On structure constants of Iwahori-Hecke algebras for Kac-Moody groups

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Abstract

We consider the Iwahori-Hecke algebra ${}^I\mathscr{H}$ associated to an almost split Kac-Moody group G (affine or not) over a nonarchimedean local field $\mathcal{K}$. It has a canonical double-coset basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$ indexed by a sub-semigroup W^+ of the affine Weyl group W. The multiplication is given by structure constants $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \in \mathbb{N} = \mathbb{Z}_{\geq 0}$: $T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u}\in P_{\mathbf{w},\mathbf{v}}} a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} T_{\mathbf{u}}$. A conjecture, by Braverman, Kazhdan, Patnaik, Gaussent and the authors, tells that $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is a polynomial, with coefficients in $\mathbb{N}$, in the parameters q_i-1, q'_i-1 of G over $\mathcal{K}$. We prove this conjecture when $\mathbf{w}$ and $\mathbf{v}$ are spherical or, more generally, when they are said to be generic: this includes all cases of $\mathbf{w}, \mathbf{v} \in W^+$ if G is of affine or strictly hyperbolic type. In the split affine case (where $q_i = q'_i = q$, $\forall i$) we get a universal Iwahori-Hecke algebra with the same basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$ over a polynomial ring $\mathbb{Z}[Q]$; it specializes to ${}^I\mathscr{H}$ when one sets $Q = q$.

Introduction

Let G be a split, semi-simple, simply connected algebraic group over a non archimedean local field $\mathcal{K}$. So $\mathcal{K}$ is complete for a discrete, non trivial valuation with a finite residue field κ. We write $\mathcal{O} \subset \mathcal{K}$ for the ring of integers and q for the cardinality of κ. Then G is locally compact. In this situation, Nagayoshi Iwahori and Hideya Matsumoto in [IM65], introduced an open compact subgroup K_I of G, now known as an Iwahori subgroup. If N is the normalizer of a suitable split maximal torus $T \simeq (\mathcal{K}^*)^n$, then (K_I, N) is a BN pair. The Iwahori-Hecke algebra of G is the algebra ${}^I\mathscr{H}_R = {}^I\mathscr{H}_R(G, K_I)$ of locally constant, compactly supported functions on G, with values in a ring R, that are bi-invariant by the left and right actions of K_I. The multiplication is given by the convolution product.

If $H \simeq (\mathcal{O}^*)^n$ is the maximal compact subgroup of T, then $H \subset K_I$ and $W = N/H$ is the affine Weyl group. One has the Bruhat decomposition $G = K_I.W.K_I = \sqcup_{\mathbf{w}\in W} K_I.\mathbf{w}.K_I$. If one considers the characteristic function $T_{\mathbf{w}}$ of $K_I.\mathbf{w}.K_I$, we get a basis of ${}^I\mathscr{H}_R$: ${}^I\mathscr{H}_R = \oplus_{\mathbf{w}\in W} R.T_{\mathbf{w}}$. The convolution product is given by $T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u}\in P_{\mathbf{w},\mathbf{v}}} a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} T_{\mathbf{u}}$, with $P_{\mathbf{w},\mathbf{v}}$ a finite subset of W. The numbers $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \in R$ are the structure constants of ${}^I\mathscr{H}_R$. The unit is $1 = T_e$.

Iwahori and Matsumoto gave a precise (and now classical) definition of ${}^I\mathscr{H}_R$ by generators and relations. The group W is an infinite Coxeter group generated by $\{r_0, \dots, r_n\}$. Then ${}^I\mathscr{H}_R$ is generated by $\{T_{r_0}, \dots, T_{r_n}\}$ with relations $T^2_{r_i} = q.1 + (q-1).T_{r_i}$ and $T_{r_i} * T_{r_j} * T_{r_i} * \cdots = T_{r_j} * T_{r_i} * T_{r_j} * \cdots$ (with $m_{i,j}$ factors on each side) for $i \neq j$, if $m_{i,j}$ is the finite order of $r_i r_j$. For $\mathbf{w} = r_{i_1}.\dots.r_{i_s}$ a reduced expression in W, one has $T_{\mathbf{w}} = T_{r_{i_1}} * \cdots * T_{r_{i_s}}$. In a Coxeter group one knows the rules to get (using the Coxeter relations between the r_i) a reduced expression from a non reduced expression (*e.g.* the product of two reduced expressions $\mathbf{w} = r_{i_1}.\dots.r_{i_s}$ and $\mathbf{v} = r_{j_1}.\dots.r_{j_t}$). So one deduces easily (using the above relations between the T_{r_i}) that each structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ (for $\mathbf{u},\mathbf{v},\mathbf{w} \in W$) is in $\mathbb{Z}[q]$. More precisely it is a polynomial in $q-1$ with coefficients in $\mathbb{N} = \mathbb{Z}_{\geq 0}$. This polynomial depends only on $\mathbf{u},\mathbf{v},\mathbf{w}$ and W.

So one has a universal description of ${}^I\mathscr{H}_{\mathbb{Z}}$ as a $\mathbb{Z}[q]-$algebra, depending only on W.

There are various generalizations of the above situation. First one may replace G by a general reductive group over $\mathcal{K}$, isotropic but potentially non split. Then one has to consider the relative affine Weyl group W, which is a Coxeter group. One may still define a compact, open Iwahori subgroup K_I and there is a Bruhat decomposition $G = K_I.W.K_I$. Now the description of ${}^I\mathscr{H}_R$ involves parameters q_i (satisfying $T_{r_i}^2 = q_i.1 + (q_i - 1).T_{r_i}$) which are potentially different from q. This gives the Iwahori-Hecke algebra with unequal parameters. There is a pleasant description of ${}^I\mathscr{H}_R$ using the Bruhat-Tits building associated to the BN pair (K_I, N), see *e.g.* [P06].

For now more than twenty years, there is an increasing interest in the study of Kac-Moody groups over local fields, see the works of Braverman, Garland, Kapranov, Kazhdan, Patnaik, Gaussent and the authors: *e.g.* [Ga95], [GaG95], [Kap01], [BrK11], [BrK14], [BrGKP14], [BrKP16], [GR14], [BaPGR16], [BaPGR19]. It has been possible to define and study for Kac-Moody groups (supposed at first affine) the spherical Hecke algebra, the Iwahori-Hecke algebra, the Satake isomorphism, ... This is also closely related to more abstract works on Hecke algebras by Cherednik and Macdonald, *e.g.* [Che92], [Che95], [Ma03].

We are mainly interested in Iwahori-Hecke algebras for Kac-Moody groups over local fields. They were introduced and described by Braverman, Kazhdan and Patnaik in the affine case [BrKP16] and then in general by Gaussent and the authors [BaPGR16]. So let us consider a Kac-Moody group G (affine or not) over the local field $\mathcal{K}$. We suppose it split (as defined by Tits [T87]) or more generally almost split [Re02]. Let us choose also a maximal split subtorus. To this situation is associated an affine (relative) Weyl group W and an Iwahori subgroup K_I (defined up to conjugacy by W), see 1.4 (5) and (7) below. This group W is not a Coxeter group but may be described as a semi-direct product $W = W^v \ltimes Y$, where W^v is a Coxeter group, the relative Weyl group, and Y is (essentially) the cocharacter group of the torus.

Unfortunately the Bruhat decomposition "$G = K_I.W.K_I$" fails to be true (even in the untwisted affine case, *i.e.* for loop groups). One has to consider the sub-semigroup $W^+ = W^v \ltimes Y^+$ (*resp.,* $W^{+g} = W^v \ltimes Y^{+g}$) of W, where Y^+ (*resp.,* Y^{+g}) is the intersection of Y with the Tits cone $\mathcal{T}$ (*resp.,* with a cone $\mathcal{T}^\circ \cup V_0 \subset \mathcal{T}$, where $\mathcal{T}^\circ$ is the open Tits cone) in $V = Y \otimes_{\mathbb{Z}} \mathbb{R}$ (see 1.2, 1.5, and 1.8 below). Then $G^+ = K_I.W^+.K_I$ (*resp.,* $G^{+g} = K_I.W^{+g}.K_I \subset G^+$) is a sub-semigroup of G: the Kac-Moody-Tits semigroup (*resp.,* the generic Kac-Moody-Tits semigroup). We may consider the characteristic functions $T_{\mathbf{w}}$ of the double cosets $K_I.\mathbf{w}.K_I$ and one proves in [BaPGR16] that:

The space ${}^I\mathscr{H}_R$ (*resp.,* ${}^I\mathscr{H}_R^g$) of $R-$valued functions with finite support on $K_I\backslash G^+/K_I$ (*resp.,* $K_I\backslash G^{+g}/K_I$) is naturally endowed with a structure of algebra (see 1.11). We get thus the Iwahori-Hecke algebra ${}^I\mathscr{H}_R = \oplus_{\mathbf{w}\in W^+} R.T_{\mathbf{w}}$ (*resp.,* the generic Iwahori-Hecke algebra ${}^I\mathscr{H}_R^g = \oplus_{\mathbf{w}\in W^{+g}} R.T_{\mathbf{w}}$). The product is given by structure constants $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \in \mathbb{N} = \mathbb{Z}_{\geq 0}$: $T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u}\in P_{\mathbf{w},\mathbf{v}}} a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} T_{\mathbf{u}}$.

Conjecture 1. *[BaPGR16, 2.5] Each $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is a polynomial, with coefficients in $\mathbb{N} = \mathbb{Z}_{\geq 0}$, in the parameters $q_i - 1, q'_i - 1$ of the situation, see 1.4.6 below. This polynomial depends only on the affine Weyl group W acting on the apartment $\mathbb{A}$ and on $\mathbf{w},\mathbf{v},\mathbf{u} \in W^+$.*

One may consider that this is a translation of the following question of Braverman, Kazhdan and Patnaik :

Question. *[BrKP16, end of 1.2.4] Has the algebra ${}^I\mathscr{H}_{\mathbb{C}}$ a purely algebraic or combinatorial description with respect to the coset basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$?*

But a more precise formulation of this question is as follows :

Conjecture 2. *The algebra ${}^I\mathscr{H}_{\mathbb{Z}}$ (or ${}^I\mathscr{H}^g_{\mathbb{Z}}$) is the specialization of an algebra ${}^I\mathscr{H}_{\mathbb{Z}[\mathscr{Q}]}$ (or ${}^I\mathscr{H}^g_{\mathbb{Z}[\mathscr{Q}]}$) with the same basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$ (or $(T_{\mathbf{w}})_{\mathbf{w}\in W^{+g}}$) over $\mathbb{Z}[\mathscr{Q}]$. Here $\mathscr{Q}$ is a set of indeterminates Q_i, Q'_i (with some equalities between them, see 1.4.6 below) and the specialization is given by $Q_i \mapsto q_i, Q'_i \mapsto q'_i, \forall i \in I$. The algebra ${}^I\mathscr{H}_{\mathbb{Z}[\mathscr{Q}]}$ (or ${}^I\mathscr{H}^g_{\mathbb{Z}[\mathscr{Q}]}$) depends only on the affine Weyl group W acting on the apartment $\mathbb{A}$.*

Let us consider the split case: G is a split Kac-Moody group, all parameters q_i, q'_i are equal to $q = |\kappa|$ and all indeterminates Q_i, Q'_i are equal to a single indeterminate Q. Then the conjecture 1 has already been proved by Gaussent and the authors [BaPGR16, 6.7] and independently by Muthiah [Mu18] if, moreover, G is untwisted affine. Actually the same proof gives also conjecture 2, see 1.4.7 below.

In the general (non split) case, weakened versions were obtained in [BaPGR16]: the $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ are Laurent polynomials in the q_i, q'_i [*l.c.* 6.7]; they are true polynomials if $\mathbf{w}, \mathbf{v} \in W^v \ltimes (Y \cap \mathcal{T}^\circ)$ and $\mathbf{v}$ is "regular" [*l.c.* 3.8].

In this article, we prove the conjecture 1 when $\mathbf{w}$ and $\mathbf{v}$ are in W^{+g} (see 3.6). We remark also that $W^+ = W^{+g}$ in the affine case (twisted or not) or the strictly hyperbolic case, even if G is not split. This is a first step towards the description of an abstract algebra ${}^I\mathscr{H}_{\mathbb{Z}[\mathscr{Q}]}$ (*resp.,* ${}^I\mathscr{H}^g_{\mathbb{Z}[\mathscr{Q}]}$) over $\mathbb{Z}[\mathscr{Q}]$ in the affine (or strictly hyperbolic) case (*resp.,* in the general case).

One should mention here that one may give a more precise description of the Iwahori-Hecke algebra using a Bernstein-Lusztig presentation (see [GaG95], [BrKP16] and [BaPGR16]). But this description is given in a new basis and the coefficients of the change of basis matrix are Laurent polynomials in the parameters q_i, q'_i. So this description is not sufficient to prove the conjecture.

Actually this article is written in a more general framework explained in Section 1: as in [BaPGR16], we work with an abstract masure $\mathscr{I}$ and we take G to be a strongly transitive group of vectorially-Weyl automorphisms of $\mathscr{I}$. In Section 2 we gather the additional technical tools (*e.g.* decorated Hecke paths) needed to improve the results of [BaPGR16, Section 3]. We get our main results about $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ in Section 3: we deal with the cases $\mathbf{w}, \mathbf{v}$ spherical. In Section 4 we deal with the remaining cases where $\mathbf{w}, \mathbf{v}$ are in W^{+g}, *i.e.* when $\mathbf{w}, \mathbf{v}$ are said generic.

1 General framework

1.1 Vectorial data

We consider a quadruple $(V, W^v, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$ where V is a finite dimensional real vector space, W^v a subgroup of $GL(V)$ (the vectorial Weyl group), I a finite set, $(\alpha_i^\vee)_{i\in I}$ a free family in V and $(\alpha_i)_{i\in I}$ a free family in the dual V^*. We ask these data to satisfy the conditions of [Ro11, 1.1]. In particular, the formula $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ defines a linear involution in V which is an element in W^v and $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system.

To be more concrete, we consider the Kac-Moody case of [*l.c.* ; 1.2]: the matrix $\mathbb{M} = (\alpha_j(\alpha_i^\vee))_{i,j\in I}$ is a generalized Cartan matrix. Then W^v is the Weyl group of the corresponding Kac-Moody Lie algebra $\mathfrak{g}_{\mathbb{M}}$ and the associated real root system is

$$\Phi = \{w(\alpha_i) \mid w \in W^v, i \in I\} \subset Q = \bigoplus_{i\in I} \mathbb{Z}.\alpha_i.$$

We set $\Phi^\pm = \Phi \cap Q^\pm$ where $Q^\pm = \pm(\bigoplus_{i\in I}(\mathbb{Z}_{\geq 0}).\alpha_i)$ and $Q^\vee = (\bigoplus_{i\in I}\mathbb{Z}.\alpha_i^\vee)$, $Q^\vee_\pm = \pm(\bigoplus_{i\in I}(\mathbb{Z}_{\geq 0}).\alpha_i^\vee)$. We have $\Phi = \Phi^+ \cup \Phi^-$ and, for $\alpha = w(\alpha_i) \in \Phi$, $r_\alpha = w.r_i.w^{-1}$ and $r_\alpha(v) = v - \alpha(v)\alpha^\vee$, where the coroot $\alpha^\vee = w(\alpha_i^\vee)$ depends only on α.

The set Φ is an (abstract, reduced) real root system in the sense of [MoP89], [MoP95] or [Ba96]. We shall sometimes also use the set $\Delta = \Phi \cup \Delta^+_{im} \cup \Delta^-_{im}$ of all roots (with $-\Delta^-_{im} = \Delta^+_{im} \subset Q^+$, W^v-stable) defined in [Ka90]. It is an (abstract, reduced) root system in the sense of [Ba96].

The *fundamental positive chamber* is $C^v_f = \{v \in V \mid \alpha_i(v) > 0, \forall i \in I\}$. Its closure $\overline{C^v_f}$ is the disjoint union of the vectorial faces $F^v(J) = \{v \in V \mid \alpha_i(v) = 0, \forall i \in J, \alpha_i(v) > 0, \forall i \in I \setminus J\}$ for $J \subset I$. We set $V_0 = F^v(I)$. The positive (resp. negative) vectorial faces are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. The support of such a face is the vector space it generates. The set J or the face $w.F^v(J)$ or an element of this face is called *spherical* if the group $W^v(J)$ generated by $\{r_i \mid i \in J\}$ (which is the fixator or stabilizer in W^v of $F^v(J)$) is finite. An element of a vectorial chamber $\pm w.C^v_f$ is called *regular.*

The *Tits cone* $\mathcal{T}$ (*resp.,* its interior $\mathcal{T}^\circ$) is the (disjoint) union of the positive (*resp.,* and spherical) vectorial faces. It is a W^v-stable convex cone in V. One has $\mathcal{T} = \mathcal{T}^\circ = V$ (*resp.,* $V_0 \subset \mathcal{T} \setminus \mathcal{T}^\circ$) in the classical (*resp.,* non classical) case, *i.e.* when W^v is finite (*resp.,* infinite). By the above characterization of spherical faces, $\mathcal{T}^\circ$ is the set of $x \in \mathcal{T}$ whose fixator in W^v is finite.

We say that $\mathbb{A}^v = (V, W^v)$ is a *vectorial apartment.*

1.2 The model apartment

As in [Ro11, 1.4] the model apartment $\mathbb{A}$ is V considered as an affine space and endowed with a family $\mathcal{M}$ of walls. These walls are affine hyperplanes directed by $\ker(\alpha)$ for $\alpha \in \Phi$. More precisely, they may be written $M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\}$, for $\alpha \in \Phi$ and $k \in \mathbb{R}$.

We ask this apartment to be **semi-discrete** and the origin 0 to be **special**. This means that these walls are the hyperplanes $M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\}$ for $\alpha \in \Phi$ and $k \in \Lambda_\alpha$, with $\Lambda_\alpha = k_\alpha.\mathbb{Z}$ a non trivial discrete subgroup of $\mathbb{R}$. Using [GR14, Lemma 1.3] (*i.e.* replacing Φ by another system Φ_1) we may (and shall) assume that $\Lambda_\alpha = \mathbb{Z}, \forall \alpha \in \Phi$.

For $\alpha = w(\alpha_i) \in \Phi$, $k \in \mathbb{Z}$ and $M = M(\alpha, k)$, the reflection $r_{\alpha,k} = r_M$ with respect to M is the affine involution of $\mathbb{A}$ with fixed points the wall M and associated linear involution r_α. The affine Weyl group W^a is the group generated by the reflections r_M for $M \in \mathcal{M}$; we assume that W^a stabilizes $\mathcal{M}$. We know that $W^a = W^v \ltimes Q^\vee$ and we write $W^a_{\mathbb{R}} = W^v \ltimes V$; here $Q^\vee$ and V have to be understood as groups of translations.

An automorphism of $\mathbb{A}$ is an affine bijection $\varphi : \mathbb{A} \to \mathbb{A}$ stabilizing the set of pairs $(M, \alpha^\vee)$ of a wall M and the coroot associated with $\alpha \in \Phi$ such that $M = M(\alpha, k)$, $k \in \mathbb{Z}$. The group $Aut(\mathbb{A})$ of these automorphisms contains W^a and normalizes it. We consider also the group $Aut^W_{\mathbb{R}}(\mathbb{A}) = \{\varphi \in Aut(\mathbb{A}) \mid \overrightarrow{\varphi} \in W^v\} = Aut(\mathbb{A}) \cap W^a_{\mathbb{R}}$.

For $\alpha \in \Phi$ and $k \in \mathbb{R}$, $D(\alpha, k) = \{v \in V \mid \alpha(v) + k \geq 0\}$ is an half-space, it is called an *half-apartment* if $k \in \mathbb{Z}$. We write $D(\alpha, \infty) = \mathbb{A}$.

The Tits cone $\mathcal{T}$ and its interior $\mathcal{T}^o$ are convex and W^v-stable cones, therefore, we can define three W^v-invariant preorder relations on $\mathbb{A}$:

$$x \leq y \ \Leftrightarrow\ y - x \in \mathcal{T}; \quad x \overset{o}{<} y \ \Leftrightarrow\ y - x \in \mathcal{T}^o; \quad x \overset{o}{\leq} y \ \Leftrightarrow\ y - x \in \mathcal{T}^o \cup V_0.$$

If W^v has no fixed point in $V \setminus \{0\}$ (*i.e.* $V_0 = \{0\}$) and no finite factor, then they are orders; but, in general, they are not.

1.3 Faces, sectors

The faces in $\mathbb{A}$ are associated to the above systems of walls and half-apartments. As in [BrT72], they are no longer subsets of $\mathbb{A}$, but filters of subsets of $\mathbb{A}$. For the definition of that notion and its properties, we refer to [BrT72] or [GR08].

If F is a subset of $\mathbb{A}$ containing an element x in its closure, the germ of F in x is the filter $\mathrm{germ}_x(F)$ consisting of all subsets of $\mathbb{A}$ which contain intersections of F and neighbourhoods of x. In particular, if $x \neq y \in \mathbb{A}$, we denote the germ in x of the segment $[x, y]$ (resp. of the interval $]x, y]$) by $[x, y)$ (resp. $]x, y)$).

For $y \neq x$, the segment germ $[x, y)$ is called of sign $\pm$ if $y - x \in \pm\mathcal{T}$. The segment $[x, y]$ (or the segment germ $[x, y)$ or the ray with origin x containing y) is called *preordered* if $x \leq y$ or $y \leq x$ and *generic* if $x \overset{o}{<} y$ or $y \overset{o}{<} x$.

Given F a filter of subsets of $\mathbb{A}$, its *strict enclosure* $cl_{\mathbb{A}}(F)$ (resp. *closure* $\overline{F}$) is the filter made of the subsets of $\mathbb{A}$ containing an element of F of the shape $\cap_{\alpha\in\Delta} D(\alpha, k_\alpha)$, where $k_\alpha \in \mathbb{Z} \cup \{\infty\}$ (resp. containing the closure $\overline{S}$ of some $S \in F$). One considers also the (larger) *enclosure* $cl^{\#}_{\mathbb{A}}(F)$ of [Ro17, 3.6.1] (introduced in [Cha10], [Cha11] and well studied in [He20], see also [He18]). It is the filter made of the subsets of $\mathbb{A}$ containing an element of F of the shape $\cap_{\alpha\in\Psi} D(\alpha, k_\alpha)$, with $\Psi \subset \Phi$ finite and $k_\alpha \in \mathbb{Z}$ (*i.e.* a finite intersection of half apartments).

A *local face* F in the apartment $\mathbb{A}$ is associated to a point $x \in \mathbb{A}$, its vertex, and a vectorial face F^v in V, its direction. It is defined as $F = germ_x(x + F^v)$ and we denote it by $F = F^\ell(x, F^v)$. Its closure is $\overline{F^\ell}(x, F^v) = germ_x(x + \overline{F^v})$. There is an order on the local faces: the assertions "F is a face of F' ", "F' covers F " and "$F \leq F'$ " are by definition equivalent to $F \subset \overline{F'}$. The dimension of a local face F is the smallest dimension of an affine space generated by some $S \in F$. The (unique) such affine space E of minimal dimension is the support of F; if $F = F^\ell(x, F^v)$, $supp(F) = x + supp(F^v)$. A local face $F = F^\ell(x, F^v)$ is spherical if the direction of its support meets the open Tits cone (*i.e.* when F^v is spherical), then its pointwise stabilizer W_F in W^a or $W^a_{\mathbb{R}}$ is finite and fixes x.

We shall actually here speak only of local faces, and sometimes forget the word local or write $F = F(x, F^v)$.

A *local chamber* is a maximal local face, *i.e.* a local face $F^\ell(x, \pm w.C^v_f)$ for $x \in \mathbb{A}$ and $w \in W^v$. The *fundamental local positive (resp., negative)* chamber is $C_0^+ = germ_0(C^v_f)$ *(resp.,* $C_0^- = germ_0(-C^v_f)$).

A *(local) panel* is a spherical local face maximal among local faces which are not chambers, or, equivalently, a spherical face of dimension $n - 1$. Its support is an hyperplane parallel to a wall.

A *sector* in $\mathbb{A}$ is a $V-$translate $\mathfrak{s} = x + C^v$ of a vectorial chamber $C^v = \pm w.C^v_f$, $w \in W^v$. The point x is its *base point* and C^v its *direction*. Two sectors have the same direction if, and only if, they are conjugate by $V-$translation, and if, and only if, their intersection contains another sector.

The *sector-germ* of a sector $\mathfrak{s} = x + C^v$ in $\mathbb{A}$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ consisting of the sets containing a $V-$translate of $\mathfrak{s}$, it is well determined by the direction C^v. So, the set of translation classes of sectors in $\mathbb{A}$, the set of vectorial chambers in V and the set of sector-germs in $\mathbb{A}$ are in canonical bijection.

A *sector-face* in $\mathbb{A}$ is a $V-$translate $\mathfrak{f} = x + F^v$ of a vectorial face $F^v = \pm w.F^v(J)$. The sector-face-germ of $\mathfrak{f}$ is the filter $\mathfrak{F}$ of subsets containing a translate $\mathfrak{f}'$ of $\mathfrak{f}$ by an element of F^v (*i.e.* $\mathfrak{f}' \subset \mathfrak{f}$). If F^v is spherical, then $\mathfrak{f}$ and $\mathfrak{F}$ are also called spherical. The sign of $\mathfrak{f}$ and $\mathfrak{F}$ is the sign of F^v.

1.4 The Masure

In this section, we recall the definition and some properties of a masure given by Guy Rousseau in [Ro11] and simplified by Auguste Hébert [He20].

1) An apartment of type $\mathbb{A}$ is a set A endowed with a set $Isom^W(\mathbb{A}, A)$ of bijections (called Weyl-isomorphisms) such that, if $f_0 \in Isom^W(\mathbb{A}, A)$, then $f \in Isom^W(\mathbb{A}, A)$ if, and only if, there exists $w \in W^a$ satisfying $f = f_0 \circ w$. An isomorphism (resp. a Weyl-isomorphism, a vectorially-Weyl isomorphism) between two apartments $\varphi : A \to A'$ is a bijection such that, for any $f \in Isom^W(\mathbb{A}, A)$, $f' \in Isom^W(\mathbb{A}, A')$, $f'^{-1} \circ \varphi \circ f \in Aut(\mathbb{A})$ (resp. $\in W^a$, $\in Aut^W_{\mathbb{R}}(\mathbb{A})$); the group of these isomorphisms is written $Isom(A, A')$ (resp. $Isom^W(A, A')$, $Isom^W_{\mathbb{R}}(A, A')$). As the filters in $\mathbb{A}$ defined in 1.3 above (*e.g.* local faces, sectors, walls,..) are permuted by $Aut(\mathbb{A})$, they are well defined in any apartment of type $\mathbb{A}$ and exchanged by any isomorphism.

A *masure* (formerly called an *ordered affine hovel*) of type $\mathbb{A}$ is a set $\mathscr{I}$ endowed with a covering $\mathcal{A}$ of subsets called apartments, each endowed with some structure of an apartment of type $\mathbb{A}$. We recall here the simplification and improvement of the original definition given by Auguste Hébert in [He20]: these data have to satisfy the following two axioms :

(MA ii) If two apartments A, A' are such that $A \cap A'$ contains a generic ray, then $A \cap A'$ is a finite intersection of half-apartments (*i.e.* $A \cap A' = cl^{\#}_A(A \cap A')$) and there exists a Weyl isomorphism $\varphi : A \to A'$ fixing $A \cap A'$.

(MA iii) If $\mathfrak{R}$ is the germ of a splayed chimney and if F is a local face or a germ of a chimney, then there exists an apartment containing $\mathfrak{R}$ and F.

Actually a filter or subset in $\mathscr{I}$ is called a preordered (or generic) segment (or segment germ), a local face, a spherical sector face or a spherical sector face germ if it is included in some apartment A and is called like that in A. We do not recall here what is (a germ of) a (splayed) chimney; it contains (the germ of) a (spherical) sector face. We shall actually use (MA iii) uniquely through its consequence b) below.

In the affine case the hypothesis "$A \cap A'$ contains a generic ray" may be omitted in (MA ii).

We list now some of the properties of masures we shall use.

a) If F is a point, a preordered segment, a local face or a spherical sector face in an apartment A and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $cl^{\#}_A(F)$ of F and there exists a Weyl-isomorphism from A onto A' fixing $cl^{\#}_A(F)$, see [He20, 5.11] or [He18, 4.4.10]. Hence any isomorphism from A onto A' fixing F fixes $\overline{F}$ (and even $cl_A^{\#}(F) \cap supp(F)$).

More generally the intersection of two apartments A, A' is always closed (in A and A'), see [He20, 3.9] or [He18, 4.2.17].

b) If $\mathfrak{F}$ is the germ of a spherical sector face and if F is a local face or a germ of a sector face, then there exists an apartment that contains $\mathfrak{F}$ and F.

c) If two apartments A, A' contain $\mathfrak{F}$ and F as in **b)**, then their intersection contains $cl_A^{\#}(\mathfrak{F} \cup F)$ and there exists a Weyl-isomorphism from A onto A' fixing $cl_A^{\#}(\mathfrak{F} \cup F)$.

d) We consider the relations $\leq$, $\overset{o}{<}$ and $\overset{o}{\leq}$ on $\mathscr{I}$ defined as follows:

$$x \leq y \text{ (\textit{resp.}, } x \overset{o}{<} y, x \overset{o}{\leq} y) \iff \exists A \in \mathcal{A} \text{ such that } x, y \in A \text{ and } x \leq_A y \text{ (resp. } x \overset{o}{<}_A y, x \overset{o}{\leq}_A y)$$

Then $\leq$ (*resp.*, $\overset{o}{<}$, $\overset{o}{\leq}$) is a well defined preorder relation, in particular transitive; it is called the *Tits preorder* (*resp., Tits open preorder, large Tits open preorder*), see [He20].

e) We ask here $\mathscr{I}$ to be thick of **finite thickness**: the number of local chambers covering a given (local) panel in a wall has to be finite ≥ 3. This number is the same for any panel F in a given wall M [Ro11, 2.9]; we denote it by $1 + q_M = 1 + q_F$.

f) An automorphism (resp. a Weyl-automorphism, a vectorially-Weyl automorphism) of $\mathscr{I}$ is a bijection $\varphi : \mathscr{I} \to \mathscr{I}$ such that $A \in \mathcal{A} \iff \varphi(A) \in \mathcal{A}$ and then $\varphi|_A : A \to \varphi(A)$ is an isomorphism (resp. a Weyl-isomorphism, a vectorially-Weyl isomorphism). We write $Aut(\mathscr{I})$ (resp. $Aut^W(\mathscr{I})$, $Aut^W_{\mathbb{R}}(\mathscr{I})$) the group of these automorphisms.

2) For $x \in \mathscr{I}$, the set $\mathcal{T}_x^+\mathscr{I}$ (resp. $\mathcal{T}_x^-\mathscr{I}$) of segment germs $[x, y)$ for $y > x$ (resp. $y < x$) may be considered as a building, the *positive* (resp. *negative*) *tangent building*. The corresponding faces are the local faces of positive (resp. negative) direction and vertex x. For such a local face F, we write sometimes $[x, y) \in F$ if $]x, y) \subset F$. The associated Weyl group is W^v. If the $W-$distance (calculated in $\mathcal{T}_x^{\pm}\mathscr{I}$) of two local chambers is $d^W(C_x, C'_x) = w \in W^v$, to any reduced decomposition $w = r_{i_1} \cdots r_{i_n}$ corresponds a unique minimal gallery from C_x to C'_x of type $(i_1, \cdots, i_n)$.

The buildings $\mathcal{T}_x^+\mathscr{I}$ and $\mathcal{T}_x^-\mathscr{I}$ are actually twinned. The codistance $d^{*W}(C_x, C'_x)$ of two opposite sign chambers C_x and C'_x is the $W-$distance $d^W(C_x, opC'_x)$, where opC'_x denotes the opposite chamber to C'_x in an apartment containing C_x and C'_x. Similarly two segment germs $\eta \in \mathcal{T}_x^+\mathscr{I}$ and $\zeta \in \mathcal{T}_x^-\mathscr{I}$ are said opposite if they are in a same apartment A and opposite in this apartment (*i.e.* in the same line, with opposite directions).

3) Lemma. *[Ro11, 2.9] Let D be an half-apartment in $\mathscr{I}$ and $M = \partial D$ its wall (i.e. its boundary). One considers a panel F in M and a local chamber C in $\mathscr{I}$ covering F. Then there is an apartment containing D and C.*

4) We assume that $\mathscr{I}$ has a strongly transitive group of automorphisms G, *i.e.* 1.a and 1.c above (after replacing $cl_A^{\#}$ by cl_A) are satisfied by isomorphisms induced by elements of G, *cf.* [Ro17, 4.10] and [CiMR20, 4.7].

We choose in $\mathscr{I}$ a fundamental apartment which we identify with $\mathbb{A}$. As G is strongly transitive, the apartments of $\mathscr{I}$ are the sets $g.\mathbb{A}$ for $g \in G$. The stabilizer N of $\mathbb{A}$ in G induces a group $W = \nu(N) \subset Aut(\mathbb{A})$ of affine automorphisms of $\mathbb{A}$ which permutes the walls, local faces, sectors, sector-faces... and contains the affine Weyl group $W^a = W^v \ltimes Q^\vee$ [Ro17, 4.13.1].

We denote the stabilizer of $0 \in \mathbb{A}$ in G by K and the pointwise stabilizer (or fixator) of C_0^+ (*resp.*, C_0^-) by $K_I = K_I^+$ (*resp.*, K_I^-). This group K_I is called the *Iwahori subgroup*.

5) We ask $W = \nu(N)$ to be **vectorially-Weyl** for its action on the vectorial faces. This means that the associated linear map $\overrightarrow{w}$ of any $w \in \nu(N)$ is in W^v. As $\nu(N)$ contains W^a and stabilizes $\mathcal{M}$, we have $W = \nu(N) = W^v \ltimes Y$, where W^v fixes the origin 0 of $\mathbb{A}$ and Y is a group of translations such that: $Q^\vee \subset Y \subset P^\vee = \{v \in V \mid \alpha(v) \in \mathbb{Z}, \forall \alpha \in \Phi\}$. An element $\mathbf{w} \in W$ will often be written $\mathbf{w} = \lambda.w$, with $\lambda \in Y$ and $w \in W^v$.

We ask Y to be **discrete** in V. This is clearly satisfied if Φ generates V^* *i.e.* $(\alpha_i)_{i\in I}$ is a basis of V^*.

6) Note that there is only a finite number of constants q_M as in the definition of thickness. Indeed, we must have $q_{wM} = q_M$, $\forall w \in \nu(N)$ and $w.M(\alpha,k) = M(w(\alpha),k), \forall w \in W^v$. So now, fix $i \in I$, as $\alpha_i(\alpha_i^\vee) = 2$ the translation by $\alpha_i^\vee$ permutes the walls $M = M(\alpha_i,k)$ (for $k \in \mathbb{Z}$) with two orbits. So, $Q^\vee \subset W^a$ has at most two orbits in the set of the constants $q_{M(\alpha_i,k)}$: one containing the $q_i = q_{M(\alpha_i,0)}$ and the other containing the $q'_i = q_{M(\alpha_i,\pm 1)}$. Hence, the number of (possibly) different q_M is at most $2.|I|$. We denote this set of parameters by $\mathcal{Q} = \{q_i, q'_i \mid i \in I\}$.

In [BaPGR16, 1.4.5] one proves the following further equalities: $q_i = q'_i$ if $\alpha_i(Y) = \mathbb{Z}$ and $q_i = q'_i = q_j = q'_j$ if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$.

We consider also the polynomial algebra $\mathbb{Z}[\mathscr{Q}]$, where $\mathscr{Q}$ is the set $\mathscr{Q} = \{Q_i, Q'_i \mid i \in I\}$ of indeterminates, satisfying the same equalities: $Q_i = Q'_i$ if $\alpha_i(Y) = \mathbb{Z}$ and $Q_i = Q'_i = Q_j = Q'_j$ if $\alpha_i(\alpha_j^\vee) = \alpha_j(\alpha_i^\vee) = -1$. See [BaPGR16, 6.1] where $Q_i = \sigma_i^2, Q'_i = (\sigma'_i)^2$.

7) Examples. The main examples of all the above situation are provided by the Kac-Moody theory, as already indicated in the introduction. More precisely let G be an almost split Kac-Moody group over a non archimedean complete field $\mathcal{K}$. We suppose moreover the valuation of $\mathcal{K}$ discrete and its residue field κ perfect. Then there is a masure $\mathscr{I}$ on which G acts strongly transitively by vectorially Weyl automorphisms. If $\mathcal{K}$ is a local field (*i.e.* κ is finite), then we are in the situation described above. This is the main result of [Cha10], [Cha11] and [Ro17].

When G is actually split, this result was known previously by [GR14] and [Ro16]. And in this case all the constants q_M, q_i, q'_i are equal to the cardinality q of the residue field κ.

We gave in [BaPGR16, 6.7] a proof of conjecture 1 for this split case; see also [Mu18]. Actually these proofs are proofs of conjecture 2, as the polynomials $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ are Laurent polynomials inherited from the description of ${}^I\mathscr{H}$ as a specialization of the associative Bernstein-Lusztig algebra over $\mathbb{Z}[\mathscr{Q}]$: the algebra ${}^I\mathscr{H}_{\mathbb{Z}[\mathscr{Q}]}$ over $\mathbb{Z}[\mathscr{Q}]$ defined by these structure constants on the basis $(T_{\mathbf{w}})_{\mathbf{w}\in W^+}$ is associative.

8) Remark. All isomorphisms in [Ro11] are Weyl-isomorphisms, and, when G is strongly transitive, all isomorphisms constructed in *l.c.* are induced by an element of G.

1.5 Type 0 vertices

The elements of Y, through the identification $Y = N.0 \subset \mathbb{A}$, are called *vertices of type* 0 in $\mathbb{A}$; they are special vertices. We note $Y^+ = Y \cap \mathcal{T}$, $Y^{+g} = Y \cap (\mathcal{T}^\circ \cup V_0)$, $Y^{+0} = Y \cap V_0$ and $Y^{++} = Y \cap \overline{C^v_f}$. The type 0 vertices in $\mathscr{I}$ are the points on the orbit $\mathscr{I}_0$ of 0 by G. This set $\mathscr{I}_0$ is often called the affine Grassmannian as it is equal to G/K, where $K = \mathrm{Stab}_G(\{0\})$. But in general, G is not equal to $KYK = KNK$ [GR08, 6.10] *i.e.* $\mathscr{I}_0 \neq K.Y$.

We know that $\mathscr{I}$ is endowed with a $G-$invariant preorder $\leq$ which induces the known one on $\mathbb{A}$. Moreover, if $x \leq y$, then x and y are in a same apartment.

We set $\mathscr{I}^+ = \{x \in \mathscr{I} \mid 0 \leq x\}$, $\mathscr{I}^+_0 = \mathscr{I}_0 \cap \mathscr{I}^+$, $G^+ = \{g \in G \mid 0 \leq g.0\}$ and $G^{+g} = \{g \in G \mid 0 \overset{o}{\leq} g.0\}$; so $\mathscr{I}^+_0 = G^+.0 = G^+/K$. As $\leq$ (*resp.*, $\overset{o}{\leq}$) is a $G-$invariant preorder, G^+ (*resp.,* G^{+g}) is a semigroup, called the *Kac-Moody-Tits semigroup* (*resp.,* the *generic Kac-Moody-Tits semigroup*).

One has $G^+ = K(N \cap G^+)K$; more precisely the map $Y^{++} \to K\backslash G^+/K$ is a bijection, if we identify $\lambda \in Y^{++} \subset W^v \ltimes Y = W = N/\ker\nu$ with its class in N modulo $\ker\nu \subset K$. Clearly $G^{+g} = K(Y^{++} \cap Y^{+g})K$.

1.6 Vectorial distance

For x in the Tits cone $\mathcal{T}$, we denote by x^{++} the unique element in $\overline{C^v_f}$ conjugated by W^v to x.

Let $\mathscr{I} \times_\leq \mathscr{I} = \{(x,y) \in \mathscr{I} \times \mathscr{I} \mid x \leq y\}$ be the set of increasing pairs in $\mathscr{I}$. Such a pair (x,y) is always in a same apartment $g.\mathbb{A}$; so $(g^{-1}).y - (g^{-1}).x \in \mathcal{T}$ and we define the *vectorial distance* $d^v(x,y) \in \overline{C^v_f}$ by $d^v(x,y) = ((g^{-1}).y - (g^{-1}).x)^{++}$. It does not depend on the choices we made (by 1.8.1 below).

For $(x,y) \in \mathscr{I}_0 \times_\leq \mathscr{I}_0 = \{(x,y) \in \mathscr{I}_0 \times \mathscr{I}_0 \mid x \leq y\}$, the vectorial distance $d^v(x,y)$ takes values in Y^{++}. Actually, as $\mathscr{I}_0 = G.0$, K is the stabilizer of 0 and $\mathscr{I}_0^+ = K.Y^{++}$ (with uniqueness of the element in Y^{++}), the map d^v induces a bijection between the set $(\mathscr{I}_0 \times_\leq \mathscr{I}_0)/G$ of $G-$orbits in $\mathscr{I}_0 \times_\leq \mathscr{I}_0$ and Y^{++}.

Further, d^v gives the inverse of the map $Y^{++} \to K\backslash G^+/K$, as any $g \in G^+$ is in $K.d^v(0,g.0).K$.

1.7 Paths and retractions

We consider piecewise linear continuous paths $\pi : [0,1] \to \mathbb{A}$ such that each (existing) tangent vector $\pi'(t)$ belongs to an orbit $W^v.\lambda$ for some $\lambda \in \overline{C^v_f}$. Such a path is called a $\lambda-$*path*; it is increasing with respect to the preorder relation $\leq$ on $\mathbb{A}$. If $\lambda \in \overline{C^v_f} \cap (\mathcal{T}^\circ \cup V_0)$, then it is increasing for $\overset{o}{\leq}$.

For any $t \neq 0$ (resp. $t \neq 1$), we let $\pi'_-(t)$ (resp. $\pi'_+(t)$) denote the derivative of π at t from the left (resp. from the right). Further, we define $w_\pm(t) \in W^v$ to be the smallest element in its $(W^v)_\lambda-$class such that $\pi'_\pm(t) = w_\pm(t).\lambda$ (where $(W^v)_\lambda$ is the stabilizer in W^v of λ).

Moreover, we denote by $\pi_-(t) = \pi(t) - [0,1)\pi'_-(t) = [\pi(t), \pi(t-\varepsilon)\,)$ (*resp.,* $\pi_+(t) = \pi(t) + [0,1)\pi'_+(t) = [\pi(t), \pi(t+\varepsilon)\,)$ (for $\varepsilon > 0$ small) the negative (*resp.,* positive) segment-germ of π at t, for $0 < t \leq 1$ (*resp.,* $0 \leq t < 1$).

Let C_z (*resp.,* $\mathfrak{S}$) be a local chamber with vertex z (*resp.,* a sector germ) in an apartment A of $\mathscr{I}$. For all $x \in \mathscr{I}_{\geq z} = \{y \in \mathscr{I} \mid y \geq z\}$ (*resp.,* $x \in \mathscr{I}$) there is an apartment A' containing x and C_z (*resp.,* $\mathfrak{S}$). And this apartment is conjugated to A by an element of G fixing C_z (*resp.,* $\mathfrak{S}$) (*cf.* 1.4.1.a and 1.4.4). So, by the usual arguments we can define the retraction $\rho = \rho_{A,C_z}$ from $\mathscr{I}_{\geq z}$ (*resp.,* $\rho = \rho_{A,\mathfrak{S}}$ from $\mathscr{I}$) onto $A_{\geq z} = A \cap \mathscr{I}_{\geq z}$ (*resp.,* onto the apartment A) with center C_z (*resp.,* $\mathfrak{S}$).

For any such retraction ρ, the image of any segment $[x,y]$ with $(x,y) \in \mathscr{I} \times_\leq \mathscr{I}$ and $d^v(x,y) = \lambda \in \overline{C^v_f}$ (with moreover $x, y \in \mathscr{I}_{\geq z}$ if $\rho = \rho_{A,C_z}$) is a $\lambda-$path [GR08, 4.4]. In particular, $\rho(x) \leq \rho(y)$. By definition, if A' is another apartment containing $\mathfrak{S}$ (*resp.,* C_z), then ρ induces an isomorphism from A' onto A. As we assume the existence of the strongly transitive group G, this isomorphism is the restriction of an automorphism of $\mathscr{I}$.

1.8 Preordered convexity

Let $\mathscr{C}^\pm$ (*resp.,* $\mathscr{C}_0^\pm$) be the set of all local chambers of direction $\pm$ (*resp.,* with moreover vertices of type 0). A positive (resp. negative) local chamber of vertex $x \in \mathscr{I}$ will often be written C_x (*resp.,* C_x^-) and its direction $C_x^v = \overrightarrow{C_x}$ (*resp.,* $C_x^{-\,v} = \overrightarrow{C_x^-}$). We consider the set $\mathscr{C}^+ \times_\leq \mathscr{C}^+ = \{(C_x, C_y) \in \mathscr{C}^+ \times \mathscr{C}^+ \mid x \leq y\}$ (*resp.,* $\mathscr{C}^+ \times_{\overset{\circ}{\leq}} \mathscr{C}^+ = \{(C_x, C_y) \in \mathscr{C}^+ \times \mathscr{C}^+ \mid x \overset{o}{\leq} y\}$). We sometimes write $C_x \leq C_y$ (*resp.,* $C_x \overset{o}{\leq} C_y$) when $x \leq y$ (*resp.,* $x \overset{o}{\leq} y$).

Proposition. *Let $x, y \in \mathscr{I}$ with $x \leq y$. We consider two local faces F_x, F_y with respective vertices x, y. Then*

(a) F_x and F_y are contained in a common apartment.

(b) If A, B are two apartments containing $\{x, y\}$ (resp., $F_x \cup F_y$), then there is a Weyl-isomorphism from A onto B, fixing the enclosure $cl_A^\#(\{x,y\}) = cl_B^\#(\{x,y\}) \supset [x,y]$ (resp., the closed convex hull $\overline{conv}_A(F_x \cup F_y) = \overline{conv}_B(F_x \cup F_y)$).

This improvement of results in [Ro11, 5.4, 5.1] and [BaPGR16, 1.10] is proved by Auguste Hébert: [He20, 5.17, 5.18], see also [He18, 4.4.16, 4.4.17]. In b) the case of $\{x, y\}$ is proved in [Ro11, 5.4] as, by [He20, 5.1] or [He18, 4.4.1], one may replace *cl* by $cl^\#$. This property is called the *preordered convexity* of intersections of apartments.

Consequence. We define $W^+ = W^v \ltimes Y^+$ (*resp.,* $W^{+g} = W^v \ltimes Y^{+g}$) which is a subsemigroup of W, and call it the *Tits-Weyl* (*resp., generic Tits-Weyl*) *semigroup.* An element $\mathbf{w} \in W^{+g}$ is called *generic* (in a large sense) and *spherical* if, moreover, $\lambda \in \mathcal{T}^\circ \cap Y^+$.

Let $\varepsilon, \eta \in \{+, -\}$. If $C_x^\varepsilon \in \mathscr{C}_0^\varepsilon$ and $0 \leq x$, we know by b) above, that there is an apartment A containing C_0^η and C_x^ε. But all apartments containing C_0^η are conjugated to $\mathbb{A}$ by K_I^η (by 1.4.1.a), so there is $k \in K_I^\eta$ with $k^{-1}.C_x^\varepsilon \subset \mathbb{A}$. Now the vertex $k^{-1}.x \in \mathscr{I}_0$ of $k^{-1}.C_x^\varepsilon$ satisfies $k^{-1}.x \geq 0$, so there is $\mathbf{w} \in W^+$ such that $k^{-1}.C_x^\varepsilon = \mathbf{w}.C_0^\varepsilon$.

When $g \in G^+$, $g.C_0^\varepsilon$ is in $\mathscr{C}_0^\varepsilon$ and there are $k \in K_I^\eta$, $\mathbf{w} \in W^+$ with $g.C_0^\varepsilon = k.\mathbf{w}.C_0^\varepsilon$, *i.e.* $g \in K_I^\eta.W^+.K_I^\varepsilon$. We have proved the *Bruhat decompositions* $G^+ = K_I^\pm.W^+.K_I^\pm$ and the *Birkhoff decompositions* $G^+ = K_I^\mp.W^+.K_I^\pm$. For uniqueness, see 1.10 below.

Similarly we have also $G^{+g} = K_I^\pm.W^{+g}.K_I^\pm$ and $G^{+g} = K_I^\mp.W^{+g}.K_I^\pm$.

Remark 1.9. If the generalized Cartan matrix $\mathbb{M}$ is of affine or strictly hyperbolic type (in the sense of [Ka90, 4.3 or Ex. 4.1]), then any non spherical vectorial face is $w.F^v(I) = F^v(I) = V_0 = \{v \in V \mid \alpha_i(v) = 0, \forall i \in I\}$. So the Tits cones satisfy $\mathcal{T} = \mathcal{T}^\circ \sqcup V_0$ and $Y^+ = Y^{+g}$, $W^+ = W^{+g}$.

1.10 $W-$distance

Let $(C_x, C_y) \in \mathscr{C}_0^+ \times_\leq \mathscr{C}_0^+$, there is an apartment A containing C_x and C_y. We identify $(\mathbb{A}, C_0^+)$ with (A, C_x) *i.e.* we consider the unique $f \in Isom_\mathbb{R}^W(\mathbb{A}, A)$ such that $f(C_0^+) = C_x$. Then $f^{-1}(y) \geq 0$ and there is $\mathbf{w} \in W^+$ such that $f^{-1}(C_y) = \mathbf{w}.C_0^+$. By 1.8.b, $\mathbf{w}$ does not depend on the choice of A.

We define the $W-$*distance* between the two local chambers C_x and C_y to be this unique element: $d^W(C_x, C_y) = \mathbf{w} \in W^+ = Y^+ \rtimes W^v$. If $\mathbf{w} = \lambda.w$, with $\lambda \in Y^+$ and $w \in W^v$, we write also $d^W(C_x, y) = \lambda$; it implies $d^v(x, y) = \lambda^{++}$. As $\leq$ is $G-$invariant, the $W-$distance is also $G-$invariant. When $\mathbf{w} = w \in W^v$ and $w = r_{i_1}.\cdots.r_{i_r}$ is a reduced decomposition, we have $d^W(C_x, C_y) = w$ if and only if there is a minimal gallery (of local chambers in $\mathcal{T}_x^+\mathscr{I}$)

from C_x to C_y of type $(i_1,\dots,i_r)$, in particular $x=y$. When $x=y$, this definition coincides with the one in 1.4.2.

Let us consider an apartment A and local chambers $C_x, C_y, C_z \in \mathscr{C}_0^+$ included in A. If $d^W(C_x,C_y)=\mathbf{w}$, we write $C_y = C_x * \mathbf{w}$. Conversely, for any $\mathbf{w}\in W^+$, there is a unique local chamber $C_z = C_x * \mathbf{w}$ in A such that $d^W(C_x,C_z)=\mathbf{w}$; actually $C_x*\mathbf{w}$ depends on A, but not on an identification of A with $\mathbb{A}$. For $x\le y\le z$, we have (in A) the Chasles relation: $d^W(C_x,C_z)=d^W(C_x,C_y).d^W(C_y,C_z)$; *i.e.* $(C_x,\mathbf{w})\mapsto C_x*\mathbf{w}$ is a right action of the semi-group W^+. When (A,C_x) is identified with $(\mathbb{A},C_0^+)$, one has $C_x*\mathbf{w}=\mathbf{w}C_x$.

When $C_x = C_0^+$ and $C_y = g.C_0^+$ (with $g\in G^+$), $d^W(C_x,C_y)$ is the only $\mathbf{w}\in W^+$ such that $g\in K_I.\mathbf{w}.K_I$. This is the uniqueness result in Bruhat decomposition: $G^+ = \coprod_{\mathbf{w}\in W^+} K_I.\mathbf{w}.K_I$. Similarly we have $G^{+g}=\coprod_{\mathbf{w}\in W^{+g}} K_I.\mathbf{w}.K_I$.

The $W-$distance classifies the orbits of K_I on $\{C_y\in\mathscr{C}_0^+ \mid y\ge 0\}$, hence also the orbits of G on $\mathscr{C}_0^+\times_\le \mathscr{C}_0^+$.

1.11 Iwahori-Hecke Algebras

We consider any commutative ring with unity R. The *Iwahori-Hecke algebra* ${}^I\mathcal{H}_R$ associated to $\mathscr{I}$ with coefficients in R introduced in [BaPGR16] is as follows:

To each $\mathbf{w}\in W^+$, we associate a function $T_\mathbf{w}$ from $\mathscr{C}_0^+\times_\le\mathscr{C}_0^+$ to R defined by

$$T_\mathbf{w}(C,C') = \begin{cases} 1 & \text{if } d^W(C,C')=\mathbf{w},\\ 0 & \text{otherwise.}\end{cases}$$

The Iwahori-Hecke algebra ${}^I\mathcal{H}_R$ is the free $R-$module

$$\{\sum_{\mathbf{w}\in W^+} a_\mathbf{w}T_\mathbf{w} \mid a_\mathbf{w}\in R,\ a_\mathbf{w}=0 \text{ except for a finite number of } \mathbf{w}\},$$

endowed with the convolution product:

$$(\varphi*\psi)(C_x,C_y)=\sum_{C_z}\varphi(C_x,C_z)\psi(C_z,C_y).$$

where $C_z\in\mathscr{C}_0^+$ is such that $x\le z\le y$.

Actually, ${}^I\mathcal{H}_R$ can be identified with the natural convolution algebra of the functions $G^+\to R$, bi-invariant under K_I and with finite support (in $K_I\backslash G^+/K_I$); this is the definition given in the introduction.

More precisely ${}^I\mathcal{H}_R$ is the space of functions $\varphi:\mathscr{C}_0^+\times_\le\mathscr{C}_0^+\to R$, that are left $G-$invariant and with support a finite union of orbits (see the last two lines of 1.10). To a $\varphi\in{}^I\mathcal{H}_R$ is associated $\varphi^G: K_I\backslash G^+/K_I\to R$ such that $\varphi^G(g)=\varphi(C_0^+,g.C_0^+)$. So, for $\varphi,\psi\in{}^I\mathcal{H}_R$,

$$\begin{aligned}(\varphi*\psi)^G(g) = (\varphi*\psi)(C_0^+,g.C_0^+) &= \textstyle\sum_{C_z}\varphi(C_0^+,C_z)\psi(C_z,g.C_0^+)\\ &= \textstyle\sum_{h\in G^+/K_I}\varphi(C_0^+,h.C_0^+)\psi(h.C_0^+,g.C_0^+)\\ &= \textstyle\sum_{h\in G^+/K_I}\varphi(C_0^+,h.C_0^+)\psi(C_0^+,h^{-1}g.C_0^+) = \sum_{h\in G^+/K_I}\varphi^G(h)\psi^G(h^{-1}g):\end{aligned}$$

we get the convolution product (in the classical case, we take a Haar measure on G with K_I of measure 1).

One considers also the subspace ${}^I\mathcal{H}_R^g=\sum_{\mathbf{w}\in W^{+g}}R.T_\mathbf{w}$. From 4.3 and Remark 3.5.2 one sees that it is a subalgebra of ${}^I\mathcal{H}_R$. We call it the *generic Iwahori-Hecke algebra* associated to $\mathscr{I}$ with coefficients in R. From 1.9 one has ${}^I\mathcal{H}_R = {}^I\mathcal{H}_R^g$ in the affine or strictly hyperbolic cases.

We recall now some useful results of [BaPGR16] in order to introduce the structure constants and a way to compute them.

Proposition 1.12. *[BaPGR16, 2.3]*

Let us fix two local chambers C_x and C_y in $\mathscr{C}_0^+$ with $x \leq y$ and $d^W(C_x, C_y) = \mathbf{u} \in W^+$. We consider $\mathbf{w}$ and $\mathbf{v}$ in W^+. Then the number $a_{\mathbf{w},\mathbf{v}}^{\mathbf{u}}$ of $C_z \in \mathscr{C}_0^+$ with $x \leq z \leq y$, $d^W(C_x, C_z) = \mathbf{w}$ and $d^W(C_z, C_y) = \mathbf{v}$ is finite (i.e. in $\mathbb{N}$).

Theorem 1.13. *[BaPGR16, 2.4]*

For any ring R, ${}^I\mathcal{H}_R$ is an algebra with identity element $Id = T_1$ such that

$$T_{\mathbf{w}} * T_{\mathbf{v}} = \sum_{\mathbf{u} \in P_{\mathbf{w},\mathbf{v}}} a_{\mathbf{w},\mathbf{v}}^{\mathbf{u}} T_{\mathbf{u}}$$

where $P_{\mathbf{w},\mathbf{v}}$ is a finite subset of W^+, such that $a_{\mathbf{w},\mathbf{v}}^{\mathbf{u}} = 0$ for $\mathbf{u} \notin P_{\mathbf{w},\mathbf{v}}$.

2 Projections and retractions

In this section we introduce the new tools that we shall use in the next section to compute the structure constants of the Iwahori-Hecke algebra.

2.1 Projections of chambers

1) Projection of a chamber C_y on a point x.

Let $x \in \mathscr{I}$, $C_y \in \mathscr{C}^+$ with $x \leq y$, $x \neq y$. We consider an apartment A containing x and C_y (by 1.8 (a) above) and write $C_y = F(y, C_y^v)$ in A. For $y' \in y + C_y^v$ sufficiently near to y, $\alpha(y' - x) \neq 0$ for any root α and $y' - x \in \mathcal{T}^\circ$. So $]x, y')$ is in a unique positive local chamber $pr_x(C_y)$ of vertex x; this chamber satisfies $[x, y) \subset \overline{pr_x(C_y)} \subset cl_A(\{x, y'\})$ and does not depend on the choice of y'. Moreover, if A' is another apartment containing x and C_y, we may suppose $y' \in A \cap A'$ and $]x, y')$, $cl_A(\{x, y'\})$, $pr_x(C_y)$ are the same in A'. The local chamber $pr_x(C_y)$ is well determined by x and C_y, it is the *projection* of C_y in $\mathcal{T}_x^+\mathscr{I}$.

The same things may be done changing $+$ to $-$ or $\leq$ to $\geq$. But, in the above situation, if $C_y \in \mathscr{C}^-$, we have to assume $x \overset{o}{<} y$ to define $pr_x(C_y) \in \mathscr{C}^+$: otherwise $]x, y')$ might be outside $x + \mathcal{T}$.

When $x = y$, we write $pr_x(C_y) = C_y$.

2) Projection of a chamber C_y on a generic segment germ

Let $x \in \mathscr{I}$, $\delta = [x, x')$ a generic segment-germ and $C_y \in \mathscr{C}$ with $x \leq y$. By 1) we can consider $pr_x(C_y) \in \mathscr{C}^+$ (with the hypothesis $x \overset{o}{<} y$ if $C_y \in \mathscr{C}^-$). We consider now an apartment A containing $[x, x')$ and $pr_x(C_y)$ (by 1.8 a) above).

We consider inside A the prism denoted by $prism_\delta(C_y)$ obtained as the intersection of all half-spaces $D(\alpha, k)$ (for $\alpha \in \Phi$ and $k \in \mathbb{R}$) that contain $pr_x(C_y)$ and such that $\delta \subset M(\alpha, k)$. We can see that if $\delta = [x, x')$ is regular, $prism_\delta(C_y) = A$. If the apartment A contains δ and C_y (hence also $pr_x(C_y)$) we may replace $pr_x(C_y)$ by C_y in the above definition of $prism_\delta(C_y)$.

Lemma 2.2. *In $prism_\delta(C_y)$, there is a unique local chamber of vertex x that contains δ in its closure. This chamber is independent of the choice of A.*

N.B. This local chamber is, by definition, the *projection* $pr_\delta(C_y)$ of the chamber C_y on the segment-germ δ. It is the local chamber containing δ in its closure which is the nearest from $pr_x(C_y)$: either $d^W(pr_x(C_y), pr_\delta(C_y))$ is minimum or $d^{*W}(pr_x(C_y), pr_\delta(C_y))$ is maximum.

The same things may be done when one supposes $y \leq x$ and $C_y \in \mathscr{C}^-$ or $y \overset{o}{<} x$ and $C_y \in \mathscr{C}^+$.

Proof. In the apartment A, we consider δ_+ the segment-germ δ if δ is in $\mathcal{T}_x^+\mathscr{I}$ and $op_A(\delta)$ if $\delta \in \mathcal{T}_x^-\mathscr{I}$ (where $op_A(\delta)$ denotes the opposite segment-germ in A). By 1.4.2, we can consider in the building $\mathcal{T}_x^+\mathscr{I}$ the minimal galleries from $pr_x(C_y)$ to δ_+ (more exactly to a chamber C such that $\delta_+ \in \bar{C}$). The last chamber of each of these galleries is the same (as it has to be on the same side as $pr_x(C_y)$ of any hyperplane of A, containing δ_+ and parallel to a wall); we denote it C_x^{++}. This chamber is associated to a positive system of roots Φ^+ and a root basis $(\alpha_1, \ldots, \alpha_\ell)$, satisfying $\alpha_i(\delta) = 0 \iff i \leq r$, where $0 \leq r < \ell$ (we identify x and 0). Then, we have the characterization of the prism : $p \in prism_\delta(C_y) \iff (\alpha_i(p) \geq 0$ for $1 \leq i \leq r)$. We consider w_r the element of highest length in the finite Weyl group $\langle (r_{\alpha_i})_{i \leq r} \rangle$.

The local chamber C_x^{++} if $\delta \in \mathcal{T}_x^+\mathscr{I}$ (*resp.*, $op_A(w_r(C_x^{++})$ if not) is the unique chamber with vertex x of $prism_\delta(C_y)$ that contains δ in its closure. Indeed, if C is such a chamber, then if $]x, p) \subset C$, we have $\alpha_i(p) > 0$ for all $i \leq r$ (because $C \subset prism_\delta(C_y)$) and $\alpha_i(p)$ of the same sign as $\alpha_i(\delta)$ if $i > r$ (because $\delta \subset \bar{C}$) . So $C = C_x^{++}$ if $\delta \in \mathcal{T}_x^+\mathscr{I}$ (*resp.*, $C = op_A(w_r(C_x^{++}))$ if $\delta \in \mathcal{T}_x^-\mathscr{I}$).

In the case $\delta \in \mathcal{T}_x^+\mathscr{I}$, the characterization of C_x^{++} in the building $\mathcal{T}_x^+\mathscr{I}$ proves that it does not depend on the choice of A.

The chamber $op_A(w_r(C_x^{++}))$ also only depends on δ and C_y if $\delta \in \mathcal{T}_x^-\mathscr{I}$. It is sufficient to prove that it intersects $conv_A(\delta \cup pr_x(C_y))$. Indeed, let us choose ξ and y such that $[x, \xi) = \delta$ and $]x, y) \subset pr_x(C_y)$. We have $\alpha_i(\xi) = 0$ for $i \leq r$, $\alpha_i(\xi) < 0$ for $i > r$ and $\alpha_i(y) > 0$ for $i \leq r$. So for t near 1 enough, $\alpha_i(t\xi + (1-t)y) > 0$ for $i \leq r$ and < 0 for $i > r$, so $]x, t\xi + (1-t)y) \subset op_A(w_r(C_x^{++})$. By Proposition 1.8, the local chamber $op_A(w_r(C_x^{++}))$ is included in all apartments containing δ and $pr_x(C_y)$, so is independent of the choice of A. □

2.3 Centrifugally folded galleries of chambers

Let z be a point in the standard apartment $\mathbb{A}$. We have twinned buildings $\mathcal{T}_z^+\mathscr{I}$ (resp. $\mathcal{T}_z^-\mathscr{I}$). As in 1.4.2, we consider their unrestricted structure, so the associated Weyl group is W^v and the chambers (resp. closed chambers) are the local chambers $C = germ_z(z + C^v)$ (resp. local closed chambers $\overline{C} = germ_z(z + \overline{C^v})$), where C^v is a vectorial chamber, *cf.* [GR08, 4.5] or [Ro11, § 5]. The distances (resp. codistances) between these chambers are written d^W (resp. d^{*W}). To $\mathbb{A}$ is associated a twin system of apartments $\mathbb{A}_z = (\mathbb{A}_z^-, \mathbb{A}_z^+)$.

Let $\mathbf{i} = (i_1, ..., i_r)$ be the type of a minimal gallery. We choose in $\mathbb{A}_z^-$ a negative (local) chamber C_z^- and denote by C_z^+ its opposite in $\mathbb{A}_z^+$. We consider now galleries of (local) chambers $\mathbf{c} = (C_z^-, C_1, ..., C_r)$ in the apartment $\mathbb{A}_z^-$ starting at C_z^- and of type $\mathbf{i}$. Their set is written $\Gamma(C_z^-, \mathbf{i})$. We consider the root β_j corresponding to the common limit hyperplane $M_j = M(\beta_j, -\beta_j(z))$ of type i_j of C_{j-1} and C_j satisfying moreover $\beta_j(C_j) \geq \beta_j(z)$.

We consider the system of positive roots Φ^+ associated to C_z^+. Actually, $\Phi^+ = w.\Phi_f^+$, if Φ_f^+ is the system Φ^+ defined in 1.1 and $C_z^+ = germ_z(z + w.C_f^v)$. We denote by $(\alpha_i)_{i \in I}$ the corresponding basis of Φ and by $(r_i)_{i \in I}$ the corresponding generators of W^v. Note that this change of notation for Φ^+ and r_i is limited to subsection 2.3.

The set $\Gamma(C_z^-,\mathbf{i})$ of galleries is in bijection with the set $\Gamma(\mathbf{i})=\{1,r_{i_1}\}\times\cdots\times\{1,r_{i_r}\}$ via the map $(c_1,...,c_r)\mapsto(C_z^-,c_1C_z^-,...,c_1\cdots c_rC_z^-)$. Moreover $\beta_j=-c_1\cdots c_j(\alpha_{i_j})$.

Definition. Let $\mathfrak{Q}$ be a chamber in $\mathbb{A}_z$. A gallery $\mathbf{c}=(C_z^-,C_1,...,C_r)\in\Gamma(C_z^-,\mathbf{i})$ is said to be *centrifugally folded* with respect to $\mathfrak{Q}$ if $C_j=C_{j-1}$ implies that M_j is a wall and separates $\mathfrak{Q}$ from $C_j=C_{j-1}$. We denote this set of centrifugally folded galleries by $\Gamma^+_{\mathfrak{Q}}(C_z^-,\mathbf{i})$. We write $\Gamma^+_{\mathfrak{Q}}(C_z^-,\mathbf{i},C)$ the subset of galleries in $\Gamma_{\mathfrak{Q}}(C_z^-,\mathbf{i})$ such that C_r is a given chamber C.

2.4 Liftings of galleries

Next, let $\rho_{\mathfrak{Q}}:\mathcal{T}_z\mathscr{I}\to\mathbb{A}_z$ be the retraction centered at $\mathfrak{Q}$. To a gallery of chambers $\mathbf{c}=(C_z^-,C_1,...,C_r)$ in $\Gamma(C_z^-,\mathbf{i})$, one can associate the set of all galleries of type $\mathbf{i}$ starting at C_z^- in $\mathcal{T}_z^-\mathscr{I}$ that retract onto $\mathbf{c}$, we denote this set by $\mathcal{C}_{\mathfrak{Q}}(C_z^-,\mathbf{c})$. We denote the set of galleries $\mathbf{c}'=(C_z^-,C'_1,...,C'_r)$ in $\mathcal{C}_{\mathfrak{Q}}(C_z^-,\mathbf{c})$ that are minimal (i.e. satisfy $C'_{j-1}\neq C'_j$ for any j) by $\mathcal{C}^m_{\mathfrak{Q}}(C_z^-,\mathbf{c})$. Recall from [GR14, Proposition 4.4], that the set $\mathcal{C}^m_{\mathfrak{Q}}(C_z^-,\mathbf{c})$ is nonempty if, and only if, the gallery $\mathbf{c}$ is centrifugally folded with respect to $\mathfrak{Q}$. Recall also from loc. cit., Corollary 4.5, that if $\mathbf{c}\in\Gamma^+_{\mathfrak{Q}}(C_z^-,\mathbf{i})$, then the number of elements in $\mathcal{C}^m_{\mathfrak{Q}}(C_z^-,\mathbf{c})$ is:

$$\sharp\mathcal{C}^m_{\mathfrak{Q}}(C_z^-,\mathbf{c})=\prod_{j\in J_1}(q_j-1)\times\prod_{j\in J_2}q_j$$

where $q_j=q_{M_j}\in\mathcal{Q}$,

$$J_1=\{j\in\{1,\cdots,r\}\mid c_j=1\}=\{j\in\{1,\cdots,r\}\mid C_{j-1}=C_j\}$$

and

$$J_2=\{j\in\{1,\cdots,r\}\mid C_{j-1}\neq C_j \text{ and } M_j \text{ is a wall separating } \mathfrak{Q} \text{ (and } C_{j-1}) \text{ from } C_j\}.$$

One may remark that $\{1,\cdots,r\}$ contains the disjoint union $J_1\sqcup J_2$, but may be different from it. The missing j are precisely those j such that M_j is not a wall (hence q_{M_j} is not defined) or that $\mathfrak{Q}$ (and C_j) are separated from C_{j-1} by M_j.

More generally let $\mathbf{c}^m=(C_z^-,C_1^m,...,C_r^m)$ be the minimal gallery in $\mathbb{A}_z^-$ of type $\mathbf{i}$. We write $\mathcal{C}^m(C_z^-,\mathbf{i})$ the set of all minimal galleries in $\mathscr{I}$ of type $\mathbf{i}$ starting from C_z^-. Its cardinality is $\prod_{j\in J_2}q_j$, where J_2 is the set of $1\leq j\leq r$ such that the hyperplane M_j separating C^m_{j-1} from C^m_j is a wall.

N.B. The $q_j=q_{M_j}$ in the above formulas are in the set $\mathcal{Q}$ of parameters. More precisely, by 1.4.6, if $M_j=M(\beta_j,k_j)$ with $\beta_j=w.\alpha_i$ (for some $w\in W^v$, $i\in I$ and $k_j\in\mathbb{Z}$), then one has $q_j=q_i$ if k_j is even and $q_j=q'_i$ if k_j is odd.

2.5 Hecke paths

The Hecke paths we consider here are slight modifications of those used in [GR14]. They were defined in [BaPGR16], or in [BCGR13] (for the classical case).

Let us fix a local chamber $C_x\in\mathscr{C}_0\cap\mathbb{A}$.

Definition. A Hecke path of shape $\lambda\in Y^{++}$ with respect to C_x in $\mathbb{A}$ is a $\lambda-$path in $\mathbb{A}$ that satisfies the following assumptions. For all $p=\pi(t)$, we ask $x\overset{o}{<}p$, so we can consider the local negative chamber $C_p^- = pr_p(C_x)$ by 2.1.1. Then we assume moreover that for all $t \in [0,1] \setminus \{0,1\}$, there exist finite sequences $(\xi_0 = \pi'_-(t), \xi_1, \dots, \xi_s = \pi'_+(t))$ of vectors in V and $(\beta_1, \dots, \beta_s)$ of real roots such that, for all $j = 1, \dots, s$:

(i) $r_{\beta_j}(\xi_{j-1}) = \xi_j$,

(ii) $\beta_j(\xi_{j-1}) < 0$,

(iii) $\beta_j(\pi(t)) \in \mathbb{Z}$, *i.e.* $\pi(t)$ is in a wall of direction $\ker \beta_j$,

(iv) $\beta_j(C^-_{\pi(t)}) < \beta_j(\pi(t))$.

One says then that these two sequences are a $(W^v_{\pi(t)}, C^-_{\pi(t)})-$chain from $\pi'_-(t)$ to $\pi'_+(t)$. Actually $W^v_{\pi(t)}$ is the subgroup of W^v generated by the r_β such that $M(\beta, -\beta(\pi(t)))$ is a wall.

When $t \in]0,1[$ is such that $s \neq 0$, one has $\pi'_-(t) \neq \pi'_+(t)$, the path is centrifugally folded with respect to C_x at $\pi(t)$.

Lemma 2.6. *Let $\pi \subset \mathbb{A}$ be a Hecke path with respect to C_x as above. Then,*

(a) For t varying in $[0,1]$ and $p = \pi(t)$, the set of vectorial rays $\mathbb{R}_+(x - \pi(t))$ is contained in a finite set of closures of (negative) vectorial chambers.

(b) There is only a finite number of pairs (M, t) with a wall M containing a point $p = \pi(t)$ for $t > 0$, such that $\pi_-(t)$ is not in M and x is not in the same side of M as $\pi_-(t)$ (but may be $x \in M$).

(c) One writes $p_0 = \pi(t_0), p_1 = \pi(t_1), \dots, p_{\ell_\pi} = \pi(t_{\ell_\pi})$ with $0 = t_0 < t_1 < \cdots < t_{\ell_\pi - 1} < 1 = t_{\ell_\pi}$ the points $p = \pi(t)$ satisfying to (b) above (or $t = 0, t = 1$). Then any point t where the path is (centrifugally) folded with respect to C_x at $\pi(t)$ appears in the set $\{t_k \mid 1 \leq k \leq \ell_\pi - 1\}$.

Proof. a) The $\lambda-$path π is a union of line segments $[p'_0, p'_1] \cup [p'_1, p'_2] \cup \cdots \cup [p'_{n-1}, p'_n]$. By hypothesis on Hecke paths, for each point $p = \pi(t)$, $x - p$ is in the open negative Tits cone $-\mathcal{T}^\circ$ (in particular only in a finite number of closures of negative vectorial chambers). Let $p \in [p'_i, p'_{i+1}]$, then $x - p = x - p'_i - (p - p'_i)$ and $\mathbb{R}_+(x-p) \subset conv(\mathbb{R}_+(x - p'_i), -\mathbb{R}_+(p - p'_i))$ and this convex hull is independent of p and only in a finite number of closures of (negative) vectorial chambers (as $(x - p'_i) \in -\mathcal{T}^\circ$ and $(p - p'_i) \in \mathbb{R}_+(p'_{i+1} - p'_i) \subset \mathcal{T}$). So (a) is proved.

b) There is only a finite number of vectorial walls separating (strictly) a chamber in the set of (a) above and a vector $p'_i - p'_{i+1}$. And, for each such vectorial wall, there is only a finite number of walls with this direction meeting the compact set $\pi([0,1])$. Moreover such a wall meets a segment $]p'_i, p'_{i+1}]$ at most once or contains $[p'_i, p'_{i+1}]$ (hence $\pi_-(t) \subset M$ for $\pi(t) \in]p'_i, p'_{i+1}]$).

c) The folding points are among $\{p_1, \dots, p_{\ell_\pi - 1}\}$ by (iv) and (ii) above for $j = 1$. □

2.7 Retractions and liftings of line segments

1) Local study.

In tangent buildings, the centrifugally folded galleries are related with retractions of opposite segment germs, by the following lemma proved in [GR14, Lemma 4.6].

We consider a point $z \in \mathbb{A}$ and a negative local chamber C_z^- in $\mathbb{A}_z^-$. Let ξ and η be two segment germs in $\mathbb{A}_z^+ = \mathbb{A} \cap \mathcal{T}_z^+ \mathscr{I}$. Let $-\eta$ and $-\xi$ opposite respectively η and ξ in $\mathbb{A}_z^-$. Let $\mathbf{i}$ be the type of a minimal gallery between C_z^- and $C_{-\xi}$, where $C_{-\xi}$ is the negative (local) chamber containing $-\xi$ such that $d^W(C_z^-, C_{-\xi})$ is of minimal length. Let $\mathfrak{Q}$ be a chamber of $\mathbb{A}_z^+$ containing η. We suppose ξ and η conjugated by W_z^v.

Lemma. *The following conditions are equivalent:*

(i) There exists an opposite ζ to η in $\mathcal{T}_z^-\mathscr{I}$ such that $\rho_{\mathbb{A}_z,C_z^-}(\zeta) = -\xi$.

(ii) There exists a gallery $\mathbf{c} \in \Gamma^+_{\mathfrak{Q}}(C_z^-, \mathbf{i})$ ending in $-\eta$.

(iii) There exists a $(W_z^v, C_z^-)-$chain from ξ to η.

Moreover the possible ζ are in one-to-one correspondence with the disjoint union of the sets $\mathcal{C}^m_{\mathfrak{Q}}(C_z^-, \mathbf{c})$ for $\mathbf{c}$ in the set $\Gamma^+_{\mathfrak{Q}}(C_z^-, \mathbf{i}, -\eta)$ of galleries in $\Gamma^+_{\mathfrak{Q}}(C_z^-, \mathbf{i})$ ending in $-\eta$.

2) Consequence. Let C_x be a positive local chamber in $\mathbb{A}$ and $z \in \mathbb{A}$ a point such that $x \overset{o}{<} z$. We consider $C_z^- = pr_z(C_x)$. Then one knows that the restriction of the retraction $\rho = \rho_{\mathbb{A},C_x}$ to the tangent twin building $\mathcal{T}_z\mathscr{I}$ is the retraction $\rho_{\mathbb{A}_z,C_z^-}$.

We consider two points y, z_0 in $\mathscr{I}$ such that $x \overset{o}{<} z_0 \le y$, with $d^v(z_0, y) = \lambda \in Y^{++}$. By 1.7, the image $\rho([z_0, y])$ is a $\lambda-$path π from $\rho(z_0)$ to $\rho(y)$. For $z \in [z_0, y[$, we consider an apartment A containing $[z, y)$ and C_x, hence also C_z^-. We write $p = \rho(z)$. The restriction $\rho|_A$ is the restriction to A of an automorphism φ of $\mathscr{I}$ fixing C_x (and an isomorphism from A to $\mathbb{A}$); φ induces an isomorphism $\varphi|_{\mathcal{T}_z\mathscr{I}}$ from $\mathcal{T}_z\mathscr{I}$ onto $\mathcal{T}_z\mathscr{I}$. One has $\rho|_{\mathcal{T}_z\mathscr{I}} = \rho_{\mathbb{A}_p,C_p^-} \circ \varphi|_{\mathcal{T}_z\mathscr{I}} = \varphi|_{A_z} \circ \rho_{A_z,C_z^-}$. So one may use the above Lemma, more precisely the implication $(i) \implies (iii)$: we get a $(W_p^v, C_p^-)-$chain from $\pi'_-(t)$ to $\pi'_+(t)$ (if $p = \pi(t)$).

We have proved that $\pi = \rho([z_0, y])$ is a Hecke path of shape λ with respect to C_x in $\mathbb{A}$. This result is a part of [BaPGR16, Theorem 3.4]. It is also a consequence of the proof of [BCGR13, Th. 3.8] which deals with the classical case of buildings.

3) Liftings of Hecke paths.

One considers in $\mathbb{A}$ a positive local chamber C_x, a Hecke path π of shape $\lambda \in Y^{++}$ with respect to C_x and the retraction $\rho = \rho_{\mathbb{A},C_x}$. Given a point $y \in \mathscr{I}$ with $\rho(y) = \pi(1)$, we consider the set $S_{C_x}(\pi, y)$ of all segment germs $[z, y]$ in $\mathscr{I}$ such that $\rho([z, y]) = \pi$. The above Lemma (essentially (ii)) is used in [BaPGR16] to compute the cardinality of $S_{C_x}(\pi, y)$.

We consider the notations of 1.7 and the numbers t_k of Lemma 2.6. Then $p_k = \pi(t_k)$, $\xi_k = -\pi_-(t_k)$, $\eta_k = \pi_+(t_k)$ and $\mathbf{i}_k$ is the type of a minimal gallery between $C_{p_k}^-$ and $C_{-\xi_k}$, where $C_{-\xi_k}$ is the negative (local) chamber such that $-\xi_k \subset \overline{C_{-\xi_k}}$ and $d^W(C_{p_k}^-, C_{-\xi_k})$ is of minimal length. Let $\mathfrak{Q}_k$ be a fixed chamber in $\mathbb{A}^+_{z_k}$ containing η_k in its closure and $\Gamma^+_{\mathfrak{Q}_k}(C_{p_k}^-, \mathbf{i}_k, -\eta_k)$ be the set of all the galleries $(C_{z_k}^-, C_1, ..., C_r)$ of type $\mathbf{i}_k$ in $\mathbb{A}^-_{z_k}$, centrifugally folded with respect to $\mathfrak{Q}_k$ and with $-\eta_k \in \overline{C_r}$.

The following result is Theorem 3.4 in [BaPGR16]. One uses the notations of 2.3 and 2.4. One considers paths π more general than Hecke paths. The idea is to lift the path π step by step starting from its end by using the above Lemma. We shall generalize it in Theorem 3.5 by lifting decorated Hecke paths (see just below).

Theorem 2.8. *The set $S_{C_x}(\pi, y)$ is non empty if, and only if, π is a Hecke path with respect to C_x. Then, we have a bijection*

$$S_{C_x}(\pi, y) \simeq \Big(\prod_{k=1}^{\ell_\pi - 1} \coprod_{\mathbf{c} \in \Gamma^+_{\mathfrak{Q}_k}(C_{p_k}^-, \mathbf{i}_k, -\eta_k)} \mathcal{C}^m_{\mathfrak{Q}_k}(C_{p_k}^-, \mathbf{c}) \Big) . \mathcal{C}^m(C_y^-, \mathbf{i}_{\ell_\pi})$$

In particular, the number of elements in this set is a polynomial in the numbers $q \in \mathcal{Q}$ with coefficients in $\mathbb{Z}$ depending only on $\mathbb{A}$.

2.9 Decorated segments and paths

Let us consider z_0 and y in $\mathscr{I}$ such that $z_0 \overset{o}{<} y$.

1) Definition. A *decorated segment* $\underline{[z_0, y]}$ is the datum of a segment $[z_0, y]$ as above and, for any $z \in [z_0, y[$ (*resp.*, $z \in]z_0, y]$) of a positive (*resp.*, negative) chamber C_z^+ (*resp.*, C_z'') with vertex z and containing the segment germ $[z, y)$ (*resp.*, $[z, z_0)$) in its closure. One asks moreover that $C_z^+ = pr_{[z,y)}(C)$ (*resp.*, $C_z'' = pr_{[z,z_0)}(C)$) for any local chamber $C = C_{z'}^+$ or $C = C_{z'}''$ as above. One may remark that, then, $C_z^+ = pr_z(C)$ (*resp.*, $C_z'' = pr_z(C)$) if $z' \in [z, y]$ (*resp.*, $z' \in [z_0, z]$).

Clearly the decorated segment $\underline{[z_0, y]}$ is entirely determined by the segment $[z_0, y]$ and any of the local chambers $C_{z'}^+$ or $C_{z'}''$. It is entirely contained in any apartment containing $[z_0, y]$ and one local chamber $C_{z'}^+$ or $C_{z'}''$ (by 2.2).

For points $z_0' \neq y'$ in $[z_0, y]$ in the order z_0, z_0', y', y (*i.e.* $z_0' \overset{o}{<} y'$) the datum $\underline{[z_0', y']} = ([z_0', y'], (C_z^+)_{z\in[z_0',y'[}, (C_z'')_{z\in]z_0',y']})$ is a decorated segment.

2) Lemma. *Let $[z_0, y]$ be a segment as above, $z_1 \in [z_0, y]$ and C_{z_1} a local chamber with vertex z_1 contained in a same apartment A as $[z_0, y]$. Let us define $C_z^+ = pr_{[z,y)}(C_{z_1})$ and $C_z'' = pr_{[z,z_0)}(C_{z_1})$. Then $\underline{[z_0, y]} = ([z_0, y], (C_z^+)_{z\in[z_0,y[}, (C_z'')_{z\in]z_0,y]})$ is a decorated segment. Moreover in A all chambers C_z^+ (resp., C_z'') are deduced from each-other by a translation.*

N.B. If z_1 is z_0 or y then any local chamber C_{z_1} with vertex z_1 is contained in a same apartment as $[z_0, y]$.

Proof. We have to prove that $C_z^+ = pr_{[z,y)}(C)$ (*resp.*, $C_z'' = pr_{[z,z_0)}(C)$) for any local chamber $C = C_{z'}^+$ or $C = C_{z'}''$. Let us recall that the chamber C_z^+ (*resp.*, C_z'') is the unique chamber, that contains $\delta = [z, y)$ (*resp.*, $\delta = [z, z_0)$) in its closure, of the prism $prism_\delta(C_{z_1})$ defined in A as the intersection of all half-spaces $D(\alpha, k)$ (for $\alpha \in \Phi$ and $k \in \mathbb{R}$) that contain C_{z_1} and such that $\delta \subset M(\alpha, k)$. In fact each prism considered to define all these chambers in these definitions is the same prism $prism_{[z_0,y]}(C_{z_1})$, as $\delta \subset M(\alpha, k) \iff [z_0, y] \subset M(\alpha, k)$. Moreover, as already partially remarked in 2.1.2, $prism_{[z_0,y]}(C_{z_1}) = prism_{[z_0,y]}(C)$ for $C = C_{z'}^+$ or $C = C_{z'}''$. Indeed, such a C is in $prism_{[z_0,y]}(C_{z_1})$ and any $M(\alpha, k)$ containing $[z_0, y]$ cannot cut C, so $prism_{[z_0,y]}(C_{z_1}) = prism_{[z_0,y]}(C)$.

It is now clear that $C_z^+ = pr_{[z,y)}(C)$ (*resp.*, $C_z'' = pr_{[z,z_0)}(C)$) for any local chamber $C = C_{z'}^+$ or $C = C_{z'}''$. Moreover the translations of vector in the direction of the line of A containing δ stabilize the prism and exchange the segment germs. So the last assertion of the lemma is clear. □

3) Definitions. A *decorated $\lambda-$path* $\underline{\pi}$ is the datum of :

- a $\lambda-$path $\{\pi(t) \mid 0 \leq t \leq 1\}$,
- a positive (*resp.*, a negative) local chamber $C_{\pi(t)}^+$ (*resp.*, $C_{\pi(t)}''$) of vertex $\pi(t)$ for $0 \leq t < 1$ (*resp.*, $0 < t \leq 1$).

such that there are numbers $0 = t_0' < t_1' < \cdots t_r' = 1$ satisfying, for any $1 \leq i \leq r$,

- $\{\pi(t) \mid t_{i-1}' \leq t \leq t_i'\}$ is a segment $[\pi(t_{i-1}'), \pi(t_i')]$,
- $\underline{[\pi(t_{i-1}'), \pi(t_i')]} = ([\pi(t_{i-1}'), \pi(t_i')], (C_{\pi(t)}^+)_{t\in[t_{i-1}',t_i'[}, (C_{\pi(t)}'')_{t\in]t_{i-1}',t_i']})$ is a decorated segment (in particular $\pi(t_{i-1}') \overset{o}{<} \pi(t_i')$), hence λ is spherical).

A *decorated Hecke path* of shape λ with respect to C_x in $\mathbb{A}$ is a decorated $\lambda-$path $\underline{\pi}$ such that the underlying path π is a Hecke path of shape λ with respect to C_x in $\mathbb{A}$. One assumes moreover that the numbers $0 < t'_1 < \cdots < t'_r = 1$ are equal to the numbers $0 < t_1 < t_2 < \cdots < t_{\ell_\pi} = 1$ of Lemma 2.6 above.

4) Proposition. *Let $\underline{[z_0, y]}$ be a decorated segment (with $d^v(z_0, y) = \lambda \in Y^{++}$ spherical), C_x a chamber of vertex x in $\mathbb{A}$ with $x \overset{o}{<} z_0$ (hence $x \overset{o}{<} z$ for any $z \in [z_0, y]$) and $\rho = \rho_{\mathbb{A},C_x}$ the associated retraction. We parametrize $[z_0, y]$ by $z(t) = z_0 + t(y - z_0)$ in any apartment containing $[z_0, y]$. Then $\rho(\underline{[z_0, y]}) = (\pi = \rho \circ z, (C^+_{\rho z(t)} = \rho C^+_{z(t)})_{t\in[0,1[}, (C^*_{\rho z(t)} = \rho C''_{z(t)})_{t\in]0,1]})$ is a decorated Hecke path of shape λ with respect to C_x in $\mathbb{A}$.*

N.B. Conversely a decorated Hecke path is not always the image by ρ of a decorated segment. But the calculations of the number of such liftings (as in Theorem 2.8) is the main ingredient of our main theorem (3.5 below) generalizing the Theorem 3.7 in [BaPGR16].

Proof. For any $z \in [z_0, y[$ (*resp.*, $z \in]z_0, y]$), we consider an apartment A^+_z (*resp.*, A''_z) containing C_x and C^+_z (*resp.*, C''_z). Then $A^+_z \cup A''_z$ (or $A^+_{z_0}$, A''_y) contains a neighbourhood of z (or z_0, y) in the segment $[z_0, y]$. By compactness of this segment we get numbers $0 = t'_0 < t'_1 < \cdots t'_r = 1$ and apartments A_i such that A_i contains C_x, $z([t'_{i-1}, t'_i])$ and either $C^+_{z(t'_{i-1})}$ or $C''_{z(t'_i)}$. By the projection properties of decorated segments, it contains all other $C^+_{z(t)}$ (*resp.*, $C''_{z(t)}$) for $t \in [t'_{i-1}, t'_i[$ (*resp.*, $t \in]t'_{i-1}, t'_i]$). As ρ sends isomorphically A_i onto $\mathbb{A}$, we get that $\rho(\underline{[z_0, y]})$ is a decorated $\lambda-$path, with underlying path a Hecke path of shape λ with respect to C_x in $\mathbb{A}$.

To get that $\rho(\underline{[z_0, y]})$ is a decorated Hecke path, we have now to prove that the t'_i may be replaced by the t_i associated to this Hecke path by Lemma 2.6. We may apply the following Lemma to $[\pi(t_{i-1}), \pi(t_i)]$. Any apartment A containing C_x and $C''_{z(t_i)}$ contains $[z(t_{i-1}), z(t_i)]$, hence also $C''_{z(t)}$ for $t_{i-1} < t \le t_i$ and $C^+_{z(t)}$ for $t_{i-1} \le t < t_i$, by the projection properties of decorated segments. But ρ induces an isomorphism from A onto $\mathbb{A}$. So $([\pi(t_{i-1}), \pi(t_i)], (\rho C^+_{z(t)})_{t_{i-1}\le t<t_i}, (\rho C''_{z(t)})_{t_{i-1}<t\le t_i})$ is a decorated segment, as expected. □

5) Lemma. *In an apartment $\mathbb{A}$ of a masure $\mathscr{I}$, we consider a local chamber C_x and a line segment $[p_0, p_1]$ with $x \overset{o}{<} p_0 \le p_1$. We suppose that, for any $p \in]p_0, p_1[$ and any wall M containing p, then $[p, p_0]$ is in the half-apartment containing C_x delimited by M. We consider the retraction $\rho = \rho_{\mathbb{A},C_x}$. Then,*

for any segment germ $[z_1, z)$ in $\mathscr{I}$ such that $\rho([z_1, z)) = [p_1, p_0)$ (hence $\rho(z_1) = p_1$), there is a unique line segment $[z_1, z_0]$ such that $[z_1, z_0) = [z_1, z)$ and $\rho([z_1, z_0]) = [p_1, p_0]$. More precisely any apartment A containing C_x and $[z_1, z)$ contains $[z_1, z_0]$.

Proof. Let A be an apartment containing C_x and $[z_1, z)$. Up to the isomorphism ρ from A onto $\mathbb{A}$, one may suppose $A = \mathbb{A}$. Then $z_1 = p_1$ and $[p_1, p_0]$ satisfies $[p_1, p_0) = [p_1, z)$, $\rho([p_1, p_0]) = [p_1, p_0]$ as expected for $[p_1, z_0]$. Let us consider another solution $[p_1, z_0]$, so $[p_1, z_0) = [p_1, p_0)$ and $\rho([p_1, z_0]) = [p_1, p_0]$. Let z' be the point satisfying $[p_1, z'] \subset [p_1, p_0] \cap [p_1, z_0]$ that is the nearest from p_0. One has $z' \neq p_1$ and one wants to prove that $z' = p_0$. If $z' \neq p_0$, one may consider a minimal gallery $\mathbf{c}'$ in $\mathcal{T}^-_{z'}\mathscr{I}$ from $C^-_{z'} = pr_{z'}(C_x)$ to the segment germ $[z', z_0)$. Clearly $\mathbf{c} = \rho(\mathbf{c}')$ is a minimal gallery in $\mathbb{A}^-_{z'}$ from $C^-_{z'}$ to the segment germ $[z', p_0)$. If we write $\mathfrak{Q} = C^-_{z'}$, we have $\mathbf{c}' \in \mathcal{C}^m_{\mathfrak{Q}}(C^-_{z'}, \mathbf{c})$, with the notations of 2.4. But by the hypotheses, no wall

M containing z' separates strictly C_x (*i.e.* $C^-_{z'}$) from $[z', p_0)$. Hence the formula in 2.4 tells that $\mathcal{C}^m_{\mathfrak{Q}}(C^-_{z'}, \mathbf{c})$ is reduced to one element : we have $\mathbf{c}' = \mathbf{c}$, $[z', z_0) = [z', p_0)$, contrary to the hypothesis on z'. □

6) Remark. *The definitions and results in 3), 4), 5) above are also true if we replace C_x by a negative sector germ $\mathfrak{S}$ in $\mathbb{A}$ and ρ by $\rho_{\mathbb{A},\mathfrak{S}}$. The corresponding results of the Lemma are more or less implicit in [BaPGR16], see the last paragraph of proof of Lemma 2.1 or of Proposition 2.3 in l.c.*

3 Structure constants in spherical cases

In this section, we compute the structure constants $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ of the Iwahori-Hecke algebra ${}^I\mathcal{H}^{\mathscr{I}}_R$, assuming that $\mathbf{v} = \mu.v$ and $\mathbf{w} = \lambda.w$ are spherical, *i.e.* μ and λ are spherical (see 1.1 for the definitions). As in [BaPGR16], we will adapt some results obtained in the spherical case in [GR14] to our situation.

These structure constants depend on the shape of the standard apartment $\mathbb{A}$ and on the numbers q_M of 1.4.6. Recall that the number of (possibly) different parameters is at most $2.|I|$. We denoted by $\mathcal{Q} = \{q_1, \cdots, q_l, q'_1 = q_{l+1}, \cdots, q'_l = q_{2l}\}$ this set of parameters.

For $\lambda \in Y^+$ spherical, we denote w_λ (*resp.,* w^+_λ) the smallest (*resp.,* longest) element $w \in W^v$ such that $w.\lambda \in \overline{C^v_f}$. We start by several lemmas.

Lemma 3.1. *[BaPGR16, 3.6] Let $C_x, C_z \in \mathscr{C}^+_0$ with $x \leq z$ and $\lambda \in Y^+$ spherical, $w \in W^v$. We write $C^-_z = pr_z(C_x)$. Then*

$$d^W(C_x, C_z) = \lambda.w \iff \begin{cases} d^W(C_x, z) = \lambda \\ d^{*W}(C^-_z, C_z) = w^+_\lambda w. \end{cases}$$

Lemma 3.2. *Let $C_z, C_y \in \mathscr{C}^+_0$ with $z \overset{o}{<} y$ and $\mu \in Y^+$ spherical, $v \in W^v$. We write $C^+_z = pr_z(C_y)$ and $C''_y = pr_{[y,z)}(C^+_z) = pr_y(C^+_z)$. Then*

$$(1) \qquad d^W(C_z, C_y) = \mu v \iff \begin{cases} d^W(C_z, C^+_z) = v(w_{v^{-1}.\mu})^{-1} \\ d^W(C^+_z, C_y) = \mu^{++} w_{v^{-1}.\mu}. \end{cases}$$

$$(2) \quad d^W(C^+_z, C_y) = \mu^{++} w_{v^{-1}.\mu} \iff d^W(C^+_z, y) = \mu^{++} \text{ and } d^{*W}(C''_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}.\mu}$$

Proof. (1) Let us fix an apartment A' containing C_z, C_y and so C^+_z and identify (A', C_z) with $(\mathbb{A}, C^+_0)$.

Let us suppose that $d^W(C_z, C_y) = \mu v$ and denote $C^+_y := C^+_z + \mu$. Clearly $d^W(C_z, C_z + \mu) = \mu$ and, by Chasles in A', $\mu.v = d^W(C_z, C_y) = d^W(C_z, C_z + \mu) d^W(C_z + \mu, C_y)$, hence $d^W(C_z + \mu, C_y) = v$ *i.e.* $C_y = (C_z + \mu) * v$ (*cf.* 1.10). By $G-$invariance of d^W and Chasles, we have $d^W(C_z, C^+_z) = d^W(C_z + \mu, C^+_y) = d^W(C_z + \mu, C_y) d^W(C_y, C^+_y) = v d^W(C_y, C^+_y)$. Among the walls containing $[z, y]$, no one separates C^+_y from C_y, so the local chamber C^+_y is the closest chamber to C_y among those containing the segment-germ $]y, y + \mu)$ in their closure, *i.e.* $C^+_y = pr_{[y,y+\mu)}(C_y)$ and $d^W(C_y, C^+_y) = w'$ where w' is the smallest $w \in W^v \subset W^+$ (for the Bruhat order of W^v) such that $]y, y + \mu) \subset \overline{C_y * w} = \overline{C_{z+\mu} * vw} = \overline{C_z * \mu vw} = \mu vw \overline{C_z}$, as we identified C_z with C^+_0 . As $\mu = y - z$, we can see w' as the smallest $w \in W^v \subset W^+$ (for the Bruhat order of W^v) such that $]z, z+\mu) \subset vw\overline{C_z}$ *i.e.* $v^{-1}\mu \in w\overline{C^v_f}$ (as we identified C_z with C_0^+), so $w' = (w_{v^{-1}.\mu})^{-1}$. Finally, we get $d^W(C_z, C_z^+) = v(w_{v^{-1}.\mu})^{-1}$ and so

$$d^W(C_z^+, C_y) = (d^W(C_z, C_z^+))^{-1} d^W(C_z, C_y) = w_{v^{-1}.\mu} v^{-1} \mu v (w_{v^{-1}.\mu})^{-1} w_{v^{-1}.\mu} = \mu^{++} w_{v^{-1}.\mu}.$$

In the same way, if we suppose that $d^W(C_z, C_z^+) = v(w_{v^{-1}.\mu})^{-1}$ and $d^W(C_z^+, C_y) = \mu^{++} w_{v^{-1}.\mu}$, by Chasles we obtain $d^W(C_z, C_y) = \mu v$.

(2) We consider now the opposite local chamber at y of C_y^+ (*resp.*, C_y) in A' which is denoted by $-C_y^+$ (*resp.*, $-C_y$). If $d^W(C_z^+, C_y) = \mu^{++} w_{v^{-1}.\mu}$, we have $d^W(C_z^+, y) = \mu^{++} = d^W(C_z^+, C_y^+)$ and $d^W(C_y^+, C_y) = w_{v^{-1}.\mu}$, so $d^{*W}(-C_y^+, C_y) = w_{v^{-1}.\mu}$. By the proof of 2.2, we see that C_y'' and $-C_y^+$ are such that $d^W(-C_y^+, C_y'') = d^W(C_y'', -C_y^+) = w^+_{\mu^{++}}$ (the longest element of $W^v_{\mu^{++}}$ the fixator of μ^{++} in W^v). By Chasles in A', we have

$$d^{*W}(C_y'', C_y) = d^W(C_y'', -C_y) = d^W(C_y'', -C_y^+) d^W(-C_y^+, -C_y) = w^+_{\mu^{++}}.w_{v^{-1}.\mu}.$$

The converse result is clear by Chasles. □

3.3 Local study

We shall need a partial generalization of Lemma 2.7.1 dealing with decorations.

We consider a point $z \in \mathbb{A}$, a negative local chamber C_z^- in $\mathbb{A}_z^-$ and the retraction $\rho = \rho_{\mathbb{A}_z, C_z^-}$ in $\mathcal{T}_z\mathscr{I}$. Let C_z^+ (*resp.*, C_z^*) be a positive (*resp.*, negative) local chamber in $\mathbb{A}_z$, we also introduce the retraction $\rho' = \rho_{\mathbb{A}_z, C_z^+}$ in $\mathcal{T}_z\mathscr{I}$. Let ξ and η be two segment germs in $\mathbb{A}_z^+ = \mathbb{A} \cap \mathcal{T}_z^+\mathscr{I}$ of the same "type" (*i.e.* $\eta = [z, z+w.\lambda)$, $\xi = [z, z+w'.\lambda)$ for some $\lambda \in Y^{++}$ and $w, w' \in W^v$). We suppose that $\overline{C_z^+}$ contains η and $\overline{C_z^*}$ contains the opposite $-\xi = [z, z-w'\lambda)$ of ξ in $\mathbb{A}_z$. We denote $-\eta = [z, z-w.\lambda)$ the opposite of η in $\mathbb{A}_z$ and $\widetilde{C}_z = pr_{-\eta}(C_z^+)$. Let $\mathbf{i}$ be the type of a minimal gallery from C_z^- to C_z^*.

Lemma. *The following conditions are equivalent:*

(i) There exists a segment germ ζ opposite η in $\mathcal{T}_z^-\mathscr{I}$ and a negative local chamber C_z'' containing ζ in its closure such that $\rho(\zeta) = -\xi$, $\rho(C_z'') = C_z^*$ and $C_z'' = pr_\zeta(C_z^+)$.

(ii) There exists a gallery $\mathbf{c} \in \Gamma^+_{C_z^+}(C_z^-, \mathbf{i})$ ending in the local chamber $\widetilde{C}_z$.

Moreover the possible (ζ, C_z'') are in one-to-one correspondence with the disjoint union of the sets $\mathcal{C}^m_{C_z^+}(C_z^-, \mathbf{c})$ for $\mathbf{c}$ in the set $\Gamma^+_{C_z^+}(C_z^-, \mathbf{i}, \widetilde{C}_z)$.

Proof. If ζ, a segment germ opposite η in $\mathcal{T}_z^-\mathscr{I}$, and C_z'', a negative local chamber containing ζ in its closure, are such that $\rho(\zeta) = -\xi$, $\rho(C_z'') = C_z^*$ and $C_z'' = pr_\zeta(C_z^+)$, there is a unique minimal gallery $\mathbf{c}'$ from C_z^- to C_z'' of type $\mathbf{i}$ (as ρ induces a bijection between the minimal galleries from C_z^- to C_z'' and the minimal galleries from C_z^- to C_z^*). The gallery $\mathbf{c} = \rho'(\mathbf{c}')$ is in $\Gamma^+_{C_z^+}(C_z^-, \mathbf{i}, \widetilde{C}_z)$. Indeed, ζ is opposite η so $\rho'(\zeta) = -\eta$, hence the image of $C_z'' = pr_\zeta(C_z^+)$ by ρ' is $\widetilde{C}_z = pr_{-\eta}(C_z^+)$.

Reciprocally, let $\mathbf{c} \in \Gamma^+_{C_z^+}(C_z^-, \mathbf{i})$ be a gallery ending in the local chamber $\widetilde{C}_z$. We can lift this gallery with respect to ρ' while preserving the first chamber C_z^- to obtain a minimal gallery $\mathbf{c}'$ of type $\mathbf{i}$. Let us call C_z'' the last chamber of the lifted gallery. The isomorphism associated to ρ' (see 1.7) between an apartment A_z containing C_z^+ and C_z'' and $\mathbb{A}_z$ enables us to say that the lifting of $-\eta$ is a segment germ ζ opposite η in A_z and $C_z'' = pr_\zeta(C_z^+)$. As the gallery $\mathbf{c}$ is of type $\mathbf{i}$, ρ sends C''_z onto the end of the minimal gallery of same type beginning at C_z^-, so $\rho(C''_z) = C_z^*$. Moreover, ζ is of the same type that $-\eta$ (and $-\xi$), so $\rho(\zeta) = -\xi$.

From the first paragraph above, we get an injective map $(\zeta, C''_z) \mapsto \mathbf{c}'$ from the set of pairs (ζ, C''_z) as in (i) and the disjoint union of the sets $\mathcal{C}^m_{C_z^+}(C_z^-, \mathbf{c})$ for $\mathbf{c}$ in the set $\Gamma^+_{C_z^+}(C_z^-, \mathbf{i}, \widetilde{C}_z)$: indeed, ζ is fully determined by C''_z (and λ). The second paragraph proves that this map is surjective. □

3.4 Opposite line segments

The following lemma will be useful in Theorem 3.5.

Lemma. *Let us consider in a masure $\mathscr{I}$ two preordered line segments or rays δ_1, δ_2 in apartments A_1, A_2, sharing the same origin x. One supposes the segments germs $germ_x(\delta_1)$ and $germ_x(\delta_2)$ opposite (in any apartment containing them both). Then there is a line in an apartment A of $\mathscr{I}$ containing δ_1 and δ_2. In particular, if δ_1, δ_2 are line segments (resp., rays), then $\delta_1 \cup \delta_2$ is also a line segment (resp., a line).*

Proof. The case of line segments is Lemma 4.9 in [GR14]. The case of rays may be deduced from the fact stated in the part 2 of the proof of [Ro11, Prop. 5.4]. As we shall not use it, we omit the details. □

3.5 The main formula

Let us fix two local chambers C_x and C_y in $\mathscr{C}_0^+$ with $x \leq y$ and $d^W(C_x, C_y) = \mathbf{u} = \nu.u \in W^+$. We consider $\mathbf{w} = \lambda.w$ and $\mathbf{v} = \mu.v$ in W^+. Then we know that the structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is the number of $C_{z_0} \in \mathscr{C}_0^+$ with $x \leq z_0 \leq y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$; moreover this number is finite, see Proposition 1.12. In Lemmas 3.1 and 3.2 we gave conditions equivalent to these $W-$distance conditions.

We choose the standard apartment $\mathbb{A}$ containing C_x and C_y, and we identify C_x with the fundamental local chamber C_0^+.

The datum of z_0 is equivalent to the datum of the segment $[z_0, y]$ or of the decorated segment $\underline{[z_0, y]}$ associated, as in 2.9.2, to $[z_0, y]$ and C_y. We consider then the decorated Hecke path $\underline{\pi}$ image of $\underline{[z_0, y]}$ by the retraction $\rho_{\mathbb{A}, C_x}$.

To the Hecke path π underlying a decorated Hecke path $\underline{\pi}$ are associated $\ell_\pi \in \mathbb{N}$ and numbers $t_0 = 0 < t_1 < t_2 < \cdots < t_{\ell_\pi} = 1$ as in Lemma 2.6 and Definition 2.9.3. We write $p_k = \pi(t_k)$. We write C_p^+ (*resp.*, C_p^* instead of C''_p) the decorations of $\underline{\pi}$ at a point p of π. We write C_z^+ (*resp.*, C''_z) the decorations of a decorated segment at one of its points z.

We use freely the notations from 2.1, 2.3 and 2.4.

Theorem. *Assume μ and λ spherical. Then the structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is given by:*

$$a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} = \sum_{\underline{\pi}} \prod_{k=0}^{\ell_\pi} a_{\underline{\pi}}(k)$$

where $\underline{\pi}$ runs over the decorated Hecke paths in $\mathbb{A}$ of shape μ^{++} with respect to C_x from $p_0 = x + \lambda = \lambda$ to $y = x + \nu = \nu$, and the integers $a_{\underline{\pi}}(k)$ are given by :

(1) $a_{\underline{\pi}}(\ell_\pi) = \sum_{\mathbf{d}\in\Gamma^+_{C_y}(C_y^-,\mathbf{i}_\ell,\tilde{C}_y)} \sharp\mathcal{C}^m_{C_y}(C_y^-,\mathbf{d})$*, where* $\mathbf{i}_\ell$ *is the type of a fixed minimal gallery from* C_y^- *to* C_y^* *and* $\tilde{C}_y$ *is the unique local chamber at* y *in* $\mathbb{A}$ *such that* $d^{*W}(\tilde{C}_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}.\mu}$ *.*

(2) For $1 \leq k \leq \ell_\pi - 1$*,* $a_{\underline{\pi}}(k) = \sum_{\mathbf{c}\in\Gamma^+_{C^+_{p_k}}(C^-_{p_k},\mathbf{i}_k,\tilde{C}_{p_k})} \sharp\mathcal{C}^m_{C^+_{p_k}}(C^-_{p_k},\mathbf{c})$*, where* $\mathbf{i}_k$ *is the type of a fixed minimal gallery from* $C^-_{p_k}$ *to* $C^*_{p_k}$ *and* $\tilde{C}_{p_k} = pr_{-\eta_k}(C^+_{p_k})$ *with* $-\eta_k$ *the segment germ of origin* p_k *in* $\mathbb{A}$ *opposite* $\eta_k = \pi_+(t_k)$*.*

(3) $a_{\underline{\pi}}(0) = \sum_{\mathbf{e}\in\Gamma^+_{C^-_{p_0}}(C^+_{p_0},\mathbf{i},C'_{p_0})} \sharp\mathcal{C}^m_{C^-_{p_0}}(C^+_{p_0},\mathbf{e})$*, where* $\mathbf{i}$ *is the type of a fixed reduced decomposition of* $w_{v^{-1}.\mu}.v^{-1}$ *and* C'_{p_0} *is the unique local chamber at* $p_0 = \pi(0)$ *in* $\mathbb{A}$ *such that* $d^{*W}(C^-_{p_0}, C'_{p_0}) = w^+_\lambda w$*.*

Remarks. 1) Actually $\prod_{k=1}^{\ell_\pi-1} a_{\underline{\pi}}(k)$ is the number of decorated segments $\underline{[z_0,y]}$ such that $\rho(\underline{[z_0,y]}) = \underline{\pi}$ and $C_y^* = C_y''$. It may be zero.

2) If $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \neq 0$, then necessarily ν is spherical (in particular $\mathbf{u} \in W^{+g}$), as then any Hecke path of shape μ^{++} is increasing for $\overset{o}{<}$ (see 1.7). The arguments of [BaPGR16] are sufficient for this result.

3) From this theorem we deduce that $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} \neq 0$ is equivalent to the following:

- there exists a Hecke path in $\mathbb{A}$ of shape μ^{++} with respect to C_x from $p_0 = x + \lambda = \lambda$ to $y = x + \nu = \nu$,
- there exists a decoration $\underline{\pi}$ of π (always true),
- for this decorated Hecke path each of the sets $\Gamma^+_{C_y}(C_y^-,\mathbf{i}_\ell,\tilde{C}_y)$, $\Gamma^+_{C^+_{p_k}}(C^-_{p_k},\mathbf{i}_k,\tilde{C}_{p_k})$ and $\Gamma^+_{C^-_{p_0}}(C^+_{p_0},\mathbf{i},C'_{p_0})$ is non empty.

4) The number of decorated Hecke paths $\underline{\pi}$ as above is finite: we know that the number of paths π is finite (it is a consequence of Theorem 3.5 in [BaPGR16]) and, as μ is spherical, the number of decorations of π is finite.

Proof. $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is the number of local chambers $C_{z_0} \in \mathscr{C}_0^+$ with $x \leq z_0 \leq y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$ (we chose C_x, C_y in $\mathbb{A}$ such that $d^W(C_x, C_y) = \mathbf{u}$). We know that this number is finite, see Proposition 1.12. The datum of z_0 is equivalent to the datum of the segment $[z_0, y]$ or of the decorated segment $\underline{[z_0,y]}$ associated, as in 2.9.2, to $[z_0, y]$ and C_y. We use now the retraction $\rho = \rho_{\mathbb{A},C_x} : \mathscr{I}_{\geq x} \to \mathbb{A}$. We have $y = \rho(y) = x + \nu$ and the condition $d^W(C_x, z_0) = \lambda$ is equivalent to $\rho(z_0) = x + \lambda = p_0$. So $\rho(\underline{[z_0,y]})$ has to be a decorated Hecke path $\underline{\pi}$ as asked in the theorem. And we get the formula:

$$a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} = \sum_{\underline{\pi}} \Bigg(\text{number of liftings of } \underline{\pi}\Bigg) \times \Bigg(\text{number of } C_{z_0} \text{ for } z_0 \text{ given}\Bigg),$$

It is possible to calculate like that for $\rho(C^+_{z_0}) = C^+_{p_0}$ is well determined by the decorated path $\underline{\pi}$. Hence (as we shall see in 2) or 3) below), the number of C_{z_0} only depends on $\underline{\pi}$ and not on the lifting of $\underline{\pi}$. In [BaPGR16, Theorem 3.7] we argued the same way, but with Hecke paths (without decoration) so we had to suppose μ^{++} regular to get that $\rho(C^+_{z_0})$ was well determined by the path π.

For short, we write $\ell = \ell_\pi$. We compute the number of liftings of $\underline{\pi}$ by looking successively at the number of liftings of $\underline{[p_{\ell-1}, p_\ell]}$, $\underline{[p_{\ell-2}, p_{\ell-1}]}$, ..., $\underline{[p_0, p_1]}$.

1) The number $a_{\underline{\pi}}(\ell)$ of liftings of $\underline{[p_{\ell-1}, p_\ell = y]}$ is the number of liftings $[z_{\ell-1}, z_\ell = y]$ of $[p_{\ell-1}, p_\ell = y]$ and C''_y of C^*_y such that $[y, z_{\ell-1}) \subset \overline{C''_y}$ and $d^{*W}(C''_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}.\mu}$ (by Lemma 3.2.2). But $[y, z_{\ell-1}]$ is determined by $[y, z_{\ell-1})$ (*cf.* Lemma 2.9.5) and $[y, z_{\ell-1})$ is determined by C''_y and μ^{++}. So we just have to count the liftings C''_y of C^*_y. By the same way as in the proof of Lemma 3.3, we are going to prove that the possible C''_y are in one-to-one correspondance with the disjoint union of the sets $\mathcal{C}^m_{C_y}(C^-_y, \mathbf{c})$ for $\mathbf{c}$ in $\Gamma^+_{C_y}(C^-_y, \mathbf{i}_\ell, \tilde{C}_y)$. In this case, the tools are $\rho = \rho_{\mathbb{A}, C_x}$, that on $\mathcal{T}_y\mathscr{I}$, coincides with $\rho = \rho_{\mathbb{A}, C^-_y}$ (2.7.2) and $\rho' = \rho_{\mathbb{A}, C_y}$.

If C''_y is given, there is a unique minimal gallery $\mathbf{c}'$ from C^-_y to C''_y of type $\mathbf{i}_\ell$ (as ρ induces a bijection between the minimal galleries from C^-_y to $C''_y = pr_{[y, z_{\ell-1})}(C_y)$ and those from C^-_y to $C^*_y = pr_{[y, p_{\ell-1})}(C_y)$). By Lemma 3.2(2) we know that $d^{*W}(C''_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}.\mu}$, so $\rho'(C''_y) = \tilde{C}_y$, and the gallery $\mathbf{c} = \rho'(\mathbf{c}')$ is in $\Gamma^+_{C_y}(C^-_y, \mathbf{i}_\ell, \tilde{C}_y)$, while $\mathbf{c}'$ is in $\mathcal{C}^m_{C_y}(C^-_y, \mathbf{c})$.

Reciprocally, if $\mathbf{c}$ is in the set $\Gamma^+_{C_y}(C^-_y, \mathbf{i}_\ell, \tilde{C}_y)$, let us consider C''_y the last chamber of $\mathbf{c}'$ a lifted gallery of $\mathbf{c}$ with respect to ρ'. The condition on $\tilde{C}_y$ enables to say that $d^{*W}(C''_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}.\mu}$ and so, by Lemma 3.2 the decoration C''_y of $\underline{[z_{\ell-1}, y]}$ at y satisfies the expected codistance condition.

2) For $1 \leq k \leq \ell - 1$, we suppose given the lifting $\underline{[z_k, y]}$ of $\underline{\pi}|_{[t_k,1]}$. The number $a_{\underline{\pi}}(k)$ of suitable liftings $\underline{[z_{k-1}, z_k]}$ of $\underline{[p_{k-1}, p_k]}$ is the number of pairs $([z_{k-1}, z_k], C''_{z_k})$ of liftings $[z_{k-1}, z_k]$ of $[p_{k-1}, p_k]$ and C''_{z_k} of $C^*_{p_k}$ such that $[z_k, z_{k-1})$ is opposite to $[z_k, z_{k+1})$ (see Lemma 3.4), $[z_k, z_{k-1}) \in \overline{C''_{z_k}}$ and C''_{z_k} is the decoration of $[z_k, z_{k-1}]$ associated to C_y. Let us consider an apartment A containing C_x and $C''_{z_{k+1}}$ hence also $[z_k, z_{k+1}]$ and $C^+_{z_k}$ (see Lemma 2.9.5). The restriction $\rho|_A$ is the restriction to A of an automorphism φ of $\mathscr{I}$ fixing C_x that induces an isomorphism $\varphi|_{\mathcal{T}_{z_k}\mathscr{I}}$ from $\mathcal{T}_{z_k}\mathscr{I}$ onto $\mathcal{T}_{p_k}\mathscr{I}$ and sends $C^+_{z_k} \subset A$ to $C^+_{p_k} = \rho(C^+_{z_k})$. So the map φ induces a bijection from the set of suitable liftings $([z_{k-1}, z_k], C''_{z_k})$ of $([p_{k-1}, p_k], C^*_{z_k})$ onto the set of pairs $([z'_{k-1}, p_k], C''_{p_k})$ such that $[p_k, z'_{k-1}) \in \overline{C''_{p_k}}$ is opposite to $[p_k, p_{k+1}]$ $(= \rho([z_k, z_{k+1}]) = \varphi([z_k, z_{k+1}]))$, $C''_{p_k} = pr_{[p_k, z'_{k-1})}(C^+_{p_k})$ and $\rho_{\mathbb{A}, C^-_{p_k}}(C''_{p_k}) = C^*_{p_k}$ (as $\rho_{\mathbb{A}, C^-_{p_k}} \circ \varphi|_{\mathcal{T}_{z_k}\mathscr{I}}(C''_{z_k}) = \rho(C''_{z_k})$).

By Lemma 3.3 the possible $([p_k, z'_{k-1}), C''_{p_k})$ (and so the possible $([p_k, z'_{k-1}], C''_{p_k})$ by Lemma 2.9.5) are in one-to-one correspondance with the union of the sets $\mathcal{C}^m_{C^+_{p_k}}(C^-_{p_k}, \mathbf{c})$ for $\mathbf{c}$ in the set $\Gamma^+_{C^+_{p_k}}(C^-_{p_k}, \mathbf{i}_\ell, \tilde{C}_{p_k})$, with $\tilde{C}_{p_k} = pr_{-\eta_k}(C^+_{p_k})$.

3) For the last step of the lifting, by the same way as before, we suppose given the lifting $\underline{[z_0, y]}$ and we suppose $z_0 = p_0$. So we know that $C^+_{p_0} = C^+_{z_0}$. The Lemma 3.1 says that $d^{*W}(C^-_{p_0}, C_{z_0}) = w^+_\lambda w$, and Lemma 3.2 that $d^W((C^+_{p_0}, C_{z_0}) = w_{v^{-1}\mu} v^{-1}$. So, as before, the number of C_{z_0} is the number of elements of the different sets $\mathcal{C}^m_{C^-_{p_0}}(C^+_{p_0}, \mathbf{e})$ where $\mathbf{e}$ is a gallery of $\Gamma^+_{C^-_{p_0}}(C^+_{p_0}, \mathbf{i}, C'_{p_0})$ as $\mathbf{i}$ is the type of a minimal gallery from $C^+_{p_0}$ to C_{z_0} that retracts by $\rho_{\mathbb{A}, C^-_{p_0}}$ to a gallery from $C^+_{p_0}$ to C'_{p_0}. □

3.6 Consequence

The above explicit formula, together with the formula for $\sharp\mathcal{C}^m_{\mathfrak{Q}}(C^-_z, \mathbf{c})$ in 2.4, tell us that the structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is a polynomial in the parameters $q_i - 1, q'_i - 1$ for $q_i, q'_i \in \mathcal{Q}$ with coefficients in $\mathbb{N} = \mathbb{Z}_{\geq 0}$ and that this polynomial depends only on $\mathbb{A}$, W, $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{u}$. So we have proved the conjecture 1 of the introduction in this generic case: when λ and μ are spherical.

Note that we have not got all the structure constants $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ for the generic Iwahori-Hecke algebra ${}^I\mathcal{H}^g_{\mathbb{Z}}$. The cases $\mathbf{w} \in W^v \ltimes V_0$ or $\mathbf{v} \in W^v \ltimes V_0$ (*i.e.* $\lambda \in V_0$ or $\mu \in V_0$ in the above notations) are missing. We deal with them in the following section.

4 Structure constants in remaining generic cases

4.1 The problem

Let us choose $C_x, C_y \in \mathscr{C}_0^+$ with $x \leq y$ and $d^W(C_x, C_y) = \mathbf{u} = \nu.u \in W^+ = W^v \ltimes Y^+$. Then the structure constant $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ (for $\mathbf{w} = \lambda.w$ and $\mathbf{v} = \mu.v$ in W^+) is the number of $C_{z_0} \in \mathscr{C}_0^+$ with $x \leq z_0 \leq y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$, see Proposition 1.12.

In Theorem 3.5, we computed $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ when $\mathbf{w}, \mathbf{v}$ are spherical (*i.e.* $\lambda, \mu \in Y \cap \mathcal{T}^\circ$). We shall compute it below in the remaining cases where $\mathbf{w}, \mathbf{v} \in W^{+g} = W^v \ltimes (Y \cap (\mathcal{T}^\circ \cup V_0))$. So, in the affine or strictly hyperbolic cases, we shall get $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ for any $\mathbf{w}, \mathbf{v} \in W^+$. But we get, in general, these structure constants for $\mathbf{w}, \mathbf{v} \in W^{+g} = W^v \ltimes Y^{+g}$, *i.e.* we get the structure constants of ${}^I\mathcal{H}^g$, see 3.6 and 4.6.

We start with a lemma analogous to lemmas 3.1 and 3.2.

Lemma 4.2. *Let $C_x, C_z \in \mathscr{C}_0^+$ with $x \leq z$ and $\lambda \in Y^{+0}$, $w \in W^v$. We write $C_x^+ = pr_x(C_z)$, then*

$$d^W(C_x, C_z) = \lambda.w \Longleftrightarrow \begin{cases} d^W(C_x, z) = \lambda \\ d^W(C_z^-, C_z) = w. \end{cases} \Longleftrightarrow \begin{cases} d^W(C_x, z) = \lambda \\ d^W(C_x, C_x^+) = w. \end{cases}$$

Actually $d^W(C_x, z) = \lambda \in V_0$ implies $x \leq z$ and $z \leq x$. So $C_z^- := pr_z(C_x)$ is well defined, by 2.1.1, and is a positive local chamber.

Proof. By definition $d^W(C_x, C_z) = \lambda.w$ implies $d^W(C_x, z) = \lambda$ (1.10). Suppose now $d^W(C_x, z) = \lambda$. Then $d^v(x, z) = \lambda \in V_0$, so any apartment A containing x or z contains z or x and, in A, one has $z = x + \lambda \leq x$; this is a consequence of 1.4.1.a, as any enclosure is stable under V_0. Hence $C_z^- = pr_z(C_x) \in A$ is well defined, by 2.1.1, and is a positive local chamber. Actually $C_z^- = C_x + \lambda$ (calculation in A). We have also $C_x^+ = C_z - \lambda$. It is now clear that $d^W(C_x, C_z) = \lambda.w \iff d^W(C_z^-, C_z) = w \iff d^W(C_x, C_x^+) = w$. □

4.3 First reduction

We consider $\mathbf{u}, \mathbf{v}, \mathbf{w} \in W^+$ and write $\mathbf{u} = \nu.u, \mathbf{v} = \mu.v, \mathbf{w} = \lambda.w$ with $\lambda, \mu, \nu \in Y^+$ and $u, v, w \in W^v$. We choose $C_x, C_y \in \mathscr{C}_0^+$ with $x \leq y$ and $d^W(C_x, C_y) = \mathbf{u}$; we may suppose $C_x, C_y \subset \mathbb{A}$. We choose $C_{z_0} \in \mathscr{C}_0^+$ with $x \leq z_0 \leq y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$.

If $\lambda \in Y^{+0} = Y \cap V_0$, one has $d^W(C_x, z_0) = \lambda$ (Lemma 4.2) and $z_0 \in \mathbb{A}$, more precisely $z_0 = x + \lambda$ (as we saw in the proof of Lemma 4.2).

If $\mu \in Y^{+0}$, then we get $z_0 \in \mathbb{A}$, more precisely $z_0 = y - \mu$, by Lemma 4.2 applied to C_{z_0}, C_y instead of C_x, C_z.

In both cases z_0 has to be a well determined point in $\mathbb{A}$ and $\nu = d^v(x, y) \in W^v\lambda + W^v\mu$. In particular, if $\mathbf{w}, \mathbf{v} \in W^{+g}$ *i.e.* $\lambda, \mu \in Y^{+g}$, one has also $\nu \in Y^{+g}$ *i.e.* $\mathbf{u} \in W^{+g}$.

We want now to compute the number $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ of $C_{z_0} \in \mathscr{C}_0^+$ with $x \leq z_0 \leq y$, $d^W(C_x, C_{z_0}) = \mathbf{w}$ and $d^W(C_{z_0}, C_y) = \mathbf{v}$. For this we separate below the cases $\lambda \in Y^{+0}$ and $\mu \in Y^{+0}$.

4.4 The case $\mu \in Y^{+0}$

We suppose $\lambda \in Y \cap \mathcal{T}^\circ$ (*resp.*, $\lambda \in Y^{+0}$). By Lemma 4.2 above and Lemma 3.1, we have to find the number $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ of $C_{z_0} \in \mathscr{C}_0^+$ satisfying (with $C^+_{z_0} = pr_{[z_0,y)}(C_y) = pr_{z_0}(C_y)$):

(a) $d^W(C_x, z_0) = \lambda$, (b) $d^W(C_{z_0}, y) = \mu$, (c) $d^W(C_{z_0}, C^+_{z_0}) = v$

and (d) $d^{*W}(C^-_{z_0}, C_{z_0}) = w^+_\lambda.w$ (*resp.*, and (d) $d^W(C^-_{z_0}, C_{z_0}) = w$).

Actually $\mu \in V_0$ is fixed by W^v and $y, C_{z_0}, C^+_{z_0}$ are in a same apartment (containing C_y and C_{z_0}), so $d^W(C_{z_0}, y) = \mu \iff d^W(C^+_{z_0}, y) = \mu$. Then $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is the number of $C_{z_0} \in \mathscr{C}_0^+$ satisfying (a), (b') $d^W(C^+_{z_0}, y) = \mu$, (c) and (d). The first two conditions involve only $z_0, C_x, C_y \in \mathbb{A}$.

Proposition. *The number* $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ *is either* 0 *(if the conditions (a), (b') above are incompatible) or*

$$\sum_{\mathbf{e}\in\Gamma^+_{C^-_{z_0}}(C^+_{z_0},\mathbf{i},C'_{z_0})} \sharp\mathcal{C}^m_{C^-_{z_0}}(C^+_{z_0}, \mathbf{e})$$

where $\mathbf{i}$ *is the type of a fixed reduced decomposition of* v^{-1} *and* C'_{z_0} *is the unique local chamber at* z_0 *in* $\mathbb{A}$ *such that* $d^{*W}(C^-_{z_0}, C'_{z_0}) = w^+_\lambda.w$ *(resp.,* $d^W(C^-_{z_0}, C'_{z_0}) = w$*).*

Remark. The coefficient $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is zero when (a) and (b') are incompatible, *i.e.* when $\nu \neq \lambda+\mu$: if in $\mathbb{A}$ we identify C_x to the fundamental chamber C^+_0, (a) is equivalent to $z_0 = x + \lambda$, (b') to $y = z_0 + \mu$ and $d^W(C_x, C_y) = \nu.u$ implies $y = x + \nu$.

But the other case where $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}} = 0$ is when $\Gamma^+_{C^-_{z_0}}(C^+_{z_0}, \mathbf{i}, C'_{z_0})$ is empty.

Proof. We have to translate the conditions (c) and (d). We consider the retraction $\rho = \rho_{\mathbb{A},C^-_{z_0}}$. The condition (c) is equivalent to the existence of a minimal gallery $\mathbf{c}$ starting from $C^+_{z_0}$, of type $\mathbf{i}$ (*i.e.* $\mathbf{c} \in \mathcal{C}^m(C^+_{z_0}, \mathbf{i})$) ending in C_{z_0}; and there is a bijection between these $\mathbf{c}$ and the C_{z_0} satisfying (c). Now the condition (d) is equivalent to $\rho(C_{z_0}) = C'_{z_0}$ (as ρ preserves the $W-$distances to $C^-_{z_0}$). Considering $\mathbf{e} = \rho(\mathbf{c})$, the proposition is now clear. □

4.5 The case $\lambda \in Y^{+0}$ (and $\mu \in Y \cap \mathcal{T}^\circ$)

By Lemma 4.2 above and Lemma 3.2, we have to find the number $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ of $C_{z_0} \in \mathscr{C}_0^+$ satisfying:

(a) $d^W(C_x, z_0) = \lambda$, (b) $d^W(C^+_{z_0}, y) = \mu^{++}$, (c) $d^{*W}(C''_y, C_y) = w^+_{\mu^{++}} w_{v^{-1}.\mu}$

(d) $d^W(C^-_{z_0}, C_{z_0}) = w$ and (e) $d^W(C^+_{z_0}, C_{z_0}) = w_{v^{-1}\mu}.v^{-1}$

But $C^+_{z_0} = pr_{z_0}(C_y)$, $C''_y = pr_y(C^+_{z_0})$ and $C_x, C_y, z_0 = x + \lambda$ are in $\mathbb{A}$. So the conditions (a), (b), (c) involve only C_x, C_y and z_0.

Proposition. *The number* $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ *is either* 0 *(if the conditions (a), (b), (c) above are incompatible) or*

$$\sum_{\mathbf{e}\in\Gamma^+_{C^-_{z_0}}(C^+_{z_0},\mathbf{i},C'_{z_0})} \sharp\mathcal{C}^m_{C^-_{z_0}}(C^+_{z_0}, \mathbf{e})$$

where $\mathbf{i}$ *is the type of a fixed reduced decomposition of* $w_{v^{-1}\mu}.v^{-1}$ *and* C'_{z_0} *is the unique local chamber at* z_0 *in* $\mathbb{A}$ *such that* $d^W(C^-_{z_0}, C'_{z_0}) = w$*.*

Remark. The coefficient $a^{\mathbf{u}}_{\mathbf{w},\mathbf{v}}$ is zero when (a), (b) and (c) are incompatible, *i.e.* when z_0, determined by (b) does not satisfy (a) and (c). But it is more difficult than in 4.4 to translate it simply. It is also zero when $\Gamma^+_{C^-_{z_0}}(C^+_{z_0}, \mathbf{i}, C'_{z_0})$ is empty.

Proof. We have to translate conditions (d) and (e). It goes the same way as in 4.4. □

4.6 Conclusion

In all cases where $\lambda, \mu \in Y^{+g} = Y \cap (\mathcal{T}^\circ \cup V_0)$, we may use the formula for $\mathcal{C}_{\mathfrak{Q}}^m(C'_z, \mathbf{c})$ in 2.4, the Theorem 3.5 and/or the Propositions 4.4, 4.5. We get the expected result: the structure constant $a_{\mathbf{w},\mathbf{v}}^{\mathbf{u}}$ is a polynomial in the parameters $q_i - 1, q'_i - 1$ for $q_i, q'_i \in \mathcal{Q}$ with coefficients in $\mathbb{N} = \mathbb{Z}_{\geq 0}$ and this polynomial depends only on $\mathbb{A}$, W, $\mathbf{w}$, $\mathbf{v}$ and $\mathbf{u}$. We have proved Conjecture 1 in these cases, in particular in the affine or strictly hyperbolic cases.

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