphic to $\mathbb{A}_{\mathbb{T}} = (\mathbb{R}, \mathbb{Z})$ (which means that the walls of $\mathbb{A}$ are the elements of $\mathbb{Z}$). Its structure of simplicial complex is given as follows. An **alcove** of $\mathbb{A}_{\mathbb{T}}$ is a set of the shape $]n, n+1[$ for some $n \in \mathbb{Z}$. A **vertex** of $\mathbb{A}_{\mathbb{T}}$ is an element of $\mathbb{Z}$. A **face** of $\mathbb{A}_{\mathbb{T}}$ is either an alcove or a vertex. The faces of a face F are the faces included in $\overline{F}$: the faces of an alcove $]n, n+1[$ are $]n, n+1[$, $\{n\}$ and $\{n+1\}$ and the face of a vertex is itself. The **fundamental positive alcove** C_0^+ is $]0, 1[$.

The **vectorial Weyl group** of $\mathbb{A}_{\mathbb{T}}$ is $\{\pm \mathrm{Id}_{\mathbb{R}}\} \simeq \{\pm 1\}$. The **affine Weyl group** of $\mathbb{A}_{\mathbb{T}}$ is $2\mathbb{Z} \rtimes \{\pm 1\}$ which is isomorphic to the infinite dihedral group.

The simple root of $\mathbb{A}_{\mathbb{T}}$ is $\alpha := \mathrm{Id}_{\mathbb{R}}$. The **fundamental chamber** $C_f^v = \mathbb{R}_+$ and $-C_f^v = \mathbb{R}_-$. The **sectors** of $\mathbb{A}_{\mathbb{T}}$ are the $a + \epsilon\mathbb{R}_+$ with $a \in \mathbb{R}$ and $\epsilon \in \{\pm 1\}$.

We identify simplicial complexes and their geometric realizations.

Definition 3.4.2. *A* ***Bruhat-Tits tree*** *(see Figure 3.4.1)* $\mathbb{T}$ *is a simplicial complex equipped with a set* $\mathcal{A}$ *of sub-simplicial complexes called apartments such that:*

(T0) Each apartment is a simplicial complex isomorphic to $\mathbb{A}_{\mathbb{T}}$.

(T1) Each pair of faces of $\mathbb{T}$ *is included in an apartment.*

(T2) If A *and* B *are apartments, there exists an isomorphism of simplicial complexes* $A \overset{A\cap B}{\to} B$ *(i.e fixing* $A \cap B$*).*

As we only consider Bruhat-Tits trees, we say tree for short. The **complete system of apartments** of $\mathbb{T}$ is the set $\mathcal{A}_c$ of sub-simplicial complexes of $\mathcal{I}$ isomorphic to $\mathbb{A}_{\mathbb{T}}$. Then $(\mathcal{I}, \mathcal{A}_c)$ is a tree.

The axiom T_1 and T_2 enable to define a distance on $\mathbb{T}$ extending the absolute value of $\mathbb{A}_{\mathbb{T}}$ and such that the isomorphism of apartments are isometries. Contrary to what appear on Figure 3.4.1, the distance between two consecutive vertices is always 1.

The tree of SL_2 Let $\mathbf{G} = \mathrm{SL}_2$ and $\mathcal{K}$ be a non-archimedean local field. Let $G = \mathbf{G}(\mathcal{K})$. Then the Bruhat-Tits building of G is the homogeneous tree with thickness $q+1$ (where q is the residue cardinal of $\mathcal{K}$) equipped with its complete system of apartments. A way to construct it is to use the lattices of $\mathcal{K}^2$ up to homothetic transformations, see [Ser77].

Let $\alpha = \mathrm{Id} : \mathbb{R} \to \mathbb{R}$. Then the root system of $\mathfrak{sl}_2$ is $\{\alpha, -\alpha\}$ and one has $\alpha^\vee = 2$.

A vertex x of $\mathbb{T}$ (that is an element x such that $d(0, x) \in \mathbb{N}$) is said to be even if $d(x, 0) \in 2\mathbb{N}$. Then G acts on $\mathcal{I}$ by automorphisms and if $g \in G$, $d(0, g.0)$ is even. Reciprocally, an isomorphism of apartments preserving even vertices is induced by an element of G. Let π be a uniformizer of $\mathcal{K}$. By Subsection 3.4.1, one has:

- $T := G \cap \begin{pmatrix} \mathcal{K} & 0 \\ 0 & \mathcal{K} \end{pmatrix} = \mathrm{Fix}_G(0)$,
- $H := G \cap \begin{pmatrix} \mathcal{O} & 0 \\ 0 & \mathcal{O} \end{pmatrix} = \mathrm{Fix}_G(\mathbb{A})$,
- $B^+ := G \cap \begin{pmatrix} \mathcal{K} & \mathcal{K} \\ 0 & \mathcal{K} \end{pmatrix} = \mathrm{Stab}_G(+\infty)$, $B^- := G \cap \begin{pmatrix} \mathcal{K} & 0 \\ \mathcal{K} & \mathcal{K} \end{pmatrix} = \mathrm{Stab}_G(-\infty)$,

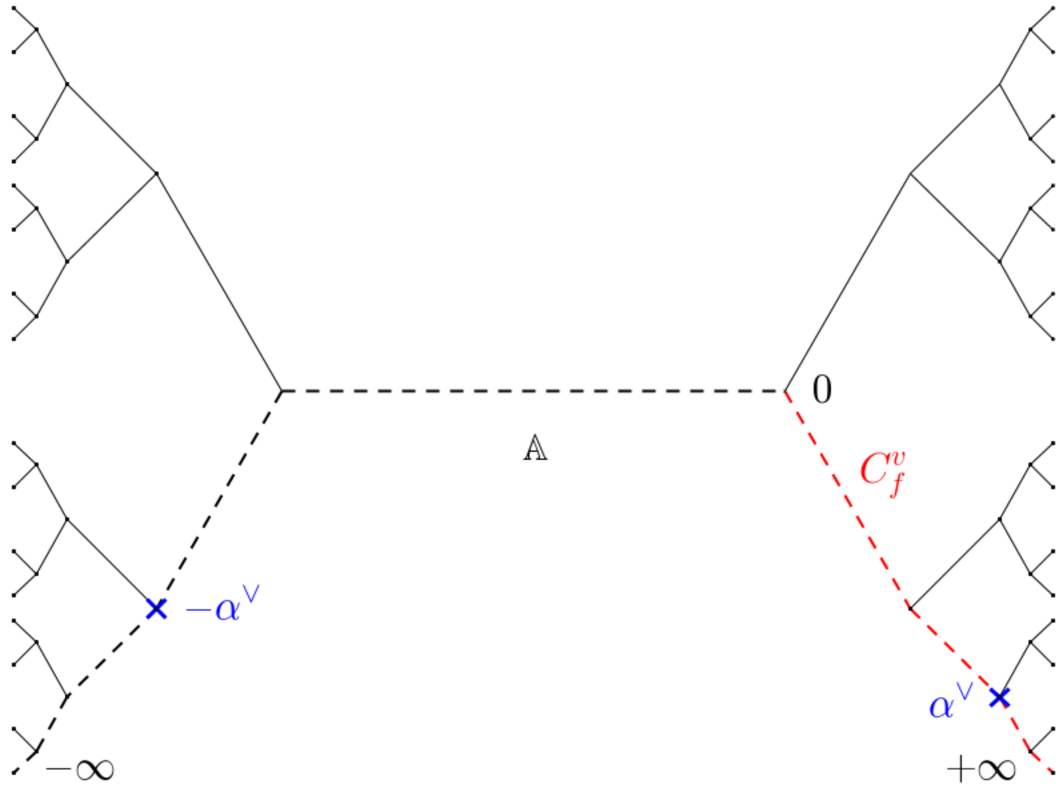

Figure 3.4.1 – Homogeneous tree with thickness 3

- if $U = U_\alpha = \begin{pmatrix} 1 & \mathcal{K} \\ 0 & 1 \end{pmatrix}$, $\mathrm{Fix}_G(+\infty) = HU$ and if $U^- = \begin{pmatrix} 1 & 0 \\ \mathcal{K} & 1 \end{pmatrix}$, $\mathrm{Fix}_G(-\infty) = HU^-$,
- if $k \in \mathbb{Z}$ and $U_{\alpha,k} := \begin{pmatrix} 1 & \pi^k \mathcal{O} \\ 0 & 1 \end{pmatrix}$, $\mathrm{Fix}_G(D(\alpha, k) = HU_{\alpha,k}$

If $k \in \mathbb{Z}$, then $\begin{pmatrix} \pi^k & 0 \\ 0 & \pi^{-k} \end{pmatrix}$ induce the translation of vector $2k$ on $\mathbb{A}_\mathbb{T}$. The matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ induces $-\mathrm{Id}$ on $\mathbb{A}_\mathbb{T}$. The fixer of C_0^+ is the Iwahori subgroup $\begin{pmatrix} \mathcal{O} & \mathcal{O} \\ \pi\mathcal{O} & \mathcal{O} \end{pmatrix}$.

Extended tree associated to a wall of a masure Let $\mathcal{I}$ be a semi-discrete masure (one can drop the semi-discreteness assumption and obtain possibly non-discrete trees). Let M be a wall of $\mathcal{I}$, M^∞ be its direction (see 3.2.4) and $\mathcal{A}(M^\infty)$ be the set of apartments containing M^∞. Concretely, an apartment A is in $\mathcal{A}(M^\infty)$ if and only if there exists a wall M' of A such that the walls M and M' are included in some apartment B and are parallel in B. One says that M and M' are parallel. Let $\mathcal{I}(M^\infty) = \bigcup_{A \in \mathcal{A}(M^\infty)} A$. One equips each apartment A with the following simplicial structure. The faces of A are the strips strictly delimited by two consecutive walls parallel to M and the walls of A parallel to M. Then by 4.9 of [Rou11], $\mathcal{I}(M^\infty)$ is an extended tree. If $x \in \mathcal{I}(M^\infty)$ and $A \in \mathcal{A}(M^\infty)$ contains x, one denotes by $x + M^\infty$ the hyperplane of A parallel to M and containing x. Then $x + M^\infty$ does not depend on the choice of A. If $x, y \in \mathcal{I}(M^\infty)$, one writes $x \sim y$ if $x + M^\infty = y + M^\infty$. Then $\mathcal{I}(M^\infty)$ can be identified to $(\mathcal{I}(M^\infty)/\sim) \times M$ and $\mathcal{I}(M^\infty)/\sim$ is a tree whose apartments are the $A/\sim$ for $A \in \mathcal{A}(M^\infty)$.

Let $\alpha \in \Phi_{re}$. If $G = \mathbf{G}(\mathcal{K})$ is a split Kac-Moody group over a local field and with the same notation as in Section 3.3, $U_\alpha.\mathbb{A} = \mathcal{I}(M^\infty)$, where $M = M(\alpha, 0)$.

3.4.3 Classical decompositions of G

In this subsection, we prove Cartan and Iwasawa decompositions for a split Kac-Moody group over a local field. For the Iwasawa decomposition, this is a circular reasoning because Rousseau uses it to prove that the masure associated to such a group satisfies its axiomatic. However this kind of reasoning is often used to obtain decompositions of G and this enables to illustrate the dictionary between quotients of subgroups of G and sets of vertices of $\mathcal{I}$, see Chapter 5 and Chapter 6.

Let $\mathcal{I}_0 = G.0$.

Lemma 3.4.3. *One has $\mathcal{I}_0 \cap \mathbb{A} = Y$.*

Proof. Let $x \in \mathcal{I}_0 \cap \mathbb{A}$. One has $x = g.0$ with $g \in G$. By (MA2), there exists a isomorphism $\phi : g.\mathbb{A} \to \mathbb{A}$ fixing x. Then $x = \phi(g.0)$ and $\phi \circ g_{|\mathbb{A}} : \mathbb{A} \to \mathbb{A}$ is a Weyl automorphism of apartment. Let $h \in G$ inducing ϕ on $g.\mathbb{A}$. Then $h.g \in N$, hence $(h.g)_{|\mathbb{A}} \in W^v \ltimes Y$ and thus $x = h.g.0 \in Y$. □

For all $y \in Y$, one chooses an element τ_y of T inducing the translation of vector y on $\mathbb{A}$. The choice has no importance because we want to consider sets of the form $\tau_y.K_s$ for $y \in Y$.

Proposition 3.4.4. *(Cartan decomposition) The map $\Gamma : K_s\backslash G^+/K_s \to Y^{++}$ mapping each $K_sgK_s \in K_s\backslash G^+/K_s$ on $d^v(0, g.0)$ is well-defined and is a bijection. Its reciprocal function is the map $\Delta : Y^{++} \to K_s\backslash G^+/K_s$ sending each $y \in Y^{++}$ on $K_s\tau_yK_s$.*

Proof. Let us prove that Γ is well-defined. Let $g \in G^+$ and $k \in K_s$. Then $d^v(0, g.0) = d^v(k.0, k.g.0) = d^v(0, k.g.0)$ (as d^v is G-invariant) and thus $\Gamma(K_sgK_s) \in \overline{C^v_f}$ is well-defined. By Remark 3.2.2, there exists $h \in G$ such that $h.g.0$, $h.0 \in \mathbb{A}$ and $h.g.0 = h.0 + d^v(0, g.0)$. Applying Lemma 3.4.3, we deduce that $d^v(0, g.0) \in Y$. Thus Γ is well-defined.

Let $y, y' \in Y^{++}$ such that $K_s\tau_yK_s = K_s\tau_{y'}K_s$. One writes $\tau_y = k.\tau_{y'}.k'$, with $k, k' \in K_s$. Then $\tau_y.0 = k.\tau_{y'}.0 = y = k.y'$. Therefore $d^v(0, y) = y = d^v(0, k.y') = y'$ (by G-invariance of d^v).

Let $y \in Y$. Let us prove that $K_s\tau_{w.y}K_s = K_s\tau_yK_s$ for all $w \in W^v$. Indeed, let $w \in W^v$. Let $k \in K_s$ inducing w^{-1} on $\mathbb{A}$. Then $k.\tau_{w.y}.k^{-1}$ induces a translation on $\mathbb{A}$. As $k.\tau_{w.y}.k^{-1}(0) = y$, $K_s\tau_{w.y}K_s = K_s\tau_yK_s$.

Let $g \in G^+$ and $y = d^v(0, g.0)$. Let us prove that $g \in K_s\tau_yK_s$. Let $A = g.\mathbb{A}$. One has $\tau_{g.0}^{-1}.g \in K_s$. Therefore $g \in \tau_{g.0}.K_s$. Hence $g \in K_s\tau_{g.0}K_s = K_s\tau_yK_s$. Consequently Δ is well-defined and Γ and Δ are inverse of each others. □

Remark 3.4.5. *If we replace K_s by the fixer of a face between 0 and C_0^+, we get similar decompositions.*

Proposition 3.4.6. *(Iwasawa decomposition) The map $\Gamma : U\backslash G/K_s \to Y$ mapping each $UgK_s \in U\backslash G/K_s$ on $\rho_{+\infty}(g.0)$ is well-defined and is a bijection. Its reciprocal function is the map $\Delta : Y \to U\backslash G/K_s$ sending each $y \in Y$ on $U\tau_yK_s$.*

Proof. Let us prove that $\rho_{+\infty}$ is U-invariant. Let $x \in \mathcal{I}$. Let $A \in \mathcal{A}(x, +\infty)$ and $\phi : A \overset{+\infty}{\to} \mathbb{A}$. Then $\rho_{+\infty}(x) = \phi(x)$. Let $u \in U$. Then u fixes $+\infty$. Let $\psi = (\phi \circ u^{-1})_{|u.\mathbb{A}}^{|\mathbb{A}}$. Then ψ fixes $+\infty$, thus $\psi(u.x) = \rho_{+\infty}(u.x)$. But $\psi(u.x) = \phi(x)$ and thus $\rho_{+\infty}(u.x) = \rho_{+\infty}(x)$. Consequently $\rho_{+\infty}$ is U-invariant.

By definition of $\rho_{+\infty}$, $\rho_{+\infty}(g.0) \in \mathcal{I}_0 \cap \mathbb{A} = Y$ and thus Γ is well-defined.

Let $y, y' \in Y$ such that $U\tau_yK_s = U\tau_{y'}K_s$. Then $\rho_{+\infty}(\tau_y.K_s.0) = \rho_{+\infty}(\tau_{y'}.K_s.0) = y = y'$.

Let $g \in G$ and $y = \rho_{+\infty}(g.0)$. Let us prove that $g \in U\tau_y K_s$. Let $A \in \mathcal{A}(g.0, +\infty)$ which exists by (MA3). Let $\phi : A \overset{+\infty}{\to} \mathbb{A}$ and $u \in U$ such that $u.A = \mathbb{A}$, which exists by (MA4). Then $u.g.0 = y$. Thus $\tau_y^{-1}.u.g \in K_s$. Therefore $g \in U\tau_y K_s$. Consequently Δ is well-defined and Γ and Δ are inverse of each others.

□

3.5 Notation

In this Section, we recall the notations introduced in the thesis.

Affine spaces and topology Let X be a finite dimensional affine space. Let $C \subset X$ be a convex set and A' be its support. The **relative interior** (resp. **relative frontier**) of C, denoted $\mathrm{Int}_r(C)$ (resp. $\mathrm{Fr}_r(C)$) is the interior (resp. frontier) of C seen as a subset of A'. A set is said to be **relatively open** if it is open in its support.

If X is an affine space and $U \subset X$, one denotes by $\mathrm{conv}(X)$ the convex hull of X. If $x, y \in \mathbb{A}$, we denote by $[x, y]$ the segment of $\mathbb{A}$ joining x and y. If A is an apartment and $x, y \in A$, we denote by $[x, y]_A$ the segment of A joining x and y.

If X is a topological space and $a \in X$, one denotes by $\mathcal{V}_X(a)$ the set of open neighborhoods of a.

If X is a subset of $\mathbb{A}$, one denotes by $\mathring{X}$ or by $\mathrm{Int}(X)$ (depending on the legibility) its interior. One denotes by $\mathrm{Fr}(X)$ the boundary (or frontier) of X: $\mathrm{Fr}(X) = \overline{X} \backslash \mathring{X}$.

If X is a topological space, $x \in X$ and Ω is a subset of X containing x in its closure, then the **germ** of Ω in x is denoted $germ_x(\Omega)$.

We use the same notation as in [Rou11] for segments and segment-germs in an affine space X. For example if $X = \mathbb{R}$ and $a, b \in \overline{\mathbb{R}} = \mathbb{R} \cup \{\pm\infty\}$, $[a, b] = \{x \in \overline{\mathbb{R}} | \ a \leq x \leq b\}$, $[a, b[= \{x \in \overline{\mathbb{R}} | \ a \leq x < b\}$, $[a, b) = germ_a([a, b])$ ...

Masure If A and B are apartments, $X \subset A \cap B$ and $\phi : A \to B$, the notation $\phi : A \overset{X}{\to} B$ means that ϕ fixes X. If such an isomorphism is unique, one sometimes denote it $A \overset{X}{\to} B$.

If $\mathfrak{q}$ is a sector germ and A is an apartment containing $\mathfrak{q}$, we let $\mathcal{I} \overset{\mathfrak{q}}{\to} A$ denote the retraction onto A centered at $\mathfrak{q}$.

If $\mathcal{X}$ is a filter, one denotes by $\mathcal{A}(\mathcal{X})$ the set of apartments containing $\mathcal{X}$.

Chapter 4

Axiomatic of masures

4.1 Introduction

4.1.1 Axiomatic of masures

Let G be a reductive group over a local field. The construction of the building $\mathcal{I}$ associated to G in [BT72] and [BT84] is rather complicate. However, we can recover many information on $\mathcal{I}$ by considering it as an abstract building, which means that $\mathcal{I}$ is a union of subsets of $\mathcal{I}$ called apartments, satisfying the following axioms:

(I0) Each apartment is a euclidean apartment.

(I1) For any two faces F and F' there exists an apartment containing F and F'.

(I2) If A and A' are apartments, their intersection is a union of faces and, for any faces F, F' in $A \cap A'$ there exists an isomorphism from A to A' fixing F and F'.

In this definition, a euclidean apartment is a euclidean space equipped with a locally finite arrangement of hyperplanes called walls satisfying some conditions. If G is a split reductive group over a local field, $T \subset G$ is a maximal split torus, and $\Phi \subset \mathbb{A}^*$ (where $\mathbb{A}$ is some affine space) is the set of roots of (G, T), then $\mathbb{A}$ equipped with the set of walls $\{\alpha^{-1}(\{k\}) | (\alpha, k) \in \Phi \times \mathbb{Z}\}$ is an example of euclidean apartment.

This definition comes from Section 6 of [Rou04] (see also [Tit86] and Section IV of [Bro89], and [Par00] or [BS14] for a definition of possibly non-discrete buildings).

Rousseau had a similar approach to masures. Inspired by the axioms above, he proposes in [Rou11] a list of five axioms (see 3.1.5) that a masure associated to an almost-split Kac-Moody group over a valued field satisfies (see 4.10 of [Rou17] for the almost-split case). He then studies the properties of abstract masures. The results of [GR14], [BPGR16], [Héb17a], [AH17] and [BPGR17] are obtained with this point of view: the expected property of Kac-Moody groups is translated as a property of the associated masure and then this property is proved for abstract masures (satisfying additional conditions such as semi-discreteness, existence of a group acting strongly transitively, thickness, finite thickness, ...).

The aim of this chapter is to simplify Rousseau's axiomatic definition of masures given in 3.1.5.

Let $\mathcal{I}$ be a Bruhat-Tits building. Then the following proposition is well-known:

Proposition 4.1.1. *(Proposition 2.5.8 of [BT72]) Let A and B be two apartments. Then there exist $k \in \mathbb{N}$ and half-apartments $D_1, \ldots, D_k$ of A such that $A \cap B = \bigcap_{i=1}^k D_i$ and there exists an isomorphism of apartments $\phi : A \to B$ fixing $A \cap B$.*

As masures generalize Bruhat-Tits buildings, it is natural to ask if this property is true for masures. Moreover, (MA2), (MA4) and (MAO) are weak versions of this property. Therefore if this property were true, one could replaces these axioms by the proposition above. We do not know if this proposition is true in general. However, we prove a weak version of it:

Theorem 8. (see Theorem 4.3.22) Let $\mathcal{I}$ be a masure. Let A and B be two apartments sharing a generic ray. Then there exist $k \in \mathbb{N}$ and half-apartments $D_1, \ldots, D_k$ of A such that $A \cap B = \bigcap_{i=1}^{k} D_i$ and there exists an isomorphism of apartments $\phi : A \to B$ fixing $A \cap B$.

In the particular case of masures associated to affine Kac-Moody group, we prove that this property is satisfied:

Theorem 9. (see Theorem 4.4.37) Let $\mathcal{I}$ be a masure of affine type. Let A and B be two apartments. Then there exist $k \in \mathbb{N}$ and half-apartments $D_1, \ldots, D_k$ of A such that $A \cap B = \bigcap_{i=1}^{k} D_i$ and there exists an isomorphism of apartments $\phi : A \to B$ fixing $A \cap B$.

We then define new series of axioms simpler and closer to the definition of Bruhat-Tits buildings: one in the general case and one in the affine case. Using Theorem 8 and 9, we prove that they are equivalent to the one given by Rousseau. We prove the following theorems:

Theorem 10. (see Theorem 4.4.1) Let $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ be an apartment and $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$ be an enclosure. Then the series of axioms (MA1, cl), (MA2, cl), (MA3, cl), (MA4, cl) and (MAO, cl) is equivalent to the series of axioms (MA1, $\mathrm{cl}^{\#}$), (MA3, $\mathrm{cl}^{\#}$) and (MA ii): "if A and B are apartments sharing a generic ray, then there exist $k \in \mathbb{N}$ and half-apartments $D_1, \ldots, D_k$ of A such that $A \cap B = \bigcap_{i=1}^{k} D_i$ and there exists an isomorphism of apartments $\phi : A \to B$ fixing $A \cap B$."

Theorem 11. (see Theorem 4.4.33) Let $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ be an apartment, where $\mathcal{S}$ is associated to an affine Kac-Moody matrix and $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$ be an enclosure. Then the series of axioms (MA1, cl), (MA2, cl), (MA3, cl), (MA4, cl) and (MAO, cl) is equivalent to the series of axioms (MA1, $\mathrm{cl}^{\#}$), (MA3, $\mathrm{cl}^{\#}$) and (MA af ii): "if A and B are apartments then there exist $k \in \mathbb{N}$ and half-apartments $D_1, \ldots, D_k$ of A such that $A \cap B = \bigcap_{i=1}^{k} D_i$ and there exists an isomorphism of apartments $\phi : A \to B$ fixing $A \cap B$."

An important result of the theorem above is that the series of axioms (MA1, cl), (MA2, cl), (MA3, cl), (MA4, cl) and (MAO, cl) does not depend on the choice of enclosure $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. Consequently, the best choice is to take the biggest one $\mathrm{cl}^{\#}$.

Frameworks Actually we do not limit our study to masures associated to Kac-Moody groups: for us a masure is a set satisfying the axioms of [Rou11] and whose apartments are associated to a root generating system (and thus to a Kac-Moody matrix). We do not neither assume that there exists a group acting strongly transitively on it. We do not either make any discreteness hypothesis for the standard apartment: if M is a wall, the set of walls parallel to it is not necessarily discrete; this enables to handle masures associated to split Kac-Moody groups over any ultrametric field. We assume that the root system Φ_{re} is reduced, which means that for all $\alpha \in \Phi_{re}$, $\mathbb{R}\alpha \cap \Phi_{re} = \{-\alpha, \alpha\}$. We can drop this hypothesis with minor changes, see Remark 4.2.2.

Organization of the chapter This chapter is organized as follows. In Section 4.2, we study the properties of the intersection of two arbitrary apartments A and B. In Section 4.3, we restrict our study to the case where $A \cap B$ contains a generic ray. The main result is Theorem 8. In Section 4.4, we simplify the axiomatic of masures, using Theorem 8.

4.1.2 Secondary results

In the proof of Theorem 10 and Theorem 11, we prove some results which could be interesting on their own.

The following proposition enables to reduce some proofs to the case where two apartments share a sector. We use it in Chapter 7 and Chapter 8. This is a simplified version of Lemma 4.2.7 and Proposition 4.2.8.

If $\mathfrak{q}$ is a sector-germ of $\mathcal{I}$ and A is an apartment, $d(\mathfrak{q}, A)$ is the minimal possible length of a gallery between $\mathfrak{q}$ a sector-germ of A of the same sign as $\mathfrak{q}$.

Proposition 4.1.2. *Let A be an apartment, $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$ such that $\mathfrak{q} \not\subseteq A$ and $n = d(\mathfrak{q}, A)$.*

1. *One can write $A = D_1 \cup D_2$, where D_1 and D_2 are opposite half-apartments of A such that for all $i \in \{1, 2\}$, there exists an apartment A_i containing D_i and such that $d(A_i, \mathfrak{q}) = n - 1$.*

2. *There exist $k \in \mathbb{N}$, enclosed subsets $P_1, \ldots, P_k$ of A such that for all $i \in [\![1, k]\!]$, there exist an apartment A_i containing $\mathfrak{q} \cup P_i$ and an isomorphism $\phi_i : A \overset{P_i}{\to} A_i$.*

The first part of the Proposition above correspond to the axiom (SC) (sundial configuration) of [BS14].

The following theorem improves Proposition 5.1, Proposition 5.2 and Proposition 5.5 of [Rou11] and Proposition 1.10 of [BPGR16]. This kind of results is useful to define Hecke algebras. Only Point 1 is used in the proof of Theorem 11.

Theorem 4.1.3. *(see Proposition 4.4.16 and Theorem 4.4.17) Let $x, y \in \mathcal{I}$ and F_x (resp. F_y) be a face based at x (resp. y).*

1. *There exists an apartment containing $F_x \cup F_y$ if and only if there exists an apartment containing $\{x, y\}$*

2. *Suppose moreover $x \leq y$ (or $y \leq x$). Let A and B be two apartments containing $F_x \cup F_y$. Then $A \cap B$ contains the convex hull $\mathrm{conv}_A(F_x \cup F_y)$ of $F_x \cup F_y$ and there exists an isomorphism from A to B fixing $\mathrm{conv}_A(F_x \cup F_y)$.*

4.1.3 Outline of the proof of Theorem 8

The main step of our proof of the equivalence of the axiomatics is Theorem 8: if A and B are two apartments sharing a generic ray then $A \cap B$ is a finite intersection of half-apartments and there exists an isomorphism $\phi : A \to B$ fixing $A \cap B$. Let us sketch its proof. We can suppose that $B = \mathbb{A}$. We first prove that its suffice to prove this theorem when $A \cap \mathbb{A}$ has nonempty interior. We prove that $A \cap \mathbb{A}$ is closed in $\mathbb{A}$. Then by using order-convexity (the fact that if $x, y \in \mathcal{I}$ such that $x \leq y$, then the segment linking x to y is independent on the choice of apartment containing x and y), we prove that $A \cap \mathbb{A}$ is connected and satisfies $A \cap \mathbb{A} = \overline{\mathrm{Int} A \cap \mathbb{A}}$. It thus suffices to study $\mathrm{Fr}(A \cap \mathbb{A})$. We then parametrize $\mathrm{Fr}(A \cap \mathbb{A})$ by a map $\mathrm{Fr} : U \to A \cap \mathbb{A}$ where U is a convex closed set of $A \cap \mathbb{A}$ with nonempty interior. The key point is then to prove that the map $\mathrm{Fr} : \mathring{U} \to \mathrm{Fr}(A \cap \mathbb{A})$ is convex. For this we prove that the maps $\mathrm{Fr}_{x,y} = \mathrm{Fr}_{|[x,y]}$ are convex for good choices of $[x, y]$. We proceed in two steps: first, we prove that they are piecewise affine and then we study the points where the slope changes, using order convexity. The fact that $\mathrm{Fr}_{x,y}$ is piecewise affine is a consequence of Section 4.2, where we prove, using Proposition 4.1.2 that the frontier of $A \cap \mathbb{A}$ is included in a finite number of walls (see Proposition 4.2.22).

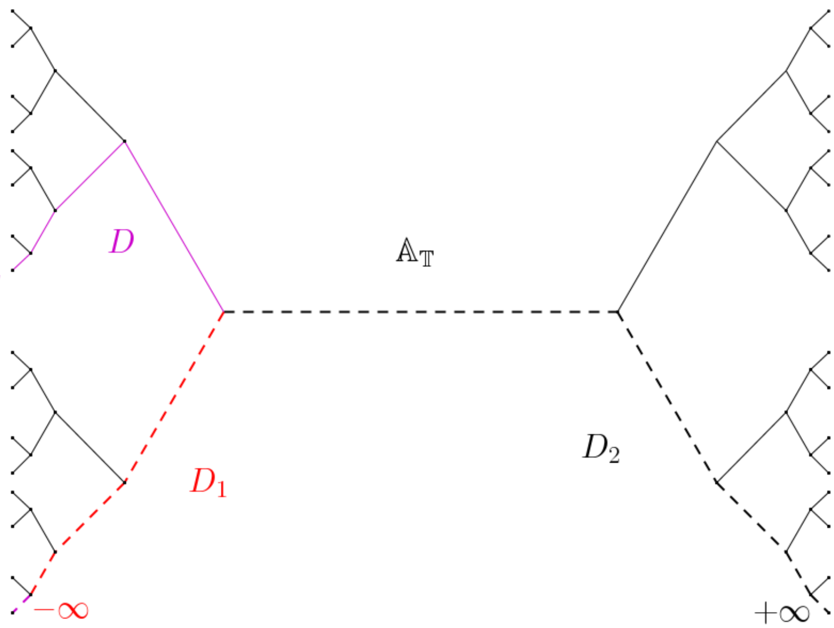

Figure 4.2.1 – Sundial configuration in a tree. In this case, all the sector-germs are adjacent to each others. One has $\mathbb{A}_{\mathbb{T}} = D_1 \cup D_2$ and $D_1 \cup D$ and $D_2 \cup D$ are apartments containing $\mathfrak{q}$.

4.2 General properties of the intersection of two apartments

In this section, we study the intersection of two apartments, without assuming that their intersection contains a generic ray. Our main result is that the intersection of two apartments can be written as a finite union of enclosed subsets (see Proposition 4.2.22.

In Subsection 4.2.1, we prove Proposition 4.1.2.

In Subsection 4.2.2, we prove some preliminary results. We prove in particular that the intersection of two apartment is closed (Proposition 4.2.17) and we give a characterization of the points of $\mathbb{A}$ by retractions (Proposition 4.2.15), which will be crucial in Subsection 4.2.3.

In Subsection 4.2.3, we write the intersection of two apartments as a finite union of enclosed parts.

In Subsection 4.2.4, we use the results of Subsection 4.2.3 to prove that if the intersection of two apartments is convex, then it is enclosed.

In Subsection 4.2.5, we study the existence of isomorphisms fixing subsets of an intersection of two apartments.

4.2.1 Sundial configuration and splitting of apartments

The aim of this subsection is to prove Proposition 4.1.2. The main step is the fact that if A is an apartment and $\mathfrak{q}$ is a sector-germ not included in A but such that $\mathfrak{q}$ is adjacent to a sector-germ of A, one can write $A = D_1 \cup D_2$, with $D_i, \mathfrak{q}$ included in some apartment A_i for all $i \in \{1, 2\}$ (this is Lemma 4.2.7).

Recall the definition of cl and $D(\alpha, k)$ for $\alpha \in \Phi_{re}$ and $k \in \mathbb{R}$ from 3.1.3.

Lemma 4.2.1. *Let $\alpha \in \Phi_{re}$ and $k \in \mathbb{R} \backslash \Lambda'_\alpha$. Then $\mathrm{cl}(D(\alpha, k)) \supsetneq D(\alpha, k)$ (where we identify $D(\alpha, k)$ and the filter composed of subsets of $\mathbb{A}$ containing $D(\alpha, k)$).*

Proof. By definition of cl, $D(\alpha, k) \in \mathrm{cl}(D(\alpha, k))$.

Let $\mathcal{X}$ be the filter of subsets of $\mathbb{A}$ containing strictly $D(\alpha,k)$. Let us prove that $\mathcal{X} \subset \mathrm{cl}(D(\alpha,k))$.

Let $E \in \mathrm{cl}(D(\alpha,k))$. Then there exists $(\ell_\beta) \in (\mathbb{R}\cup\{+\infty\})^{\Phi_{all}}$ such that $E \supset \bigcap_{\beta\in\Phi_{all}} D(\beta,\ell_\beta) \supset D(\alpha,k)$. Let $\beta \in \Phi_{all}\backslash\{\alpha \cup -\alpha\}$. Then $D(\beta,\ell_\beta) \supset D(\alpha,k)$ and thus $\ell_\beta = +\infty$. As $D(-\alpha,\ell_{-\alpha}) \supset D(\alpha,k)$, $\ell_{-\alpha} = +\infty$. Therefore $E \supset D(\alpha,\ell_\alpha) \supset D(\alpha,k)$. We deduce that $\ell_\alpha \geq k$ and by hypothesis, $\ell_\alpha > k$. Consequently, $E \supsetneq D(\alpha,k)$: $E \in \mathcal{X}$.

As $D(\alpha,k) \notin \mathcal{X}$, we deduce that $D(\alpha,k) \subsetneq \mathcal{X} \subset \mathrm{cl}(D(\alpha,k))$ and the lemma is proved. □

Remark 4.2.2. *If the system of roots Φ_{re} is not reduced, one has to replace Λ'_α by $\Lambda'_\alpha \cup \frac{1}{2}\Lambda'_{2\alpha}$ or by $2\Lambda'_{\frac{1}{2}\alpha} \cup \Lambda'_\alpha$ in the lemma above.*

Lemma 4.2.3. *Let A, B be two distinct apartments of $\mathcal{I}$ containing a half-apartment D. Then $A \cap B$ is a half-apartment.*

Proof. Using an isomorphism of apartments if necessary, one can suppose that $B = \mathbb{A}$. Let $\alpha \in \Phi_{re}$ and $k \in \mathbb{R}$ such that $\mathbb{A} \cap A \supset D(\alpha,k)$. Let us first prove that $\mathbb{A} \cap A = D(\alpha,\ell)$ for some $\ell \in \mathbb{R}$. Let $\vec{M} = \alpha^{-1}(\{0\})$, $D_0 = D(\alpha,0)$ and Q be a sector of $\mathbb{A}$ such that $\overline{Q} \cap \vec{M}$ is a sector panel $\vec{P}$ of $\vec{M}$. Let $\vec{P'} = -\vec{P}$. Let $\mathfrak{p}$, $\mathfrak{p}'$, $\mathfrak{q}$ denote the germs of $\vec{P}$, $\vec{P'}$ and Q. Let $x \in \mathbb{A} \cap A$. Then $\mathbb{A} \cap A$ contains $x + \mathfrak{q}$ and $x + \mathfrak{p}'$, which are splayed chimneys. By (MA4), $\mathbb{A} \cap A$ contains the closure of the convex hull of $x + \mathfrak{q}$ and $x + \mathfrak{p}'$ and hence it contains $x + D_0$. Consequently, there exists $\ell \in \mathbb{R}$ such that $\mathbb{A} \cap A$ is either $\mathbb{A} \cap A = D(\alpha,\ell)$ or $\mathbb{A} \cap A = D^\circ(\alpha,\ell)$. Let us prove that the second case cannot occur. Let $x \in \mathbb{A}$ such that $\alpha(x) = -\ell$. Let $y \in x \pm \mathcal{T}$ such that $\alpha(y) > -\ell$. Then $\mathbb{A} \cap A \supset germ_x(]x,y])$. By (MA2), $\mathbb{A} \cap A$ contains $\mathrm{cl}(germ_x(]x,y])) \supset germ_x([x,y])$ and hence $\mathbb{A} \cap A = D(\alpha,\ell)$.

Let $x \in \mathbb{A}$ such that $\alpha(x) = -\ell$. By (MA4) applied to $x + \mathfrak{q}$ and $x + \mathfrak{p}'$, $\mathbb{A} \cap A$ contains $\mathrm{cl}(D(\alpha,\ell))$. By Lemma 4.2.1, we deduce that $\ell \in \Lambda'_\alpha$ and thus $\mathbb{A} \cap A$ is a half-apartment. □

Lemma 4.2.4. *Let M be a wall of $\mathbb{A}$ and $\phi \in W$ be an element fixing M. Then $\phi \in \{\mathrm{Id}, s\}$, where s is the reflection of W with respect to M.*

Proof. One writes $\phi = \tau \circ u$, with $u \in W^v$ and τ a translation of $\mathbb{A}$. Then $u(M)$ is a wall parallel to M. Let $\vec{M}$ be the wall parallel to M containing 0. Then $u(\vec{M})$ is a wall parallel to $\vec{M}$ and containing 0: $u(\vec{M}) = \vec{M}$. Let C be a vectorial chamber adjacent to $\vec{M}$. Then $u(C)$ is a chamber adjacent to C: $u(C) \in \{C, \vec{s}(C)\}$, where $\vec{s}$ is the reflection of W with respect to $\vec{M}$. Maybe composing u by $\vec{s}$, one can suppose that $u(C) = C$ and thus $u = \mathrm{Id}$ (because the action of W^v on the set of chambers is simply transitive). □

If D is a half-apartment of $\mathcal{I}$, one sets $\mathfrak{D}$, the filter of subsets of $\mathcal{I}$ containing a sub-half-apartment of D. If D_1 and D_2 are two half-apartments, one says that $D_1 \sim D_2$ if $\mathfrak{D}_1 = \mathfrak{D}_2$ and one says that D_1 and D_2 have opposite directions if there exists an apartment A containing a shortening of them, if their walls are parallel, and if D_1 and D_2 are not equivalent. One says that D_1 and D_2 are opposite if they have opposite directions and if $D_1 \cap D_2$ is a wall.

Lemma 4.2.5. *Let A_1, A_2, A_3 be distinct apartments. Suppose that $A_1 \cap A_2$, $A_1 \cap A_3$ and $A_2 \cap A_3$ are half-apartments such that $A_1 \cap A_3$ and $A_2 \cap A_3$ have opposite directions. Then $A_1 \cap A_2 \cap A_3 = M$ where M is the wall of $A_1 \cap A_3$, and for all $(i,j,k) \in \{1,2,3\}^3$ such that $\{i,j,k\} = \{1,2,3\}$, $A_i \cap A_j$ and $A_i \cap A_k$ are opposite. Moreover, if $s : A_3 \to A_3$ is the reflection with respect to M, $\phi_1 : A_3 \overset{A_1\cap A_3}{\to} A_1$, $\phi_2 : A_3 \overset{A_2\cap A_3}{\to} A_2$ and $\phi_3 : A_1 \overset{A_2\cap A_1}{\to} A_2$, then* *the following diagram is commutative:*

$$\begin{array}{ccc} A_3 & \xrightarrow{s} & A_3 \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} \\ A_2 & \xrightarrow{\phi_3} & A_1 \end{array}$$

Proof. By "Propriété du Y" (Section 4.9 of [Rou11]), $A_1 \cap A_2 \cap A_3$ is nonempty. Let $x \in A_1 \cap A_2 \cap A_3$. Let N be the wall parallel to the wall of $A_1 \cap A_3$ containing x. Then $A_1 \cap A_2 \cap A_3 \supset N$. Let $D_1 = A_1 \cap A_3$ and $D_2 = A_2 \cap A_3$. Then D_1 and D_2 have opposite directions and are not disjoint: they contain M and thus $A_1 \cap A_2 \cap A_3 \supset M$. Let us show that $\phi_2^{-1} \circ \phi_3 \circ \phi_1 = s$ The half-apartments $D_3 = A_1 \cap A_2$ and $D_1 = A_1 \cap A_3$ contains M and thus their walls are parallel. Suppose D_3 and D_2 are equivalent. Then $D_3 \supset \mathfrak{D}_2$ and $D_1 \cap D_2 = A_1 \cap A_2 \cap A_3 = D_3 \cap D_2 \supset \mathfrak{D}_2$. Therefore, D_1 and D_2 are equivalent, which is absurd. The half-apartments $\phi_1(D_1) = D_1$ and $\phi_1(D_2)$ have opposite directions in A_1, hence $\phi_1(D_2) \sim D_3$. Consequently $\phi_3 \circ \phi_1(D_2) \sim \phi_3(D_3) = D_3$. We also have $\phi_2(D_2) = D_2$. Therefore, $\phi_2^{-1} \circ \phi_3 \circ \phi_1 \neq \mathrm{Id}$. By Lemma 4.2.4, $\phi_2^{-1} \circ \phi_3 \circ \phi_1 = s$. As s fixes $A_1 \cap A_2 \cap A_3$, $A_1 \cap A_2 \cap A_3 \subset M$ and thus $A_1 \cap A_2 \cap A_3 = M$. □

Recall the definition of $\mathcal{I}^\infty$ and of the direction M^∞ of a wall M from Subsection 3.2.

Lemma 4.2.6. *Let M^∞ be a wall of $\mathcal{I}^\infty$. Let $\mathfrak{q}$ be a sector-germ dominating a sector-panel of direction $\mathfrak{F}^\infty \subset M^\infty$. Let A_1 (resp. A_2) be an apartment containing a wall M_1 (resp. M_2) of direction M^∞ and $\mathfrak{q}$. Then either $A_1 = A_2$ or $A_1 \cap A_2$ is a half-apartment.*

Proof. Let Q be a sector of germ $\mathfrak{q}$ and included in $A_1 \cap A_2$. Let N_1 be a wall parallel to M_1 and such that $N_1 \cap Q \neq \emptyset$. Let $\phi : A_1 \overset{A_1 \cap A_2}{\to} A_2$ and $N_2 = \phi(N_1)$. Let $f \subset N_1 \cap Q$ be a sector-panel of direction $\mathfrak{F}^\infty$. Then $N_2 \supset f$. Let H be a wall of A_2 parallel to M_2 such that $H \supset f$. Then H is the affine space of A_2 spanned by f and N_2 too: $H = N_2$. Therefore, $N_2^\infty = M^\infty = N_1^\infty$. By Proposition 4.8 2) of [Rou11] $N_1 = N_2$ (this is the unique wall of direction M^∞ containing $\mathfrak{F} = germ_\infty(f)$). Let $\mathfrak{F}'$ be the sector-panel germ of N_1 opposite to $\mathfrak{F} = germ_\infty(f)$. Then by (MA4), $A_1 \cap A_2 \supset cl(\mathfrak{F}' \cup \mathfrak{q})$, which is a half-apartment and one concludes with Lemma 4.2.3. □

The following lemma is similar to Proposition 2.9.1) of [Rou11].

Lemma 4.2.7. *Let A be an apartment, M be a wall of A and M^∞ be its direction. Let $\mathfrak{F}_\infty$ be the direction of a sector-panel of M^∞ and $\mathfrak{q}$ be a sector-germ dominating $\mathfrak{F}^\infty$ and not included in A. Then there exists a unique pair $\{D_1, D_2\}$ of half-apartments of A such that:*

- *D_1 and D_2 are opposite with common wall N parallel to M*
- *for all $i \in \{1, 2\}$, D_i and $\mathfrak{q}$ are in some apartment A_i.*

Moreover such apartments A_1 and A_2 are unique and if D is the half-apartment of A_1 opposite to D_1, then $D \cap D_2 = D_1 \cap D_2$ is a wall and $A_2 = D_2 \cup D$.

Proof. Let us first show the existence of D_1 and D_2. Let $\mathfrak{F}'^\infty$ be the sector-panel of M^∞ opposite to $\mathfrak{F}^\infty$. Let $\mathfrak{q}_1'$ and $\mathfrak{q}_2'$ be the sector-germ of A containing $\mathfrak{F}'^\infty$. For $i \in \{1, 2\}$, let A_i be an apartment of $\mathcal{I}$ containing $\mathfrak{q}_i'$ and $\mathfrak{q}$. Let $i \in \{1, 2\}$ and $x \in A \cap A_i$. Then $x + \mathfrak{q}_i' \subset A \cap A_i$ and $E_i = \bigcup_{y \in x + \mathfrak{q}_i'} y + \mathfrak{F}^\infty \subset A \cap A_i$ is a half-apartment of A and A_i.

Suppose $A_1 = A_2$. Then $A_1 \supset \bigcup_{x \in E_1} x + \mathfrak{q}'_2 = A$ and thus $A_1 = A \supset \mathfrak{q}$, which is absurd. For $i \in \{1, 2\}$, E_i and thus A_i contains a wall of direction M^∞. By Lemma 4.2.6, $A_1 \cap A_2$ is a half-apartment.

By Lemma 4.2.5, $A_1 \cap A_2 \cap A = N$, where N is a wall of A parallel to M, and if $D_i = A \cap A_i$ for all $i \in \{1, 2\}$, $\{D_1, D_2\}$ fulfills the conditions of the lemma. Let E_1, E_2 be another pair of opposite half-apartments of A such that for all $i \in \{1, 2\}$, E_i and $\mathfrak{q}$ are included in some apartment B_i and such that $E_1 \cap E_2$ is parallel to M. One can suppose $E_1 \sim D_1$ and $E_2 \sim D_2$. Suppose for example that $E_1 \subset D_1$. By the same reasoning as in the proof of the existence of D_1 and D_2, $B_1 \cap A_2$ is a half-apartment. Thus by Lemma 4.2.5, $B_1 \cap A_2 \cap A$ is a wall of A. But $B_1 \cap A_2 \cap A = E_1 \cap D_2 \subset D_1 \cap D_2 = N$, thus $E_1 \cap D_2 = N$ and hence $E_1 \supset D_1$. Therefore, $E_1 = D_1$ and by symmetry, $E_2 = D_2$. This proves the uniqueness of such a pair. The fact that D_1 and D_2 are half-apartments comes from Lemma 4.2.3.

Let us show the uniqueness of such apartments A_1 and A_2. Let C_1 be an apartment containing D_1 and $\mathfrak{q}$. Let $\mathfrak{F}$ be the sector-panel germ of N dominated by $\mathfrak{q}$. Let $\mathfrak{F}'$ be the sector-panel germ of N opposite to $\mathfrak{F}$. Then by (MA4), $C_1 \supset cl(\mathfrak{F}', \mathfrak{q}) = D$ and thus $C_1 \supset A_1$. Therefore, $A_1 = C_1$. By symmetry, we get the lemma. □

We now generalize Lemma 4.2.7. We show that if $\mathfrak{q}$ is a sector germ of $\mathcal{I}$ and if A is an apartment of $\mathcal{I}$, then A is the union of a finite number of enclosed subsets P_i of A such that for all i, P_i and $\mathfrak{q}$ are included in some apartment.

Let $\mathfrak{q}$ be a sector-germ and $\epsilon \in \{+, -\}$ be its sign. Let A be an apartment of $\mathcal{I}$. Then one sets $d_{\mathfrak{q}}(A) = \min\{d(\mathfrak{q}, \mathfrak{q}') | \mathfrak{q}' \text{ is a sector germ of } A \text{ of sign } \epsilon\}$. Let $\mathcal{D}_A$ be the set of half-apartments of A. One sets $\mathcal{P}_{A,0} = \{A\}$ and for all $n \in \mathbb{N}^*$, $\mathcal{P}_{A,n} = \{\bigcap_{i=1}^n D_i | (D_i) \in (\mathcal{D}_A)^n\}$. The following proposition is very similar to Proposition 4.3.1 of [Cha10].

Proposition 4.2.8. *Let A be an apartment of $\mathcal{I}$, $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$ and $n = d_{\mathfrak{q}}(A)$. Then there exist $P_1, \ldots, P_k \in \mathcal{P}_{A,n}$, with $k \leq 2^n$ such that $A = \bigcup_{i=1}^k P_i$ and for each $i \in [\![1, k]\!]$, P_i and $\mathfrak{q}$ are contained in some apartment A_i. Moreover, for all $i \in [\![1, k]\!]$, there exists an isomorphism $\psi_i : A_i \overset{P_i}{\to} A$.*

Proof. We do it by induction on n. This is clear if $n = 0$. Suppose this is true for all apartment B such that $d_{\mathfrak{q}}(B) \leq n - 1$. Let B be an apartment such that $d_{\mathfrak{q}}(B) = n$. Let $\mathfrak{s}$ be a sector-germ of A such that there exists a minimal gallery $\mathfrak{s} = \mathfrak{q}_0, \ldots, \mathfrak{q}_n = \mathfrak{q}$ from $\mathfrak{s}$ to $\mathfrak{q}$. Let M be a wall of A containing a sector-panel $\mathfrak{F}$ dominated by $\mathfrak{q}_0$ and $\mathfrak{q}_1$. Let D_1, D_2 be a pair of opposite half-apartments whose wall is parallel to M and such that for all $i \in \{1, 2\}$, $D_i, \mathfrak{q}_1$ is included in an apartment B_i (such a pair exists by Lemma 4.2.7). Let $i \in \{1, 2\}$. One has $d_{\mathfrak{q}}(B_i) = n - 1$ and thus $B_i = \bigcup_{j=1}^{k_i} P_j^{(i)}$, with $k_i \leq 2^{n-1}$, for all $j \in [\![1, k_i]\!]$, $P_j^{(i)} \in \mathcal{D}_{B_i, n-1}$ and $\mathfrak{q}, P_j^{(i)}$ is contained in some apartment $A_j^{(i)}$. One has

$$A = D_1 \cap B_1 \cup D_2 \cap B_2 = \bigcup_{i \in \{1,2\}, j \in [\![1, k_i]\!]} P_j^{(i)} \cap D_i.$$

Let $i \in \{1, 2\}$ and $\phi_i : B_i \overset{A \cap B_i}{\to} A$. Let $j \in [\![1, k_i]\!]$. Then we still have $P_j^{(i)} \cap D_i \subset A_j^{(i)}$. One writes $P_j^{(i)} = \bigcap_{l=1}^{n-1} E_\ell$ with $(E_\ell) \in D_{B_i}^{n-1}$. We have $P_j^{(i)} \cap D_i = \phi_i(P_j^{(i)} \cap D_i) = \phi_i(P_j^{(i)}) \cap D_i$, and thus

$$P_j^{(i)} \cap D_i = D_i \cap \bigcap_{l=1}^{n-1} \phi_i(E_\ell) \in \mathcal{P}_{A,n}.$$

This shows the first part of this proposition. Let $i \in \{1,2\}$ and $j \in [\![1, k_i]\!]$. Let $f : A_i^{(j)} \overset{P_i^{(j)}}{\to} B_i$. Let $\psi = \phi_i \circ f$. One has $\psi_{|P_j^{(i)} \cap D_i} = \phi_i \circ \mathrm{Id}_{P_j^{(i)} \cap D_i} = \mathrm{Id}_{P_j^{(i)} \cap D_i}$. Thus $\psi : A \overset{P_j^{(i)} \cap D_i}{\to} A$, which completes the proof. □

If A is an apartment and $x, y \in A$, one denotes by $[x, y]_A$ the segment joining x and y in A.

We deduce a corollary which was already known for masures associated to split Kac-Moody groups over ultrametric fields by Section 4.4 of [GR08]:

Corollary 4.2.9. *Let $\mathfrak{q}$ be a sector-germ, A be an apartment and $x, y \in A$. Then there exists $x = x_1, \ldots, x_k = y \in A$ such that $[x, y]_A = \bigcup_{i=1}^{k-1} [x_i, x_{i+1}]_A$ and such that for all $i \in [\![1, k-1]\!]$, $[x_i, x_{i+1}]_A$ and $\mathfrak{q}$ are included in some apartment.*

4.2.2 Topological results

4.2.2.1 Definition of y_ν and T_ν

We now define for all $\nu \in C_f^v$ some projection $y_\nu : \mathcal{I} \to \mathbb{A}$ along $\mathbb{R}_+\nu$ and some distance $T_\nu : \mathcal{I} \to \mathbb{R}_+$ along $\mathbb{R}_+\nu$ (see Figure 4.2.2).

If $\nu \in C_f^v$, one writes δ_ν the ray $\mathbb{R}_+\nu$ of $\mathbb{A}$. This is a generic ray. Recall the definition of $x + \delta_\nu$ from 3.2.4.

Lemma 4.2.10. *Let $\nu \in C_f^v$ and $x \in \mathcal{I}$. Then there exists a unique point $y_\nu(x) \in \mathbb{A}$ such that $(x + \delta_\nu) \cap \mathbb{A} = y_\nu(x) + \mathbb{R}_+\nu$.*

Proof. By axiom (MA3) applied to x and the splayed chimney C_f^v, there exists an apartment A containing x and $+\infty$. Then A contains $x + \delta_\nu$. The set $(x + \delta_\nu) \cap \mathbb{A}$ is nonempty. Let $z \in (x + \delta_\nu) \cap \mathbb{A}$. Then $A \cap \mathbb{A}$ contains z, $+\infty$ and by (MA4), $A \cap \mathbb{A}$ contains $\mathrm{cl}(z, +\infty)$. As $\mathrm{cl}(z, +\infty) \supset z + \overline{C_f^v}$, $A \cap \mathbb{A} \supset z + \delta$ and thus $(x + \delta_\nu) \cap \mathbb{A} = y + \delta_\nu$ or $(x + \delta_\nu) \cap \mathbb{A} = y + \mathring{\delta}_\nu$ for some $y \in x + \delta_\nu$, where $\mathring{\delta}_\nu = \mathbb{R}_+^*\nu$.

Suppose $(x+\delta_\nu) \cap \mathbb{A} = y + \mathring{\delta}_\nu$. Let $z \in y + \mathring{\delta}_\nu$. Then by (MA2) applied to $germ_y([y, z] \backslash \{y\})$, $\mathbb{A} \cap A \supset \mathrm{cl}(germ_y([y, z] \backslash \{y\})) \ni y$ because $\mathrm{cl}(germ_y([y, z] \backslash \{y\}))$ contains the closure of $germ_y([y, z] \backslash \{y\})$. This is absurd and thus $(x + \delta_\nu) \cap \mathbb{A} = y + \delta_\nu$, with $y \in \mathbb{A}$, which proves the existence of such $y_\nu(x) \in \mathbb{A}$. The uniqueness of $y_\nu(x)$ is clear because a ray of an affine space has a unique origin. □

Actually the map $y_\nu : \mathcal{I} \to \mathbb{A}$ defined by Lemma 4.2.10 only depends on δ_ν.

If $x \in \mathcal{I}$, one has $\rho_{+\infty}(x + \delta_\nu) = \rho_{+\infty}(x) + \delta_\nu$ and $y_\nu(x) \in \rho_{+\infty}(x) + \delta_\nu$. We define $T_\nu(x)$ as the unique element T of $\mathbb{R}_+$ such that $y_\nu(x) = \rho_{+\infty}(x) + T\nu$.

Our proof of the Gindikin-Karpelevich finiteness relies on the study of the maps y_ν and T_ν, see Chapter 6.

Lemma 4.2.11. *Let $x \in \mathcal{I}$ and $\nu \in C_f^v$. Then $x \leq y_\nu(x)$ and $d^v(x, y_\nu(x)) = T_\nu(x)\nu$.*

Proof. Let $\delta_\nu = \mathbb{R}_+\nu$. By definition, $y_\nu(x) \in x + \delta_\nu$ and thus $x \leq y_\nu(x)$. Let A be an apartment containing x and $+\infty$ and $\phi : A \overset{+\infty}{\to} \mathbb{A}$, which exists by (MA4). One has $\phi(x) = \rho_{+\infty}(x) = y_\nu(x) + T_\nu(x)\nu = \phi(y_\nu(x)) + T_\nu(x)\nu$ and hence by Remark 3.2.2, $d^v(x, y_\nu(x)) = T_\nu(x)\nu$.

□

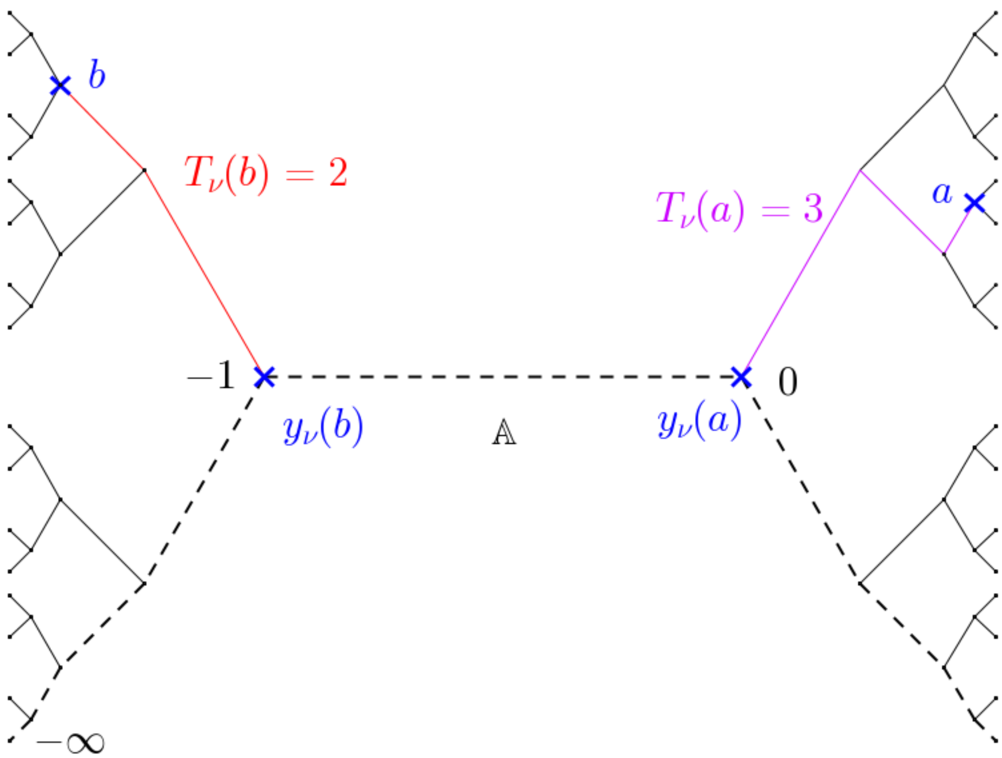

Figure 4.2.2 – The maps T_ν and y_ν on $\mathbb{T}$, when ν is the vector joining 0 to -1.

4.2.2.2 A characterization of the points of $\mathbb{A}$

Image of a preordered segment by a retraction In Theorem 6.2 of [GR08], Gaussent and Rousseau give a very precise description of the image of a preordered segment by a retraction centered at a sector-germ, see 6.4. However they suppose that a group acts strongly transitively on $\mathcal{I}$. Without this assumption, they prove a simpler property of these images. We recall it here.

Let $\lambda \in \overline{C_f^v}$. A **$\lambda$-path** π in $\mathbb{A}$ is a map $\pi : [0,1] \to \mathbb{A}$ such that there exists $n \in \mathbb{N}$ and $0 \leq t_1 < \ldots < t_n \leq 1$ such that for all $i \in [\![1, n-1]\!]$, π is affine on $[t_i, t_{i+1}]$ and $\pi'(t) \in W^v.\lambda$ for all $t \in [t_i, t_{i+1}]$.

Lemma 4.2.12. *Let A be an apartment of $\mathcal{I}$, $x, y \in A$ such that $x \leq y$ and $\rho : \mathcal{I} \to \mathbb{A}$ be a retraction of $\mathcal{I}$ onto $\mathbb{A}$ centered at a sector-germ $\mathfrak{q}$ of $\mathbb{A}$. Let $\tau : [0,1] \to A$ defined by $\tau(t) = (1-t)x + ty$ for all $t \in [0,1]$ and $\lambda = d^v(x,y)$. Then $\rho \circ \tau$ is a λ-path between $\rho(x)$ and $\rho(y)$.*

Proof. We rewrite the proof of the beginning of Section 6 of [GR08]. Let $\phi : A \to \mathbb{A}$ be an isomorphism such that $\phi(y) - \phi(x) = \lambda$, which exists by Remark 3.2.2. By Corollary 4.2.9, there exist $n \in \mathbb{N}$, $A_1, \ldots, A_n \in \mathcal{A}(\mathfrak{q})$ and $0 = t_1 < \ldots < t_n = 1$ such that $\tau([t_i, t_{i+1}]) \subset A_i$ for all $i \in [\![1, n-1]\!]$.

Using Proposition 5.4 of [Rou11], for all $i \in [\![1, n-1]\!]$, one chooses an isomorphism $\psi_i : A \overset{\tau([t_i, t_{i+1}])}{\to} A_i$. Let $\phi_i : A_i \overset{A_i \cap \mathbb{A}}{\to} \mathbb{A}$. For all $t \in [t_i, t_{i+1}]$, $\rho(\tau(t)) = \phi_i \circ \psi_i(\tau(t))$. Moreover, $\phi_i \circ \psi_i : A \to \mathbb{A}$ and by (MA1), there exists $w_i \in W$ such that $\phi_i \circ \psi_i = w_i \circ \phi$. Therefore for all $t \in [t_i, t_{i+1}]$, $(\rho \circ \tau)'(t) = w_i.\lambda$, which proves that $\rho \circ \tau$ is a λ-path. □

If $x \in \mathbb{A}$ and $\lambda \in \overline{C_f^v}$, one defines $\pi_\lambda^a : [0,1] \to \mathbb{A}$ by $\pi_\lambda^a(t) = a + t\lambda$ for all $t \in [0,1]$.

Lemma 4.2.13. *Let $\lambda \in \overline{C_f^v}$ and $a \in \mathbb{A}$. Then the unique λ-path from a to $a + \lambda$ is π_λ^a.*

Proof. Let π be a λ-path from a to $a+\lambda$. One chooses a subdivision $0 = t_1 < \ldots < t_n = 1$ of $[0,1]$ such that for all $i \in [\![1, n-1]\!]$, there exists $w_i \in W^v$ such that $\pi'_{|[t_i,t_{i+1}]}(t) = w_i.\lambda$. By Lemma 2.2.9, $w_i.\lambda \leq_{Q^\vee_\mathbb{R}} \lambda$ for all $i \in [\![1, n-1]\!]$. Let $h : Q^\vee_\mathbb{R} \to \mathbb{R}$ defined by $h(\sum_{i\in I} u_i \alpha_i^\vee) = \sum_{i\in I} u_i$ for all $(u_i) \in \mathbb{R}^I$. Suppose that for some $i \in [\![1, n-1]\!]$, $w_i.\lambda \neq \lambda$. Then $h(w_i.\lambda - \lambda) < 0$ and for all $j \in [\![1, n-1]\!]$, $h(w_j.\lambda - \lambda) \leq 0$. By integrating, we get that $h(0) < 0$: a contradiction. Therefore $\pi(t) = a + t\lambda = \pi^a_\lambda(t)$ for all $t \in [0,1]$, which is our assertion. □

The following lemma is Lemma 3.6 of [Héb17b]. As we want to use it to simplify the axiomatic of masures and in particular to prove that (MAO) is a consequence of other axioms, we do not use (MAO) (see 4.4.1) in its proofs and we use weaker versions of (MA2) and (MA3). We can replace axioms (MA2') and (MA iii') by (MA2) and (MA3). As we do not assume (MAO), the Tits preorder $\leq$ is not defined on $\mathcal{I}$ a priori but we have the Tits preorder $\leq_A$ on each apartment of $\mathcal{I}$. We assume that for all $x, y \in \mathcal{I}$, if there exists $A \in \mathcal{A}(\{x,y\})$ such that $x \leq_A y$, then $x \leq_B y$ for all $B \in \mathcal{A}(\{x,y\})$. Thus this defines a relation $\leq$ on $\mathcal{I}$. However, we do not know yet that $\leq$ is a preorder because the proof of Théorème 5.9 of [Rou11] uses (MAO).

Lemma 4.2.14. *Let $\tau : [0,1] \to \mathcal{I}$ be a segment such that $\tau(0) \leq \tau(1)$, such that $\tau(1) \in \mathbb{A}$ and such that there exists $\nu \in \overline{C^v_f}$ such that $(\rho_{-\infty} \circ \tau)' = \nu$. Then $\tau([0,1]) \subset \mathbb{A}$ and thus $\rho_{-\infty} \circ \tau = \tau$.*

Proof. Let A be an apartment such that τ is a segment of A. Then τ is increasing for $\leq_A$ and thus τ is increasing for $\leq$.

Suppose that $\tau([0,1]) \not\subset \mathbb{A}$. Let $u = \sup\{t \in [0,1] | \tau(t) \notin \mathbb{A}\}$. Let us prove that $\tau(u) \in \mathbb{A}$. If $u = 1$, this is our hypothesis. Suppose $u < 1$. Then by (MA2') applied to $]\tau(u), \tau(1))$, $\mathbb{A}$ contains $\mathrm{cl}_A(]\tau(u), \tau(1)))$ and thus $\mathbb{A}$ contains $\tau(u)$.

By (MA iii'), there exists an apartment B containing $\tau((0,u]) \cup -\infty$ and by (MA4), there exists an isomorphism $\phi : B \overset{\tau(u)-\overline{C^v_f}}{\to} \mathbb{A}$. For all $t \in [0,u]$, near enough from u, $\phi(\tau(t)) = \rho_{-\infty}(\tau(t))$. By hypothesis, for all $t \in [0,u]$, $\rho_{-\infty}(\tau(t)) \in \tau(u) - \overline{C^v_f}$. Therefore for $t \leq u$ near enough from u, $\phi(\tau(t)) = \tau(t) \in \mathbb{A}$: this is absurd by choice of u and thus $\tau([0,1]) \subset \mathbb{A}$. □

The following proposition corresponds to Corollary 4.4 of [Héb17b], but the proof here is different.

Proposition 4.2.15. *Let $x \in \mathcal{I}$ such that $\rho_{+\infty}(x) = \rho_{-\infty}(x)$. Then $x \in \mathbb{A}$.*

Proof. Let $x \in \mathcal{I}$ such that $\rho_{+\infty}(x) = \rho_{-\infty}(x)$. Suppose that $x \in \mathcal{I}\backslash\mathbb{A}$. By Lemma 4.2.11, one has $x \leq y_\nu(x)$ and $d^v(x, y_\nu(x)) = \lambda$, with $\lambda = y_\nu(x) - \rho_{+\infty}(x) \in \mathbb{R}^*_+\nu$. Let A be an apartment containing x and $+\infty$, which exists by (MA3). Let $\tau : [0,1] \to A$ be defined by $\tau(t) = (1-t)x + t y_\nu(x)$ for all $t \in [0,1]$ (this does not depend on the choice of A by Proposition 5.4 of [Rou11]) and $\pi = \rho_{-\infty} \circ \tau$. Then by Lemma 4.2.12, π is a λ-path from $\rho_{-\infty}(x) = \rho_{+\infty}(x)$ to $y_\nu(x) = \rho_{+\infty}(x) + \lambda$.

By Lemma 4.2.13, $\pi(t) = \rho_{+\infty}(x) + t\lambda$ for all $t \in [0,1]$. By Lemma 4.2.14, $\tau([0,1]) \subset \mathbb{A}$: $x = \tau(0) \in \mathbb{A}$: this is absurd. Therefore $x \in \mathbb{A}$, which is our assertion. □

4.2.2.3 Topological considerations on apartments

Proposition 4.2.16. *Let $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$ and A be an apartment of $\mathcal{I}$. Let $\rho : \mathcal{I} \overset{\mathfrak{q}}{\to} \mathbb{A}$. Then $\rho_{|A} : A \to \mathbb{A}$ is continuous (for the affine topologies on A and $\mathbb{A}$).*

Proof. Using Proposition 4.1.2 2, one writes $A = \bigcup_{i=1}^n P_i$ where the P_i's are closed sets of A such that for all $i \in [\![1,n]\!]$, there exists an apartment A_i containing P_i and $\mathfrak{q}$ and an isomorphism $\psi_i : A \overset{P_i}{\to} A_i$. For all $i \in [\![1,n]\!]$, one denotes by ϕ_i the isomorphism $A_i \overset{\mathfrak{q}}{\to} \mathbb{A}$. Then $\rho_{|P_i} = \phi_i \circ \psi_{i|P_i}$ for all $i \in [\![1,n]\!]$.

Let $(x_k) \in A^{\mathbb{N}}$ be a converging sequence and $x = \lim x_k$. Then for all $k \in \mathbb{N}$, $\rho(x_k) \in \{\phi_i \circ \psi_i(x_k) | \, i \in [\![1,n]\!]\}$ and thus $(\rho(x_n))$ is bounded. Let $(x_{\sigma(k)})$ be a subsequence of (x_k) such that $(\rho(x_{\sigma(k)})$ converges. Maybe extracting a subsequence of $(x_{\sigma(k)})$, one can suppose that there exists $i \in [\![1,n]\!]$ such $x_{\sigma(k)} \in P_i$ for all $k \in \mathbb{N}$. One has $(\rho(x_{\sigma(k)}) = (\phi_i \circ \psi_i(x_{\sigma(k)}))$ and thus $\rho(x_{\sigma(k)}) \to \phi_i \circ \psi_i(x) = \rho(x)$ (because P_i is closed) and thus $(\rho(x_k))$ converges towards $\rho(x)$: $\rho_{|A}$ is continuous. □

The following proposition generalizes Corollary 5.10 of [Héb16] to our context.

Proposition 4.2.17. *Let A be an apartment. Then $A \cap \mathbb{A}$ is closed.*

Proof. By Proposition 4.2.15, $A \cap \mathbb{A} = \{x \in A | \; \rho_{+\infty}(x) = \rho_{-\infty}(x)\}$, which is closed by Proposition 4.2.16. □

4.2.3 Decomposition of the intersection of two apartments in enclosed subsets

The aim of this subsection is to show that $\mathbb{A} \cap A$ is a finite union of enclosed subsets of $\mathbb{A}$.

If X is an affine space, one denotes by $\mathring{X}$ or by $\mathrm{Int}(X)$ (depending on the legibility) its interior. One denotes by $\mathrm{Fr}(X)$ the boundary (or frontier) of X: $\mathrm{Fr}(X) = \overline{X} \backslash \mathring{X}$.

We first suppose that A and $\mathbb{A}$ share a sector. One can suppose that $+\infty \subset A \cap \mathbb{A}$. By Proposition 4.1.2, one has $A = \bigcup_{i=1}^k P_i$, for some $k \in \mathbb{N}$, where the P_i's are enclosed and $P_i, -\infty$ is included in some apartment A_i for all $i \in [\![1,k]\!]$.

Lemma 4.2.18. *Let X be a finite dimensional affine space, $U \subset X$ be a set such that $U \subset \overline{\mathring{U}}$ and suppose that $U = \bigcup_{i=1}^n U_i$, where for all $i \in [\![1,n]\!]$ U_i is the intersection of U and of a finite number of half-spaces. Let $J = \{j \in [\![1,n]\!] | \mathring{U}_j \neq \emptyset\}$. Then $U = \bigcup_{j \in J} U_j$.*

Proof. Let $j \in [\![1,n]\!]$. Then $\mathrm{Fr}(U_j) \cap \mathring{U}$ is included in a finite number of hyperplanes. Therefore, if one chooses a Lebesgue measure on X, $\bigcup_{i \in [\![1,n]\!]} \mathring{U} \cap \mathrm{Fr}(U_i)$ has measure 0, hence $\mathring{U} \backslash \bigcup_{i \in [\![1,n]\!]} \mathrm{Fr}(U_i)$ is dense in $\mathring{U}$ and thus in U. Let $x \in U$. Let $(x_k) \in (\mathring{U} \backslash \bigcup_{i \in [\![1,n]\!]} \mathrm{Fr}(U_i))^{\mathbb{N}}$ converging towards x. Extracting a sequence if necessary, one can suppose that for some $i \in [\![1,n]\!]$, $x_k \in U_i$ for all $k \in \mathbb{N}$. By definition of the frontier, $x_k \in \mathring{U}_i$ for all $k \in \mathbb{N}$. As U_i is closed in U, $x \in U_i$ and the lemma follows. □

Lemma 4.2.19. *Let $i \in [\![1,k]\!]$ such that $A \cap \mathbb{A} \cap P_i$ has nonempty interior in $\mathbb{A}$. Then $A \cap \mathbb{A} \supset P_i$.*

Proof. One chooses an apartment A_i containing $P_i, -\infty$ and $\phi_i : A \overset{P_i}{\to} A_i$. Let $\psi_i : A_i \overset{A_i \cap \mathbb{A}}{\to} \mathbb{A}$ (ψ_i exists and is unique by Remark 3.2.4). By definition of $\rho_{-\infty}$, if $x \in P_i$, $\rho_{-\infty}(x) = \psi_i(x)$ and thus $\rho_{-\infty}(x) = \psi_i \circ \phi_i(x)$.

Let $f : A \overset{A \cap \mathbb{A}}{\to} \mathbb{A}$. One has $\rho_{+\infty}(x) = f(x)$ for all $x \in A$. By Proposition 4.2.15, $A \cap \mathbb{A} \cap P_i = \{x \in P_i | \rho_{+\infty}(x) = \rho_{-\infty}(x)\} = P_i \cap (f - \psi_i \circ \phi_i)^{-1}(\{0\})$.

As $f - \psi_i \circ \phi_i$ is affine, $(f - \psi_i \circ \phi_i)^{-1}(\{0\})$ is an affine subspace of A and as it has nonempty interior, $(f - \psi_i \circ \phi_i)^{-1}(\{0\}) = A$. Therefore $P_i \subset \mathbb{A} \cap A$.

□

We recall the definition of $x+\infty$, if $x\in\mathcal{I}$ (see 3.2.4). Let $x\in\mathcal{I}$ and B be an apartment containing x and $+\infty$. Let Q be a sector of $\mathbb{A}$, parallel to C_f^v and such that $Q\subset B\cap\mathbb{A}$. Then $x+\infty$ is the sector of B based at x and parallel to Q. This does not depend on the choice of B.

Lemma 4.2.20. *One has* $A\cap\mathbb{A}=\overline{\mathrm{Int}(A\cap\mathbb{A})}$*.*

Proof. By Proposition 4.2.17, $A\cap\mathbb{A}$ is closed and thus $\overline{\mathrm{Int}(A\cap\mathbb{A})}\subset A\cap\mathbb{A}$.

Let $x\in A\cap\mathbb{A}$. By (MA4) $x+\infty\subset A\cap\mathbb{A}$. The fact that there exists $(x_n)\in\mathrm{Int}(x+\infty)^{\mathbb{N}}$ such that $x_n\to x$ proves the lemma. □

Lemma 4.2.21. *Let* $J=\{i\in[\![1,k]\!]|\ \mathrm{Int}_{\mathbb{A}}(P_i\cap A\cap\mathbb{A})\neq\emptyset\}$*. Then* $A\cap\mathbb{A}=\bigcup_{j\in J}P_j$*.*

Proof. Let $U=A\cap\mathbb{A}$. Then by Lemma 4.2.20 and Lemma 4.2.18, $U=\bigcup_{j\in J}U\cap P_j$ and Lemma 4.2.19 completes the proof. □

We no more suppose that A contains $+\infty$. We say that $(\bigcup_{i=1}^k P_i,(\phi_i)_{i\in[\![1,k]\!]})$ is a **decomposition of $A\cap\mathbb{A}$ in enclosed subsets** if:

1. $k\in\mathbb{N}$ and for all $i\in[\![1,k]\!]$, P_i is enclosed,
2. $A\cap\mathbb{A}=\bigcup_{i=1}^k P_i$,
3. for all $i\in[\![1,k]\!]$, ϕ_i is an isomorphism $\mathbb{A}\overset{P_i}{\to}A$.

When we do not want to precise the ϕ_i's, we simply say "$\bigcup_{i=1}^k P_i$ is a decomposition of $A\cap\mathbb{A}$ in enclosed subsets".

If G is a group of automorphisms of $\mathcal{I}$, a decomposition $(\bigcup_{i=1}^k P_i,(\phi_i)_{i\in[\![1,k]\!]})$ of $A\cap\mathbb{A}$ in enclosed subsets is said to be G-compatible if the ϕ_i's are induced by elements of G.

Proposition 4.2.22. *1. Let A be an apartment. Then there exists a decomposition $\bigcup_{i=1}^k P_i$ of $A\cap\mathbb{A}$ in enclosed subsets.*

As a consequence there exists a finite set $\mathscr{M}$ of walls such that $\mathrm{Fr}(A\cap\mathbb{A})\subset\bigcup_{M\in\mathscr{M}}M$*.*

If moreover $A\cap\mathbb{A}$ is convex, one has $A\cap\mathbb{A}=\bigcup_{j\in J}P_j$, where $J=\{j\in[\![1,k]\!]|\ \mathrm{supp}(P_j)=\mathrm{supp}(A\cap\mathbb{A})\}$*.*

2. Let $G\subset\mathrm{Aut}(\mathcal{I})$. Suppose that each isomorphism of apartments fixing a sector-germ is induced by an element of G. Then if A is an apartment, there exists a G-compatible decomposition $(\bigcup_{i=1}^k P_i,(\phi_i)_{i\in[\![1,k]\!]})$ of $A\cap\mathbb{A}$ in enclosed subsets.

Proof. We prove 1 and 2 at the same time. For 2, it suffices to require that all the isomorphisms of this proof are induced by some $g\in G$. Let $n\in\mathbb{N}$ and $\mathcal{P}_n$: "for all apartment B such that $d(B,\mathbb{A})\leq n$, there exists a decomposition $\bigcup_{i=1}^{\ell}Q_i$ of $\mathbb{A}\cap B$ in enclosed subsets". The property $\mathcal{P}_0$ is true by Lemma 4.2.21. Let $n\in\mathbb{N}$ and suppose that $\mathcal{P}_n$ is true. Suppose that there exists an apartment B such that $d(B,\mathbb{A})=n+1$. Using Proposition 4.1.2, one writes $B=D_1\cup D_2$ where D_1, D_2 are opposite half-apartments such that for all $i\in\{1,2\}$, D_i is included in a apartment B_i satisfying $d(B_i,\mathbb{A})=n$. If $i\in\{1,2\}$, let $(\bigcup_{j=1}^{l_j}Q_j^i,(\psi_j^i))$ be a decomposition of $B_i\cap\mathbb{A}$ in enclosed subsets. Then $B\cap\mathbb{A}=\bigcup_{j=1}^{l_1}(D_1\cap Q_j^1)\cup\bigcup_{j=1}^{l_2}(D_2\cap Q_j^2)$. If $i\in\{1,2\}$, one denotes by f^i the isomorphism $B\overset{D_i}{\to}B_i$. Then if $j\in[\![1,l_i]\!]$, $(\psi_j^i)^{-1}\circ f^i$ fixes $Q_j^i\cap D_i$ and thus $\mathcal{P}_{n+1}$ is true.

Therefore $A\cap\mathbb{A}=\bigcup_{i=1}^k P_i$ where the P_i's are enclosed. One has $\mathrm{Fr}(A\cap\mathbb{A})\subset\bigcup_{i=1}^k \mathrm{Fr}(P_i)$, which is included in a finite union of walls.

Suppose that $A\cap\mathbb{A}$ is convex. Let $X=\mathrm{supp}(A\cap\mathbb{A})$. By Lemma 4.2.18 applied with $U=A\cap\mathbb{A}$,

$$A\cap\mathbb{A}=\bigcup_{i\in[\![1,k]\!],\ \mathrm{Int}_X(P_i)\neq\emptyset} P_i,$$

which completes the proof. □

4.2.4 Encloseness of a convex intersection

In this subsection, we prove Proposition 4.2.30: if A is an apartment such that $A\cap\mathbb{A}$ is convex, then $A\cap\mathbb{A}$ is enclosed. For this we study the "gauge" of $A\cap\mathbb{A}$, which is a map parameterizing the frontier of $A\cap\mathbb{A}$.

Let X be a finite dimensional affine space. Let $C\subset X$ be a convex set and A' be its support. The **relative interior** (resp. **relative frontier**) of C, denoted $\mathrm{Int}_r(C)$ (resp. $\mathrm{Fr}_r(C)$) is the interior (resp. frontier) of C seen as a subset of A'. A set is said to be **relatively open** if it is open in its support.

If X is a topological space and $a\in X$, one denotes by $\mathcal{V}_X(a)$ the set of open neighborhoods of a.

Lemma 4.2.23. *Let A be a finite dimensional affine space, $k\in\mathbb{N}^*$, $D_1,\ldots,D_k$ be half-spaces of A and $M_1,\ldots,M_k$ be their hyperplanes. Then their exists $J\subset[\![1,k]\!]$ (maybe empty) such that* $\mathrm{supp}(\bigcap_{i=1}^k D_i)=\bigcap_{j\in J}M_j$

Proof. Let $d\in\mathbb{N}^*$ and $\ell\in\mathbb{N}$. Let $\mathcal{P}_{d,\ell}$: "for all affine space X such that $\dim X\leq d$ and for all half-spaces $E_1,\ldots,E_\ell$ of X, there exists $J\subset[\![1,\ell]\!]$ such that $\mathrm{supp}(\bigcap_{i=1}^\ell E_i)=\bigcap_{j\in J}H_j$ where for all $j\in J$, H_j is the hyperplane of E_j".

It is clear that for all $\ell\in\mathbb{N}$, $\mathcal{P}_{1,\ell}$ is true and that for all $d\in\mathbb{N}$, $\mathcal{P}_{d,0}$ and $\mathcal{P}_{d,1}$ is true. Let $d\in\mathbb{N}_{\geq 2}$ and $\ell\in\mathbb{N}$ and suppose that (for all $d'\leq d-1$ and $\ell'\in\mathbb{N}$, $\mathcal{P}_{d',\ell'}$ is true) and that (for all $\ell'\in[\![0,\ell]\!]$, $\mathcal{P}_{d,\ell'}$ is true).

Let X be a d dimensional affine space, $E_1,\ldots,E_{\ell+1}$ be half-spaces of X and $H_1,\ldots,H_{\ell+1}$ be their hyperplanes. Let $L=\bigcap_{j=1}^l E_j$ and $S=\mathrm{supp}\,L$. Then $E_{\ell+1}\cap S$ is either S or a half-space of S. In the first case, $E_{\ell+1}\supset S\supset L$, thus $\bigcap_{i=1}^{\ell+1}E_i=L$ and thus by $\mathcal{P}_{d,\ell}$, $\mathrm{supp}(\bigcap_{i=1}^{\ell+1}E_i)=\bigcap_{j\in J}H_j$ for some $J\subset[\![1,\ell]\!]$.

Suppose that $E_{\ell+1}\cap S$ is a half-space of S. Then either $\mathring{E}_{\ell+1}\cap L\neq\emptyset$ or $\mathring{E}_{\ell+1}\cap L=\emptyset$. In the first case, one chooses $x\in\mathring{E}_{\ell+1}\cap L$ and a sequence $(x_n)\in(\mathrm{Int}_r(L))^{\mathbb{N}}$ converging towards x. Then for $n\gg 0$, $x_n\in\mathring{E}_{\ell+1}\cap\mathrm{Int}_r(L)$. Consequently, $L\cap E_{\ell+1}$ has nonempty interior in S. Thus $\mathrm{supp}(\bigcap_{i=1}^{\ell+1}E_i)=S$ and by $\mathcal{P}_{d,\ell}$, $\mathrm{supp}(\bigcap_{i=1}^{\ell+1}E_i)=\bigcap_{j\in J}H_j$ for some $J\subset[\![1,\ell]\!]$.

Suppose now that $\mathring{E}_{\ell+1}\cap L$ is empty. Then $L\cap E_{\ell+1}\subset H_{\ell+1}$, where $H_{\ell+1}$ is the hyperplane of $E_{\ell+1}$. Therefore $\bigcap_{i=1}^{\ell+1}E_i=\bigcap_{i=1}^{\ell+1}(E_i\cap H_{\ell+1})$ and thus by $\mathcal{P}_{d-1,\ell+1}$, $\mathrm{supp}(\bigcap_{i=1}^{\ell+1}E_i)=\bigcap_{j\in J}H_j$ for some $J\subset[\![1,\ell+1]\!]$.

□

Lemma 4.2.24. *Let A be an apartment such that $A\cap\mathbb{A}$ is convex. Then* $\mathrm{supp}(A\cap\mathbb{A})$ *is enclosed.*

Proof. Using Proposition 4.2.22, one writes $A\cap\mathbb{A}=\bigcup_{i=1}^k P_i$, where the P_i's are enclosed and $\mathrm{supp}(P_i)=\mathrm{supp}(A\cap\mathbb{A})$ for all $i\in[\![1,k]\!]$. By Lemma 4.2.23, if $i\in[\![1,k]\!]$, $\mathrm{supp}(P_i)$ is a finite intersection of walls, which proves the lemma. □

Gauge of a convex Let A be a finite dimensional affine space. Let C be a closed and convex subset of A with nonempty interior. One chooses $x \in \mathring{C}$ and one fixes the origin of A in x. Let $j_{C,x} : A \to \mathbb{R}_+ \cup \{+\infty\}$ defined by $j_{C,x}(s) = \inf\{t \in \mathbb{R}_+^* | s \in tC\}$. The map $j_{C,x}$ is called the **gauge** of C based at x. In the sequel, we will fix some $x \in \mathring{C}$ and we will denote j_C instead of $j_{C,x}$. Then by Theorem 1.2.5 of [HUL12] and discussion at the end of Section 1.2 of loc cit, $j_C(A) \subset \mathbb{R}_+$ and j_C is continuous.

The following lemma is easy to prove:

Lemma 4.2.25. *Let C be a convex closed set with nonempty interior. Fix the origin of A in a point of $\mathring{C}$. Then $C = \{x \in A | j_C(x) \leq 1\}$ and $\mathring{C} = \{x \in A | j_C(x) < 1\}$.*

Lemma 4.2.26. *Let C be a convex closed set with nonempty interior. Fix the origin of A in $\mathring{C}$. Let $U = U_C = \{s \in A |\ j_C(s) \neq 0\}$. Let $\mathrm{Fr} = \mathrm{Fr}_C : U \to \mathrm{Fr}(C)$ defined by $\mathrm{Fr}(s) = \frac{s}{j_C(s)}$ for all $s \in U$. Then Fr is well defined, continuous and surjective.*

Proof. If $s \in U$, then $j_C(\mathrm{Fr}(s)) = \frac{j_C(s)}{j_C(s)} = 1$ and thus Fr takes it values in $\mathrm{Fr}(C)$ by Lemma 4.2.25. The continuity of Fr is a consequence of the one of j_C.

Let $f \in \mathrm{Fr}(C)$. Then $\mathrm{Fr}(f) = f$ and thus Fr is surjective. □

Let A be an apartment such that $A \cap \mathbb{A}$ is convex and nonempty. Let X be the support of $A \cap \mathbb{A}$ in $\mathbb{A}$. By Lemma 4.2.24, if $A \cap \mathbb{A} = X$, then $A \cap \mathbb{A}$ is enclosed. One now supposes that $A \cap \mathbb{A} \neq X$. One chooses $x_0 \in \mathrm{Int}_X(A \cap \mathbb{A})$ and consider it as the origin of $\mathbb{A}$. One defines $U = U_{A\cap\mathbb{A}}$ and $\mathrm{Fr} : U \to \mathrm{Fr}_r(A \cap \mathbb{A})$ as in Lemma 4.2.26. The set U is open and nonempty. Using Proposition 4.2.22, one writes $A \cap \mathbb{A} = \bigcup_{i=1}^r P_i$, where $r \in \mathbb{N}$, the P_i's are enclosed and $\mathrm{supp}(P_i) = X$ for all $i \in [\![1, r]\!]$. Let $M_1, \ldots, M_k$ be distinct walls not containing X such that $\mathrm{Fr}_r(A \cap \mathbb{A}) \subset \bigcup_{i=1}^k M_i$, which exists because the P_i's are intersections of half-spaces of X and $A \cap \mathbb{A} \neq X$. Let $\mathscr{M} = \{M_i \cap X | i \in [\![1, k]\!]\}$. If $M \in \mathscr{M}$, one sets $U_M = \mathrm{Fr}^{-1}(M)$.

Lemma 4.2.27. *Let $U' = \{x \in U | \exists (M, V) \in \mathscr{M} \times \mathcal{V}_U(x) | \mathrm{Fr}(V) \subset M\}$. Then U' is dense in U.*

Proof. Let $M \in \mathscr{M}$. By Lemma 4.2.26, U_M is closed in U. Let $V' \subset U$ be nonempty and open. Then $V' = \bigcup_{M\in\mathscr{M}} U_M \cap V'$. By Baire's Theorem, there exists $M \in \mathscr{M}$ such that $V' \cap U_M$ has nonempty interior and hence U' is dense in U. □

Lemma 4.2.28. *Let $x \in U'$ and $V \in \mathcal{V}_U(x)$ such that $\mathrm{Fr}(V) \subset M$ for some $M \in \mathscr{M}$. Then M is the unique hyperplane H such that $\mathrm{Fr}(V) \subset H$. Moreover M does not depend on the choice of $V' \in \mathcal{V}_U(x)$ such that $\mathrm{Fr}(V')$ is included in a hyperplane.*

Proof. Suppose that $\mathrm{Fr}(V) \subset M \cap H$, where H is a hyperplane of X. Let $f, f' \in X^*$, $k, k' \in \mathbb{R}$ such that $M = f^{-1}(\{k\})$ and $H = f'^{-1}(\{k'\})$. By definition of U, for all $y \in V$, $\mathrm{Fr}(y) = \lambda(y)y$ for some $\lambda(y) \in \mathbb{R}_+^*$. Suppose that $k = 0$. Then $f(y) = 0$ for all $y \in V$, which is absurd because $f \neq 0$. By the same reasoning, $k' \neq 0$.

If $y \in V \backslash \big(f^{-1}(\{0\}) \cup f'^{-1}(\{0\})\big)$, $\mathrm{Fr}(y) = \lambda(y)y$ for some $\lambda(y) \in \mathbb{R}_+^*$ and thus $\mathrm{Fr}(y) = \frac{k}{f(y)}y = \frac{k'}{f'(y)}y$. As $V \backslash \big(f^{-1}(\{0\}) \cup f'^{-1}(\{0\})\big)$ is dense in V, $kf'(y) - k'f(y) = 0$ for all $y \in V$ and thus M and H are parallel. Therefore $M = H$. It remains to show that M does not depend on V. Let $V_1 \in \mathcal{V}_U(x)$ such that $\mathrm{Fr}(V_1) \subset H_1$ for some hyperplane H_1. By the unicity we just proved applied to $V \cap V_1$, $M = H_1$, which completes the proof. □

If $x \in U'$, one denotes by M_x the wall defined by Lemma 4.2.28.

Lemma 4.2.29. *Let $x \in U'$ and D_1, D_2 be the two half-spaces of X defined by M_x. Then $A \cap \mathbb{A} \subset D_i$, for some $i \in \{1, 2\}$.*

Proof. Let $V \in \mathcal{V}_U(x)$ such that $\mathrm{Fr}(V) \subset M_x$. Let us prove that $\mathrm{Fr}(V) = \mathbb{R}_+^* V \cap M_x$. As $\mathrm{Fr}(y) \in \mathbb{R}_+^* y$ for all $y \in V$, $\mathrm{Fr}(V) \subset \mathbb{R}_+^* V \cap M_x$. Let f be a linear form on X such that $M_x = f^{-1}(\{k\})$ for some $k \in \mathbb{R}$. If $k = 0$, then for all $v \in V$, $f(v) = 0$, and thus $f = 0$: this is absurd and $k \neq 0$. Let $a \in \mathbb{R}_+^* V \cap M_x$. One has $a = \lambda \mathrm{Fr}(v)$, for some $\lambda \in \mathbb{R}_+^*$ and $v \in V$. Moreover $f(\mathrm{Fr}(v)) = k = f(a)$ and as $k \neq 0$, $a = \mathrm{Fr}(v) \in \mathrm{Fr}(V)$. Thus $\mathrm{Fr}(V) = \mathbb{R}_+^* V \cap M_x$ and $\mathrm{Fr}(V)$ is an open set of M_x. Suppose there exists $(x_1, x_2) \in (\mathring{D}_1 \cap A \cap \mathbb{A}) \times (\mathring{D}_2 \cap A \cap \mathbb{A})$. Then $\mathrm{conv}(x_1, x_2, \mathrm{Fr}(V)) \subset A \cap \mathbb{A}$ is an open neighborhood of $\mathrm{Fr}(V)$ in X. This is absurd because Fr takes it values in $\mathrm{Fr}_r(A \cap \mathbb{A})$. Thus the lemma is proved. □

If $x \in U'$, one denotes by D_x the half-space delimited by M_x and containing $A \cap \mathbb{A}$.

Proposition 4.2.30. *Let A be an apartment such that $A \cap \mathbb{A}$ is convex. Then $A \cap \mathbb{A}$ is enclosed.*

Proof. If $u \in U'$, then $A \cap \mathbb{A} \subset D_u$ and thus $A \cap \mathbb{A} \subset \bigcap_{u \in U'} D_u$.

Let $x \in U' \cap \bigcap_{u \in U'} D_u$. One has $0 \in A \cap \mathbb{A}$ and thus $0 \in D_x$. Moreover $\mathrm{Fr}(x) \in M_x \cap A \cap \mathbb{A}$ and thus $x \in [0, \mathrm{Fr}(x)] \subset A \cap \mathbb{A}$. Therefore $U' \cap \bigcap_{x \in U'} D_x \subset A \cap \mathbb{A}$.

Let $x \in \mathrm{Int}_X(\bigcap_{u \in U'} D_u)$. If $x \notin U$, then $x \in A \cap \mathbb{A}$. Suppose $x \in U$. Then by Lemma 4.2.27, there exists $(x_n) \in (U' \cap \mathrm{Int}_X(\bigcap_{u \in U'} D_u))^{\mathbb{N}}$ such that $x_n \to x$. But then for all $n \in \mathbb{N}$, $x_n \in A \cap \mathbb{A}$ and by Proposition 4.2.17, $x \in \overline{A \cap \mathbb{A}}$. As a consequence, $A \cap \mathbb{A} \supset \mathrm{Int}_X(\bigcap_{u \in U'} D_u)$ and as $A \cap \mathbb{A}$ is closed, $A \cap \mathbb{A} \supset \overline{\mathrm{Int}_X(\bigcap_{u \in U'} D_u)} = \bigcap_{u \in U'} D_u$ because $\bigcap_{u \in U'} D_u$ is closed, convex with nonempty interior in X. Thus we have proved $A \cap \mathbb{A} = \bigcap_{u \in U'} D_u$.

Let $M'_1, \ldots, M'_k$ be walls of $\mathbb{A}$ such that for all $x \in U'$, there exists $i(x) \in [\![1, k]\!]$ such that $M'_{i(x)} \cap X = M_x$. One sets $M'_x = M'_{i(x)}$ for all $x \in U'$ and one denotes by D'_x the half-apartment of $\mathbb{A}$ delimited by M'_x and containing D_x. Then $X \cap \bigcap_{x \in U'} D'_x = A \cap \mathbb{A}$. Lemma 4.2.24 completes the proof.

□

4.2.5 Existence of isomorphisms of apartments fixing a convex set

In this section, we study, if A is an apartment and $P \subset \mathbb{A} \cap A$, the existence of isomorphisms of apartments $\mathbb{A} \stackrel{P}{\to} A$. We give a sufficient condition of existence of such an isomorphism in Proposition 4.2.34. The existence of an isomorphism $A \stackrel{A \cap \mathbb{A}}{\to} \mathbb{A}$ when A and $\mathbb{A}$ share a generic ray will be a particular case of this proposition, see Theorem 4.3.22. In the affine case, this will be a first step to prove that for all apartment A, there exists an isomorphism $A \stackrel{A \cap \mathbb{A}}{\to} \mathbb{A}$.

Lemma 4.2.31. *Let A be an apartment of $\mathcal{I}$ and $\phi : \mathbb{A} \to A$ be an isomorphism of apartments. Let $P \subset \mathbb{A} \cap A$ be a nonempty relatively open convex set, $Z = \mathrm{supp}(P)$ and suppose that ϕ fixes P. Then ϕ fixes $P + (\mathcal{T} \cap \vec{Z}) \cap A$, where $\mathcal{T}$ is the Tits cone.*

Proof. Let $x \in P + (\mathcal{T} \cap \vec{Z}) \cap A$, $x = p + t$, where $p \in P$ and $t \in \mathcal{T}$. Suppose $t \neq 0$. Let $L = p + \mathbb{R}t$. Then L is a preordered line in $\mathcal{I}$ and ϕ fixes $L \cap P$. Moreover, $p \leq x$ and thus by Proposition 5.4 of [Rou11], there exists an isomorphism $\psi : \mathbb{A} \stackrel{[p,x]}{\to} A$. In particular, $\phi^{-1} \circ \psi : \mathbb{A} \to \mathbb{A}$ fixes $L \cap P$. But then $\phi^{-1} \circ \psi_{|L}$ is an affine isomorphism fixing a nonempty open set of L: this is the identity. Therefore $\phi^{-1} \circ \psi(x) = x = \phi^{-1}(x)$, which shows the lemma. □

Lemma 4.2.32. *Let A be an apartment of $\mathcal{I}$. Let $U \subset \mathbb{A} \cap A$ be a nonempty relatively open set and $X = \mathrm{supp}(U)$. Then there exists a nonempty open subset V of U (in X) such that there exists an isomorphism $\phi : \mathbb{A} \overset{V}{\to} A$.*

Proof. Let $\bigcup_{i=1}^k P_i$ be a decomposition in enclosed subsets of $A \cap \mathbb{A}$. Let $i \in [\![1, k]\!]$ such that $P_i \cap U$ has nonempty interior in X and $\phi : \mathbb{A} \overset{P_i}{\to} A$. Then ϕ fixes a nonempty open set of U, which proves the lemma. □

Lemma 4.2.33. *Let A be an apartment of $\mathcal{I}$ and $\phi : \mathbb{A} \to A$ be an isomorphism. Let $F = \{z \in A | \phi(z) = z\}$. Then F is closed in $\mathbb{A}$.*

Proof. By Proposition 4.2.16, $\rho_{+\infty} \circ \phi : \mathbb{A} \to \mathbb{A}$ and $\rho_{-\infty} \circ \phi : \mathbb{A} \to \mathbb{A}$ are continuous. Let $(z_n) \in F^{\mathbb{N}}$ such that (z_n) converges in $\mathbb{A}$ and $z = \lim z_n$.

For all $n \in \mathbb{N}$, one has $\rho_{+\infty}(\phi(z_n)) = z_n = \rho_{-\infty}(\phi(z_n)) \to \rho_{+\infty}(\phi(z)) = z = \rho_{-\infty}(\phi(z))$ and by Proposition 4.2.15, $z = \phi(z)$, which proves the lemma. □

Proposition 4.2.34. *Let A be an apartment of $\mathcal{I}$ and $P \subset \mathbb{A} \cap A$ be a convex set. Let $X = \mathrm{supp}(P)$ and suppose that $\mathcal{T} \cap \vec{X}$ has nonempty interior in $\vec{X}$. Then there exists an isomorphism of apartments $\phi : \mathbb{A} \overset{P}{\to} A$.*

Proof. (see Figure 4.2.3) Let $V \subset P$ be a nonempty open set of X such that there exists an isomorphism $\phi : \mathbb{A} \overset{V}{\to} A$ (such a V exists by Lemma 4.2.32). Let us show that ϕ fixes $\mathrm{Int}_r(P)$.

Let $x \in V$. One fixes the origin of $\mathbb{A}$ in x and thus X is a vector space. Let $(e_j)_{j \in J}$ be a basis of $\mathbb{A}$ such that for some subset $J' \subset J$, $(e_j)_{j \in J'}$ is a basis of X and $(x + \mathcal{T}) \cap X \supset \bigoplus_{j \in J'} \mathbb{R}_+^* e_j$. For all $y \in X$, $y = \sum_{j \in J'} y_j e_j$ with $y_j \in \mathbb{R}$ for all $j \in J'$, one sets $|y| = \max_{j \in J'} |y_j|$. If $a \in A$ and $r > 0$, one sets $B(a, r) = \{y \in X | \ |y - a| < r\}$.

Suppose that ϕ does not fix $\mathrm{Int}_r(P)$. Let $y \in \mathrm{Int}_r(P)$ such that $\phi(y) \neq y$. Let $s = \sup\{t \in [0, 1] | \exists U \in \mathcal{V}_X([0, ty]) | \ \phi \text{ fixes } U\}$ and $z = sy$. Then by Lemma 4.2.33, $\phi(z) = z$.

By definition of z, for all $r > 0$, ϕ does not fix $B(z, r)$. Let $r > 0$ such that $B(z, 5r) \subset \mathrm{Int}_r P$. Let $z_1 \in B(z, r) \cap [0, z)$ and $r_1 > 0$ such that ϕ fixes $B(z_1, r_1)$ and $z_2' \in B(z, r)$ such that $\phi(z_2') \neq z_2'$. Let $r_2' \in (0, r)$ such that for all $a \in B(z_2', r_2')$, $\phi(z) \neq z$. Let $z_2 \in B(z_2', r_2')$ such that for some $r_2 \in (0, r_2')$, $B(z_2, r_2) \subset B(z_2', r_2')$ and such that there exists an isomorphism $\psi : \mathbb{A} \overset{B(z_2, r_2)}{\to} A$ (such z_2 and r_2 exists by Lemma 4.2.32). Then $|z_1 - z_2| < 3r$.

Let us prove that $(z_1 + \mathcal{T} \cap X) \cap (z_2 + \mathcal{T} \cap X) \cap \mathrm{Int}_r(P)$ contains a nonempty open set $U \subset X$. One identifies X and $\mathbb{R}^{J'}$ thanks to the basis $(e_j)_{j \in J'}$. One has $z_2 - z_1 \in]-3, 3[^{J'}$ and thus $(z_1 + \mathcal{T}) \cap (z_2 + \mathcal{T}) = (z_1 + \mathcal{T}) \cap (z_1 + z_2 - z_1 + \mathcal{T}) \supset z_1 +]3, 4[^{J'}$. As $P \supset B(z_1, 4r)$, $(z_1 + \mathcal{T} \cap X) \cap (z_2 + \mathcal{T} \cap X) \cap \mathrm{Int}_r(P)$ contains a nonempty open set $U \subset X$.

By Lemma 4.2.31, ϕ and ψ fix U. Therefore, $\phi^{-1} \circ \psi$ fixes U and as it is an isomorphism of affine space of A, $\phi^{-1} \circ \psi$ fixes X. Therefore $\phi^{-1} \circ \psi(z_2) = \phi^{-1}(z_2) = z_2$ and thus $\phi(z_2) = z_2$: this is absurd. Hence ϕ fixes $\mathrm{Int}_r(P)$. By Lemma 4.2.33, ϕ fixes $\overline{\mathrm{Int}_r(P)} = \overline{P}$ and thus ϕ fixes P, which shows the proposition. □

4.3 Intersection of two apartments sharing a generic ray

The aim of this section is to prove Theorem 4.3.22: let A and B be two apartments sharing a generic ray. Then $A \cap B$ is enclosed and there exists an isomorphism $\phi : A \overset{A \cap B}{\to} B$.

We first reduce our study to the case where $A \cap B$ has nonempty interior by the following lemma:

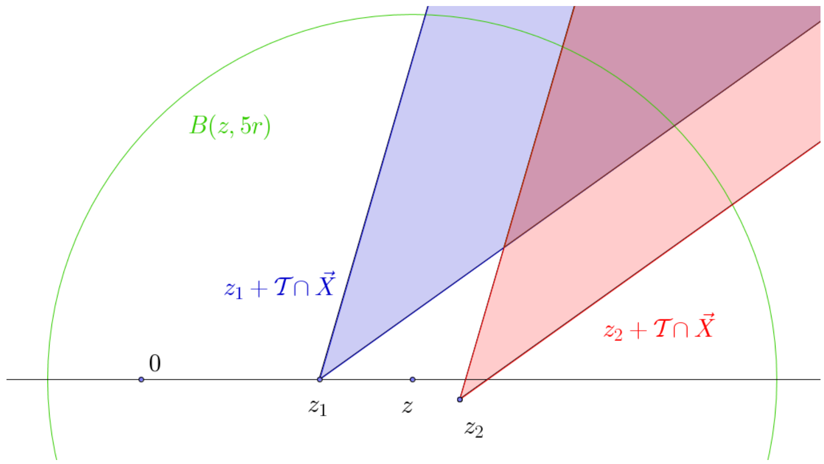

Figure 4.2.3 – Proof of Proposition 4.2.34

Lemma 4.3.1. *Suppose that for all apartments A, B such that $A \cap B$ contains a generic ray and has nonempty interior, $A \cap B$ is convex. Then if A_1 and A_2 are two apartments containing a generic ray, $A_1 \cap A_2$ is enclosed and there exists an isomorphism $\phi : A_1 \overset{A_1 \cap A_2}{\to} A_2$.*

Proof. Let us prove that $A_1 \cap A_2$ is convex. Let δ be the direction of a generic ray shared by A_1 and A_2. Let $x_1, x_2 \in A_1 \cap A_2$ and $\mathfrak{F}^\infty$ be the vectorial face direction containing δ. Let $\mathfrak{F}'^\infty$ be the vectorial face direction of A_1 opposite to $\mathfrak{F}^\infty$. Let C_1 be a chamber of A_1 containing x_1, C_2 be a chamber of A_2 containing x_2, $\mathfrak{r}_1 = \mathfrak{r}(C_1, \mathfrak{F}'^\infty) \subset A_1$, $\mathfrak{r}_2 = \mathfrak{r}(C_2, \mathfrak{F}^\infty) \subset A_2$, $\mathfrak{R}_1 = germ(\mathfrak{r}_1)$ and $\mathfrak{R}_2 = germ(\mathfrak{r}_2)$. By (MA3) there exists an apartment A_3 containing $\mathfrak{R}_1$ and $\mathfrak{R}_2$.

Let us prove that A_3 contains x_1 and x_2. One identifies A_1 and $\mathbb{A}$. Let $F^v = 0 + \mathfrak{F}^\infty$ and $F'^v = 0 + \mathfrak{F}'^\infty$. As $A_3 \supset \mathfrak{R}_1$, there exists $f' \in F'^v$ such that $A_3 \supset x_1 + f' + F'^v$. Moreover $A_3 \supset \mathfrak{F}^\infty$ and thus it contains $x_1 + f' + \mathfrak{F}^\infty$. By Proposition 4.7.1 of [Rou11] $x_1 + f' + \mathfrak{F}^\infty = x_1 + f' + F^v$ and thus $A_3 \ni x_1$. As $A_3 \supset \mathfrak{R}_2$, there exists $f \in F^v$ such that $A_3 \supset x_2 + f$. As $A_3 \supset \mathfrak{F}'^\infty$, $A_3 \supset x_2 + f + \mathfrak{F}' = x_2 + f + F'^v$ by Proposition 4.7.1 of [Rou11] and thus $A_3 \ni x_2$.

If $i \in \{1, 2\}$, each element of $\mathfrak{R}_i$ has nonempty interior in A_i and thus $A_i \cap A_3$ has nonempty interior. By hypothesis, $A_1 \cap A_3$ and $A_2 \cap A_3$ are convex. By Proposition 4.2.34, there exist $\phi : A_1 \overset{A_1 \cap A_3}{\to} A_3$ and $\psi : A_2 \overset{A_2 \cap A_3}{\to} A_3$. Therefore $[x_1, x_2]_{A_1} = [x_1, x_2]_{A_3} = [x_1, x_2]_{A_2}$ and thus $A_1 \cap A_2$ is convex.

The existence of an isomorphism $A_1 \overset{A_1 \cap A_2}{\to} A_2$ is a consequence of Proposition 4.2.34 because the direction X of $A_1 \cap A_2$ meets $\mathring{\mathcal{T}}$ and thus $\vec{X} \cap \mathcal{T}$ spans $\mathcal{T}$.

The fact that $A_1 \cap A_2$ is enclosed is a consequence of Proposition 4.2.30. □

4.3.1 Definition of the frontier maps

The aim of 4.3.1 to 4.3.5 is to prove that if A and B are two apartments containing a generic ray and such that $A \cap B$ has nonempty interior, $A \cap B$ is convex. There is no loss of generality in assuming that $B = \mathbb{A}$ and that the direction $\mathbb{R}_+\nu$ of δ is included in $\pm\overline{C^v_f}$. As the roles of C^v_f and $-C^v_f$ are similar, one supposes that $\mathbb{R}_+\nu \subset \overline{C^v_f}$ and that $A \neq \mathbb{A}$. These hypothesis run until the end of 4.3.5.

In this subsection, we parametrize $\text{Fr}(A \cap \mathbb{A})$ by a map and describe $A \cap \mathbb{A}$ using the values of this map.

Lemma 4.3.2. *Let V be a bounded subset of $\mathbb{A}$. Then there exists $a \in \mathbb{R}$ such that for all $u \in [a, +\infty[$ and $v \in V$, $v \leq u\nu$.*

Proof. Let $a \in \mathbb{R}_+^*$ and $v \in V$, then $a\nu - v = a(\nu - \frac{1}{a}v)$. As $\nu \in \mathring{\mathcal{T}}$ and V is bounded, there exists $b > 0$ such that for all $a > b$, $\nu - \frac{1}{a}v \in \mathring{\mathcal{T}}$, which proves the lemma because $\mathring{\mathcal{T}}$ is a cone. □

Lemma 4.3.3. *Let $y \in A \cap \mathbb{A}$. Then $A \cap \mathbb{A}$ contains $y + \mathbb{R}_+\nu$.*

Proof. Let $x \in \mathbb{A}$ such that $A \cap \mathbb{A} \supset x + \mathbb{R}_+\nu$. The ray $x + \mathbb{R}_+\nu$ is generic and by (MA4), if $y \in \mathbb{A}$, $A \cap \mathbb{A}$ contains the convex hull of y and $x + [a, +\infty[\nu$, for some $a \gg 0$. In particular it contains $y + \mathbb{R}_+\nu$. □

Let $U = \{y \in \mathbb{A} | y + \mathbb{R}\nu \cap A \neq \emptyset\} = (A \cap \mathbb{A}) + \mathbb{R}\nu$.

Lemma 4.3.4. *The set U is convex.*

Proof. Let $u, v \in U$. Let $u' \in u + \mathbb{R}_+\nu \cap A$. By Lemma 4.3.2 and Lemma 4.3.3, there exists $v' \in v + \mathbb{R}_+\nu$ such that $u' \leq v'$. By order-convexity, $[u', v'] \subset A \cap \mathbb{A}$. By definition of U, $[u', v'] + \mathbb{R}\nu \subset U$ and in particular $[u, v] \subset U$, which is the desired conclusion. □

There are two possibilities: either there exists $y \in \mathbb{A}$ such that $y + \mathbb{R}\nu \subset A$ or for all $y \in \mathbb{A}$, $y + \mathbb{R}\nu \not\subset A$. The first case is the easiest and we treat it in the next lemma.

Lemma 4.3.5. *Suppose that for some $y \in \mathbb{A}$, $y - \mathbb{R}_+\nu \subset A \cap \mathbb{A}$. Then $A \cap \mathbb{A} = U$. In particular, $A \cap \mathbb{A}$ is convex.*

Proof. By Lemma 4.3.3, $A \cap \mathbb{A} = (A \cap \mathbb{A}) + \mathbb{R}_+\nu$. By symmetry and by hypothesis on $A \cap \mathbb{A}$, $(A \cap \mathbb{A}) + \mathbb{R}_-\nu = A \cap \mathbb{A}$. Therefore $A \cap \mathbb{A} = (A \cap \mathbb{A}) + \mathbb{R}\nu = U$. □

Definition of the frontier Until the end of 4.3.5, we suppose that for all $y \in \mathbb{A}$, $y + \mathbb{R}\nu \not\subset A$. Let $u \in U$. Then by Lemma 4.3.3, $u + \mathbb{R}\nu \cap A$ is of the form $a + \mathbb{R}_+^*\nu$ or $a + \mathbb{R}_+\nu$ for some $a \in \mathbb{A}$. As $A \cap \mathbb{A}$ is closed (by Proposition 4.2.17), the first case cannot occur. One sets $\mathrm{Fr}_\nu(u) = a \in \mathbb{A} \cap A$. One fixes ν until the end of 4.3.5 and one writes Fr instead of Fr_ν.

Lemma 4.3.6. *The map* Fr *takes it values in* $\mathrm{Fr}(\mathbb{A} \cap A)$ *and* $\mathbb{A} \cap A = \bigcup_{x \in U} \mathrm{Fr}(x) + \mathbb{R}_+\nu$.

Proof. Let $u \in U$. Then $\mathrm{Fr}(u) + \mathbb{R}_+\nu = (u + \mathbb{R}\nu) \cap A$. Thus $\mathrm{Fr}(u) \notin \mathrm{Int}(A \cap \mathbb{A})$. By Proposition 4.2.17, $\mathrm{Fr}(u) \in \mathrm{Fr}(A \cap \mathbb{A})$ and hence $\mathrm{Fr}(U) \subset \mathrm{Fr}(A \cap \mathbb{A})$.

Let $u \in A \cap \mathbb{A}$. One has $u \in A \cap (u + \mathbb{R}\nu) = \mathrm{Fr}(u) + \mathbb{R}_+\nu$ and we deduce that $\mathbb{A} \cap A \subset \bigcup_{x \in U} \mathrm{Fr}(x) + \mathbb{R}_+\nu$. The reverse inclusion is a consequence of Lemma 4.3.3, which finishes the proof. □

Let us sketch the proof of the convexity of $A \cap \mathbb{A}$ (which is Lemma 4.3.21). If $x, y \in \mathring{U}$, one defines $\mathrm{Fr}_{x,y} : [0, 1] \to \mathrm{Fr}(A \cap \mathbb{A})$ by $\mathrm{Fr}_{x,y}(t) = \mathrm{Fr}((1-t)x + ty)$ for all $t \in [0, 1]$. For all $t \in [0, 1]$, there exists a unique $f_{x,y}(t) \in \mathbb{R}$ such that $\mathrm{Fr}_{x,y}(t) = (1-t)x + ty + f_{x,y}(t)\nu$. We prove that for "almost" all $x, y \in \mathring{U}$, $f_{x,y}$ is convex. Let $x, y \in \mathring{U}$. We first prove that $f_{x,y}$ is continuous and piecewise affine. We prove the continuity by using order-convexity and the piecewise affineness by using the fact that there exists a finite set $\mathscr{M}$ of walls such that $\mathrm{Fr}(A \cap \mathbb{A}) \subset \bigcup M \in \mathscr{M} M$ (see Proposition 4.2.22). This enables to reduce the study of the convexity of $f_{x,y}$ to the study of $f_{x,y}$ at the points where the slope changes. Using order-convexity, we prove that if $\{x, y\}$ is such that for each point $u \in]0, 1[$ at which the slope changes, $\mathrm{Fr}_{x,y}(u)$ is contained in exactly two walls of $\mathscr{M}$, then $f_{x,y}$ is convex. We then prove that there are "enough" such pairs and conclude by an argument of density.

4.3.2 Continuity of the frontier

In this subsection, we prove that Fr is continuous on $\mathring{U}$, using order-convexity.

Let $\lambda : U \to \mathbb{R}$ be such that for all $x \in U$, $\mathrm{Fr}(x) = x + \lambda(x)\nu$. We prove the continuity of $\mathrm{Fr}_{|\mathring{U}}$ by proving the continuity of $\lambda_{|\mathring{U}}$. For this, we begin by majorizing $\lambda([x,y])$ if $x, y \in \mathring{U}$ (see Lemma 4.3.7) by a number depending on x and y. We use it to prove that if $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathring{U}$, $\lambda\big(\mathrm{conv}(\{a_1, \dots, a_n\})\big)$ is majorized and then deduce that $\mathrm{Fr}_{|\mathring{U}}$ is continuous (which is Lemma 4.3.12).

Lemma 4.3.7. *Let $x, y \in U$, $M = \max\{\lambda(x), \lambda(y)\}$ and $k \in \mathbb{R}_+$ such that $x + k\nu \geq y$. Then for all $u \in [x, y]$, $\lambda(u) \leq k + M$.*

Proof. By Lemma 4.3.3, $x + M\nu, y + M\nu \in A$. By hypothesis, $x + k\nu + M\nu \geq y + M\nu$. Let $t \in [0,1]$ and $u = tx + (1-t)y$. By order-convexity $t(x + k\nu + M\nu) + (1-t)(y + M\nu) \in A$. Therefore $\lambda(u) \leq M + tk \leq M + k$, which is our assertion. □

Lemma 4.3.8. *Let $d \in \mathbb{N}$, X be a d dimensional affine space and $P \subset X$. One sets $\mathrm{conv}_0(P) = P$ and for all $k \in \mathbb{N}$, $\mathrm{conv}_{k+1}(P) = \{(1-t)p + tp' | t \in [0,1] \text{ and } (p, p') \in \mathrm{conv}_k(P)^2\}$. Then $\mathrm{conv}_d(P) = \mathrm{conv}(P)$.*

Proof. By induction,

$$\mathrm{conv}_k(P) = \{\sum_{i=1}^{2^k} \lambda_i p_i | (\lambda_i) \in [0,1]^{2^k},\ \sum_{i=1}^{2^k} \lambda_i = 1 \text{ and } (p_i) \in P^{2^k}\}.$$

This is thus a consequence of Carathéodory's Theorem. □

Lemma 4.3.9. *Let P be a bounded subset of $\mathring{U}$ such that $\lambda(P)$ is majorized. Then $\lambda(\mathrm{conv}_1(P))$ is majorized.*

Proof. Let $M = \sup_{x \in P} \lambda(x)$ and $k \in \mathbb{R}_+$ such that for all $x, x' \in P$, $x' + k\nu \geq x$, which exists by Lemma 4.3.2. Let $u \in \mathrm{conv}_1(P)$ and $x, x' \in P$ such that $u \in [x, x']$. By Lemma 4.3.7, $\lambda(u) \leq k + M$ and the lemma follows.

□

Lemma 4.3.10. *Let $x \in \mathring{U}$. Then there exists $V \in \mathcal{V}_{\mathring{U}}(x)$ such that V is convex and $\lambda(V)$ is majorized.*

Proof. Let $n \in \mathbb{N}$ and $a_1, \dots, a_n \in \mathring{U}$ such that $V = \mathrm{conv}(a_1, \dots, a_n)$ contains x in its interior. Let $M \in \mathbb{R}_+$ such that for all $y, y' \in V$, $y + M\nu \geq y'$, which is possible by Lemma 4.3.2. One sets $P = \{a_1, \dots, a_n\}$ and for all $k \in \mathbb{N}$, $P_k = \mathrm{conv}_k(P)$. By induction using Lemma 4.3.9, $\lambda(P_k)$ is majorized for all $k \in \mathbb{N}$ and we conclude with Lemma 4.3.8. □

Lemma 4.3.11. *Let $V \subset \mathring{U}$ be open, convex, bounded and such $\lambda(V)$ is majorized by some $M \in \mathbb{R}_+$. Let $k \in \mathbb{R}_+$ be such that for all $x, x' \in V$, $x + k\nu \geq x'$. Let $a \in V$ and $u \in \mathbb{A}$ be such that $a + u \in V$. Then for all $t \in [0,1]$, $\lambda(a + tu) \leq (1-t)\lambda(a) + t(M + k)$.*

Proof. By Lemma 4.3.3, $a + u + (M+k)\nu \in A$. Moreover $a + u + (M+k)\nu \geq a + M\nu$, $a + M\nu \geq a + \lambda(a)\nu = \mathrm{Fr}(a)$ and thus $a + u + (M+k)\nu \geq \mathrm{Fr}(a)$.

Let $t \in [0,1]$. Then by order-convexity,

$$(1-t)(a + \lambda(a)\nu) + t(a + u + (M+k)\nu = a + tu + \big((1-t)\lambda(a) + t(M+k)\big)\nu \in A.$$

Therefore $\lambda(a + tu) \leq (1-t)\lambda(a) + t(M+k)$, which is our assertion. □

Lemma 4.3.12. *The map* Fr *is continuous on* $\mathring{U}$.

Proof. Let $x \in \mathring{U}$ and $V \in \mathcal{V}_{\mathring{U}}(x)$ be convex, open, bounded and such that $\lambda(V)$ is majorized by some $M \in \mathbb{R}_+$, which exists by Lemma 4.3.10. Let $k \in \mathbb{R}_+$ such that for all $v, v' \in V$, $v + k\nu \geq v'$. Let $|\ |$ be a norm on $\mathbb{A}$ and $r > 0$ such that $B(x,r) \subset V$, where $B(x,r) = \{u \in \mathbb{A} |\ |x-u| \leq r\}$. Let $S = \{u \in \mathbb{A} |\ |u - x| = r\}$. Let $N = M + k$.

Let $y \in S$ and $t \in [0,1]$. By applying Lemma 4.3.11 with $a = x$ and $u = y - x$, we get that

$$\lambda((1-t)x + ty) \leq \lambda(x) + tN.$$

By applying Lemma 4.3.11 with $a = (1-t)x + ty$ and $u = x - y$, we obtain that

$$\lambda(x) = \lambda\big((1-t)x + ty + t(x-y)\big) \leq \lambda\big((1-t)x + ty\big) + tN.$$

Therefore for all $t \in [0,1]$ and $y \in S$,

$$\lambda(x) - tN \leq \lambda\big((1-t)x + ty\big) \leq \lambda(x) + tN.$$

Let $(x_n) \in B(x,r)^{\mathbb{N}}$ such that $x_n \to x$. Let $n \in \mathbb{N}$. One sets $t_n = \frac{|x_n - x|}{r}$. If $t_n = 0$, one chooses $y_n \in S$. It $t_n \neq 0$, one sets $y_n = x + \frac{1}{t_n}(x_n - x) \in S$. Then $x_n = t_n y_n + (1-t_n)x$ and thus $|\lambda(x_n) - \lambda(x)| \leq t_n N \to 0$. Consequently $\lambda_{|\mathring{U}}$ is continuous and we deduce that $\mathrm{Fr}_{|\mathring{U}}$ is continuous. □

4.3.3 Piecewise affineness of $\mathrm{Fr}_{x,y}$

In this subsection, we prove that there exists a finite set $\mathscr{H}$ of hyperplanes of $\mathbb{A}$ such that Fr is affine on each connected component of $\mathring{U} \backslash \bigcup_{H \in \mathscr{H}} H$.

Let $\mathscr{M}$ be a finite set of walls such that $\mathrm{Fr}(A \cap \mathbb{A})$ is included in $\bigcup_{M \in \mathscr{M}} M$, whose existence is provided by Proposition 4.2.22. Let $r = |\mathscr{M}|$. Let $\beta_1, \ldots, \beta_r \in \Phi_{re}^r$ and $(l_1, \ldots, \ell_r) \in \prod_{i=1}^r \Lambda'_{\beta_i}$ such that $\mathscr{M} = \{M_i |\ i \in [\![1, r]\!]\}$ where $M_i = \beta_i^{-1}(\{l_i\})$ for all $i \in [\![1, r]\!]$.

If $i, j \in [\![1, r]\!]$, with $i \neq j$, $\beta_i(\nu)\beta_j(\nu) \neq 0$ and M_i and M_j are not parallel, one sets $H_{i,j} = \{x \in \mathbb{A} | \frac{l_i - \beta_i(x)}{\beta_i(\nu)} = \frac{l_j - \beta_j(x)}{\beta_j(\nu)}\}$ (this definition will appear naturally in the proof of the next lemma). Then $H_{i,j}$ is a hyperplane of $\mathbb{A}$. Indeed, otherwise $H_{i,j} = \mathbb{A}$. Hence $\frac{\beta_j(x)}{\beta_j(\nu)} - \frac{\beta_i(x)}{\beta_i(\nu)} = \frac{l_j}{\beta_j(\nu)} - \frac{l_i}{\beta_i(\nu)}$, for all $x \in \mathbb{A}$. Therefore $\frac{\beta_j(x)}{\beta_j(\nu)} - \frac{\beta_i(x)}{\beta_i(\nu)} = 0$ for all $x \in \mathbb{A}$ and thus M_i and M_j are parallel: a contradiction. Let $\mathscr{H} = \{H_{i,j} | i \neq j,\ \beta_i(\nu)\beta_j(\nu) \neq 0 \text{ and } M_i \nparallel M_j\} \cup \{M_i | \beta_i(\nu) = 0\}$.

Even if the elements of $\mathscr{H}$ can be walls of $\mathbb{A}$, we will only consider them as hyperplanes of $\mathbb{A}$. To avoid confusion between elements of $\mathscr{M}$ and elements of $\mathscr{H}$, we will try to use the letter M (resp. H) in the name of objects related to $\mathscr{M}$ (resp. $\mathscr{H}$).

Lemma 4.3.13. *Let* $M_\cap = \bigcup_{M \neq M' \in \mathscr{M}} M \cap M'$. *Then* $\mathrm{Fr}^{-1}(M_\cap) \subset \bigcup_{H \in \mathcal{H}} H$.

Proof. Let $x \in \mathrm{Fr}^{-1}(M_\cap)$. One has $\mathrm{Fr}(x) = x + \lambda\nu$, for some $\lambda \in \mathbb{R}$. There exists $i, j \in [\![1, r]\!]$ such that $i \neq j$, $\beta_i(\mathrm{Fr}(x)) = \ell_i$ and $\beta_j(\mathrm{Fr}(x)) = \ell_j$ and M_i and M_j are not parallel. Therefore if $\beta_i(\nu)\beta_j(\nu) \neq 0$, $\lambda = \frac{l_i - \beta_i(x)}{\beta_i(\nu)} = \frac{l_j - \beta_j(x)}{\beta_j(\nu)}$ and thus $x \in H_{i,j}$, and if $\beta_i(\nu)\beta_j(\nu) = 0$, $x \in M_k$, where $k \in \{i, j\}$ such that $\beta_k(\nu) = 0$, which proves the lemma. □

Lemma 4.3.14. *One has* $A \cap \mathbb{A} = \overline{\mathrm{Int}(A \cap \mathbb{A})}$.

Proof. By Proposition 4.2.17, $A\cap\mathbb{A}$ is closed and thus $\overline{\mathrm{Int}(A\cap\mathbb{A})}\subset A\cap\mathbb{A}$.

Let $x\in A\cap\mathbb{A}$. Let V be an open bounded set included in $A\cap\mathbb{A}$. By Lemma 4.3.2 applied to $x-V$, there exists $a>0$ such that for all $v\in V$, $v+a\nu\geq x$. One has $V+a\nu\subset A\cap\mathbb{A}$ and by order convexity (Conséquence 2 of Proposition 5.4 in [Rou11]), $\mathrm{conv}(V+a\nu,x)\subset A\cap\mathbb{A}$. As $\mathrm{conv}(V+a\nu,x)$ is a convex set with nonempty interior, there exists $(x_n)\in\mathrm{Int}(\mathrm{conv}(V+a\nu,x))^{\mathbb{N}}$ such that $x_n\to x$, and the lemma follows. □

Let $f_1,\ldots,f_s$ be affine forms on $\mathbb{A}$ such that $\mathscr{H}=\{f_i^{-1}(\{0\})|i\in[\![1,s]\!]\}$ for some $s\in\mathbb{N}$. Let $R=(R_i)\in\{\leq,\geq,<,>\}^s$. One sets $P_R=\mathring{U}\cap\{x\in\mathbb{A}|\ (f_i(x)\ R_i\ 0)\ \forall i\in[\![1,s]\!]\}$. If $R=(R_i)\in\{\leq,\geq\}^s$, one defines $R'=(R'_i)\in\{<,>\}^s$ by $R'_i=$ " $<$" if $R_i=$ " $\leq$" and $R'_i=$ " $>$" otherwise (one replaces large inequalities by strict inequalities). If $R\in\{\leq,\geq\}^s$, then $\mathrm{Int}(P_R)=P_{R'}$.

Let $X=\{R\in\{\leq,\geq\}^s|\mathring{P}_R\neq\emptyset\}$. By Lemma 4.3.14 and Lemma 4.2.18, $\mathring{U}=\bigcup_{R\in X}P_R$ and for all $R\in X$, $\mathring{P}_R\subset\mathbb{A}\backslash\bigcup_{H\in\mathscr{H}}H$.

Lemma 4.3.15. *Let $R\in X$. Then there exists $M\in\mathscr{M}$ such that $\mathrm{Fr}(P_R)\subset M$.*

Proof. Let $x\in\mathring{P}_R$. Let $M\in\mathscr{M}$ such that $\mathrm{Fr}(x)\in M$. Let us show that $\mathrm{Fr}(P_R)\subset M$. By continuity of Fr (by Lemma 4.3.12), it suffices to prove that $\mathrm{Fr}(\mathring{P}_R)\subset M$. By connectedness of $\mathring{P}_R$ it suffices to prove that $\mathrm{Fr}^{-1}(M)\cap\mathring{P}_R$ is open and closed. As Fr is continuous, $\mathrm{Fr}^{-1}(M)\cap\mathring{P}_R$ is closed (in $\mathring{P}_R$).

Suppose that $\mathrm{Fr}^{-1}(M)\cap\mathring{P}_R$ is not open. Then there exists $y\in\mathring{P}_R$ such that $\mathrm{Fr}(y)\in M$ and a sequence $(y_n)\in\mathring{P}_R^{\ \mathbb{N}}$ such that $y_n\to y$ and such that $\mathrm{Fr}(y_n)\notin M$ for all $n\in\mathbb{N}$. For all $n\in\mathbb{N}$, $\mathrm{Fr}(y_n)\in\bigcup_{M'\in\mathscr{M}}M'$, and thus, maybe extracting a subsequence, one can suppose that for some $M'\in\mathscr{M}$, $y_n\in M'$ for all $n\in\mathbb{N}$.

As Fr is continuous (by Lemma 4.3.12), $\mathrm{Fr}(y)\in M'$. Thus $\mathrm{Fr}(y)\in M\cap M'$ and by Lemma 4.3.13, $y\in\bigcup_{H\in\mathscr{H}}H$, which is absurd by choice of y. Therefore, $\mathrm{Fr}^{-1}(M)\cap\mathring{P}_R$ is open, which completes the proof of the lemma. □

Lemma 4.3.16. *Let $R\in X$ and $M\in\mathscr{M}$ such that $\mathrm{Fr}(P_R)\subset M$. Then $\nu\notin\vec{M}$ and there exists a (unique) affine morphism $\psi:\mathbb{A}\to M$ such that $\mathrm{Fr}_{|P_R}=\psi_{|P_R}$. Moreover ψ induces an isomorphism $\overline{\psi}:\mathbb{A}/\mathbb{R}\nu\to M$.*

Proof. If $y\in\mathring{U}$, $\mathrm{Fr}(y)=y+k(y)\nu$ for some $k(y)\in\mathbb{R}$. Let $\alpha\in\Phi_{re}$ such that $M=\alpha^{-1}(\{u\})$ for some $u\in-\Lambda'_\alpha$. For all $y\in P_R$, one has $\alpha(\mathrm{Fr}(y))=\alpha(y)+k(y)\alpha(\nu)=u$ and $\alpha(\nu)\neq0$ because α is not constant on P_R. Consequently $\nu\notin\vec{M}$ and $\mathrm{Fr}(y)=y+\frac{u-\alpha(y)}{\alpha(\nu)}\nu$. One defines $\psi:\mathbb{A}\to M$ by $\psi(y)=y+\frac{u-\alpha(y)}{\alpha(\nu)}\nu$ for all $y\in\mathbb{A}$ and ψ has the desired properties. □

4.3.4 Local convexity of $\mathrm{Fr}_{x,y}$

Let $M\in\mathscr{M}$ and $\vec{M}$ be its direction. Let $\mathcal{T}_M=\mathring{\mathcal{T}}\cap\vec{M}$ and D_M be the half-apartment containing a shortening of $\mathbb{R}_+\nu$ and whose wall is M.

Lemma 4.3.17. *Let $a\in\mathrm{Fr}(\mathring{U})$ and suppose that for some $\mathcal{Y}\in\mathcal{V}_{\mathring{U}}(a)$, $\mathrm{Fr}(\mathcal{Y})\subset M$ for some $M\in\mathscr{M}$. Then $\mathrm{Fr}\big((a\pm\mathring{\mathcal{T}}_M)\cap\mathring{U}\big)\subset D_M$.*

Proof. Let $u\in\mathring{U}\cap(a-\mathring{\mathcal{T}}_M)$, $u\neq a$. Suppose $\mathrm{Fr}(u)\notin D_M$. Then $\mathrm{Fr}(u)=u-k\nu$, with $k\geq0$. Then $\mathrm{Fr}(u)\leq u\mathring{\leq}a$ (which means that $a-u\in\mathring{\mathcal{T}}$). Therefore for some $\mathcal{Y}'\in\mathcal{V}_M(a)$ such that $\mathcal{Y}'\subset\mathcal{Y}$, one has $\mathrm{Fr}(u)\mathring{\leq}u'$ for all $u'\in\mathcal{Y}'$. As a consequence $\mathbb{A}\cap A\supset\mathrm{conv}(\mathcal{Y}',\mathrm{Fr}(u))$ and thus $\mathrm{Fr}(u')\notin M$ for all $u'\in\mathcal{Y}'$. This is absurd and hence $\mathrm{Fr}(u)\in D_M$.

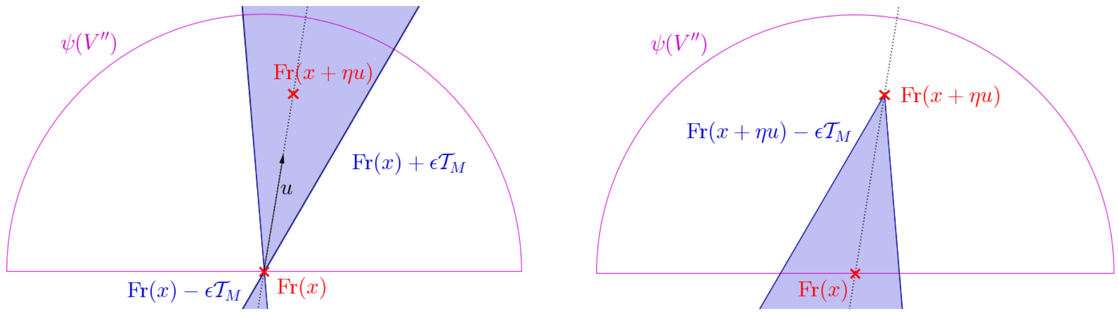

Figure 4.3.1 – Proof of Lemma 4.3.18 when $\dim H = 2$ (the illustration is made in M)

Let $v \in \mathring{U} \cap (a + \mathring{\mathcal{T}}_M)$, $v \neq a$ and suppose that $\mathrm{Fr}(v) \notin D_M$. Then for $v' \in [\mathrm{Fr}(v), v)$ near enough from v, $a \leq v'$. Therefore, $[a, v'] \subset \mathbb{A} \cap A$. Thus for all $t \in]a, v[$, $\mathrm{Fr}(t) \notin D_M$, a contradiction. Therefore $\mathrm{Fr}(v) \in D_M$ and the lemma follows. □

The following lemma is crucial to prove the local convexity of $\mathrm{Fr}_{x,y}$ for good choices of x and y. This is mainly here that we use that $A \cap \mathbb{A}$ have nonempty interior.

Let $H_\cap = \bigcup_{H \neq H' \in \mathscr{H}} H \cap H'$.

Lemma 4.3.18. *Let $x \in \mathring{U} \cap (\bigcup_{H \in \mathscr{H}} H) \backslash H_\cap$ and $H \in \mathscr{H}$ such that $x \in H$. Let C_1 and C_2 be the half-spaces defined by H. Then there exists $V \in \mathcal{V}_{\mathring{U}}(x)$ satisfying the following conditions:*

1. *For $i \in \{1, 2\}$, let $V_i = V \cap \mathring{C}_i$. Then $V_i \subset \mathring{P}_{R_i}$ for some $R_i \in X$.*
2. *Let M be a wall containing $\mathrm{Fr}(P_{R_1})$. Then $\mathrm{Fr}(V) \subset D_M$.*

Proof. (see Figure 4.3.1) The set $\mathring{U} \backslash \bigcup_{H \in \mathscr{H} \backslash \{H\}} H$ is open in $\mathring{U}$. Let $V' \in \mathcal{V}_{\mathring{U}}(x)$ such that $V' \cap \bigcup_{H' \in \mathscr{H} \backslash \{H\}} H' = \emptyset$ and such that V' is convex. Let $i \in \{1, 2\}$ and $V'_i = V' \cap \mathring{C}_i$. Then $V'_i \subset \mathring{U} \backslash \bigcup_{H \in \mathcal{H}} H$ and V'_i is connected. As the connected components of $\mathring{U} \backslash \bigcup_{H \in \mathcal{H}} H$ are the $\mathring{P}_R$'s for $R \in X$, V' satisfies 1.

Let $\psi : \mathbb{A} \to M$ be the affine morphism such that $\psi_{|P_{R_1}} = \mathrm{Fr}_{|P_{R_1}}$ and $\overline{\psi} : \mathbb{A}/\mathbb{R}\nu \to M$ be the induced isomorphism, which exists by Lemma 4.3.16. Let $\pi : \mathbb{A} \twoheadrightarrow \mathbb{A}/\mathbb{R}\nu$ be the canonical projection. As C_1 is invariant under translation by ν (by definition of the elements of $\mathscr{H}$) $\psi(C_1) = \overline{\psi}(\pi(C_1))$ is a half-space D of M. Let $V'' = V' \cap C_1$. Then $\psi(V'') = \overline{\psi}(C_1) \cap \overline{\psi}(\pi(V')) \in \mathcal{V}_D(\mathrm{Fr}(x))$.

Let $g : \vec{M} \to \mathbb{R}$ be a linear form such that $D = g^{-1}([b, +\infty[)$, for some $b \in \mathbb{R}$. Let $\epsilon \in \{-1, 1\}$ such that for some $u \in \epsilon\mathcal{T}_M$ one has $g(u) > 0$. Let $\eta > 0$. Then $\mathrm{Fr}(x + \eta u) \in x + \eta u + \mathbb{R}\nu$ and thus $\mathrm{Fr}(x + \eta u) = \mathrm{Fr}(x) + \eta u + k\nu$ for some $k \in \mathbb{R}$. If η is small enough that $x + \eta u \in V''$, $k\nu = \mathrm{Fr}(x + \eta u) - (\mathrm{Fr}(x) + \eta u) \in \vec{M}$ and hence $k = 0$ (by Lemma 4.3.16). Let $\mathcal{Y} = \psi(V'') + \mathbb{R}\nu$ and $a = \mathrm{Fr}(x) + \eta u$. Then $\mathcal{Y} \in \mathcal{V}_{\mathring{U}}(a)$ and for all $v \in \mathcal{Y}$, $\mathrm{Fr}(v) \in M$. By Lemma 4.3.17, $\mathrm{Fr}(\mathring{U} \cap (a - \epsilon\mathcal{T}_M)) = \mathrm{Fr}(\mathring{U} \cap (a - \epsilon\mathcal{T}_M + \mathbb{R}\nu)) \subset D_M$. Moreover, $a - \epsilon\mathcal{T}_M + \mathbb{R}\nu \in \mathcal{V}_{\mathring{U}}(x)$ and thus if one sets $V = V' \cap (a - \epsilon\mathcal{T}_M + \mathbb{R}\nu)$, V satisfies 1 and 2.

□

4.3.5 Convexity of $A \cap \mathbb{A}$

Let $\vec{\mathscr{H}} = \bigcup_{H \in \mathscr{H}} \vec{H}$ be the set of directions of the hyperplanes of $\mathscr{H}$.

Lemma 4.3.19. *Let $x, y \in \mathring{U} \cap A \cap \mathbb{A}$ such that $y - x \notin \vec{\mathscr{H}}$ and such that the line spanned by $[x, y]$ does not meet any point of $H_\cap$. Then $[x, y] \subset \mathring{U} \cap A \cap \mathbb{A}$.*

Proof. Let $\pi : [0, 1] \to \mathbb{A}$ defined by $\pi(t) = tx + (1-t)y$ for all $t \in [0, 1]$ and $g = \mathrm{Fr} \circ \pi$. Let $f_1, \ldots, f_s$ be affine forms on $\mathbb{A}$ such that $\mathscr{H} = \{f_i^{-1}(\{0\}) |\ i \in [\![1, s]\!]\}$. As $y - x \notin \vec{\mathscr{H}}$, for all $i \in [\![1, s]\!]$, the map $f_i \circ g$ is strictly monotonic and $\pi^{-1}(\bigcup_{H \in \mathscr{H}} H)$ is finite. Therefore, there exist $k \in \mathbb{N}$ and open intervals $T_1, \ldots, T_k$ such that $[0, 1] = \bigcup_{i=1}^{k} \overline{T_i}$, $T_1 < \ldots < T_k$ and $\pi(T_i) \subset \mathring{P}_{R_i}$ for some $R_i \in X$ for all $i \in [\![1, k]\!]$. For all $t \in [0, 1]$, $g(t) = \pi(t) + f(t)\nu$ for some $f(t) \in \mathbb{R}$. By Lemma 4.3.16 this equation uniquely determines $f(t)$ for all $t \in [0, 1]$. By Lemma 4.3.12, f is continuous and by Lemma 4.3.16, f is affine on each T_i.

Let us prove that f is convex. Let $i \in [\![1, k-1]\!]$. One writes $T_i =]a, b[$. Then for $\epsilon > 0$ small enough, one has $f(b + \epsilon) = f(b) + \epsilon c_+$ and $f(b - \epsilon) = f(b) - \epsilon c_-$. To prove the convexity of f, it suffices to prove that $c_- < c_+$. Let M be a wall containing $\mathrm{Fr}(P_{R_i})$. As $\pi(b) \in \mathring{U} \cap \bigcup_{H \in \mathscr{H}} H \backslash H_\cap$, we can apply Lemma 4.3.18 and there exists $V \in \mathcal{V}_{[0,1]}(b)$ such that $g(V) \subset D_M$. Let $h : \mathbb{A} \to \mathbb{R}$ be a linear map such that $D_M = h^{-1}([a, +\infty[)$. For $\epsilon > 0$ small enough, one has $h(g(b + \epsilon)) \geq a$ and $h(g(b - \epsilon)) = a$.

For $\epsilon > 0$ small enough, one has

$$\begin{aligned} h(g(b + \epsilon)) &= h(\pi(b) + \epsilon(y - x) + (f(b) + \epsilon c_+)\nu) \\ &= h(g(b) + \epsilon(y - x + c_+\nu)) \\ &= a + \epsilon(h(y - x) + c_+ h(\nu)) \geq a, \end{aligned}$$

and similarly, $h(g(b - \epsilon)) = a - \epsilon(h(y - x) + c_- h(\nu)) = a$.

Therefore $h(y - x) + c_+ h(\nu) \geq 0$, $h(y - x) + c_- h(\nu) = 0$ and hence $(c_+ - c_-)h(\nu) \geq 0$. As D_M contains a shortening of $\mathbb{R}_+\nu$, $h(\nu) \geq 0$ and by Lemma 4.3.16, $h(\nu) > 0$. Consequently, $c_- \leq c_+$ and, as $i \in [\![1, k-1]\!]$ was arbitrary, f is convex.

For all $t \in [0, 1]$, $f(t) \leq (1 - t)f(0) + tf(1)$. Therefore

$$(1 - t)g(0) + tg(1) = \pi(t) + ((1 - t)f(0) + tf(1))\nu \in \pi(t) + f(t)\nu + \mathbb{R}_+\nu = g(t) + \mathbb{R}_+\nu.$$

By definition of Fr, if $t \in [0, 1]$, $(1 - t)g(0) + tg(1) \in A \cap \mathbb{A}$. Moreover, there exist $\lambda, \mu \geq 0$ such that $x = g(0) + \lambda\nu$ and $y = g(1) + \mu\nu$. Then $\pi(t) = (1 - t)x + ty = (1 - t)g(0) + tg(1) + ((1 - t)\lambda + t\mu)\nu \in A \cap \mathbb{A}$ and hence $[x, y] \subset A \cap \mathbb{A}$.

□

Lemma 4.3.20. *Let $x, y \in \mathrm{Int}(\mathbb{A} \cap A)$ and $\vec{\mathscr{H}} = \bigcup_{H \in \mathscr{H}} \vec{H}$. Then there exists $(x_n), (y_n) \in \mathrm{Int}(A \cap \mathbb{A})^{\mathbb{N}}$ satisfying the following conditions:*

1. *$x_n \to x$ and $y_n \to y$*
2. *for all $n \in \mathbb{N}$, $y_n - x_n \notin \vec{\mathscr{H}}$*
3. *the line spanned by $[x_n, y_n]$ does not meet any point of $H_\cap$.*

Proof. Let $(x_n) \in (\mathrm{Int}(A \cap \mathbb{A}) \backslash H_\cap)^{\mathbb{N}}$ such that $x_n \to x$. Let $|\ |$ be a norm on $\mathbb{A}$. Let $n \in \mathbb{N}$. Let F be the set of points z such that the line spanned by $[x_n, z]$ meets $H_\cap$. Then F is a finite union of hyperplanes of $\mathbb{A}$ (because $H_\cap$ is a finite union of spaces of dimension at most $\dim \mathbb{A} - 2$). Therefore $\mathbb{A} \backslash (F \cup x_n + \vec{\mathscr{H}})$ is dense in $\mathbb{A}$ and one can choose $y_n \in \mathbb{A} \backslash (F \cup x_n + \vec{\mathscr{H}})$ such that $|y_n - y| \leq \frac{1}{n+1}$. Then (x_n) and (y_n) satisfy the conditions of the lemma. □

Lemma 4.3.21. *The set $A \cap \mathbb{A}$ is convex.*

Proof. Let $x, y \in \text{Int}(A \cap \mathbb{A})$. Let $(x_n), (y_n)$ be as in Lemma 4.3.20. Let $t \in [0, 1]$. As $\text{Int}(A \cap \mathbb{A}) \subset \mathring{U}$, for all $n \in \mathbb{N}$, $tx_n + (1-t)y_n \in A \cap \mathbb{A}$ by Lemma 4.3.19. As $A \cap \mathbb{A}$ is closed (by Proposition 4.2.17), $tx + (1-t)y \in A \cap \mathbb{A}$. Therefore $\text{Int}(A \cap \mathbb{A})$ is convex. Consequently $A \cap \mathbb{A} = \overline{\text{Int}(A \cap \mathbb{A})}$ (by Lemma 4.3.14) is convex. □

We thus proved the following theorem:

Theorem 4.3.22. *Let A and B be two apartments sharing a generic ray. Then $A \cap B$ is enclosed and there exists an isomorphism $\phi : A \overset{A\cap B}{\to} B$.*

Proof. By Lemma 4.3.21 and Lemma 4.3.1, $A \cap B$ is convex. By Proposition 4.2.30, $A \cap B$ is enclosed and by Proposition 4.2.34, there exists an isomorphism $\phi : A \overset{A\cap B}{\to} B$. □

4.3.6 A partial reciprocal

We now prove a kind of weak reciprocal of Theorem 4.3.22 when some group G acts strongly transitively on $\mathcal{I}$ and when $\mathcal{I}$ is thick, which means that each panel is included in at least three chambers. For example if G is a quasi-split Kac-Moody group over a ultrametric field $\mathcal{K}$, it acts strongly transitively on the thick masure $\mathcal{I}(G, \mathcal{K})$ associated.

Lemma 4.3.23. *Let P be an enclosed subset of $\mathbb{A}$ and suppose that $\mathring{P} \neq \emptyset$. One fixes the origin of $\mathbb{A}$ in some point of $\mathring{P}$. Let j_P be the gauge of P defined in Section 4.2.4. Let $U = \{x \in \mathbb{A} | j_P(x) \neq 0\}$. One defines $\text{Fr} : U \to P$ as in Lemma 4.2.26. One writes $P = \bigcap_{i=1}^k D_i$, where the D_i's are half-apartments of $\mathbb{A}$. Let $j \in [\![1, k]\!]$, M_j be the wall of D_j and suppose that for all open subset V of U, $\text{Fr}(V) \not\subseteq M_j$. Then $P = \bigcap_{i \in [\![1,k]\!] \setminus \{j\}} D_i$.*

Proof. Suppose that $P \not\supseteq \bigcap_{i \in [\![1,k]\!] \setminus \{j\}} D_i$. Let V be a nonempty open and bounded subset included in $\bigcap_{i \in [\![1,k]\!] \setminus \{j\}} D_i \backslash P$. Let $n \in \mathbb{N}^*$ such that $\frac{1}{n} V \subset P$. Let $v \in V$. Then $[\frac{1}{n} v, v] \cap \text{Fr}(P) = \{\text{Fr}(v)\}$. Moreover for all $i \in [\![1, k]\!] \backslash \{j\}$, $[\frac{1}{n} v, v] \subset \mathring{D}_i$. As $\text{Fr}(P) \subset \bigcup_{i \in [\![1,k]\!]} M_i$, $\text{Fr}(v) \in M_j$: this is absurd and thus $P = \bigcap_{i \in [\![1,k]\!] \setminus \{j\}} D_i$.

□

Lemma 4.3.24. *Suppose that $\mathcal{I}$ is thick. Let D be a half-apartment of $\mathbb{A}$. Then there exists an apartment A of $\mathbb{A}$ such that $D = A \cap \mathbb{A}$.*

Proof. Let F be a panel of the wall of D. As $\mathcal{I}$ is thick, there exists a chamber C dominating F and such that $C \not\subseteq \mathbb{A}$. By Proposition 2.9 1) of [Rou11], there exists an apartment A containing D and C. The set $\mathbb{A} \cap A$ is a half-apartment by Lemma 4.2.3 and thus $\mathbb{A} \cap A = D$, which proves the lemma. □

Corollary 4.3.25. *Suppose that $\mathcal{I}$ is thick and that some group G acts strongly transitively on $\mathcal{I}$. Let P be an enclosed subset of $\mathbb{A}$ containing a generic ray and having nonempty interior. Then there exists an apartment A such that $A \cap \mathbb{A} = P$.*

Proof. One writes $P = D_1 \cap \ldots \cap D_k$, where the D_i's are half-apartments of $\mathbb{A}$. One supposes that k is minimal for this writing, which means that for all $i \in [\![1, n]\!]$, $P \neq \bigcap_{j \in [\![1,k]\!] \setminus \{i\}} D_j$. For all $i \in [\![1, n]\!]$, one chooses an apartment A_i such that $\mathbb{A} \cap A_i = D_i$. Let $\phi_i : \mathbb{A} \overset{D_i}{\to} A_i$ and $g_i \in G$ inducing ϕ_i.

Let $g = g_1 \ldots g_k$ and $A = g.\mathbb{A}$. Then $A \cap \mathbb{A} \supset D_1 \cap \ldots \cap D_k$ and g fixes $D_1 \cap \ldots \cap D_k$. Let us show that $A \cap \mathbb{A} = \{x \in \mathbb{A} | g.x = x\}$. By Theorem 4.3.22, there exists $\phi : \mathbb{A} \overset{A\cap\mathbb{A}}{\to} A$. Moreover

$g_{|\mathbb{A}}^{-1} \circ \phi : \mathbb{A} \to \mathbb{A}$ fixes $D_1 \cap \ldots \cap D_k$, which has nonempty interior and thus $g_{|\mathbb{A}}^{-1} \circ \phi = \mathrm{Id}_{\mathbb{A}}$, which proves that $A \cap \mathbb{A} = \{x \in \mathbb{A} | g.x = x\}$.

Suppose that $A \cap \mathbb{A} \supsetneq D_1 \cap \ldots \cap D_k$. Let $i \in [\![1, k]\!]$ such that there exists $a \in A \cap \mathbb{A} \backslash D_i$.

One fixes the origin of $\mathbb{A}$ in some point of $\mathring{P}$, one sets $U = \{x \in \mathbb{A} |\ j_P(x) \neq 0\}$ and one defines $\mathrm{Fr} : U \to \mathrm{Fr}(P)$ as in Lemma 4.2.26. By minimality of k and Lemma 4.3.23, there exists a nonempty open set V of U such that $\mathrm{Fr}(V) \subset M_i$.

By the same reasoning as in the proof of Lemma 4.2.29, $\mathrm{Fr}(V) \cap M_i$ is open in M_i. Consequently, there exists $v \in \mathrm{Fr}(V)$ such that $v \notin \bigcup_{j \in [\![1,k]\!] \backslash \{i\}} M_j$. Let $V' \in \mathcal{V}_U(v)$ such that $V' \cap \bigcup_{j \in [\![1,k]\!] \backslash \{i\}} M_j = \emptyset$ and such that V' is convex. Then $V' \subset \bigcap_{j \in [\![1,k]\!] \backslash \{i\}} \mathring{D}_j$. Let $V'' = \mathrm{Fr}(V) \cap V'$. By Theorem 4.3.22, $[a, v] \subset A \cap \mathbb{A}$ and hence g fixes $[a, v]$. Moreover for $u \in [a, v]$ near v, $u \in \bigcap_{j \in [\![1,k]\!] \backslash \{i\}} D_j$. Then $g.u = g_1 \ldots g_i.(g_{i+1} \ldots g_k.u) = g_1 \ldots g_i.u$. Moreover, $g_i.u = g_{i-1}^{-1}. \ldots .g_1^{-1}.u = u$. Therefore $u \in D_i$, which is absurd by choice of u. Therefore $A \cap \mathbb{A} = D_1 \cap \ldots \cap D_k = P$. □

Remark 4.3.26. 1. *In the proof above, the fact that P contains a generic ray is only used to prove that $A \cap \mathbb{A}$ is convex and that there exists an isomorphism $\phi : A \overset{A \cap \mathbb{A}}{\to} \mathbb{A}$. When G is an affine Kac-Moody group and $\mathcal{I}$ is its masure, we will see that these properties are true without assuming that $A \cap \mathbb{A}$ contains a generic ray. Therefore, for all enclosed subset P of $\mathbb{A}$ having nonempty interior, there exists an apartment A such that $A \cap \mathbb{A} = P$*

2. *Suppose that all panel of $\mathcal{I}$ is contained in at most three chambers. Then for all wall M of $\mathcal{I}$, there exists no pair (A, A') of apartments such that $A \cap A' = M$. Indeed, let A be an apartment containing M, P be a panel of M and C_1, C_2 be the alcoves of A dominating P. Let A' be an apartment containing P. If A' does not contain C_1, it contains C_2 and thus $A \cap A' \neq M$. Therefore the hypothesis "P has nonempty interior" is necessary in Corollary 4.3.25.*

Proposition 4.3.27. *Let $\mathcal{I}$ be a masure such that all panel is dominated by at most three chambers. Then for all wall M of $\mathbb{A}$, there exist no apartments A such that $A \cap \mathbb{A} = M$.*

Proof. Let A be an apartment containing M. Let P be a panel of $\mathbb{A}$. Let C_1, C_2 be the chambers of $\mathbb{A}$ dominating P. By hypothesis, there exists $i \in \{1, 2\}$ such that A contains C_i. □

4.4 Axiomatic of masures

In this section, we simplify the axiomatic of masures.

In Subsection 4.4.1, we simplify it in the general case, using mainly Theorem 4.3.22.

In Subsection 4.4.2, we study the existence, for a given pair of faces, of an apartment containing it.

In Subsection 4.4.3, we study the existence, for a given pair of faces included in two apartments A and B, of an isomorphism $A \to B$ fixing this pair.

In Subsection 4.4.4, we simplify the axiomatic of masures in the affine case.

In Subsection 4.4.5, we give other definitions of strongly transitivity for the action of a group on a masure (see Corollary 4.4.40 and Proposition 4.4.44).

4.4.1 Axiomatic of masures in the general case

The aim of this section is to give an other axiomatic of masure than the one of [Rou11] and [Rou17]. For this, we mainly use Theorem 4.3.22.

We fix an apartment $\mathbb{A} = (\mathcal{S}, W, \Lambda')$. A **construction** of type $\mathbb{A}$ is a set endowed with a covering of subsets called apartments and satisfying (MA1).

Let $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. Let (MA i)=(MA1).

Let (MA ii) : if two apartments A, A' contain a generic ray, then $A \cap A'$ is enclosed and there exists an isomorphism $\phi : A \overset{A\cap A'}{\to} A'$.

Let (MA iii, cl): if $\mathfrak{R}$ is the germ of a splayed chimney and if F is a face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

It is easy to see that the axiom (MA ii) implies (MA4, cl) for all $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. If $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$, (MA iii, cl) is equivalent to (MA3, cl) because each chimney is included in a solid chimney.

Let $\mathcal{I}$ be a construction of type $\mathbb{A}$ and $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. One says that $\mathcal{I}$ is a **masure of type** $(1, \mathrm{cl})$ if it satisfies the axioms of [Rou11]: (MA2, cl), (MA3, cl), (MA4, cl) and (MAO). One says that $\mathcal{I}$ is a **masure of type** $(2, \mathrm{cl})$ if it satisfies (MA ii) and (MA iii, cl).

The aim of the next two subsections is to prove the following theorem:

Theorem 4.4.1. *Let $\mathcal{I}$ be a construction of type $\mathbb{A}$ and $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. Then $\mathcal{I}$ is a masure of type $(1, \mathrm{cl})$ if and only if $\mathcal{I}$ is a masure of type $(1, \mathrm{cl}^\#)$ if and only if $\mathcal{I}$ is a masure of type $(2, \mathrm{cl})$ if and only if $\mathcal{I}$ is a masure of type $(2, \mathrm{cl}^\#)$.*

Let us introduce some other axioms and definitions. An **extended chimney** of $\mathbb{A}$ is associated to a local face $F^\ell = F^\ell(x, F^v_0)$ (its **basis**) and a vectorial face (its **direction**) F^v, this is the filter $\mathfrak{r}_e(F^\ell, F^v) = F^\ell + F^v$. Similarly to classical chimneys, we define shortenings and germs of extended chimneys. We use the same vocabulary for extended chimneys as for classical: splayed, solid, full, ... We use the isomorphisms of apartments to extend these notions in constructions. Actually each classical chimney is of the shape $\mathrm{cl}(\mathfrak{r}_e)$ for some extended chimney $\mathfrak{r}_e$.

Let $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. Let (MA2', cl): if F is a point, a germ of a preordered interval or a splayed chimney in an apartment A and if A' is another apartment containing F then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists an isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$.

Let (MA2", cl): if F is a solid chimney in an apartment A and if A' is an other apartment containing F then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists an isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$.

The axiom (MA2, cl) is a consequence of (MA2', cl), (MA2", cl) and (MA ii).

Let (MA iii'): if $\mathfrak{R}$ is the germ of a splayed extended chimney and if F is a local face or a germ of an extended chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

Let $\mathcal{I}$ be a construction. Then $\mathcal{I}$ is said to be a **masure of type** 3 if it satisfies (MA ii) and (MA iii').

In order to prove Theorem 4.4.1, we will in fact prove the following stronger theorem:

Theorem 4.4.2. *Let $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$ and $\mathcal{I}$ be a construction of type $\mathbb{A}$. Then $\mathcal{I}$ is a masure of type $(1, \mathrm{cl})$ if and only $\mathcal{I}$ is a masure of type $(2, \mathrm{cl})$ if and only if $\mathcal{I}$ is a masure of type 3.*

The proof of this theorem will be divided in two steps. In the first step, we prove that (MAO) is a consequence of variants of (MA1), (MA2), (MA3) and (MA4) (see Proposition 4.4.3 for a precise statement). This uses paths but not Theorem 4.3.22. In the second step, we prove the equivalence of the three definitions. One implication relies on Theorem 4.3.22.

4.4.1.1 Dependency of (MAO)

The aim of this subsection is to prove the following proposition:

Proposition 4.4.3. *Let $\mathcal{I}$ be a construction of type $\mathbb{A}$ satisfying (MA2'), (MA iii') and (MA4). Then $\mathcal{I}$ satisfies (MAO).*

We now fix a construction $\mathcal{I}$ of type $\mathbb{A}$ satisfying (MA2'), (MA iii') and (MA4). To prove proposition above, the key step is to prove that if B is an apartment and if $x, y \in \mathbb{A} \cap B$ is such that $x \leq_{\mathbb{A}} y$, the image by $\rho_{-\infty}$ of the segment of B joining x to y is a $(y-x)^{++}$-path.

Let $a, b \in \mathbb{A}$. An **(a, b)-path of** $\mathbb{A}$ is a continuous piecewise linear map $[0, 1] \to \mathbb{A}$ such that for all $t \in [0, 1)$, $\pi'(t)^+ \in W^v.(b-a)$. When $a \leq b$, the (a, b)-paths are the $(b-a)^{++}$-paths defined in 4.2.2.2.

Let A be an apartment an $\pi : [0, 1] \to A$ be a map. Let $a, b \in A$. One says that π is an (a, b)**-path** of A if there exists $\Upsilon : A \to \mathbb{A}$ such that $\Upsilon \circ \pi$ is a $(\Upsilon(a), \Upsilon(b))$-path of $\mathbb{A}$.

Lemma 4.4.4. *Let A be an apartment and $a, b \in A$. Let $\pi : [0, 1] \to A$ be an (a, b)-path in A and $f : A \to B$ be an isomorphism of apartments. Then $f \circ \pi$ is an $(f(a), f(b))$-path.*

Proof. Let $\Upsilon : A \to \mathbb{A}$ be an isomorphism such that $\Upsilon \circ \pi$ is a $(\Upsilon(a), \Upsilon(b))$-path in $\mathbb{A}$. Then $\Upsilon' = \Upsilon \circ f^{-1} : B \to \mathbb{A}$ is an isomorphism, $\Upsilon' \circ f \circ \pi$ is a $\big(\Upsilon'(f(a)), \Upsilon'(f(b))\big)$-path in $\mathbb{A}$ and we get the lemma. □

The following lemma slightly improves Proposition 2.7 1) of [Rou11]. We recall that if A is an affine space and $x, y \in A$, $[x, y)$ means the germ $germ_x([x, y])$, $(x, y]$ means $germ_y([x, y])$, ..., see 3.5.

Lemma 4.4.5. *Let $\mathfrak{R}$ be the germ of a splayed extended chimney, A be an apartment of $\mathcal{I}$ and $x^-, x^+ \in A$ such that $x^- \leq_A x^+$. Then there exists a subdivision $z_1 = x^-, \ldots, z_n = x^+$ of $[x^-, x^+]_A$ such that for all $i \in [\![1, n-1]\!]$ there exists an apartment A_i containing $[z_i, z_{i+1}]_A \cup \mathfrak{R}$ such that there exists an isomorphism $\phi_i : A \overset{[z_i, z_{i+1}]_{A_i}}{\to} A_i$.*

Proof. Let $u \in [x^-, x^+]$. By (MA iii'), applied to $(x^-, u])$ and $[u, x+)$ there exist apartments A_u^- and A_u^+ containing $\mathfrak{R} \cup (x^-, u]$ and $\mathfrak{R} \cup [u, x^+)$ and by (MA2'), there exist isomorphisms $\phi_u^+ : A \overset{(x^-, u]}{\to} A_u^-$ and $\phi_u^- : A \overset{[u, x^+)}{\to} A_u^+$. For all $u \in [x^-, x^+]$ and $\epsilon \in \{-, +\}$, one chooses a convex set $V_u^\epsilon \in [u, x^\epsilon)$ such that $V_u^\epsilon \subset A \cap A_u^\epsilon$ and V_u^ϵ is fixed by ϕ_u^ϵ. If $u \in [x^-, x^+]$, one sets $V_u = \mathrm{Int}_{[x^-, x^+]_A}(V_u^+ \cup V_u^-)$. By compactness of $[x^-, x^+]$, there exists a finite set K and a map $\epsilon : K \to \{-, +\}$ such that $[x^-, x^+] = \bigcup_{k \in K} V_k^{\epsilon(k)}$ and the lemma follows. □

Let $\mathfrak{q}$ be a sector-germ. Then $\mathfrak{q}$ is an extended chimney. Let A be an apartment containing $\mathfrak{q}$. The axioms (MA2'), (MA iii') and (MA4) enable to define a retraction $\rho : \mathcal{I} \overset{\mathfrak{q}}{\to} \mathbb{A}$ as in 2.6 of [Rou11].

Lemma 4.4.6. *Let A and B be two apartments, $\mathfrak{q}$ be a sector-germ of B and $\rho : \mathcal{I} \overset{\mathfrak{q}}{\to} B$. Let $x, y \in A$ such that $x \leq_A y$, $\tau : [0, 1] \to A$ mapping each $t \in [0, 1]$ on $(1-t)x +_A ty$ and $f : A \to B$ be an isomorphism. Then $\rho \circ \tau$ is an $(f(x), f(y))$-path of B.*

Proof. By Lemma 4.4.5, there exist $k \in \mathbb{N}$, $t_1 = 0 < \ldots < t_k = 1$ such that for all $i \in [\![1, k-1]\!]$, there exists an apartment A_i containing $\tau([t_i, t_{i+1})]) \cup \mathfrak{q}$ such that there exists an isomorphism $\phi_i : A \overset{\tau([t_i, t_{i+1}])}{\to} A_i$.

If $i \in [\![1, k-1]\!]$, one denotes by ψ_i the isomorphism $A_i \overset{\mathfrak{q}}{\to} B$. Then for $t \in [t_i, t_{i+1}]$, one has $\rho(\tau(t)) = \psi_i \circ \phi_i(\tau(t))$. Let $\Upsilon : B \to \mathbb{A}$ be an isomorphism. By (MA1), for all $i \in [\![1, k]\!]$, there exists $w_i \in W$ such that $\Upsilon \circ \psi_i \circ \phi_i = w_i \circ \Upsilon \circ f$.

Let $i \in [\![1, k-1]\!]$ and $t \in [t_i, t_{i+1}]$. Then

$$\Upsilon \circ \rho \circ \tau(t) = \Upsilon \circ \psi_i \circ \phi_i \circ \tau(t) = (1-t)w_i \circ \Upsilon \circ f(x) + tw_i \circ \Upsilon \circ f(y).$$

Therefore $\rho \circ \tau$ is an $(f(x), f(y))$-path in B. □

Lemma 4.4.7. *Let $\lambda \in \overline{C^v_f}$ and $\pi : [0,1] \to \mathbb{A}$ be a λ-path. Then $\pi(1) - \pi(0) \leq_{Q^\vee_\mathbb{R}} \lambda$.*

Proof. By definition, there exists $k \in \mathbb{N}$, $(t_i) \in [0,1]^k$ and $(w_i) \in (W^v)^k$ such that $\sum_{i=1}^k t_i = 1$ and $\pi(1) - \pi(0) = \sum_{i=1}^k t_i.w_i.\lambda$. Therefore $\pi(1) - \pi(0) - \lambda = \sum_{i=1}^k t_i(w_i.\lambda - \lambda)$ and thus $\pi(1) - \pi(0) - \lambda \leq_{Q^\vee_\mathbb{R}} 0$ by Lemma 2.2.9. □

Lemma 4.4.8. *Let $x, y \in \mathbb{A}$ such that $x \leq_\mathbb{A} y$ and B be an apartment containing x, y. Let $\tau_B : [0,1] \to B$ defined by $\tau_B(t) = (1-t)x +_B ty$, $\mathfrak{s}$ be a sector-germ of $\mathbb{A}$ and $\rho_\mathfrak{s} : \mathcal{I} \overset{\mathfrak{s}}{\to} \mathbb{A}$. Then $x \leq_B y$ and $\pi_\mathbb{A} := \rho_\mathfrak{s} \circ \tau_B$ is an (x, y)-path of $\mathbb{A}$.*

Proof. Maybe changing the choice of $\overline{C^v_f}$, one can suppose that $y - x \in \overline{C^v_f}$. Let $\mathfrak{q}$ be a sector-germ of B, $\rho_B : \mathcal{I} \overset{\mathfrak{q}}{\to} B$ and $\tau_\mathbb{A} : [0,1] \to \mathbb{A}$ defined by $\tau_\mathbb{A}(t) = (1-t)x + ty$. Let $\phi : \mathbb{A} \to B$. By Lemma 4.4.6, $\pi_B := \rho_B \circ \tau_\mathbb{A}$ is a $(\phi(x), \phi(y))$-path of B from x to y. Therefore $x \leq_B y$. Let $\psi = \phi^{-1} : B \to \mathbb{A}$. Composing ϕ by some $w \in W^v$ if necessary, one can suppose that $\psi(y) - \psi(x) \in \overline{C^v_f}$.

By Lemma 4.4.6, $\pi_\mathbb{A}$ is a $(\psi(x), \psi(y))$-path of $\mathbb{A}$. By Lemma 4.4.7, we deduce that $y - x \leq_{Q^\vee_\mathbb{R}} \psi(y) - \psi(x)$.

By Lemma 4.4.4, $\psi \circ \pi_B$ is an (x, y)-path of $\mathbb{A}$ from $\psi(x)$ to $\psi(y)$. By Lemma 4.4.7, we deduce that $\psi(y) - \psi(x) \leq_{Q^\vee_\mathbb{R}} y - x$. Therefore $x - y = \psi(x) - \psi(y)$ and $\pi_\mathbb{A}$ is an (x, y)-path of $\mathbb{A}$. □

If $x, y \in \mathcal{I}$, one says that $x \leq y$ if there exists an apartment A containing x, y and such that $x \leq_A y$. By Lemma 4.4.8, this does not depend on the choice of A: if $x \leq y$ then for all apartment B containing x, y, one has $x \leq_B y$. Thus we can apply Lemma 4.2.14.

We can now prove Proposition 4.4.3: $\mathcal{I}$ satisfies (MAO).

Proof. Let $x, y \in \mathbb{A}$ be such that $x \leq_\mathbb{A} y$ and B be an apartment containing $\{x, y\}$. We suppose that $y - x \in \overline{C^v_f}$. Let $\pi_\mathbb{A} : [0,1] \to \mathbb{A}$ mapping each $t \in [0,1]$ on $\rho_{-\infty}((1-t)x +_B ty)$. By Lemma 4.4.8, $\pi_\mathbb{A}$ is an (x, y)-path from x to y. By Lemma 4.2.13, $\pi_\mathbb{A}(t) = x + t(y - x)$ for all $t \in [0,1]$. Then by Lemma 4.2.14, $\pi_\mathbb{A}(t) = (1-t)x +_B ty$ for all $t \in [0,1]$. In particular $[x, y] = [x, y]_B$ and thus $\mathcal{I}$ satisfies (MAO). □

4.4.1.2 Equivalence of the axiomatics

As each chimney or face contains an extended chimney or a local face of the same type, if $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$, (MA iii, cl) implies (MA iii'). Therefore a masure of type $(2, \mathrm{cl})$ is also a masure of type 3.

If A is an apartment and F is a filter of A, then $\mathrm{cl}_A(F) \subset \mathrm{cl}^\#_A(F)$. Therefore for all $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$, (MA2', $\mathrm{cl}^\#$) implies (MA2', cl) and (MA iii, $\mathrm{cl}^\#$) implies (MA iii, cl).

Lemma 4.4.9. *Let $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$ and $\mathcal{I}$ be a masure of type $(1, \mathrm{cl})$. Then $\mathcal{I}$ is a masure of type $(2, \mathrm{cl})$.*

Proof. By Theorem 4.3.22, $\mathcal{I}$ satisfies (MA ii). By conséquence 2.2 3) of [Rou11], $\mathcal{I}$ satisfies (MA iii, cl). $\square$

By abuse of notation if $\mathcal{I}$ is a masure of any type and if $\mathfrak{q}$, $\mathfrak{q}'$ are adjacent sectors of $\mathcal{I}$, we denote by $\mathfrak{q} \cap \mathfrak{q}'$ the maximal face of $\overline{\mathfrak{q}} \cap \overline{\mathfrak{q}'}$. This has a meaning by Section 3 of [Rou11] for masures of type 1 and by (MA ii) for masures of type 2 and 3.

Lemma 4.4.10. *Let $\mathcal{I}$ be a masure of type 3. Let A be an apartment, and $\mathcal{X}$ be a filter of A such that for all sector-germ $\mathfrak{s}$ of $\mathcal{I}$, there exists an apartment containing $\mathcal{X}$ and $\mathfrak{s}$. Then if B is an apartment containing $\mathcal{X}$, B contains $\mathrm{cl}^{\#}(\mathcal{X})$ and there exists an isomorphism $\phi : A \overset{\mathrm{cl}^{\#}(\mathcal{X})}{\to} B$.*

Proof. Let $\mathfrak{q}$ and $\mathfrak{q}'$ be sector-germs of A and B of the same sign. By (MA iii'), there exists an apartment C containing $\mathfrak{q}$ and $\mathfrak{q}'$. Let $\mathfrak{q}_1 = \mathfrak{q}, \ldots, \mathfrak{q}_n = \mathfrak{q}'$ be a gallery of sector-germs from $\mathfrak{q}$ to $\mathfrak{q}'$ in C. One sets $A_1 = A$ and $A_{n+1} = B$. By hypothesis, for all $i \in [\![2, n]\!]$ there exists an apartment A_i containing $\mathfrak{q}_i$ and $\mathcal{X}$. For all $i \in [\![1, n-1]\!]$, $\mathfrak{q}_i \cap \mathfrak{q}_{i+1}$ is a splayed chimney and $A_i \cap A_{i+1} \supset \mathfrak{q}_i \cap \mathfrak{q}_{i+1}$. Therefore $A_i \cap A_{i+1}$ is enclosed and there exists $\phi_i : A_i \overset{A_i \cap A_{i+1}}{\to} A_{i+1}$. The set $A_n \cap A_{n+1}$ is also enclosed and there exists $\phi_n : A_n \overset{A_n \cap A_{n+1}}{\to} A_{n+1}$.

If $i \in [\![1, n+1]\!]$, one sets $\psi_i = \phi_{i-1} \circ \ldots \circ \phi_1$. Then ψ_i fixes $A_1 \cap \ldots \cap A_i$.

Let $i \in [\![1, n]\!]$ and suppose that $A_1 \cap \ldots \cap A_i$ is enclosed in A. As ψ_i fixes $A_1 \cap \ldots \cap A_i$, one has $A_1 \cap \ldots \cap A_i = \psi_i(A_1 \cap \ldots \cap A_i)$ is enclosed in A_i. Moreover, $A_i \cap A_{i+1}$ is enclosed in A_i and thus $A_1 \cap \ldots \cap A_{i+1}$ is enclosed in A_i. Consequently $A_1 \cap \ldots \cap A_{i+1} = \psi_i^{-1}(A_1 \cap \ldots \cap A_{i+1})$ is enclosed in A. Let $X = A_1 \cap \ldots \cap A_{n+1}$. By induction, X is enclosed in A and $\phi := \psi_n$ fixes X. As $X \supset \mathcal{X}$, $X \in \mathrm{cl}^{\#}(\mathcal{X})$ and we get the lemma. $\square$

Lemma 4.4.11. *Let $\mathcal{I}$ be a masure of type 3. Then for all $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$, $\mathcal{I}$ satisfies (MA iii, cl).*

Proof. Each face is included in the finite enclosure of a local face and each chimney is included in the finite enclosure of an extended chimney. Thus by Lemma 4.4.10, applied when $\mathcal{X}$ is a local face and a germ of a chimney, $\mathcal{I}$ satisfies (MA iii, $\mathrm{cl}^{\#}$). Consequently for all $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$, $\mathcal{I}$ satisfies (MA iii, cl), hence (MA3, cl) and the lemma is proved. $\square$

Lemma 4.4.12. *Let $\mathcal{I}$ be a masure of type 3 and $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. Then $\mathcal{I}$ satisfies (MA2', cl).*

Proof. If A is an apartment and F is a filter of A, then $\mathrm{cl}(F) \subset \mathrm{cl}^{\#}(F)$. Therefore it suffices to prove that $\mathcal{I}$ satisfies (MA2', $\mathrm{cl}^{\#}$). We conclude the proof by applying Lemma 4.4.10 applied when $\mathcal{X}$ is a point or a germ of a preordered segment. $\square$

Using Proposition 4.4.3, we deduce that a masure of type 2 or 3 satisfies (MAO), as (MA4) is a consequence of (MA ii).

Lemma 4.4.13. *Let $\mathcal{I}$ be a masure of type 3. Let $\mathfrak{r}$ be a chimney of $\mathbb{A}$, $\mathfrak{r} = \mathfrak{r}(F^\ell, F^v)$, where F^ℓ (resp. F^v) is a local face (resp. vectorial face) of A. Let $\mathfrak{R}^{\#} = germ_\infty(\mathrm{cl}^{\#}(F^\ell, F^v))$. Let A be an apartment containing $\mathfrak{r}$ and $\mathfrak{R}^{\#}$ and such that there exists $\phi : \mathbb{A} \overset{\mathfrak{R}^{\#}}{\to} A$. Then $\phi : \mathbb{A} \overset{\mathfrak{r}}{\to} A$.*

Proof. One can suppose that $F^v \subset \overline{C}^v_f$. Let $U \in \mathfrak{R}^{\#}$ such that U is enclosed, $U \subset A \cap \mathbb{A}$ and such that U is fixed by ϕ. One writes $U = \bigcap_{i=1}^k D(\beta_i, k_i)$, with $\beta_1, \ldots, \beta_k \in \Phi_{re}$ and $(k_1, \ldots, k_r) \in \prod_{i=1}^r \Lambda'_{\beta_i}$.

Let $\xi \in F^v$ such that $U \in \mathrm{cl}(F^\ell + F^v + \xi)$. Let $J = \{i \in [\![1,k]\!] |\ \beta_i(\xi) \neq 0\}$. As for all $i \in [\![1,r]\!]$, $D(\beta_i, k_i) \supset n\xi$ for $n \gg 0$, one has $\beta_i(\xi) > 0$ for all $i \in J$. One has $U - \xi = \bigcap_{i=1}^k D(\beta_i, k_i + \beta_i(\xi))$. Let $\lambda \in [1, +\infty[$ such that for all $i \in J$, there exists $\widetilde{k_i} \in \Lambda'_{\beta_i}$ such that $k_i + \beta_i(\xi) \leq \widetilde{k_i} \leq k_i + \lambda\beta_i(\xi)$. Let $\widetilde{U} = \bigcap_{i=1}^k D(\beta_i, \widetilde{k_i})$. Then $U - \xi \subset \widetilde{U} \subset U - \lambda\xi$. Therefore, $\widetilde{U} \in \mathfrak{r}$. Let $V' \in \mathfrak{r}$ such that $V' \subset A \cap \mathbb{A}$ and such that $V' + F^v \subset V'$. Then $V := \widetilde{U} \cap V' \in \mathfrak{r}$. Let $v \in V$ and $\delta \subset F^v$ be the ray based at 0 and containing ξ. By the proof of Proposition 5.4 of [Rou11] (which uses only (MA1), (MA2'), (MA3), (MA4) and (MAO)), there exists $g_v : \mathbb{A} \overset{v+\delta}{\to} A$. As $V \subset U - \lambda\xi$, there exists a shortening δ' of $v + \delta$ included in U. Then $g_v^{-1} \circ \phi : \mathbb{A} \to \mathbb{A}$ fixes δ'. Consequently, $g_v^{-1} \circ \phi$ fixes the support of δ' and thus ϕ fixes v: ϕ fixes V. Therefore ϕ fixes $\mathfrak{r}$ and the lemma follows. □

Lemma 4.4.14. *Let $\mathcal{I}$ be a masure of type* 3 *and* $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. *Then $\mathcal{I}$ satisfies (MA2",* cl*).*

Proof. Let $\mathfrak{r} = \mathrm{cl}(F^\ell, F^v)$ be a solid chimney of an apartment A and A' be an apartment containing $\mathfrak{r}$. One supposes that $A = \mathbb{A}$. Let $\mathfrak{r}^\# = \mathrm{cl}^\#(F^\ell, F^v)$ (resp. $\mathfrak{r}_e = F^\ell + F^v$) and $\mathfrak{R}^\#$ (resp. $\mathfrak{R}_e$) be the germ of $\mathfrak{r}^\#$ (resp. $\mathfrak{r}_e$). By Lemma 4.4.10 applied with $\mathcal{X} = \mathfrak{R}_e$, there exists $\phi : A \overset{\mathfrak{R}^\#}{\to} A'$. By Lemma 4.4.13, ϕ fixes $\mathfrak{r}$ and thus $\mathcal{I}$ satisfies (MA2", cl).

□

We can now prove Theorem 4.4.2: let $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. By Lemma 4.4.9, a masure of type $(1, \mathrm{cl})$ is also a masure of type $(2, \mathrm{cl})$ and thus it is a masure of type 3. By Lemma 4.4.11, Lemma 4.4.12 and Lemma 4.4.14, a masure of type 3 is a masure of type $(1, \mathrm{cl})$ which concludes the proof of the theorem.

4.4.2 Friendly pairs in $\mathcal{I}$

Let $\mathbb{A} = (\mathbb{A}, W, \Lambda')$ be an apartment. Let $\mathcal{I}$ be a masure of type $\mathbb{A}$. We now use the finite enclosure $\mathrm{cl} = \mathrm{cl}^\#_{\Lambda'}$, which makes sense by Theorem 4.4.1. A family $(F_j)_{j\in J}$ of filters in $\mathcal{I}$ is said to be **friendly** if there exists an apartment containing $\bigcup_{j\in J} F_j$. In this section we obtain friendliness results for pairs of faces, improving results of Section 5 of [Rou11]. We will use it to give a very simple axiomatic of masures in the affine case. These kinds of results also have an interest on their own: the definitions of the Iwahori-Hecke algebra of [BPGR16] and of the parahoric Hecke algebras of [AH17] (see Chapter 5) relies on the existence of apartments containing pairs of faces.

If $x \in \mathcal{I}$, $\epsilon \in \{-, +\}$ and A is an apartment, one denotes by $\mathcal{F}_x$ (resp. $\mathcal{F}^\epsilon$, $\mathcal{F}^\epsilon(A)$, $\mathcal{C}_x, \ldots$) the set of faces of $\mathcal{I}$ based at x (resp. and of sign ϵ, and included in A, the set of chambers of $\mathcal{I}$ based at $x, \ldots$). If $\mathcal{X}$ is a filter, one denotes by $\mathcal{A}(\mathcal{X})$ the set of apartments containing $\mathcal{X}$.

Lemma 4.4.15. *Let A be an apartment of $\mathcal{I}$, $a \in A$ and $C_1, C_2 \in \mathcal{C}_a(A)$. Let $\mathcal{D}_a$ be the set of half-apartments of A whose wall contains a. Suppose that $C_1 \neq C_2$. Then there exists $D \in \mathcal{D}_a$ such that $D \supset C_1$ and $D \not\supset C_2$.*

Proof. Let C_1^v and C_2^v be vectorial chambers of A such that $C_1 = F(a, C_1^v)$ and $C_2 = F(a, C_2^v)$. Suppose that for all $D \in \mathcal{D}_a$ such that $D \supset C_1$, one has $D \supset C_2$. Let $X \in C_1$. There exists half-apartments $D_1, \ldots, D_k$ and $\Omega \in \mathcal{V}_A(a)$ such that $X \supset \bigcap_{i=1}^k D_i^\circ \supset \Omega \cap (a + C_1^v)$.

Let $J = \{j \in [\![1,k]\!] |\ D_j \notin \mathcal{D}_a\}$. For all $j \in J$, one chooses $\Omega_j \in \mathcal{V}_A(a)$ such that $D_j^\circ \supset \Omega_j$. If $j \in [\![1,k]\!] \backslash J$, $D_j \supset C_1$, thus $D_j \supset C_2$ and hence $D_j^\circ \supset C_2$. Therefore, there exists $\Omega_j \in \mathcal{V}_A(a)$ such that $D_j^\circ \supset \Omega_j \cap (x + C_2^v)$. Hence, $X \supset \bigcap_{j=1}^k D_j^\circ \supset (\bigcap_{j=1}^k \Omega_j) \cap (x + C_2^v)$, thus $X \in C_2$ and $C_1 \supset C_2$

Let $D \in \mathcal{D}_a$ such that $D \supset C_2$. Suppose that $D \not\supset C_1$. Let D' be the half-apartment opposite to D. Then $D' \supset C_1$ and therefore $D' \supset C_2$: this is absurd. Therefore for all $D \in \mathcal{D}_a$ such that $D \supset C_2$, $D \supset C_1$. By the same reasoning we just did, we deduce that $C_2 \supset C_1$ and thus $C_1 = C_2$. This is absurd and the lemma is proved.

□

The following proposition improves Proposition 5.1 of [Rou11]. It is the analogue of the axiom (I1) of buildings (see the introduction).

Proposition 4.4.16. *Let $\{x, y\}$ be a friendly pair in $\mathcal{I}$.*

1. *Let $A \in \mathcal{A}(\{x, y\})$ and δ be a ray of A based at x and containing y (if $y \neq x$, δ is unique) and $F_x \in \mathcal{F}_x$. Then (δ, F_x) is friendly. Moreover, there exists $A' \in \mathcal{A}(\delta \cup F_x)$ such that there exists an isomorphism $\phi : A \overset{\delta}{\to} A'$.*

2. *Let $(F_x, F_y) \in \mathcal{F}_x \times \mathcal{F}_y$. Then (F_x, F_y) is friendly.*

Proof. We begin by proving 1. Let C_x be a chamber of $\mathcal{I}$ containing F_x. Let C be a chamber of A based at x and having the same sign as C_x. By Proposition 5.1 of [Rou11], there exists an apartment B containing C_x and C. Let $C_1 = C, \ldots, C_n = C_x$ be a gallery in B from C to C_x. If $i \in [\![1, n]\!]$, one sets $\mathcal{P}_i$: "there exists an apartment A_i containing C_i and δ such that there exists an isomorphism $\phi : A \overset{\delta}{\to} A_i$". The property $\mathcal{P}_1$ is true by taking $A_1 = A$. Let $i \in [\![1, n-1]\!]$ such that $\mathcal{P}_i$ is true. If $C_{i+1} = C_i$, then $\mathcal{P}_{i+1}$ is true. Suppose $C_i \neq C_{i+1}$. Let A_i be an apartment containing C_i and δ. By Lemma 4.4.15, there exists a half-apartment D of A whose wall contains x and such that $C_i \subset D$ and $C_{i+1} \not\subset D$. As C_i and C_{i+1} are adjacent, the wall M of D is the wall separating C_i and C_{i+1}. By (MA2), there exists an isomorphism $\phi : B \overset{C_i}{\to} A_i$. Let $M' = \phi(M)$ and D_1, D_2 be the half-apartments of A_i delimited by M'. Let $j \in \{1, 2\}$ such that $D_j \supset \delta$. By Proposition 2.9 1) of [Rou11], there exists an apartment A_{i+1} containing D_j and C_{i+1}. Let $\psi_i : A \overset{\delta}{\to} A_i$ and $\psi : A_i \overset{D_j}{\to} A_{i+1}$. Then $\psi \circ \psi_i : A \overset{\delta}{\to} A_{i+1}$. Therefore $\mathcal{P}_{i+1}$ is true. Consequently, $\mathcal{P}_n$ is true, which proves 1.

Let us prove 2, which is very similar to 1. As a particular case of 1, there exists an apartment A' containing F_x and y. Let C_y be a chamber of $\mathcal{I}$ containing F_y. Let C be a chamber of A' based at y and of the same sign as F_y. Let $C_1 = C, \ldots, C_n = C_y$ be a gallery of chambers from C to C_y (which exists by Proposition 5.1 of [Rou11]). By the same reasoning as above, for all $i \in [\![1, n]\!]$, there exists an apartment containing F_x and C_i, which proves 2. □

4.4.3 Existence of isomorphisms fixing preordered pairs of faces

A filter $\mathcal{X}$ of an apartment A is said to be **intrinsic** if for all apartment $B \in \mathcal{A}(\mathcal{X})$ (which means that B contains $\mathcal{X}$), $\mathrm{conv}_A(\mathcal{X}) \subset A \cap B$ and there exists an isomorphism $\phi : A \overset{\mathrm{conv}_A(\mathcal{X})}{\to} B$.

The aim of this subsection is to prove the theorem below. It improves Proposition 5.2, Proposition 5.5 of [Rou11] and Proposition 1.10 of [BPGR16]. We will not use it in the simplification of the axioms of masures in the affine case.

Theorem 4.4.17. *Let $x, y \in \mathcal{I}$ such that $x \leq y$ and $(F_x, F_y) \in \mathcal{F}_x \times \mathcal{F}_y$. Then $F_x \cup F_y$ is intrinsic.*

To prove this our main tool will be Lemma 4.4.19, which establishes that under some conditions, if P is included in two apartments and is fixed by an isomorphism of apartments, the convex hull of P is also included in these apartments. We first treat the case where one of the faces is a chamber and the other one is spherical.

4.4.3.1 Convex hull of a set fixed by an isomorphism

The following lemma is stated in the proof of Proposition 5.4 of [Rou11]:

Lemma 4.4.18. *Let $x, y \in \mathcal{I}$ such that $x \leq y$ and $x \neq y$ and $A, B \in \mathcal{A}(\{x, y\})$. Let δ_A be the ray of A based at x and containing y and δ_B be the ray of B based at y and containing x. Then there exists an apartment containing $L = \delta_A \cup \delta_B$ and in this apartment, L is a line.*

If A is an affine space and $\mathcal{X}, \mathcal{X}'$ are two segment-germs, infinite intervals or rays, one says that $\mathcal{X}$ and $\mathcal{X}'$ are parallel if the line spanned by $\mathcal{X}$ is parallel to the line spanned by $\mathcal{X}'$.

The following lemma is a kind of reciprocal to Proposition 4.2.34.

If $P \subset \mathbb{A}$ and $k \in \mathbb{N}$, $\mathrm{conv}_k(P)$ was defined in Lemma 4.3.8.

Lemma 4.4.19. *Let A be an apartment. Suppose that there exists $P \subset A \cap \mathbb{A}$ satisfying the following conditions:*

1. *there exists $u \in \mathring{\mathcal{T}}$ such that for all $x \in P$, there exists $U_x \in \mathcal{V}_{[-1,1]}(0)$ such that $x + U_x.u \subset P$*
2. *there exists an isomorphism $\phi : \mathbb{A} \overset{P}{\to} A$.*

Then $\overline{\mathrm{conv}(P)} \subset A \cap \mathbb{A}$ and ϕ fixes $\overline{\mathrm{conv}(P)}$.

Proof. If $k \in \mathbb{N}$, one sets $\mathcal{P}_k$: "$\mathrm{conv}_k(P) \subset A \cap \mathbb{A}$ and $\mathrm{conv}_k(P)$ satisfies 1 and 2". Let us prove that $\mathcal{P}_k$ is true for all $k \in \mathbb{N}$. As if $k \in \mathbb{N}$, $\mathrm{conv}_{k+1}(P) = \mathrm{conv}_1(\mathrm{conv}_k(P))$, it suffices to prove that $\mathcal{P}_1$ is true.

Let $x, y \in P$. Let us prove that $[x, y] \subset A \cap \mathbb{A}$. If $\epsilon \in \{-, +\}$, one denotes by $\delta_{\mathbb{A}}^{\epsilon\infty}$ the direction of $\mathbb{R}_\epsilon^*\nu$ and by $\delta_A^{\epsilon\infty}$ the direction of $\phi(\mathbb{R}_\epsilon^* u)$. Let $x' \in (x + \delta_{\mathbb{A}}^{+\infty}) \backslash \{x\}$ (resp. $y' \in (y + \delta_{\mathbb{A}}^{+\infty}) \backslash \{y\}$) such that $[x, x']$ (resp. $[y, y']$) is included in A and fixed by ϕ. By Lemma 4.4.18, $L_x = (x + \delta_{\mathbb{A}}^{+\infty}) \cup (x' + \delta_A^{-\infty})$ and $L_y = (y + \delta_{\mathbb{A}}^{+\infty}) \cup (y' + \delta_A^{-\infty})$ are lines of some apartments of $\mathcal{I}$.

Let $\mathfrak{F}_{\mathbb{A}}^{+\infty}$ (resp. $\mathfrak{F}_A^{-\infty}$) be the face of $\mathcal{I}^\infty$ containing $\delta_{\mathbb{A}}^{+\infty}$ (resp. $\delta_A^{-\infty}$). By (MA3), there exists an apartment A' containing the germs of $x + \mathfrak{F}_{\mathbb{A}}^{+\infty}$ and of $y + \mathfrak{F}_A^{-\infty}$. Then A' contains shortenings $x'' + \delta_{\mathbb{A}}^{+\infty}$ and $y'' + \delta_A^{-\infty}$ of $x + \delta_{\mathbb{A}}^{+\infty}$ and of $y + \delta_A^{-\infty}$. Then A' contains $(x'' + \delta_{\mathbb{A}}^{+\infty}) + \delta_A^{-\infty}$ and $(y'' + \delta_A^{-\infty}) + \delta_{\mathbb{A}}^{+\infty}$.

Let A_x be an apartment containing L_x. The apartment A_x contains $\delta_{\mathbb{A}}^{+\infty}$ and $\delta_A^{-\infty}$, and $\delta_{\mathbb{A}}^{+\infty}$ is opposite to $\delta_A^{-\infty}$ in A_x. Therefore $(x'' + \delta_{\mathbb{A}}^{+\infty}) + \delta_A^{-\infty}$ is a line of A_x and as $x'' + \delta_{\mathbb{A}}^{+\infty} \subset L_x$, $(x'' + \delta_{\mathbb{A}}^{+\infty}) + \delta_A^{-\infty} = L_x$. By the same reasoning, $A' \supset L_y$. In particular, A' contains x and y. By Theorem 4.3.22, $[x, y] = [x, y]_{A'} = [x, y]_A$. Therefore, for all $x, y \in P$, $[x, y] \subset A \cap \mathbb{A}$.

Consequently if $x, y \in P$ and $U \in \mathcal{V}_{[-1,1]}(0)$ is convex such that $x + U.u \subset P$, $y + U.u \subset P$, then $\mathrm{conv}_1(x + U.u, y + U.u) = \mathrm{conv}(x + U.u, y + U.u) \subset A \cap \mathbb{A}$. By Proposition 4.2.34, there exists $\psi : \mathbb{A} \overset{\mathrm{conv}(x+U.u, y+U.u)}{\to} A$. The isomorphism $\phi^{-1} \circ \psi : \mathbb{A} \to \mathbb{A}$ fixes $x + U.u \cup y + U.u$, hence it fixes its support; it fixes in particular $\mathrm{conv}(x + U.u, y + U.u)$. Thus ϕ fixes $\mathrm{conv}_1(P)$: $\mathrm{conv}_1(P)$ satisfies 2. Moreover $\mathrm{conv}_1(P)$ satisfies 1 and it follows that for all $k \in \mathbb{N}$, $\mathcal{P}_k$ is true.

By Lemma 4.3.8, $\mathrm{conv}(P) \subset A \cap \mathbb{A}$ and ϕ fixes $\mathrm{conv}(P)$. We conclude by using Proposition 4.2.17 and Lemma 4.2.33. $\square$

Lemma 4.4.20. *Let A be an apartment and $\mathfrak{q}, \mathfrak{q}'$ be opposite sector germs of A. Then A is the unique apartment containing $\mathfrak{q}$ and $\mathfrak{q}'$.*

Proof. One identifies A and $\mathbb{A}$. Let $Q = 0+\mathfrak{q}$ and $Q' = 0+\mathfrak{q}'$. One has $Q' = -Q$. Let $(e_j)_{j\in J}$ be a basis of $\mathbb{A}$ such that $\sum_{j\in J} \mathbb{R}_+^* e_j \subset Q$ and (e_j^*) be the dual basis of $\mathbb{A}$. Let A' be an apartment containing $\mathfrak{q}$ and $\mathfrak{q}'$. Let $M \in \mathbb{R}_+^*$ such that $A' \supset \pm\{x \in \mathbb{A} | \forall j \in J,\ e_j^*(x) \geq M\}$. Let $a \in \mathbb{A}$. There exists $b \in \{x \in \mathbb{A} | \forall j \in J,\ e_j^*(x) \geq M\}$ such that $a \in b + \mathfrak{q}'$. Therefore $a \in A'$ and $A' \supset \mathbb{A}$. By (MA ii), there exists an isomorphism $\phi : \mathbb{A} \overset{\mathbb{A}\cap A'}{\to} A'$. Therefore $\mathfrak{q}, \mathfrak{q}'$ are two opposite sector-germs of A' and by symmetry, $\mathbb{A} \supset A'$. Thus $\mathbb{A} = A'$. □

Proposition 4.4.21. *Let A and B be two apartments such that $A\cap B$ has nonempty interior. Then $A \cap B$ is convex if and only if there exists an apartment A' such that $d_+(A, A') = d_-(A', B) = 0$ and such that $A \cap B \subset A'$.*

Proof. The implication $\Leftarrow$ is a corollary of Theorem 4.3.22. Suppose that $A \cap B$ is convex. One identifies A and $\mathbb{A}$. By Proposition 4.2.34, there exists an isomorphism $\phi : \mathbb{A} \overset{\mathbb{A}\cap B}{\to} B$. Let $\nu \in C_f^v$ and $\delta_{\mathbb{A}}^{+\infty}$ (resp. $\delta_B^{-\infty}$) be the direction of $\mathbb{R}_+^*\nu$ (resp. of $\phi(\mathbb{R}_-^*\nu)$). By Lemma 4.4.18, if $x \in \mathrm{Int}(\mathbb{A} \cap B)$, $L_x = x + \delta_{\mathbb{A}}^{+\infty} \cup x + \delta_B^{-\infty}$ is a line of some apartment A_x. One has $\mathrm{cl}(x + \delta_{\mathbb{A}}^{+\infty}) \supset x + C_f^v$ and $\mathrm{cl}(x + \delta_A^{-\infty}) \supset \phi(x - C_f^v)$. Moreover, $x + C_f^v$ and $\phi(x - C_f^v)$ are opposite in A_x and thus $A_x = \mathrm{conv}(+\infty, germ_\infty(\phi(x - C_f^v)))$. In particular A_x is an apartment containing $+\infty$ and $germ_\infty(\phi(-C_f^v))$. By Lemma 4.4.20, A_x does not depend on x and we denote it A'. If $x \in \mathrm{Int}(\mathbb{A} \cap B)$, $x \in L_x \subset A_x = A'$ and thus $\mathrm{Int}(\mathbb{A} \cap B) \subset A'$. By Proposition 4.2.17, $\mathbb{A} \cap B \subset A'$, which proves the proposition. □

4.4.3.2 Case of a pair chamber-spherical face

Lemma 4.4.22. *Let A be an apartment, $x, y \in A$ such that $x \leq y$ and $C \in \mathcal{C}_x^+(A)$. Then if $B \in \mathcal{A}(C \cup [x, y))$, there exists $C' \in \mathcal{C}_x^+(A)$ such that $C' \subset B$, $\overline{C'} \supset [x, y)$ and such that there exists a sector Q of A based at x and satisfying $\overline{Q} \supset [x, y]$ and $C' = germ_x(Q)$.*

Proof. One identifies A and $\mathbb{A}$. We call a subset of $\mathbb{A}$ a face of $\mathcal{T}_x$ if it is of the shape $x + F^v$ for some positive vectorial face F^v. Let $X \in C$ such that X is open and convex. Let $z' \in X$ such that $x \mathring{<} z'$. Let $y' \in]x, y]$ such that $y' \mathring{<} z'$ and $z \in X$ such that $y' \mathring{<} z$ and such that $z - y'$ is not included in a direction of a wall of $\mathbb{A}$. By Proposition 4.4.13 of [Bar96], there exists a finite number $F_1, \ldots, F_k$ of faces of $\mathcal{T}_x$ such that $[z, y']$ is included in $F_1 \cup \ldots \cup F_k$. One identifies $[0, 1]$ and $[z, y']$. Then maybe renumbering and decreasing k, one can suppose that there exists $t_1 = 0 = z < t_2 < \ldots < t_{k+1} = 1 = y'$ such that for all $i \in [\![1, k]\!]$, $[t_i, t_{i+1}] \subset F_i$. As $[t_k, t_{k+1}]$ is not included in any wall, F_k is necessarily a chamber. Let F_k^v be the vectorial face such that $F_k = x + F_k^v$. By order convexity, $[z, y'] \subset \mathbb{A} \cap B$ and thus $\frac{1}{2}(t_k + t_{k+1}) \in \mathbb{A} \cap B$. As $x \leq \frac{1}{2}(t_k + t_{k+1})$, $[x, \frac{1}{2}(t_k + t_{k+1})] \subset \mathbb{A} \cap B$ and by Proposition 5.4 of [Rou11], $\mathrm{cl}([x, \frac{1}{2}(t_k + t_{k+1})]) \subset \mathbb{A} \cap B$. Let Q be the sector containing the ray based at x and containing $\frac{1}{2}(t_k + t_{k+1})$. As $\mathrm{cl}([x, \frac{1}{2}(t_k + t_{k+1})])$ contains the chamber $C' = F(x, F_k^v)$, we get the lemma. □

Lemma 4.4.23. *Let A be an apartment, $x \in A$, Q a sector of A based at x and $y \in A$ such that $x \leq y$ and $[x, y] \subset \overline{Q}$. Let $C' = germ_x(Q) \in \mathcal{C}_x^+(A)$. Then if $B \in \mathcal{A}(C' \cup [x, y])$, $A \cap B$ contains $\mathrm{conv}_A(C', [x, y])$.*

Proof. Let $X \in C'$ such that $X \subset A \cap B$ and $z \in X \cap Q$. Let δ_A^x be the ray of A based at x and containing z and δ_B^z be the ray of B based at z and containing $[x, z]$. By Lemma 4.4.18, $L = \delta_A^x \cup \delta_B^z$ is a line of some apartment A'. By Theorem 4.3.22, $A \cap A'$ is enclosed. As the enclosure of δ_A^x contains $\overline{Q}$, $A' \supset [x,y] \cup C'$ and $\mathrm{conv}_A(C',[x,y]) = \mathrm{conv}_{A'}(C',[x,y])$. Using Theorem 4.3.22 again, we get that $\mathrm{conv}_{A'}(C',[x,y]) = \mathrm{conv}_B(C',[x,y])$ and the lemma follows. □

Lemma 4.4.24. *Let A be an affine space and $U, V \subset A$ be two convex subsets. Then* $\mathrm{conv}_1(U,V) = \mathrm{conv}(U,V)$.

Proof. The inclusion $\mathrm{conv}_1(U,V) \subset \mathrm{conv}(U,V)$ is clear. Let $a \in \mathrm{conv}(U,V)$. There exist $k, \ell \in \mathbb{N}$ $u_1, \ldots, u_k \in U$, $v_{k+1}, \ldots, v_{k+\ell} \in V$ and $(\lambda_i) \in [0,1]^{k+\ell}$ such that $\sum_{i=1}^{k+\ell} \lambda_i = 1$ and $a = \sum_{i=1}^k \lambda_i u_i + \sum_{i=k+1}^{k+\ell} \lambda_i v_i$. Let $t = \sum_{i=1}^k \lambda_i$. If $t \in \{0,1\}$, $a \in U \cup V$. Suppose $t \notin \{0,1\}$. Then $a = t \sum_{i=1}^k \frac{\lambda_i}{t} u_i + (1-t) \sum_{i=k+1}^{k+\ell} \frac{\lambda_i}{(1-t)} \in \mathrm{conv}_1(U,V)$, which proves the lemma. □

Lemma 4.4.25. *Let $x, y \in \mathcal{I}$ such that $x \leq y$, $(F_x, C_y) \in \mathcal{F}_x^+ \times \mathcal{C}_y^+$ with F_x spherical. Then $F_x \cup C_y$ is intrinsic.*

Proof. Let $A, B \in \mathcal{A}(F_x, C_y)$. Let $X \in C_y$ be convex, open, such that $X \subset A \cap B$ and such that there exists an isomorphism $\psi : A \overset{X}{\to} B$. Let $z \in \mathring{X}$ such that $y \mathring{<} z$. Then $x \mathring{<} z$. Let $X' \in F_x$ such that $X' \subset A \cap B$, X' is convex, relatively open and such that there exists an isomorphism $\psi' : A \overset{X'}{\to} B$. Let $U \in \mathcal{V}_{X'}(x)$ be convex and $V \in \mathcal{V}_A(z)$ be convex and open and such that for all $(u,v) \in U \times V$, $u \mathring{<} v$. Let $P = \mathrm{conv}(U,V) = \mathrm{conv}_1(U,V)$. Then $P \subset A \cap B$ and by Proposition 4.2.34 there exists an isomorphism $\phi : A \overset{P}{\to} B$. As $P \cap V$ has nonempty interior, $\phi = \psi$. As $\phi_{|U} = \psi'_{|U}$, ϕ fixes X'. Therefore ϕ fixes $\mathrm{Int}_r(X') \cup \mathrm{Int}(X)$. As F_x is spherical, one can apply Lemma 4.4.19 and we deduce that $\overline{\mathrm{conv}_A(\mathrm{Int}_r(X'), \mathrm{Int}(X))} \subset A \cap B$ and is fixed by ϕ and the lemma follows. □

Lemma 4.4.26. *Let A be an apartment, $x, y \in \mathcal{I}$ such that $x \leq y$ and Q be a sector of A based at x and such that $[x,y] \subset \overline{Q}$. Let $C_x = F(x, F^\ell(x,Q))$ and $F_y \in \mathcal{F}_y$ spherical. Then $C_x \cup F_y$ is intrinsic.*

Proof. Let $B \in \mathcal{A}(C_x \cup F_y)$. Let $X_1 \in \mathrm{conv}_A(C_x, [x,y])$ such that X_1 is convex and $X_1 \subset A \cap B$, which exists by Lemma 4.4.23. Let $P_1 = \mathring{X}_1$. By Proposition 4.2.34, there exists an isomorphism $\phi_1 : A \overset{P_1}{\to} B$.

If F_y is positive, one sets $\mathring{R} =$ "$\mathring{<}$" and if F_y is negative, one sets $\mathring{R} =$ "$\mathring{>}$". Let $X_2 \in F_y$ such that $X_2 \subset A \cap B$, such that there exists $\phi_2 : A \overset{X_2}{\to} B$ and such that X_2 is convex and $P_2 = \mathrm{Int}_r(X_2)$. Let $z \in P_2$ such that $y \mathring{R} z$. Let (U,V) be such that $V \in \mathcal{V}_{P_2}(z)$ is convex and relatively open and $U \subset P_1$ is open and convex and such that for all $u \in U$ and $v \in V$, $u \mathring{R} v$. Then $P_3 = \mathrm{conv}(U,V) = \mathrm{conv}_1(U,V)$ is included in $A \cap B$. By Proposition 4.2.34, there exists $\phi_3 : A \overset{P_3}{\to} B$.

The map ϕ_1 fixes P_1, ϕ_3 fixes P_3 and $P_1 \cap P_3$ has nonempty interior. Therefore $\phi_1 = \phi_3$. As $\phi_{3|V} = \phi_{2|V}$, $\phi_{3|\mathrm{supp}(V)} = \phi_{2|\mathrm{supp}(V)}$ and ϕ_3 fixes P_2. Consequently, ϕ_1 fixes $P_1 \cup P_2$.

One identifies A and $\mathbb{A}$. Let $u_1 \in \overrightarrow{\mathrm{supp}}(P_2) \cap \mathring{\mathcal{T}}$, which exists because F_y is spherical. As P_2 is relatively open and P_1 is open, $P_1 \cup P_2$ satisfies conditions 1 and 2 of Lemma 4.4.19 (with $u = u_1$ and $P = P_1 \cup P_2$). Consequently $\overline{\mathrm{conv}(P_1 \cup P_2)} \subset A \cap B$ and ϕ_1 fixes $\overline{\mathrm{conv}(P_1 \cup P_2)}$. Moreover, $P_1 \in C_x$ and

$$\overline{\mathrm{conv}(P_1 \cup P_2)} \supset \mathrm{conv}(P_1 \cup \overline{P_2}) = \mathrm{conv}(P_1 \cup X_2) \supset \mathrm{conv}(C_x \cup F_y).$$

Therefore $C_x \cup F_y$ is intrinsic, which is our assertion. □

Lemma 4.4.27. *Let $x, y \in \mathcal{I}$ such that $x \leq y$, $C_x \in \mathcal{C}_x^+$ and $F_y \in \mathcal{F}_y$ spherical. Then $C_x \cup F_y$ is intrinsic.*

Proof. Let $A, B \in \mathcal{A}(C_x \cup F_y)$, which exist by Proposition 4.4.16. Let Q be a sector of A based at x such that $[x, y] \subset \overline{Q}$ and such that $C' = F(x, F^\ell(x, Q)) \subset A \cap B$, which is possible by Lemma 4.4.22. Let $X_2 \in F_y$ (resp. $X' \in C'$) such that X_2 is relatively open, included in $A \cap B$ and such that there exists $\phi : A \overset{X' \cup X_2}{\to} B$, which exists by Lemma 4.4.26. Let $X \in C$ open, included in $A \cap B$ and such that there exists an isomorphism $\psi : A \overset{X \cup X'}{\to} B$, which is possible (reducing X' if necessary) by Proposition 5.2 of [Rou11]. Then $\phi = \psi$ and we conclude with Lemma 4.4.19. $\square$

4.4.3.3 Conclusion

In order to deduce Theorem 4.4.17 from Lemmas 4.4.25 and 4.4.27, we first prove that if C and C' are chambers of the same sign dominating some face F, there exists a gallery of chambers dominating F from C to C', which is Lemma 4.4.31.

Lemma 4.4.28. *Let C^v (resp. F^v) be a positive chamber (resp. positive face) of $\mathbb{A}$. Then $\mathrm{conv}(C^v, F^v)$ contains a generic ray δ based at 0 such that $\mathrm{cl}(\delta) \supset F^v$.*

Proof. Let $(a, b) \in C^v \times F^v$ such that $b - a$ is not included in any wall direction of $\mathbb{A}$ and such that $\mathrm{cl}(\mathbb{R}_+^* b) = F^v$. By Proposition 4.4.13 of [Bar96], there exists a finite number $F_1^v, \ldots, F_k^v$ of vectorial faces such that $[a, b]$ is included in $F_1^v \cup \ldots \cup F_k^v$. One identifies $[0, 1]$ and $[a, b]$. Then maybe renumbering and decreasing k, one can suppose that there exists $t_1 = 0 = a < t_2 < \ldots < t_{k+1} = 1 = b$ such that for all $i \in [\![1, k]\!]$, $[t_i, t_{i+1}] \subset F_i^v$. As $[t_k, t_{k+1}]$ is not included in any wall, F_k^v is necessarily a chamber. Let $\delta = \mathbb{R}_+^*(t_k + t_{k+1})$. Then $\mathrm{cl}(\delta) = \overline{F_k^v} \supset F^v$, which is our assertion. $\square$

Lemma 4.4.29. *Let $x \in \mathbb{A}$, $\epsilon \in \{-, +\}$ and $C, C' \in \mathcal{C}_x^\epsilon$. Suppose that $C \supset C'$. Then $C = C'$.*

Proof. Suppose that $C \neq C'$. Then there exists $X \in C' \backslash C$. There exists disjoint finite sets J,J° and a family $(D_j)_{j \in J \cup J^\circ}$ of half-apartments such that $X \supset \bigcap_{j \in J} D_j \cap \bigcap_{j \in J^\circ} D_j^\circ$ and such that $D_j \in C'$ for all $j \in J$ and $D_{j'}^\circ \in C'$ for all $j' \in J^\circ$. There exists $j \in J \cup J^\circ$ such that D_j or D_j° is not in C. Let D_j' be the half-apartment of $\mathbb{A}$ opposite to D_j. Then $D_j' \in C \backslash C'$: a contradiction. Thus $C = C'$. $\square$

Lemma 4.4.30. *Let $a \in \mathbb{A}$ and $F = F(a, F^v)$ be a positive face based at a, with F^v a vectorial face of $\mathbb{A}$. Let C be a positive chamber based at a and dominating F. Then there exists a vectorial chamber C^v dominating F^v such that $C = F(a, C^v)$.*

Proof. Let $\mathcal{D}$ (resp. $\mathcal{D}_a$) be the set of half-apartments of $\mathbb{A}$ (resp. whose wall contains a). Let C'^v be a vectorial chamber such that $C = F(a, C'^v)$. Let $\delta \subset \mathrm{conv}(C'^v, F^v)$ such that $\mathrm{cl}(\delta)$ is a vectorial chamber C^v containing F^v, which exists by Lemma 4.4.28. Let us prove that $C = F(a, C^v)$.

Let $X' \in C$. Then there exists disjoint finite sets J and J° and a family $(D_j) \in \mathcal{D}^{J \cup J^\circ}$ such that if $X = \bigcap_{j \in J} D_j \cap \bigcap_{k \in J^\circ} D_k^\circ$, $X' \supset X \supset F^\ell(a, C'^v)$. One has $\overline{X} \in \overline{C}$ and thus $\overline{X} \supset F$. Moreover, as X is nonempty, $\overline{X} = \bigcap_{j \in J} D_j \cap \bigcap_{k \in J^\circ} D_k$. Let $L = \{j \in J \cup J^\circ | \ D_j \in \mathcal{D}_a\}$. One has $X = \overline{\Omega} \cap \bigcap_{l \in L} D_\ell$, where $\Omega \in \mathcal{V}_{\mathbb{A}}(a)$ is a finite intersection of open half-apartments. Let $\ell \in L$. Then $D_\ell \supset a + C'^v \cup a + F^v$ and thus $D_\ell \supset a + \mathrm{conv}(C'^v, F^v)$. Therefore $D_\ell \supset a + \delta$ and hence $D_\ell \supset a + C^v$. As C^v does not meet any direction of a wall of $\mathbb{A}$, $a + C^v \cap M_\ell = \emptyset$, where M_ℓ is the wall of D_ℓ. Consequently, $D_\ell^\circ \supset a + C^v$ and thus $X \in F(a, C^v)$. Therefore $F(a, C^v) \subset F(a, C'^v) = C$. By Lemma 4.4.29, $C = F(a, C^v)$ and the lemma follows. $\square$

Type of a vectorial face Let F^v be a positive vectorial face. Then $F^v = w.F^v(J)$ for some $w \in W^v$ and $J \subset I$ (see 2.2.2.2). The **type** of F^v is J. This does not depend on the choice of w by Section 1.3 of [Rou11].

If $x \in \mathcal{I}$ and $F \in \mathcal{F}_x$, we denote by $\mathcal{C}_F$ the set of chambers of $\mathcal{I}$ dominating F.

Lemma 4.4.31. *Let $x \in \mathcal{I}$ and $F \in \mathcal{F}_x^+$. Then if $C, C' \in \mathcal{C}_F$, there exists a gallery $C = C_1, \ldots, C_n = C'$ of elements of $\mathcal{C}_F$ from C to C'.*

Proof. By Proposition 5.1 of [Rou11] (or Proposition 4.4.16), there exists $A \in \mathcal{A}(C \cup C')$. One identifies A and $\mathbb{A}$. One writes $F = F(x, F^v)$, where F^v is a positive vectorial face of $\mathbb{A}$. By Lemma 4.4.30, one can write $C = F(x, C^v)$ and $C' = F(x, C'^v)$, where C^v and C'^v are two positive vectorial chambers dominating F^v. Let $J \subset I$ be the type of F^v. There exists $w \in W^v$ such that $C'^v = w.C'^v$. Then $w.F^v$ is the face of C'^v of type J: $w.F^v = F^v$. By Section 1.3 of [Rou11], the stabilizer W_{F^v} of F^v is conjugated to $W^v(J) = \langle r_j |\ j \in J\rangle$. As $(W^v(J), \{r_j |\ j \in J\})$ is a Coxeter system, there exists a gallery $C_1^v = C^v, \ldots, C_n^v = C'^v$ from C^v to C'^v such that $F^v \subset \overline{C_i^v}$ for all $i \in [\![1, n]\!]$. We set $C_i = F(x, C_i^v)$ for all $i \in [\![1, n]\!]$ and $C_1, \ldots, C_n$ has the desired property. □

If $x \in \mathcal{I}$, $\epsilon \in \{-, +\}$ and $C, C' \in \mathcal{C}_x^\epsilon$ are different and adjacent, one denotes by $C \cap C'$ the face between C and C' in any apartment containing C and C'. This is well defined by Proposition 5.1 and Proposition 5.2 of [Rou11].

Lemma 4.4.32. *Let $a, b \in \mathcal{I}$, $\epsilon \in \{-, +\}$ and $C_a \in \mathcal{C}_a$. Suppose that for all $F_b \in \mathcal{F}_b^\epsilon$ spherical, $C_a \cup F_b$ is intrinsic. Then for all $F_b \in \mathcal{F}_b^\epsilon$, $C_a \cup F_b$ is intrinsic.*

Proof. Let $F_b \in \mathcal{F}_b^\epsilon$. Let $A, B \in \mathcal{A}(C_a \cup F_b)$. Let $C_b^A \in \mathcal{C}_{F_b}^\epsilon(A)$ and $C_b^B \in \mathcal{C}_{F_b}^\epsilon(B)$. Let $C_1 = C_b^A, \ldots, C_n = C_b^B$ be a gallery of chambers of $\mathcal{C}_{F_b}^\epsilon$ from C_b^A to C_b^B, which exists by Lemma 4.4.31. One sets $A_1 = A$ and $A_n = B$. By Proposition 4.4.16, for all $i \in [\![2, n-1]\!]$, there exists an apartment A_i containing C_a and C_i. For all $i \in [\![1, n-1]\!]$, $A_i \cap A_{i+1} \supset C_a \cup C_i \cap C_{i+1}$ and by hypothesis, $\mathrm{conv}_{A_i}(C_i \cap C_{i+1}, C_a) \subset A_i \cap A_{i+1}$ and there exists an isomorphism $\phi_i : A_i \to A_{i+1}$ fixing $\mathrm{conv}_{A_i}(C_i \cap C_{i+1}, C_a)$. In particular, $\mathrm{conv}_{A_i}(C_a, F_b) = \mathrm{conv}_{A_{i+1}}(C_a, F_b) \subset A_i \cap A_{i+1}$ and ϕ_i fixes $\mathrm{conv}_{A_i}(C_a, F_b)$. By induction $A \cap B$ contains $\mathrm{conv}_A(C_a, F_b)$ and $\phi = \phi_{n-1} \circ \ldots \circ \phi_1 : A \overset{\mathrm{conv}_A(C_a, F_b)}{\to} B$ and the lemma is proved. □

We now prove Theorem 4.4.17:

Let $x, y \in \mathcal{I}$ such that $x \leq y$. Let $(F_x, F_y) \in \mathcal{F}_x \times \mathcal{F}_y$. Then $F_x \cup F_y$ is intrinsic.

Proof. We have four cases to treat, depending on the signs of F_x and F_y. The case where F_x is negative and F_y is positive is a particular case of Proposition 5.5 of [Rou11].

Suppose that F_x and F_y are positive. Let $(F_x, F_y) \in \mathcal{F}_x^+ \times \mathcal{F}_y^+$ and $A, B \in \mathcal{A}(F_x \cup F_y)$. Let $(C_x^A, C_y^B) \in \mathcal{C}_{F_x}(A) \times \mathcal{C}_{F_y}(B)$. Let $A' \in \mathcal{A}(C_x^A \cup C_y^B)$. By Lemma 4.4.25, Lemma 4.4.27 and Lemma 4.4.32, $C_x^A \cup F_y$ and $F_x \cup C_y^B$ are intrinsic. Therefore there exists $\phi : A \overset{\mathrm{conv}_A(C_x^A, F_y)}{\to} A'$ and $\psi : A' \overset{\mathrm{conv}_A(F_x, C_y^B)}{\to} B$. Then $\mathrm{conv}_A(F_x, F_y) = \mathrm{conv}_{A'}(F_x, F_y) = \mathrm{conv}_B(F_x, F_y)$ and $\psi \circ \phi : A \overset{F_x \cup F_y}{\to} B$. Consequently $F_x \cup F_y$ is intrinsic.

Suppose F_x and F_y are negative. We deduce the fact that $F_x \cup F_y$ is intrinsic from the previous case by exchanging the signs. Indeed, let $\mathbb{A}'$ be the vectorial space $\mathbb{A}$ equipped with a structure of apartment of type $-\mathbb{A}$: the Tits cone $\mathcal{T}'$ of $\mathbb{A}'$ is $-\mathcal{T}$, ... We obtain a masure $\mathcal{I}'$ of type $\mathbb{A}'$ with underlying set $\mathcal{I}$. Let $(F_x, F_y) \in \mathcal{F}_x^- \times \mathcal{F}_y^-$. One has $y \leq' x$ and F_x and F_y are positive for $\mathcal{I}'$. Therefore $F_x \cup F_y$ is intrinsic in $\mathcal{I}'$ and thus in $\mathcal{I}$.

Suppose that F_x is positive and F_y is negative. Let $A, B \in \mathcal{A}(F_x, F_y)$ and $(C_x^A, C_y^B) \in \mathcal{C}_{F_x}(A) \times \mathcal{C}_{F_y}(B)$. By Lemma 4.4.27 and Lemma 4.4.32, $C_x^A \cup F_y$ is intrinsic. As the roles of $\mathcal{T}$ and $-\mathcal{T}$ are similar, $F_x \cup C_y^B$ is also intrinsic and by the reasoning above, $F_x \cup F_y$ is intrinsic. □

4.4.4 Axiomatic of masures in the affine case

In this section, we study the particular case of a masure associated to irreducible affine Kac-Moody matrix A, which means that A satisfies condition (aff) of Theorem 4.3 of [Kac94].

Let $\mathcal{S}$ be a generating root system associated to an irreducible and affine Kac-Moody matrix and $\mathbb{A} = (\mathcal{S}, W, \Lambda')$ be an apartment. By Subsection 2.3.1, one has $\mathring{\mathcal{T}} = \{v \in \mathbb{A} | \delta(v) > 0\}$ for some imaginary root $\delta \in Q_{\mathbb{N}} \backslash \{0\}$ and $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$, where $\mathbb{A}_{in} = \bigcap_{i \in I} \ker(\alpha_i)$.

We fix an apartment $\mathbb{A}$ of affine type.

Let (MA af i)=(MA1).

Let (MA af ii) : let A and B be two apartments. Then $A \cap B$ is enclosed and there exists $\phi : A \overset{A\cap B}{\to} B$.

Let (MA af iii)= (MA iii).

The aim of this subsection is to prove the following theorem:

Theorem 4.4.33. *Let $\mathcal{I}$ be a construction of type $\mathbb{A}$ and $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$. Then $\mathcal{I}$ is a masure for* cl *if and only if $\mathcal{I}$ satisfies (MA af i), (MA af ii) and (MA af iii,* cl*) if and only if $\mathcal{I}$ satisfies (MA af i), (MA af ii) and (MA af iii,* $\mathrm{cl}^\#$*).*

Remark 4.4.34. *Actually, we do not know if this axiomatic is true for masures associated to indefinite Kac-Moody groups. We do not know if the intersection of two apartments is always convex in a masure.*

The fact that we can exchange (MA af iii, $\mathrm{cl}^\#$) and (MA af iii, cl) for all $\mathrm{cl} \in \mathcal{CL}_{\Lambda'}$ follows from Theorem 4.4.2. The fact that a construction satisfying (MA af ii) and (MA af iii, $\mathrm{cl}^\#$) is a masure is clear and does not use the fact that $\mathbb{A}$ is associated to an affine Kac-Moody matrix. It remains to prove that a masure of type $\mathbb{A}$ satisfies (MA af ii), which is the aim of this section.

Lemma 4.4.35. *Let A and B be two apartments such that there exist $x, y \in A \cap B$ such that $x \mathring{\leq} y$ and $x \neq y$. Then $A \cap B$ is convex.*

Proof. One identifies A and $\mathbb{A}$. Let $a, b \in \mathbb{A} \cap B$. If $\delta(a) \neq \delta(b)$, then $a \leq b$ or $b \leq a$ and $[a, b] \subset B$ by (MAO). Suppose $\delta(a) = \delta(b)$. As $\delta(x) \neq \delta(y)$, one can suppose that $\delta(a) \neq \delta(x)$. Then $[a, x] \subset B$. Let $(a_n) \in [a, x]^{\mathbb{N}}$ such that $\delta(a_n) \neq \delta(a)$ for all $n \in \mathbb{N}$ and $a_n \to a$. Let $t \in [0, 1]$. Then $ta_n + (1-t)b \in B$ for all $n \in \mathbb{N}$ and by Proposition 4.2.17, $ta + (1-t)b \in B$: $\mathbb{A} \cap B$ is convex. □

Lemma 4.4.36. *Let A and A' be two apartments of $\mathcal{I}$. Then $A \cap A'$ is convex. Moreover, if $x, y \in A \cap A'$, there exists an isomorphism $\phi : A \overset{[x,y]_A}{\to} A'$.*

Proof. Let $x, y \in A \cap A'$ such that $x \neq y$. Let C_x be a chamber of A based at x and C_y be a chamber of A' based at y. Let B be an apartment containing C_x and C_y, which exists by Proposition 4.4.16. By Lemma 4.4.35, $A \cap B$ and $A' \cap B$ are convex and by Proposition 4.2.34, there exist isomorphisms $\psi : A \overset{A\cap B}{\to} B$ and $\psi' : B \overset{A'\cap B}{\to} A'$. Therefore $[x, y]_A = [x, y]_B = [x, y]_{A'}$. Moreover, $\phi = \psi' \circ \psi$ fixes $[x, y]_A$ and the lemma is proved. □

Theorem 4.4.37. *Let A and B be two apartments. Then $A \cap B$ is enclosed and there exists an isomorphism $\phi : A \overset{A\cap B}{\to} B$.*

Proof. The fact that $A \cap B$ is enclosed is a consequence of Lemma 4.4.36 and Proposition 4.2.30. By Proposition 4.2.22, there exist $\ell \in \mathbb{N}$, enclosed subsets $P_1, \ldots, P_\ell$ of A such that $\mathrm{supp}(A \cap B) = \mathrm{supp}(P_j)$ and isomorphisms $\phi_j : A \overset{P_j}{\to} B$ for all $j \in [\![1, \ell]\!]$. Let $x \in \mathrm{Int}_r(P_1)$ and $y \in A \cap B$. By Lemma 4.4.36, there exists $\phi_y : A \overset{[x,y]}{\to} B$. Then $\phi_y^{-1} \circ \phi_1$ fixes a neighborhood of x in $[x, y]$ and thus ϕ_1 fixes y, which proves the theorem. □

4.4.5 Group acting strongly transitively on a masure

In this subsection, we give equivalent definitions of strongly transitivity for a group acting on a masure. We begin by a definition adapted to the new axiomatic of masures and then give a simpler definition when $\mathcal{I}$ is thick.

4.4.5.1 Definition adapted to the new axiomatics

Lemma 4.4.38. *Let $G \subset \mathrm{Aut}(\mathcal{I})$. Then G acts strongly transitively on $\mathcal{I}$ if and only if all the isomorphisms of apartments fixing a sector-germ are induced by elements of G.*

Proof. The implication $\Rightarrow$ follows directly from (MA4).

Reciprocally, suppose that all the isomorphisms of apartments fixing a sector-germ are induced by an element of G. Let A and B be two apartments. Let $X \subset A \cap B$ be a convex set such that there exists $\phi : A \overset{X}{\to} B$. Let $(\bigcup_{i=1}^k P_i, (\phi_i)_{i\in I})$ be a G-compatible decomposition of $A \cap B$ in enclosed subsets, which exists by Proposition 4.2.22. Let $i \in [\![1, k]\!]$ such that $X \cap P_i$ has nonempty interior in X. Let $g \in G$ inducing ϕ_i on A. Then $\phi_i^{-1} \circ \phi$ is an affine automorphism of A fixing $X \cap P_i$ and thus it fixes X. Therefore g induces ϕ on A. Consequently all the isomorphisms of (MA2) are induced by elements of G, thus G acts strongly transitively on $\mathcal{I}$ and the lemma is proved. □

Remark 4.4.39. *The lemma above is also proved in [CMR17] but in a different way (see Proposition 4.7). It will be improved in chapter 8 (see Proposition 8.4.3).*

Corollary 4.4.40.
1. *Let $\mathcal{I}$ be a masure. A group G acts strongly transitively on $\mathcal{I}$ if and only if all the isomorphisms of apartments involved in (MA ii) are induced by an element of G.*
2. *Let $\mathcal{I}$ be a masure of affine type. A group G acts strongly transitively on $\mathcal{I}$ if and only if all the isomorphism of apartments involved in (MA af ii) are induced by an element of G.*

4.4.5.2 Strongly transitive action and thickness

We now give a simpler definition of strongly transitivity when $\mathcal{I}$ is thick.

Lemma 4.4.41. *Suppose that there exists a group G acting strongly transitively on $\mathcal{I}$. Then for all pair (A, B) of apartments, there exists $g \in G$ such that $g.A = B$.*

Proof. Let $\mathfrak{q}_A, \mathfrak{q}_B$ be sector-germs of A and B and A' be an apartment containing $\mathfrak{q}_A$ and $\mathfrak{q}_B$. By (MA4) and Remark 3.2.4, there exists $g_1, g_2 \in G$ inducing $A \overset{\mathfrak{q}_A}{\to} A'$ and $A' \overset{\mathfrak{q}_B}{\to} B$. Then $g_2.g_1.A = B$, and the lemma follows. □

Lemma 4.4.42. *Suppose that a group G acts strongly transitively on $\mathcal{I}$. Let A be an apartment, M be a wall of A, s be the reflexion of A fixing M and D_1 be a half-apartment of A delimited by M. Suppose that some panel $P \subset M$ is dominated by a chamber C not included in A. Then there exists $g \in G$ inducing s on A.*

Proof. By Lemma 4.4.41, one can suppose that $A = \mathbb{A}$. Let D_2 be the half-apartment of $\mathbb{A}$ opposite to M. By Proposition 2.9 1) of [Rou11], there exists an apartment A_2 containing C and D. By (MA ii), $\mathbb{A} \cap A_2 = D_1$. Let D_3 be the half-apartment of A_2 opposed to D_1. By Proposition 2.9 2) of [Rou11], $A_1 := D_2 \cup D_3$ is an apartment. Let $\phi_{32} : \mathbb{A} \overset{D_1}{\to} A_2$, $\phi_{21} : A_2 \overset{D_3}{\to} A_1$ and $\phi_{13} : A_1 \overset{D_2}{\to} \mathbb{A}$. By Remark 3.2.4 and (MA4), there exist $g_{32}, g_{21}, g_{13} \in G$ inducing ϕ_{32}, ϕ_{21} and ϕ_{13}. By Lemma 4.2.5, the following diagram is commutative:

$$\begin{array}{ccc} \mathbb{A} & \xrightarrow{\phi_{32}} & A_2 \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle \phi_{21}} \\ \mathbb{A} & \xleftarrow[\phi_{13}]{} & A_1. \end{array}$$

and thus $g_{13}.g_{21}.g_{32}$ induces s on A. □

Remark 4.4.43. *Suppose that a group G acts strongly on $\mathcal{I}$. The choice of the Weyl group W (and thus of $Q^\vee$) imposes restrictions on the walls that can delimit the intersection of two apartments. Let A be an apartment and suppose that $A \cap \mathbb{A} = D(\alpha, k)$ for some $\alpha \in \Phi_{re}$ and $k \in \Lambda'_\alpha$. Then $k \in \frac{1}{2}\alpha(Q^\vee)$. Indeed, by Lemma 4.4.42, there exists $g \in G$ inducing the reflexion s of $\mathbb{A}$ fixing $M(\alpha, k)$ and thus $s \in \mathrm{Isom}(\mathbb{A}, \mathbb{A}) = W$. The vectorial part $\vec{s}$ of s fixes $M(\alpha, 0)$. As $W = W^v \ltimes Q^\vee$, $s = t \circ \vec{s}$, where t is a translation of vector $q^\vee$ in $Q^\vee$. If $y \in M(\alpha, k)$, one has $\alpha(s(y)) = k = \alpha(q^\vee) - k$ and therefore $k \in \frac{1}{2}\alpha(Q^\vee)$. In particular if $\mathcal{I}$ is thick, then $\Lambda'_\alpha \subset \frac{1}{2}\alpha(Q^\vee)$ for all $\alpha \in \Phi_{re}$. This could enable to be more precise in Proposition 4.2.8.*

Proposition 4.4.44. *Suppose that $\mathcal{I}$ is thick. Then a group G of automorphisms of $\mathcal{I}$ acts strongly transitively on it if and only if for all apartments A, B and all (Weyl)-isomorphism of apartments $\phi : A \to B$, there exists $g \in G$ inducing ϕ.*

Proof. The implication $\Leftarrow$ is clear without assumption of thickness of $\mathcal{I}$. Reciprocally, suppose that G acts strongly transitively on $\mathcal{I}$.

By Lemma 4.4.42 and definition of the Weyl group W, if A is an apartment, all element of $\mathrm{Isom}(A, A)$ is induced by an element of G.

Let A and B be two apartments and $\phi : A \to B$ be an isomorphism. By Lemma 4.4.41, there exists $g \in G$ such that $g.B = A$. Let $\psi = g_{|B}^{|A}$. Then $\psi \circ \phi \in \mathrm{Isom}(A, A)$, thus there exists $h \in G$ inducing $\psi \circ \phi$ on A. Then $g^{-1}.h$ induces ϕ on A, which proves $\Rightarrow$ and the proposition is proved. □

Remark 4.4.45. *In [CMR17], Ciobotaru, Mühlherr and Rousseau study criteria of strong transitivity. They give a criteria involving the action on the twin building at infinity, see Theorem 1.7 of loc. cit.*

Chapter 5

Hecke algebras for Kac-Moody groups over local fields

5.1 Introduction

Let G be a split reductive group over a local field. The theory of smooth representations of G is strongly related to the representation theory of Hecke algebras of G, which are algebras associated to open compact subgroups of G.

When G is a split Kac-Moody group over a local field (not reductive), there is up to now no definition of smoothness for the representations of G. However Braverman, Kazhdan and Patnaik (in the affine case) and Bardy-Panse, Gaussent and Rousseau (in the general case) define certain Hecke algebras of G: the spherical Hecke algebra $\mathcal{H}_s$ ([BK11] and [GR14]), associated to the spherical subgroup $K_s = \mathbf{G}(\mathcal{O})$, which is the fixer of some vertex 0 in the masure $\mathcal{I}$ of G and the Iwahori-Hecke algebra $\mathcal{H}$ ([BKP16] and [BPGR16]) associated to the Iwahori subgroup K_I, which is the fixer of some alcove C_0^+ of $\mathcal{I}$ based at 0. In this chapter, we associate Hecke algebras to fixers of faces between 0 and C_0^+ and study the center of the Iwahori-Hecke algebra of G. Let us be more precise.

Smooth representations and Hecke algebras in the reductive case Our reference for this paragraph is Chapter 4 of [BH06].

Let $\mathbf{G}$ be a connected split reductive group, $\mathcal{K}$ be a non-archimedean local field and $G = \mathbf{G}(\mathcal{K})$. Then G is locally profinite and in particular the set of compact open subgroups is a neighborhood basis of the identity. A **smooth representation of** G (over $\mathbb{C}$) is a couple (π, V) where V is a vector space over $\mathbb{C}$ and $\pi : G \to \mathrm{GL}(V)$ is a morphism such that for all $v \in V$, $\{g \in G | \pi(g)(v) = v\}$ is open in G. In order to study these representations, one introduces Hecke algebras. Let $\mathcal{H}(G) = \mathcal{C}_c^\infty(G)$ be the set of functions $f : G \to \mathbb{C}$ which are locally constant and of compact support. Let μ be a Haar measure on G. If $f_1, f_2 \in \mathcal{H}(G)$ and $g \in G$, one sets

$$f_1 * f_2(g) = \int_G f_1(x) f_2(x^{-1}g) d\mu(x).$$

Then $f_1 * f_2 \in \mathcal{H}(G)$ and $(\mathcal{H}(G), *)$ is an associative algebra, the **global Hecke algebra of** G **over** $\mathbb{C}$. A left $\mathcal{H}(G)$-module M is said to be **smooth** if it satisfies $\mathcal{H}(G).M = M$. Then the category of "smooth representations of G over $\mathbb{C}$" is equivalent to the category of "smooth $\mathcal{H}(G)$-modules".

Let $K \subset G$ be an open and compact subgroup. Let $e_K = \frac{1}{\mu(K)} \mathbb{1}_K : G \to \mathbb{C}$. Then $e_K \in \mathcal{H}(G)$ and $e_K * e_K = e_K$. Then $\mathcal{H}_K(G) := e_K * \mathcal{H}(G) * e_K$ is a subalgebra of $\mathcal{H}(G)$ with unit e_K. This is the **Hecke algebra of G over $\mathbb{C}$ relative to K.**

Let (π, V) be a smooth representation of G. One denotes by V^K the space $\{v \in V | \forall k \in K,\ \pi(k)(v) = v\}$. Then V^K is an $\mathcal{H}_K(G)$-module. The process $(\pi, V) \mapsto V^K$ induces a bijection between the following sets of objects:

- equivalence classes of irreducible smooth representations (π, V) of G such that $V^K \neq 0$

- equivalence classes of simple $\mathcal{H}_K(G)$-modules.

If $\mathcal{I}$ is the Bruhat-Tits building of G, the fixers of faces between a special vertex 0 and an alcove C_0^+ based at 0 are open and compact.

Parahoric Hecke algebras Suppose now that G is a split Kac-Moody group over a local field. There is up to now no global Hecke algebra of G and no more topology analogous to the topology of reductive groups. Let $\mathcal{I}$ be the masure of G, 0 be a special vertex of $\mathcal{I}$ and C_0^+ be an alcove based at 0. Let K be the fixer of some face between $\{0\}$ and C_0^+. We begin by proving that we cannot turn G into a topological group in such a way that K is open and compact (see Proposition 5.2.8). This applies in particular to K_s and K_I.

We then construct a Hecke algebra associated to each fixer of a spherical face F between $\{0\}$ and C_0^+, spherical meaning that the fixer of F in the Weyl group is finite (see Theorem 5.3.14). Our construction is very close to the one of Bardy-Panse, Gaussent and Rousseau of the Iwahori-Hecke algebra. When F is no more spherical and different from $\{0\}$ (this case does not occur when G is affine), we prove that this construction fails: the structure constants are infinite (see Proposition 5.3.22).

Spherical and Iwahori-Hecke algebras in the reductive case Suppose G reductive. Let us recall some properties of the spherical Hecke algebra $\mathcal{H}_s$ and the Iwahori-Hecke algebra $\mathcal{H}$ of G. By Satake isomorphism, if W^v is the Weyl group of G and $Q_\mathbb{Z}^\vee$ is the coweight lattice of G, $\mathcal{H}_s$ is isomorphic to $\mathbb{C}[Q_\mathbb{Z}^\vee]^{W^v}$, which is the sub-algebra of W^v-invariant elements of the algebra of the group $(Q_\mathbb{Z}^\vee, +)$.

The Iwahori-Hecke algebra $\mathcal{H}$ has a basis indexed by the affine Weyl group of G - the Bernstein-Lusztig basis - and the product of two elements of this basis can be expressed with the Bernstein-Lusztig presentation. It enables to determine the center of $\mathcal{H}$ and one sees that it is isomorphic to the spherical Hecke algebra of G. We summarize these results as follows:

$$\mathcal{H}_s \xrightarrow[S]{\backsim} \mathbb{C}[Q_\mathbb{Z}^\vee]^{W^v} \underset{g}{\hookrightarrow} \mathcal{H}, \text{ and } \mathrm{Im}(g) = \mathcal{Z}(\mathcal{H})$$

where S is the Satake isomorphism and g comes from the Bernstein-Lusztig basis.

Completed Iwahori-Hecke algebra We no more suppose G to be reductive. Braverman, Kazhdan and Patnaik and Bardy-Panse, Gaussent and Rousseau obtain a Satake isomorphism and Bernstein-Lusztig relations. In this framework, the Satake isomorphism is an isomorphism between $\mathcal{H}_s := \mathcal{H}_{K_s}$ and $\mathbb{C}[[Y]]^{W^v}$, where Y is a lattice which can be thought of as the coweight lattice in a first approximation (but it can be bigger, notably when G is affine) and $\mathbb{C}[[Y]]$ is the Looijenga's algebra of Y, which is some completion of the group algebra $\mathbb{C}[Y]$ of Y. Let $\mathcal{H}$ be the Iwahori-Hecke algebra of G. As we shall see (Theorem 5.5.19), the center of $\mathcal{H}$ is more or less trivial. Moreover, $\mathbb{C}[[Y]]^{W^v}$ is a set of infinite formal series and there is no obvious injection from $\mathbb{C}[[Y]]$ to $\mathcal{H}$. For these reasons, we define a "completion" $\widehat{\mathcal{H}}$ of $\mathcal{H}$. More precisely, let $(Z^\lambda H_w)_{\lambda \in Y^+, w \in W^v}$, where Y^+ is a sub-monoid of Y, be the Bernstein-Lusztig basis of $\mathcal{H}$. Then $\widehat{\mathcal{H}}$ is the set of formal series $\sum_{w \in W^v, \lambda \in Y^+} c_{w,\lambda} Z^\lambda H_w$, whose support satisfies some conditions similar to what appears in the definition of $\mathbb{C}[[Y]]$. We equip it with a convolution compatible with the inclusion $\mathcal{H} \subset \widehat{\mathcal{H}}$. The fact that this product is well defined is not obvious and this is our main result: Theorem 5.5.10. We then determine the center of $\widehat{\mathcal{H}}$ and we show that it is isomorphic to $\mathbb{C}[[Y]]^{W^v}$ (Theorem 5.5.19), which is similar to the classical case. We thus get the following diagram:

$$\mathcal{H}_s \xrightarrow[S]{\simeq} \mathbb{C}[[Y]]^{W^v} \underset{g}{\hookrightarrow} \widehat{\mathcal{H}}, \text{ and } \mathrm{Im}(g) = \mathcal{Z}(\widehat{\mathcal{H}}),$$

where S is the Satake isomorphism (see Section 8 of [BK11] or Theorem 5.4 of [GR14]), and g comes from the Bernstein-Lusztig basis.

Framework Actually, this chapter is written in a more general framework: we ask $\mathcal{I}$ to be an abstract masure and G to be a strongly transitive group of (vectorially Weyl) automorphisms of $\mathcal{I}$. This applies in particular to almost-split Kac-Moody groups over local fields. We assume that $\mathcal{I}$ is semi-discrete (which means that if M is a wall of $\mathbb{A}$, the set of wall parallel to M is discrete) and that $\mathcal{I}$ has finite thickness (which means that for each panel, the number of chamber containing it is finite). The group G is a group acting strongly transitively on $\mathcal{I}$. Let N be the stabilizer of $\mathbb{A}$ in G and $\nu : N \to \mathrm{Aut}(\mathbb{A})$ be the induced morphism. We assume moreover that $\nu(N) = Y \rtimes W^v$.

Organization of the chapter In Section 5.2, we prove that if K is the fixer of some face between 0 and C_0^+, there exists no topology of topological group on G for which K is open and compact (see Proposition 5.2.8).

In Section 5.3, we associate Hecke algebras to spherical faces (see Theorem 5.3.14) and we prove that the same definition leads to infinite structure coefficients when $F \neq 0$ is not spherical (see Proposition 5.3.22).

In Section 5.4, we introduce the Looijenga's algebra of Y and recall the definition of the spherical Hecke algebra and of the Satake isomorphism in the Kac-Moody framework.

In Section 5.5, we recall the Bernstein Lusztig presentation in the Kac-Moody framework, define the completed Iwahori-Hecke algebra (see Theorem 5.5.10) and determine the center of the Iwahori-Hecke algebra and of its completion (see Theorem 5.5.19).

5.2 A topological restriction on parahoric subgroups

In this section, we prove that if G is not reductive, there exists no topology of topological group on G for which K_s or K_I are compact and open. We begin by recalling general facts concerning thickness properties of a masure acted upon strongly transitive group. We then prove the topology result.

5.2.1 Thickness of a wall

In this chapter, we ask $\mathcal{I}$ to be a semi-discrete thick masure of finite thickness. The walls of $\mathbb{A}$ are the $\alpha^{-1}(\{k\})$ such that $\alpha \in \Phi_{re}$ and $k \in \mathbb{Z}$. One has $Q^\vee = Q^\vee_{\mathbb{Z}}$ and $W = W^v \ltimes Y$. Let N be the fixer of $\mathbb{A}$ and $\nu : N \to \mathrm{Aut}(\mathbb{A})$ be the restriction morphism. We assume that $\nu(N) = W$.

Proposition 5.2.1. *Let M a wall. Then if P is a panel of M, the number of chambers dominating P only depends on M.*

Proof. Let A be an apartment containing M and P' be a panel of M. Let D be a half-apartment of A delimited by M. Let $\mathcal{C}_P$ (resp. $\mathcal{C}_{P'}$) be the set of chambers dominating P (resp. P') and not included in D. If E is a half-apartment, one denotes by $C_{P,E}$ (resp. $C_{P',E}$) the chamber of E dominating P (resp. P').

Let $f_{P,P'} : \mathcal{C}_P \to \mathcal{C}_{P'}$ defined as follows. Let $C \in \mathcal{C}_P$. Let B be an apartment containing D and C, which exists by Proposition 2.9 1) of [Rou11]. Let E be the half-apartment of B opposite to D. One sets $f_{P,P'}(C) = C_{P',E}$. Then f is well defined. Indeed, let B' be an apartment containing D and C and E' be the half-apartment of B' opposite to D. Then by (MA ii), $B \cap B'$ is a half-apartment of B containing D and C. Therefore $B \cap B'$ contains a neighborhood of M and thus it contains $C_{P',E}$. Consequently, $f_{P,P'}$ is well defined.

Similarly, one defines $f_{P',P} : \mathcal{C}_{P'} \to \mathcal{C}_P$. Then $f_{P,P'}$ and $f_{P',P}$ are reciprocal bijections, which proves the proposition.

□

Proposition 5.2.2. *Let $\alpha \in \Phi_{re}$ and $i \in I$ such that $\alpha = w.\alpha_i$ for some $w \in W^v$. If $k \in \mathbb{Z}$, one denotes by $1 + q_\alpha(k)$ the number of chambers containing any panel of $M(\alpha, k)$. Then $q_\alpha(k) = q_{\alpha_i}(k)$ and this number only depends on the parity of k.*

Proof. Let $g \in G$ inducing $w \in W^v$. Then $g.M(\alpha, k) = M(\alpha_i, k)$, thus $q_\alpha(k) = q_{\alpha_i}(k)$. For all $k' \in \mathbb{Z}$, there exists $g \in G$ inducing the translation of vector $k'\alpha_i^\vee$ on $\mathbb{A}$. As $\alpha_i(\alpha_i^\vee) = 2$, one gets the proposition. □

5.2.2 Topological restriction

Let F be a type 0 face of $\mathbb{A}$, i.e a local face whose vertex is in Y. Maybe considering $h.F$ for some $h \in G$, one can suppose that $F \subset \pm C_0^+$. One supposes that $F \subset C_0^+$ as the other case is similar. Let K_F be the fixer of F in G. In this subsection, we show that if W^v is infinite, there exists no topology of topological group on G such that K_F is open and compact. For this, we show that there exists $g \in G$ such that $K_F/(K_F \cap g.K_F.g^{-1})$ is infinite.

Let $\alpha \in \Phi_{re}^+$ and $i \in I$ such that $\alpha = w.\alpha_i$, for some $w \in W^v$. For $\ell \in \mathbb{Z}$, one sets $M_\ell = \{t \in \mathbb{A} | \alpha(t) = \ell\}$ and $D_\ell = \{t \in \mathbb{A} | \alpha(t) \leq l\}$ and one denotes by $K_{\alpha,\ell}$ the fixer of D_ℓ in G. For $\ell \in \mathbb{Z}$, one chooses a panel P_ℓ in M_ℓ and a chamber C_ℓ dominating P_ℓ and included in $\mathrm{conv}(M_\ell, M_{\ell+1})$.

For $i \in I$, one denotes by $1 + q_i$ the number of chambers containing P_0 and by $1 + q_i'$ the number of chambers containing P_1.

Let us explain the basic idea of the proof. Let $g \in G$ such that $g.0 \in C_f^v$ and $F' = g.F$. Then $K_F/(K_F \cap K_{F'})$ is in bijection with $K_F.F'$. Let $\widetilde{K_\alpha} = \bigcup_{l \in \mathbb{Z}} K_{\alpha,\ell}$, then $\mathbb{T}_\alpha := \widetilde{K_\alpha}.\mathbb{A}$ is a semi-homogeneous extended tree with parameters q_i and q_i'. Let $K_\alpha = K_{\alpha,1}$. We deduce from the thickness of $\mathcal{I}$ that if n_α is the number of walls parallel to $\alpha^{-1}(\{0\})$ between a and $g.a$, $|K_\alpha.g.a| \geq 2^{n_\alpha}$ and hence that $|K_F.g.a| \geq 2^{n_\alpha}$. As W^v is infinite, n_α can be made arbitrarily large by changing α.

One chooses a sector-germ $\mathfrak{q}$ included in D_0. Let $\rho_\mathfrak{q} : \mathcal{I} \overset{\mathfrak{q}}{\to} \mathbb{A}$. The restriction of $\rho_\mathfrak{q}$ to $\mathbb{T}_\alpha$ does not depend on the choice of $\mathfrak{q} \subset D_0$.

Lemma 5.2.3. *Let $v \in \widetilde{K}_\alpha$ and $x \in \mathbb{A}$. Suppose that $v.x \in \mathbb{A}$. Then v fixes $D_{\lceil \alpha(x) \rceil}$.*

Proof. Let $A = v.\mathbb{A}$. There exists $\ell \in \mathbb{Z}$ such that $A \cap \mathbb{A}$ contains D_ℓ. By (MA ii), $A \cap \mathbb{A}$ is a half-apartment: there exists $k \in \mathbb{Z}$ such that $A \cap \mathbb{A} = D_k$. Then v fixes $A \cap \mathbb{A}$. Indeed, let $\psi : A \overset{A \cap \mathbb{A}}{\to} \mathbb{A}$ (which exists by (MA ii)) and $\phi : A \to \mathbb{A}$ be the isomorphism induced by

v. Then ϕ and ψ coincide on a set with nonempty interior and thus $\phi = \psi$ and our claim follows.

One has $v.x \in A \cap \mathbb{A}$. Consequently $v.(v.x) = v.x$ and thus $v.x = x$. Therefore $x \in D_k$, thus $D_{\lceil \alpha(x) \rceil} \subset D_k$ and hence v fixes $D_{\lceil \alpha(x) \rceil}$. $\square$

Lemma 5.2.4. *Let $x \in \mathbb{A}$ and $M_x = \{t \in \mathbb{A} | \alpha(t) = \lceil \alpha(x) \rceil\}$. Then the map $f : \begin{array}{r} K_\alpha.x \to K_\alpha.M_x \\ u.x \mapsto u.M_x \end{array}$ is well defined and is a bijection.*

Proof. Let $u, u' \in K_\alpha$ such $u.x = u'.x$. By Lemma 5.2.3, $u'^{-1}.u$ fixes M_x and thus f is well-defined.

Let $u, u' \in K_\alpha$ such that $u.M_x = u'.M_x$, $v = u'^{-1}.u$ and $A = v.\mathbb{A}$. Then $v.M_x = M_x \subset A \cap \mathbb{A}$ and by Lemma 5.2.3, v fixes $D_{\lceil \alpha(x) \rceil}$. In particular $v.x = x$ and f is injective. By definition f is surjective and the lemma follows. $\square$

Let $\alpha_{\mathcal{I}} = \alpha \circ \rho_{\mathfrak{q}}$. If $\ell \in \mathbb{N}$, one denotes by $\mathcal{C}_\ell$ the set of chambers C dominating an element of $K_\alpha.P_\ell$ and satisfying $\alpha_{\mathcal{I}}(C) > \ell$ (which means that there exists $X \in C$ such that $\alpha_{\mathcal{I}}(x) > \ell$ for all $x \in X$). Let C_ℓ be the chamber of $\mathbb{A}$ dominating P_ℓ and not included in D_ℓ.

Lemma 5.2.5. *Let $\ell \in \mathbb{N}$. Then the map $g : \begin{array}{r} K_\alpha.M_{\ell+1} \to \mathcal{C}_\ell \\ u.M_{\ell+1} \to u.C_\ell \end{array}$ is well defined and is a bijection.*

Proof. Let $u, u' \in K_\alpha$ such that $u.M_{\ell+1} = u'.M_{\ell+1}$. Let $v = u'^{-1}.u$. Then $v.M_{\ell+1} \subset \mathbb{A}$ and by Lemma 5.2.3, $v.C_\ell = C_\ell$. Moreover $\alpha_{\mathcal{I}}(u.C_\ell) = \alpha(C_\ell) > \ell$ and thus g is well defined.

Let $u, u' \in K_\alpha$ such that $u.C_\ell = u'.C_\ell$. Let $v = u'^{-1}.u$ and $X \in C_\ell$ such that v fixes X. Let $x \in X$ such that $\alpha(x) > \ell$. Then by Lemma 5.2.3, v fixes $M_{\ell+1}$ and thus g is injective. It remains to show that $\mathcal{C}_\ell = K_\alpha.C_\ell$. Let $C \in \mathcal{C}_\ell$. Then C dominates $u.P_\ell$ for some $u \in K_\alpha$. By Proposition 2.9 1 of [Rou11], there exists an apartment A containing $u.D_\ell$ and C. Let $\phi : A \overset{A \cap \mathbb{A}}{\to} \mathbb{A}$ and $v \in K_\alpha$ inducing ϕ. Then $\alpha_{\mathcal{I}}(C) = \alpha(v.C)$, thus $\alpha(v.C) > \ell$. Moreover $v.C \subset \mathbb{A}$ dominates P_ℓ and thus $v.C = C_\ell$, which concludes the proof. $\square$

By combining Lemma 5.2.4 and Lemma 5.2.5, we get the following corollary:

Corollary 5.2.6. *Let $x \in \mathbb{A}$. Then if $\ell = \max(1, \lceil \alpha(x) \rceil)$, $|K_\alpha.x| = q_i' q_i q_i' \ldots (l-1$ factors$)$*

We now suppose that W^v is infinite.

Lemma 5.2.7. *Let F be a type 0 face of $\mathbb{A}$. Then there exists $g \in G$ such that if $F' = g.F$, $K_F / K_F \cap K_{F'}$ is infinite.*

Proof. Let $g \in G$ such $a := g.0 \in C_f^v$. Let $(\beta_k) \in (\Phi_{re}^+)^{\mathbb{N}}$ be an injective sequence. For all $k \in \mathbb{N}$, $K_{\beta_k} \subset K_F$ and thus $|K_F.F'| \geq |K_{\alpha_k}.a|$. By Corollary 5.2.6, it suffices to show that $\beta_k(a) \to +\infty$ (by thickness of $\mathcal{I}$).

One has $\beta_k = \sum_{i \in I} \lambda_{i,k} \alpha_i$, with $\lambda_{i,k} \in \mathbb{N}$ for all $(i,k) \in I \times \mathbb{N}$. By injectivity of (β_k), $\sum_{i \in I} \lambda_{i,k} \to +\infty$. Therefore, $\beta_k(a) \to +\infty$, which proves the lemma. $\square$

Proposition 5.2.8. *Let F be a type 0 face of $\mathcal{I}$. Then there is no way to turn G into a topological group in such a way that K_F is open and compact.*

Proof. Suppose that such a topology exists. Let $g \in G$ and $F' = g.F$. The group $K_{F'} = g.K_F.g^{-1}$ is open and compact and thus $K_F \cap K_{F'}$ is open and compact. Therefore $K_F / K_F \cap K_{F'}$ is finite: a contradiction with Lemma 5.2.7. $\square$

This proposition applies to $K = K_s = K_{F_0}$ and to the Iwahori group $K_I = K_{C_0^+}$, which shows that reductive groups and (non-reductive) Kac-Moody groups are very different from this point of view.

5.3 Hecke algebra associated to a parahoric subgroup

In this section, we associate Hecke algebras to fixers of faces between F_0 and C_0^+. This generalizes constructions of [BKP16] and [BPGR16].

5.3.1 The reductive case

This subsection uses I 3.3 of [Vig96]. Assume that G is reductive and let K be an open compact subgroup of G. Let us give an other equivalent definition of the relative Hecke algebras of G than in the introduction, more combinatoric and thus more adapted for our purpose. Let $\mathbb{Z}_c(G/K)$ be the space of functions from G to $\mathbb{Z}$ which are K-invariant under right multiplication and have compact support. One defines an action of G on this set as follows: $g.f(x) = f(g.x)$ for all $g \in G$, $f \in \mathbb{Z}_c(G/K)$ and $x \in G/K$. The **Hecke algebra of G relative to K** is the algebra $H(G,K) = \mathrm{End}_G \mathbb{Z}_c(G/K)$ of G-equivariant endomorphisms of $\mathbb{Z}_c(G/K)$. Let $\mathbb{Z}_c(G//K)$ be the ring of functions from G to $\mathbb{Z}$, with compact support, which are invariant under the action of K on the left and on the right. We have an isomorphism of $\mathbb{Z}$-modules $\Upsilon : H(G,K) \to \mathbb{Z}_c(G//K)$ defined by $\Upsilon(\phi) = \phi(\mathbb{1}_K)$ for all $\phi \in H(G,K)$. Therefore, $H(G,K)$ is a free $\mathbb{Z}$-algebra, with canonical basis $(e_g)_{g\in K\backslash G/K}$, where $e_g = \mathbb{1}_{KgK}$ for all $g \in G$. If $\mathcal{R}$ is a commutative ring, one defines $H_{\mathcal{R}}(G,K) = H(G,K) \otimes_{\mathbb{Z}} \mathcal{R}$: this is the **Hecke algebra over $\mathcal{R}$ of G relative to K**.

If $g, g' \in K\backslash G/K$, $e_g e_{g'} = \sum_{g''\in K\backslash G/K} m(g,g';g'')e_{g''}$, where

$$m(g,g';g'') = |(KgK \cap g''Kg'^{-1}K)/K|$$

for all $g'' \in K\backslash G/K$ ($m(g,g';g'') \neq 0$ implies $Kg''K \subset KgKg'K$ for all $g, g', g'' \in K\backslash G/K$). When $\mathcal{R} = \mathbb{C}$, these algebras are the same as the algebras of Section 5.1:

Proposition 5.3.1. *Let K be an open and compact subgroup of G. Then the morphism $\Gamma : H_{\mathbb{C}}(G,K) \to \mathcal{H}_K(G)$ defined by $\Gamma(e_g) = \frac{1}{\mu(K)}\mathbb{1}_{KgK}$ for all $g \in K\backslash G/K$ is well defined and is an isomorphism of algebras.*

Proof. By (2) of Proposition of 4.1 of [BH06], an element $f \in \mathcal{H}(G)$ satisfies $e_K * f = f$ if and only if $f(kg) = f(g)$ for all $k \in K$ and $g \in G$. A direct computation proves that if $f \in \mathcal{H}(G)$, then $f * e_k = f$ if and only if $f(gk) = f(g)$ for all $k \in K$ and $g \in G$. Therefore $\frac{1}{\mu(K)}\mathbb{1}_{KgK} = e_K * \frac{1}{\mu(K)}\mathbb{1}_{KgK} * e_K \in \mathcal{H}_K(G)$ and Γ is well defined. Moreover the elements of $\mathcal{H}_K(G)$ are K-bi-invariant. Thus the condition of compactness of the support implies that $(\frac{1}{\mu(K)}\mathbb{1}_{KgK})_{g\in K\backslash G/K}$ is a basis of $\mathcal{H}_K(G)$.

Let $g'' \in G$. Then if $x \in G$, $\mathbb{1}_{KgK}(x)\mathbb{1}_{Kg'K}(x^{-1}g'') \neq 0$ if and only if $x \in KgK \cap g''Kg'^{-1}K$. Therefore

$$\mathbb{1}_{KgK} * \mathbb{1}_{Kg'K}(g'') = \int_G \mathbb{1}_{KgK}(x)\mathbb{1}_{Kg'K}(x^{-1}g'')d\mu(x) = \mu(KgK \cap g''Kg'^{-1}K).$$

Moreover

$$\mu(KgK \cap g''Kg'^{-1}K) = \sum_{u\in KgK\cap g''Kg'^{-1}K/K} \mu(uK) = \mu(K)m(g,g';g'').$$

Consequently, $\frac{1}{\mu(K)}\mathbb{1}_{KgK} * \frac{1}{\mu(K)}\mathbb{1}_{Kg'K} = \Gamma(e_g) * \Gamma(e_{g'}) = \sum_{g''\in K\backslash G/K} m(g,g';g'')\frac{1}{\mu(K)}\mathbb{1}_{Kg''K} = \Gamma(e_g * e_{g'})$ and the proposition is proved. □

Remark 5.3.2. *If $g, g' \in G$ and $h \in K\backslash G/K$, the finiteness of $m(g, g', h)$ and of $\{g'' \in K\backslash G/K |\ m(g, g'; g'') \neq 0\}$ follow from the fact that K is open and compact. Indeed, $KgK \cap hKg'^{-1}K$ is compact. Moreover*

$$KgK \cap hKg'^{-1}K = \bigsqcup_{u \in KgK \cap hKg'^{-1}K/K} uK.$$

By Borel-Lebesgue's property, as the uK are open (and nonempty), $KgK \cap hKg'^{-1}K/K$ is finite. A similar argument proves that $\{g'' \in K\backslash G/K |\ m(g, g'; g'')\}$ is finite.

We no more suppose G to be reductive. We want to define Hecke algebras relative to some subgroups of G. As there is (up to now?) no topology on G similar to the topology of reductive groups, we cannot define "open compact" in G. However we can still define special parahoric subgroups, which are fixers of type 0 faces (whose vertices are in $G.0$) in the masure $\mathcal{I}$. Let $K = K_F$ be the fixer in G of some type 0 face F such that $F_0 \subset \overline{F} \subset \overline{C_0^+}$, where $F_0 = F^\ell(0, \mathbb{A}_{in})$ and $C_0^+ = F^\ell(0, C_f^v)$. We know from Proposition 5.2.8 that we cannot use the same argument as in Remark 5.3.2. We use the method of Bardy-Panse, Gaussent and Rousseau of [GR14] and [BPGR16]: we view the $(KgK \cap g''Kg'^{-1}K)/K$ as intersections of "spheres" in $\mathcal{I}$. We prove that when F is spherical, these intersections are finite. We then define the Hecke algebra ${}^F\mathcal{H}$ of G relative to K as follows: ${}^F\mathcal{H}$ is the free module over $\mathbb{Z}$ with basis $e_g = \mathbb{1}_{KgK}$, $g \in G^+$, where $G^+ = \{g \in G | g.0 \geq 0\}$, with convolution product $e_g * e'_g = \sum_{g'' \in K\backslash G^+/K} m(g, g'; g'') e_{g''}$, where $m(g, g'; g'') = |(KgK \cap g''Kg'^{-1}K)/K|$ for all $g'' \in K\backslash G^+/K$. To prove that this formula indeed defines an algebra, we need to prove finiteness results. We prove these results by using the fact that they are true when F is a type 0 chamber, which was proved by Bardy-Panse, Gaussent and Rousseau to define the Iwahori-Hecke algebra, and the fact that the number of type 0 chambers dominating F is finite. The reason why one uses G^+ instead of G is linked to the fact that two points are not always in a same apartment. This was already done in [BK11], [GR14], [BKP16] and [BPGR16].

We also prove that when F is non-spherical type 0 face such that $F_0 \subset \overline{F} \subset \overline{C_0^+}$, there exists $g \in G$ such that $(KgK \cap g''Kg'^{-1}K)/K$ is infinite and thus this method fails to associate a Hecke algebra to F.

5.3.2 Distance and spheres associated to a type 0 face

In this subsection, we define an "F-distance" (or a W_F-distance, where W_F is the fixer of F in W^v) for each type 0 face F between F_0 and C_0^+, generalizing the W^v-distance of [GR14] and the W-distance of [BPGR16].

If A (resp. A') is an apartment of $\mathcal{I}$, $E_1, \ldots, E_k$ (resp. $E'_1, \ldots, E'_k$) are subsets or filters of A (resp. A'), the notation $\phi : (A, E_1, \ldots, E_k) \mapsto (A', E'_1, \ldots, E'_k)$ means that ϕ is an isomorphism from A to A' induced by an element of G and that $\phi(E_i) = E'_i$ for all $i \in [\![1, k]\!]$. When we do not want to precise A and A', we write $\phi : (E_1, \ldots, E_k) \to (E'_1, \ldots, E'_k)$.

Let F be type 0 face of $\mathcal{I}$ such that $F \subset \overline{C_0^+}$. Let W_F (resp. K_F) be the fixer of F in W^v (resp. in G). Let $\Delta_F = G.F$. We have a bijection $\Upsilon : G/K_F \to \Delta_F$ mapping each $g.K_F$ to $g.F$.

If $F_1, F_2 \in \Delta_F$ are based in $a_1, a_2 \in \mathcal{I}$, one writes $F_1 \leq F_2$ if $a_1 \leq a_2$. One denotes by $\Delta_F \times_\leq \Delta_F$ the set $\{(F_1, F_2) \in \Delta_F^2 | F_1 \leq F_2\}$.

Definition/Proposition 5.3.3. *If $F' \in \Delta_F$ such that $F' \subset \mathbb{A}$, one sets $[F'] = W_F.F'$.*

Let $(F_1, F_2) \in \Delta_F \times_\leq \Delta_F$. Then there exists an apartment A containing F_1 and F_2 and an isomorphism $\phi : (A, F_1) \mapsto (\mathbb{A}, F)$. One sets $d^F(F_1, F_2) = [\phi(F_2)]$. This does not depend on the choices we made.

Proof. Proposition 5.1 of [Rou11] (or Proposition 4.4.16) yields the existence of A.

By definition, $F_1 = g.F$ for some $g \in G$. Let $A' = g.\mathbb{A}$. By (MA2), there exists $\psi : (A, F_1) \mapsto (A', F_1)$ and if $\psi' = g_{|\mathbb{A}}^{|A'}$, then $\phi := \psi'^{-1} \circ \psi : (A, F_1) \mapsto (\mathbb{A}, F)$: ϕ has the desired property.

Suppose A_1 is an apartment containing F_1, F_2 and $\phi_1 : (A_1, F_1) \mapsto (\mathbb{A}, F)$. By Theorem 4.4.17, there exists $f : (A, F_1, F_2) \mapsto (A_1, F_1, F_2)$. Then one has the following commutative diagram:

$$\begin{array}{ccc} (\mathbb{A}, F_1, F_2) & \xrightarrow{f} & (A_1, F_1, F_2) \\ \downarrow \phi & & \downarrow \phi_1 \\ (\mathbb{A}, F, \phi(F_2)) & \longrightarrow & (\mathbb{A}, F, \phi_1(F_2)), \end{array}$$

and the lower horizontal arrow is in W_F, which completes the proof. □

Remark 5.3.4. *Suppose that $F = F_0$. Using the natural bijection $\Delta_{F_0} \simeq \mathcal{I}_0$, where $\mathcal{I}_0 = G.0$ and $Y^{++} \simeq Y^+/W^v$, we get that d^{F_0} is the "vectorial distance" d^v of [GR14].*

Suppose that $F = C_0^+$. Then $W_{C_0^+} = \{1\}$. One has $[C] = \{C\}$ for all $C \in \Delta_{C_0^+}$. Therefore the distance $d^{C_0^+}$ is the distance d^W of [BPGR16], by identifying each element w of W to the type 0 chamber $w.C_0^+$.

Let $\Delta_{\geq F}^{\mathbb{A}} = \{E \in \Delta_F | E \subset \mathbb{A} \text{ and } E \geq F\}$. Let $[\Delta_F] = \{[F'] | F' \in \Delta_{\geq F}^{\mathbb{A}}\}$. If $E \in \Delta_F$ and $[R] \in [\Delta_F]$, one sets $\mathcal{S}^F(E, [R]) = \{E' \in \Delta_F | E' \geq E \text{ and } d^F(E, E') = [R]\}$ and $\mathcal{S}_{op}^F(E, [R]) = \{E' \in \Delta_F | E' \leq E \text{ and } d^F(E', E) = [R]\}$. If $E \in \Delta_{\geq F}^{\mathbb{A}}$, one chooses $g_E \in N$ such that $E = g_E.F$. Such a g_E exists: let $g \in G$ such that $E = g.F$ and $A = g.\mathbb{A}$. By (MA2) and 2.2.1) of [Rou11], there exists $\phi : (A, g.F) \mapsto (\mathbb{A}, g.F)$. Let $\psi = g_{|\mathbb{A}}^{|A}$. Then $\phi \circ \psi \in N$ and $\phi \circ \psi(F) = \phi(E) = E$.

Lemma 5.3.5. *Let $[R] \in [\Delta_F]$ and $\Upsilon : G/K_F \xrightarrow{\sim} \Delta_F$. Then $\Upsilon^{-1}(\mathcal{S}^F(F, [R])) = K_F g_R K_F/K_F$ and $\Upsilon^{-1}(\mathcal{S}_{op}^F(F, [R])) = K_F g_R^{-1} K_F/K_F$.*

Proof. Let $E \in \mathcal{S}^F(F, [R])$. Then there exists $g \in K_F$ such that $g.E = R = g_R.F$. Thus $\Upsilon^{-1}(E) \in K_F g_R K_F/K_F$. Let $x \in K_F g_R K_F$, $x = k_1 g_R k_2$, with $k_1, k_2 \in K_F$. Then $\Upsilon(x) = k_1 g_R.F = k_1.R$. As d^F is G-invariant, $d^F(k_1.F, k_1.R) = d^F(F, R) = d^F(F, k_1.R)$, and thus $x \in \Upsilon^{-1}(S^F(F, [R]))$. The proof of the second statement is similar. □

If $r \in [\Delta_F]$, one chooses $n_r \in N$ such that $n_r.F \in r$. Then $K_F.n_r.K_F$ does not depend on the choice of n_r. Indeed let $n' \in N$ such that $n'.F \in r$. Then $n'.F = w.n_r.F$ for some $w \in W_F$. Le $g \in N$ inducing w on $\mathbb{A}$. Then $n'^{-1}.g.n_r \in K_F$ and thus $n_r \in K_F n' K_F$.

The following proposition generalizes Cartan decomposition (Proposition 3.4.4).

Proposition 5.3.6. *The map $\Gamma : K_F \backslash G^+/K_F \to [\Delta_F]$ mapping each $K_F g K_F \in K_F \backslash G^+/K_F$ on $d^F(F, g.F)$ is well-defined and is a bijection. Its reciprocal function is the map $[\Delta_F] \to K_F \backslash G^+/K_F$ sending each $r \in [\Delta_F]$ on $K_F n_r K_F$.*

Proof. Let us prove that Γ is well-defined. Let $g \in G^+$ and $k \in K_F$. Then $d^F(F, k.F) = d^F(k.F, k.g.F) = d^F(F, k.g.F)$ (as d^F is G-invariant) and thus $\Gamma(K_F g K_F) \in [\Delta_F]$ is well-defined.

Let $r, r' \in Y^{++}$ such that $K_F n_r K_F = K_F n_{r'} K_F$. One writes $n_r = k.n_{r'}.k'$, with $k, k' \in K_F$. Then $d^F(F, kn_rk'.F) = r = d^F(F, kn_{r'}k'.F) = d^F(F, n_{r'}.F) = [r']$ (by G-invariance of d^F).

Let $g \in G^+$ and $r = d^F(F, g.F)$. Let us prove that $g \in K_F n_r K_F$. Let $A \in \mathcal{A}(F, g.F)$. Let $h \in G$ inducing a morphism $A \overset{F}{\to} \mathbb{A}$. Then $hg.F \in r$ by definition of d^F. Thus one can choose $n_r = hg$. Thus $K_F n_r K_F = K_F g K_F$ and the proposition follows. □

5.3.3 Hecke algebra associated to a spherical type 0 face

In this subsection we define the Hecke algebra associated to a spherical type 0 face F between F_0 and C_0^+ (or to K_F).

Let C, C' be two positive type 0 chambers based at some $x \in \mathcal{I}_0$. One identifies the type 0 chambers of $\mathbb{A}$ whose vertices are in Y^+ and $W^+ := W \ltimes Y^+$. Then $d^W(C, C')(=d^{C_0^+}(C, C'))$ is in W^v. One sets $d(C, C') = \ell(d^W(C, C'))$.

Lemma 5.3.7. *Let C be a positive type 0 chamber of $\mathcal{I}$, $n \in \mathbb{N}$ and x be the vertex of C. Let $B_n(C)$ be the set of positive type 0 chambers C' of $\mathcal{I}$ based at x and such that $d(C, C') \leq n$. Then $B_n(C)$ is finite.*

Proof. We do it by induction on n. The set $B_1(F')$ is finite for all $F' \in G.C_0^+$ because $\mathcal{I}$ is of finite thickness. Let $n \in \mathbb{N}$. Suppose that $B_k(F')$ is finite for all $k \leq n$ and $F' \in G.C_0^+$. Let $C' \in B_{n+1}(C)$. Let A be an apartment containing C' and C, which exists by Proposition 5.1 of [Rou11] and $\phi : A \to \mathbb{A}$ be an isomorphism of apartments such that $\phi(C) = C_0^+$. One has $\phi(C') = w.C_0^+$, with $w \in W^v$ and $\ell(w) = n + 1$. Let $\widetilde{w} \in W^v$ such that $\ell(\widetilde{w}) = n$ and $d(\widetilde{w}.C_0^+, \phi(C')) = 1$. Let $\widetilde{C} = \phi^{-1}(\widetilde{w}.C_0^+)$. Then $d(C, \widetilde{C}) = 1$ and thus $B_{n+1}(C) \subset \bigcup_{C'' \in B_n(C)} B_1(C'')$, which is finite. □

Type of a type 0 face Let $\mathfrak{F}^v_{\mathbb{A}}$ be the set of positive vectorial faces of $\mathbb{A}$ and $\mathfrak{F}^0_{\mathbb{A}}$ be the set of positive type 0 faces of $\mathbb{A}$ based at 0.

Lemma 5.3.8. *The map $f : \mathfrak{F}^v_{\mathbb{A}} \to \mathfrak{F}^0_{\mathbb{A}}$ mapping each $F^v \in \mathfrak{F}^v_{\mathbb{A}}$ on $F^\ell(0, F^v)$ is a bijection.*

Proof. By definition of local faces, f is surjective. Let $F_1^v, F_2^v \in \mathfrak{F}^v_{\mathbb{A}}$ such that $F_1^v \neq F_2^v$. As 0 is special, $F_1^v \in f(F_1^v)$ and $F_2^v \in f(F_2^v)$. As $F_1^v \cap F_2^v = \emptyset$, $f(F_1^v) \neq f(F_2^v)$ (for otherwise, one would have $\emptyset \in f(F_1^v)$) and thus f is a bijection. □

Let F be a positive type 0 face of $\mathcal{I}$. One has $F = g_1.F_1$ for some type 0 face F_1 dominated by C_0^+. Let $J \subset I$ such that $F_1 = F^\ell(0, F^v(J))$ (see Subsection 2.2.2.2 for the definition of $F^v(J)$). The **type of** F, denoted type(F) is J and it is well-defined. Indeed, suppose $F = g_2.F^\ell(0, F^v(J_2))$, for some $g_2 \in G$ and $J_2 \subset I$. Let $g = g_2^{-1}.g_1$. One has $F^\ell(0, F^v(J)) = g.F^\ell(0, F^v(J_2))$. By (MA2) and 2.2.1) of [Rou11], one can suppose that $g \in N$, thus $g_{|\mathbb{A}} \in W^v$ and by Lemma 5.3.8, $F^v(J) = g.F^v(J_2)$. By Section 1.3 of [Rou11] $J = J_2$ and the type is well defined.

The type is invariant under the action of G and if C is a type 0 chamber, there exists exactly one sub-face of C of type J for each $J \subset I$.

We now fix a spherical type 0 face F between F_0 and C_0^+.

Lemma 5.3.9. *Let $F' \in \Delta_F$ and $\mathcal{C}_{F'}$ be the set of type 0 chambers of $\mathcal{I}$ containing F'. Then $\mathcal{C}_{F'}$ is finite.*

Proof. We fix $C \in \mathcal{C}_{F'}$. Let x be the vertex of C and $C' \in \mathcal{C}_{F'}$. Let A be an apartment containing C and C' (such an apartment exists by Proposition 5.1 of [Rou11]). We identify A and $\mathbb{A}$ and we fix the origin of $\mathbb{A}$ in x. There exists $w \in W^v$ such that $C' = w.C$. Let J be the type of F'. Then $w.F'$ is the face of C' of type J and thus $w.F' = F'$. Therefore $w \in W_{F'}$ where $W_{F'}$ is the fixer of F' in W^v. Therefore $d(C, C') \leq \max\{\ell(x) | \ x \in W_{F'}\}$ ($W_{F'}$ is finite by definition of spherical). Lemma 5.3.7 concludes the proof. □

Lemma 5.3.10. *Let $(E_1, E_2), (E_1', E_2') \in \Delta_F \times_\leq \Delta_F$. Then $d^F(E_1, E_2) = d^F(E_1', E_2')$ if and only if there exists an isomorphism $\phi : (E_1, E_2) \mapsto (E_1', E_2')$.*

Proof. Suppose that $d^F(E_1, E_2) = d^F(E_1', E_2') = [R]$. Let $\psi : (E_1, E_2) \mapsto (F, R)$ and $\psi' : (E_1', E_2') \mapsto (F, R)$. Then $\phi = \psi'^{-1} \circ \psi : (E_1, E_2) \mapsto (E_1', E_2')$.

Suppose that there exists an isomorphism $\phi : (E_1, E_2) \mapsto (E_1', E_2')$. Let $R \in d^F(E_1, E_2)$ and $\psi : (E_1, E_2) \mapsto (F, R)$. Then $\phi^{-1} \circ \psi : (E_1', E_2') \mapsto (F, R)$ and thus $d^F(E_1', E_2') = [R] = d^F(E_1, E_2)$, which proves the lemma. □

Lemma 5.3.11. *Let $(F_1, F_2) \in \Delta_F \times_\leq \Delta_F$ and $r := d^F(F_1, F_2)$. Let $R \in \Delta^{\mathbb{A}}_{\geq F}$ such that $r = W_F.R = [R]$. Then if C_1, C_2 are two type 0 chambers dominating F_1 and F_2, $d^W(C_1, C_2) \in \mathcal{C}_{\mathbb{A}}(r)$ where $\mathcal{C}_{\mathbb{A}}(r)$ is the set of chambers of $\mathbb{A}$ containing an element of $r = W_F.R$. Moreover $\mathcal{C}_{\mathbb{A}}(r)$ is finite.*

Proof. Let A be an apartment containing C_1 and C_2 and $\phi : (A, C_1) \mapsto (\mathbb{A}, C_0^+)$. Then $\phi(F_1)$ is the face of C_0^+ of type type(F): $\phi(F_1) = F$. Therefore $\phi(F_2) \in W_F.R = r$ and thus $d^W(C_1, C_2) \in \mathcal{C}_{\mathbb{A}}(r)$.

Using the type, we get that if $w \in W_F$, the set of type 0 chambers of $\mathbb{A}$ containing $w.R$ is in bijection with the fixer of R in W, which is conjugated to W_F and the lemma follows. □

Lemma 5.3.12. *Let $(F_1, F_2) \in \Delta_F \times_\leq \Delta_F$ and $r_1, r_2 \in [\Delta_F]$. Then the set $\mathcal{S}^F(F_1, r_1) \cap \mathcal{S}^F_{op}(F_2, r_2)$ is finite. Its cardinal depends only on r_1, r_2 and $r := d^F(F_1, F_2)$ and we denote it by $a^r_{r_1,r_2}$.*

Proof. Let $\mathcal{S}$ be the set of type 0 chambers containing an element of $\mathcal{S}^F(F_1, r_1) \cap \mathcal{S}^F_{op}(F_2, r_2)$. Let C_1 (resp. C_2) be a type 0 chamber containing F_1 (resp. F_2).

By Lemma 5.3.11, if $C \in \mathcal{S}$, one has $d^W(C_1, C) \in \mathcal{C}_{\mathbb{A}}(r_1)$ and $d^W(C, C_2) \in \mathcal{C}_{\mathbb{A}}(r_2)$. Consequently,

$$\mathcal{S} \subset \bigcup_{w_1 \in \mathcal{C}_{\mathbb{A}}(r_1), w_2 \in \mathcal{C}_{\mathbb{A}}(r_2)} \{C \in \mathcal{C}_0^+ | \ C_1 \leq C \leq C_2, \ d^W(C_1, C) = w_1 \text{ and } d^W(C, C_2) = w_2\}.$$

By Lemma 5.3.11 and Proposition 2.3 of [BPGR16], $\mathcal{S}$ is finite. Hence

$$\mathcal{S}^F(F_1, r_1) \cap \mathcal{S}^F_{op}(F_2, r_2)$$

is finite.

It remains to prove the invariance of the cardinals. Let $(F_1', F_2') \in \Delta_F \times_\leq \Delta_F$ such that $d^F(F_1', F_2') = r$ and $\phi : (F_1, F_2) \mapsto (F_1', F_2')$, which exists by Lemma 5.3.10. Then $\mathcal{S}^F(F_1', r_1) \cap \mathcal{S}^F_{op}(F_2', r_2) = \phi\big(\mathcal{S}^F(F_1, r_1) \cap \mathcal{S}^F_{op}(F_2, r_2)\big)$, which proves the lemma. □

Lemma 5.3.13. *Let $r_1, r_2 \in [\Delta_F]$ and*

$$P_{r_1,r_2} := \{d^F(F_1, F_2) | (F_1, F', F_2) \in \Delta_F \times_\leq \Delta_F \times_\leq \Delta_F, d^F(F_1, F') = r_1 \text{ and } d^F(F', F_2) = r_2\}.$$

Then P_{r_1,r_2} is finite.

Proof. Let $\mathcal{E}$ be the set of triples (C_1, C', C_2) of type 0 chambers such that for some faces F_1, F' and F_2 (with $F_1 \subset C_1$, ...) of these chambers, $d^F(F_1, F') = r_1$ and $d^F(F', F_2) = r_2$.

Let $(C_1, C', C_2) \in \mathcal{E}$. By Lemma 5.3.11, $d^W(C_1, C') \in \mathcal{C}_{\mathbb{A}}(r_1)$ and $d^W(C', C_2) \in \mathcal{C}_{\mathbb{A}}(r_2)$. Therefore,

$$P := \{d^W(C_1, C_2) | (C_1, C', C_2) \in \mathcal{E}\} \subset \bigcup_{\mathbf{w_1} \in \mathcal{C}_{\mathbb{A}}(r_1), \mathbf{w_2} \in \mathcal{C}_{\mathbb{A}}(r_2)} P_{\mathbf{w_1}, \mathbf{w_2}},$$

where the $P_{\mathbf{w_1}, \mathbf{w_2}}$ are as in Proposition 2.2 of [BPGR16] (or in the statement of this lemma). Thus P is finite.

Let $(F_1, F', F_2) \in \Delta_F \times_{\leq} \Delta_F \times_{\leq} \Delta_F$ such that $d^F(F_1, F') = r_1$ and $d^F(F', F_2) = r_2$. Then F_1 and F_2 are some faces of C_1 and C_2, for some $(C_1, C', C_2) \in \mathcal{E}$. The distance $d^F(F_1, F_2)$ is $W_F.F''$ for some face F'' of $d^W(C_1, C_2)$, which proves the lemma. □

Let $\mathcal{R}$ be a unitary and commutative ring and let ${}^F\mathcal{H} = {}^F\mathcal{H}^{\mathcal{I}}_{\mathcal{R}}$ be the set of functions from $G\backslash\Delta_F \times_{\leq} \Delta_F$ to $\mathcal{R}$. Let $r \in [\Delta_F]$. One defines $T_r : \Delta_F \times_{\leq} \Delta_F \to \mathcal{R}$ by $T_r(F_1, F_2) = \delta_{d^F(F_1, F_2), r}$ for all $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$. Then ${}^F\mathcal{H}$ is a free $\mathcal{R}$-module with basis T_r, for $r \in [\Delta_F]$.

Theorem 5.3.14. *We equip ${}^F\mathcal{H}$ with a product $* : {}^F\mathcal{H} \times {}^F\mathcal{H} \to {}^F\mathcal{H}$ defined as follows: if $\phi_1, \phi_2 \in {}^F\mathcal{H}^{\mathcal{I}}_{\mathcal{R}}$,*

$$\phi_1 * \phi_2(F_1, F_2) = \sum_{F' \in \Delta_F | F_1 \leq F' \leq F_2} \phi_1(F_1, F')\phi_2(F', F_2)$$

for all $(F_1, F_2) \in \Delta_F \times_{\leq} \Delta_F$. This product is well defined and equips ${}^F\mathcal{H}$ with a structure of associative algebra with identity element $T_{\{F\}}$. Moreover, if $r_1, r_2 \in [\Delta_F]$,

$$T_{r_1} * T_{r_2} = \sum_{r \in P_{r_1, r_2}} a^r_{r_1, r_2} T_r.$$

Proof. The fact that $*$ is well defined and the expression of $T_{r_1} * T_{r_2}$ are consequences of Lemma 5.3.12 and of Lemma 5.3.13. The associativity is clear from the definition. The fact that $T_{\{F\}}$ is the identity element comes from the fact that $\mathcal{S}^F(F_1, [F]) = \{F_1\}$ for all $F_1 \in \Delta_F$. □

Definition 5.3.15. *The algebra ${}^F\mathcal{H} = {}^F\mathcal{H}^{\mathcal{I}}_{\mathcal{R}}$ is the **Hecke algebra of $\mathcal{I}$ associated to** F **over** $\mathcal{R}$. When $F = C_O^+$, ${}^{C_0^+}\mathcal{H} = {}^{C_0^+}\mathcal{H}^{\mathcal{I}}_{\mathcal{R}}$ is the **Iwahori-Hecke algebra of $\mathcal{I}$ over** $\mathcal{R}$*

Remark 5.3.16. *One has a bijection $f : G\backslash\Delta_F \times_{\leq} \Delta_F \xrightarrow{\sim} K_F\backslash G^+/K_F$. This map is defined as follows: let $(F_1, F_2) \in G\backslash\Delta_F \times_{\leq} \Delta_F$. One can suppose that $F_1 = F$. One has $F_2 = g.F$ for some $g \in G$ and one sets $f(g) = K_F g K_F$. Then it is easy to see that f is well defined and is a bijection. One identifies ${}^F\mathcal{H}$ and the set of functions from $K_F\backslash G^+/K_F$ to $\mathcal{R}$.*

Through this identification, $e_g = \mathbb{1}_{K_F g K_F}$ corresponds to $T_{[g.F]}$ for all $g \in G^+$. If $g, g' \in K_F\backslash G^+/K_F$, one has $e_g * e_{g'} = \sum_{g'' \in K_F\backslash G^+/K_F} m(g, g'; g'') e_{g''}$, where $m(g, g'; g'') = a^{[g''.F]}_{[g.F],[g'.F]}$ for all $g'' \in K_F\backslash G^+/K_F$.

By Lemma 5.3.5 and Lemma 5.3.12, $m(g, g'; g'') = |(K_F g K_F \cap g'' K_F g'^{-1} K_F)/K_F|$ for all $g'' \in K_F\backslash G^+/K_F$.

5.3.4 Case of a non-spherical type 0 face

In [GR14], Gaussent and Rousseau associate an algebra (the spherical Hecke algebra) to the type 0 face F_0. By Remark 5.3.4, their distance d^v correspond to d^{F_0}. It seems natural to try to associate a Hecke algebra to each type 0 face F between F_0 and C_0^+. Let F be a non-spherical type 0 face such that $F_0 \subsetneq F \subsetneq C_0^+$ (as a consequence A is an indefinite Kac-Moody matrix of size at least 3 because if A is of finite type, all type 0 faces are spherical and if A is of affine type, F_0 is the only non-spherical type 0 face of C_0^+). As we will see in this section, the definition of the product as above leads to infinite coefficients. To prove this, we use the fact that the restriction map which associates $w_{|Q_{\mathbb{Z}}^\vee}$ to each $w \in W^v$ is injective.

Lemma 5.3.17. *The map* $\begin{array}{c} W^v \to \mathrm{Aut}_{\mathbb{Z}}(Q_{\mathbb{Z}}^\vee) \\ w \mapsto w_{|Q_{\mathbb{Z}}^\vee} \end{array}$ *is injective.*

Proof. Using Lemma 2.2.1,one writes $\mathbb{A} = \mathbb{A}_0 \oplus \mathbb{A}'$, with $\mathbb{A}_0 \supset \mathbb{Q}_{\mathbb{R}}^\vee$ the minimal free realization of the Kac-Moody matrix A, and $\mathbb{A}' \subset \mathbb{A}_{in}$. For all $x \in \mathbb{A}$ and $w \in W^v$, $w(x) - x \in Q_{\mathbb{R}}^\vee$. Therefore, $\mathbb{A}'$ is stable by W^v. Moreover, for all $x \in \mathbb{A}_{in}$, $w(x) = x$. Hence the restriction map $W^v \to W_{\mathbb{A}_0}^v$ is a an isomorphism. As a consequence, one can suppose that $\mathbb{A} = \mathbb{A}_0$. But by the assertion (3.12.1) of proof of Proposition 3.12 of [Kac94] (applied to $\Delta^\vee$ instead of Δ), if $w \in W_{\mathbb{A}_0}^v$ satisfies $w_{|\Delta^\vee} = 1$ then $w = 1$ (where $\Delta^\vee$ is a set included in $Q^\vee$). This proves the lemma. □

We suppose that there exists a non-spherical type 0 face F of $\mathbb{A}$ satisfying $F_0 \subsetneq F \subsetneq C_0^+$.

Lemma 5.3.18. *Let F be a non-spherical type* 0 *face of* $\mathbb{A}$ *of type* 0*. Then its fixer W_F in W is infinite.*

Proof. One can suppose that the vertex of F is 0. Then $W_F \subset W^v$. Let F^v be the vectorial face such that $F = F^\ell(0, F^v)$. Let us prove that W_F is the fixer W_{F^v} of F^v. Let $w \in W_F$. Let $X \in F$ such that w fixes X. Then w fixes $\mathbb{R}_+^*.X \supset F^v$ and thus $w \in W_{F^v}$. Let $w \in W_{F^v}$. As 0 is special, $F^v \in F$. Therefore $W_{F^v} \subset W_F$ and thus $W_F = W_{F^v}$. As F^v is non spherical, W_{F^v} is infinite and the lemma follows. □

Remark 5.3.19. *The vectorial faces based at* 0 *form a partition of the Tits cone. Therefore, if F^v is a vectorial face and if for some $u \in F^v$ and $w \in W^v$, $w.u \in F^v$, then $w.F^v = F^v$. Hence if $W' \subset W^v$, $W'.F^v$ is infinite if and only if $W'.u$ is infinite for some $u \in F^v$ if and only if $W'.u$ is infinite for all $u \in F^v$.*

For the next proposition, we use the graph of the matrix A, whose vertices are the $i \in I$ and whose arrows are the $\{i, j\}$ such that $a_{i,j} \neq 0$.

Lemma 5.3.20. *Suppose that the matrix A is indecomposable. Let F be a non-spherical type* 0 *face F of $\mathbb{A}$ satisfying $F_0 \subsetneq F \subsetneq C_0^+$. Then there exists $w \in W^v$ such that $W_F.w.F$ is infinite.*

Proof. One writes $F = F^\ell(0, F^v)$, with $F^v = \{u \in \mathbb{A} | \alpha_i(x) > 0\ \forall i \in J$ and $\alpha_i(x) = 0\ \forall i \in I \backslash J\}$ for some $J \subset I$. By Lemma 5.3.17, there exists $k \in I$ such that $W_F.\alpha_k^\vee$ is infinite.

Let $i \in J$ ($J \neq \emptyset$ because $F_0 \subsetneq F$) . By 4.7 of [Kac94], the graph of A is connected. Therefore, there exists a sequence $j_1 = i, \ldots, j_\ell = k$ such that $a_{j_1,j_2} a_{j_2,j_3} \ldots a_{j_{l-1},j_\ell} \neq 0$.

Let $u \in F^v$. Let us show that there exists $w \in W^v$ such that $\alpha_k(w.u) \neq 0$. If $x \in \mathbb{A}$ and $m \in [\![1, \ell]\!]$, one says that x satisfies P_m if for all $m' \in [\![m+1, \ell]\!]$, $\alpha_{j_{m'}}(x) = 0$ and $\alpha_{j_m}(x) \neq 0$. Let $x \in \mathbb{A}$, $m \in [\![1, \ell - 1]\!]$ and suppose that x satisfies P_m. Let $x' = r_{j_m}(x) = x - \alpha_{j_m}(x)\alpha_{j_m}^\vee$. Then $\alpha_{j_{m+1}}(x') = -\alpha_{j_m}(x) a_{j_m, j_{m+1}} \neq 0$ and thus x' satisfies $P_{m'}$ for some $m' \in [\![m+1, l]\!]$. As $i = j_1 \in J$, $\alpha_{j_1}(u) \neq 0$ and hence u satisfies P_m for some $m \in [\![1, \ell]\!]$. Therefore, there exists $w \in W^v$ such that $w(u)$ satisfies P_ℓ: $\alpha_k(w(u)) \neq 0$.

If $W_F.w(u)$ is finite, $W_F.r_k(w(u)) = W_F.(u - \alpha_k(w(u))\alpha_k^\vee)$ is infinite and thus at least one of the sets $W_F.w(u)$ and $W_F.r_k(w(u))$ is infinite. This proves the lemma by Remark 5.3.19. □

Let $A_1, \dots, A_r$ be the indecomposable components of the Kac-Moody matrix A. One writes $\mathbb{A} = \mathbb{A}_1 \oplus \dots \oplus \mathbb{A}_r$, where $\mathbb{A}_i$ is a realization of A_i for all $i \in [\![1, r]\!]$. One has $W^v = W_1^v \times \dots \times W_r^v$, where for all $i \in [\![1, r]\!]$, W_i^v is the vectorial Weyl group of $\mathbb{A}_i$.

Proposition 5.3.21. *Let F be a type 0 face of $\mathbb{A}$. One writes $F = \bigoplus_{i=1}^r F_i$. Then there exists $w \in W^v$ such that $W_F.w.F$ is infinite if and only if there exists $i \in [\![1, r]\!]$ such that F_i is non-spherical and different from $F_{i,0}$, where $F_{i,0}$ is the minimal type 0 face based at 0 of $\mathbb{A}_i$ ($F_{i,0} = F^\ell(0, \mathbb{A}_{i,in})$).*

Proof. With obvious notation, one has $W_F = W_{F_1} \times \dots \times W_{F_r}$. Suppose that there exists $w \in W^v$ such that $W_F.w.F$ is infinite. One writes $w = (w_1, \dots, w_r)$. Then $W_F.w.F = W_{F_1}.w_1.F_1 \oplus \dots \oplus W_{F_r}.w_r.F_r$ and thus for some $i \in [\![1, r]\!]$, $W_{F_i}.w_i.F_i$ is infinite. If F_i is spherical, W_{F_i} is finite and if $F_i = F_{i,0}$, $W_{F_i}.w_i.F_i = F_{i,0}$ and thus F_i is non-spherical and different from $F_{i,0}$.

The reciprocal is a consequence of Lemma 5.3.20. □

The following proposition is a counterexample to Lemma 5.3.12 when we do not consider spherical type 0 faces.

Proposition 5.3.22. *Let F be a face based at 0 such that for some $w \in W^v$, $W_F.w.F$ is infinite. Then*

$$\mathcal{S}^F(F, [w.F]) \cap \mathcal{S}^F_{op}(F, [w^{-1}.F])$$

is infinite.

Proof. Let us prove that $W_F.w.F \subset \mathcal{S}^F(F, [w.F]) \cap \mathcal{S}^F_{op}(F, [w^{-1}.F])$. Let $E \in W_F.w.F$. Then $F \leq E \leq F$. By definition, $d^F(F, E) = [E] = [w.F]$. One writes $E = w_F.w.F$, with $w_F \in W_F$. The map $(w_F.w)^{-1} : \mathbb{A} \to \mathbb{A}$ sends E on F and F on $w^{-1}.w_F^{-1}.F = w^{-1}.F$ and thus $d^F(E, F) = [w^{-1}.F]$. Therefore $W_F.w.F \subset \mathcal{S}^F(F, [w.F]) \cap \mathcal{S}^F_{op}(F, [w^{-1}.F])$ and the proposition follows. □

Let $A_1, \dots, A_r$ be the indecomposable components of the matrix A. Let J^f be the set of $j \in [\![1, r]\!]$ such that A_j is of finite type (see Theorem 4.3 of [Kac94]) and $J^\infty = [\![1, r]\!] \backslash J^f$. Let $\mathbb{A}^f = \bigoplus_{j \in J^f} \mathbb{A}_j$. If $j \in [\![1, r]\!]$, one sets $\mathbb{A}_{j,in} = \bigcap_{\alpha \in \Phi_{re,j}} \ker \alpha$, where $\Phi_{re,j}$ is the root system of $\mathbb{A}_j$. One sets $\mathbb{A}_{in}^\infty = \bigoplus_{j \in J^\infty} \mathbb{A}_{j,in}$, $Y^f = Y \cap \mathbb{A}^f$ and $Y_{in}^\infty = Y \cap \mathbb{A}_{in}^\infty$.

Corollary 5.3.23. *Let $\lambda \in Y^+$. Then $W^v.\lambda$ is finite if and only if $\lambda \in Y^f \oplus Y_{in}^\infty$.*

Proof. Let $\lambda \in Y^+$. One writes $\lambda = \sum_{j \in [\![1,r]\!]} \lambda_j$, with $\lambda_j \in \mathbb{A}_j$ for all $j \in [\![1, r]\!]$.

Suppose that $\lambda \in Y^f \oplus Y_{in}^\infty$. Then $W^v.\lambda = \bigoplus_{j \in J^f} W_j^v.\lambda_j \oplus \bigoplus_{j \in J^\infty} \lambda_j$. As W_j^v is finite for all $j \in J^f$, one gets one implication.

Suppose $\lambda \notin Y^f \oplus Y_{in}^\infty$. Let $j \in J^\infty$ such that $\lambda_j \notin \mathbb{A}_{j,in}$. Let F_j^v be the vectorial face of $\mathbb{A}_j$ containing λ_j. By Remark 5.3.19, the map $\begin{matrix} W_j^v.F_j^v \to W_j^v.\lambda_j \\ w.F_j^v \to w.\lambda_j \end{matrix}$ is well defined and is a bijection. If F_j^v is spherical, $W_j^v.F_j^v$ is infinite because the stabilizer of F_j^v is finite. Suppose F_j^v non-spherical. Then by Lemma 5.3.20, there exists $w_j \in W_j^v$ such that $W_{F_j}.w_j.F_j^v$ is infinite, where W_{F_j} is the fixer of F_j^v in W_j^v. Therefore $W_j^v.F_j^v$ is infinite and the lemma follows. □

5.4 Almost finite sets and Looijenga's algebra

In Subsection 5.4.1, we define the almost finite sets of Y and Y^+, which are involved in the definition of the Looijenga's algebra, in the definition of the spherical Hecke algebra and in the definition of the completed Iwahori-Hecke algebra.

In Subsection 5.4.2, we define the Looijenga's algebra of Y, which is some kind of completion of the group algebra of Y.

In Subsection 5.4.3, we define the spherical Hecke algebra of G and the Satake isomorphism.

5.4.1 Almost finite sets

Definition 5.4.1. *A set $E \subset Y$ is said to be **almost finite** if there exists a finite set $J \subset Y$ such that for all $\lambda \in E$ there exists $\nu \in J$ such that $\lambda \leq_{Q^\vee_\mathbb{Z}} \nu$.*

Remark 5.4.2. *We will also use almost finite sets included in Y^+ and thus we could define an almost finite set of Y^+ as follows: a set $E \subset Y^+$ is almost finite if there exists a finite set $J \subset Y^+$ such that for all $\lambda \in E$, there exists $\nu \in J$ such that $\lambda \leq_{Q^\vee_\mathbb{Z}} \nu$. This is actually the same definition by Lemma 5.4.4 below applied to $F = Y^+$.*

Let $\ell \in \mathbb{N}^*$. Let us define a binary relation $\prec$ on $\mathbb{N}^\ell$. Let $x, y \in \mathbb{N}^\ell$, and $x = (x_1, \dots, x_\ell)$, $y = (y_1, \dots, y_\ell)$, then one says $x \prec y$ if $x \neq y$ and for all $i \in [\![1, \ell]\!]$, $x_i \leq y_i$.

Lemma 5.4.3. *Let $\ell \in \mathbb{N}^*$ and F be a subset of $\mathbb{N}^\ell$ satisfying property (INC(ℓ)):*
for all $x, y \in F$, x and y are not comparable for $\prec$.
Then F is finite.

Proof. This is clear for $\ell = 1$ because a set F satisfying INC(1) is a singleton or $\emptyset$.

Suppose that $\ell > 1$ and that we have proved that all set satisfying INC($\ell - 1$) is finite.

Let F be a set satisfying INC(ℓ) and suppose F infinite. Let $(\lambda_n)_{n\in\mathbb{N}}$ be an injective sequence of F. One writes $(\lambda_n) = (\lambda_n^1, \dots, \lambda_n^\ell)$. Let $M = \max \lambda_0$ and, for any $n \in \mathbb{N}$, $m_n = \min(\lambda_n)$. Then for all $n \in \mathbb{N}$, $m_n \leq M$ (if $m_n > M$, we would have $\lambda_0 \prec \lambda_n$). Maybe extracting a sequence of λ, one can suppose that $(m_n) = (\lambda_n^i)$ for some $i \in [\![1, l]\!]$ and that (m_n) is constant and equal to $m_0 \in [\![0, M]\!]$. For $x \in \mathbb{N}^\ell$, we define $\widetilde{x} = (x_j)_{j \in [\![1,\ell]\!] \setminus \{i\}} \in \mathbb{N}^{l-1}$.

Set $\widetilde{F} = \{\widetilde{\lambda_n} | n \in \mathbb{N}\}$. The set $\widetilde{F}$ satisfies INC($\ell - 1$). By the induction hypothesis, $\widetilde{F}$ is finite and thus F is finite, which is absurd. Hence F is finite and the lemma is proved. □

Lemma 5.4.4. *Let $E \subset Y$ be an almost finite set and $F \subset Y$. Then there exists a finite set $J \subset F$ such that $F \cap E \subset \bigcup_{j \in J} j - Q^\vee_\mathbb{N}$.*

Proof. One can suppose that $E \subset y - Q^\vee_\mathbb{N}$, for some $y \in Y$. Let J be the set of elements of $F \cap E$ which are maximal in $F \cap E$ for $\leq_{Q^\vee_\mathbb{Z}}$. As E is almost finite, for all $x \in E$, there exists $\nu \in J$ such that $x \leq_{Q^\vee_\mathbb{Z}} \nu$. It remains to prove that J is finite. Let $J' = \{u \in Q^\vee_\mathbb{N} | y - u \in J\}$. One identifies $Q^\vee_\mathbb{Z}$ and $\mathbb{Z}^I$. If $x = (x_i)_{i\in I}$ and $x' = (x'_i)_{i \in I}$ one says that $x \prec x'$ if $x_i \leq x'_i$ for all $i \in I$ and $x \neq x'$. Then the elements of J' are pairwise non comparable. Therefore J' is finite by Lemma 5.4.3, which completes the proof. □

We will use almost finite sets of Y^+ in the definition of the completed Iwahori-Hecke algebra. We now describe the almost finite sets of Y^+ in the affine case and in the dimension 2 indefinite case.

Affine case Suppose that $\mathbb{A}$ is associated to an affine Kac-Moody matrix A. By 2.3.1, if δ is the smallest positive imaginary root of A, $\mathcal{T} = \delta^{-1}(\mathbb{R}_+^*) \sqcup \mathbb{A}_{in}$. Moreover, δ is W^v-invariant (see 5.10 of [Rou11]) and thus $\delta(\alpha_i^\vee) = 0$ for all $i \in I$. Therefore an almost finite set of Y^+ is a set E such that for some $k \in \mathbb{N}$ and $y_1, \ldots, y_k \in Y^+$, $E \subset \bigcup_{i=1}^k y_i - Q_\mathbb{N}^\vee$.

Indefinite case On the contrary to the finite and affine case, when $\mathbb{A}$ is associated to an indefinite Kac-Moody matrix, $y - Q_\mathbb{N}^\vee \nsubseteq Y^+$ for all $y \in Y$. Indeed, by the Lemma of 2.9 of [GR14] and its proof, there exists a linear form δ on $\mathbb{A}$ such that $\delta(\mathcal{T}) \geq 0$ and $\delta(\alpha_i^\vee) < 0$ for all $i \in I$. If $y \in Y$ and $i \in I$, then $\delta(y - n\alpha_i^\vee) < 0$ for $n \in \mathbb{N}$ large enough and thus $y - Q_\mathbb{N}^\vee \nsubseteq Y^+$. However, the inclusion $-Q_\mathbb{N}^\vee \supset Y^+$ can occur.

Lemma 5.4.5. *One has $-Q_\mathbb{N}^\vee \supset Y^+$ if and only if $-Q_\mathbb{N}^\vee \supset Y^{++}$.*

Proof. One implication is clear because $Y^{++} \subset Y^+$. Suppose $-Q_\mathbb{N}^\vee \supset Y^{++}$. Let $\lambda \in Y^+$. By Lemma 2.2.9, $\lambda \leq_{Q_\mathbb{Z}^\vee} \lambda^{++}$ and thus $\lambda \in -Q_\mathbb{N}^\vee$. □

Suppose that $\mathbb{A}$ is essential and associated to a size 2 indefinite matrix A. One writes $A = \begin{pmatrix} 2 & a_{1,2} \\ a_{2,1} & 2 \end{pmatrix}$, with $a_{1,2}, a_{2,1} \in -\mathbb{N}^*$. One has $\mathbb{A} = \mathbb{R}\alpha_1^\vee \oplus \mathbb{R}\alpha_2^\vee$. If $(\lambda, \mu) \in \mathbb{Z}^2$ such that $\lambda\mu < 0$ then $(2\lambda + a_{1,2}\mu)(a_{2,1}\lambda + 2\mu) < 0$ and thus $Y^{++} \subset Q_\mathbb{N}^\vee \cup -Q_\mathbb{N}^\vee$. By Theorem 4.3 of [Kac94], $Q_\mathbb{N}^\vee \cap Y^{++} = \{0\}$, thus $-Q_\mathbb{N}^\vee \supset Y^{++}$ and hence $-Q_\mathbb{N}^\vee \supset Y^+$. Therefore an almost finite set of Y^+ is just a subset of Y^+.

However this property is not true in general when the size of A is greater or equal to 3. For example if $A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -5 & 0 & 2 \end{pmatrix}$ and $\mathbb{A}$ is the essential realization of A, $-2\alpha_1^\vee + \alpha_2^\vee - \alpha_3^\vee \in Y^{++} \backslash -Q_\mathbb{N}^\vee$.

5.4.2 Looijenga's algebra

Let $\mathcal{R}$ be a ring.

Definition 5.4.6. *Let $\mathcal{R}[[Y]]$ be the vector space of formal series $\sum_{\lambda \in Y} a_\lambda e^\lambda$, with $(a_\lambda) \in \mathcal{R}^Y$ such that* $\mathrm{supp}((a_\lambda)) \subset Y$ *is almost finite. One equips it with the product $\sum_{\lambda \in Y} a_\lambda e^\lambda . \sum_{\lambda \in Y} b_\mu e^\mu = \sum_{\nu \in Y}(\sum_{\lambda+\mu=\nu} a_\lambda b_\mu) e^\nu$. Then $(\mathcal{R}[[Y]], .)$ is an associative algebra: this is the* ***Looijenga's algebra of*** Y ***over*** $\mathcal{R}$*. It was introduced in [Loo80].*

For all $\lambda \in Y$, one defines $\pi_\lambda : \mathcal{R}[[Y]] \to \mathcal{R}$ by $\pi_\lambda(\sum_{\mu \in Y} a_\mu e^\mu) = a_\lambda$. One sets $\mathcal{R}[[Y^+]] = \{a \in \mathcal{R}[[Y]] \mid \pi_\lambda(a) = 0 \ \forall \lambda \in Y \backslash Y^+\}$. One also sets $\mathcal{R}[[Y]]^{W^v} = \{a \in \mathcal{R}[[Y]] \mid \pi_\lambda(a) = \pi_{w(\lambda)}(a) \ \forall (\lambda, w) \in Y \times W^v\}$. Then $\mathcal{R}[[Y^+]]$ and $\mathcal{R}[[Y]]^{W^v}$ are sub-algebras of $\mathcal{R}[[Y]]$.

One denotes by $\mathrm{AF}_\mathcal{R}(Y^{++})$ the set of elements of $\mathcal{R}^{Y^{++}}$ having almost finite support. A family $(a_j)_{j \in J} \in (\mathcal{R}[[Y]])^J$ is said to be summable if:

- for all $\lambda \in Y$, $\{j \in J | \pi_\lambda(a_j) \neq 0\}$ is finite
- the set $\{\lambda \in Y \ | \exists j \in J | \pi_\lambda(a_j) \neq 0\}$ is almost finite.

In this case, one sets $\sum_{j \in J} a_j = \sum_{\lambda \in Y} b_\lambda e^\lambda \in R[[Y]]$, where $b_\lambda = \sum_{j \in J} \pi_\lambda(a_j)$ for all $\lambda \in Y$.

For all $\lambda \in Y^{++}$, one sets $E(\lambda) = \sum_{\mu \in W^v.\lambda} e^\mu \in \mathcal{R}[[Y]]$ (this is well defined by Lemma 2.2.9). Let $\lambda \in \mathcal{T}$. There exists a unique $\mu \in \overline{C_f^v}$ such that $W^v.\lambda = W^v.\mu$. One defines $\lambda^{++} = \mu$. The following two results are proved (but not stated) in the proof of Theorem 5.4 of [GR14].

Lemma 5.4.7. *Let $x \in Y$. Then $W^v.x$ is majorized for $\leq_{Q^\vee_\mathbb{Z}}$ if and only if $x \in Y^+$.*

Proof. If $x \in Y^+$, then $W.x$ is majorized by x^{++} by Lemma 2.2.9.

Let $x \in Y$ such that $W^v.x$ is majorized. Let $y \in W^v.x$ be maximal for $\leq_{Q^\vee_\mathbb{Z}}$ and $i \in I$. One has $r_i(y) \leq_{Q^\vee_\mathbb{Z}} y$ and thus $\alpha_i(y) \geq 0$. Therefore $y \in \overline{C^v_f}$, which proves that $x \in Y^+$. □

Proposition 5.4.8. *The map $E : \mathrm{AF}_\mathcal{R}(Y^{++}) \to \mathcal{R}[[Y]]^{W^v}$ defined by*

$$E((x_\lambda)) = \sum_{\lambda \in Y^{++}} x_\lambda E(\lambda)$$

is well defined and is a bijection. In particular, $\mathcal{R}[[Y]]^{W^v} \subset \mathcal{R}[[Y^+]]$.

Proof. Let $(x_\lambda) \in \mathrm{AF}_\mathcal{R}(Y^{++})$ and J be a finite set such that for all $\mu \in \mathrm{supp}((x_\lambda))$ there exists $j \in J$ such that $\mu \leq_{Q^\vee_\mathbb{Z}} j$. Let us prove that $(x_\lambda E(\lambda))_{\lambda \in Y^+}$ is summable. Let $\nu \in Y$ and $F_\nu = \{\lambda \in Y^{++} |\ \pi_\nu(x_\lambda E(\lambda)) \neq 0\}$. Let $\lambda \in F_\nu$. Then $\nu \in W^v.\lambda$ and by Lemma 2.2.9, $\nu \leq_{Q^\vee_\mathbb{Z}} \lambda$. Moreover, $\lambda \leq_{Q^\vee_\mathbb{Z}} j$ for some $j \in J$, which proves that F_ν is finite.

Let $F = \{\nu \in Y |\ \exists \lambda \in Y^{++} |\ \pi_\nu(x_\lambda E(\lambda)) \neq 0\}$. If $\nu \in F$ then $\nu \leq_{Q^\vee_\mathbb{Z}} j$ for some $j \in J$ and thus F is almost finite: the family $(x_\lambda E(\lambda))_{\lambda \in Y^{++}}$ is summable.

As for all $\lambda \in Y^{++}$, $E(\lambda) \in \mathcal{R}[[Y]]^{W^v}$, $\sum_{\lambda \in Y^{++}} x_\lambda E(\lambda) \in \mathcal{R}[[Y]]^{W^v}$. Therefore, E is well-defined.

Let $(x_\lambda) \in \mathrm{AF}_\mathcal{R}(Y^{++}) \backslash \{0\}$ and $\mu \in Y^{++}$ such that x_μ is different from 0 and maximal for this property for the $Q^\vee_\mathbb{Z}$-order. Then $\pi_\mu\big(E((x_\lambda))\big) = x_\mu \neq 0$: $E((x_\lambda) \neq 0$. Therefore E is injective.

Let $u = \sum_{\lambda \in Y} u_\lambda e^\lambda \in \mathcal{R}[[Y]]^{W^v}$ and $\lambda \in \mathrm{supp}\ u$. As $\mathrm{supp}\ u$ is almost finite, $W^v.\lambda$ is majorized and by Lemma 5.4.7, $\lambda \in Y^+$. Consequently $\mathrm{supp}\ u \subset Y^+$. One has $u = E((u_\lambda)_{\lambda \in (\mathrm{supp}\ u)^{++}})$, and the proof is complete. □

5.4.3 Spherical Hecke algebra and Satake isomorphism

We now briefly recall the definitions of the spherical Hecke algebra and of the Satake isomorphism, see[GR14] for more details. Their construction uses Hecke paths, which are some particular case of λ-paths (see 4.2.2.2) and are more or less the images by retractions of segments in $\mathcal{I}$ (see 6.4 for a definition). Contrary to the case of Iwahori-Hecke and parahoric Hecke algebras, the definition of the spherical Hecke algebra involves infinite sums.

Let $\widehat{\mathcal{H}}_s = \{\varphi : \mathcal{I}_0 \times_\leq \mathcal{I}_0 \to \mathbb{C} |\ \varphi(g.x, g.y) = \varphi(x,y) \forall\ (x,y,g) \in Y \times Y \times G\}$. In [GR14], Gaussent and Rousseau define the spherical Hecke algebra of $\mathcal{I}$ as a subspace of $\widehat{\mathcal{H}}_s$. A natural convolution product on this space would be the product defined by the following formula: if $\varphi, \psi \in \widehat{\mathcal{H}}_s$,

$$(\varphi * \psi)(x,y) = \sum_{x \leq z \leq y} \varphi(x,z)\psi(z,y), \forall (x,y) \in \mathcal{I}_0. \tag{5.1}$$

For this formula to be defined, we need to impose some conditions on the support of φ and ψ. Indeed, when $\mathcal{I}$ is associated to an affine Kac-Moody group, if $x, y \in Y$ such that $\delta(x) + 2 \leq \delta(y)$ (where $\delta : \mathbb{A} \to \mathbb{R}$ is the imaginary root defined in 2.3.1.1), there exists $z \in Y$ such that $\delta(x) < \delta(z) < \delta(y)$ (one can take $z = x + d$ with the notation of 2.3.1.1). As $\delta_{|Q^\vee_\mathbb{R}} = 0$, and as $\mathcal{T} \supset \delta^{-1}(\mathbb{R}^*_+)$, for all $z' \in z + Q^\vee_\mathbb{Z}$, $x \leq z' \leq y$. Therefore $\{z' \in \mathcal{I}_0 |\ x \leq z' \leq y\}$ is infinite.

Let $K_s = \mathrm{Fix}_G(\{0\})$. Let $\widehat{\mathcal{H}}'_s = \{\varphi : Y^{++} = K \backslash G^+ / K \to \mathbb{C}\}$. Let $\Xi : \widehat{\mathcal{H}}_s \to \widehat{\mathcal{H}}'_s$ defined by $\Xi(\varphi)(K_s g K_s) = \varphi(0, g.0)$ for all $\varphi \in \widehat{\mathcal{H}}_s$ and $K_s g K_s \in K_s \backslash G^+ / K_s$. Then Ξ is an isomorphism of vector spaces and if $\varphi \in \widehat{\mathcal{H}}'_s$, $x, y \in \mathcal{I}_0 \times_\leq \mathcal{I}_0$, then $\Xi^{-1}(\varphi)(x,y) = \varphi(d^v(x,y))$. Using Ξ, we identify $\widehat{\mathcal{H}}_s$ and $\widehat{\mathcal{H}}'_s$. Then the formula for the convolution becomes, for $\varphi, \psi \in \widehat{\mathcal{H}}_s$:

$$\varphi * \psi(y) = \sum \varphi(d^v(0,z))\psi(d^v(z,y)),\ \forall y \in Y^{++}, \tag{5.2}$$

where the sum runs over the $z \in \mathcal{I}_0$ such that $0 \leq z \leq y$.

Let $\mathcal{H}_s$ be the subspace of $\mathcal{H}_s$ of functions φ with almost finite support.

By Theorem 2.6 of [GR14], the formula 5.2 equips $\mathcal{H}_s$ with a structure of an associative algebra.

If $\lambda \in Y^{++}$, one denotes by c_λ the map $Y^{++} \to \mathbb{C}$ mapping λ to 1 and each $\mu \neq \lambda$ to 0.

If $\lambda, \mu \in Y^{++}$, $c_\lambda * c_\mu = \sum_{\nu \leq_{Q^\vee_\mathbb{Z}} \lambda+\mu} m_{\lambda,\mu}(\nu)c_\nu$, where for all $\nu \in Y^{++}$,

$$m_{\lambda,\mu}(\nu) = |\{z \in \mathcal{I}_0 |\ d^v(0,z) = \lambda \text{ and } d^v(z,\nu) = \mu\}|.$$

Remark 5.4.9. *By 2.9 of [GR14], formula 5.1 defines a structure of algebra on $\widehat{\mathcal{H}}_s$ when G is associated to an indefinite Kac-Moody matrix, see also 6.6.8.*

Let $h : \mathbb{Q}^\vee_\mathbb{R} \to \mathbb{R}$ defined by $h(\sum_{i\in I} a_i\alpha_i^\vee) = \sum_{i\in I} a_i$ for all $(a_i) \in \mathbb{R}^I$.

Theorem 5.4.10. *1. Let $\mu, \lambda \in Y$. Then the number*

$$n_\lambda(\mu) := |\{x \in \mathcal{I}_0 |\ d^v(x,\mu) = \lambda \text{ and } \rho_{-\infty}(x) = 0\}|$$

is finite. Moreover $n_\lambda(\mu) \neq 0$ implies $\mu \leq_{Q^\vee_\mathbb{Z}} \lambda$.

2. Let $S : \mathcal{H}_s \to \mathbb{C}[[Y]]$ be the vector space morphism defined by

$$S(c_\lambda) = \sum_{\mu \leq_{Q^\vee_\mathbb{Z}} \lambda} n_\lambda(\mu) q^{h(\mu)} e^\mu, \forall \lambda \in Y.$$

Then S is a morphism of algebra and it induces an isomorphism $\mathcal{H}_s \xrightarrow{\sim} \mathbb{C}[[Y]]^{W^v}$. In particular $\mathcal{H}_s$ is commutative.

Proof. Point 1 is proved in Definition/Proposition 5.2 of [GR14]. To prove it, Gaussent and Rousseau use the theory of Hecke paths. Let $\Gamma(\lambda,\mu)$ be the set of preordered segments $\tau : [0,1] \to \mathcal{I}$ such that $\tau(1) = \mu$, $d^v(\tau(0),\mu) = \lambda$ and $\rho_{-\infty}(\tau(0)) = 0$. Then $n_\lambda(\mu) = |\Gamma(\lambda,\mu)|$. They prove that the image of a preordered segment $\tau : [0,1] \to \mathcal{I}$ by $\rho_{-\infty}$ is a Hecke path of shape $d^v(\tau(0),\tau(1))$ (Theorem 6.2 of [GR08]). Therefore if $\tau \in \Gamma(\lambda,\mu)$, $\rho_{-\infty} \circ \tau$ is a Hecke path of shape λ between 0 and μ. The non vanishing condition on $n_\lambda(\mu)$ is thus a consequence of Lemma 2.2.9. They prove that the number of such paths is finite (Corollary 5.9 of [GR08]). They conclude by using the fact that if $\pi : [0,1] \to \mathbb{A}$ is a Hecke path, the number of segments $\tau : [0,1] \to \mathbb{A}$ retracting on π and such that $\tau(1) = \pi(1)$ is finite (Theorem 6.3 of [GR08]).

Point 2 is Theorem 5.4 of [GR14]. The injectivity of S is a consequence of the fact that $n_\lambda(\lambda) = 1$ for all $\lambda \in Y^{++}$ and $n_\lambda(\mu) = 0$ for all $\mu \in Y \backslash (\lambda - Q^\vee_\mathbb{N})$. The most difficult part is the fact that S takes it values in $\mathbb{C}[[Y]]^{W^v}$. They prove that its is equivalent to the formula:

$$n_\lambda(r_i.\mu) = q^{\alpha_i(\mu)} n_\lambda(\mu),\ \forall i \in I, \forall (\lambda,\mu) \in Y^2.$$

By working on the extended tree $\mathbb{T}_{\alpha_i} = U_{\alpha_i}.\mathbb{A}$, for all $i \in I$, they prove that it suffices to check this formula on a homogeneous tree with valence $1+q$, which is easy.

The surjectivity of S is then obtained by constructing an antecedent to each element of $\mathbb{C}[[Y]]^{W^v}$ by some explicit procedure. $\square$

The isomorphism S above is called the **Satake isomorphism**.

5.5 Completed Iwahori-Hecke algebra

5.5.1 Iwahori-Hecke algebra

When $F = C_0^+$, the Hecke algebra of Definition 5.3.15 is the **Iwahori-Hecke** algebra of $\mathcal{I}$. In [BPGR16], following [BKP16], the authors give a Bernstein-Lusztig presentation of this algebra. We begin this section by recalling this presentation.

Let $(\sigma_i)_{i\in I}, (\sigma'_i)_{i\in I}$ be indeterminates satisfying the following relations:

- if $\alpha_i(Y) = \mathbb{Z}$, then $\sigma_i = \sigma'_i$
- if r_i, r_j $(i, j \in I)$ are conjugate (i.e if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$), then $\sigma_i = \sigma_j = \sigma'_i = \sigma'_j$.

Let $\mathcal{R}_1 = \mathbb{Z}[\sigma_i, \sigma'_i | i \in I]$. In order to define the Iwahori-Hecke algebra $\mathcal{H}$ associated to $\mathbb{A}$ and $(\sigma_i)_{i\in I}$, $(\sigma'_i)_{i\in I}$, we first introduce the Bernstein-Lusztig-Hecke algebra ${}^{BL}\mathcal{H}$. Let ${}^{BL}\mathcal{H}$ be the free $\mathcal{R}_1$-module with basis $(Z^\lambda H_w)_{\lambda\in Y, w\in W^v}$. For short, we write $H_i = H_{r_i}$, $H_w = Z^0 H_w$ and $Z^\lambda H_1 = Z^\lambda$, for $i \in I$, $\lambda \in Y^+$ and $w \in W^v$. The Iwahori-Hecke algebra ${}^{BL}\mathcal{H}$ is the module ${}^{BL}\mathcal{H}$ equipped with the unique product $*$ which makes it an associative algebra and satisfying the following relations (the Bernstein-Lusztig relations):

- (BL1) $\forall \lambda \in Y$, $Z^\lambda * H_w = Z^\lambda H_w$,
- (BL2) $\forall i \in I$, $\forall w \in W^v$, $H_i * H_w = \begin{cases} H_{r_i w} & \text{if } \ell(r_i w) = \ell(w) + 1 \\ (\sigma_i - \sigma_i^{-1}) H_w + H_{r_i w} & \text{if } \ell(r_i w) = \ell(w) - 1 \end{cases}$,
- (BL3) $\forall (\lambda, \mu) \in Y^2$, $Z^\lambda * Z^\mu = Z^{\lambda+\mu}$,
- (BL4) $\forall \lambda \in Y$, $\forall i \in I$, $H_i * Z^\lambda - Z^{r_i(\lambda)} * H_i = b(\sigma_i, \sigma'_i; Z^{-\alpha_i^\vee})(Z^\lambda - Z^{r_i(\lambda)})$, where $b(t, u; z) = \frac{(t-t^{-1}) + (u-u^{-1})z}{1-z^2}$.

The existence and unicity of such a product is Theorem 6.2 of [BPGR16]. The Iwahori-Hecke algebra $\mathcal{H}_{\mathcal{R}_1}$ associated to $\mathbb{A}$ and $(\sigma_i)_{i\in I}$, $(\sigma'_i)_{i\in I}$ over $\mathcal{R}_1$ is the submodule spanned by $(Z^\lambda H_w)_{\lambda\in Y^+, w\in W^v}$, where $Y^+ = Y \cap \mathcal{T}$ (where $\mathcal{T}$ is the Tits cone). It is isomorphic to the Iwahori-Hecke algebra defined in Definition 5.3.15. When G is reductive, we find the usual Iwahori-Hecke algebra of G.

Extension of scalars Let $(\mathcal{R}, \phi)$ be a pair such that $\mathcal{R}$ is a ring containing $\mathbb{Z}$, $\phi : \mathcal{R}_1 \to \mathcal{R}$ is a ring morphism and the $\phi(\sigma_i)$ and $\phi(\sigma'_i)$ are invertible in $\mathcal{R}$ for all $i \in I$. The Iwahori-Hecke algebra associated to $\mathbb{A}$ and $(\phi(\sigma_i))_{i\in I}$, $(\phi(\sigma'_i))_{i\in I}$ over $\mathcal{R}$ is $\mathcal{H}_\mathcal{R} = \mathcal{R} \otimes_{\mathcal{R}_1} \mathcal{H}_{\mathcal{R}_1}$.

Split Kac-Moody case When G is a split Kac-Moody group over a local field $\mathcal{K}$ with residue cardinal q, one can choose $\sigma_i = \sigma'_i = \sqrt{q}$, for all $i \in I$ and $\mathcal{R} = \mathbb{Z}[\sqrt{q}^{\pm 1}]$.

5.5.2 Completed Iwahori-Hecke algebra

In this subsection, we define the completed Iwahori-Hecke algebra $\widehat{\mathcal{H}}$. We equip W^v with its Bruhat order $\leq$. One has $1 \leq w$ for all $w \in W^v$. If $u \in W^v$, one sets $[1, u] = \{w \in W^v | w \leq u\}$.

Let $\mathcal{B} = \prod_{w\in W^v, \lambda\in Y^+} \mathcal{R}$. If $f = (a_{\lambda,w}) \in \mathcal{B}$, the set $\{(\lambda, w) \in W^v \times Y^+ | a_{\lambda,w} \neq 0\}$ is called the support of f and is denoted by $\mathrm{supp} f$, the set $\{w \in W^v | \exists \lambda \in Y^+ | a_{\lambda,w} \neq 0\}$ is called the **support of f along** W^v and denoted $\mathrm{supp}_{W^v} f$, and the set $\{\lambda \in Y^+ | \exists w \in W^v | a_{\lambda,w} \neq 0\}$

is called the **support of** f **along** Y and denoted $\mathrm{supp}_Y f$. A set $Z \subset Y^+ \times W^v$ is said to be **almost finite** if $\{w \in W^v | \exists \lambda \in Y^+ | (\lambda, w) \in Z\}$ is finite and for all $w \in W^v$, $\{\lambda \in Y^+ | (\lambda, w) \in Z\}$ is almost finite.

Let $\widehat{\mathcal{H}}$ be the set of $a \in \mathcal{B}$ such that supp a is almost finite. If $a = (a_{\lambda,w}) \in \widehat{\mathcal{H}}$, one writes $a = \sum_{(\lambda,w)\in Y^+\times W^v} a_{\lambda,w} Z^\lambda H_w$. For $(\lambda, w) \in Y^+ \times W^v$, we define $\pi_{\lambda,w} : \widehat{\mathcal{H}} \to \mathcal{R}$ by $\pi_{\lambda,w}(\sum a_{\lambda',w'} Z^{\lambda'} H_{w'}) = a_{\lambda,w}$. In order to extend $*$ to $\widehat{\mathcal{H}}$, we first prove that if $\sum a_{\lambda,w} Z^\lambda H_w$, $\sum b_{\lambda,w} Z^\lambda H_w \in \widehat{\mathcal{H}}$ and $(\mu, v) \in Y^+ \times W^v$,

$$\sum_{(\lambda,w),(\lambda',w')\in Y^+\times W^v} \pi_{\mu,v}(a_{\lambda,w} b_{\lambda',w'} Z^\lambda H_w * Z^{\lambda'} H_{w'})$$

is well defined, i.e that only a finite number of $\pi_{\mu,v}(a_{\lambda,w} b_{\lambda',w'} Z^\lambda H_w * Z^{\lambda'} H_{w'})$ are non-zero. The key ingredient to prove this is the fact that if $w \in W^v$ and $\lambda \in Y$, the support of $H_w * Z^\lambda$ along Y^+ is in the convex hull of the $u.\lambda$'s, for $u \leq w$ for the Bruhat order (this is Lemma 5.5.3).

For $E \subset Y$ and $i \in I$, one sets $R_i(E) = \mathrm{conv}(\{E, r_i(E)\}) \subset E + Q^\vee_\mathbb{Z}$. If $E = \{\lambda\}$, one writes $R_i(\lambda)$ for short. Let $w \in W^v$, one sets $R_w(\lambda) = \bigcup R_{i_1}(R_{i_2}(\ldots(R_{i_k}(\lambda)\ldots))$ where the union is taken over all the reduced writings of w.

Remark 5.5.1. *Let $\lambda \in Y$ and $w \in W^v$, $R_w(\lambda)$ is finite. Indeed, if E is a finite set and $i \in I$, $R_i(E)$ is bounded and included in $E + Q^\vee_\mathbb{Z}$. Therefore $R_i(E)$ is finite and by induction, if $k \in \mathbb{N}^*$ and $i_1, \ldots, i_k \in I$, $R_{i_1}(R_{i_2}(\ldots(R_{i_k}(E))\ldots)$ is finite. As there are at most $|I|^{\ell(w)}$ reduced writings of w, we deduce that $R_w(\lambda)$ is finite.*

Lemma 5.5.2. *Let $\nu \in Y$ and $i \in I$. Then*

$$H_i * Z^\nu \in \bigoplus_{\nu'\in R_i(\nu), t'\in\{1,r_i\}} \mathcal{R}.Z^{\nu'} H_{t'}.$$

Proof. Suppose $\sigma_i = \sigma'_i$. Then by (BL4), one has

$$H_i * Z^\nu = Z^{r_i(\nu)} * H_i + (\sigma_i - \sigma_i^{-1}) Z^\nu \frac{1 - Z^{-\alpha_i(\nu)\alpha_i^\vee}}{1 - Z^{-\alpha_i^\vee}}.$$

If $\alpha_i(\nu) = 0$, $H_i * Z^\nu = Z^\nu * H_i$.

If $\alpha_i(\nu) > 0$, $H_i * Z^\nu = Z^{r_i(\nu)} * H_i + (\sigma_i - \sigma_i^{-1}) \sum_{h=0}^{\alpha_i(\nu)-1} Z^{\nu - h\alpha_i^\vee}$ and $r_i(\nu), \nu - h\alpha_i^\vee \in R_i(\nu)$ for all $h \in [\![0, \alpha_i(\nu) - 1]\!]$.

If $\alpha_i(\nu) < 0$, $H_i * Z^\nu = Z^{r_i(\nu)} * H_i + (\sigma_i - \sigma_i^{-1}) \sum_{h=1}^{-\alpha_i(\nu)} Z^{\nu + h\alpha_i^\vee}$ and $r_i(\nu), \nu + h\alpha_i^\vee \in R_i(\nu)$ for all $h \in [\![1, -\alpha_i(\nu)]\!]$.

Consequently if $\sigma_i = \sigma'_i$, $H_i * Z^\nu \in \sum_{\nu'\in R_i(\nu), t'\in\{1,r_i\}} \mathcal{R}.Z^{\nu'} H_t$.

Suppose $\sigma_i \neq \sigma'_i$. Then $\alpha_i(Y) = 2\mathbb{Z}$. One has

$$H_i * Z^\lambda = Z^{r_i(\lambda)} * H_i + Z^\lambda \big((\sigma_i - \sigma_i^{-1}) + (\sigma'_i - \sigma_i'^{-1}) Z^{-\alpha_i^\vee}\big) \frac{1 - Z^{-\alpha_i(\lambda)\alpha_i^\vee}}{1 - Z^{-2\alpha_i^\vee}}$$

The same computations as above complete the proof.

□

Lemma 5.5.3. *For all $u, v \in W^v$ and $\mu \in Y$,*

$$H_u * Z^\mu H_v \in \bigoplus_{\nu\in R_u(\mu), t\in[1,u].v} \mathcal{R}.Z^\nu H_t.$$

Proof. We do it by induction on $\ell(u)$. Let $v \in W^v$ and $\mu \in Y$. Let $k \in \mathbb{N}^*$ and suppose that for all $w \in W^v$ such that $\ell(w) \leq k-1$, $H_w * Z^\mu H_v \in \bigoplus_{\nu \in R_w(\mu), t \in [1,w].v} \mathcal{R}.Z^\nu H_t$. Let $u \in W^v$ and suppose that $\ell(u) = k$. One writes $u = r_i w$, with $i \in I$ and $w \in W^v$ such that $\ell(w) = k-1$. One has

$$H_u * Z^\mu H_v = H_i * H_w * Z^\mu H_v \in \sum_{\nu \in R_w(\mu), t \in [1,w].v} \mathcal{R}.H_i * Z^\nu H_t.$$

By Lemma 5.5.2,

$$\sum_{\nu \in R_w(\mu), t \in [1,w].v} \mathcal{R}.H_i * Z^\nu H_t \subset \sum_{\nu \in R_w(\mu), t \in [1,w].v} \sum_{\nu' \in R_i(\nu), t' \in \{1, r_i\}} \mathcal{R}.Z^{\nu'} * H_{t'} * H_t.$$

By (BL2), if $t' \in \{1, r_i\}$ and $t \in [1,w].v$,

$$H_{t'} * H_t \in \mathcal{R}.H_{r_i.t} \oplus \mathcal{R}.H_t \subset \sum_{u' \in [1,u].v} \mathcal{R}.H_{u'},$$

and the lemma follows.

□

Lemma 5.5.4. *Let $u \in W^v$ and $\mu \in Y$. Then for all $\nu \in R_u(\mu)$, there exists $(\lambda_{u'})_{u' \leq u} \in [0,1]^{[1,u]}$ such that $\sum_{u' \leq u} \lambda_{u'} = 1$ and $\nu = \sum_{u' \leq u} \lambda_{u'} u'.\mu$.*

Proof. We do it by induction on $\ell(u)$. Let $k \in \mathbb{N}$ and suppose this is true for all u having length k. Let $\widetilde{u} \in W^v$ such that $\ell(\widetilde{u}) = k+1$. Let $\nu \in R_{\widetilde{u}}(\mu)$. Then $\nu \in R_i(\nu')$ for some $i \in I$ and $\nu' \in R_u(\mu)$, for some $u \in W^v$ having length k. One writes $\nu = s\nu' + (1-s) r_i.\nu'$, with $s \in [0,1]$. One writes $\nu' = \sum_{u' \leq u} \lambda_{u'} u'.\mu$. One has $\nu = s \sum_{u' \leq u} \lambda_{u'} u'.\mu + (1-s) \sum_{u' \leq u} \lambda_{u'} r_i.u'.\mu$. As $r_i.u' \leq \widetilde{u}$ for all $u' \leq u$, one gets the lemma. □

Lemma 5.5.5. *1. Let $\lambda, \mu \in Y^+$. Then $(\lambda + \mu)^{++} \leq_{Q^\vee_\mathbb{Z}} \lambda^{++} + \mu^{++}$.*

2. Let $\mu \in Y^+$ and $v \in W^v$. Then for all $\nu \in R_v(\mu)$, $\nu^{++} \leq_{Q^\vee_\mathbb{Z}} \mu^{++}$.

Proof. Let $\lambda, \mu \in Y^+$. Let $w \in W^v$ such that $(\lambda+\mu)^{++} = w.(\lambda+\mu)$. By Lemma 2.2.9, $w.\lambda \leq_{Q^\vee_\mathbb{Z}} \lambda^{++}$ and $w.\mu \leq_{Q^\vee_\mathbb{Z}} \mu^{++}$ and thus we get 1.

The point 2 is a consequence of Remark 5.5.1, Lemma 5.5.4, Lemma 2.2.9 and point 1. □

If $x = \sum_{i \in I} x_i \alpha_i^\vee \in Q^\vee_\mathbb{R}$, one sets $h(x) = \sum_{i \in I} x_i$. If $\lambda \in Y^+$, we denote by w_λ the element w of W^v of minimal length such that $w^{-1}.\lambda \in \overline{C^v_f}$. One has $\lambda = w_\lambda.\lambda^{++}$ for all $\lambda \in Y^+$.

Lemma 5.5.6. *Let $\lambda \in Y^{++}$ and $(\mu_n) \in (W^v.\lambda)^\mathbb{N}$ such that $\ell(w_{\mu_n}) \to +\infty$. Then $h(\mu_n - \lambda) \to -\infty$.*

Proof. The fact that $h(\mu_n - \lambda)$ is well defined is a consequence of Lemma 2.2.9. Suppose that $h(\mu_n - \lambda)$ does not converge to $-\infty$. By Lemma 2.2.9, $h(\mu_n - \lambda) \leq 0$ for all $n \in \mathbb{N}$ and thus, maybe considering a subsequence of (μ_n), one can suppose that (μ_n) is injective and that $h(\mu_n - \lambda) \to k$, for some $k \in \mathbb{Z}$. Let $(h_i)_{i \in I}$ be the dual basis of the basis $(\alpha_i^\vee)$ of $Q^\vee_\mathbb{R}$ and $\underline{h} = (h_i)_{i \in I}$. For all $i \in I$ and $n \in \mathbb{N}$, $h_i(\mu_n - \lambda) \leq 0$ and $(\underline{h}(\mu_n - \lambda))$ is injective. This is absurd and thus $h(\mu_n - \lambda) \to -\infty$. □

Lemma 5.5.7. *Let $\lambda \in Y^{++}$, $\nu \in Y^+$ and $u \in W^v$. Then $F = \{\mu \in W^v.\lambda | \nu \in R_u(\mu)\}$ is finite.*

Proof. Let $N \in \mathbb{N}$ such that for all $\nu' \in W^v.\lambda$ satisfying $\ell(w_{\nu'}) \geq N$, $h(\nu'-\lambda) < h(\nu-\lambda)$ (N exists by Lemma 5.5.6).

Let $\mu \in F$ and $w = w_\mu$. One writes $\nu = \sum_{x \leq u} \lambda_x x.\mu$, with $\lambda_x \in [0,1]$ for all $x \leq u$ and $\sum_{x\leq u} \lambda_x = 1$, which is possible by Lemma 5.5.4.

If $u' \leq u$, one sets $v(u') = w_{u'.\mu}$. Suppose that for all $u' \leq u$, $\ell(v(u')) \geq N$. One has

$$\nu - \lambda = \sum_{u' \leq u} \lambda_{u'}(u'.\mu - \lambda) = \sum_{u'\leq u} \lambda_{u'}(v(u').\mu - \lambda)$$

and thus

$$h(\nu-\lambda) = \sum_{u'\leq u} \lambda_{u'} h(v(u') - \lambda) < \sum_{u' \leq u} \lambda_{u'} h(\nu - \lambda) = h(\nu-\lambda),$$

which is absurd. Therefore, $\ell(v(u')) < N$, for some $u' \leq u$.

One has $u'.\mu = v(u').\mu^{++}$, thus $u'^{-1}.v(u').\mu^{++} = \mu$ and hence $\ell(u'^{-1}v(u')) \geq \ell(w)$, by definition of w. Therefore, $\ell(v(u')) + \ell(u) \geq \ell(v(u')) + \ell(u') \geq \ell(w)$. As a consequence, $\ell(w) \leq N + \ell(u)$ and thus F is finite. □

Definition 5.5.8. *A family $(a_j)_{j\in J} \in \widehat{\mathcal{H}}^J$ is said to be **summable** if:*

- *for all $\lambda \in Y^+$, $\{j \in J | \exists w \in W^v | \pi_{w,\lambda}(a_j) \neq 0\}$ is finite*
- *$\bigcup_{j\in J} \mathrm{supp}\ a_j$ is almost finite.*

When $(a_j)_{j\in J} \in \widehat{\mathcal{H}}^J$ is summable, one defines $\sum_{j\in J} a_j \in \widehat{\mathcal{H}}$ as follows: $\sum_{j\in J} a_j = \sum_{\lambda,w} x_{\lambda,w} Z^\lambda H_w$ where $x_{\lambda,w} = \sum_{j\in J} \pi_{\lambda,w}(a_j)$ for all $(\lambda,w) \in Y^+ \times W^v$.

Lemma 5.5.9. *Let $(a_j)_{j\in J} \in (\mathcal{H})^J$, $(b_k)_{k\in K} \in (\mathcal{H})^K$ be two summable families. Then $(a_j * b_k)_{(j,k)\in J\times K}$ is summable. Moreover $\sum_{(j,k)} a_j * b_k$ depends only on $\sum_{j\in J} a_j$ and $\sum_{k\in K} b_k$ and we denote it by $a * b$, if $a = \sum_{j\in J} a_j$ and $b = \sum_{k\in K} b_k$.*

Proof. For $j \in J$, $k \in K$, one writes

$$a_j = \sum_{(\lambda,u)\in Y^+\times W^v} x_{j,\lambda,u} Z^\lambda H_u,\ b_k = \sum_{(\mu,v)\in Y^+\times W^v} y_{k,\mu,v} Z^\mu H_v.$$

If $(u,\mu,v) \in W^v \times Y \times W^v$, one denotes by $(z^{u,\mu,v}_{\nu,t})_{\nu\in R_u(\mu), t\in[1,u].v} \in \mathcal{R}^{R_u(\mu)\times([1,u].v)}$ the family such that

$$H_u * Z^\mu H_v = \sum_{\nu\in R_u(\mu), t\in[1,u].v} z^{u,\mu,v}_{\nu,t} Z^\nu H_t,$$

which exists by Lemma 5.5.3. One has

$$a_j * b_k = \sum_{(\lambda,u),(\mu,v)\in Y^+\times W^v,\ \nu\in R_u(\mu),\ t\in[1,u].v} x_{j,\lambda,u} y_{k,\mu,v} z^{u,\mu,v}_{\nu,t} Z^{\lambda+\nu} H_t.$$

As a consequence, if $S^a_j = \bigcup_{u\in \mathrm{supp}_{W^v} a_j} [1,u]$ and $S^b_k = \bigcup_{v\in\mathrm{supp}_{W^v} b_k} [1,v]$,
$\mathrm{supp}_{W^v}(a_j * b_k) \subset S^a_j.S^b_k$.
Thus

$$S_{W^v} := \bigcup_{(j',k')\in J\times K} \mathrm{supp}_{W^v}(a_{j'} * b_{k'}) \subset (\bigcup_{j'\in J} S^a_{j'}).(\bigcup_{k'\in K} S^b_{k'})$$

is finite.

Let $(\rho, s) \in Y^+ \times W^v$. One has

$$\pi_{\rho,s}(a_j * b_k) = \sum_{(\lambda,u)\in Y^+\times W^v,(\mu,v)\in Y^+\times W^v,\ \nu\in R_u(\mu),\ \lambda+\nu=\rho} x_{j,\lambda,u} y_{k,\mu,v} z_{\nu,s}^{u,\mu,v}$$

Let $S = \bigcup_{j\in J} \text{supp } a_j \cup \bigcup_{k\in K} \text{supp } b_k$ and $S_Y = \pi_Y(S)$, where $\pi_Y : Y \times W^v \to Y$ is the projection on the first coordinate. By hypothesis, S and S_Y are almost finite.

Let $k \in \mathbb{N}$ and $\kappa_1, \ldots, \kappa_k \in Y^{++}$ such that for all $\lambda \in S_Y$, $\lambda^{++} \leq_{Q^\vee_\mathbb{Z}} \kappa_i$, for some $i \in [\![1, k]\!]$.

Let $F(\rho) = \{(\lambda, \nu) \in S_Y \times Y^+ | \ \exists(\mu, u) \in S_Y \times S_{W^v} | \nu \in R_u(\mu)$, and $\lambda + \nu = \rho\}$. Let $(\lambda, \nu) \in F(\rho)$, $(\mu, u) \in S_Y \times S_{W^v}$ such that $\nu \in R_u(\mu)$. By Lemma 5.5.5, one has $\lambda \leq_{Q^\vee_\mathbb{Z}} \lambda^{++} \leq_{Q^\vee_\mathbb{Z}} \kappa_i$ and $\nu \leq_{Q^\vee_\mathbb{Z}} \mu^{++} \leq_{Q^\vee_\mathbb{Z}} \kappa_j$ for some $i, j \in [\![1, k]\!]$. Therefore, $F(\rho)$ is finite.

Let $F'(\rho) = \{\mu \in S_Y | \ \exists(u, \lambda, \nu) \in S_{W^v} \times F(\rho) | \nu \in R_u(\mu)\}$. Let $\mu \in F'(\rho)$ and $(u, \lambda, \nu) \in S_{W^v} \times F(\rho)$ such that $\nu \in R_u(\mu)$. Then by Lemma 5.5.5, $\nu^{++} \leq_{Q^\vee_\mathbb{Z}} \mu^{++} \leq_{Q^\vee_\mathbb{Z}} \kappa_i$, for some $i \in [\![1, k]\!]$. As a consequence, $F'(\rho)^{++}$ is finite and by Lemma 5.5.7, $F'(\rho)$ is finite.

If $\lambda \in Y^+$, one sets $J(\lambda) = \{j \in J | \ \exists u \in W^v | \ x_{j,\lambda,u} \neq 0\}$ and $K(\lambda) = \{k \in K | \ \exists u \in W^v | \ y_{k,\lambda,u} \neq 0\}$. Let $F_1(\rho) = \{\lambda \in Y^+ | \ \exists \nu \in Y^+ | \ (\lambda, \nu) \in F(\rho)\}$ and $L(\rho) = \bigcup_{(\lambda,\mu)\in F_1(\rho)\times F'(\rho)} J(\lambda) \times K(\mu)$. Then $L(\rho)$ is finite and for all $(j, k) \in J \times K$, $\pi_{\rho,s}(a_j * b_k) \neq 0$ implies that $(j, k) \in L(\rho)$.

Let $(\rho, s) \in \bigcup_{(j,k)\in J\times K} \text{supp}(a_j * b_k)$. Then there exist $(\lambda, \mu) \in S_Y^2$, $u \in S_{W^v}$ and $\nu \in R_u(\mu)$ such that $\lambda + \nu = \rho$. Thus $\rho^{++} \leq_{Q^\vee_\mathbb{Z}} \lambda^{++} + \mu^{++} \leq_{Q^\vee_\mathbb{Z}} \kappa_i + \kappa_{i'}$ for some $i, i' \in [\![1, k]\!]$. Consequently, $\bigcup_{(j,k)\in J\times K} \text{supp}(a_j * b_k)$ is almost finite and $(a_j * b_k)$ is summable.

Moreover,

$$\begin{aligned}\pi_{\rho,s}\Big(\sum_{(j,k)\in J\times K} a_j * b_k\Big) &= \sum_{(\lambda,u)\in Y^+\times W^v,(\mu,v)\in Y^+\times W^v,\nu\in R_u(\mu),\lambda+\nu=\rho} \Big(\sum_{(j,k)\in J\times K} x_{j,\lambda,u} y_{k,\mu,v} z_{\nu,s}^{u,\mu,v}\Big) \\ &= \sum_{(\lambda,u)\in Y^+\times W^v,(\mu,v)\in Y^+\times W^v,\nu\in R_u(\mu),\lambda+\nu=\rho} x_{\lambda,u} y_{\mu,v} z_{\nu,s}^{u,\mu,v}\end{aligned},$$

where $a = \sum_{(\lambda,u)\in Y^+\times W^v} x_{\lambda,u} Z^\lambda H_u$ and $b = \sum_{(\mu,v)\in Y^+\times W^v} y_{\mu,v} Z^\mu H_v$, which completes the proof. □

Theorem 5.5.10. *The convolution $*$ equips $\widehat{\mathcal{H}}$ with a structure of associative algebra.*

Proof. By Lemma 5.5.9, $(\widehat{\mathcal{H}}, *)$ is an algebra. The associativity comes from Lemma 5.5.9 and from the associativity of $\mathcal{H}$. □

Definition 5.5.11. *The algebra $\widehat{\mathcal{H}} = \widehat{\mathcal{H}}_\mathcal{R}$ is the* ***completed Iwahori-Hecke algebra of*** $(\mathbb{A}, (\sigma_i)_{i\in I}, (\sigma'_i)_{i\in I})$ ***over*** $\mathcal{R}$.

Example 5.5.12. *Let $\mathcal{I}$ be a masure and suppose that $\mathcal{I}$ is thick of finite thickness and that a group G acts strongly transitively on $\mathcal{I}$. Let $i \in I$ and P_i (resp. P'_i) be a panel of $\{x \in \mathbb{A} | \ \alpha_i(x) = 0\}$ (resp.$\{x \in \mathbb{A} | \ \alpha_i(x) = 1\}$). One denotes by $1 + q_i$ (resp. $1 + q'_i$) the number of chambers containing P_i (resp. P'_i). One sets $\sigma_i = \sqrt{q_i}$ and $\sigma'_i = \sqrt{q'_i}$ for all $i \in I$. Then $(\sigma_i)_{i\in I}$, $(\sigma'_i)_{i\in I}$ satisfy the conditions of the beginning of Section 5.5 and the completed Iwahori-Hecke algebra over $\mathcal{R}$ associated to $(\mathbb{A}, (\sigma_i)_{i\in I}, (\sigma'_i)_{i\in I})$ is the completed Iwahori-Hecke algebra of $\mathcal{I}$ over $\mathcal{R}$.*

5.5.3 Center of Iwahori-Hecke algebras

In this subsection, we determine the center $\mathcal{Z}(\widehat{\mathcal{H}})$ of $\widehat{\mathcal{H}}$. For this we adapt the proof of Theorem 1.4 of [NR03].

5.5.3.1 Completed Bernstein-Lusztig bimodule

In order to determine $\mathcal{Z}(\widehat{\mathcal{H}})$, we would like to compute $Z^\mu * z * Z^{-\mu}$ if $z \in \mathcal{Z}(\widehat{\mathcal{H}})$ and $\mu \in Y^+$. However, left and right multiplication by Z^μ is defined in $\widehat{\mathcal{H}}$ only when $\mu \in Y^+$. We need to extend this multiplication to arbitrary $\mu \in Y$ in a compatible way with the multiplication in $\widehat{\mathcal{H}}$. Obviously, multiplication by Z^μ cannot stabilize $\widehat{\mathcal{H}}$ (because $Z^\mu * 1 = Z^\mu \notin \widehat{\mathcal{H}}$ if $\mu \in Y \backslash Y^+$). Thus we define a "completion" ${}^{BL}\overline{\mathcal{H}}$ of ${}^{BL}\mathcal{H}$ containing $\widehat{\mathcal{H}}$. We do not equip ${}^{BL}\overline{\mathcal{H}}$ with a structure of algebra but we equip it with a structure of Y-bimodule compatible with the convolution product on $\widehat{\mathcal{H}}$.

If $a = (a_{\lambda,w}) \in \mathcal{R}^{Y \times W^v}$, one writes $a = \sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w$. The **support of a along W^v** is $\{w \in W^v |\ \exists \lambda \in Y |\ a_{\lambda,w} \neq 0\}$ and is denoted $\mathrm{supp}_{W^v}(a)$.

Let ${}^{BL}\overline{\mathcal{H}} = \{a \in \mathcal{R}^{W^v \times Y} |\ \mathrm{supp}_{W^v}(a) \text{ is finite }\}$. If $(\rho, s) \in Y \times W^v$, one defines $\pi_{\rho,s}$: ${}^{BL}\overline{\mathcal{H}} \to \mathcal{R}$ by $\pi_{\rho,s}(\sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w) = a_{\rho,s}$ for all $\sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w \in {}^{BL}\overline{\mathcal{H}}$.

One considers $\widehat{\mathcal{H}}$ as a subspace of ${}^{BL}\overline{\mathcal{H}}$.

Definition 5.5.13. *A family $(a_j)_{j\in J} \in ({}^{BL}\overline{\mathcal{H}})^J$ is said to be* ***summable*** *if:*

- *for all $(s, \rho) \in W^v \times Y, \{j \in J|\ \pi_{s,\rho}(a_j) \neq 0\}$ is finite*
- *$\bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j)$ is finite.*

When $(a_j)_{j\in J}$ is summable, one defines $\sum_{j\in J} a_j \in {}^{BL}\overline{\mathcal{H}}$ by

$$\sum_{j\in J} a_j = \sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w,$$

with $a_{\lambda,w} = \sum_{j\in J} \pi_{\lambda,w}(a_j)$ for all $(\lambda, w) \in Y \times W^v$.

Lemma 5.5.14. *Let $(a_j) \in ({}^{BL}\mathcal{H})^J$ be a summable family of ${}^{BL}\overline{\mathcal{H}}$, $\mu \in Y$ and $a = \sum_{j\in J} a_j$. Then $(a_j * Z^\mu)$ and $(Z^\mu * a_j)$ are summable and $\sum_{j\in J} a_j * Z^\mu$, $\sum_{j\in J} Z^\mu * a_j$ depends on a (and μ), but not on the choice of the family $(a_j)_{j\in J}$.*

One sets $a \mathbin{\overline{*}} Z^\mu = \sum_{j\in J} a_j * Z^\mu$ and $Z^\mu \mathbin{\overline{*}} a = \sum_{j\in J} Z^\mu * a_j$. Then this convolution equips ${}^{BL}\overline{\mathcal{H}}$ with a structure of Y-bimodule.

Proof. Let $S = \bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j)$. If $(\lambda, w) \in Y \times W^v$, one sets $J(\lambda, w) = \{j \in J|\pi_{\lambda,w}(a_j) \neq 0\}$.

Let $(\rho, s) \in Y \times W^v$. Let $j \in J$. One has $\pi_{\rho,s}(Z^\mu * a_j) = \pi_{\rho-\mu,s}(a_j)$, therefore $\bigcup_{j\in J} \mathrm{supp}_{W^v}(Z^\mu * a_j) = \bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j) = S$ is finite and $\{j \in J|\ \pi_{\rho,s}(Z^\mu * a_j) \neq 0\} = J(\rho - \mu, s)$ is finite. Consequently $(Z^\mu * a_j)$ is summable. Moreover $\pi_{\rho,s}(\sum_{j\in J} Z^\mu * a_j) = \pi_{\rho-\mu,s}(a)$, which depends only on a.

Let $w \in W^v$. By Lemma 5.5.3, there exists $(z^w_{\nu,t})_{(\nu,t)\in R_w(\mu)\times[1,w]} \in \mathcal{R}^{R_w(\mu)\times[1,w]}$ such that

$$H_w * Z^\mu = \sum_{\nu\in R_w(\mu), t\in[1,w]} z^w_{\nu,t} Z^\nu H_t.$$

Let $j \in J$. One writes $a_j = \sum_{(\lambda,w)\in Y\times W^v} a_{j,\lambda,w} Z^\lambda H_w$, with $(a_{j,\lambda,w}) \in \mathcal{R}^{Y\times W^v}$.

One has:

$$\pi_{\rho,s}(a_j * Z^\mu) = \pi_{\rho,s}\Big(\sum_{(\lambda,w)\in Y\times S} a_{j,\lambda,w} Z^\lambda H_w * Z^\mu\Big)$$
$$= \pi_{\rho,s}\Big(\sum_{(\lambda,w)\in Y\times S} \big(\sum_{\nu\in R_w(\mu), t\in[1,w]} a_{j,\lambda,w} z^w_{\nu,t} Z^{\nu+\lambda} H_t\big)\Big)$$
$$= \sum_{(\lambda,w)\in Y\times S}\Big(\sum_{\nu\in R_w(\mu),\nu+\lambda=\rho} a_{j,\lambda,w} z^w_{\nu,s}\Big).$$

Let $F_{\rho,s} = \{j \in J | \pi_{\rho,s}(a_j * Z^\mu) \neq 0\}$. Then $F_{\rho,s} \subset \bigcup_{w\in S,\nu\in R_w(\mu)} J(\rho-\nu, w)$, which is finite. Moreover $\mathrm{supp}_{W^v}(a_j * Z^\mu) \subset \bigcup_{w\in S}[1,w]$ and thus $\bigcup_{j\in J} \mathrm{supp}_{W^v}(a_j * Z^\mu)$ is finite: $(a_j * Z^\mu)$ is summable. One has:

$$\pi_{\rho,s}(\sum_{j\in J} a_j * Z^\mu) = \sum_{j\in J}\Big(\sum_{(\lambda,w)\in Y\times S}\big(\sum_{\nu\in R_w(\mu),\nu+\lambda=\rho} a_{j,\lambda,w} z^w_{\nu,s}\big)\Big)$$
$$= \sum_{(\lambda,w)\in Y\times S}\Big(\sum_{\nu\in R_w(\mu),\nu+\lambda=\rho}\big(\sum_{j\in J} a_{j,\lambda,w} z^w_{\nu,s}\big)\Big)$$
$$= \sum_{(\lambda,w)\in Y\times S}\ \sum_{\nu\in R_w(\mu),\nu+\lambda=\rho} a_{\lambda,w} z^w_{\nu,s},$$

if $a = \sum_{(\lambda,w)\in Y\times W^v} a_{\lambda,w} Z^\lambda H_w$.

Let b, $\mu, \mu' \in Y$. It remains to show that $Z^\mu \overline{*} (Z^{\mu'} \overline{*} b) = (Z^{\mu+\mu'}) \overline{*} b$, $(b \overline{*} Z^\mu) \overline{*} Z^{\mu'} = b \overline{*} (Z^{\mu+\mu'})$ and $Z^\mu \overline{*} (b \overline{*} Z^{\mu'}) = (Z^\mu \overline{*} b) \overline{*} Z^{\mu'}$. One writes $b = \sum_{(w,\lambda)\in W^v\times Y} b_{w,\lambda} Z^\lambda H_w$ and one applies the first part of the lemma with $J = W^v \times Y$, using the fact that if $x \in {}^{BL}\mathcal{H}$, $Z^\mu * (Z^{\mu'} * x) = (Z^{\mu+\mu'}) * x$, $(x * Z^\mu) * Z^{\mu'} = x * (Z^{\mu+\mu'})$ and $Z^\mu * (x * Z^{\mu'}) = (Z^\mu * x) * Z^{\mu'}$, which is a consequence of the associativity of $({}^{BL}\mathcal{H}, *)$. □

Corollary 5.5.15. *Let $a \in \widehat{\mathcal{H}}$ and $\mu \in Y^+$. Then $Z^\mu * a = Z^\mu \overline{*} a$ and $a * Z^\mu = a \overline{*} Z^\mu$.*

5.5.3.2 Center of Iwahori-Hecke algebras

We now write $*$ instead of $\overline{*}$. Let $\mathcal{Z}(\widehat{\mathcal{H}})$ be the center of $\widehat{\mathcal{H}}$.

Lemma 5.5.16. *Let $a \in \mathcal{Z}(\widehat{\mathcal{H}})$ and $\mu \in Y$. Then $a * Z^\mu = Z^\mu * a$.*

Proof. One writes $\mu = \mu_+ - \mu_-$, with $\mu_+, \mu_- \in Y^+$.

One has $Z^{\mu_-} * (Z^{-\mu_-} * a) = a$ and $Z^{\mu_-} * (a * Z^{-\mu_-}) = a$. Therefore $Z^{-\mu_-} * a = a * Z^{-\mu_-}$. Consequently, $Z^\mu * a = Z^{\mu_+} * a * Z^{-\mu_-} = a * Z^\mu$. □

Let $w \in W^v$. Let ${}^{BL}\overline{\mathcal{H}}_{\not\geq w} = \{\sum_{(\lambda,v)\in Y\times W^v} a_{\lambda,v} Z^\lambda H_v \in {}^{BL}\overline{\mathcal{H}} \mid a_{\lambda,v} \neq 0 \Rightarrow v \not\geq w\}$, $\widehat{\mathcal{H}}_{\not\geq w} = \widehat{\mathcal{H}} \cap {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$, ${}^{BL}\overline{\mathcal{H}}_{=w} = \{\sum_{(\lambda,v)\in Y\times W^v} a_{\lambda,v} Z^\lambda H_v \in {}^{BL}\overline{\mathcal{H}} \mid a_{\lambda,v} \neq 0 \Rightarrow w = v\}$ and $\widehat{\mathcal{H}}_{=w} = \widehat{\mathcal{H}} \cap {}^{BL}\overline{\mathcal{H}}_{=w}$.

Lemma 5.5.17. *Let $w \in W^v$. Then :*

1. *For all $\lambda \in Y$,*

   - ${}^{BL}\overline{\mathcal{H}}_{\not\geq w} * Z^\lambda \subset {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$

- $Z^\lambda * {}^{BL}\overline{\mathcal{H}}_{\not\geq w} \subset {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$
- $Z^\lambda * {}^{BL}\overline{\mathcal{H}}_{=w} \subset {}^{BL}\overline{\mathcal{H}}_{=w}$

2. *Let $\lambda \in Y$. Then there exists $S \in {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$ such that $H_w * Z^\lambda = Z^{w(\lambda)} H_w + S$.*

Proof. This is a consequence of Theorem 6.2 of [BPGR16] or of Lemma 5.5.3 and of Lemma 5.5.14. □

Lemma 5.5.18. *One has $\mathcal{Z}(\mathcal{H}) = \mathcal{Z}(\widehat{\mathcal{H}}) \cap \mathcal{H}$.*

Proof. Let $a \in \mathcal{Z}(\mathcal{H})$. Then $a * Z^\lambda H_w = Z^\lambda H_w * a$ for all $(\lambda, w) \in Y \times W^v$. By Lemma 5.5.9, $a \in \mathcal{Z}(\widehat{\mathcal{H}})$. The other inclusion is clear. □

Let $A_1, \ldots, A_r$ be the indecomposable components of the matrix A. Let J^f be the set of $j \in [\![1, r]\!]$ such that A_j is of finite type (see Theorem 4.3 of [Kac94]) and $J^\infty = [\![1, r]\!] \backslash J^f$. Let $\mathbb{A}^f = \bigoplus_{j \in J^f} \mathbb{A}_j$. If $j \in [\![1, r]\!]$, one sets $\mathbb{A}_{j,in} = \bigcap_{\alpha \in \Phi_{re,j}} \ker \alpha$, where $\Phi_{re,j}$ is the root system of $\mathbb{A}_j$. One sets $\mathbb{A}_{in}^\infty = \bigoplus_{j \in J^\infty} \mathbb{A}_{j,in}$, $Y^f = Y \cap \mathbb{A}^f$ and $Y_{in}^\infty = Y \cap \mathbb{A}_{in}^\infty$.

The following theorem is a generalization of a well-known theorem of Bernstein, whose first published version seems to be Theorem 8.1 of [Lus83].

Theorem 5.5.19. 1. *The center $\mathcal{Z}(\widehat{\mathcal{H}})$ of $\widehat{\mathcal{H}}$ is $\mathcal{R}[[Y]]^{W^v}$.*

2. *The center $\mathcal{Z}(\mathcal{H})$ of $\mathcal{H}$ is $\mathcal{R}[Y^f \oplus Y_{in}^\infty]$.*

Proof. We first prove 1. Let $z \in \mathcal{R}[[Y]]^{W^v} \subset \widehat{\mathcal{H}}$, $z = \sum_{\lambda \in Y^+} a_\lambda Z^\lambda$. Let $i \in I$. One has $z = x + y$, with $x = \sum_{\lambda \in Y^+ | \alpha_i(\lambda) = 0} a_\lambda Z^\lambda$ and $y = \sum_{\lambda \in Y^+ | \alpha_i(\lambda) > 0} a_\lambda (Z^\lambda + Z^{r_i(\lambda)})$. As $H_i * x = x * H_i$ and $H_i * y = y * H_i$, we get that $z \in \mathcal{Z}(\widehat{\mathcal{H}})$ and thus $\mathcal{R}[[Y]]^{W^v} \subset \mathcal{Z}(\widehat{\mathcal{H}})$.

Let $z \in \mathcal{Z}(\widehat{\mathcal{H}})$. One writes $z = \sum_{\lambda \in Y, w \in W^v} c_{\lambda,w} Z^\lambda H_w \in {}^{BL}\overline{\mathcal{H}}$. Suppose that there exists $w \in W^v \backslash \{1\}$ such that for some $\lambda \in Y$, $\pi_{\lambda,w}(z) \neq 0$. Let $m \in W^v$ be maximal (for the Bruhat order) for this property. One writes $z = x + y$ with $x \in \widehat{\mathcal{H}}_{=m}$ and $y \in \widehat{\mathcal{H}}_{\not\geq m}$. One writes $x = \sum_{\lambda \in Y} c_{\lambda,m} Z^\lambda H_m$. By Lemma 5.5.16 and Lemma 5.5.17, if $\mu \in Y$,

$$z = Z^\mu * z * Z^{-\mu} = \sum_{\lambda \in Y} c_{\lambda,m} Z^{\lambda + \mu - m(\mu)} H_m + y',$$

for some $y' \in {}^{BL}\overline{\mathcal{H}}_{\not\geq m}$.

By projecting on ${}^{BL}\overline{\mathcal{H}}_{=m}$, we get that $x = \sum_{\lambda \in Y} c_{\lambda,m} Z^{\lambda + \mu - m(\mu)} H_m \in \widehat{\mathcal{H}}_{=m}$. Let $J \subset Y$ finite such that for all $(w, \lambda) \in W^v \times Y$, $c_{\lambda,w} \neq 0$ implies that there exists $\nu \in J$ such that $\lambda \leq \nu$. Let $\gamma \in Y$ such that $c_{\gamma,m} \neq 0$. For all $\mu \in Y$, one has $\pi_{\gamma + \mu - m(\mu), m}(z) \neq 0$ therefore $\gamma + \mu - m(\mu) \leq_{Q_\mathbb{Z}^\vee} \nu(\mu)$ for some $\nu(\mu) \in J$ for all $\mu \in Y$. Let $\mu \in Y \cap C_f^v$. Let $\nu \in J$ such that for some $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\sigma(n) \to +\infty$, $\gamma + \sigma(n)(\mu - m(\mu)) \leq_{Q_\mathbb{Z}^\vee} \nu$ for all $n \in \mathbb{N}$. In particular $\gamma + \sigma(1)(\mu - m(\mu)) - \nu \in Q_\mathbb{Z}^\vee$. By Lemma 2.2.9, $\mu - m(\mu) \in Q_\mathbb{N}^\vee \backslash \{0\}$ and thus for n large enough $\gamma + \sigma(n)(\mu - m(\mu)) = \gamma + \sigma(1)(\mu - m(\mu)) + (\sigma(n) - \sigma(1))(\mu - m(\mu)) >_{Q_\mathbb{Z}^\vee} \nu$, which is absurd. Therefore $\mathcal{Z}(\widehat{\mathcal{H}}) \subset \mathcal{R}[[Y]]$.

Let $z \in \mathcal{Z}(\widehat{\mathcal{H}})$. One writes $z = \sum_{\lambda \in Y} c_\lambda Z^\lambda$. Let $w \in W^v$. By Lemma 5.5.17, one has $H_w z = \sum_{\lambda \in Y} Z^{w(\lambda)} H_w + y$, with $y \in {}^{BL}\overline{\mathcal{H}}_{\not\geq w}$. But $H_w * z = z * H_w = \sum_{\lambda \in Y} c_\lambda Z^\lambda H_w$. By projecting on $\widehat{\mathcal{H}}_{=w}$, we get that $\sum_{\lambda \in Y} c_\lambda Z^{w(\lambda)} H_w = \sum_{\lambda \in Y} c_\lambda Z^\lambda H_w$. Therefore, $z \in \mathcal{R}[[Y]]^{W^v}$.

To prove 2, Lemma 5.5.18 shows that $\mathcal{Z}(\mathcal{H}) = \mathcal{H} \cap \mathcal{R}[[Y]]^{W^v}$. We then use Corollary 5.3.23 to conclude. □

Module over the center When W^v is finite, it is well known that $\mathcal{H}$ is a finitely generated $\mathcal{Z}(\mathcal{H})$-module. Suppose that W^v is infinite and let $\mathcal{Z} = \mathcal{Z}(\widehat{\mathcal{H}})$. Then $\widehat{\mathcal{H}}$ is not a finitely generated $\mathcal{Z}$-module. Indeed, let J be a finite set and $(h_j) \in \widehat{\mathcal{H}}^J$. Then for all $(a_j) \in \mathcal{Z}^J$, $\mathrm{supp}_{W^v}(\sum_{j\in J} a_j h_j) \subset \bigcup_{j\in J} \mathrm{supp}_{W^v}(h_j) \subsetneq W^v$ and thus (h_j) does not span $\widehat{\mathcal{H}}$.

5.5.4 Case of a reductive group

In this subsection, we study the case where G is reductive.

In [GR14], an almost finite set is a set E such that $E \subset (\bigcup_{i=1}^k y_i - Q^\vee_{\mathbb{N}}) \cap Y^{++}$ for some $y_1, \ldots, y_k \in Y$. If G is reductive, then such a set is finite. Indeed the Kac-Moody matrix A of G is a Cartan matrix: it satisfies condition (FIN) of Theorem 4.3 of [Kac94]. In particular, $Y^{++} \subset Q^\vee_{\mathbb{N}} \oplus \mathbb{A}_{in}$, where $\mathbb{A}_{in} = \bigcap_{i\in I} \ker \alpha_i$, which proves our claim.

However, the algebra $\widehat{\mathcal{H}}$ that we define is different from $\mathcal{H}$ even in the reductive case. If G is reductive, $\mathcal{T} = \mathbb{A}$ and thus $Y^+ = Y$. For instance, $\sum_{\mu\in Q^\vee_{\mathbb{N}}} Z^{-\mu} \in \widehat{\mathcal{H}} \backslash \mathcal{H}$.

Proposition 5.5.20. *Let $\mathcal{R}$ be a ring. Then $\mathcal{R}[[Y]]^{W^v} = \mathcal{R}[Y]^{W^v}$ if and only if W^v is finite.*

Proof. Suppose that W^v is infinite. Let $y \in Y \cap C^v_f$. Then $\sum_{w\in W^v} e^{w.y} \in \mathcal{R}[[Y]]^{W^v} \backslash \mathcal{R}[Y]^{W^v}$.

Suppose that W^v is finite. Let w_0 be the longest element of W^v. By the paragraph after Theorem of Section 1.8 of [Hum92], $w_0.Q^\vee_{\mathbb{N}} = -Q^\vee_{\mathbb{N}}$. Let $E \subset Y$ be an almost finite set invariant under the action of W^v. One has $E \subset \bigcup_{j\in J} y_j - Q^\vee_{\mathbb{N}}$ for some finite set J. Therefore $E = w_0.E \subset \bigcup_{j\in J} w_0.y_j + Q^\vee_{\mathbb{N}}$. Consequently, for all $x \in E$, there exists $j, j' \in J$ such that $w_0.y_{j'} \leq_{Q^\vee_{\mathbb{Z}}} x \leq_{Q^\vee_{\mathbb{Z}}} y_j$ and hence E is finite, which completes the proof. □

By Theorem 8.1 of [Lus83] and Theorem 5.5.19, when W^v is finite, one has:

$$\mathcal{Z}(\widehat{\mathcal{H}}) = \mathcal{R}[Y]^{W^v} = \mathcal{Z}(\mathcal{H}).$$

Chapter 6

Gindikin-Karpelevich finiteness

6.1 Introduction

The classical Gindikin-Karpelevich formula was introduced in 1962 by Gindikin and Karpelevich. It applies to real semi-simple Lie groups and it enables to compute certain Plancherel densities for semi-simple Lie groups. This formula was established in the non-archimedean case by Langlands in [Lan71]. In 2014, in [BGKP14], Braverman, Garland, Kazhdan an Patnaik obtained a generalization of this formula in the affine Kac-Moody case.

Let $\mathbf{G}$ be an affine Kac-Moody group associated to a simply connected semi-simple algebraic group. Let $\mathcal{K}$ be a local field, $G = \mathbf{G}(\mathcal{K})$ and $T \subset G$ be a maximal torus. Let $\mathcal{O}$ be the ring of integers of $\mathcal{K}$, π be a generator of the maximal ideal of $\mathcal{O}$ and $q = |\mathcal{O}/\pi\mathcal{O}|$. Choose a pair B, B^- of opposite Borel subgroups such that $B \cap B^- = T$ and let U, U^- be their "unipotent radicals". Let $Q_{\mathbb{Z}}$ and $Q^\vee_{\mathbb{Z}}$ be the root lattice and the coroot lattice of T. Let Φ_{all} be the set of roots of $\mathbf{G}$ and Φ^+_{all} be the set of positive roots. Let $K_s = \mathbf{G}(\mathcal{O})$. Let $(\alpha_i^\vee)_{i\in I}$ denote the simple coroots. For a coroot $\nu = \sum_{i\in I} n_i\alpha_i^\vee \in Q^\vee_{\mathbb{Z}}$, one sets $h(\nu) = \sum_{i\in I} n_i$. Let $Q^\vee_{\mathbb{N}} = \bigoplus_{i\in I} \mathbb{N}\alpha_i^\vee$. Let $\mathbb{C}[[Y]]$ be the Looijenga's coweight algebra of $\mathbf{G}$, where Y is some lattice containing $Q^\vee_{\mathbb{Z}}$, with generators e^λ, for $\lambda \in Y$. Then the Gindikin-Karpelevich formula reads as follows:

$$\forall \lambda \in Q^\vee_{\mathbb{Z}},\ \sum_{\mu\in Q^\vee_{\mathbb{Z}}} |K_s\backslash K_s\pi^\lambda U^- \cap K_s\pi^{\lambda-\mu}U| q^{h(\lambda-\mu)} e^{\lambda-\mu} = \frac{1}{H_0}\prod_{\alpha\in\Phi^+_{all}} (\frac{1-q^{-1}e^{-\alpha^\vee}}{1-e^{-\alpha^\vee}})^{m_\alpha} \tag{6.1}$$

where for $\alpha \in \Phi^+_{all}$, m_α denotes the multiplicity of the coroot $\alpha^\vee$ in the Lie algebra $\mathfrak{g}$ of $\mathbf{G}$, and H_0 is some term (see Section 6.8) for an explicit formula) depending on $\mathfrak{g}$. When $\mathbf{G}$ is a reductive group, $H_0 = 1$, the m_α are equal to 1 and this formula is equivalent to the Gindikin-Karpelevich formula.

The aim of this chapter is to establish the following theorems:

Theorem 6.5.1: Let $\mu \in Q^\vee_{\mathbb{Z}}$. Then if $\mu^\vee \notin -Q^\vee_{\mathbb{N}}$, $K_s\pi^\lambda U^- \cap K_s\pi^{\lambda+\mu}U$ is empty for all $\lambda \in Q^\vee_{\mathbb{Z}}$. If $\mu \in Q^\vee_{\mathbb{N}}$, then for $\lambda \in Q^\vee_{\mathbb{Z}}$ sufficiently dominant,

$$K_s\pi^\lambda U^- \cap K_s\pi^{\lambda+\mu}U \subset K_s\pi^\lambda K_s \cap K_s\pi^{\lambda+\mu}U.$$

Theorem 6.6.7: Let $\mu \in -Q^\vee_{\mathbb{N}}$ and $\lambda \in Q^\vee_{\mathbb{Z}}$. Then

$$|K_s\backslash K_s\pi^\lambda U^- \cap K_s\pi^{\lambda+\mu}U| = |K_s\backslash K_s\pi^0 U^- \cap K_s\pi^{\mu^\vee}U|$$

and these sets are finite.

Theorem 6.7.1: Let $\mu \in Q^\vee_\mathbb{Z}$. Then for $\lambda \in Q^\vee_\mathbb{Z}$ sufficiently dominant,

$$K_s\pi^\lambda U^- \cap K_s\pi^{\lambda+\mu}U = K_s\pi^\lambda K_s \cap K_s\pi^{\lambda+\mu}U.$$

These theorems correspond to Theorem 1.9 of [BGKP14] and this is a positive answer to Conjecture 4.4 of [BK14].

Theorem 6.6.7 is named "Gindikin-Karpelevich finiteness" in [BKP16] and it enables to make sense to the left side of formula 6.1. Thus this could be a first step to a generalization of Gindikin-Karpelevich formula to general split Kac-Moody groups. However, we do not know yet how to express the term H_0 when $\mathbf{G}$ is not affine.

These finiteness results are a key tool in the definition by Patnaik and Puskás of Whittaker functions in the Kac-Moody framework, see [Pat17] for the affine case and [PP17] for the general case.

In order to prove these theorems, we use the masure of $\mathbf{G}(\mathcal{K})$. We interpret the sets involved in these theorems as sets of vertices of $\mathcal{I}$, using retractions $\rho_{+\infty}$, $\rho_{-\infty}$ and the vectorial distance d^v on $\mathcal{I}$: for all $\lambda \in Q^\vee_\mathbb{Z}$, $K_s\backslash K_s\pi^\lambda U^-$ corresponds to $\rho_{-\infty}^{-1}(\{\lambda\})$, $K_s\backslash K_s\pi^\lambda U$ corresponds to $\rho_{+\infty}^{-1}(\{\lambda\})$ and $K_s\backslash K_s\pi^\lambda K_s$ corresponds to

$$\mathcal{S}^v(0,\lambda) = \{x \in \mathcal{I} |\ d^v(0,x) \text{ is defined and } d^v(0,x) = \lambda\}.$$

This enables to use properties of "Hecke paths" to prove these theorems. These paths are more or less the images of segments in the masure by retractions. They were first defined by Kapovich and Millson in [KM08]. We also use finiteness results of [GR14].

Actually, when we will write Theorem 6.5.1 and Theorem 6.7.1 using the masure, we will show that these inclusion or equality are true modulo K. But as if $X \subset G$ is invariant by left multiplication by K, $X = \bigcup_{x\in K\backslash X} Kx$, this will be sufficient.

In the sequel, the results are not stated as in this introduction. Their statements use retractions and vectorial distance, see Section 6.2 for a dictionary. They are also a bit more general: they take into account the inessential part of the standard apartment $\mathbb{A}$. Corollary 6.6.2, as it is stated in this introduction was proved in Section 5 of [GR14] and it is slightly generalized in the following.

Framework Actually, this chapter is written in a more general framework: we ask $\mathcal{I}$ to be an abstract masure and G to be a strongly transitive group of (positive, type-preserving) automorphisms of $\mathcal{I}$. This applies in particular to almost-split Kac-Moody groups over local fields. We assume that $\mathcal{I}$ is semi-discrete (which means that if M is a wall of $\mathbb{A}$, the set of wall parallel to M is discrete) and that $\mathcal{I}$ has finite thickness (which means that for each panel, the number of chambers containing it is finite). The group G is a group acting strongly transitively on $\mathcal{I}$. Let N be the stabilizer of $\mathbb{A}$ in G and $\nu : N \to \mathrm{Aut}(\mathbb{A})$ be the induced morphism. We assume moreover that $\nu(N) = Y \rtimes W^v$.

Organization of the chapter We first prove Theorem 6.5.1 by studying paths in $\mathcal{I}$. Using finiteness results of [GR14], we deduce that if $\mu \in Q^\vee_\mathbb{Z}$, then for $\lambda \in Q^\vee_\mathbb{Z}$ sufficiently dominant $K_s\backslash K_s\pi^\lambda U^- \cap K_s\pi^{\lambda+\mu}U$ is finite. Using some kind of translations of $\mathcal{I}$, we prove that $|K_s\backslash K_s\pi^\lambda U^- \cap K_s\pi^{\lambda+\mu}U|$ is independent of λ and deduce Theorem 6.6.7. We then prove Theorem 6.7.1 by proving that Y^{++} is finitely generated monoid and using Theorem 6.6.7.

In Section 6.2, we explain the dictionary between G/K_s and the vertices of $\mathcal{I}$.

In Section 6.3, we prove the Gindikin-Karpelevich finiteness when $\mathcal{I}$ is a tree and outline the proof when $\mathcal{I}$ is a masure.

In Section 6.4, we define Hecke paths.

In Section 6.5, we bound the map T_ν introduced in 4.2.2.2 (and which somehow estimates the distance between points of $\mathcal{I}$ and $\mathbb{A}$) by a function of $\rho_{+\infty} - \rho_{-\infty}$. We then prove Theorem 6.5.1.

In Section 6.6, we study some kind of translations of $\mathcal{I}$ and prove the Gindikin-Karpelevich finiteness.

In Section 6.7, we prove Theorem 6.7.1.

In Section 6.8, we relate Macdonald's formula of [BKP16] and [BPGR17] with the Gindikin-Karpelevich formula.

6.2 Dictionary

In this section, we show the correspondence between the quotient of subgroups of the introduction and sets of vertices of $\mathcal{I}$.

Let $\mathbf{G}$ be a split Kac-Moody group over a ultrametric field $\mathcal{K}$ with ring of integer $\mathcal{O}$ and $G = \mathbf{G}(\mathcal{K})$. We use notation of the introduction or of Section 3.3. We associate a masure to G as in [Rou16].

The group G acts strongly transitively on $\mathcal{I}$. The group K_s is the fixer of 0 in G and HU (resp. HU^-) is the fixer of $+\infty$ (resp. $-\infty$) in G, where $H = \mathbf{T}(\mathcal{O})$. The action of T on $\mathbb{A}$ is as follows: if $t = \pi^{\lambda^\vee} \in T$ for some $\lambda^\vee \in \Lambda^\vee$, t acts on $\mathbb{A}$ by the translation of vector $-\lambda^\vee$ and thus $\lambda^\vee = \pi^{-\lambda^\vee}.0$ for all $\lambda^\vee \in \Lambda^\vee$.

Let $\mathcal{I}_0 = G.0$ be the set of **vertices of type** 0. The map $G \to \mathcal{I}_0$ sending $g \in G$ to $g.0$ induces a bijection $\phi : G/K \to \mathcal{I}_0$. If $g \in G$, then $\phi^{-1}(g.0) = gK_s$. If $x \in \mathbb{A}$, then $\rho_{+\infty}^{-1}(\{x\}) = U.x$ and $\rho_{-\infty}^{-1}(\{x\}) = U^-.x$. By Remark 3.2.2, if $\lambda \in Q^\vee_{\mathbb{Z}}$, then $\mathcal{S}^v(0, \lambda^\vee) = K_s.\lambda$.

Therefore, for all $\lambda, \mu \in Q^\vee$, $\phi^{-1}\big(\mathcal{S}^v(0, \lambda) \cap \rho_{+\infty}^{-1}(\{\mu\})\big) = (K_s\pi^{-\lambda}K_s \cap U\pi^{-\mu}K_s)/K_s$ and $\phi^{-1}\big(\rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\mu\})\big) = (U^-\pi^{-\lambda}K_s \cap U\pi^{-\mu}K_s)/K_s$.

We then use the bijection $\psi : K_s\backslash G \to G/K_s$ defined by $\psi(K_sg) = g^{-1}K_s$ to obtain the sets considered in the introduction.

6.3 Case of a tree and idea of the proof

When $\mathcal{I} = \mathbb{T}$ is a tree of finite thickness, it is easy to prove Gindikin-Karpelevich finiteness directly. Indeed, let $y : \mathbb{T} \to \mathbb{A}_{\mathbb{T}}$ sending each $x \in \mathcal{I}$ on the unique point $z \in \mathbb{A}_{\mathbb{T}}$ such that $d(x, z) = d(x, \mathbb{A}_{\mathbb{T}})$. Let $T : \mathbb{T} \to \mathbb{R}_+$ mapping each $x \in \mathbb{T}$ on $d(x, y(x))$. Then $\rho_{+\infty}(x) = y(x) - T(x)$ and $\rho_{-\infty}(x) = y(x) + T(x)$ for all $x \in \mathbb{T}$ (see Figure 6.3.1). Therefore for all $\lambda, \mu \in \mathbb{R}$, $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) = \{x \in \mathbb{T} |\ y(x) = \lambda + \frac{\mu}{2} \text{ and } T(x) = \frac{\mu}{2}\}$. Therefore $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ is finite and if $\mathbb{T}$ is semi-homogeneous, $|\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})|$ does not depend on $\lambda \in \mathbb{Z}$ if $\mu \in 2\mathbb{Z}$.

The crucial fact for our method is that if μ is fixed, then for λ sufficiently dominant (that is for λ sufficiently large)

$$\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) \subset \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \mathcal{S}(0, \{\lambda\}),$$

where $\mathcal{S}(0, \lambda)$ is the sphere of $\mathbb{T}$ of center 0 and radius λ. Indeed, if $x \in \mathbb{T}$, $d(0, x) = d(0, y(x)) + T(x) = \begin{cases} -\rho_{+\infty}(x) & \text{if } y(x) \leq 0 \\ \rho_{-\infty}(x) & \text{if } y(x) \geq 0 \end{cases}$. If $x \in \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$, $y(x) =$

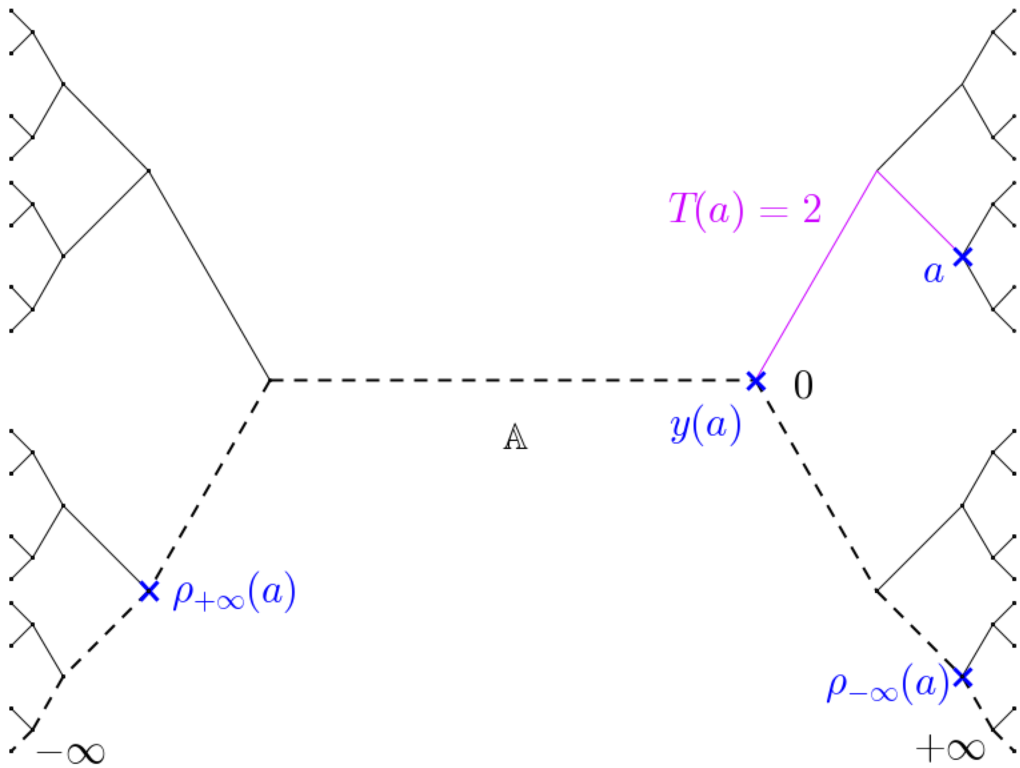

Figure 6.3.1 – Images by $\rho_{-\infty}$ and $\rho_{-\infty}$ of a point a

$\lambda + \frac{\mu}{2} > 0$ when $\lambda \gg 0$ and thus we have the desired inclusion (and even an equality). As the right term of this inclusion is finite, we deduce the Gindikin-Karpelevich finiteness.

Outline of the proof in the general case We no more suppose $\mathcal{I}$ to be a tree. We begin by proving that $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) \subset \mathcal{S}^v(0,\lambda)$. In 4.2.2.1, we defined two maps $y_\nu : \mathcal{I} \to \mathbb{A}$ and $T_\nu : \mathcal{I} \to \mathbb{A}$ which are analogous to y and T. To prove the inclusion, it suffices to prove that for $x \in \rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$, $y_\nu(x) \in C_f^v$ (by Lemma 6.5.5). We prove this fact by bounding T_ν by a map of $\rho_{+\infty} - \rho_{-\infty}$ (see Corollary 6.5.3).

Using Theorem 5.4.10 (which is proved in [GR14]), we deduce that if $\mu \in Q_{\mathbb{Z}}^\vee$, then $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ is finite for $\lambda \in Q_{\mathbb{Z}}^\vee$ sufficiently dominant. Using automorphisms of $\mathcal{I}$ inducing translations on $\mathbb{A}$, we show that $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ does not depend on $\lambda \in Q_{\mathbb{Z}}^\vee$ and thus deduce Theorem 6.6.7.

In order to prove the asymptotic equality (Theorem 6.7.1), we use the fact that Y^{++} is a finitely generated monoid and a lemma similar to Corollary 6.5.3 (Lemma 6.5.2).

The main technical lemma (Lemma 6.5.2) is obtained by using Hecke paths on $\mathcal{I}$, that we now define.

6.4 Hecke paths

We now define Hecke paths. They are more or less the images by $\rho_{-\infty}$ of preordered segments in $\mathcal{I}$. The definition is a bit technical but it expresses the fact that the image of such a path "goes nearer to $+\infty$" when it crosses a wall. A consequence of that is Remark 6.4.2 and we will not use directly this definition in the following.

We consider piecewise linear continuous paths $\pi : [0,1] \to \mathbb{A}$ such that the values of π' belong to some orbit $W^v.\lambda$ for some $\lambda \in \overline{C_f^v}$. Such a path is called a λ-**path**. It is increasing with respect to the preorder relation $\leq$ on $\mathbb{A}$. For any $t \neq 0$ (resp. $t \neq 1$), we let $\pi'_-(t)$ (resp. $\pi'_+(t)$) denote the derivative of π at t from the left (resp. from the right).

Definition 6.4.1. *A Hecke path of shape λ with respect to $-C^v_f$ is a λ-path such that $\pi'_+(t) \leq_{W^v_{\pi(t)}} \pi'_-(t)$ for all $t \in [0,1]\backslash\{0,1\}$, which means that there exists a $W^v_{\pi(t)}$-chain from $\pi'_-(t)$ to $\pi'_+(t)$, i.e., a finite sequence $(\xi_0 = \pi'_-(t), \xi_1, \ldots, \xi_s = \pi'_+(t))$ of vectors in V and $(\beta_1, \ldots, \beta_s) \in (\Phi_{re})^s$ such that, for all $i \in [\![1, s]\!]$,*

1. *$r_{\beta_i}(\xi_{i-1}) = \xi_i$.*
2. *$\beta_i(\xi_{i-1}) < 0$.*
3. *$r_{\beta_i} \in W^v_{\pi(t)}$; i.e., $\beta_i(\pi(t)) \in \mathbb{Z}$: $\pi(t)$ is in a wall of direction $\ker(\beta_i)$.*
4. *Each β_i is positive with respect to $-C^v_f$; i.e., $\beta_i(C^v_f) > 0$.*

Remark 6.4.2. *Let $\pi : [0,1] \to \mathbb{A}$ be a Hecke path of shape $\lambda \in \overline{C^v_f}$ with respect to $-C^v_f$. Then if $t \in [0,1]$ such that π is differentiable in t and $\pi'(t) \in \overline{C^v_f}$, then for all $s \geq t$, π is differentiable in s and $\pi'(s) = \lambda$.*

Hecke paths of $\mathbb{A}_\mathbb{T}$ Let $\lambda \in \mathbb{N}^*$. A Hecke path of shape λ of $\mathbb{A}_\mathbb{T}$ is a map $\pi : [0,1] \to \mathbb{R} = \mathbb{A}_\mathbb{T}$ such that there exists $i \in [\![0, \lambda]\!]$ such that $\pi_{|[0,\frac{i}{\lambda}]}$ is differentiable and $(\pi_{|[0,\frac{i}{\lambda}]})' = -1$ and $\pi_{|[\frac{i}{\lambda},1]}$ is differentiable and $(\pi_{|[0,\frac{i}{\lambda}]})' = 1$. If $a \in \mathbb{Z}$, there are exactly $\lambda + 1$ Hecke paths π of shape λ such that $\pi(0) = a$ (see Figure 6.4.1).

6.5 Bounding of T_ν and asymptotic inclusion in the sphere

Let $Q^\vee_{\mathbb{R}_+} = \bigoplus_{i\in I} \mathbb{R}_+\alpha^\vee_i$ and $Q^\vee_{\mathbb{R}_-} = -Q^\vee_{\mathbb{R}_+}$. In this section, we prove the following theorem:

Theorem 6.5.1. *Let $\mu \in \mathbb{A}$. Then if $\mu \notin Q^\vee_{\mathbb{R}_-}$, $\rho^{-1}_{+\infty}(\{\lambda + \mu\}) \cap \rho^{-1}_{-\infty}(\{\lambda\})$ is empty for all $\lambda \in \mathbb{A}$. If $\mu \in Q^\vee_{\mathbb{R}_-}$, then for $\lambda \in \mathbb{A}$ sufficiently dominant, $\rho^{-1}_{+\infty}(\{\lambda + \mu\}) \cap \rho^{-1}_{-\infty}(\{\lambda\}) \subset \mathcal{S}^v(0, \lambda) \cap \rho^{-1}_{+\infty}(\{\lambda + \mu\})$.*

In order to prove this theorem, we prove that for all $x \in \mathcal{I}$, $T_\nu(x)$ is bounded by a function of $\rho_{+\infty}(x) - \rho_{-\infty}(x)$.

One defines $h : Q^\vee_\mathbb{R} \to \mathbb{R}$ by $h(x) = \sum_{i\in I} x_i$, for all $x = \sum_{i\in I} x_i\alpha^\vee_i \in Q^\vee_\mathbb{R}$.

Lemma 6.5.2. *Let $T \in \mathbb{R}_+$, $\mu \in \mathbb{A}$, $a \in \mathbb{A}$, $\nu \in Y^{++} = Y \cap \overline{C^v_f}$ and suppose there exists a Hecke path π from a to $a + T\nu - \mu$ of shape $T\nu$. Then:*

1. *$\mu \in Q^\vee_{\mathbb{R}_+}$. Consequently $h(\mu)$ is well defined.*
2. *If $T > h(\mu)$, there exists $t \in [0,1]$ such that π is differentiable on $]t, 1]$ and $\pi'_{|]t,1]} = T\nu$. Furthermore, let t^* be the smallest $t \in [0,1]$ having this property, then $t^* \leq \frac{h(\mu)}{T}$.*

Proof. The main idea of 2 is to use the fact that during the time when $\pi'(t) \neq T\nu$, $\pi'(t) = T\nu - T\lambda(t)$ with $\lambda(t) \in Q^\vee_\mathbb{N}\backslash\{0\}$. Hence for T large, π decreases quickly for the $Q^\vee_\mathbb{Z}$ order, but it cannot decrease too much because μ is fixed.

Let $t_0 = 0$, $t_1, \ldots, t_n = 1$ be a subdivision of $[0,1]$ such that for all $i \in [\![0, n-1]\!]$, $\pi_{|]t_i,t_{i+1}[}$ is differentiable and let $w_i \in W^v$ be such that $\pi'_{|]t_i,t_{i+1}[} = w_i.T\nu$. If $w_i.\nu = \nu$, one chooses $w_i = 1$.

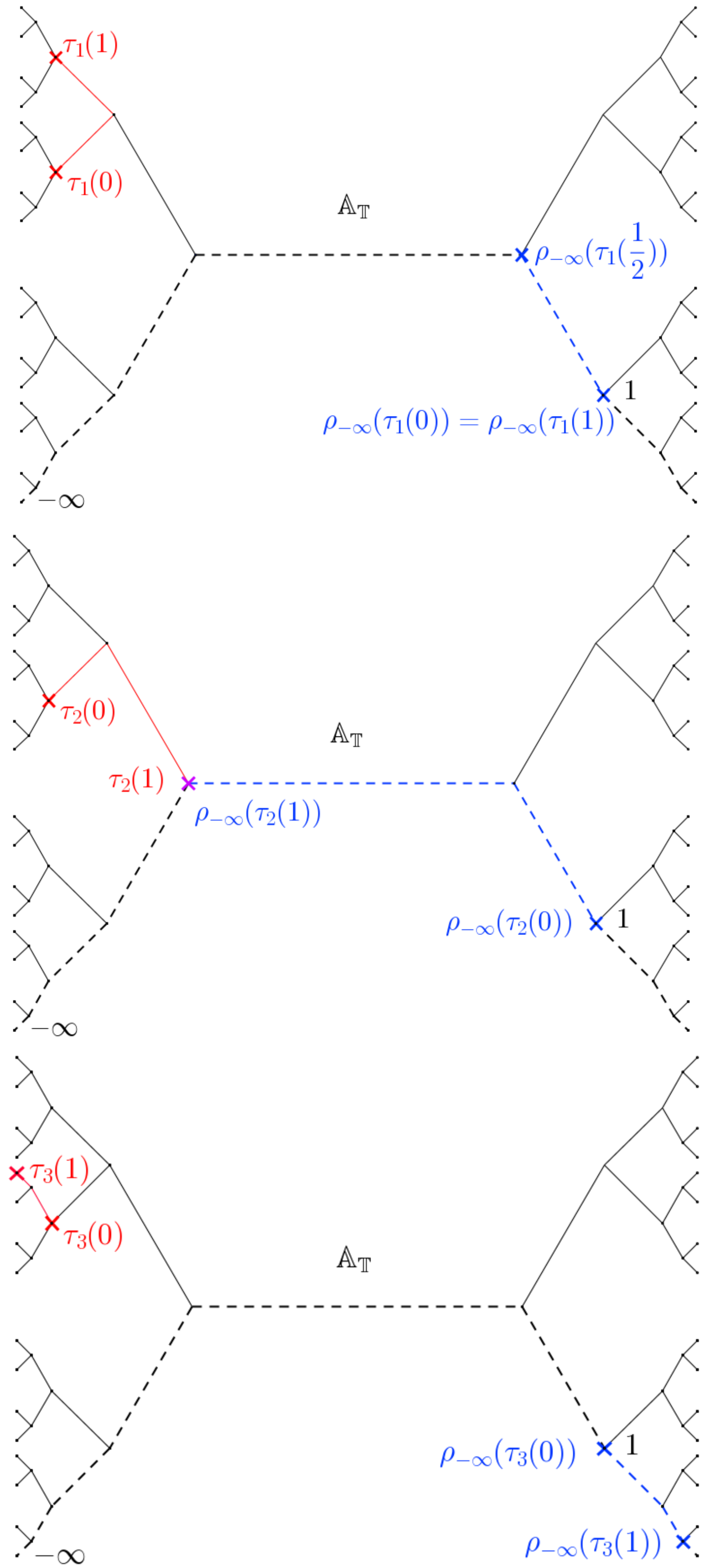

Figure 6.4.1 – Hecke paths of shape of 2 starting from 1 and segments retracting on them.

For $i \in [\![0, n-1]\!]$, according to Lemma 2.2.9, $w_i.\nu = \nu - \lambda_i$, with $\lambda_i \in Q^\vee_{\mathbb{N}}$ and if $w_i \neq 1$, $\lambda_i \neq 0$. One has

$$\pi(1) - \pi(0) = T\nu - \sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i) T\lambda_i = T\nu - \mu$$

and one deduces 1.

Suppose now $T > h(\mu)$. Let us show that there exists $i \in [\![0, n-1]\!]$ such that $w_i = 1$. Let $i \in [\![0, n-1]\!]$.

For all i such that $w_i \neq 1$, one has $h(\lambda_i) \geq 1$. Hence $T \sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i) \leq h(\mu)$, and $\sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i) < 1 = \sum_{i=0}^{n-1} (t_{i+1} - t_i)$. Thus there exists $i \in [\![0, n-1]\!]$ such that $w_i = 1$.

By Remark 6.4.2, if $w_i = 1$ for some i, then $w_j = 1$ for all $j \geq i$. This shows the existence of t^*. We also have $t^* \leq \sum_{i=0, w_i \neq 1}^{n-1} (t_{i+1} - t_i)$ and hence the claimed inequality follows. □

From now on and until the end of this subsection, ν will be a fixed element of $C^v_f \cap Y$. We define $T_\nu : \mathcal{I} \to \mathbb{R}_+$ and $y_\nu : \mathcal{I} \to \mathbb{A}$ as in 4.2.2.1. Let $\delta^-_\nu = -\mathbb{R}_+ \nu$ and $x \in \mathcal{I}$. Similarly, one defines y^-_ν as the first point of $x + \delta^-_\nu$ meeting $\mathbb{A}$ and $T^-_\nu(x)$ as the element T of $\mathbb{R}_+$ such that $\rho_{-\infty}(x) = y + T\nu$.

Corollary 6.5.3. *Let $x \in \mathcal{I}$ and $\mu = \rho_{-\infty}(x) - \rho_{+\infty}(x)$. Then $\mu \in Q^\vee_{\mathbb{R}_+}$ and $T_\nu(x) \leq h(\mu)$.*

Proof. Let $y = y_\nu(x)$ and $T = T_\nu(x)$. Let π be the image by $\rho_{-\infty}$ of $[x, y]$. This is a Hecke path from $\rho_{-\infty}(x)$ to $y = \rho_{+\infty}(x) + T\nu$, of shape $T\nu$. The minimality of T and Lemma 4.2.14 imply that $\pi'(t) \neq \nu$ for all $t \in [0, 1]$ where π is differentiable. By applying Lemma 6.5.2 2, we deduce that $T \leq h(\mu)$. Lemma 6.5.2 1 applied to $a = \rho_{-\infty}(x)$ and $\mu = \rho_{-\infty}(x) - \rho_{+\infty}(x)$ completes the proof. □

This corollary gives a simple proof of Proposition 4.2.15 in this frameworks: let $x \in \mathcal{I}$ such that $\rho_{+\infty}(x) = \rho_{-\infty}(x)$. Then by Corollary 6.5.3, $T_\nu(x) = 0$ and thus $x \in \mathbb{A}$.

Remark 6.5.4. *With the same reasoning but by considering Hecke with respect to C^v_f we obtain an analogous bounding for T^-_ν: for all $x \in \mathcal{I}$, $T^-_\nu(x) \leq h(\rho_{-\infty}(x) - \rho_{+\infty}(x))$.*

Lemma 6.5.5. *Let $x \in \mathcal{I}$ such that $y^-_\nu(x) \in C^v_f$. Then $0 \leq x$ and $\rho_{-\infty}(x) = d^v(0, x)$.*

Proof. Let $y^- = y^-_\nu(x)$. Let A be an apartment containing x and $-\infty$. By (MA ii) there exists $g \in G$ such that $A = g^{-1}.\mathbb{A}$ and g fixes $\mathrm{cl}(y, -\infty) \supset y - \overline{C^v_f} \ni 0$. Then $g.x - g.y^- = \rho_{-\infty}(x) - y^- = T^-\nu \in C^v_f$ and $g.y^- - g.0 = y^-$. Thus $g.x - g.0 = \rho_{-\infty}(x) \in C^v_f$ and we can conclude by Remark 3.2.2. □

We can now prove Theorem 6.5.1:

Let $\mu \in \mathbb{A}$. Then if $\mu \notin Q^\vee_{\mathbb{R}_-}$, $\rho^{-1}_{+\infty}(\{\lambda + \mu\}) \cap \rho^{-1}_{-\infty}(\{\lambda\})$ is empty for all $\lambda \in \mathbb{A}$. If $\mu \in Q^\vee_{\mathbb{R}_-}$, then for $\lambda \in \mathbb{A}$ sufficiently dominant, $\rho^{-1}_{+\infty}(\{\lambda + \mu\}) \cap \rho^{-1}_{-\infty}(\{\lambda\}) \subset \mathcal{S}^v(0, \lambda) \cap \rho^{-1}_{+\infty}(\{\lambda + \mu\})$.

Proof. The condition on μ comes from Corollary 6.5.3.

Suppose $\mu \in Q^\vee_{\mathbb{R}_-}$. Let $\lambda \in \mathbb{A}$. Let $y^- = y^-_\nu$ and $T^- = T^-_\nu$. By Corollary 6.5.3 and remark 6.5.4, if $x \in \rho^{-1}_{+\infty}(\{\lambda + \mu\}) \cap \rho^{-1}_{-\infty}(\{\lambda\})$, then $y^-(x) \in [\lambda - T^-(x)\nu, \lambda] \subset \lambda - [0, h(-\mu)]\nu$. For all $i \in I$, $\alpha_i([0, h(-\mu)]\nu)$ is bounded. Consequently for λ sufficiently dominant, $\alpha_i(\lambda - [0, h(-\mu)]\nu) \subset \mathbb{R}^*_+$ for all $i \in I$. For such a λ, $y^-\big(\rho^{-1}_{+\infty}(\{\lambda + \mu\}) \cap \rho^{-1}_{-\infty}(\{\lambda\})\big) \subset C^v_f$. We conclude the proof with Lemma 6.5.5.

□

6.6 The Gindikin-Karpelevich finiteness

6.6.1 Translations and invariance of the cardinals

In this subsection, we study automorphisms of $\mathcal{I}$ inducing translations on $\mathbb{A}$ and prove that they commute with maps on $\mathcal{I}$. Let $\mu \in \mathbb{A}$. In particular we prove that there exists an automorphism n of $\mathcal{I}$ such that for all $\lambda \in Y+\mathbb{A}_{in}$, $\rho_{+\infty}^{-1}(\{\lambda+\mu\})\cap\rho_{-\infty}^{-1}(\{\lambda\}) = n\big(\rho_{+\infty}^{-1}(\{\mu\})\cap \rho_{-\infty}^{-1}(\{0\})\big)$ (see Lemma 6.6.5).

We defined in 3.2.3 an action of $\mathbb{A}_{in}$ on $\mathcal{I}$ as follows: if $x \in \mathcal{I}$ and $\nu \in \mathbb{A}_{in}$, $x+\nu = \phi(\phi^{-1}(x)+\nu)$ where ϕ is any isomorphism of apartment sending x in $\mathbb{A}$. For $\nu \in \mathbb{A}_{in}$, one defines $\tau_\nu : \mathcal{I} \to \mathcal{I}$ by $\tau_\nu(x) = x+\nu$.

Then the following lemma is easy to prove:

Lemma 6.6.1. *Let $\nu \in \mathbb{A}_{in}$. Then:*

1. *The map τ_ν is an automorphism of $\mathcal{I}$ stabilizing every apartment of $\mathcal{I}$. Its inverse is $\tau_{-\nu}$.*
2. *For all $g \in G$, $g \circ \tau_\nu = \tau_\nu \circ g$.*
3. *For all retraction ρ centered at a sector germ, $\rho \circ \tau_\nu = \tau_\nu \circ \rho$.*
4. *For all $x, y \in \mathcal{I}$ such that $x \leq y$, then $d^v(x, \tau_\nu(y)) = d^v(\tau_{-\nu}(x), y) = d^v(x,y)+\nu$.*

Corollary 6.6.2. *Let $\lambda \in \mathbb{A}$ and $\mu \in Y+\mathbb{A}_{in}$. Then for all $\lambda_{in} \in \mathbb{A}_{in}$,*

$$\tau_{\lambda_{in}}\big(\mathcal{S}^v(0,\lambda)\cap\rho_{+\infty}^{-1}(\{\mu\})\big) = \mathcal{S}^v(0,\lambda+\lambda_{in})\cap\rho_{+\infty}^{-1}(\{\mu+\lambda_{in}\}).$$

In particular, for all $\lambda \in Y+\mathbb{A}_{in}$, $\mathcal{S}^v(0,\lambda)\cap\rho_{+\infty}^{-1}(\{\mu+\lambda\})$ is finite and is empty if $\mu \notin -Q_{\mathbb{N}}^\vee$.

Proof. The first assertion is a consequence of Lemma 6.6.1.

Let $\lambda \in Y+\mathbb{A}_{in}$ and $\lambda_{in} \in \mathbb{A}_{in}$ such that $\tau(\lambda) \in Y$, with $\tau = \tau_{\lambda_{in}}$. Then $\tau\big(\mathcal{S}^v(0,\lambda)\cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})\big) = \mathcal{S}^v(0,\tau(\lambda))\cap\rho_{+\infty}^{-1}(\{\tau(\lambda+\mu)\})$. Consequently, one can assume $\lambda \in Y$.

Suppose $\mathcal{S}^v(0,\lambda)\cap\rho_{+\infty}^{-1}(\{\mu+\lambda\})$ is nonempty. Let x be in this set. Then there exists $g, h \in G$ such that $g.x = \lambda$ and $h.x = \mu+\lambda$. Thus $\lambda+\mu = h.g^{-1}.\lambda \in \mathcal{I}_0 \cap \mathbb{A} = Y$ and therefore, $\mu \in Y$. We can now conclude because the finiteness and the condition on μ are shown in [GR14], Section 5, see also Theorem 5.4.10 1 (by exchanging all the signs). □

Lemma 6.6.3. *Let $n \in G$ inducing a translation on $\mathbb{A}$. Then $n \circ \rho_{+\infty} = \rho_{+\infty}\circ n$ and $n \circ \rho_{-\infty} = \rho_{-\infty}\circ n$.*

Proof. Let $x \in \mathcal{I}$ and A be an apartment containing x and $+\infty$. Then $n.A$ is an apartment containing $+\infty$. Let $\phi : A \overset{+\infty}{\to} \mathbb{A}$. We have $n.x \in n.A$, and $n\circ\phi\circ n^{-1} : n.A \to \mathbb{A}$ fixes $+\infty$. Hence $\rho_{+\infty}(n.x) = n\circ\phi\circ n^{-1}(n.x) = n\circ\phi(x) = n\circ\rho_{+\infty}(x)$ and thus $n\circ\rho_{+\infty} = \rho_{+\infty}\circ n$. By the same reasoning applied to $\rho_{-\infty}$, we get the lemma. □

Lemma 6.6.4. *Let $n \in G$ inducing a translation on $\mathbb{A}$. Let $\lambda_{in} \in \mathbb{A}_{in}$. Set $\tau = \tau_{\lambda_{in}}\circ n$. Let $\nu \in C_f^v$ and $y^- = y_\nu^-$. Then $\tau\circ y^- = y^-\circ\tau$.*

Proof. If $x \in \mathbb{A}$, then $y^-(x) = x$, $y^-(\tau(x)) = \tau(x)$ and there is nothing to prove.

Suppose $x \notin \mathbb{A}$. Then $[x, y^-(x)]\backslash\{y^-(x)\} \subset (x-\mathbb{R}_+\nu)\backslash\mathbb{A}$, thus $\tau([x,y^-(x)]\backslash\{y^-(x)\}) \subset (\tau(x)-\mathbb{R}_+\nu)\backslash\mathbb{A}$ and $\tau(y^-(x)) \in \mathbb{A}$. and the lemma follows. □

Lemma 6.6.5. *Let $\mu \in \mathbb{A}$ and $\lambda \in Y + \mathbb{A}_{in}$. One writes $\lambda = \lambda_{in} + \Lambda$, with $\lambda_{in} \in \mathbb{A}_{in}$ and $\Lambda \in Y$. Let $n \in G$ inducing the translation of vector Λ and $\tau = \tau_{\lambda_{in}} \circ n$. Then $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) = n\big(\rho_{+\infty}^{-1}(\{\mu\}) \cap \rho_{-\infty}^{-1}(\{0\})\big)$.*

Proof. This is a consequence of Lemma 6.6.3 and Lemma 6.6.1.

□

6.6.2 The Gindikin-Karpelevich finiteness

Lemma 6.6.6. *Let $x \in \mathcal{I}$. One has $\rho_{+\infty}(x) \in Y$ if and only if $\rho_{-\infty}(x) \in Y$ if and only if $x \in \mathcal{I}_0$. In this case, $\rho_{+\infty}(x) \leq_{Q^\vee_\mathbb{Z}} \rho_{-\infty}(x)$.*

Proof. For $x \in \mathcal{I}$, there exists $g_-, g_+ \in G$ such that $\rho_{-\infty}(x) = g_-.x$ and $\rho_{+\infty}(x) = g_+.x$, which shows the claimed equivalence because $Y = G.0 \cap \mathbb{A}$ (see Lemma 3.4.3).

Therefore if $x \in \mathcal{I}_0$, then $\rho_{+\infty}(x) - \rho_{-\infty}(x) \in Q^\vee_\mathbb{Z}$. Moreover, by Corollary 6.5.3, $\rho_{+\infty}(x) - \rho_{-\infty}(x) \in Q^\vee_{\mathbb{R}_-}$ and the lemma follows. □

Theorem 6.6.7. *Let $\mu \in \mathbb{A}$ and $\lambda \in Q^\vee_\mathbb{Z} + \mathbb{A}_{in}$. Then $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ is finite. If $\mu \notin -Q^\vee_\mathbb{N}$, this set is empty. Moreover*

$$|\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) = |\rho_{+\infty}^{-1}(\{\mu\}) \cap \rho_{-\infty}^{-1}(\{0\})|.$$

Proof. The condition on μ is a consequence of Lemma 6.6.6. By Theorem 6.5.1, for $\lambda \in Y \cap C^v_f$ sufficiently dominant, $\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\}) \subset \mathcal{S}^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})$. By Corollary 6.6.2, these sets are finite for λ sufficiently dominant. But by Lemma 6.6.5,

$$|\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})| = |\rho_{+\infty}^{-1}(\{\mu\}) \cap \rho_{-\infty}^{-1}(\{0\})|,$$

and hence $|\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})|$ is independent of $\lambda \in Y + \mathbb{A}_{in}$. Therefore, these sets are finite, which proves the theorem. □

Corollary 6.6.8. *Suppose that $\mathcal{I}$ is associated to an indefinite Kac-Moody matrix. Then if $x, y \in \mathcal{I}_0$, the set $\{z \in \mathcal{I}_0 |\ x \leq z \leq y\}$ is finite.*

Proof. By Lemma in 2.9 of [GR14], $\{z \in \mathcal{I}_0 |\ x \leq z \leq y\}$ is finite. By Corollaire 2.8 of [Rou11], if $z \in \mathcal{I}$ such that $x \leq z \leq y$, then $x \leq \rho_{-\infty}(z) \leq y$ and $x \leq \rho_{+\infty}(z) \leq y$. This is thus a consequence of Theorem 6.6.7 . □

Remark 6.6.9. *This corollary is specific to the indefinite case, see 5.4.3 for the affine case.*

6.7 Asymptotic equality

In this section, we prove the following theorem:

Theorem 6.7.1. *Let $\mu \in Q^\vee_\mathbb{Z}$. Then for $\lambda \in Y^{++} + \mathbb{A}_{in}$ sufficiently dominant, $\mathcal{S}^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\}) = \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\lambda+\mu\})$.*

The proof is postponed to the end of this section. Let us sketch it. Using the fact that Y^{++} is finitely generated (Lemma 6.7.3) we prove that there exists a finite set $F \subset Y^{++}$ such that for all $\lambda \in Y^{++} + \mathbb{A}_{in}$,

$$\rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \mathcal{S}^v(0,\lambda) \subset \bigcup_{f \in F | \lambda - f \in \overline{C^v_f}} \rho_{+\infty}^{-1}(\{\lambda+\mu\}) \cap \mathcal{S}^v(\lambda - f, f)$$

(this is Corollary 6.7.5, which generalizes Lemma 6.7.2). Then we use Subsection 6.6.1 to show that $\rho_{+\infty}^{-1}(\{\lambda+\mu\})\cap\mathcal{S}^v(\lambda-f,f)$ is the image of $\rho_{+\infty}^{-1}(\mu+f)\cap\mathcal{S}^v(0,f)$ by a "translation" $\tau_{\lambda-f}$ of G of vector $\lambda-f$ (which means that $\tau_{\lambda-f}$ induces the translation of vector $\lambda-f$ on $\mathbb{A}$). We fix $\nu\in C_f^v$ and set $y^-=y_\nu^-$. By Lemma 6.6.4,

$$y^-\big(\rho_{+\infty}^{-1}(\{\lambda+\mu\})\cap\mathcal{S}^v(0,\lambda)\big)\subset\bigcup_{f\in F}\tau_{\lambda-f}\circ y^-\big(\mathcal{S}^v(0,f)\cap\rho_{+\infty}^{-1}(\{\mu+f\})\big).$$

According to Section 5 of [GR14] (or Corollary 6.6.2, for all $f,\mu\in Y$, $\mathcal{S}^v(0,f)\cap\rho_{+\infty}^{-1}(\{\mu+f\})$ is finite. Consequently, for λ sufficiently dominant, $\bigcup_{f\in F}\tau_{\lambda-f}\big(y^-(\mathcal{S}^v(0,f)\cap\rho_{+\infty}^{-1}(\{\mu+f\})\subset C_f^v$ and one concludes with Lemma 6.5.5.

Lemma 6.7.2. *Let $\mu\in-Q_{\mathbb{N}}^\vee$ and $H=-h(\mu)+1\in\mathbb{N}$. Let $a\in Y$, $T\in[H,+\infty[$, $\nu\in Y^{++}$ and $x\in\mathcal{S}^v(a,T\nu)\cap\rho_{+\infty}^{-1}(\{a+T\nu+\mu\})$. Let $g\in G$ such that $g.a=a$ and $g.x=T\nu+a$. Then g fixes $[a,a+(T-H)\nu]$ and in particular $x\in\mathcal{S}^v(a+(T-H)\nu,H\nu)$.*

Proof. Let $\tau:[0,1]\to\mathbb{A}$ defined by $\tau(t)=a+(1-t)T\nu$. The main idea is to apply Lemma 6.5.2 to $\rho_{+\infty}\circ(g^{-1}.\tau)$ but we cannot do it directly because $\rho_{+\infty}\circ\tau$ is not a Hecke path with respect to $-C_f^v$. Let $\mathbb{A}'$ be the vectorial space $\mathbb{A}$ equipped with a structure of apartment of type $-\mathbb{A}$: the fundamental chamber of $\mathbb{A}'$ is $C_f'^v=-C_f^v$ etc ... Let $\mathcal{I}'$ be the set $\mathcal{I}$, whose apartments are the $-A$ where A runs over the apartment of $\mathcal{I}$. Then $\mathcal{I}'$ is a masure of standard apartment $\mathbb{A}'$. We have $a\le x$ in $\mathcal{I}$ and so $x\le' a$ in $\mathcal{I}'$.

Then the image π of $g^{-1}.\tau$ by $\rho_{+\infty}$ is a Hecke path of shape $-T\nu$ from $\rho_{+\infty}(x)=a+T\nu+\mu$ to a. By Lemma 6.5.2 (for $\mathcal{I}'$), for $t>-h(\mu)/T$, $\pi'(t)=-T\nu$, and thus Lemma 4.2.14 (for $\mathcal{I}'$ and $\mathbb{A}'$) implies $\rho_{+\infty}(g^{-1}.\tau(t))=g^{-1}.\tau(t)$ for all $t>-h(\mu)/T$. Therefore, $g^{-1}.\tau_{|]-h(\mu)/T,1]}$ is a segment in $\mathbb{A}$ ending in a, with derivative $-T\nu$ and thus, for all $t\in]\frac{-h(\mu)}{T},1]$, $g^{-1}.\tau(t)=\tau(t)$. In particular, g fixes $[a,\tau(\frac{H}{T})]=[a,a+(T-H)\nu]$, and $d^v(a+(T-H)\nu,x)=d^v(g^{-1}.(a+(T-H)\nu),g^{-1}.x)=d^v(a+(T-H)\nu,a+T\nu)=H\nu$. □

Lemma 6.7.3. *There exists a finite set $E\subset Y$ such that $Y^{++}=\sum_{e\in E}\mathbb{N}e$.*

Proof. The set $Y_{in}=Y\cap\mathbb{A}_{in}$ is a lattice in the vectorial space it spans. Consequently, it is a finitely generated $\mathbb{Z}$-module and thus a finitely generated monoid. Let E_1 be a finite set generating Y_{in} as a monoid.

Recall the definition of $\prec$ of Subsection 5.4.1. Let $Y_{\succ 0}=Y^{++}\backslash Y_{in}$. Let $\mathcal{P}=\{a\in Y_{\succ 0}|a\neq b+c\ \forall b,c\in Y_{\succ 0}\}$. Let $\alpha:Y^{++}\to\mathbb{N}^I$ such that $\alpha(x)=(\alpha_i(x))_{i\in I}$ for all $x\in Y^{++}$. Let $a,b\in\mathcal{P}$. If $\alpha(a)\prec\alpha(b)$, then $b=b-a+a$, with $a,b-a\in Y_{\succ 0}$, which is absurd and by symmetry we deduce that $\alpha(a)$ and $\alpha(b)$ are not comparable for $\prec$. Therefore, by Lemma 5.4.3, $\alpha(\mathcal{P})$ is finite. Let E_2 be a finite set of $Y_{\succ 0}$ such that $\alpha(\mathcal{P})=\{\alpha(x)|x\in E_2\}$. Then $Y^{++}=\sum_{e\in E_2}\mathbb{N}e+Y_{in}=\sum_{e\in E}\mathbb{N}e$, where $E=E_1\cup E_2$. □

Lemma 6.7.4. *Let $\mu\in-Q_{\mathbb{N}}^\vee$, $H=-h(\mu)+1\in\mathbb{N}$ and $a\in Y$. Let $\lambda\in Y^{++}$. One writes $\lambda=\sum_{e\in E}\lambda_e e$ with $\lambda_e\in\mathbb{N}$ for all $e\in E$. Let $e\in E$. Then if $\lambda_e\geq H$,*

$$\mathcal{S}^v(a,\lambda)\cap\rho_{+\infty}^{-1}(\{a+\lambda+\mu\})\subset\mathcal{S}^v(a+(\lambda_e-H)e,\lambda-(\lambda_e-H)e).$$

Proof. Let $x\in\mathcal{S}^v(a,\lambda)\cap\rho_{+\infty}^{-1}(\{a+\lambda+\mu\})$ and $g\in G$ fixing a such that $g.x=a+\lambda$. Let $z=g^{-1}(a+\lambda_e e)$.

Then one has $d^v(a,z)=\lambda_e e$ and $d^v(z,x)=\lambda-\lambda_e e$.

According to Lemma 2.4.b) of [GR14] (adapted because one considers Hecke paths with respect to C_f^v), one has:

$\rho_{+\infty}(z) - a \leq_{Q^\vee_\mathbb{Z}} d^v(a,z) = \lambda_e e$ and $\rho_{+\infty}(x) - \rho_{+\infty}(z) \leq_{Q^\vee_\mathbb{Z}} d^v(z,x) = \lambda - \lambda_e e$. Therefore,

$$a + \lambda + \mu = \rho_{+\infty}(x) \leq_{Q^\vee_\mathbb{Z}} \rho_{+\infty}(z) + \lambda - \lambda_e e \leq_{Q^\vee_\mathbb{Z}} a + \lambda.$$

Hence, $\rho_{+\infty}(z) = a + \lambda_e e + \mu'$, with $\mu \leq_{Q^\vee_\mathbb{Z}} \mu' \leq_{Q^\vee_\mathbb{Z}} 0$. One has $-h(\mu') + 1 \leq H$. By Lemma 6.7.2, g fixes $[a, a + (\lambda_e - H)e]$ and thus g fixes $a + (\lambda_e - H)e$.

As $d^v(g^{-1}.(a + (\lambda_e - H)e), x) = \lambda - (\lambda_e - H)e$, $x \in \mathcal{S}^v(a + (\lambda_e - H)e, \lambda - (\lambda_e - H)e)$. □

Corollary 6.7.5. *Let $\mu \in -Q^\vee_\mathbb{N}$. Let $H = -h(\mu) + 1$. Let $\lambda \in Y^{++}$. We fix a writing $\lambda = \sum_{e \in E} \lambda_e e$, with $\lambda_e \in \mathbb{N}$ for all $e \in E$. Let $J = \{e \in E | \lambda_e \geq H\}$. Then $\mathcal{S}^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \subset \mathcal{S}^v(\lambda - H\sum_{e \in J} e - \sum_{e \notin J} \lambda_e e, H\sum_{e \in J} e + \sum_{e \notin J} \lambda_e)$.*

Proof. This is a generalization by induction of Lemma 6.7.4. □

We now prove Theorem 6.7.1:

Let $\mu \in Q^\vee_\mathbb{Z}$. Then for $\lambda \in Y^{++} + \mathbb{A}_{in}$ sufficiently dominant, $\mathcal{S}^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\}) = \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})$.

Proof. Theorem 6.5.1 yields one inclusion. It remains to show that $\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \mathcal{S}^v(0,\lambda) \subset \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})$ for λ sufficiently dominant.

Let $H = -h(\mu) + 1$ and $F = \{\sum_{e \in E} \nu_e e | (\nu_e) \in [\![0, H]\!]^E\}$. This set is finite. Let $\lambda \in Y^{++} + \mathbb{A}_{in}$, $\lambda = \lambda_{in} + \Lambda$, with $\lambda_{in} \in \mathbb{A}_{in}$ and $\Lambda \in Y^{++}$.

Let $x \in \mathcal{S}^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})$. Then by Corollary 6.7.5, there exists $f \in F$ such that $\lambda - f \in \overline{C^v_f}$ and $x \in \mathcal{S}^v(\lambda - f, f)$ (one can take $f = H\sum_{e \in J} e + \sum_{e \notin J} \lambda_e$ where J is as in Corollary 6.7.5). Let n be an element of G inducing the translation of vector $\Lambda - f = \lambda - \lambda_{in} - f$ on $\mathbb{A}$ and $\tau_{\lambda,f} = \tau_{\lambda_{in}} \circ n$. Then $x \in \tau_{\lambda,f}(B_f)$ where $B_f = \mathcal{S}^v(0,f) \cap \rho_{+\infty}^{-1}(\{\mu + f\})$.

Let $B = \bigcup_{f \in F} B_f$. Then one has proved that

$$\mathcal{S}^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\}) \subset \bigcup_{f \in F} \tau_{\lambda,f}(B).$$

By Section 5 of [GR14] (or Corollary 6.6.2), B_f is finite for all $f \in F$ and thus $B = \bigcup_{f \in F} B_f$ is finite. Let $\nu \in C^v_f$ and $y^- = y^-_\nu$. Then $y^-(B)$ is finite and for λ sufficiently dominant, $\bigcup_{f \in F} \tau_{\lambda,f} \circ y^-(B) \subset C^v_f$. Moreover, according to Lemma 6.6.4, $\bigcup_{f \in F} \tau_{\lambda,f} \circ y^-(B) = \bigcup_{f \in F} y^- \circ \tau_{\lambda,f}(B)$. Hence

$$y^-\big(\mathcal{S}^v(0,\lambda) \cap \rho_{+\infty}^{-1}(\{\lambda + \mu\})\big) \subset C^v_f$$

for λ sufficiently dominant. Eventually one concludes with Lemma 6.5.5. □

6.8 The Gindikin-Karpelevich formula

We now suppose that $G = \mathbf{G}(\mathcal{K})$, where $\mathbf{G}$ is a split Kac-Moody group and $\mathcal{K}$ is a local field. Let $h : Q^\vee_\mathbb{Z} \to \mathbb{Z}$ defined by $h(\sum_{i \in I} n_i \alpha_i^\vee) = \sum_{i \in I} n_i$ for all $(n_i) \in \mathbb{Z}^I$. Let q be the residue cardinal of $\mathcal{K}$. By Theorem 6.6.7, if $\lambda \in Q^\vee_\mathbb{Z}$,

$$\mathcal{G} := \sum_{\mu \in Q^\vee_\mathbb{Z}} |\rho_{+\infty}^{-1}(\{\lambda + \mu\}) \cap \rho_{-\infty}^{-1}(\{\lambda\})| q^{h(\mu)} e^\mu \in \mathbb{Z}[[Y]]$$

is well defined and does not depend on λ. In [BGKP14], Braverman, Kazhdan, Garland and Patnaik compute $\mathcal{G}$ when $\mathbf{G}$ is an untwisted affine Kac-Moody groups. For this they use a generalization of Macdonald formula for these groups. This formula explicits the image of certain elements of the spherical Hecke algebra by the Satake isomorphism. Let us review these results.

6.8.1 Macdonald's formula

In [Mac71], Macdonald establishes a formula which computes $S(c_\lambda)$ for $\lambda \in Y^{++}$ when G is a reductive group, where S is the Satake isomorphism (see Theorem 5.4.10 for the definition) and $c_\lambda = \mathbb{1}_{K\lambda K} \in \mathcal{H}_s$. Braverman, Kazhdan and Patnaik generalized this formula in Theorem of 7.2.3 of [BKP16] when G is an affine group. Bardy-Panse, Gaussent and Rousseau generalized this formula to the general Kac-Moody setting, using masures and paths, see Theorem 7.3 of [BPGR17]. In this subsection, we give this formula.

Let σ be an indeterminate (one will specialize σ at $q^{-\frac{1}{2}}$) and let $\mathbb{Z}_\sigma$ denote $\mathbb{Z}[\sigma, \sigma^{-1}]$. If $\alpha^\vee \in (\Phi_{re})^\vee := W^v.\{\alpha_i^\vee | i \in I\}$, one sets $c'(\alpha^\vee) = \frac{1-\sigma^2 e^{-\alpha^\vee}}{1-e^{-\alpha^\vee}}$. This defines an element of $\mathbb{Z}_\sigma[[Y]]$. Indeed, $(\Phi_{re})^\vee \subset Q^\vee_\mathbb{N} \cup -Q^\vee_\mathbb{N}$. Thus if $\alpha^\vee \in Q^\vee_\mathbb{N}$, $c'(\alpha^\vee) = (1-\sigma^2 e^{-\alpha^\vee}) \sum_{n\in\mathbb{N}} e^{-n\alpha^\vee} \in \mathbb{Z}_\sigma[[Y]]$ and if $\alpha^\vee \in -Q^\vee_\mathbb{N}$, $c'(\alpha^\vee) := \frac{e^{\alpha^\vee}-\sigma^2}{e^{\alpha^\vee}-1} = -(e^{\alpha^\vee}-\sigma^2)\sum_{n\in\mathbb{N}} e^{k\alpha^\vee} \in \mathbb{Z}_\sigma[[Y]]$.

Let

$$\mathbb{Z}_\sigma(Y) = \{\sum_{\lambda\in Y} a_\lambda e^\lambda \in \mathbb{Z}_\sigma[[Y]] \cap \mathrm{Fr}\big(\mathbb{Z}_\sigma[Y]\big) \text{ such that } \sum_{\lambda \in Y} a_{w.\lambda} e^\lambda \in \mathbb{C}[[Y]]\ \forall w \in W^v\},$$

where $\mathrm{Fr}\big(\mathbb{Z}_\sigma[Y]\big)$ is the field of fractions of $\mathbb{Z}_\sigma[Y]$. The group W^v acts on $\mathbb{Z}_\sigma(Y)$ by setting $w.\sum_{\lambda\in Y} a_\lambda e^\lambda = \sum_{\lambda\in Y} a_\lambda e^{w.\lambda}$ for all $\sum_{\lambda\in Y} a_\lambda e_\lambda \in \mathbb{Z}_\sigma(Y)$. A sequence $(\phi_n) \in (\mathbb{Z}_\sigma(Y))^\mathbb{N}$ is said to be convergent if for all $w \in W^v$, $(w.\phi_n)$ converges in $\mathbb{Z}_\sigma(Y)^\mathbb{N}$. Let $\mathbb{Z}_\sigma(\!(Y)\!)$ denote the completion of $\mathbb{Z}_\sigma(Y)$ for this notion of convergence.

Let $\Delta = \prod_{\alpha\in\Phi^+_{re}} c'(-\alpha^\vee)$. By 5.3 of [BPGR17], $\Delta \in \mathbb{Z}_\sigma(\!(Y)\!)$ and is invertible.

If $W' \subset W^v$, one sets $W'(\sigma^2) = \sum_{w\in W'} \sigma^{2\ell(w)} \in \mathbb{Z}(\!(\sigma)\!)$ and if $\lambda \in Y^{++}$, one denotes by W^v_λ its fixer in W^v.

By Theorem 7.3 of [BPGR17], if $\lambda \in Y^{++}$, we have the **Macdonald's formula**:

$$S(c_\lambda) = q^{h(\lambda)} \left(\frac{W^v(\sigma^2)}{\sum_{w\in W^v} w.\Delta}\right)_{\sigma^2=q^{-1}} . \left(\frac{\sum_{w\in W^v}(w.\Delta)e^{w.\lambda}}{W^v_\lambda(\sigma^2)}\right)_{\sigma^2=q^{-1}}.$$

The elements $W^v(\sigma^2)$, $\sum_{w\in W^v} w.\Delta$, $\sum_{w\in W^v}(w.\Delta)e^{w.\lambda}$ and $W^v_\lambda(\sigma^2)$ are defined in $\mathbb{Z}(\!(\sigma)\!)[[Y]]$ and not in $\mathbb{Z}_\sigma[[Y]]$ a priori. However $\frac{W^v(\sigma^2)}{\sum_{w\in W^v} w.\Delta}$ and $\frac{\sum_{w\in W^v}(w.\Delta)e^{w.\lambda}}{W^v_\lambda(\sigma^2)}$ are in $\mathbb{Z}_\sigma[[Y]]$ and can be specialized at $\sigma = q^{-\frac{1}{2}}$.

6.8.2 Relation between H_0 and $\mathcal{G}$

Let $H_0 := \frac{\sum_{w\in W^v} w.\Delta}{W^v(\sigma^2)}$. Let $\Delta_q = \Delta(\sigma^2 = q^{-1})$ and $H_{0,q} = H_0(\sigma^2 = q^{-1})$. The aim of this subsection is to prove the following theorem.

Theorem 6.8.1. *One has*

$$\mathcal{G} := \sum_{\mu\in Q^\vee_\mathbb{Z}} |\rho^{-1}_{+\infty}(\{\lambda+\mu\}) \cap \rho^{-1}_{-\infty}(\{\lambda\})| q^{h(\mu)} e^\mu = \frac{\Delta_q}{H_{0,q}}.$$

This is Theorem 1.13 of [BGKP14]. We now detail the proof of Braverman, Garland, Kazhdan and Patnaik.

Let $\nu \in Y$. One sets $\pi_\nu : \begin{matrix} \mathbb{C}[[Y]] \to \mathbb{C} \\ \sum_{\mu\in Y} x_\mu e^\mu \mapsto x_\nu \end{matrix}$ and $h : \begin{matrix} Q^\vee_\mathbb{Z} \to \mathbb{Z} \\ \sum x_i \alpha_i^\vee \mapsto \sum x_i \end{matrix}$.

Let $\mathbb{C}[[-Q^\vee_\mathbb{N}]] = \{P \in \mathbb{C}[[Y]] | \pi_\nu(P) = 0\ \forall \nu \in Y \setminus -Q^\vee_\mathbb{N}\}$. Then $\mathbb{C}[[-Q^\vee_\mathbb{N}]]$ is a subalgebra of $\mathbb{C}[[Y]]$.

Lemma 6.8.2. *Let $w \in W^v$. Then $w.\Delta_q \in \mathbb{C}[[-Q^\vee_{\mathbb{N}}]]$.*

Proof. Let $\Phi_w = \{\alpha \in \Phi^+_{all} |\ w.\alpha^\vee < 0\}$. One has

$$w.\Delta_q = \prod_{\alpha\in\Phi_w} w.c'_q(\alpha^\vee) \prod_{\alpha\in\Phi^+_{re}\backslash\Phi_w} w.c'_q(\alpha^\vee) = \prod_{\alpha\in\Phi_w} c'_q(w.\alpha^\vee) \prod_{\alpha\in\Phi^+_{re}\backslash\Phi_w} c'_q(w.\alpha^\vee),$$

where if $\alpha^\vee \in Q^\vee_{\mathbb{N}} \cup -Q^\vee_{\mathbb{N}}$, $c'_q(\alpha^\vee) = \frac{e^{\alpha^\vee}-q^{-1}}{e^{\alpha^\vee}-1} \in \mathbb{C}[[-Q^\vee_{\mathbb{N}}]]$. If $\alpha \in \Phi^+_{re}\backslash\Phi_w$, $c'_q(w.\alpha) \in \mathbb{C}[[-Q^\vee_{\mathbb{N}}]]$. Let $\alpha \in \Phi_w$. Then $c'_q(w.\alpha) = \frac{e^{w.\alpha^\vee}-q^{-1}}{e^{w.\alpha^\vee}-1} \in \mathbb{C}[[-Q^\vee_{\mathbb{N}}]]$ and the lemma follows. □

Lemma 6.8.3. *Let $\gamma \in Y$. Then for $\lambda \in Y^{++}$ sufficiently dominant,*

$$n_\lambda(\lambda+\gamma) = |\rho^{-1}_{+\infty}(\{\gamma\}) \cap \rho^{-1}_{-\infty}(\{0\})|.$$

Proof. For $\lambda \in C^v_f$, one sets $\mathcal{S}^v_{op}(0,\lambda) = \{x \in \mathcal{I}|\ x \leq 0 \text{ and } d^v(x,0) = \lambda\}$. Using the masure $\mathcal{I}'$ of Lemma 6.7.2, Theorem 6.7.1 yields the equality:

$$|\mathcal{S}^v_{op}(0,\lambda) \cap \rho^{-1}_{-\infty}(\{-\lambda-\gamma\})| = |\rho^{-1}_{+\infty}(\{-\lambda\}) \cap \rho^{-1}_{-\infty}(\{-\lambda-\gamma\})|,$$

for λ sufficiently dominant. By using $g \in G$ inducing a translation of vector $\lambda+\gamma$ on $\mathbb{A}$, Lemma 6.6.3 and the G-invariance of d^v, we deduce that

$$|n_\lambda(\lambda+\gamma) = \mathcal{S}^v_{op}(\lambda+\gamma,\lambda) \cap \rho^{-1}_{-\infty}(\{0\})| = |\rho^{-1}_{+\infty}(\{\gamma\}) \cap \rho^{-1}_{-\infty}(\{0\})|,$$

which is our assertion. □

Lemma 6.8.4. *Let $\gamma \in Y$, $\lambda \in C^v_f \cap Y$, $P \in \mathbb{C}[[Y]]$ and $k \in \mathbb{N}^*$. Then $\sum_{w\in W}(w.\Delta)e^{w(k\lambda)} \in \mathbb{Z}_\sigma[[Y]]$ and for k large enough,*

$$\pi_{k\lambda+\gamma}(P\sum_{w\in W}(w.\Delta_q)e^{w(k\lambda)}) = \pi_{k\lambda+\gamma}(P\Delta_q e^{k\lambda})$$

Proof. As $k\lambda \in C^v_f$, $W^v_{k\lambda} = \{1\}$ and thus $\frac{\sum_{w\in W^v}(w.\Delta)e^{w.k\lambda}}{W^v_{k\lambda}(\sigma^2)} = \sum_{w\in W^v}(w.\Delta)e^{w.k\lambda} \in \mathbb{Z}_\sigma[[Y]]$.

Let $w \in W$. By Lemma 2.2.9, one has $w\lambda = \lambda - \mu_w$, with $\mu_w \in Q^\vee_{\mathbb{N}}$. When $w \neq 1$, $\mu_w \neq 0$ and thus $h(\mu_w) \geq 1$.

Let $k \in \mathbb{N}$ and $w \in W\backslash\{1\}$. Then $\pi_{k\lambda+\gamma}\big(P(w.\Delta_q)e^{w(k\lambda)}\big) = \pi_\gamma\big(P(w.\Delta_q)e^{-k\mu_w}\big)$.

One writes $P = \sum_{\lambda\in Y} a_\lambda e^\lambda$. Let $J \subset Y$ be a finite set such that for all $\lambda \in \mathrm{supp}(P)$, there exists $j \in J$ such that $\lambda \leq_{Q^\vee_{\mathbb{Z}}} j$. By Lemma 6.8.2, $w.\Delta_q \in \mathbb{Z}_q[[-Q^\vee_{\mathbb{N}}]]$ and thus $P(w.\Delta_q) = \sum_{\mu\in Q^\vee_{\mathbb{Z}}} b_\mu e^\mu$, with $(b_\mu) \in \mathbb{C}^Y$ such that $b_\mu \neq 0$ implies $\mu \leq_{Q^\vee_{\mathbb{Z}}} j$ for some $j \in J$. Suppose $k \geq |h(\gamma)| + \max\{|h(j)|\ |j \in J\} + 1$. Then, one has $\pi_\gamma\big(P(w.\Delta_q)e^{-k\mu_w}\big) = 0$ and the lemma follows. □

Proof of Theorem 6.8.1: Let $\lambda \in C^v_f \cap Y$ and $\gamma \in Y$. Then by Lemma 6.8.4 and Macdonald's formula, for k sufficiently dominant, one has :

$$\pi_{k\lambda+\gamma}(S(c_{k\lambda})) = q^{h(k\lambda+\gamma)}n_{k\lambda}(k\lambda+\gamma) = \pi_{k\lambda+\gamma}(q^{h(k\lambda)}\frac{\Delta_q}{H_{0,q}}e^{k\lambda}) = \pi_\gamma(q^{h(k\lambda)}\frac{\Delta_q}{H_{0,q}}).$$

By Lemma 6.8.3, for k sufficiently large,

$$q^{h(k\lambda+\gamma)}n_{k\lambda}(k\lambda+\gamma) = q^{h(k\lambda)}\pi_\nu(\mathcal{G})$$

and thus $\pi_\nu(\mathcal{G}) = \pi_\nu(\frac{\Delta_q}{H_{0,q}})$. As this is true for all ν in Y, $\mathcal{G} = \frac{\Delta_q}{H_{0,q}}$. □

6.8.3 Expression for H_0?

Suppose that G is reductive. Then $H_0 = 1$ by [Mac71], see also Corollary 2.17 of [NR03].

Let Φ_{im} be the set of imaginary roots of G, Φ_{all} be the set of all roots of G, $\Phi_{all}^+ = \Phi_{all} \cap Q_{\mathbb{N}}$ and $\Phi_{im}^+ = \Phi_{im} \cap Q_{\mathbb{N}}$.

If $\alpha \in \Phi_{im}$, one denotes by m_α the multiplicity of α (see 2.2.1.2).

Let $\Delta^{im} = \prod_{\alpha \in \Phi_{im}^+} c'(\alpha^\vee) \in \mathbb{Z}_\sigma(\!(Y)\!)$ and $\Delta^{all} = \Delta.\Delta^{im}$. Let $w \in W^v$. By Lemma 1.3.14 of [Kum02], $w.\Phi_{im}^+ = \Phi_{im}^+$, $w.\Delta^{im} = \Delta^{im}$. Therefore $w.\Delta^{all} = (w.\Delta)\Delta^{im}$ for all $w \in W^v$.

Let
$$H_0^{all} := \frac{\sum_{w \in W^v} w.\Delta^{all}}{W^v(\sigma^2)} = \Delta^{im}.H_0.$$

By Theorem 6.8.1, one has
$$\mathcal{G} = \frac{\Delta_q^{all}}{H_{0,q}^{all}}.$$

In [BGKP14], using 3.8 of [Mac03] and [Che95], the authors give an expression for H_0, when G is affine and the underlying semi-simple Lie algebra $\mathring{\mathfrak{g}}$ is of type A, D or E (see Chapter 4 of [Kac94]). One writes $I = [\![0, \ell]\!]$ with $\ell \in \mathbb{N}$. Then

$$H_{0,q}^{all} = \prod_{j=1}^{\ell} \prod_{i=1}^{\infty} \frac{1 - q^{-m_j} e^{-i\delta}}{1 - q^{-(m_j+1)} e^{-i\delta}},$$

where δ is the smallest positive imaginary root and the m_j are the exponents of $\mathring{\mathfrak{g}}$, defined by the relation:
$$\mathring{W}^v(\sigma^2) = \prod_{i=1}^{\ell} \frac{1 - \sigma^{2(m_j+1)}}{1 - \sigma^2},$$

where $\mathring{W}^v$ is the vectorial Weyl group of $\mathring{\mathfrak{g}}$.

When G is indefinite however, it seems that there is currently no simple expression of H_0, even on simple examples.

In [Fei80], Feingold computes $\prod_{\alpha \in \Phi_{all}^+} (1 - e^{-\alpha})^{m_\alpha}$ for $\mathfrak{g}$ associated to the indefinite (hyperbolic) matrix $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$. He also gives explicitly the set of imaginary roots and its formula gives a way to compute the multiplicities of imaginary roots. It would be interesting to compute H_0^{all} in this case. When G is affine, the set of imaginary roots is easy to describe and the multiplicities of these roots are known, by Corollary 2.3.4: the multiplicities are all equal to $|I| - 1$. A difficulty in the indefinite case is that Φ_{im} is harder to describe and the multiplicities of imaginary roots are not easy to compute, see [CFL14] for a survey on the problem of multiplicities in the indefinite case.

Chapter 7

Distances on a masure

7.1 Introduction

Let $\mathcal{I}$ be a Bruhat-Tits building associated to a split reductive group G over a local field. Then $\mathcal{I}$ is equipped with a distance d such that G acts isometrically on $\mathcal{I}$ and such that the restriction of d to each apartment is a euclidean distance. This distance is an important tool in the study of $\mathcal{I}$. We will show that we cannot equip masures which are not buildings with distances having these properties but it seems natural to ask whether we can define distances on a masure having "good" properties. We limit our study to distances inducing the affine topology on each apartment. We show that under assumptions of continuity for retractions, $\mathcal{I}$ cannot be complete nor locally compact (see Section 7.2). We show that there is no distance on $\mathcal{I}$ such that the restriction to each apartment is a norm. However, we prove the following theorems (Corollary 7.4.8, Lemma 7.3.9, Corollary 7.4.9 and Theorem 7.4.14): let $\mathfrak{q}$ be a sector germ of $\mathcal{I}$, then there exists a distance d on $\mathcal{I}$ having the following properties:

- the topology induced on each apartment is the affine topology
- each retraction with center $\mathfrak{q}$ is 1-Lipschitz continuous
- each retraction with center a sector-germ of the same sign as $\mathfrak{q}$ is Lipschitz continuous
- each $g \in G$ is Lipschitz continuous when we see it as an automorphism of $\mathcal{I}$.

We call the distances constructed in the proof of this theorem distances of **positive** or of **negative type**, depending on the sign of $\mathfrak{q}$. A distance of positive or negative type is called a signed distance. We prove that all distances of positive type on a masure (resp. of negative type) are equivalent, where we say that two distances d_1 and d_2 are equivalent if there exist $k, \ell \in \mathbb{R}_+^*$ such that $kd_1 \leq d_2 \leq ld_1$ (this is Theorem 7.4.7). We thus get a **positive topology** $\mathscr{T}_+$ and a **negative topology** $\mathscr{T}_-$. We prove (Corollary 7.5.4) that these topologies are different when $\mathcal{I}$ is not a building. When $\mathcal{I}$ is a building these topologies are the usual topology on a building (Proposition 7.4.15).

Let $\mathcal{I}_0$ be the orbit of some special vertex under the action of G. If $\mathcal{I}$ is not a building, $\mathcal{I}_0$ is not discrete for $\mathscr{T}_-$ and $\mathscr{T}_+$. We also prove that if ρ is a retraction centered at a positive (resp. negative) sector-germ, ρ is not continuous for $\mathscr{T}_-$ (resp. $\mathscr{T}_+$), see Proposition 7.5.3. For these reasons we introduce **mixed distances**, which are the sum of a distance of positive type and of a distance of negative type. We then have the following theorem (Theorem 7.5.7): all the mixed distances on $\mathcal{I}$ are equivalent; moreover, if d is a mixed distance and $\mathcal{I}$ is equipped with d we have:

- each $g : \mathcal{I} \to \mathcal{I} \in G$ is Lipschitz continuous
- each retraction centered at a sector-germ is Lipschitz continuous
- the topology induced on each apartment is the affine topology
- the set $\mathcal{I}_0$ is discrete.

The topology $\mathscr{T}_c$ associated to mixed distances is the initial topology with respect to the retractions of $\mathcal{I}$ (see Corollary 7.5.10).

Let us explain how we define distances of positive or negative type. Let A be an apartment and Q be a sector of A. Maybe considering $g.A$ for some g in G, one can suppose that $A = \mathbb{A}$, the standard apartment of $\mathcal{I}$ and $Q = C_f^v$, the fundamental chamber of $\mathbb{A}$ (or $Q = -C_f^v$ but this case is similar). Let N be a norm on $\mathbb{A}$. If $x \in \mathcal{I}$, there exists an apartment A_x containing x and $+\infty$ (which means that A_x contains a sub-sector of C_f^v). For $q \in \overline{C_f^v}$, we define $x + q$ as the translate of x by q in A_x. When q is made more and more dominant, $x + q \in C_f^v$. Therefore, for all $x, x' \in \mathcal{I}$, there exists $q, q' \in C_f^v$ such that $x + q = x' + q'$. We then define $d(x, x')$ to be the minimum of the $N(q) + N(q')$ for such couples q, q'.

We thus obtain a distance for each sector Q and for each norm N on an apartment containing Q. We show that this distance only depends on the germ of Q and on N (in Subsection 7.3.3) .

Framework We ask $\mathcal{I}$ to be an abstract masure and G to be a strongly transitive group of (positive, type-preserving) automorphisms of $\mathcal{I}$. This applies in particular to almost-split Kac-Moody groups over local fields. We assume that $\mathcal{I}$ is semi-discrete (which means that if M is a wall of $\mathbb{A}$, the set of wall parallel to M is discrete) and that $\mathcal{I}$ is thick of finite thickness (which means that for each panel, the number of chamber containing it is finite and greater or equal to three). The group G is a group acting strongly transitively on $\mathcal{I}$. Let N be the stabilizer of $\mathbb{A}$ in G and $\nu : N \to \mathrm{Aut}(\mathbb{A})$ be the induced morphism. We assume moreover that $\nu(N) = Y \rtimes W^v$.

Organization of the chapter In Section 7.2, we study the properties that distances on a masure cannot satisfy.

In Section 7.3, we construct the signed distances.

In Section 7.4, we prove that all the distances of the same sign are equivalent.

In Section 7.5, we prove that when $\mathcal{I}$ is not a building, the distances of opposed signs are non equivalent, introduce the mixed distances and study their properties.

In Section 7.6 we prove that for the distances we introduced, the masure is contractible.

7.2 Restrictions on the distances

7.2.1 Study of metric properties of W^v

In this subsection we prove that when W^v is infinite there exists no norm on $\mathbb{A}$ such that W^v is a group of isometries.

Let $\mathbb{A}_{in} = \bigcap_{i \in I} \ker \alpha_i = \bigcap_{w \in W^v} \ker(\mathrm{Id} - w)$. Let $\mathbb{A}_{es} = \mathbb{A}/\mathbb{A}_{in}$. If $x \in \mathbb{A}$, one denotes by $\overline{x}$ its image in $\mathbb{A}_{es}$. Each $w \in W^v$ induces an automorphism $\overline{w}$ of $\mathbb{A}_{es}$ by the formula: $\overline{w}(\overline{x}) = \overline{w(x)}$ for all $x \in \mathbb{A}$. By Lemma 2.2.10, the map $W^v \to \mathrm{Aut}(\mathbb{A}_{es})$ mapping each $w \in W^v$ on $\overline{w}$ is injective. Let $\overline{Y} = \{\overline{y} | y \in Y\}$.

Lemma 7.2.1. *The set $\overline{Y}$ is a lattice of $\mathbb{A}_{es}$.*

Proof. As Y spans $\mathbb{A}$, $\overline{Y}$ spans $\mathbb{A}_{es}$. Let $|\ |$ be a norm on $\mathbb{A}$. One equips $\mathbb{A}_{es}$ with the quotient norm $|\ |$: for all $x \in \mathbb{A}$, $|\overline{x}| = \inf_{y \in \overline{x}} |y|$.

Let $(z_n) \in \overline{Y}^{\mathbb{N}}$ be a sequence converging towards 0. For all $n \in \mathbb{N}$, one writes $z_n = \overline{y_n}$ with $y_n \in Y$, one chooses $u_n \in \overline{y_n}$ such that $|u_n| \leq 2|z_n|$ and one writes $u_n = x_n + y_n$, with $x_n \in \mathbb{A}_{in}$.

As $\mathbb{A}$ is finite dimensional, there exists $w_1, \ldots, w_\ell \in W^v$ such that $\mathbb{A}_{in} = \bigcap_{i=1}^{\ell} \ker(w_i - \mathrm{Id})$. For all $i \in [\![1, \ell]\!]$, $w_i.u_n = x_n + w_i.y_n \to 0$. Consequently $y_n - w_i.y_n \to 0$ for all $i \in [\![1, \ell]\!]$. As $y_n - w_i.y_n \in Y$ for all $(i, n) \in [\![1, \ell]\!] \times \mathbb{N}$, $y_n \in \mathbb{A}_{in}$ for n large enough. Thus $z_n = \overline{y_n} = 0$ for n large enough, which shows that $\overline{Y}$ is a lattice of $\mathbb{A}_{es}$. □

Lemma 7.2.2. *Let Z be a finite dimensional vectorial space on $\mathbb{R}$ and H be a subgroup of* $\mathrm{GL}(Z)$. *Suppose that H is infinite and stabilizes a lattice L. Then for all norm on Z, H is not a group of isometries.*

Proof. Let $(e_1, \ldots, e_k)$ be a basis of Z such that $L = \mathbb{Z}e_1 \oplus \ldots \oplus \mathbb{Z}e_k$. Consider $f : H \to L^k$ defined by $f(h) = (h.e_1, \ldots, h.e_k)$ for all $h \in H$. Then f is injective and thus there exists $i \in [\![1, k]\!]$ such that $\{h.e_i | h \in H\}$ is infinite. As $\{h.e_i | h \in H\} \subset L$, $\{h.e_i | h \in H\}$ is not bounded, which shows the lemma. □

Corollary 7.2.3. *Suppose that there exists a norm $|\ |$ on $\mathbb{A}$ such that W is a group of isometries. Then W^v is finite.*

7.2.2 Restrictions on the distances

In this section, we show that some properties cannot be satisfied by distances on masures. If A is an apartment of $\mathcal{I}$, we show that there exist apartments branching at all wall of A (this is Lemma 7.2.4). This implies that if $\mathcal{I}$ is not a building the interior of each apartments is empty for the distances we study. We write $\mathcal{I}$ as a countable union of apartment and then use Baire's Theorem to show that under rather weak assumption of regularity for retractions, a masure cannot be complete or locally compact for the distances we study.

Let us show a slight refinement of Corollaire 2.10 of [Rou11]:

Lemma 7.2.4. *Let A be an apartment of $\mathcal{I}$ and D be a half-apartment of A. Then there exists an apartment B such that $A \cap B = D$.*

Proof. Let P be a panel of the wall of A. Let C be a chamber containing P and not included in A, which exists by thickness of $\mathcal{I}$. By Proposition 2.9 1) of [Rou11], there exists an apartment B containing D and C. By (MA ii), $A \cap B$ is a half-apartment containing D. Moreover $A \cap B$ does not contain C and thus $A \cap B = D$.

□

Proposition 7.2.5. *Suppose that there exists a distance $d_{\mathcal{I}}$ on $\mathcal{I}$ such that for each apartment A, $d_{\mathcal{I}|A^2}$ is induced by some norm. Then $\mathcal{I}$ is a building and $d_{\mathcal{I}|\mathbb{A}^2}$ is W-invariant.*

Proof. Let A and B be two apartments sharing a sector and $\phi : AA \overset{\to}{\cap} BB$. Let us prove that $\phi : (A, d_{\mathcal{I}}) \to (B, d_{\mathcal{I}})$ is an isometry. Let $d' : A \times A \to \mathbb{R}_+$ defined by $d'(x, y) = d_{\mathcal{I}}(\phi(x), \phi(y))$ for all $x, y \in A$. Then d' is induced by some norm. Moreover $d'_{|(A \cap B)^2} = d_{\mathcal{I}|(A \cap B)^2}$. As $A \cap B$ has nonempty interior, we deduce that $d' = d_{\mathcal{I}}$ and thus that $\phi : (A, d_{\mathcal{I}}) \to (B, d_{\mathcal{I}})$ is an isometry.

Let M be a wall of $\mathbb{A}$ and $s : \mathbb{A} \to \mathbb{A}$ be the reflection fixing s. By Lemma 4.4.42 and its proof, s is an isometry of $(\mathbb{A}, d_{\mathcal{I}})$. Thus W is a group of isometries for $d_{\mathcal{I}|\mathbb{A}^2}$. By Corollary 7.2.3, W^v is finite and by [Rou11] 2.2 6), $\mathcal{I}$ is a building. □

Lemma 7.2.6. *Let $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$ and d be a distance on $\mathcal{I}$ inducing the affine topology on each apartment and such that there exists a continuous retraction ρ of $\mathcal{I}$ centered at $\mathfrak{q}$. Then each apartment containing $\mathfrak{q}$ is closed.*

Proof. Let A be an apartment containing $\mathfrak{q}$ and $B = \rho(\mathcal{I})$. Let $\phi : B \overset{\mathfrak{q}}{\to} A$ and $\rho_A : \mathcal{I} \overset{\mathfrak{q}}{\to} A$. Then $\rho_A = \phi \circ \rho$ is continuous because ϕ is an affine map. Let $(x_n) \in A^{\mathbb{N}}$ be a converging sequence and $x = \lim x_n$. Then $x_n = \rho_A(x_n) \to \rho_A(x)$ and thus $x = \rho(x) \in A$. □

Proposition 7.2.7. *Suppose $\mathcal{I}$ is not a building. Let d be a distance on $\mathcal{I}$ inducing the affine topology on each apartment. Then the interior of each apartment of $\mathcal{I}$ is empty.*

Proof. Let U be a nonempty open set of $\mathcal{I}$. Let A be an apartment of $\mathcal{I}$ such that $A \cap U \neq \emptyset$. By Proposition 3.1.1, there exists a wall M of A such that $M \cap U \neq \emptyset$. Let D be a half-apartment delimited by M. Let B be an apartment such that $A \cap B = D$, which exists by Lemma 7.2.4. Then $B \cap U$ is an open set of B containing $M \cap U$ and thus $E \cap U \neq \emptyset$, where E is the half-apartment of B opposite to D. Therefore $U \backslash A \neq \emptyset$ and we get the proposition. □

One sets $\mathcal{I}_0 = G.0$ where $0 \in \mathbb{A}$. This is the set of **vertices of type 0**. Let $-\infty = germ_\infty(-C_f^v)$. One sets $\rho_{-\infty} : \mathcal{I} \overset{-\infty}{\to} \mathbb{A}$.

Lemma 7.2.8. *The set $\mathcal{I}_0$ is countable.*

Proof. Let $i \in \{-\infty, +\infty\}$. By definition of ρ_i, $\rho_i(x) \in \mathcal{I}_0$ for all $x \in \mathcal{I}_0$ and thus (by Lemma 3.4.3), $\rho_i(x) \in Y$ for all $x \in \mathcal{I}_0$. Therefore $\mathcal{I}_0 = \bigcup_{(\lambda,\mu) \in Y^2} \rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\mu\})$. By Theorem 6.6.7, $\rho_{-\infty}^{-1}(\{\lambda\}) \cap \rho_{+\infty}^{-1}(\{\mu\})$ is finite for all $(\lambda, \mu) \in Y^2$, which completes the proof. □

Let $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$. For all $z \in \mathcal{I}_0$, one chooses an apartment $A(z)$ containing z and $\mathfrak{q}$. Let $x \in \mathcal{I}$ and A be an apartment containing x and $\mathfrak{q}$. There exists $z \in \mathcal{I}_0 \cap A$ such that $x \in z + \mathfrak{q}$ and thus $x \in A(z)$. Therefore $\mathcal{I} = \bigcup_{z \in \mathcal{I}_0} A(z)$.

Proposition 7.2.9. *Let d be a distance on $\mathcal{I}$. Suppose that there exists a sector-germ $\mathfrak{q}$ such that each apartment containing $\mathfrak{q}$ is closed and with empty interior. Then $(\mathcal{I}, d)$ is not complete and the interior of each compact set of $\mathcal{I}$ is empty.*

Proof. One has $\mathcal{I} = \bigcup_{z \in \mathcal{I}_0} A(z)$, with $\mathcal{I}_0$ countable by Lemma 7.2.8. Thus by Baire's Theorem, $(\mathcal{I}, d)$ is not complete.

Let K be a compact of $\mathcal{I}$ and $U \subset K$ be open. Then $\overline{U}$ is compact and thus complete. One has $\overline{U} = \bigcup_{z \in \mathcal{I}_0} \overline{U} \cap A(z)$ and thus $\overline{U}$ has empty interior. Thus K has empty interior. □

7.3 Construction of signed distances

In this section we construct distances on $\mathcal{I}$. To each sector-germ $\mathfrak{q}$ and to each norm on an apartment containing $\mathfrak{q}$, we associate a distance on $\mathcal{I}$. Let us be more precise.

Let A be an apartment of $\mathcal{I}$. Let d be a distance on A. One says that d is a norm if there exists an isomorphism $\phi : \mathbb{A} \to A$ and a norm $|\ |$ on $\mathbb{A}$ such that $d(x, y) = |\phi^{-1}(x) - \phi^{-1}(y)|$

for all $(x, y) \in A^2$. A distance d on A is a norm if and only if for all isomorphism $\phi : \mathbb{A} \to A$, there exists a norm $|\ |$ on $\mathbb{A}$ such that $d(x, y) = |\phi^{-1}(x) - \phi^{-1}(y)|$ for all $x, y \in A$. Let $\mathcal{N}(A)$ be the set of norms on A. Let $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$ and $\mathcal{A}(\mathfrak{q})$ be the set of apartments containing $\mathfrak{q}$. Let $A \in \mathcal{A}(\mathfrak{q})$ and $d \in \mathcal{N}(A)$. Then one sets $\theta_{\mathfrak{q}}(d) = (d_B)_{B \in \mathcal{A}(\mathfrak{q})}$, where for all $B \in \mathcal{A}(\mathfrak{q})$ and all $x, y \in B$, $d_B(x, y) = d(\phi^{-1}(x), \phi^{-1}(y))$, where $\phi : B \overset{A \cap B}{\to} A$. Let $\Theta(\mathfrak{q}) = \{\theta_{\mathfrak{q}}(d) |\ d \in \bigcup_{A \in \mathcal{A}(\mathfrak{q})} \mathcal{N}(A)\}$. In this section, we fix $\mathfrak{q}$ and we associate to each $\theta \in \Theta(\mathfrak{q})$ a distance d_θ on $\mathcal{I}$.

7.3.1 Translation in a direction

In this subsection we define for all sector Q of direction $\mathfrak{q}$ a map $+ : \mathcal{I} \times \overline{Q}$ such that for all $x \in \mathcal{I}$ and $q \in \overline{Q}$, $x + q$ is the "translate of x by q" (see Figure 7.3.1).

Let $A \in \mathcal{A}(\mathfrak{q})$. One chooses an isomorphism $\phi : \mathbb{A} \to A$. We consider A as a vectorial space over $\mathbb{R}$ via this isomorphism. Let $Q = 0 + \mathfrak{q} \subset A$. Let $\rho : \mathcal{I} \overset{\mathfrak{q}}{\to} A$.

Lemma 7.3.1. *Let $x \in \mathcal{I}$. Then $\rho_x : x + \overline{\mathfrak{q}} \to \rho(x) + \overline{\mathfrak{q}}$ sending each $u \in x + \overline{\mathfrak{q}}$ on $\rho(u)$ is well defined and is a bijection.*

Proof. Let B be an apartment containing $x + \overline{\mathfrak{q}}$ and $\phi : B \overset{\mathfrak{q}}{\to} A$. Then $\phi(x + \overline{\mathfrak{q}}) = \phi(x) + \overline{\mathfrak{q}} = \rho(x) + \overline{\mathfrak{q}}$. Let $u, u' \in x + \overline{\mathfrak{q}}$ be such that $\rho(u) = \rho(u')$. Then $u, u' \in B$, thus $\phi(u) = \phi(u') = \rho(u) = \rho(u')$ and thus $u = u'$. □

Let $x \in \mathcal{I}$ and $u \in \overline{Q}$. One sets $x + u = \rho_x^{-1}(\rho(x) + u)$. If B is an apartment containing x and $\mathfrak{q}$, then for all $u \in \overline{Q}$, $x + u \in B$ (because $B \supset x + \mathfrak{q}$).

Lemma 7.3.2. *Let $x \in \mathcal{I}$. Then for all $u, v \in \overline{Q}$, $(x + u) + v = x + (u + v)$ and $x + u + v = x + v + u$.*

Proof. Let B be an apartment containing $\mathfrak{q}$ and x. Let $\phi : A \overset{A \cap B}{\to} B$. This isomorphism enables to consider B as an affine space under the action of A. For $u \in \overline{Q}$, one denotes by $\tau_u : B \to B$ the translation of vector u. Then for all $u \in \overline{Q}$, $\tau_u(x) = x + u$, which proves the lemma. □

7.3.2 Definition of a distance

Let $\theta = (d_B)_{B \in \mathcal{A}(\mathfrak{q})} \in \Theta(\mathfrak{q})$. For all $u \in A$, one sets $|u| = d_A(u, 0)$. For $x, y \in \mathcal{I}$, one sets $T(x, y) = \{(u, v) \in \overline{Q}^2 | x + u = y + v\}$. One defines $d_\theta(x, y) = \inf_{(u,v) \in T(x,y)} |u| + |v|$. Until the end of this section, we will write d instead of d_θ.

Let $\mathbb{A}_{in} = \bigcap_{\alpha \in \Phi_{re}} \ker \alpha \subset \mathbb{A}$. Let $A_{in} = \phi(\mathbb{A}_{in})$.

Lemma 7.3.3. *Let $a, b \in \mathcal{I}$. Then $T(a, b)$ is not empty.*

Proof. Let $(e_i)_{i \in J}$ be a basis of A such that for some $J' \subset J$, $(e_i)_{i \in J \setminus J'}$ is a basis of A_{in} and such that $Q = A_{in} \oplus \bigoplus_{i \in J'} \mathbb{R}_+^* e_i$. Let (e_i^*) be the dual basis of (e_i). Let $B \in \mathcal{A}(\mathfrak{q})$ and $\phi : A \overset{\mathfrak{q}}{\to} B$. For $i \in J$, one defines $e_{i,B}^* : B \to \mathbb{R}$ by $e_{i,B}^*(x) = e_i^*(\phi^{-1}(x))$ for all $x \in B$. Then for $i \in J$, $x \in B$ and $q \in Q$, $e_{i,B}^*(x + q) = e_{i,B}^*(x) + e_i^*(q)$.

For $x \in \mathcal{I}$ one chooses $B_x \in \mathcal{A}(\mathfrak{q})$ containing x. Let $Q_x \subset Q$ be a sector of direction $\mathfrak{q}$ included in $A \cap B_x$ and $M_x \in \mathbb{R}$ such that for all $y \in A$, $\min_{i \in J'} e_i^*(y) \geq M_x$ implies $y \in Q_x \subset A \cap B_x$. One chooses $q_x \in Q$ such that $\min_{i \in J'}(e_{i,B_x}^*(x) + e_i^*(q_x)) \geq M_x$. Then $x + q_x \in A$ for all $x \in \mathcal{I}$. Therefore $a + q_a$ and $b + q_b$ are in A. Thus there exists $r, s \in Q$ such that $a + q_a + r = b + q_b + s$ and $(q_a + r, q_b + s) \in T(a, b)$. □

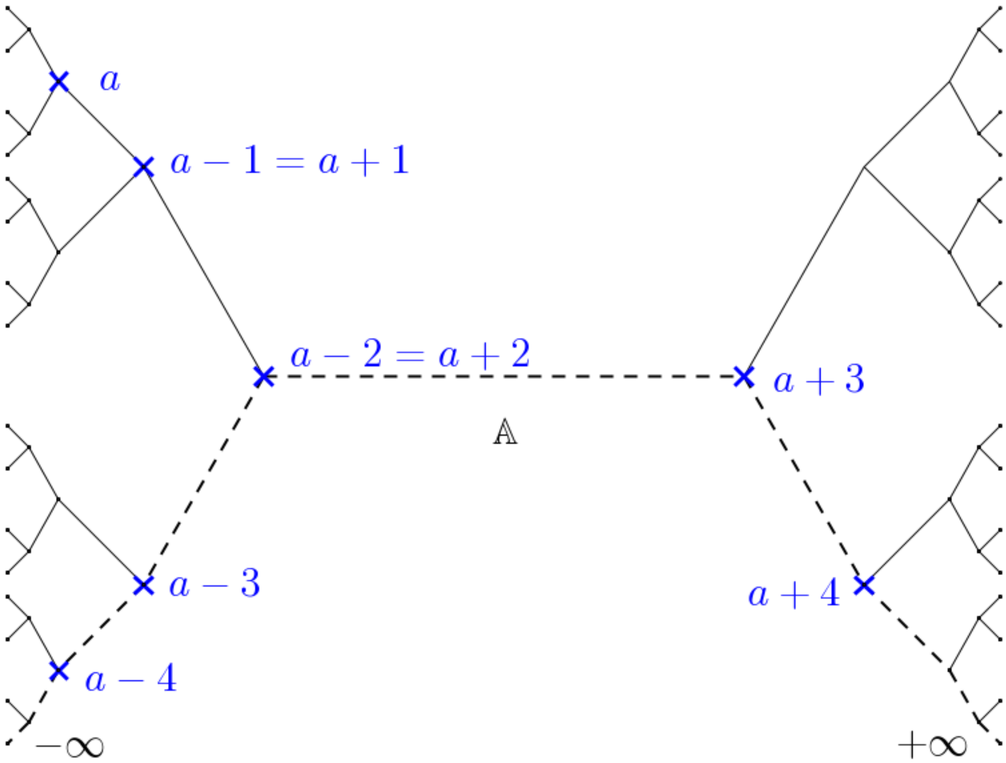

Figure 7.3.1 – Action of $\mathbb{N}$ (which is an element of $+\infty$) and $-\mathbb{N}$ (which is an element of $-\infty$) on $\mathbb{T}$.

Proposition 7.3.4. *The function $d : \mathcal{I}^2 \to \mathbb{R}_+$ is a distance.*

Proof. The function d is clearly symmetric. Let us show that d satisfies the triangular inequality. Let $x, y, z \in \mathcal{I}$. Let $\epsilon > 0$ and $(u, v) \in T(x, y)$, $(\mu, \nu) \in T(y, z)$ be such that $|u| + |v| \leq d(x, y) + \epsilon$ and $|\mu| + |\nu| \leq d(y, z) + \epsilon$. One has $x + u = y + v$ and $y + \mu = z + \nu$. Thus $x+u+\mu = y+\mu+v = z+\nu+v$ (by Lemma 7.3.2) and hence $(u+\mu, \nu+v) \in T(x, z)$. Consequently, $d(x, z) \leq |u+\mu| + |\nu+v| \leq |u| + |v| + |\mu| + |\nu| \leq d(x, y) + d(y, z) + 2\epsilon$, which proves the triangular inequality.

Let $x, y \in \mathcal{I}$ be such that $d(x, y) = 0$. Let $(u_n, v_n) \in T(x, y)^{\mathbb{N}}$ be such that $u_n \to 0$ and $v_n \to 0$. Let $n \in \mathbb{N}$. One has $y+\mathfrak{q} \supset y+v_n+\mathfrak{q} = x+u_n+\mathfrak{q}$ and thus $y+\mathfrak{q} \supset \bigcup_{n\in\mathbb{N}} x+u_n+\mathfrak{q} = x+\mathfrak{q}$. By symmetry, $x+\mathfrak{q} \supset y+\mathfrak{q}$ and hence $x+\mathfrak{q} = y+\mathfrak{q}$. Let B be an apartment containing x and $\mathfrak{q}$. By (MA ii), $B \supset cl(x + \mathfrak{q}) = cl(y + \mathfrak{q}) \ni y$. Therefore, $x = y$. □

One equips $\mathcal{I} \times \overline{Q}$ with a distance d defined by $d((x, q), (x', q')) = d(x, x') + |q - q'|$.

Lemma 7.3.5. *The map* $\begin{array}{c} \mathcal{I} \times \overline{Q} \to \mathcal{I} \\ (x, \nu) \mapsto x + \nu \end{array}$ *is Lipschitz continuous.*

Proof. By the fact that all norms are equivalent on an affine space of finite dimension, one can choose a particular $\theta \in \Theta(\mathfrak{q})$. Let $(e_i)_{i\in J}$ be a basis of A such that for some $J' \subset J$, $(e_i)_{i\in J\setminus J'}$ is a basis of A_{in} and $Q = A_{in} \oplus \bigoplus_{i\in J\setminus J'} \mathbb{R}_+^* e_i$ and (e_i^*) be the dual basis to (e_i). For $x \in A$, one sets $|x| = \sum_{i\in I} |e_i^*(x)|$ and one supposes that θ is associated to $|\ |$. Let us show that $+$ is 1-Lipschitz continuous.

Let $x, x' \in \mathcal{I}$ and $\epsilon > 0$. Let $(u, u') \in T(x, x')$ such that $|u| + |u'| \leq d(x, x') + \epsilon$. Let $\nu, \nu' \in \overline{Q}$. One sets $\lambda_i = e_i^*(\nu - \nu')$ for all $i \in J$. Let $U = \{i \in J |\ \lambda_i > 0\}$ and $V = \{i \in J |\ \lambda_i < 0\}$. Let $\mu' = \sum_{i\in U} \lambda_i e_i$ and $\mu = -\sum_{i\in V} \lambda_i e_i$. One has $\nu + \mu = \nu' + \mu'$ and

$|\mu| + |\mu'| = d(\nu, \nu')$. One has $x + \nu + u + \mu = x' + \nu' + u' + \mu'$ and thus $d(x + \nu, x' + \nu') \leq |u| + |u'| + |\mu| + |\mu'| \leq d(x, x') + d(\nu, \nu') + \epsilon$, which enables to conclude. □

Remark 7.3.6. *A consequence of Lemma 7.3.5 is the fact that for all $x, y \in \overline{Q}$, there exists $(u_0, v_0) \in T(x, y)$ such that $d(x, y) = |u_0| + |v_0|$. Indeed, let $x, y \in \mathcal{I}$ and let $(a_n, b_n) \in T(x, y)^{\mathbb{N}}$ such that $|a_n| + |b_n| \to d(x, y)$. Obviously $(a_n), (b_n)$ are bounded and thus extracting subsequences if necessary, one can suppose that (a_n) and (b_n) converge in $\overline{Q}$. The continuity of $+$ implies that $(\lim a_n, \lim b_n) \in T(x, y)$, which proves our assertion.*

Proposition 7.3.7. *For all $x, y \in \mathcal{I}$, there exists a geodesic from x to y.*

Proof. Let $x, y \in \mathcal{I}$ and $(u, v) \in T(x, y)$ such that $d(x, y) = |u| + |v|$. One defines $\tau : [0, 1] \to \mathcal{I}$ by $\tau(t) = x + 2tu$ if $t \in [0, \frac{1}{2}]$ and $\tau(t) = y + 2(1 - t)v$ if $t \in [\frac{1}{2}, 1]$, and τ is a geodesic from x to y. □

Remark 7.3.8. *If $\dim \mathbb{A} \geq 2$, for any choice of θ, there exists two points in $\mathcal{I}$ such that there exist infinitely many geodesics between them. For example, if we choose the norm on A as in the proof of Lemma 7.3.5, we have that $d_{|\mathbb{A}^2}$ is the distance induced by $|\ |$. If $x = \sum_{i \in I} x_i e_i \in A$ and if for all $i \in I$, $f_i : [0, 1] \to \mathbb{R}$ is a continuous monotonic function such that $f_i(0) = 0$ and $f_i(1) = x_i$, then $f = (f_i)_{i \in I}$ is a geodesic from 0 to x.*

Lemma 7.3.9. *Let B and C be two apartments containing $\mathfrak{q}$. Then:*

(i) the retraction $\rho_B : (\mathcal{I}, d) \overset{\mathfrak{q}}{\to} (B, d_{|B^2})$ is 1-Lipschitz continuous.

(ii) the isomorphism $\varphi : (B, d_{|B^2}) \overset{\mathfrak{q}}{\to} (C, d_{|C^2})$ is an isometry.

Proof. Let $\rho : \mathcal{I} \overset{\mathfrak{q}}{\to} A$. Let $x, y \in \mathcal{I}$ and $(u, v) \in T(x, y)$. One has $x + u = y + v$, thus $\rho(x + u) = \rho(x) + u = \rho(y + v) = \rho(y) + v$ and hence $(u, v) \in T(\rho(x), \rho(y))$, which proves that ρ is 1-Lipschitz continuous.

First suppose that $C = A$. Let $x, y \in B$ and $(u, v) \in T(\rho(x), \rho(y))$. Then $x + u = \rho_x^{-1}(\rho(x) + u) = \varphi^{-1}(\rho(x) + u) = \varphi^{-1}(\rho(y) + v) = y + v$. Hence $T(x, y) \supset T(\rho(x), \rho(y))$ and thus $T(x, y) = T(\rho(x), \rho(y))$. Therefore, φ is an isometry. Suppose now $C \neq A$. Let $\varphi_1 : B \overset{\mathfrak{q}}{\to} A$, $\varphi_2 : A \overset{\mathfrak{q}}{\to} C$ and $\varphi : B \overset{\mathfrak{q}}{\to} C$. Then $\varphi = \varphi_2 \circ \varphi_1$ is an isometry. One has $\rho_B = \varphi_1^{-1} \circ \rho$ and thus ρ_B is 1-Lipschitz continuous.

□

7.3.3 Independence of the choices of apartments and isomorphisms

Let us show that the distance we defined only depends on $\theta \in \Theta(\mathfrak{q})$. For this we have to show that d is independent of the choice of $A \in \mathcal{A}(\mathfrak{q})$ and of the isomorphism $\phi : \mathbb{A} \to A$. Let $A' \in \mathcal{A}(\mathfrak{q})$, $f : A \overset{\mathfrak{q}}{\to} A'$ and $\phi' = f \circ \phi$. One considers A' as a vectorial space over $\mathbb{R}$ by saying that f is an isomorphism of vectorial space. Objects or operations in A' are denoted with a $'$.

Lemma 7.3.10. *Let $x \in \mathcal{I}$ and $u \in \overline{Q}$. Then $x + u = x +' f(u)$.*

Proof. One has $\rho'(x+u) = f \circ \rho(x+u) = f(\rho(x)+u) = f(\rho(x)) +' f(u) = \rho'(x) +' f(u)$. Let B be an apartment containing x and $\mathfrak{q}$. Then $B \ni x+u, x+' f(u)$ and thus $x+u = x+' f(u)$. □

Lemma 7.3.11. *One has $d = d'$.*

Proof. Let $x, y \in \mathcal{I}$. By Lemma 7.3.10, $T_{A'}(x,y) = f(T_A(x,y))$. Let $u \in A$. Then $|f(u)'| = d_{A'}(0', f(u)) = d_{A'}(f(0), f(u)) = d_A(0,u) = |u|$ by definition of $\Theta(\mathfrak{q})$. Therefore, $d = d'$. □

Let now $\psi : \mathbb{A} \to A$ be an other isomorphism of apartments. This defines an other structure of vectorial space on A. We put a subscript ψ or ϕ to make the difference between these structures.

Lemma 7.3.12. *One has $d_\phi = d_\psi$.*

Proof. One has $\psi = \phi \circ w$ with $w \in W$. One writes $w = \tau \circ \vec{w}$, where $\vec{w}$ is the vectorial part of w and τ is a translation of $\mathbb{A}$. Let us show that if $\widetilde{\tau} = \phi \circ \tau^{-1} \circ \phi^{-1}$, $x +_\psi u = x +_\phi \widetilde{\tau}(u)$ for all $(x,u) \in \mathcal{I} \times \overline{Q_\psi}$. Let $x, u \in A$. One has $x +_\psi u = \psi(\psi^{-1}(x) + \psi^{-1}(u)) = \phi\big(\phi^{-1}(x) + \vec{w}(w^{-1} \circ \phi^{-1}(u))\big) = \phi\big(\phi^{-1}(x) + \phi^{-1}(\widetilde{\tau}(x))\big) = x +_\phi \widetilde{\tau}(u)$. Therefore, for all $(x,u) \in \mathcal{I} \times \overline{Q_\psi}$, $x +_\psi u = \rho_x^{-1}(x +_\psi u) = \rho_x^{-1}(x +_\phi \widetilde{\tau}(u)) = x +_\phi \widetilde{\tau}(u)$. Let us show that $|u|_\psi = |\widetilde{\tau}(u)|_\phi$ for all $u \in \overline{Q_\psi}$. Let $x \in A$. Then $x +_\psi 0_\psi = x +_\phi \widetilde{\tau}(0_\psi) = x +_\phi 0_\phi$ and thus $0_\phi = \widetilde{\tau}(0_\psi)$. Let $u \in A$. Then $|u|_\psi = d(u, 0_\psi) = d(\widetilde{\tau}(u), \widetilde{\tau}(0_\psi)) = d(\widetilde{\tau}(u), 0_\phi) = |\widetilde{\tau}(u)|_\phi$ because $\widetilde{\tau}$ is a translation of A. Let $x, y \in \mathcal{I}$. Then $T_\psi(x,y) = \widetilde{\tau}(T_\phi(x,y))$ and thus $d_\psi(x,y) = d_\phi(x,y)$. □

Thus we have constructed a distance d_θ for all $\theta \in \Theta(\mathfrak{q})$. When $\mathcal{I}$ is a tree, we obtain the usual distance.

7.4 Comparison of distances of the same sign

The aim of this section is to show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are sector-germs of $\mathcal{I}$ of the same sign and if $\theta \in \Theta(\mathfrak{q})$, $\theta' \in \Theta(\mathfrak{q}')$ then d_θ and $d_{\theta'}$ are equivalent, which means that there exists $k, \ell \in \mathbb{R}_+^*$ such that $kd_\theta \leq d_{\theta'} \leq ld_\theta$. To prove this we make an induction on the distance between $\mathfrak{q}$ and $\mathfrak{q}'$. In the next subsection, we treat the case where $\mathfrak{q}$ and $\mathfrak{q}'$ are adjacent. We use Proposition 4.1.2.

Lemma 7.4.1. *Let $\mathfrak{q}$ be a sector germ of $\mathcal{I}$. Then for all $\theta \in \Theta(\mathfrak{q})$ and all $A \in \mathcal{A}(\mathfrak{q})$, $d_{\theta|A^2} \in \mathcal{N}(A)$*

Proof. Let $A \in \mathcal{A}(\mathfrak{q})$ and $\theta \in \Theta(\mathfrak{q})$. Let $\phi : \mathbb{A} \to A$ be an isomorphism of apartments. This equips A with a structure of vectorial space. Let $N : \mathbb{A} \to \mathbb{R}_+$ defined by $N(x) = d_\theta(\phi(x), \phi(0))$ for all $x \in A$. Let $x, y \in A$. Then $T(x,y) = T(0, y-x)$ and thus $d_\theta(x,y) = N(\phi^{-1}(y) - \phi^{-1}(x))$. Let $\lambda \in \mathbb{R}^*$ and $x \in A$. Then $T(0, \lambda x) = |\lambda| T(0,x)$ and thus $N(\lambda x) = |\lambda| N(x)$. Therefore N is a norm on $\mathbb{A}$ and the lemma follows. □

7.4.1 Comparison of distances for adjacent sector-germs

Let A be an apartment of $\mathcal{I}$ and $\mathfrak{q}$, $\mathfrak{q}'$ be two adjacent sector-germs of A. Let d be a norm on A, $\theta = \theta_{\mathfrak{q}}(d)$ and $\theta' = \theta_{\mathfrak{q}'}(d)$. Let $d_{\mathfrak{q}} = d_\theta$ and $d_{\mathfrak{q}'} = d_{\theta'}$. We fix a vectorial structure on A. One sets $|x| = d(0,x)$ for all $x \in A$. Let $Q = 0 + \mathfrak{q}$ and $Q' = 0 + \mathfrak{q}'$. For all $x, y \in \mathcal{I}$ and $R \in \{Q, Q'\}$, one sets $T_R(x,y) = \{u, v \in \overline{R} | x + u = y + v\}$. The aim of this subsection is to show that there exists $k \in \mathbb{R}$ such that $d_{\mathfrak{q}'} \leq kd_{\mathfrak{q}}$. Let $\rho_q : \mathcal{I} \overset{\mathfrak{q}}{\to} A$ and $\rho_{q'} : \mathcal{I} \overset{\mathfrak{q}'}{\to} A$.

Lemma 7.4.2. *There exists $\ell \in \mathbb{R}_+^*$ such that for all $B \in \mathcal{A}(\mathfrak{q}) \cap \mathcal{A}(\mathfrak{q}')$ we have: for all $x, y \in B$, $d_{\mathfrak{q}'}(x,y) \leq ld_{\mathfrak{q}}(x,y)$.*

Proof. Let $x, y \in B$. By Lemma 7.3.9, $d_{\mathfrak{q}}(x,y) = d_{\mathfrak{q}}(\rho_{\mathfrak{q}}(x), \rho_{\mathfrak{q}}(y))$ and $d_{\mathfrak{q}'}(\rho_{\mathfrak{q}'}(x), \rho_{\mathfrak{q}'}(y)) = d_{\mathfrak{q}'}(x,y)$. As $\rho_{\mathfrak{q}'|B} = \rho_{\mathfrak{q}|B}$, one can suppose that $x, y \in A$. Then this is a consequence of Lemma 7.4.1. □

Let B be an apartment containing $\mathfrak{q}$ but not $\mathfrak{q}'$. Let $\mathfrak{F}^\infty$ be the direction of sector-panel dominated by $\mathfrak{q}$ and $\mathfrak{q}'$. Let $x \in B$ and N be a wall containing $x + \mathfrak{F}^\infty$. By Proposition 4.1.2, one can write $B = D_1 \cup D_2$, with D_1 and D_2 two opposite half-apartments having a wall H parallel to N, such that D_i and $\mathfrak{q}'$ are included in some apartment B_i, for all $i \in \{1, 2\}$. One supposes that $D_1 \supset \mathfrak{q}$.

Let M be the wall of A containing $0 + \mathfrak{F}^\infty$, $t_0 : A \to A$ be the reflection with respect to this wall and $T \in \mathbb{R}_+$ such that $t_0 : (A, d_\mathfrak{q}) \to (A, d_\mathfrak{q})$ is T-Lipschitz continuous (such a T exists by Lemma 7.4.1). As t_0 is an involution, $T \geq 1$.

Lemma 7.4.3. *There exists a translation τ of A such that if $\widetilde{t} = \tau \circ t_0$, one has for all $x \in B$:*

- *if $x \in D_1$, $\rho_\mathfrak{q}(x) = \rho_{\mathfrak{q}'}(x)$*
- *if $x \in D_2$, $\rho_\mathfrak{q}(x) = \widetilde{t} \circ \rho_{\mathfrak{q}'}(x)$*

Proof. Let $\phi_i : B \overset{B \cap B_i}{\to} B_i$, for $i \in \{1, 2\}$ and $\phi : B_2 \overset{B_1 \cap B_2}{\to} B_1$. Let t be the reflection of B_1 with respect to H. By Lemma 4.2.5, one has the following commutative diagram:

$$\begin{array}{ccc} B & \xrightarrow{\phi_2} & B_2 \\ \downarrow{\scriptstyle \phi_1} & & \downarrow{\scriptstyle \phi} \\ B_1 & \xrightarrow{t} & B_1 \end{array} .$$

Let $x \in D_1$. Then $\rho_{\mathfrak{q},B_1}(x) = x = \rho_{\mathfrak{q}',B_1}(x)$. Let $\psi : B_1 \overset{B_1 \cap A}{\to} A$. Then $\rho_\mathfrak{q}(x) = \psi(\rho_{\mathfrak{q},B_1}(x)) = \psi(\rho_{\mathfrak{q}',B_1}(x)) = \rho_{\mathfrak{q}'}(x)$.

Let $x \in D_2$. One has $\rho_{\mathfrak{q},B_1}(x) = \phi_1(x)$ and $\rho_{\mathfrak{q}',B_1}(x) = \phi(x)$ and thus $\rho_{\mathfrak{q},B_1}(x) = t \circ \rho_{\mathfrak{q}',B_1}(x)$. Let $\widetilde{t}$ making the following diagram commute:

$$\begin{array}{ccc} B_1 & \xrightarrow{t} & B_1 \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle \psi} \\ A & \xrightarrow{\widetilde{t}} & A \end{array} .$$

Then $\rho_\mathfrak{q}(x) = \widetilde{t} \circ \rho_{\mathfrak{q}'}(x)$. Moreover $\widetilde{t}$ fixes $\psi(H)$, which is parallel to M. Therefore, $\widetilde{t} = \tau \circ t_0$ for some translation τ of A (by Lemma 4.2.4).

□

Lemma 7.4.4. *Let ℓ be as in Lemma 7.4.2. Let $x, y \in B$ be such that $x, y \in D_i$ for some $i \in \{1, 2\}$. Then $d_{\mathfrak{q}'}(x, y) \leq \ell T d_\mathfrak{q}(x, y)$.*

Proof. By Lemma 7.3.9, $d_\mathfrak{q}(x, y) = d_\mathfrak{q}(\rho_\mathfrak{q}(x), \rho_\mathfrak{q}(y))$ and $d_{\mathfrak{q}'}(x, y) = d_{\mathfrak{q}'}(\rho_{\mathfrak{q}'}(x), \rho_{\mathfrak{q}'}(y))$. Lemma 7.4.3 completes the proof. □

Lemma 7.4.5. *Let (X, d) be a metric space, $f : (\mathcal{I}, d_\mathfrak{q}) \to (X, d)$ be a map and $k \in \mathbb{R}_+$. Then f is k-Lipschitz continuous if and only if for all apartment A containing $\mathfrak{q}$, $f_{|A}$ is k-Lipschitz continuous.*

Proof. One implication is clear. Suppose that for all apartment A containing $\mathfrak{q}$, $f_{|A}$ is k-Lipschitz continuous. Let $x, y \in \mathcal{I}$. Let $(u, v) \in T(x, y)$ such that $|u| + |v| = d(x, y)$. One has $d(f(x), f(y)) \leq d(f(x), f(x + u)) + d(f(y + v), f(y)) \leq k(|u| + |v|) \leq kd(x, y)$. □

Lemma 7.4.6. *One has $d_{\mathfrak{q}'} \leq \ell T d_\mathfrak{q}$.*

Proof. Let us prove that $\mathrm{Id} : (\mathcal{I}, d_{\mathfrak{q}}) \to (\mathcal{I}, d_{\mathfrak{q}'})$ is ℓT-Lipschitz continuous. Let $x, y \in B$. We already know that if $x, y \in D_i$ for some $i \in \{1, 2\}$, $d_{\mathfrak{q}'}(x, y) \leq \ell T d_{\mathfrak{q}}(x, y)$. Suppose now $x \in D_1$ and $y \in D_2$. Let $u \in [x, y] \cap H$. By Lemma 7.4.1, one has:

$$d_{\mathfrak{q}'}(x, y) \leq d_{\mathfrak{q}'}(x, u) + d_{\mathfrak{q}'}(u, y) \leq \ell T(d_{\mathfrak{q}}(x, u) + d_{\mathfrak{q}}(u, y)) = \ell T d_{\mathfrak{q}}(x, y).$$

As B is an arbitrary apartment containing $\mathfrak{q}$ and not $\mathfrak{q}'$, Lemma 7.4.2 and Lemma 7.4.5 completes the proof. □

7.4.2 Comparison of distances for sector-germs of the same sign

In this subsection, we show that if $\mathfrak{q}$ and $\mathfrak{q}'$ are sector-germs of the same sign, d_θ and $d_{\theta'}$ are equivalent. We then deduce corollaries about the induced topologies on each apartment and on retractions centered at a sector-germ.

Let Θ be the disjoint union of the $\Theta(\mathfrak{s})$ for $\mathfrak{s}$ a sector-germ of $\mathcal{I}$. Let $\theta \in \Theta$, $\mathfrak{q}$ such that $\theta \in \Theta(\mathfrak{q})$ and $\epsilon \in \{-, +\}$. One says that θ is of sign ϵ if $\mathfrak{q}$ is of sign ϵ. For $\epsilon \in \{-, +\}$, one denotes by Θ_ϵ the set of elements of Θ of sign ϵ.

Theorem 7.4.7. *Let $\mathfrak{q}_1$, $\mathfrak{q}_2$ be two sector germs of $\mathcal{I}$ of the same sign. For $i \in \{1, 2\}$, let $\theta_i \in \Theta(\mathfrak{q}_i)$. Then there exists $k \in \mathbb{R}_+^*$ such that $d_{\theta_1} \leq k d_{\theta_2}$. In particular for all $\theta \in \Theta$, the topology of $(\mathcal{I}, d_\theta)$ only depends on the sign of θ.*

Proof. Let A be an apartment containing $\mathfrak{q}_1$ and $\mathfrak{q}_2$, and $d \in \mathcal{N}(A)$. Let $\mathfrak{s}_0 = \mathfrak{q}_1, \ldots, \mathfrak{s}_n = \mathfrak{q}_2$ be a gallery joining $\mathfrak{q}_1$ and $\mathfrak{q}_2$. For all $i \in [\![0, n]\!]$, one sets $u_i = \theta_{\mathfrak{s}_i}(d)$. By an induction using Lemma 7.4.6, there exists $a \in \mathbb{R}_+^*$ such that $d_{u_0} \leq a d_{u_n}$. As every norms are equivalent on A, there exists $b, c \in \mathbb{R}_+^*$ such that $d_{\theta_1} \leq b d_{u_0}$ and $d_{u_n} \leq c d_{\theta_2}$, which concludes the proof of the theorem. □

We thus obtain (at most) two topologies on $\mathcal{I}$: the topology $\mathscr{T}_+$ obtained by taking a positive $\theta \in \Theta$ and the topology $\mathscr{T}_-$ obtained by taking a negative $\theta \in \Theta$. We will see that when $\mathcal{I}$ is not a building, these topologies are different.

Corollary 7.4.8. *Let A be an apartment of $\mathcal{I}$ and $\theta \in \Theta$. Then the topology of A induced by the topology of $(\mathcal{I}, d_\theta)$ is the affine topology on A.*

Proof. By Theorem 7.4.7, one can suppose $\theta \in \Theta(\mathfrak{q})$ where $\mathfrak{q}$ is a sector germ of A of the same sign as θ. Then Lemma 7.4.1 concludes the proof. □

Corollary 7.4.9. *Let $\mathfrak{q}$ be a sector-germ. Let ρ be a retraction with center $\mathfrak{q}$, $A = \rho(\mathcal{I})$ and $d \in \mathcal{N}(A)$. Then:*

(i) for each $\theta \in \Theta$ of the sign of $\mathfrak{q}$, $\rho : (\mathcal{I}, d_\theta) \to (A, d)$ is Lipschitz continuous

(ii) if B is an apartment and $d' \in \mathcal{N}(B)$, $\rho_{|B} : (B, d') \to (A, d)$ is Lipschitz continuous.

Proof. Let $\theta \in \Theta(\mathfrak{q})$. Then by Lemma 7.3.9, $\rho : (\mathcal{I}, d_\theta) \to (A, d_\theta)$ is Lipschitz continuous and one concludes the proof of (i) with Theorem 7.4.7 and Lemma 7.4.1.

Let $\mathfrak{q}'$ be a sector-germ of B of the same sign as $\mathfrak{q}$ and $\theta' \in \Theta(\mathfrak{q}')$. Then $\rho : (\mathcal{I}, d_{\theta'}) \to (A, d)$ is Lipschitz continuous by (i). Thus $\rho_{|B} : (B, d_{\theta'}) \to (A, d)$ is Lipschitz continuous and one concludes with Lemma 7.4.1. □

We now give an other proof of Proposition 4.2.17 (in our less general frameworl).

Corollary 7.4.10. *Let A, B be two apartments of $\mathcal{I}$. Then $A \cap B$ is a closed subset of A (seen as an affine space).*

Proof. By Lemma 7.2.6, A and B are closed for $\mathscr{T}_+$ (or $\mathscr{T}_-$) and thus $A \cap B$ is closed for $\mathscr{T}_+$. Consequently it is closed for the topology induced by $\mathscr{T}_+$ on A, and Corollary 7.4.8 completes the proof. □

Remark 7.4.11. *Suppose that $\mathcal{I}$ is not a building. Then by Subsection 7.2.2, for all $\theta \in \Theta$, $(\mathcal{I}, d_\theta)$ is not complete.*

*Let $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$ and (Q_n) be an increasing sequence of sectors with germ $\mathfrak{q}$. One says that (Q_n) is converging if there exists a retraction onto an apartment $\rho : \mathcal{I} \overset{\mathfrak{q}}{\to} \rho(\mathcal{I})$ such that $(\rho(x_n))$ converges, where x_n is the base point of Q_n for all $n \in \mathbb{N}$ and we call **limit** of (Q_n) the set $\bigcup_{n \in \mathbb{N}} Q_n$. One can show that the fact that $\mathcal{I}$ is not complete implies that there exists a converging sequence of direction $\mathfrak{q}$ whose limit is not a sector of $\mathcal{I}$, which is impossible in a discrete building. To prove this one can associate to each Cauchy sequence (x_n) a sequence (z_n) such that $d(z_n, x_n) \to 0$ and such that $z_n + \mathfrak{q} \subset z_{n+1} + \mathfrak{q}$ for all $n \in \mathbb{N}$. Then we show that (z_n) converges in $(\mathcal{I}, d_\theta)$ if an only if the limit of $(z_n + \mathfrak{q})$ is a sector of $\mathcal{I}$.*

7.4.3 Study of the action of G

In this subsection, we show that for each $g \in G$, $g : \mathcal{I} \to \mathcal{I}$ is Lipschitz continuous for the distances we constructed. We begin by treating the case where g stabilizes a sector-germ. After this, we treat the case where g stabilizes an apartment and then we conclude.

Lemma 7.4.12. *Let $g \in G$ and suppose that g stabilizes some sector-germ $\mathfrak{q}$. Let $\theta \in \Theta(\mathfrak{q})$. Then $g : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_\theta)$ is an isometry.*

Proof. Let us prove that g is 1-Lipschitz continuous. Let A be an apartment containing $\mathfrak{q}$ and $\phi : A \overset{\mathfrak{q}}{\to} g.A$. Let τ making the following diagram commute:

$$\begin{array}{ccc} (A, d_\theta) & \xrightarrow{\;g\;} & (g.A, d_\theta)\;. \\ & \searrow^{\phi} & \downarrow \tau \\ & & (g.A, d_\theta) \end{array}$$

Then τ is an automorphism of $g.A$ stabilizing $\mathfrak{q}$ and thus τ is a translation. As $g.A \supset \mathfrak{q}$, τ is an isometry of $(g.A, d_\theta)$. By Lemma 7.3.9, ϕ is an isometry. Thus $g_{|A}^{|g.A}$ is an isometry and it is in particular 1-Lipschitz continuous. By Lemma 7.4.5, g is 1-Lipschitz continuous. As g^{-1} stabilizes $\mathfrak{q}$, it is also 1-Lipschitz continuous, which shows the lemma. □

Lemma 7.4.13. *Let A be an apartment of $\mathcal{I}$. We fix a structure of vectorial space on A. Let Q be a sector of $\mathcal{I}$ with base point 0. Let $g \in G$ stabilizing A. Let $w = g_{|A}^{|A}$ and $\vec{w}$ be the linear part of w. Then for all $x \in \mathcal{I}$ and $u \in Q$, $g.(x+u) = g.x + \vec{w}(u)$.*

Proof. Let $\rho_\mathfrak{q} : \mathcal{I} \overset{\mathfrak{q}}{\to} A$ and $\rho_{g.\mathfrak{q}} : \mathcal{I} \overset{g.\mathfrak{q}}{\to} A$. Let us show that $\rho_{g.\mathfrak{q}} = g.\rho_\mathfrak{q}.g^{-1}$. Let $x \in \mathcal{I}$ and B be an apartment containing x and $\mathfrak{q}$. Let $\phi : B \overset{B \cap A}{\to} A$. One has $\rho_\mathfrak{q}(x) = \phi(x)$. As $g.\phi.g^{-1} : g.B \overset{g.\mathfrak{q}}{\to} A$, $\rho_{g.\mathfrak{q}}(g.x) = g.\phi(g^{-1}.g.x) = g.\rho_\mathfrak{q}(x)$. Therefore $\rho_{g.\mathfrak{q}} = g.\rho_\mathfrak{q}.g^{-1}$. Let $x \in \mathcal{I}$ and $u \in Q$. One has

$$\rho_{g.\mathfrak{q}}(g(x+u)) = g.\rho_\mathfrak{q}(x+u) = w.(\rho_\mathfrak{q}(x) + u) = w(\rho_\mathfrak{q}(x)) + \vec{w}(u).$$

We also have

$$\rho_{g.\mathfrak{q}}(g.x+\vec{w}(u)) = \rho_{g.\mathfrak{q}}(g.x)+\vec{w}(u) = g.\rho_{\mathfrak{q}}(x)+\vec{w}(u) = \rho_{g.\mathfrak{q}}(g(x+u)).$$

Then $B \ni x+u$ and $g.B \ni g.(x+u), g.x+\vec{w}(u)$. As $\rho_{g.\mathfrak{q}|g.B}$ is an isomorphism, $g.(x+u) = g.x+\vec{w}(u)$. □

Theorem 7.4.14. *Let $g \in G$ and $\theta \in \Theta$. Then $g : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_\theta)$ is Lipschitz continuous.*

Proof. Let $\mathfrak{q}$ be such that $\theta \in \Theta(\mathfrak{q})$. Let A be an apartment containing $\mathfrak{q}$ and $g.\mathfrak{q}$. Let $\phi : g.A \overset{g.\mathfrak{q}}{\to} A$ and $h \in G$ inducing ϕ on $g.A$. Then $g = h^{-1} \circ f$, where $f = hg$. As h^{-1} is Lipschitz continuous by Lemma 7.4.12 (and by Theorem 7.4.7), it suffices to show that f is Lipschitz continuous. One has $f(A) = A$. One fixes a vectorial structure on A. Let $x, y \in \mathcal{I}$ and $\vec{w}$ be the linear part of $f_{|A}^{|A}$.

One writes $\theta = (d_B)_{B\in\mathcal{A}(\mathfrak{q})}$. Let $\theta' = \theta_{f.\mathfrak{q}}(d_A)$. For $x \in A$, one sets $|x| = d_A(x, 0)$.

Let $k \in \mathbb{R}_+^*$ be such that $\vec{w} : (A, |\ |) \to (A, |\ |)$ is k-Lipschitz continuous. Let $x, y \in \mathcal{I}$. Let $(u, v) \in T_Q(x, y)$ such that $|u| + |v| = d_\theta(x, y)$.

By Lemma 7.4.13, $(\vec{w}(u), \vec{w}(v)) \in T_{\vec{w}.Q}(f(x), f(y))$. Therefore, $d_{\theta'}(f(x), f(y)) \le |\vec{w}(u)| + |\vec{w}(v)| \le kd_\theta(x, y)$. Therefore $f : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_{\theta'})$ is k-Lipschitz continuous and one concludes with Theorem 7.4.7. □

7.4.4 Case of a building

In this subsection we suppose that $\mathcal{I}$ is a building. We show that the distances we constructed and the usuals distance are equivalent.

If d is a W^v-invariant euclidean norm on $\mathbb{A}$, one denotes by $d_\mathcal{I}$ the extension of d to $\mathcal{I}$ in the usual manner (see [Bro89] VI.3 for example). For $x \in \mathcal{I}$, one denotes by $\mathrm{st}(x) = \bigcup_{C\in A(x)} \overline{C}$, where $A(x)$ is the set of alcoves that contains x. Then by a Lemma in VI.3 of [Bro89], $\mathrm{st}(x)$ contains x in its interior.

Proposition 7.4.15. *Let d be a W^v-invariant euclidean norm on $\mathbb{A}$ and $\theta \in \Theta$. Then there exist $k, \ell \in \mathbb{R}_+^*$ such that $kd_\mathcal{I} \le d_\theta \le ld_\mathcal{I}$.*

Proof. Let $\mathfrak{q}$ be such that $\theta \in \Theta(\mathfrak{q})$. By Theorem 7.4.7, one can suppose that $\mathfrak{q} \subset \mathbb{A}$ and that $\theta = \theta_\mathfrak{q}(d)$. Let $k, \ell \in \mathbb{R}_+^*$ such that $kd_{|\mathbb{A}^2} \le d_{\theta|\mathbb{A}^2} \le ld_{|\mathbb{A}^2}$. Let us first show that $\mathrm{Id} : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_\mathcal{I})$ is $\frac{1}{k}$-Lipschitz continuous. Let A be an apartment containing $\mathfrak{q}$. Let $x, y \in A$. Then $d_\theta(x, y) = d_\theta(\rho(x), \rho(y)) \ge kd_\mathcal{I}(\rho(x), \rho(y)) = kd_\mathcal{I}(x, y)$, where $\rho : \mathcal{I} \overset{\mathfrak{q}}{\to} \mathbb{A}$. Thus $\mathrm{Id} : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_\mathcal{I})$ is $\frac{1}{k}$-Lipschitz continuous by Lemma 7.4.5.

Let $x, y \in \mathcal{I}$. As $[x, y]$ is compact and thanks to the Lemma recalled before the proposition, there exists $n \in \mathbb{N}^*$ and $x_0 = x, x_1, \ldots, x_n = y \in [x, y]$ such that $x_{i+1} \in \mathrm{st}(x_i)$ for all $i \in [\![0, n-1]\!]$. In particular, for all $i \in [\![0, n-1]\!]$, there exists an apartment A_i containing x_i, x_{i+1} and $\mathfrak{q}$. One has

$$\begin{aligned} d_\theta(x, y) \le \sum_{i=0}^{n-1} d_\theta(x_i, x_{i+1}) &= \sum_{i=0}^{n-1} d_\theta(\rho(x_i), \rho(x_{i+1})) \\ &\le \ell \sum_{i=0}^{n-1} d_\mathcal{I}(\rho(x_i), \rho(x_{i+1})) = \ell \sum_{i=0}^{n-1} d_\mathcal{I}(x_i, x_{i+1}) = \ell d_\mathcal{I}(x, y), \end{aligned}$$

which proves the proposition. □

7.5 Mixed distances

7.5.1 Comparison of positive and negative topologies

In this subsection, we show that $\mathscr{T}_+$ and $\mathscr{T}_-$ are different when $\mathcal{I}$ is not a building. Let $\mathcal{I}_0 = G.0$ be the set of **vertices of type** 0. For this we prove that $\mathcal{I}_0$ is composed of limit points when $\mathcal{I}$ is not a building and then we apply finiteness results of Chapter 6.

Proposition 7.5.1. *Let $\theta \in \Theta$. Then $\mathcal{I}_0$ is discrete in $(\mathcal{I}, d_\theta)$ if and only if $\mathcal{I}$ is a building.*

Proof. Suppose that $\mathcal{I}$ is a building. By Proposition 7.4.15, we can replace d_θ by a usual distance on $\mathcal{I}$. By Lemma 3.4.3 one has $\mathcal{I}_0 \cap \mathbb{A} = Y$, which is a lattice of $\mathbb{A}$. Let $\eta > 0$ such that for all $x, x' \in Y$, $d(x, x') > \eta$ implies $x = x'$. Let $x, x' \in \mathcal{I}_0$ such that $d(x, x') < \eta$. Let A be an apartment of $\mathcal{I}$ containing x and x' and $g \in G$ such that $g.A = \mathbb{A}$. Then $d(g.x, g.x') < \eta$ and thus $x = x'$.

Suppose now that $\mathcal{I}$ is not a building and thus that W^v is infinite. By Theorem 7.4.7, one can suppose that $\theta \in \Theta(\mathfrak{q})$, where $\mathfrak{q}$ is a sector-germ of $\mathbb{A}$. Let $\epsilon > 0$. Let us show that there exists $x \in \mathcal{I}_0$ such that $d_\theta(x, 0) < 2\epsilon$ and $x \neq 0$. Let M be a wall such that $d_\theta(0, M) < \epsilon$ and such that $0 \notin M$, which exists by Proposition 3.1.1. Let D be the half-apartment of $\mathbb{A}$ delimited by M and containing $\mathfrak{q}$. We assume that D does not contain 0 (which is possible, maybe considering $-M$ instead of M). By Lemma 7.2.4, there exists an apartment A such that $A \cap \mathbb{A} = D$. Let $\phi : \mathbb{A} \overset{A\cap\mathbb{A}}{\to} A$ and $x = \phi(0)$. Let $y \in M$ such that $d_\theta(0, y) < \epsilon$. Then by Lemma 7.3.9, $d_\theta(x, y) = d_\theta(0, y)$ and thus $d(x, 0) < 2\epsilon$. As $x \notin \mathbb{A}$, $x \neq 0$ and we get the proposition. □

Remark 7.5.2. *In fact, by Theorem 7.4.14, we have shown that when $\mathcal{I}$ is not a building, each point of $\mathcal{I}_0$ is a limit point.*

If B is an apartment and $(x_n) \in B^{\mathbb{N}}$, one says that (x_n) converges towards ∞, if for some (for each) norm $|\ |$ on B, $|x_n| \to +\infty$.

Proposition 7.5.3. *Suppose that $\mathcal{I}$ is not a building. Let $\epsilon \in \{-, +\}$ and $\delta = -\epsilon$. Let $\mathfrak{q}$ (resp. $\mathfrak{s}$) be a sector-germ of $\mathcal{I}$ of sign ϵ (resp. δ). Let $\theta \in \Theta(\mathfrak{q})$. We equip $\mathcal{I}$ with d_θ. Let ρ be a retraction centered at $\mathfrak{s}$ and $(x_n) \in \mathcal{I}_0^{\mathbb{N}}$ be an injective and converging sequence. Then $\rho(x_n) \to \infty$ in $\rho(\mathcal{I})$. In particular ρ is not continuous.*

Proof. By Theorem 7.4.7, one can suppose that there exists an apartment A containing $\mathfrak{q}$ and $\mathfrak{s}$ such that $\mathfrak{q}$ and $\mathfrak{s}$ are opposite in A. Maybe composing ρ by an isomorphism of apartments fixing $\mathfrak{s}$, one can suppose that $\rho(\mathcal{I}) = A$. Let $\rho_\mathfrak{q} : \mathcal{I} \overset{\mathfrak{q}}{\to} A$. Let $x = \lim x_n$ and $y = \rho_\mathfrak{q}(x)$. Then by Corollary 7.4.9, $\rho_\mathfrak{q}(x_n) \to y$. Let $Y_A = \mathcal{I}_0 \cap A$. Then Y_A is a lattice of A by Lemma 3.4.3. As $\rho_\mathfrak{q}(x_n) \in \mathcal{I}_0 \cap A$ for all $n \in \mathbb{N}$, $\rho_\mathfrak{q}(x_n) = y$ for n large enough.

We also have $\rho(x_n) \in Y_A$ for all $n \in \mathbb{N}$. By Theorem 6.6.7, for all $a \in Y_A$, $\rho^{-1}(\{a\}) \cap \rho_\mathfrak{q}^{-1}(\{y\})$ is finite. This concludes the proof of this proposition. □

Corollary 7.5.4. *If $\mathcal{I}$ is not a building, $\mathscr{T}_+$ and $\mathscr{T}_-$ are different.*

Remark 7.5.5. *Proposition 7.5.3 shows that if $\theta, \theta' \in \Theta$ have opposite signs, then each open subset of $(\mathcal{I}, d_\theta)$ containing a point of $\mathcal{I}_0$ is not bounded for $d_{\theta'}$.*

7.5.2 Mixed distances

In this section we define and study mixed distances.

Let $\Theta_c = \Theta_+ \times \Theta_-$. Let $\theta = (\theta_+, \theta_-) \in \Theta_c$. One sets $d_\theta = d_{\theta_+} + d_{\theta_-}$.

Remark 7.5.6. *Let $\theta \in \Theta_c$. Let $f : \mathcal{I} \to \mathcal{I}$. Then if for some $\theta_+ \in \Theta_+$ and $\theta_- \in \Theta_-$, $f : (\mathcal{I}, d_{\theta_+}) \to (\mathcal{I}, d_{\theta_+})$ and $f : (\mathcal{I}, d_{\theta_-}) \to (\mathcal{I}, d_{\theta_-})$ are Lipschitz continuous (resp. continuous), then $f : (\mathcal{I}, d_\theta) \to (\mathcal{I}, d_\theta)$ is Lipschitz continuous (resp. continuous).*

Let A be an apartment of $\mathcal{I}$ and $f : \mathcal{I} \to A$. Suppose that for some $\theta' \in \Theta_\pm$, $f : (\mathcal{I}, d_{\theta'}) \to (A, d_{\theta'})$ is Lipschitz continuous (resp. continuous). Then $f : (\mathcal{I}, d_\theta) \to (A, d_\theta)$ is Lipschitz continuous (resp. continuous).

Theorem 7.5.7. *Let $\theta \in \Theta_c$. We equip $\mathcal{I}$ with d_θ. Then:*

(i) For all $\theta' \in \Theta_c$, d_θ and $d_{\theta'}$ are equivalent.

(ii) For all $g \in G$, g is Lipschitz-continuous.

(iii) Each retraction of $\mathcal{I}$ centered at a sector-germ is Lipschitz continuous.

(iv) The topology induced on each apartment is the affine topology.

(v) The set $\mathcal{I}_0$ is discrete.

Proof. The assertions (i) to (iv) are consequences of Remark 7.5.6, Theorem 7.4.7, Corollary 7.4.9, Theorem 7.4.14 and Corollary 7.4.8. Let us prove (v). Let $x \in \mathcal{I}_0$ and set $x_+ = \rho_{+\infty}(x)$ and $x_- = \rho_{-\infty}(x)$. By Theorem 6.6.7, $\rho_{+\infty}^{-1}(\{x_+\}) \cap \rho_{-\infty}^{-1}(\{x_-\})$ is finite and thus there exists $r > 0$ such that $B(x, r) \cap \rho_{+\infty}^{-1}(\{x_+\}) \cap \rho_{-\infty}^{-1}(\{x_-\}) = \{x\}$, where $B(x, r)$ is the open ball of radius r and center x. Let $k \in \mathbb{R}_+^*$ such that $\rho_{+\infty}$ and $\rho_{-\infty}$ are k-Lipschitz continuous. Let $\alpha > 0$ such that for all $a, a' \in Y$, $a \neq a'$ implies $d(a, a') \geq \alpha$. Let $r' = \min(r, \frac{\alpha}{k})$. Let us prove that $B(x, r') \cap \mathcal{I}_0 = \{x\}$. Let $a \in B(x, r') \cap \mathcal{I}_0$. Suppose $\rho_{\epsilon\infty}(a) \neq x_\epsilon$, for some $\epsilon \in \{-, +\}$. Then

$$kd(a, x) \geq d(\rho_{\epsilon\infty}(a), \rho_{\epsilon\infty}(x)) \geq \alpha,$$

thus $a \notin B(x, r')$, a contradiction. Therefore $\rho_{+\infty}(a) = x_+$ and $\rho_{-\infty}(a) = x_-$, hence $a = x$ by choice of r, which completes the proof of the theorem.

□

Lemma 7.5.8. *Let $\theta \in \Theta_c$ and $a \in \mathcal{I}$. Let A be an apartment of $\mathcal{I}$ containing a. We fix an origin in A. Let Q_+ and Q_- be opposite sector-germs of A based at 0 and $\rho_+ : \mathcal{I} \overset{\mathfrak{q}_+}{\to} A$, $\rho_- : \mathcal{I} \overset{\mathfrak{q}_-}{\to} A$, where $\mathfrak{q}_-, \mathfrak{q}_+$ are the germs of Q_- and Q_+. Let d be a distance induced by a norm on A. Then there exists $k \in \mathbb{R}_+^*$ such that for all $x \in \mathcal{I}$, $d_\theta(a, x) \leq k(d(a, \rho_-(x)) + d(a, \rho_+(x))$.*

Proof. One writes $\theta = (\theta_+, \theta_-)$. By Theorem 7.5.7 (i) and Lemma 7.4.1, one can suppose that $\theta_- = \theta_{\mathfrak{q}_-}(d)$, $\theta_+ = \theta_{\mathfrak{q}_+}(d)$ and $d = d_{\theta|A^2}$.

Let $\nu \in Q$. Let $T_+ = T_\nu : \mathcal{I} \to \mathbb{R}$ and $T_- = T_{-\nu} : \mathcal{I} \to \mathbb{R}$ be the maps of 4.2.2.3. By Corollary 6.5.3 and Remark 6.5.4, $T_+(x), T_-(x) \leq h\big(\rho_-(x) - \rho_+(x)\big)$ with $h : A \to \mathbb{R}$ a linear function. Therefore, there exists $\ell \in \mathbb{R}_+^*$ such that $T_+(x), T_-(x) \leq ld(\rho_-(x), \rho_+(x))$ for all $x \in \mathcal{I}$.

One sets $d_+ = d_{\theta_+}$ and $d_- = d_{\theta_-}$. By definition of T_+, T_- and $+$, one has $x + T_+(x)\nu = \rho_+(x) + T_+(x)\nu$ and $x + T_-(x)(-\nu) = \rho_-(x) + T_-(x)(-\nu)$ for all $x \in \mathcal{I}$ and thus

$$d_i(x, \rho_i(x)) \leq 2T_i(x)d(0, \nu) \leq 2ld(0, \nu)d(\rho_-(x), \rho_+(x)) \leq 2ld(0, \nu)\big(d(\rho_-(x), a) + d(\rho_+(x), a)\big)$$

for all $x \in \mathcal{I}$ and $i \in \{-,+\}$.

One has,

$$d(a,x) = d_+(a,x) + d_-(a,x) \leq d_-(a,\rho_-(x)) + d_-(\rho_-(x),x) + d_+(a,\rho_+(x)) + d_+(\rho_+(x),x)$$

and thus $d(a,x) \leq (4ld(0,\nu)+1)\big(d(a,\rho_-(x)) + d(a,\rho_+(x))\big)$.

□

Corollary 7.5.9. *Let $\theta \in \Theta_c$ and let us equip $\mathcal{I}$ with $d = d_\theta$. Then if $X \subset \mathcal{I}$ the following assertions are equivalent:*

(i) X is bounded

(ii) for all retractions ρ centered at a sector of $\mathcal{I}$, $\rho(X)$ is bounded

(iii) there exist two opposite sectors $\mathfrak{q}_+$ and $\mathfrak{q}_-$ such that if $\rho_{\mathfrak{q}_-}$ and $\rho_{\mathfrak{q}_+}$ are retractions centered at $\mathfrak{q}_-$ and $\mathfrak{q}_+$, $\rho_{\mathfrak{q}_-}(X)$ and $\rho_{\mathfrak{q}_+}(X)$ are bounded.

Moreover each bounded subset of $\mathcal{I}_0$ is finite.

Proof. By Theorem 7.5.7, (i) implies (ii), and it is clear that (ii) implies (iii). The implication (iii) implies (i) is a consequence of Lemma 7.5.8. The last assertion is a consequence of (iii) and of Theorem 6.6.7. □

Corollary 7.5.10. *The $\rho_+^{-1}(U) \cap \rho_-^{-1}(U)$ such that U is an open set of an apartment A and ρ_- and ρ_- are retractions onto A centered at opposite sectors of A form a basis of $\mathcal{T}_c$. In particular $\mathscr{T}_c$ is the initial topology with respect to retractions centered at sector-germs.*

Proof. This is a consequence of Lemma 7.5.8. □

7.6 Contractibility of $\mathcal{I}$

In this section we prove the contractibility of $\mathcal{I}$ for $\mathscr{T}_+$, $\mathscr{T}_-$ and $\mathscr{T}_c$. By Theorem 7.4.7, by symmetry of the roles of the Tits cone and of its opposite, it suffices to prove that there exists $\theta \in \Theta(+\infty)$ (where $+\infty = germ_\infty(C_f^v)$) such that $(\mathcal{I}, d_\theta)$ is contractible and Remark 7.5.6 will conclude for the contractibility of $(\mathcal{I}, \mathscr{T}_c)$.

One chooses a basis $(e_i)_{i\in J}$ of $\mathbb{A}$ such that for some $J' \subset J$, $(e_i)_{i \in J\setminus J'}$ is a basis of $\mathbb{A}_{in}$ and $C_f^v = \bigoplus_{i\in I} \mathbb{R}_+^* e_i \oplus \mathbb{A}_{in}$. Let (e_i^*) be the dual basis of (e_i). For $x \in \mathbb{A}$, one sets $|x| = \sum_{i\in J} |e_i^*(x)|$. Let $d_\mathbb{A}$ be the distance on $\mathbb{A}$ induced by $|\ |$ and $\theta = \theta_{+\infty}(d_\mathbb{A})$. Let $d = d_\theta$. One has $d_\mathbb{A} = d_{\theta|\mathbb{A}^2}$.

One uses the maps $y_\nu : \mathcal{I} \to \mathbb{A}$ and $T_\nu : \mathcal{I} \to \mathbb{R}$, for $\nu \in C_f^v$ defined in 4.2.2.3.

Lemma 7.6.1. *Let $\nu \in C_f^v$ and $\eta = \min_{i\in I} e_i^*(\nu)$. Then T_ν is $\frac{1}{\eta}$-Lipschitz continuous and $y_\nu : (\mathcal{I}, d) \to (\mathbb{A}, d_{|\mathbb{A}^2})$ is $(\frac{1}{\eta}|\nu| + 1)$-Lipschitz continuous.*

Proof. Let $x, x' \in \mathcal{I}$ and $(u,u') \in T(x,x')$ such that $d(x,x') = |u| + |u'|$. Then $x + T_\nu(x)\nu \in \mathbb{A}$ and thus $x' + u' + T_\nu(x)\nu = x + u + T_\nu(x)\nu \in \mathbb{A}$. As $\frac{1}{\eta}|u'|\nu - u' \in C_f^v$, $x' + (T_\nu(x) + \frac{1}{\eta}|u'|)\nu \in \mathbb{A}$. Consequently, $T_\nu(x') \leq T_\nu(x) + \frac{d(x,x')}{\eta}$ and we get the first part of lemma.

One has

$$\begin{aligned} d(y_\nu(x), y_\nu(x')) &= d(\rho_{+\infty}(x) + T_\nu(x)\nu, \rho_{+\infty}(x') + T_\nu(x')\nu) \\ &\leq d(\rho_{+\infty}(x) + T_\nu(x)\nu, \rho_{+\infty}(x') + T_\nu(x)\nu) + d(\rho_{+\infty}(x') + T_\nu(x)\nu, \rho_{+\infty}(x') + T_\nu(x')\nu) \\ &= d(\rho_{+\infty}(x), \rho_{+\infty}(x')) + d(T_\nu(x)\nu, T_\nu(x')\nu), \end{aligned}$$

because d is invariant by translation on $\mathbb{A}$.

By Lemma 7.3.9, $d(\rho_{+\infty}(x), \rho_{+\infty}(x')) \leq d(x, x')$. One also has $d(T_\nu(x)\nu, T_\nu(x')\nu) \leq |T_\nu(x) - T_\nu(x')||\nu| \leq \frac{1}{\eta}|\nu|d(x, x')$ and we can conclude. □

Remark 7.6.2. *One can also prove that the maps $(\nu, x) \mapsto T_\nu(x)$ and $(\nu, x) \mapsto y_\nu(x)$ are locally Lipschitz continuous.*

Proposition 7.6.3. *Let $\nu \in C_f^v$. One defines*

$$\phi_\nu : \begin{cases} \mathcal{I} \times [0,1] \to \mathcal{I} \\ \quad (x,t) \mapsto x + \dfrac{t}{1-t}\nu \text{ if } \dfrac{t}{1-t} < T_\nu(x) \\ \quad (x,t) \mapsto y_\nu(x) \text{ if } \dfrac{t}{1-t} \geq T_\nu(x) \end{cases}$$

(where we consider that $\frac{1}{0} = +\infty > t$ for all $t \in \mathbb{R}$). Then ϕ_ν is a strong deformation retract on $\mathbb{A}$.

Proof. Let $x \in \mathbb{A}$ and $t \in [0,1]$. Then $T_\nu(x) = 0$ and thus $\phi_\nu(x) = y_\nu(x) = x$. Let $x \in \mathcal{I}$. Then $\phi_\nu(x, 0) = x$ and $\phi_\nu(x, 1) = y_\nu(x) \in \mathbb{A}$. It remains to show that ϕ_ν is continuous. Let $(x_n, t_n) \in (\mathcal{I} \times [0,1])^{\mathbb{N}}$ be a converging sequence and $(x, t) = \lim(x_n, t_n)$. Suppose for example that $\frac{t}{1-t} < T_\nu(x)$ (the case $\frac{t}{1-t} = T_\nu(x)$ and $\frac{t}{1-t} > T_\nu(x)$ are analogous). Then by Lemma 7.6.1, $\frac{t_n}{1-t_n} < T_\nu(x_n)$ for n large enough and thus by continuity of addition (Lemma 7.3.5), $\phi_\nu((x_n, t_n)) = x_n + \frac{t_n}{1-t_n}\nu \to x + \frac{t}{1-t}\nu = \phi_\nu(x, t)$. Therefore, ϕ_ν is continuous, which concludes the proof. □

Corollary 7.6.4. *The masure $(\mathcal{I}, d)$ is contractible.*

Proof. Let $\nu \in Q$. One sets $\psi_\nu : \begin{cases} \mathcal{I} \times [0,1] \to \mathcal{I} \\ \quad (x,t) \mapsto \phi_\nu(x, 2t) \text{ if } t \leq \dfrac{1}{2} \\ \quad (x,t) \mapsto 2(1-t)y_\nu(x) \text{ if t} > \dfrac{1}{2} \end{cases}$. Then ψ_ν is a strong deformation retract on $\{0\}$, which proves the corollary. □

Chapter 8

Tits preorder on a masure of affine type - Préordre de Tits sur une masure de type affine

8.1 Introduction en français

Supposons que G est un groupe de Kac-Moody affine sur un corps local. Dans cette situation, $\mathring{\mathcal{T}}$ est bien compris : c'est un demi-espace ouvert de $\mathbb{A}$ défini par une certaine forme linéaire $\delta_{\mathbb{A}} : \mathbb{A} \to \mathbb{R}$ (la plus petite racine imaginaire de G). En utilisant l'action de G, on étend naturellement $\delta_{\mathbb{A}}$ en une application $\delta : \mathcal{I} \to \mathbb{A}$. Le but de ce court chapitre est de montrer le théorème suivant :

Théorème 1. Soient $x, y \in \mathcal{I}$. Alors $\delta(x) < \delta(y)$ si et seulement si $x \mathring{<} y$.

Comme en restriction à chaque appartement, δ est une forme affine, cela prouve que « presque toutes » les paires de points sont incluses dans un appartement. Cela répond à la question du dernier paragraphe de la section 5 de [Rou11].

Idée de la preuve On définit une distance d sur l'ensemble des appartements contenant $+\infty$: si A, B contiennent $+\infty$, $d(A, B)$ est la longueur minimal possible d'une suite d'appartements $A_1, \ldots, A_n$ contenant $+\infty$ telle que $A_1 = A$, $A_n = B$ et pour tout $i \in [\![1, n-1]\!]$, $A_i \cap A_{i+1}$ est un demi-appartement (on prouve qu'une telle suite existe). On fait ensuite une récurrence sur cette distance.

Cadre Dans ce chapitre, $\mathcal{I}$ est une masure abstraite associée à un système générateur de racines.

Organisation du chapitre Dans la section 8.3, on étend la racine $\delta_A : \mathbb{A} \to \mathbb{R}$ en une fonction $\delta : \mathcal{I} \to \mathbb{R}$.

Dans la section 8.4, on introduit une distances sur les appartements contenant $+\infty$.

Dans la section 8.5, on prouve le théorème.

8.2 Introduction

Suppose that G is an affine Kac-Moody group over a local field. In this situation, $\mathring{\mathcal{T}}$ is well understood: this is an open half-space of $\mathbb{A}$ defined by some linear form $\delta_{\mathbb{A}} : \mathbb{A} \to \mathbb{R}$ (the smallest imaginary root of G). Using the action of G, one naturally extends $\delta_{\mathbb{A}}$ to a map $\delta : \mathcal{I} \to \mathbb{A}$. The aim of this short chapter is to prove the following theorem:

Theorem 12. Let $x, y \in \mathcal{I}$. Then $\delta(x) < \delta(y)$ if and only if $x \mathring{<} y$.

As in restriction to each apartments, δ is an affine map, this proves that "almost all" pair of points is included in an apartment. This answers the question of the last paragraph of Section 5 of [Rou11].

Outline of the proof We define a distance d on the set of apartments containing $+\infty$: if A, B contains $+\infty$, $d(A, B)$ is the minimal possible length of a sequence $A_1, \ldots, A_n$ of apartments containing $+\infty$ such that $A_1 = A$, $A_n = B$ and for all $i \in [\![1, n-1]\!]$, $A_i \cap A_{i+1}$ is a half-apartment (we prove that such a sequence exists). We then make an induction on this distance.

Framework In this chapter, $\mathcal{I}$ is an abstract masure associated to a root generating system of affine type.

Organization of the chapter In Section 8.3, we extend the root $\delta_{\mathbb{A}} : \mathbb{A} \to \mathbb{R}$ to a map $\delta : \mathcal{I} \to \mathbb{R}$.

In Section 8.4, we introduce a distance between the apartments containing $+\infty$.

In Section 8.5, we prove the theorem.

8.3 Standard apartment in the affine case

We now suppose that the Kac-Moody matrix involved in the definition of $\mathbb{A}$ is indecomposable and affine. Then there exists $\delta_{\mathbb{A}} \in \bigoplus_{i \in I} \mathbb{N}^* \alpha_i$ such that $\mathring{\mathcal{T}} = \delta^{-1}(\mathbb{R}_+^*)$ and $\delta_{\mathbb{A}}$ is W^v-invariant, see 2.3.1. One has $\mathcal{T} = \mathring{\mathcal{T}} \cup \mathbb{A}_{in}$.

Lemma 8.3.1. *Let $w \in W$ such that w fixes a point of $\mathbb{A}$. Then $\delta_{\mathbb{A}} \circ w = \delta_{\mathbb{A}}$.*

Proof. One embeds W in $W_{\mathbb{A}} = W^v \ltimes \mathbb{A}$. Let $a \in \mathbb{A}$ such that $w(a) = a$. Let τ be the translation of $\mathbb{A}$ sending a to 0. Let $w' = \tau \circ w \circ \tau^{-1} \in W_{\mathbb{A}}$. Then $w' = \tau_1 \circ w_1$, where τ_1 is a translation of $\mathbb{A}$ and $w_1 \in W^v$. As $w'(0) = 0$, τ_1 is the identity and $w' \in W^v$. Let $x \in \mathbb{A}$. Then $w(x) = a + w'(x) - w'(a)$. As $\delta_{\mathbb{A}}$ is W^v-invariant, $\delta(w(x)) = \delta(x) + \delta(a) - \delta(a) = \delta(x)$, which is our assertion. □

Proposition 8.3.2.

1. *There exists a unique map $\delta : \mathcal{I} \to \mathbb{R}$ such that $\delta_{|\mathbb{A}} = \delta_{\mathbb{A}}$ and such that the restriction of δ to each apartment is an affine form.*

2. *Let $\mathfrak{q}$ be a sector-germ of $\mathbb{A}$ and $\rho_{\mathfrak{q}} = \rho_{\mathfrak{q},\mathbb{A}} : \mathcal{I} \to \mathbb{A}$ be the retraction on $\mathbb{A}$ centered at $\mathfrak{q}$. Then $\delta = \delta_{\mathbb{A}} \circ \rho_{\mathfrak{q}}$.*

Proof. Suppose that there exist $\delta_1, \delta_2 : \mathcal{I} \to \mathbb{R}$ satisfying 1. Let $x \in \mathcal{I}$. By (MA iii), there exists an apartment A containing x and $+\infty$. As $\delta_{1|A}, \delta_{2|A}$ coincide on a nonempty open set and are affine, $\delta_{1|A} = \delta_{2|A}$ and thus $\delta_1(x) = \delta_2(x)$. Therefore, $\delta_1 = \delta_2$.

Let $\mathfrak{q}$ be a sector-germ of $\mathbb{A}$ and $\delta_{\mathfrak{q}} = \delta \circ \rho_{\mathfrak{q}}$. Let A be an apartment of $\mathcal{I}$. By Proposition 4.1.2, there exists $n \in \mathbb{N}$ and $P_1, \ldots, P_n$ non-empty enclosed subsets of A such that $A = \bigcup_{i=1}^n P_i$, where for all $i \in [\![1, n]\!]$, there exists an apartment A_i containing $\mathfrak{q} \cup P_i$. If

$i \in \llbracket 1, n \rrbracket$, let $\psi_i : A_i \overset{\mathbb{A}\cap A_i}{\to} \mathbb{A}$ and $\phi_i : A \overset{A\cap A_i}{\to} A_i$, which exist by (MA af ii). Let $i \in \llbracket 1, n \rrbracket$ and $x \in P_i$. Then $\rho_q(x) = \psi_i(x) = \psi_i \circ \phi_i(x)$.

Let us prove that for all $i, j \in \llbracket 1, n \rrbracket$, $\delta_{\mathbb{A}} \circ \psi_i \circ \phi_i = \delta_{\mathbb{A}} \circ \psi_j \circ \phi_j$. Let $i, j \in \llbracket 1, n \rrbracket$. Let $x \in P_i$ and $y \in P_j$. One identifies $[x, y]$ and $[0, 1]$. Then there exists $k \in \llbracket 1, n \rrbracket$, a map $\sigma : \llbracket 1, k \rrbracket \to \llbracket 1, n \rrbracket$ and $t_1 = 0 < t_2 < \ldots < t_k = 1$ such that for all $\ell \in \llbracket 1, k-1 \rrbracket$, $[t_\ell, t_{\ell+1}] \subset P_{\sigma(l)}$. Let $\ell \in \llbracket 1, k-1 \rrbracket$. There exists $w \in W$ such that $\psi_{\sigma(\ell+1)} \circ \phi_{\sigma(\ell+1)} = w \circ \psi_{\sigma(l)} \circ \phi_{\sigma(l)}$. Moreover if $a = \psi_{\sigma(\ell+1)} \circ \phi_{\sigma(\ell+1)}(t_{\ell+1}) = \psi_{\sigma(l)} \circ \phi_{\sigma(l)}(t_{\ell+1})$, w fixes a. By Lemma 8.3.1, $\delta_{\mathbb{A}}(\psi_{\sigma(l)} \circ \phi_{\sigma(l)}) = \delta_{\mathbb{A}}(\psi_{\sigma(\ell+1)} \circ \phi_{\sigma(\ell+1)})$. By induction, we get that $\delta_{\mathbb{A}} \circ \psi_i \circ \phi_i = \delta_{\mathbb{A}} \circ \psi_j \circ \phi_j$. Therefore, for all $i \in \llbracket 1, n \rrbracket$, $\delta_{\mathfrak{q}|A} = \delta_{\mathbb{A}} \circ \psi_i \circ \phi_i$. In particular, $\delta_{\mathfrak{q}}$ is an affine form on each apartment and we get 2 and 1. □

8.4 Distance between apartments containing $+\infty$

One denotes by $\mathcal{A}(+\infty)$ the set of apartments containing $+\infty$. In this subsection, we define a distance on $\mathcal{A}(+\infty)$. This subsection is not specific to the affine case. We deduce an other characterization of strong transitivity for the action of a group on $\mathcal{I}$.

Lemma 8.4.1. *Let $A \in \mathcal{A}(+\infty)$ and $\mathfrak{q}$ be a negative sector-germ of $\mathcal{I}$. Then there exist $n \in \mathbb{N}^*$ and a sequence $(A_i) \in \mathcal{A}(+\infty)^n$ such that $A_1 = A$, $A_n \supset \mathfrak{q}$ and $A_i \cap A_{i+1}$ is a half-apartment for all $i \in \llbracket 1, n-1 \rrbracket$.*

Proof. Let $\mathfrak{q}'$ be a negative sector-germ of A. By (MA af iii), there exists an apartment B containing $\mathfrak{q}$ and $\mathfrak{q}'$. Let $\mathfrak{q}_1 = \mathfrak{q}', \ldots, \mathfrak{q}_n = \mathfrak{q}$ be a gallery of sector-germs from $\mathfrak{q}'$ to $\mathfrak{q}$. Let $A_1 = A$. Let $i \in \llbracket 1, n-1 \rrbracket$ and suppose that we have constructed $(A_j) \in \mathcal{A}(+\infty)^i$ such that for all $j \in \llbracket 1, i-1 \rrbracket$, $A_j \cap A_{j+1}$ is a half-apartment and $A_j \supset \mathfrak{q}_j$. By Proposition 4.1.2, there exist two half-apartments D_1, D_2 of A_i such that for all $k \in \{1, 2\}$, $D_k \cup \mathfrak{q}_{i+1}$ is included in an apartment B_k. Let $k \in \{1, 2\}$ such that $D_k \supset +\infty$. One sets $A_{i+1} = B_k$. By induction, we get a sequence (A_i) satisfying the desired property. □

Proposition 8.4.2. *Let $A, B \in \mathcal{A}(+\infty)$. Then there exists $n \in \mathbb{N}^*$ and a sequence $(A_i) \in \mathcal{A}(+\infty)^n$ such that $A_1 = A$, $A_n = B$ and for all $i \in \llbracket 1, n-1 \rrbracket$, $A_i \cap A_{i+1}$ is a half-apartment.*

Proof. We apply Lemma 8.4.1 taking $\mathfrak{q}$ to be the sector-germ of B opposite to $+\infty$ and conclude with Lemma 4.4.20. □

Distance on $\mathcal{A}(+\infty)$ Let $A, B \in \mathcal{A}(+\infty)$. One defines $d(A, B) \in \mathbb{N}$ as the minimal possible $k \in \mathbb{N}$ such that there exists a sequence $(A_i) \in \mathcal{A}(+\infty)^{k+1}$ such that $A_1 = A$, $A_{k+1} = B$ and $A_i \cap A_{i+1}$ is a half-apartment for all $i \in \llbracket 1, k-1 \rrbracket$. This is well-defined by Proposition 8.4.2 and it is easy to see that it is a distance on $\mathcal{A}(+\infty)$.

Proposition 8.4.3. *Let $\mathcal{I}$ be a masure, not necessarily of affine type and G be a group of automorphisms of $\mathcal{I}$. Then G acts strongly transitively on $\mathcal{I}$ if and only if for all pairs A, B of apartments such that $A \cap B$ is a half-apartment, there exists $g \in G$ inducing $A \overset{A\cap B}{\to} B$.*

Proof. The implication $\Rightarrow$ is a consequence of (MA4) and Remark 3.2.4. Suppose that all pairs A, B of apartments such that $A \cap B$ is a half-apartment, there exists $g \in G$ inducing $A \overset{A\cap B}{\to} B$. Let $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$ and $A, B \in \mathcal{A}(\mathfrak{q})$. Let d be the distance on $\mathcal{A}(\mathfrak{q})$ defined similarly as the distance on $\mathcal{A}(+\infty)$. Let $k = d(A, B)$ and $(A_1, \ldots, A_k) \in \mathcal{A}(\mathfrak{q})$ such that $A_1 = A$, $A_k = B$ and for all $i \in \llbracket 1, k-1 \rrbracket$, $A_i \cap A_{i+1}$ is a half-apartment. For all $i \in \llbracket 1, k-1 \rrbracket$, one chooses $g_i \in G$ inducing $A_i \overset{A_i\cap A_{i+1}}{\to} A_{i+1}$. Let $g = g_{k-1} \ldots g_1$. Then $g.A = B$ and moreover g fixes $\mathfrak{q}$. Therefore g induces $A \overset{\mathfrak{q}}{\to} B$ on A. We conclude with Lemma 4.4.38. □

8.5 Induction on the distance

Lemma 8.5.1. *Let $x, y \in \mathbb{A}$ such that $x \mathring{<} y$ and $w \in W$ such that w fixes a point of $\mathbb{A}$. Then $x \mathring{<} w(y)$.*

Proof. By Lemma 8.3.1 $\delta_{\mathbb{A}}(w(y)) = \delta_{\mathbb{A}}(y)$, thus $\delta_{\mathbb{A}}(x) < \delta_{\mathbb{A}}(w(y))$ and hence $x \mathring{<} w(y)$. □

Lemma 8.5.2. *Let $A, B \in \mathcal{A}(+\infty)$ such that $A \cap B$ is a half-apartment D_1 and $\phi : A \overset{D_1}{\to} B$. Let $x, y \in A$ such that $x \mathring{<} y$. Then $x \mathring{<} \phi(y)$ and $\phi(x) \mathring{<} y$.*

Proof. If $\phi(y) = y$ this is clear. If $\phi(x) = x$, then $x = \phi(x) \mathring{<} \phi(y)$ by invariance of the Tits preorder. Suppose that $\phi(y) \neq y$ and $\phi(x) \neq x$. Let D_2 be the half-apartment of A opposed to D_1, $D_3 = \phi(D_2)$ and M be the wall of D_1. Let $A_1 = D_2 \cup D_3$. By Proposition 2.9 2 of [Rou11], A_1 is an apartment of $\mathcal{I}$. One has $B = D_1 \cup D_3$. Let $s : A_1 \to A_1$ be the reflexion of A_1 fixing M. Then by Lemma 4.2.5, the following diagram is commutative:

$$\begin{array}{ccc} A & \xrightarrow{\phi} & B \\ \downarrow{\scriptstyle \phi_2} & & \downarrow{\scriptstyle \phi_1} \\ A_1 & \xrightarrow{s} & A_1 \end{array},$$

where $\phi_2 : A \overset{D_2}{\to} A_1$ and $\phi_1 : A \overset{D_1}{\to} B$. By hypothesis, $x, y \in D_2$ and thus $\phi(x) = s(x)$ and $\phi(y) = s(y)$.

Let $\psi : A \to \mathbb{A}$ be an isomorphism. Let $s' = \psi \circ s \circ \psi^{-1} : \mathbb{A} \to \mathbb{A}$. Then $s' \in W$ and s' fixes $\psi(M)$. By Lemma 8.5.1, $\psi(x) \mathring{<} s'(\psi(y)) = \psi(s(y))$. Therefore $x \mathring{<} s(y) = \phi(y)$.

It remains to prove that $\phi(x) \mathring{<} y$. By applying the result we just proved with $\phi^{-1} : B \to A$ instead of ϕ, we get that $x \mathring{<} \phi^{-1}(y)$. Therefore $\phi(x) \mathring{<} \phi(\phi^{-1}(y)) = y$ and the lemma is proved. □

Theorem 8.5.3. *Let $x, y \in \mathcal{I}$. Then $\delta(x) < \delta(y)$ if and only if $x \mathring{<} y$.*

Proof. By Corollaire 2.8 of [Rou11], for all $x, y \in \mathcal{I}$, if $x \mathring{<} y$, then $\rho(x) \mathring{<} \rho(y)$ (resp. $\rho(y) \mathring{<} \delta(x)$) (there is no need to assume that $\mathbb{A}$ is of affine type) and thus $\delta(x) < \delta(y)$.

If $x, y \in \mathcal{I}$, one sets

$$d(x, y) = \min\{d(A, B) |\ (A, B) \in \mathcal{A}(+\infty)^2, A \ni x \text{ and } B \ni y\}$$

(d is not a distance on $\mathcal{I}$). If $n \in \mathbb{N}$, one sets $\mathcal{P}_n$:

$$\forall (x, y) \in \mathcal{I}^2 |\ d(x, y) \leq n, \delta(x) < \delta(y) \Rightarrow x \mathring{<} y.$$

The property $\mathcal{P}_0$ is true because δ determines the Tits preorder on $\mathbb{A}$. Let $n \in \mathbb{N}$ such that $\mathcal{P}_n$ is true and $x, y \in \mathcal{I}$ such that $d(x, y) = n + 1$ and $\delta(x) < \delta(y)$. Let $A, B \in \mathcal{A}(+\infty)$ such that $d(A, B) = n + 1$, $A \ni x$ and $B \ni y$.

Let $z \in A$ such that $\delta(x) < \delta(z) < \delta(y)$. Let $A' \in \mathcal{A}(+\infty)$ such that $d(A, A') = 1$ and $d(A', B) = n$, and $\phi : A \overset{A \cap A'}{\to} A'$. One has $\delta(\phi(z)) = \delta(z)$ and thus $\phi(z) \mathring{<} y$ by $\mathcal{P}_n$. Moreover, $\phi(x) \mathring{<} \phi(z)$ (by $\mathcal{P}_0$) and hence by Lemma 8.5.2, $x = \phi^{-1}(\phi(x)) \mathring{<} \phi(z)$. As $\mathring{<}$ is a transitive (by Théorème 5.9 of [Rou11]), we deduce that $x \mathring{<} y$ and thus $\mathcal{P}_{n+1}$ is true, which proves the theorem. □

Recall the action of $\mathbb{A}_{in}$ on $\mathcal{I}$ from 3.2.

Corollary 8.5.4. *For all $x, y \in \mathcal{I}$ such that $x + \mathbb{A}_{in} \neq y + \mathbb{A}_{in}$. Then:*

1. *$x \leq y$ if and only if $\delta(x) < \delta(y)$*

2. *x and y are not comparable for $\leq$ if and only if $\delta(x) = \delta(y)$.*

For $x, y \in \mathcal{I}$, one denotes $x\ NC\ y$ if x and y are not comparable for $\leq$ or $x + \mathbb{A}_{in} = y + \mathbb{A}_{in}$.

Corollary 8.5.5. 1. *The relation NC is an equivalence relation on $\mathcal{I}$ whose classes are the level sets of δ.*

2. *Let $x, y \in \mathcal{I}$. Let A be an apartment containing x. Then $x\ NC\ y$ if and only if there exists $(x_n^+) \in A^{\mathbb{N}}$ and $(x_n^-) \in A^{\mathbb{N}}$ such that $x_n^+ \to x$, $x_n^- \to x$ and for all $n \in \mathbb{N}$, $x_n^- \mathring{<} y \mathring{<} x_n^+$.*

3. *Let $x, y, z \in \mathcal{I}$ such that $y\ NC\ z$. Then $x \mathring{<} y$ if and only if $x \mathring{<} z$.*

4. *Let $x, y \in \mathcal{I}$. Then $x \mathring{<} y$ if and only if $\rho_{+\infty}(x) \mathring{<} \rho_{+\infty}(y)$.*

Remark 8.5.6.
- *Point 4 of Corollary 8.5.5 is specific to the affine case. Indeed suppose that G is associated to an indefinite Kac-Moody group. By the lemma of Section 2.9 of [GR14] and Proposition 5.8 c) of [Kac94], for all $i \in I$, $\alpha_i^\vee \in \mathbb{A} \backslash \overline{\mathcal{T}}$ and as $r_i.\alpha_i^\vee = -\alpha_i^\vee$, we deduce that $\alpha_i^\vee \in \mathbb{A} \backslash (\overline{\mathcal{T}} \cup \overline{-\mathcal{T}})$.*

  Let $x = \alpha_i^\vee$. Let us show that there exists $y \in \mathbb{A}$ satisfying $x \mathring{<} y$ but not $x \mathring{<} r_i(y)$. Let $u \in \mathcal{T}$ and $y_n = x + \frac{1}{n} u$ for all $n \in \mathbb{N}^*$. One has $r_i(y_n) - x = -2\alpha_i^\vee + \frac{1}{n} r_i(u)$ for all $n \in \mathbb{N}^*$ and thus $r_i(y_n) - x \notin \mathcal{T}$ for n large enough. Thus by the same reasoning as in the proof of Lemma 8.5.2, there exists an apartment $A \in \mathcal{A}(+\infty)$ such that if $\phi : \mathbb{A} \overset{A \cap \mathbb{A}}{\to} A$, one has $x \mathring{<} y_n$ but not $x \mathring{<} r_i(y_n) = \rho_{+\infty}(\phi(y_n))$ for n large enough.

- NC *is not an equivalence relation in the indefinite case. Indeed, let $u \in \mathcal{T}$, $x = 0$, $y_n = \alpha_i^\vee + \frac{1}{n} u$ and $z_n = \frac{2}{n} u$. For n large enough, x* NC *y_n and y_n* NC *z_n but $x < z_n$.*

Appendix A

A short definition of masures

In this chapter, we give a short definition of semi-discrete abstract masures. The masure associated to a split Kac-Moody group over a local field is an example of such a masure.

A.1 Vectorial apartment

A.1.1 Root generating system

A Kac-Moody matrix (or generalized Cartan matrix) is a square matrix $A = (a_{i,j})_{i,j \in I}$ with integers coefficients, indexed by a finite set I and satisfying:

1. $\forall i \in I,\ a_{i,i} = 2$
2. $\forall (i,j) \in I^2 | i \neq j,\ a_{i,j} \leq 0$
3. $\forall (i,j) \in I^2,\ a_{i,j} = 0 \Leftrightarrow a_{j,i} = 0$.

A **root generating system** is a 5-tuple $\mathcal{S} = (A, X, Y, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ made of a Kac-Moody matrix A indexed by I, of two dual free $\mathbb{Z}$-modules X (of characters) and Y (of cocharacters) of finite rank $\mathrm{rk}(X)$, a family $(\alpha_i)_{i\in I}$ (of simple roots) in X and a family $(\alpha_i^\vee)_{i\in I}$ (of simple coroots) in Y. They have to satisfy the following compatibility condition: $a_{i,j} = \alpha_j(\alpha_i^\vee)$ for all $i, j \in I$. We also suppose that the family $(\alpha_i)_{i\in I}$ is free in X and that the family $(\alpha_i^\vee)_{i\in I}$ is free in Y.

We now fix a Kac-Moody matrix A and a root generating system with matrix A.

Let $\mathbb{A} = Y \otimes \mathbb{R}$. Every element of X induces a linear form on $\mathbb{A}$. We will consider X as a subset of the dual $\mathbb{A}^*$ of $\mathbb{A}$: the α_i's, $i \in I$ are viewed as linear forms on $\mathbb{A}$. For $i \in I$, we define an involution r_i of $\mathbb{A}$ by $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ for all $v \in \mathbb{A}$. Its space of fixed points is $\ker \alpha_i$. The subgroup of $\mathrm{GL}(\mathbb{A})$ generated by the α_i's for $i \in I$ is denoted by W^v and is called the Weyl group of $\mathcal{S}$. The system $(W^v, \{r_i | i \in I\})$ is a Coxeter system.

Let $Q^\vee_\mathbb{Z} = \bigoplus_{i\in I} \mathbb{Z}\alpha_i^\vee$ be the **coroot lattice of** $\mathbb{A}$.

One defines an action of the group W^v on $\mathbb{A}^*$ by the following way: if $x \in \mathbb{A}$, $w \in W^v$ and $\alpha \in \mathbb{A}^*$ then $(w.\alpha)(x) = \alpha(w^{-1}.x)$. Let $\Phi_{re} = \{w.\alpha_i | (w,i) \in W^v \times I\}$, Φ_{re} is the set of **real roots**. Then $\Phi_{re} \subset Q$, where $Q = \bigoplus_{i\in I} \mathbb{Z}\alpha_i$ is the **root lattice of**$\mathbb{A}$. Let $W = Q^\vee \rtimes W^v \subset \mathrm{GA}(\mathbb{A})$ be the **affine Weyl group** of $\mathcal{S}$, where $\mathrm{GA}(\mathbb{A})$ is the group of affine isomorphisms of $\mathbb{A}$.

A.1.2 Vectorial faces and Tits cone

Define $C_f^v = \{v \in \mathbb{A} |\ \alpha_i(v) > 0,\ \forall i \in I\}$. We call it the **fundamental chamber**. For $J \subset I$, one sets $F^v(J) = \{v \in \mathbb{A} |\ \alpha_i(v) = 0\ \forall i \in J, \alpha_i(v) > 0\ \forall i \in J\backslash I\}$. Then the closure $\overline{C_f^v}$ of C_f^v is the union of the $F^v(J)$ for $J \subset I$. The **positive** (resp. **negative**) **vectorial faces** are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. A **vectorial face** is either a positive vectorial face or a negative vectorial face. We call **positive chamber** (resp. **negative**) every cone of the shape $w.C_f^v$ for some $w \in W^v$ (resp. $-w.C_f^v$). For all $x \in C_f^v$ and for all $w \in W^v$, $w.x = x$ implies that $w = 1$. In particular the action of w on the positive chambers is simply transitive. The **Tits cone** $\mathcal{T}$ is defined by $\mathcal{T} = \bigcup_{w \in W^v} w.\overline{C_f^v}$. We also consider the negative cone $-\mathcal{T}$. We define a W^v-invariant relation $\leq$ on $\mathbb{A}$ by: $\forall (x, y) \in \mathbb{A}^2$, $x \leq y \ \Leftrightarrow\ y - x \in \mathcal{T}$.

A.2 Masure

A.2.1 Filters

Definition A.2.1. *A filter in a set E is a nonempty set F of nonempty subsets of E such that, for all subsets S, S' of E, if S, $S' \in F$ then $S \cap S' \in F$ and, if $S' \subset S$, with $S' \in F$ then $S \in F$.*

If F is a filter in a set E, and E' is a subset of E, one says that F contains E' if every element of F contains E'. If E' is nonempty, the set $F_{E'}$ of subsets of E containing E' is a filter. By abuse of language, we will sometimes say that E' is a filter by identifying $F_{E'}$ and E'. If F is a filter in E, its closure $\overline{F}$ (resp. its convex hull) is the filter of subsets of E containing the closure (resp. the convex hull) of some element of F. A filter F is said to be contained in an other filter F': $F \subset F'$ (resp. in a subset Z in E: $F \subset Z$) if and only if any set in F' (resp. if Z) is in F.

If $x \in \mathbb{A}$ and Ω is a subset of $\mathbb{A}$ containing x in its closure, then the **germ** of Ω in x is the filter $germ_x(\Omega)$ of subsets of $\mathbb{A}$ containing a neighborhood in Ω of x.

A **sector** in $\mathbb{A}$ is a set of the shape $\mathfrak{s} = x + C^v$ with $C^v = \pm w.C_f^v$ for some $x \in \mathbb{A}$ and $w \in W^v$. The point x is its **base point** and C^v is its **direction**. The intersection of two sectors of the same direction is a sector of the same direction.

The **sector-germ** of a sector $\mathfrak{s} = x + C^v$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ containing an $\mathbb{A}$-translate of $\mathfrak{s}$. It only depends on the direction C^v. We denote by $+\infty$ (resp. $-\infty$) the sector-germ of C_f^v (resp. of $-C_f^v$).

A ray δ with base point x and containing $y \neq x$ (or the interval $\mathring{]}x, y] = [x, y]\backslash\{x\}$ or $[x, y]$) is called **preordered** if $x \leq y$ or $y \leq x$ and **generic** if $y - x \in \pm\mathring{\mathcal{T}}$, the interior of $\pm\mathcal{T}$.

A.2.2 Definitions of walls, enclosures, faces and related notions

Enclosure A hyperplane of the form $\alpha^{-1}(\{k\})$ with $\alpha \in \Phi_{re}$ and $k \in \mathbb{Z}$ is called a **wall**. A half-space of $\mathbb{A}$ delimited by a wall is a **half-apartment**. If $\alpha \in \Phi_{re}$ and $k \in \mathbb{Z}$, one sets $M(\alpha, k) = \{x \in \mathbb{A} |\ \alpha(x) + k = 0\}$, $D(\alpha, k) = \{x \in \mathbb{A} | \alpha(x) + k \geq 0\}$ and $D^\circ(\alpha, k) = \mathring{D}(\alpha, k)$.

A set $P \subset \mathbb{A}$ is said to be **enclosed** if there exist $n \in \mathbb{N}$ and half-apartments $D_1, \ldots, D_n$ such that $P = \bigcap_{i=1}^n D_i$.

If $\mathcal{X}$ is a filter of $\mathbb{A}$, its **enclosure** is the filter $\mathrm{cl}(\mathcal{X})$ defined as follows. A set E is in $\mathrm{cl}(\mathcal{X})$ if and only if there exists an enclosed set $E' \subset E$ such that E' is enclosed and $E' \in \mathcal{X}$.

Faces A **local face** F^ℓ in $\mathbb{A}$ is a filter associated to a point $x \in \mathbb{A}$, its **vertex** and a vectorial face $F^v \subset \mathbb{A}$, its **direction**. It is defined by $F^\ell = germ_x(x + F^v)$ and we denote it by $F^\ell(x, F^v)$. A **face** F in $\mathbb{A}$ is a filter associated to a point $x \in \mathbb{A}$ and a vectorial face $F^v \subset \mathbb{A}$. More precisely, a subset S of $\mathbb{A}$ is an element of the face $F = F(x, F^v)$ if and only if it contains a finite intersection of half-apartments or open half-apartments containing $F^\ell(x, F^v)$.

There is an order on the (local) faces: if $F \subset \overline{F'}$ we say that "F is a face of F' " or "F' contains F" or "F' dominates F". The dimension of a face F is the smallest dimension of an affine space generated by some $S \in F$. Such an affine space is unique and is called its support. A face is said to be **spherical** if its direction is spherical; then its point-wise stabilizer W_F in W^v is finite.

As W^v stabilizes Φ_{re}, any element of W^v permutes the sets of the shape $D(\alpha, k)$ where α runs over Φ_{re} and $k \in \mathbb{Z}$. Thus W permutes the enclosures, faces, ... of $\mathbb{A}$.

A **chamber** (or alcove) is a maximal face, i.e a face $F^\ell(x, \pm w.C^v_f)$ for $x \in \mathbb{A}$ and $w \in W^v$.

A **panel** is a spherical face maximal among faces that are not chambers or equivalently a spherical face of dimension $\dim \mathbb{A} - 1$.

Chimneys A **chimney** in $\mathbb{A}$ is associated to a face $F = F(x, F^v_0)$ and to a vectorial face F^v; it is the filter $\mathfrak{r}(F, F^v) = \mathrm{cl}(F + F^v)$. The face F is the basis of the chimney and the vectorial face F^v its direction. A chimney is **splayed** if F^v is spherical.

A **shortening** of a chimney $\mathfrak{r}(F, F^v)$, with $F = F(x, F^v_0)$ is a chimney of the shape $\mathfrak{r}(F(x + \xi, F^v_0), F^v)$ for some $\xi \in \overline{F^v}$. The **germ** of a chimney $\mathfrak{r}$ is the filter of subsets of $\mathbb{A}$ containing a shortening of $\mathfrak{r}$.

A.2.3 Masure

An apartment of type $\mathbb{A}$ is a set A with a nonempty set $\mathrm{Isom}(\mathbb{A}, A)$ of bijections (called Weyl-isomorphisms) such that if $f_0 \in \mathrm{Isom}(\mathbb{A}, A)$ then $f \in \mathrm{Isom}(\mathbb{A}, A)$ if and only if, there exists $w \in W$ satisfying $f = f_0 \circ w$. We will say isomorphism instead of Weyl-isomorphism in the sequel. An isomorphism between two apartments $\phi : A \to A'$ is a bijection such that ($f \in \mathrm{Isom}(\mathbb{A}, A)$ if, and only if, $\phi \circ f \in \mathrm{Isom}(\mathbb{A}, A')$). We extend all the notions that are preserved by W to each apartment. Thus sectors, enclosures, faces and chimneys are well defined in any apartment of type $\mathbb{A}$.

If A, A' are apartments, $\phi : A \to B$ is an isomorphism of apartments and $E \subset A \cap A'$, the notation $\phi : A \stackrel{E}{\to} A'$ means that ϕ fixes E.

Definition A.2.2. *A* ***masure of type*** $\mathbb{A}$ *is a set* $\mathcal{I}$ *endowed with a covering* $\mathcal{A}$ *of subsets called apartments such that:*

(MA i): Any $A \in \mathcal{A}$ *admits a structure of an apartment of type* $\mathbb{A}$.

(MA ii): if two apartments A, A' *contain a generic ray, then* $A \cap A'$ *is enclosed and there exists an isomorphism* $\phi : A \stackrel{A\cap A'}{\to} A'$.

(MA iii): if $\mathfrak{R}$ *is the germ of a splayed chimney and if* F *is a face or a germ of a chimney, then there exists an apartment containing* $\mathfrak{R}$ *and* F.

In this definition, one says that an apartment contains a germ of a filter if it contains at least one element of this germ. One says that a map fixes a germ if it fixes at least one element of this germ.

When $\mathcal{I}$ is associated to an affine Kac-Moody group, one can replace (MA ii) by (MA af ii):

If A, A' are two apartments then $A \cap A'$ is enclosed and there exists an isomorphism $\phi : A \stackrel{A\cap A'}{\to} A'$.

Group acting strongly transitively on a masure Let $\mathcal{I}$ be a masure. An **endomorphism of $\mathcal{I}$** is a map $\phi : \mathcal{I} \to \mathcal{I}$ such that for each apartment A, $\phi(A)$ is an apartment and $\phi_{|A}^{|\phi(A)}$ is an isomorphism of apartments. An **automorphism of $\mathcal{I}$** is a bijective endomorphism ϕ of $\mathcal{I}$ such that ϕ^{-1} is an endomorphism of $\mathcal{I}$. Let $\mathrm{Aut}(\mathcal{I})$ be the group of automorphisms of $\mathcal{I}$. A group $G \subset \mathrm{Aut}(\mathcal{I})$ is said to act **strongly transitively** if all the isomorphisms of apartment of (MA ii) are induced by elements of G.

If G is a split Kac-Moody group over a local field, then G acts strongly transitively on its masure $\mathcal{I}$, see 3.4.1 for more details on this action.

Retraction centered at a sector-germ Let $\mathcal{I}$ be a masure. Let $\mathfrak{q}$ be a sector-germ of $\mathcal{I}$ and A be an apartment containing $\mathfrak{q}$. If $x \in \mathcal{I}$, there exists an apartment A_x containing x and $\mathfrak{q}$ (by (MA iii)). By (MA ii), there exists an isomorphism $\phi_x : A_x \stackrel{A_x\cap A}{\to} A$. Then it is easy to see that $\phi_x(x)$ does not depend on the choices we made. One sets $\rho_{\mathfrak{q},A} = \phi_x(x)$. Then $\rho_{\mathfrak{q},A}$ is the **retraction of $\mathcal{I}$ onto A centered at $\mathfrak{q}$**.

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