KAC-MOODY LIE ALGEBRAS GRADED BY KAC-MOODY ROOT SYSTEMS

HECHMI BEN MESSAOUD AND GUY ROUSSEAU

Abstract. We look to gradations of Kac-Moody Lie algebras by Kac-Moody root systems with finite dimensional weight spaces. We extend, to general Kac-Moody Lie algebras, the notion of $C-$admissible pair as introduced by H. Rubenthaler and J. Nervi for semi-simple and affine Lie algebras. If $\mathfrak{g}$ is a Kac-Moody Lie algebra (with Dynkin diagram indexed by I) and (I, J) is such a $C-$admissible pair, we construct a $C-$admissible subalgebra $\mathfrak{g}^J$, which is a Kac-Moody Lie algebra of the same type as $\mathfrak{g}$, and whose root system Σ grades finitely the Lie algebra $\mathfrak{g}$. For an admissible quotient $\rho : I \to \overline{I}$ we build also a Kac-Moody subalgebra $\mathfrak{g}^\rho$ which grades finitely the Lie algebra $\mathfrak{g}$. If $\mathfrak{g}$ is affine or hyperbolic, we prove that the classification of the gradations of $\mathfrak{g}$ is equivalent to those of the $C-$admissible pairs and of the admissible quotients. For general Kac-Moody Lie algebras of indefinite type, the situation may be more complicated; it is (less precisely) described by the concept of generalized $C-$admissible pairs.

2000 Mathematics Subject Classification. 17B67.

Key words and phrases. *Kac-Moody algebra, $C-$admissible pair, gradation.*

Introduction. The notion of gradation of a Lie algebra $\mathfrak{g}$ by a finite root system Σ was introduced by S. Berman and R. Moody [8] and further studied by G. Benkart and E. Zelmanov [5], E. Neher [15], B. Allison, G. Benkart and Y. Gao [1] and J. Nervi [16]. This notion was extended by J. Nervi [17] to the case where $\mathfrak{g}$ is an affine Kac-Moody algebra and Σ the (infinite) root system of an affine Kac-Moody algebra; in her two articles she uses the notion of $C-$admissible subalgebra associated to a $C-$admissible pair for the Dynkin diagram, as introduced by H. Rubenthaler [21].

We consider here a general Kac-Moody algebra $\mathfrak{g}$ (indecomposable and symmetrizable) and the root system Σ of a Kac-Moody algebra. We say that $\mathfrak{g}$ is finitely $\Sigma-$graded if $\mathfrak{g}$ contains a Kac-Moody subalgebra $\mathfrak{m}$ (the grading subalgebra) whose root system relatively to a Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is Σ and moreover the action of $ad(\mathfrak{a})$ on $\mathfrak{g}$ is diagonalizable with weights in $\Sigma \cup \{0\}$ and finite dimensional weight spaces, see Definition 1.4. The finite dimensionality of weight spaces is a new condition, it was fulfilled by the non-trivial examples of J. Nervi [17] but it excludes the gradings of infinite dimensional Kac-Moody algebras by finite root systems as in [5]. Many examples of these gradations are provided by the almost split real forms of $\mathfrak{g}$, cf. 1.7. We are interested in describing the possible gradations of a given Kac-Moody algebra (as in [16], [17]), not in determining all the Lie algebras graded by a given root system Σ (as e.g. in [1] for Σ finite). We carry out completely this project when $\mathfrak{g}$ is affine or hyperbolic.

Let I be the index set of the Dynkin diagram of $\mathfrak{g}$, we generalize the notion of $C-$admissible pair (I,J) as introduced by H. Rubenthaler [21] and J. Nervi [16], [17], cf. Definition 2.1. For each Dynkin diagram I the classification of the $C-$admissible pairs (I,J) is easy to deduce from the list of irreducible $C-$admissible pairs due to these authors. We are able then to generalize in section 2 their construction of a $C-$admissible subalgebra (associated to a $C-$admissible pair) which grades finitely $\mathfrak{g}$:

Theorem 1. (cf. 2.6, 2.11, 2.14) *Let $\mathfrak{g}$ be an indecomposable and symmetrizable Kac-Moody algebra, associated to a generalized Cartan matrix $A=(a_{i,j})_{i,j\in I}$. Let $J\subset I$ be a subset of finite type such that the pair (I,J) is $C-$admissible. There is a generalized Cartan matrix $A^J=(a'_{k,l})_{k,l\in I'}$ with index set $I'=I\setminus J$ and a Kac-Moody subalgebra $\mathfrak{g}^J$ of $\mathfrak{g}$ associated to A^J, with root system Δ^J. Then $\mathfrak{g}$ is finitely Δ^J-graded with grading subalgebra $\mathfrak{g}^J$.*

For a general finite gradation of $\mathfrak{g}$ with grading subalgebra $\mathfrak{m}$, we prove (in section 3) that $\mathfrak{m}$ also is indecomposable, symmetrizable and the restriction to $\mathfrak{m}$ of the invariant bilinear form of $\mathfrak{g}$ is non-degenerate (3.11 and 3.17). The Kac-Moody algebras $\mathfrak{g}$ and $\mathfrak{m}$ have the same type: finite, affine or indefinite; the first two types correspond to the cases already studied e.g. by J. Nervi. Moreover if $\mathfrak{g}$ is indefinite Lorentzian or hyperbolic, then so is $\mathfrak{m}$ (Propositions 3.6 and 3.27). We get also the following precise structure result for this general situation :

Theorem 2. *Let $\mathfrak{g}$ be an indecomposable and symmetrizable Kac-Moody algebra, finitely graded by a root system Σ of Kac-Moody type with grading subalgebra $\mathfrak{m}$.*
1) We may choose the Cartan subalgebras $\mathfrak{a}$ of $\mathfrak{m}$, $\mathfrak{h}$ of $\mathfrak{g}$ such that $\mathfrak{a}\subset\mathfrak{h}$. Then there is a surjective map $\rho_a:\Delta\cup\{0\}\to\Sigma\cup\{0\}$ between the corresponding root systems. We may choose the bases $\Pi_a=\{\gamma_s\mid s\in\overline{I}\}\subset\Sigma$ and $\Pi=\{\alpha_i\mid i\in I\}\subset\Delta$ of these root systems such that $\rho_a(\Delta^+)\subset\Sigma^+\cup\{0\}$ and $\{\alpha\in\Delta\mid\rho_a(\alpha)=0\}=\Delta_J:=\Delta\cap(\sum_{j\in J}\mathbb{Z}\alpha_j)$ for some subset $J\subset I$ of finite type.
2) Let $I'_{re}=\{i\in I\mid\rho_a(\alpha_i)\in\Pi_a\}$, $I'_{im}=\{i\in I\mid\rho_a(\alpha_i)\notin\Pi_a\cup\{0\}\}$. Then $J=\{i\in I\mid\rho_a(\alpha_i)=0\}$. We note I_{re} (resp. J°) the union of the connected components of $I\setminus I'_{im}=I'_{re}\cup J$ meeting I'_{re} (resp. contained in J), and $J_{re}=J\cap I_{re}$. Then the pair (I_{re},J_{re}) is $C-$admissible (eventually decomposable).
3) There is a Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ of $\mathfrak{g}$, associated to I_{re}, which contains $\mathfrak{m}$. This Lie algebra is finitely $\Delta(I_{re})^{J_{re}}-$graded, with grading subalgebra $\mathfrak{g}(I_{re})^{J_{re}}$. Both algebras $\mathfrak{g}(I_{re})$ and $\mathfrak{g}(I_{re})^{J_{re}}$ are finitely $\Sigma-$graded with grading subalgebra $\mathfrak{m}$.

It may happen that I'_{im} is non-empty, we then say that (I,J) is a generalized $C-$admissible pair and the gradation is imaginary. We give and explain precisely an example in section 5.

When I'_{im} is empty (i.e. when the gradation is real : 3.16), $I_{re}=I$, $J_{re}=J$, $\mathfrak{g}(I_{re})=\mathfrak{g}$, $(I,J)=(I_{re},J_{re})$ is a $C-$admissible pair and the situation looks much like the one described by J. Nervi in the finite [16] or affine [17] cases. Actually we prove that this is always true when $\mathfrak{g}$ is of finite type, affine or hyperbolic (Proposition 3.26). In this real case we get the gradation of $\mathfrak{g}$ with two levels: $\mathfrak{g}$ is finitely Δ^J-graded with grading subalgebra $\mathfrak{g}^J$ as in Theorem 1 and $\mathfrak{g}^J$ is finitely $\Sigma-$graded with grading subalgebra $\mathfrak{m}$. But the gradation of $\mathfrak{g}^J$ by Σ and $\mathfrak{m}$ is such that the corresponding set "J" described as in Theorem 2 is empty; we say (following [16], [17]) that it is a maximal gradation, cf. Definition 3.16 and Proposition 3.21.

To get a complete description of the real gradations, it remains to describe the maximal gradations; this is done in section 4. We prove in Proposition 4.1 that a maximal gradation $(\mathfrak{g}, \Sigma, \mathfrak{m})$ is entirely described by a quotient map $\rho : I \to \overline{I}$ which is admissible i.e. satisfies two simple conditions (MG1) and (MG2) with respect to the generalized Cartan matrix $A = (a_{i,j})_{i,j\in I}$. Conversely for any admissible quotient map ρ, it is possible to build a maximal gradation of $\mathfrak{g}$ associated to this map, cf. Proposition 4.5 and Remark 4.7.

1. Preliminaries

We recall the basic results on the structure of Kac-Moody Lie algebras and we set the notations. More details can be found in the book of Kac [12]. We end by the definition of finitely graded Kac-Moody algebras.

1.1. **Generalized Cartan matrices.** Let I be a finite index set. A matrix $A = (a_{i,j})_{i,j\in I}$ is called a *generalized Cartan matrix* if it satisfies :
(1) $a_{i,i} = 2 \qquad (i \in I)$
(2) $a_{i,j} \in \mathbb{Z}^- \qquad (i \neq j)$
(3) $a_{i,j} = 0$ implies $a_{j,i} = 0$.

The matrix A is called *decomposable* if for a suitable permutation of I it takes the form $\begin{pmatrix} B & 0 \\ 0 & C \end{pmatrix}$ where B and C are square matrices. If A is not decomposable, it is called *indecomposable*.
The matrix A is called *symmetrizable* if there exists an invertible diagonal matrix $D = \mathrm{diag}(d_i,\ i \in I)$ such that DA is symmetric. The entries $d_i,\ i \in I$, can be chosen to be positive rationals and if moreover the matrix A is indecomposable, then these entries are unique up to a constant factor.
Any indecomposable generalized Cartan matrix is of one of three mutually exclusive types : *finite*, *affine* and *indefinite* ([12, Chap. 4]). A generalized Cartan matrix is said of *finite type* if each of its indecomposable factors is of finite type.
An indecomposable and symmetrizable generalized Cartan matrix A is called *Lorentzian* if it is non-singular and the corresponding symmetric matrix has signature $(++ ... +-)$; it is then of indefinite type.
An indecomposable generalized Cartan matrix A is called *strictly hyperbolic* (resp. *hyperbolic*) if the deletion of any one vertex, and the edges connected to it, of the corresponding Dynkin diagram yields a disjoint union of Dynkin diagrams of finite (resp. finite or affine) type.
Note that a symmetrizable hyperbolic generalized Cartan matrix is non-singular and Lorentzian (cf. [14]).

1.2. **Kac-Moody algebras and groups.** (See [12] and [18]).
Let $A = (a_{i,j})_{i,j\in I}$ be a symmetrizable generalized Cartan matrix. Let $(\mathfrak{h}_{\mathbb{R}},\ \Pi = \{\alpha_i,\ i \in I\},\ \Pi\check{} = \{\check{\alpha_i},\ i \in I\})$ be a realization of A over the real field $\mathbb{R}$: thus $\mathfrak{h}_{\mathbb{R}}$ is a real vector space such that $\dim(\mathfrak{h}_{\mathbb{R}}) = |I| + \mathrm{corank}(A)$, Π and $\Pi\check{}$ are linearly independent in $\mathfrak{h}_{\mathbb{R}}^*$ and $\mathfrak{h}_{\mathbb{R}}$ respectively such that $\langle \alpha_j, \check{\alpha_i} \rangle = a_{i,j}$. Let $\mathfrak{h} = \mathfrak{h}_{\mathbb{R}} \otimes \mathbb{C}$, then $(\mathfrak{h},\ \Pi,\ \Pi\check{})$ is a realization of A over the complex field $\mathbb{C}$.

It follows that, if A is non-singular, then $\Pi\check{}$ (resp. Π) is a basis of $\mathfrak{h}$ (resp. $\mathfrak{h}^*$); moreover $\mathfrak{h}_{\mathbb{R}} = \{h \in \mathfrak{h} \mid \alpha_i(h) \in \mathbb{R}, \forall i \in I\}$ is well defined by the realization $(\mathfrak{h},\ \Pi,\ \Pi\check{})$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the complex Kac-Moody Lie algebra associated to A : it is generated by $\{\mathfrak{h}, e_i, f_i,\ i \in I\}$ with the following relations

$$
\begin{array}{llll}
 & [\mathfrak{h},\mathfrak{h}] = 0, & [e_i, f_j] = \delta_{i,j}\alpha\check{_i} & (i,j \in I); \\
(1.1) & [h, e_i] = \langle \alpha_i, h\rangle e_i, & [h, f_i] = -\langle \alpha_i, h\rangle f_i & (h \in \mathfrak{h}); \\
 & (\mathrm{ad} e_i)^{1-a_{i,j}}(e_j) = 0, & (\mathrm{ad} f_i)^{1-a_{i,j}}(f_j) = 0 & (i \neq j).
\end{array}
$$

The Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$ decomposes as a direct sum of factors $\mathfrak{g}(A_i)$, where $A_1, \cdots, A_r$ are the indecomposable factors of A. It is said indecomposable if the corresponding generalized Cartan matrix A is indecomposable and of finite, affine or indefinite type if A is.

The derived algebra $\mathfrak{g}'$ of $\mathfrak{g}$ is generated by the *Chevalley generators* $e_i, f_i,\ i \in I$, and the center $\mathfrak{c}$ of $\mathfrak{g}$ lies in $\mathfrak{h}' = \mathfrak{h} \cap \mathfrak{g}' = \sum_{i\in I} \mathbb{C}\alpha\check{_i}$. If the generalized Cartan matrix A is indecomposable and non-singular, then $\mathfrak{g} = \mathfrak{g}'$ is a (finite or infinite)-dimensional simple Lie algebra, and the center $\mathfrak{c}$ is trivial.

The subalgebra $\mathfrak{h}$ is a maximal $\mathrm{ad}(\mathfrak{g})-$diagonalizable subalgebra of $\mathfrak{g}$, it is called the *standard Cartan subalgebra* of $\mathfrak{g}$. Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ be the corresponding root system; then Π is a root basis of Δ and $\Delta = \Delta^+ \cup \Delta^-$, where $\Delta^{\pm} = \Delta \cap \mathbb{Z}^{\pm}\Pi$ is the set of positive (or negative) roots relative to the basis Π. For $\alpha \in \Delta$, let $\mathfrak{g}_\alpha$ be the root space of $\mathfrak{g}$ corresponding to the root α; then $\mathfrak{g} = \mathfrak{h} \oplus (\underset{\alpha\in\Delta}{\oplus} \mathfrak{g}_\alpha)$.

The *Weyl group* W of $(\mathfrak{g}, \mathfrak{h})$ is generated by the fundamental reflections r_i $(i \in I)$ such that $r_i(h) = h - \langle\alpha_i, h\rangle\alpha\check{_i}$ for $h \in \mathfrak{h}$, it is a Coxeter group on $\{r_i,\ i \in I\}$ with length function $w \mapsto l(w)$, $w \in W$. The Weyl group W acts on $\mathfrak{h}^*$ and Δ, we set $\Delta^{re} = W(\Pi)$ (the real roots) and $\Delta^{im} = \Delta \setminus \Delta^{re}$ (the imaginary roots). If the generalized Cartan matrix A is indecomposable, then any root basis of Δ is $W-$conjugate to Π or $-\Pi$.

A *Borel subalgebra* of $\mathfrak{g}$ is a maximal completely solvable subalgebra. A *parabolic subalgebra* of $\mathfrak{g}$ is a (proper) subalgebra containing a Borel subalgebra. The *standard positive (or negative) Borel subalgebra* is $\mathfrak{b}^{\pm} := \mathfrak{h} \oplus (\oplus_{\alpha\in\Delta^{\pm}}\mathfrak{g}_\alpha)$. A parabolic subalgebra $\mathfrak{p}^+$ (resp. $\mathfrak{p}^-$) containing $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is called *positive (resp. negative) standard parabolic subalgebra* of $\mathfrak{g}$; then there exists a subset J of I (called the type of $\mathfrak{p}^{\pm}$) such that $\mathfrak{p}^{\pm} = \mathfrak{p}^{\pm}(J) := (\underset{\alpha\in\Delta_J}{\oplus} \mathfrak{g}_\alpha) + \mathfrak{b}^{\pm}$, where $\Delta_J = \Delta \cap (\oplus_{j\in J}\mathbb{Z}\alpha_j)$ (cf. [13]).

In [18], D.H. Peterson and V.G. Kac construct a group G, which is the connected and simply connected complex algebraic group associated to $\mathfrak{g}$ when $\mathfrak{g}$ is of finite type, depending only on the derived Lie algebra $\mathfrak{g}'$ and acting on $\mathfrak{g}$ via the adjoint representation $\mathrm{Ad} : G \to \mathrm{Aut}(\mathfrak{g})$. It is generated by the one-parameter subgroups $U_\alpha = \exp(\mathfrak{g}_\alpha)$, $\alpha \in \Delta^{re}$, and $\mathrm{Ad}(U_\alpha) = \exp(\mathrm{ad}\mathfrak{g}_\alpha))$. In the definitions of J. Tits [22] G is the group of complex points of $\mathfrak{G}_D$ where D is the datum associated to A and the $\mathbb{Z}-$dual Λ of $\bigoplus_{i\in I}\mathbb{Z}\alpha_i^\vee$.

The Cartan subalgebras of $\mathfrak{g}$ are $G-$conjugate. If $\mathfrak{g}$ is indecomposable and not of finite type, there are exactly two conjugate classes (under the adjoint action of G) of Borel subalgebras : $G.\mathfrak{b}^+$ and $G.\mathfrak{b}^-$. A Borel subalgebra $\mathfrak{b}$ of $\mathfrak{g}$ which is $G-$conjugate to $\mathfrak{b}^+$ (resp. $\mathfrak{b}^-$) is called positive (resp. negative). It follows that any parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $G-$conjugate to a standard positive (or negative) parabolic subalgebra, in which case, we say that $\mathfrak{p}$ is positive (or negative).

1.3. **Standard Kac-Moody subalgebras and subgroups.** Let J be a non-empty subset of I. Consider the generalized Cartan matrix $A_J = (a_{i,j})_{i,j\in J}$.

Definition 1.1. The subset J is called of finite type if the corresponding generalized Cartan matrix A_J is. We say also that J is connected, if the Dynkin subdiagram, with vertices indexed by J, is connected or, equivalently, the corresponding generalized Cartan submatrix A_J is indecomposable.

Proposition 1.2. *Let $\Pi_J = \{\alpha_j,\ j \in J\}$ and $\Pi\check{_J} = \{\alpha\check{_j},\ j \in J\}$. Let $\mathfrak{h}'_J$ be the subspace of $\mathfrak{h}$ generated by $\Pi\check{_J}$, and $\mathfrak{h}^J = \Pi_J^\perp = \{h \in \mathfrak{h}, \langle\alpha_j, h\rangle = 0,\ \forall j \in J\}$. Let $\mathfrak{h}''_J$ be a supplementary subspace of $\mathfrak{h}'_J + \mathfrak{h}^J$ in $\mathfrak{h}$ and let*

$$\mathfrak{h}_J = \mathfrak{h}'_J \oplus \mathfrak{h}''_J,$$

then, we have :
1) $(\mathfrak{h}_J,\ \Pi_J,\ \Pi\check{_J})$ is a realization of the generalized Cartan matrix A_J. Hence $\mathfrak{h}''_J = \{0\}$, $\mathfrak{h}_J = \mathfrak{h}'_J$ when A_J is regular (e.g. when J is of finite type).
2) The subalgebra $\mathfrak{g}(J)$ of $\mathfrak{g}$, generated by $\mathfrak{h}_J$ and the $e_j,\ f_j,\ j \in J$, is the Kac-Moody Lie algebra associated to the realization $(\mathfrak{h}_J,\ \Pi_J,\ \Pi\check{_J})$ of A_J.
3) The corresponding root system $\Delta(J) = \Delta(\mathfrak{g}(J), \mathfrak{h}_J)$ can be identified with $\Delta_J := \Delta \cap (\oplus_{j\in J}\mathbb{Z}\alpha_j)$.

N.B. The derived algebra $\mathfrak{g}'(J)$ of $\mathfrak{g}(J)$ is generated by the e_j, f_j for $j \in J$; it does not depend of the choice of $\mathfrak{h}''_J$.

Proof. We may assume $\mathfrak{g}$ indecomposable.
1) Note that $\dim(\mathfrak{h}''_J) = \dim(\mathfrak{h}'_J \cap \mathfrak{h}^J) = corank(A_J)$. In particular, $\dim(\mathfrak{h}_J) - |J| = corank(A_J)$. If $\alpha \in Vect(\alpha_j,\ j \in J)$, then α is entirely determined by its restriction to $\mathfrak{h}_J$ and hence Π_J defines, by restriction, a linearly independent set in $\mathfrak{h}_J^*$. As $\Pi\check{_J}$ is linearly independent, assertion 1) holds.
Assertions 2) and 3) are straightforward. □

In the same way, the subgroup G_J of G generated by $U_{\pm\alpha_j},\ j \in J$, is equal to the Kac-Moody group associated to the generalized Cartan matrix A_J: it is clearly a quotient; the well known equality is proven explicitly in [20, 5.15.2], it may be deduced from [22, th. 1], see also [19, 8.4.2].

1.4. **The invariant bilinear form.** (See [12]).
We recall that the generalized Cartan matrix A is supposed symmetrizable. There exists a non-degenerate $\mathrm{ad}(\mathfrak{g})-$ invariant symmetric $\mathbb{C}-$bilinear form $(.\,,.)$ on $\mathfrak{g}$, which is entirely determined by its restriction to $\mathfrak{h}$, such that

$$(\alpha\check{_i}, h) = \frac{(\alpha\check{_i}, \alpha\check{_i})}{2}\langle\alpha_i, h\rangle, \quad i \in I,\ h \in \mathfrak{h},$$

and we may thus assume that

$$(\alpha\check{_i}, \alpha\check{_i}) \text{ is a positive rational for all } i. \tag{1.2}$$

The non-degenerate invariant bilinear form $(.\,,.)$ induces an isomorphism $\nu : \mathfrak{h} \to \mathfrak{h}^*$ such that $\alpha_i = \dfrac{2\nu(\alpha\check{_i})}{(\alpha\check{_i}, \alpha\check{_i})}$ and $\alpha\check{_i} = \dfrac{2\nu^{-1}(\alpha_i)}{(\alpha_i, \alpha_i)}$ for all i.
There exists a totally isotropic subspace $\mathfrak{h}''$ of $\mathfrak{h}$ (relative to the invariant bilinear

form $(.\,,.)$) which is in duality with the center $\mathfrak{c}$ of $\mathfrak{g}$. In particular, $\mathfrak{h}''$ defines a supplementary subspace of $\mathfrak{h}'$ in $\mathfrak{h}$.
Note that any invariant symmetric bilinear form b on $\mathfrak{g}$ satisfying $b(\alpha_i^\vee, \alpha_i^\vee) > 0$, $\forall i \in I$, is non-degenerate and $b(\alpha_i^\vee, h) = \frac{b(\alpha_i^\vee,\alpha_i^\vee)}{2}\langle \alpha_i, h\rangle$, $\forall i \in I$, $\forall h \in \mathfrak{h}$. It follows that, if $\mathfrak{g}$ is indecomposable, the restriction of b to $\mathfrak{g}'$ is proportional to that of $(.\,,.)$. In particular, if moreover A is non-singular, then the invariant bilinear form $(.\,,.)$ satisfying the condition 1.2 is unique up to a positive rational factor.

1.5. **The Tits cone.** (See [12, Chap. 3 and 5]).
Let $C := \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha_i, h\rangle \geq 0, \forall i \in I\}$ be the fundamental chamber (relative to the root basis Π) and let $X := \bigcup_{w \in W} w(C)$ be the Tits cone. We have the following description of the Tits cone:
(1) $X = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h\rangle < 0$ only for a finite number of $\alpha \in \Delta^+\}$.
(2) $X = \mathfrak{h}_{\mathbb{R}}$ if and only if the generalized Cartan matrix A is of finite type.
(3) If A is indecomposable of affine type, then $X = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \delta, h\rangle > 0\} \cup \mathbb{R}\nu^{-1}(\delta)$, where δ is the lowest imaginary positive root of Δ^+.
(4) If A is indecomposable of indefinite type, then the closure of the Tits cone, for the metric topology on $\mathfrak{h}_{\mathbb{R}}$, is $\bar{X} = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h\rangle \geq 0,\ \forall \alpha \in \Delta^+_{im}\}$.
(5) If $h \in X$, then h lies in the interior $\overset{\circ}{X}$ of X if and only if the fixer W_h of h, in the Weyl group W, is finite. Thus $\overset{\circ}{X}$ is the union of finite type facets of X.
(6) If A is hyperbolic, then $\bar{X} \cup (-\bar{X}) = \{h \in \mathfrak{h}_{\mathbb{R}}; (h,h) \leq 0\}$ and the set of imaginary roots is $\Delta^{im} = \{\alpha \in Q \setminus \{0\}; (\alpha,\alpha) \leq 0\}$, where $Q = \mathbb{Z}\Pi$ is the root lattice.

Remark 1.3. Combining (3) and (4) one obtains that if A is not of finite type then $\bar{X} = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h\rangle \geq 0,\ \forall \alpha \in \Delta^+_{im}\}$.

1.6. **Graded Kac-Moody Lie algebras.**

Definition 1.4. Let Σ be a root system of Kac-Moody type. The Kac-Moody Lie algebra $\mathfrak{g}$ is said to be finitely $\Sigma-$graded if :
(i) $\mathfrak{g}$ contains, as a subalgebra, a Kac-Moody algebra $\mathfrak{m}$ whose root system relative to a Cartan subalgebra $\mathfrak{a}$ is equal to Σ.
(ii) $\mathfrak{g} = \sum_{\alpha \in \Sigma \cup \{0\}} V_\alpha$, with $V_\alpha = \{x \in \mathfrak{g}\,;\ [a,x] = \langle \alpha, a\rangle x,\ \forall a \in \mathfrak{a}\}$.
(iii) V_α is finite dimensional for all $\alpha \in \Sigma \cup \{0\}$.

We say that $\mathfrak{m}$ (as in (i) above) is a grading subalgebra, and $(\mathfrak{g}, \Sigma, \mathfrak{m})$ a gradation with finite multiplicities (or, to be short, a finite gradation).

Note that from (ii) the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is $\mathrm{ad}(\mathfrak{g})-$diagonalizable, and we may assume that $\mathfrak{a}$ is contained in the standard Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Lemma 1.5. *Let $\mathfrak{g}$ be a Kac-Moody algebra finitely $\Sigma-$graded, with grading subalgebra $\mathfrak{m}$. If $\mathfrak{m}$ itself is finitely $\Sigma'-$graded (for some root system Σ' of Kac-Moody type), then $\mathfrak{g}$ is finitely $\Sigma'-$graded.*

Proof. If $\mathfrak{m}'$ is the grading subalgebra of $\mathfrak{m}$, we may suppose the Cartan subalgebras such that $\mathfrak{a}' \subset \mathfrak{a} \subset \mathfrak{h}$, with obvious notations. Conditions (i) and (ii) are clearly satisfied for $\mathfrak{g}$, $\mathfrak{m}'$ and $\mathfrak{a}'$. Condition (iii) for $\mathfrak{m}$ and Σ' tells that, for all $\alpha' \in \Sigma'$, the set $\{\alpha \in \Sigma \mid \alpha_{|\mathfrak{a}'} = \alpha'\}$ is finite. But $V_{\alpha'} = \oplus_{\alpha_{|\mathfrak{a}'}=\alpha'} V_\alpha$, so each $V_{\alpha'}$ is finite dimensional if this is true for each V_α. $\square$

1.7. **Examples of gradations.**
1) Let $\Delta = \Delta(\mathfrak{g}, \mathfrak{h})$ the root system of $\mathfrak{g}$ relative to $\mathfrak{h}$, then $\mathfrak{g}$ is finitely $\Delta-$graded : this is the trivial gradation of $\mathfrak{g}$ by its own root system.
2) Let $\mathfrak{g}_{\mathbb{R}}$ be an almost split real form of $\mathfrak{g}$ (see [2]) and let $\mathfrak{t}_{\mathbb{R}}$ be a maximal split toral subalgebra of $\mathfrak{g}_{\mathbb{R}}$. Suppose that the restricted root system $\Delta' = \Delta(\mathfrak{g}_{\mathbb{R}}, \mathfrak{t}_{\mathbb{R}})$ is reduced of Kac-Moody type. In [4, §9], N. Bardy constructed a split real Kac-Moody subalgebra $\mathfrak{l}_{\mathbb{R}}$ of $\mathfrak{g}_{\mathbb{R}}$ such that $\Delta' = \Delta(\mathfrak{l}_{\mathbb{R}}, \mathfrak{t}_{\mathbb{R}})$, then $\mathfrak{g}$ is obviously finitely $\Delta'-$graded.

We get thus many examples coming from known tables for almost split real forms: see [2] in the affine case and [6] in the hyperbolic case.
3) When $\mathfrak{g}_{\mathbb{R}}$ is an almost compact real form of $\mathfrak{g}$, the same constructions should lead to gradations by finite root systems, as in [5] e.g.

2. Gradations associated to $C-$admissible pairs.

In this section, we suppose the Kac-Moody Lie algebra $\mathfrak{g}$ indecomposable and symmetrizable, see however Remark 2.15. We shall build a finite gradation of $\mathfrak{g}$ associated to some good subset of I.

We recall some definitions introduced by H. Rubenthaler ([21]) and J. Nervi ([16], [17]). Let J be a subset of I of finite type. For $k \in I \setminus J$, we denote by I_k the connected component, containing k, of the Dynkin subdiagram corresponding to $J \cup \{k\}$, and let $J_k := I_k \setminus \{k\}$.

We are interested in the case where I_k is of finite type for all $k \in I \setminus J$: that is always true if $\mathfrak{g}$ is of affine type and $|I \setminus J| \geq 2$ or if $\mathfrak{g}$ is of hyperbolic type and $|I \setminus J| \geq 3$.
For $k \in I \setminus J$, let $\mathfrak{g}(I_k)$ be the simple subalgebra generated by $\mathfrak{g}_{\pm\alpha_i}$, $i \in I_k$, then $\mathfrak{h}_{I_k} = \mathfrak{h} \cap \mathfrak{g}(I_k) = \sum_{i \in I_k} \mathbb{C}\alpha_i^{\vee}$ is a Cartan subalgebra of $\mathfrak{g}(I_k)$. Let H_k be the unique element of $\mathfrak{h}_{I_k}$ such that $\langle \alpha_i, H_k \rangle = 2\delta_{i,k}$, $\forall i \in I_k$.

Definition 2.1. We suppose the Dynkin diagram indexed by I connected and consider a subset J of finite type. We preserve the notations introduced above.
1) Let $k \in I \setminus J$.
(i) The pair (I_k, J_k) is called admissible if I_k is of finite type and there exist $E_k, F_k \in \mathfrak{g}(I_k)$ such that (E_k, H_k, F_k) is an $\mathfrak{sl}_2-$triple.
(ii) The pair (I_k, J_k) is called $C-$admissible if it is admissible and the simple Lie algebra $\mathfrak{g}(I_k)$ is A_1-graded by the root system, of type A_1, associated to the $\mathfrak{sl}_2-$triple (E_k, H_k, F_k).
2) The pair (I, J) is called $C-$admissible if the pairs (I_k, J_k) are $C-$admissible for all $k \in I \setminus J$. It is said irreducible if, moreover, $|I \setminus J| = 1$.

Schematically, any $C-$admissible pair (I, J) is represented by the Dynkin diagram, corresponding to A, on which the vertices indexed by J are denoted by white circles $\circ$ and those of $I \setminus J$ are denoted by black circles $\bullet$.

Remark 2.2. 1) The admissibility of each (I_k, J_k) is essential to build (in 2.6, 2.11) the grading subalgebra $\mathfrak{g}^J$ and its grading root system Δ^J.

2) As $\mathfrak{g}(J)$ will be in the eigenspace V_0 of weight 0 for the grading by Δ^J, it is necessary to assume J of finite type to get a finite gradation.

3) I_k is of finite type if, and only if, $\mathfrak{g}(I_k)$ is finite dimensional, and this is equivalent to the alternative assumption in (ii) that the A_1-gradation has finite

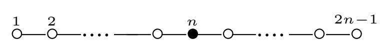
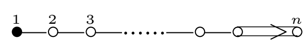
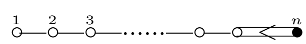
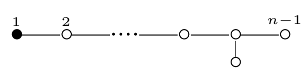
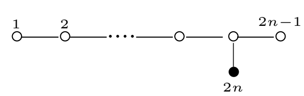
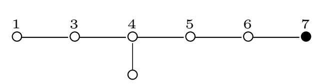
multiplicities. It is clearly necessary to get, in Theorem 2.14, a finite gradation of $\mathfrak{g}$ by the root system Δ^J. Moreover, even in a more general situation, the condition I_k of finite type will naturally appear (3.14).

4) Note that the definition presented here, for $C-$admissible pairs, is equivalent to that introduced by Rubenthaler and Nervi (see [21], [16]) in terms of prehomogeneous spaces of parabolic type : if (I_k, J_k) is $C-$admissible, define for $p \in \mathbb{Z}$, the subspace $d_{k,p} := \{X \in \mathfrak{g}(I_k)\,;\ [H_k, X] = 2pX\}$; then $(d_{k,0}, d_{k,1})$ is an irreducible regular and commutative prehomogeneous space of parabolic type, and $d_{k,p} = \{0\}$ for $|p| \geq 2$. Then (I_k, J_k) is an irreducible $C-$admissible pair. According to Rubenthaler and Nervi ([21, Table 1] or [16, Table 2]) the irreducible $C-$admissible pair (I_k, J_k) should be among the list in Table 1 below.

5) Along our study of general finite gradations in section 3, we shall meet a situation of "generalized $C-$admissible pair" (I, J) (3.16) where $J \subset I$ is of finite type and I_k (for $k \in I' = I \setminus J$) is defined as above but perhaps not of finite type. When k is in some subset I'_{re} of I', (I_k, J_k) is $C-admissible$ and the $k \in I'_{im} = I' \setminus I'_{re}$ do not contribute to the root system Σ grading $\mathfrak{g}$. But we do not know the good assumptions on these (I_k, J_k) for $k \in I'_{im}$ to get, conversely, a finite gradation of $\mathfrak{g}$ by some root system. So we give no precise definition; it is expected in the work in preparation [7].

Table 1

List of irreducible $C-$admissible pairs

$A_{2n-1},\ n\geq 1$	
$B_n,\ n\geq 3$	
$C_n,\ n\geq 2$	
$D_{n,1},\ n\geq 4$	
$D_{2n,2},\ n\geq 2$	
E_7	

Definition 2.3. Let J be a subset of I and let $i, k \in I \setminus J$. We say that i and k are $J-$connected relative to A if there exist $j_0, j_1, ..., j_{p+1} \in I$ such that $j_0 = i$, $j_{p+1} = k$, $j_s \in J$, $\forall s = 1, 2, ..., p$, and $a_{j_s, j_{s+1}} \neq 0$, $\forall s = 0, 1, ..., p$.

Remark 2.4. Note that the relation " to be $J-$connected " is symmetric on i and k. As the generalized Cartan matrix A is assumed to be indecomposable, for any vertices $i, k \in I \setminus J$ there exist $i_0, i_1, ..., i_{p+1} \in I \setminus J$ such that $i_0 = i$, $i_{p+1} = k$ and i_s and i_{s+1} are $J-$connected for all $s = 0, 1, ..., p$.

Let us assume from now on that (I, J) is a $C-$admissible pair and let $I' := I \setminus J$. For $k \in I'$, let (E_k, H_k, F_k) be an $\mathfrak{sl}_2-$triple associated to the irreducible $C-$admissible pair (I_k, J_k).

Lemma 2.5. *Let $k \neq l \in I'$, then :*
1) $\langle \alpha_l, H_k \rangle \in \mathbb{Z}^-$.
2) the following assertions are equivalent :
i) k, l are $J-$connected
ii) $\langle \alpha_l, H_k \rangle$ is a negative integer
iii) $\langle \alpha_k, H_l \rangle$ is a negative integer

Proof. 1) One can write $H_k = \sum_{i \in I_k} n_{i,k} \check{\alpha_i}$, where $n_{i,k}$ are positive integers (see [21] or [17, 1.4.1.2]). As $l \notin I_k$, we have that $\langle \alpha_l, H_k \rangle = \sum_{i \in I_k} n_{i,k} \langle \alpha_l, \check{\alpha_i} \rangle \in \mathbb{Z}^-$.
2) In view of Remark 2.4, it suffices to prove the equivalence between i) and ii). Since I_k is the connected component of $J \cup \{k\}$ containing k, the assertion i) is equivalent to say that the vertex l is connected to I_k, so there exists $i_k \in I_k$ such that $\langle \alpha_l, \check{\alpha_{i_k}} \rangle < 0$ and hence $\langle \alpha_l, H_k \rangle < 0$. □

Proposition 2.6. *Let $\mathfrak{h}^J = \Pi_J^\perp = \{h \in \mathfrak{h}, \langle \alpha_j, h \rangle = 0, \ \forall j \in J\}$. For $k \in I'$, denote by $\alpha'_k = \alpha_k/\mathfrak{h}^J$ the restriction of α_k to the subspace $\mathfrak{h}^J$ of $\mathfrak{h}$, and $\Pi^J = \{\alpha'_k;\ k \in I'\}$, $\Pi^{J\vee} = \{H_k;\ k \in I'\}$. For $k, l \in I'$, put $a'_{k,l} = \langle \alpha_l, H_k \rangle$ and $A^J = (a'_{k,l})_{k,l \in I'}$. Then A^J is an indecomposable and symmetrizable generalized Cartan matrix, $(\mathfrak{h}^J, \Pi^J, \Pi^{J\vee})$ is a realization of A^J and $corank(A^J) = corank(A)$.*

Proof. The fact that $a'_{k,k} = 2$ follows from the definition of H_k for $k \in I'$. If $k \neq l \in I'$, then by lemma 2.5, $a'_{k,l} \in \mathbb{Z}^-$ and $a'_{k,l} \neq 0$ if and only if $a'_{l,k} \neq 0$. Hence A^J is a generalized Cartan matrix. As the matrix A is indecomposable, A_J is also indecomposable (see Remark 2.4). Clearly $\Pi^J = \{\alpha'_k;\ k \in I'\}$ is a linearly independent subset of the dual space ${\mathfrak{h}^J}^*$ of $\mathfrak{h}^J$, $\Pi^{J\vee} = \{H_k;\ k \in I'\}$ is a linearly independent subset of $\mathfrak{h}^J$ and by construction $\langle \alpha_l, H_k \rangle = a'_{k,l}$, $\forall k, l \in I'$.
We have to prove that $\dim(\mathfrak{h}^J) - |I'| = corank(A^J)$. As J is of finite type, the restriction of the invariant bilinear form $(.\,, .)$ to $\mathfrak{h}_J$ is non-degenerate and $\mathfrak{h}_J$ is contained in $\mathfrak{h}' = \underset{i \in I}{\oplus} \mathbb{C}\check{\alpha_i}$. Therefore

$$\mathfrak{h} = \mathfrak{h}^J \overset{\perp}{\oplus} \mathfrak{h}_J$$

and

$$\mathfrak{h}' = (\mathfrak{h}' \cap \mathfrak{h}^J) \oplus \mathfrak{h}_J.$$

It follows that $\dim(\mathfrak{h}' \cap \mathfrak{h}^J) = |I'| = \dim(\underset{k \in I'}{\oplus} \mathbb{C}H_k)$. As the subspace $\underset{k \in I'}{\oplus} \mathbb{C}H_k$ is contained in $\mathfrak{h}' \cap \mathfrak{h}^J$, we deduce that $\mathfrak{h}' \cap \mathfrak{h}^J = \underset{k \in I'}{\oplus} \mathbb{C}H_k$. Note that any supplementary subspace ${\mathfrak{h}^J}''$ of $\mathfrak{h}' \cap \mathfrak{h}^J$ in $\mathfrak{h}^J$ is also a supplementary of $\mathfrak{h}'$ in $\mathfrak{h}$; hence, we have that $corank(A) = \dim({\mathfrak{h}^J}'') = \dim(\mathfrak{h}^J) - |I'|$. Let $\mathfrak{c} := \underset{i \in I}{\cap} \ker(\alpha_i)$ be the center of $\mathfrak{g}$ and let $\mathfrak{c}^J = \underset{k \in I'}{\cap} \ker(\alpha'_k)$. Recall that $corank(A) = \dim(\mathfrak{c})$ and

$corank(A^J) = \dim(\mathfrak{c}^J)$. It's clear that $\mathfrak{c}^J = \mathfrak{c}$; hence $corank(A^J) = \dim(\mathfrak{c}^J) = corank(A) = \dim(\mathfrak{h}^J) - |I'|$.
It remains to prove that A^J is symmetrizable. For $k \in I'$, let R_k^J be the fundamental reflection of $\mathfrak{h}^J$ such that $R_k^J(h) = h - \langle \alpha'_k, h\rangle H_k$, $\forall h \in \mathfrak{h}^J$. Let W^J be the Weyl group of A^J generated by R_k^J, $k \in I'$. Let $(.\,,.)^J$ be the restriction to $\mathfrak{h}^J$ of the invariant bilinear form $(.\,,.)$ on $\mathfrak{h}$. Then $(.\,,.)^J$ is a non-degenerate symmetric bilinear form on $\mathfrak{h}^J$ which is W^J-invariant (see the lemma hereafter). From the relation $(R_k^J(H_k), R_k^J(H_l))^J = (H_k, H_l)^J$ one can deduce that :

$$(H_k, H_l)^J = \frac{(H_k, H_k)^J}{2} a'_{l,k}, \ \forall k, l \in I',$$

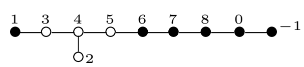
but $(H_k, H_k)^J > 0$, $\forall k \in I'$; hence ${}^tA^J$ (and so A^J) is symmetrizable. □

Lemma 2.7. *For $k \in I' := I \setminus J$, let w_k^J be the longest element of the Weyl group $W(I_k)$ generated by the fundamental reflections r_i, $i \in I_k$. Then w_k^J stabilizes $\mathfrak{h}^J$ and induces the fundamental reflection R_k^J of $\mathfrak{h}^J$ associated to H_k.*

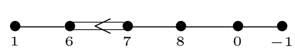
Proof. If one looks at the list above of the irreducible $C-$admissible pairs, one can see that $w_k^J(\alpha_k) = -\alpha_k$ and that $-w_k^J$ permutes the α_j, $j \in J_k$. Clearly $w_k^J(\alpha_j) = \alpha_j$, $\forall j \in J \setminus J_k$. Hence w_k^J stabilizes $\mathfrak{h}_J$ and its orthogonal subspace $\mathfrak{h}_J^\perp = \mathfrak{h}^J$. Note that $-w_k^J(H_k) \in \mathfrak{h}_{I_k}$ and it satisfies the same equations defining H_k. Hence $-w_k^J(H_k) = H_k = -R_k^J(H_k)$. Recall that $\ker(\alpha'_k) = \ker(\alpha_k) \cap (\bigcap_{j\in J} \ker(\alpha_j))$; thus it is fixed by R_k^J and W_k^J. Since $\mathfrak{h}^J = \ker(\alpha'_k) \oplus \mathbb{C}H_k$, the reflection R_k^J coincides with W_k^J on $\mathfrak{h}^J$. □

Remark 2.8. Actually we can now rediscover the list of irreducible $C-$admissible pairs given in Table 1. The black vertex k should be invariant under $-w_k^J$ and the corresponding coefficient of the highest root of I_k should be 1 (an easy consequence of the definition 2.1 1) (ii)).

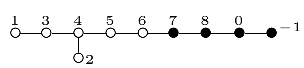
Example 2.9. *Consider the hyperbolic generalized Cartan matrix A of type $HE_8^{(1)} = E_{10}$ indexed by $I = \{-1, 0, 1, ..., 8\}$.*
The following two choices for J define $C-$admissible pairs :
1) $J = \{2, 3, 4, 5\}$.



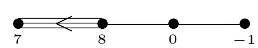
The corresponding generalized Cartan matrix A^J is hyperbolic of type $HF_4^{(1)}$:



2) $J = \{1, 2, 3, 4, 5, 6\}$.



The corresponding generalized Cartan matrix A^J is hyperbolic of type $HG_2^{(1)}$:



Note that the first example corresponds to an almost split real form of the Kac-Moody Lie algebra $\mathfrak{g}(A)$ and A^J is the generalized Cartan matrix associated to the corresponding (reduced) restricted root system (see [6]) whereas the second example does not correspond to an almost split real form of $\mathfrak{g}(A)$.

Lemma 2.10. *For $k \in I'$, set $\mathfrak{s}(k) = \mathbb{C}E_k \oplus \mathbb{C}H_k \oplus \mathbb{C}F_k$. Then, the Kac-Moody algebra $\mathfrak{g}$ is an integrable $\mathfrak{s}(k)-$module via the adjoint representation of $\mathfrak{s}(k)$ on $\mathfrak{g}$.*

Proof. Note that $\mathfrak{s}(k)$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ with standard basis (E_k, H_k, F_k). It is clear that $\mathrm{ad}(H_k)$ is diagonalizable on $\mathfrak{g}$ and $E_k = \sum_\alpha e_\alpha \in d_{k,1}$, where α runs over the set $\Delta_{k,1} = \{\alpha \in \Delta(I_k); \langle \alpha, H_k \rangle = 2\}$, $e_\alpha \in \mathfrak{g}_\alpha$ for $\alpha \in \Delta(I_k)$, and $d_{k,1} := \{X \in \mathfrak{g}(I_k)\,;\ [H_k, X] = 2X\}$. Since $\Delta_{k,1} \subset \Delta^{re}$, $\mathrm{ad}(e_\alpha)$ is locally nilpotent for $\alpha \in \Delta_{k,1}$. As $d_{k,1}$ is commutative (see Remark 2.2) we deduce that $\mathrm{ad}(E_k)$ is locally nilpotent on $\mathfrak{g}$. The same argument shows that $\mathrm{ad}(F_k)$ is also locally nilpotent. Hence, the Kac-Moody algebra $\mathfrak{g}$ is an integrable $\mathfrak{s}(k)-$module. □

Proposition 2.11. *Let $\mathfrak{g}^J$ be the subalgebra of $\mathfrak{g}$ generated by $\mathfrak{h}^J$ and E_k, F_k, $k \in I'$. Then $\mathfrak{g}^J$ is the Kac-Moody Lie algebra associated to the realization $(\mathfrak{h}^J, \Pi^J, \Pi^{J\vee})$ of the generalized Cartan matrix A^J.*

Proof. It is not difficult to check that the following relations hold in the Lie subalgebra $\mathfrak{g}^J$:

$$\begin{array}{lll} [\mathfrak{h}^J, \mathfrak{h}^J] = 0, & [E_k, F_l] = \delta_{k,l} H_k & (k, l \in I'); \\ [h, E_k] = \langle \alpha'_k, h \rangle E_k, & [h, F_k] = -\langle \alpha'_k, h \rangle F_k & (h \in \mathfrak{h}^J, k \in I'). \end{array}$$

We have to prove the Serre's relations :

$$(\mathrm{ad}E_k)^{1-a'_{k,l}}(E_l) = 0, \quad (\mathrm{ad}F_k)^{1-a'_{k,l}}(F_l) = 0 \quad (k \neq l \in I').$$

For $k \in I'$, let $\mathfrak{s}(k) = \mathbb{C}F_k \oplus \mathbb{C}H_k \oplus \mathbb{C}E_k$ be the Lie subalgebra of $\mathfrak{g}$ isomorphic to $\mathfrak{sl}_2(\mathbb{C})$. Let $l \neq k \in I'$; note that $[H_k, F_l] = -a'_{k,l}F_l$ and $[E_k, F_l] = 0$, which means that F_l is a primitive weight vector for $\mathfrak{s}(k)$. As $\mathfrak{g}$ is an integrable $\mathfrak{s}(k)-$module (see Lemma 2.10) the primitive weight vector F_l is contained in a finite dimensional $\mathfrak{s}(k)-$submodule (see [12, 3.6]). The relation $(\mathrm{ad}F_k)^{1-a'_{k,l}}(F_l) = 0$ follows from the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ (see[12, 3.2]). By similar arguments we prove that $(\mathrm{ad}E_k)^{1-a'_{k,l}}(E_l) = 0$.

Now $\mathfrak{g}^J$ is a quotient of the Kac-Moody algebra associated to A^J and $(\mathfrak{h}^J, \Pi^J, \Pi^{J\vee})$. By [12, 1.7] it is equal to it. □

Definition 2.12. The Kac-Moody Lie algebra $\mathfrak{g}^J$ is called the $C-$admissible algebra associated to the $C-$admissible pair (I, J).

Proposition 2.13. *The Kac-Moody algebra $\mathfrak{g}$ is an integrable $\mathfrak{g}^J-$module with finite multiplicities.*

Proof. The $\mathfrak{g}^J-$module $\mathfrak{g}$ is clearly $\mathrm{ad}(\mathfrak{h}^J)-$diagonalizable and $\mathrm{ad}(E_k)$, $\mathrm{ad}(F_k)$ are locally nilpotent on $\mathfrak{g}$ for $k \in I'$ (see Lemma 2.10). Hence, $\mathfrak{g}$ is an integrable $\mathfrak{g}^J-$module. For $\alpha \in \Delta$, let $\alpha' = \alpha_{|\mathfrak{h}^J}$ be the restriction of α to $\mathfrak{h}^J$. Set $\Delta' = \{\alpha';\, \alpha \in \Delta\} \setminus \{0\}$. Then the set of weights, for the $\mathfrak{g}^J-$module $\mathfrak{g}$, is exactly $\Delta' \cup \{0\}$. Note that for $\alpha \in \Delta$, $\alpha' = 0$ if and only if $\alpha \in \Delta(J)$. In particular, the weight space $V_0 = \mathfrak{h} \oplus (\underset{\alpha \in \Delta(J)}{\oplus} \mathfrak{g}_\alpha)$ corresponding to the null weight is finite dimensional. Let $\alpha = \sum\limits_{i \in I} n_i \alpha_i \in \Delta$ such that $\alpha' \neq 0$. We will see that the corresponding weight space $V_{\alpha'}$ is finite dimensional. Note that $V_{\alpha'} = \underset{\beta' = \alpha'}{\oplus} \mathfrak{g}_\beta$. Let $\beta = \sum\limits_{i \in I} m_i \alpha_i \in \Delta$ such that $\beta' = \alpha' = \sum_{k\in I'} n_k\alpha'_k$, then $m_k = n_k$, $\forall k \in I'$, since $\Pi^J = \{\alpha'_k, k \in I'\}$ is free in $(\mathfrak{h}^J)^*$. In particular, β and α are of the same sign, and we may assume $\alpha \in \Delta^+$. Let $ht_J(\beta) = \sum_{j\in J} m_j$ be the height of β relative to J, and let W_J be the finite subgroup of W generated by r_j, $j \in J$. Since W_J fixes pointwise $\mathfrak{h}^J$, we deduce that $\gamma' = \beta'$, $\forall \gamma \in W_J\beta$, and so we may assume that $ht_J(\beta)$ is minimal among the roots in $W_J\beta$. From the inequality $ht_J(\beta) \leq ht_J(r_j(\beta))$, $\forall j \in J$, we get $\langle \beta, \check{\alpha_j}\rangle \leq 0$, $\forall j \in J$. Let $\check{\rho_J}$ be the half sum of positive coroots of $\Delta(J)$. It is known that $\langle \alpha_j, \check{\rho_J}\rangle = 1$, $\forall j \in J$. Note that $\langle \beta, \check{\rho_J}\rangle = \sum_{j\in J} m_j + \sum_{k\in I'} n_k\langle \alpha_k, \check{\rho_J}\rangle = ht_J(\beta) + \sum_{k\in I'} n_k\langle \alpha_k, \check{\rho_J}\rangle$. Hence, the condition $(\langle \beta, \check{\rho_J}\rangle \leq 0)$ implies $(ht_J(\beta) \leq \sum_{k\in I'} -n_k\langle \alpha_k, \check{\rho_J}\rangle)$. Thus there is just a finite number of possibilities for β. It follows that α' is of finite multiplicity. □

Theorem 2.14. *Let Δ^J be the root system of the pair $(\mathfrak{g}^J, \mathfrak{h}^J)$, then the Kac-Moody Lie algebra $\mathfrak{g}$ is finitely Δ^J-graded, with grading subalgebra $\mathfrak{g}^J$.*

Proof. Let $\Delta' = \{\alpha', \alpha \in \Delta\} \setminus \{0\}$ be the set of non-null weights of the $\mathfrak{g}^J-$module $\mathfrak{g}$ relative to $\mathfrak{h}^J$. Let $\Delta'_+ = \{\alpha' \in \Delta', \alpha \in \Delta^+\}$ and Δ^J_+ the set of positive roots of Δ^J relative to the root basis Π^J. We have to prove that $\Delta' = \Delta^J$ or equivalently $\Delta'_+ = \Delta^J_+$. Let $Q^J = \mathbb{Z}\Pi^J$ be the root lattice of Δ^J and $Q^J_+ = \mathbb{Z}^+\Pi^J$. It is known that the positive root system Δ^J_+ is uniquely defined by the following properties (see [12, Ex. 5.4]) :
(i) $\Pi^J \subset \Delta^J_+ \subset Q^J_+$, $2\alpha'_i \notin \Delta^J_+$, $\forall i \in I'$;
(ii) if $\alpha' \in \Delta^J_+$, $\alpha' \neq \alpha'_i$, then the set $\{\alpha' + k\alpha'_i; k \in \mathbb{Z}\} \cap \Delta^J_+$ is a string $\{\alpha' - p\alpha'_i, ..., \alpha' + q\alpha'_i\}$, where $p, q \in \mathbb{Z}^+$ and $p - q = \langle \alpha', H_i\rangle$;
(iii) if $\alpha' \in \Delta^J_+$, then $\mathrm{supp}(\alpha')$ is connected.
We will see that Δ'_+ satisfies these three properties and hence $\Delta'_+ = \Delta^J_+$. Clearly $\Pi^J \subset \Delta'_+ \subset Q^J_+$. For $\alpha \in \Delta$ and $k \in I'$, the condition $\alpha' \in \mathbb{N}\alpha_k$ implies $\alpha \in \Delta(I_k)^+$. As (I_k, J_k) is $C-$admissible for $k \in I'$, the highest root of $\Delta(I_k)^+$ has coefficient 1 on the root α_k (cf. Remark 2.8). It follows that $2\alpha'_k \notin \Delta'_+$ and (i) is satisfied. By Proposition 2.13, $\mathfrak{g}$ is an integrable $\mathfrak{g}^J-$module with finite multiplicities. Hence, the propriety (ii) follows from [12, 3.6]. Let $\alpha \in \Delta_+$, then $\mathrm{supp}(\alpha)$ is connected and $\mathrm{supp}(\alpha') \subset \mathrm{supp}(\alpha)$. Let $k, l \in \mathrm{supp}(\alpha')$; if k, l are $J-$ connected in $\mathrm{supp}(\alpha)$ relative to the generalized Cartan matrix A (cf. 2.3), then by lemma 2.5, k, l are linked in I' relative to the generalized Cartan matrix A^J. Hence, the connectedness of $\mathrm{supp}(\alpha')$, relative to A^J, follows from that of $\mathrm{supp}(\alpha)$ relative to A (see Remark 2.4) and (iii) is satisfied. □

Remark 2.15. Note that the definition of $C-$admissible pair can be extended to decomposable Kac-Moody Lie algebras : thus if $I^1, I^2, ..., I^m$ are the connected components of I and $J^k = J \cap I^k$, $k = 1, 2, ..., m$, then (I, J) is $C-$admissible if and only if (I^k, J^k) is for all $k = 1, 2, ..., m$. In particular, the corresponding $C-$admissible algebra is $\mathfrak{g}^J = \overset{m}{\underset{k=1}{\oplus}} \mathfrak{g}(I^k)^{J^k}$, where $\mathfrak{g}(I^k)^{J^k}$ is the $C-$admissible subalgebra of $\mathfrak{g}(I^k)$ corresponding to the $C-$admissible pair (I^k, J^k), $k = 1, 2, ..., m$.

3. Real gradations.

From now on we suppose that the Kac-Moody Lie algebra $\mathfrak{g}$ is symmetrizable and, starting from 3.5, indecomposable.

Let $\mathfrak{m}$ be a Kac-Moody subalgebra of $\mathfrak{g}$ and let $\mathfrak{a}$ be a Cartan subalgebra of $\mathfrak{m}$. Put $\Sigma = \Delta(\mathfrak{m}, \mathfrak{a})$ the corresponding root system. We assume that $\mathfrak{a} \subset \mathfrak{h}$ and that $\mathfrak{g}$ is finitely $\Sigma-$graded with $\mathfrak{m}$ as grading subalgebra. Thus $\mathfrak{g} = \sum\limits_{\gamma \in \Sigma \cup \{0\}} V_\gamma$, with $V_\gamma = \{x \in \mathfrak{g}\,;\ [a, x] = \langle \gamma, a \rangle x,\ \forall a \in \mathfrak{a}\}$ is finite dimensional for all $\gamma \in \Sigma \cup \{0\}$. For $\alpha \in \Delta$, denote by $\rho_a(\alpha)$ the restriction of α to $\mathfrak{a}$. As $\mathfrak{g}$ is $\Sigma-$graded, one has $\rho_a(\Delta \cup \{0\}) = \Sigma \cup \{0\}$.

Lemma 3.1.
1) Let $\mathfrak{c}$ be the center of $\mathfrak{g}$ and denote by $\mathfrak{c}_a$ the center of $\mathfrak{m}$. Then $\mathfrak{c}_a = \mathfrak{c} \cap \mathfrak{a}$. In particular, if $\mathfrak{g}$ is perfect, then the grading subalgebra $\mathfrak{m}$ is also perfect.
2) Suppose that $\Delta^{im} \neq \emptyset$, then $\rho_a(\Delta^{im}) \subset \Sigma^{im}$.

Proof.
1) It is clear that $\mathfrak{c} \cap \mathfrak{a} \subset \mathfrak{c}_a$. Since $\mathfrak{g}$ is $\Sigma-$ graded, we deduce that $\mathfrak{c}_a$ is contained in the center $\mathfrak{c}$ of $\mathfrak{g}$, hence $\mathfrak{c}_a \subset \mathfrak{c} \cap \mathfrak{a}$. If $\mathfrak{g}$ is perfect, then $\mathfrak{g} = \mathfrak{g}'$, $\mathfrak{h} = \mathfrak{h}'$, $\mathfrak{c} = \{0\}$; so $\mathfrak{c}_a = \{0\}$, $\mathfrak{a} = \mathfrak{a}'$ and $\mathfrak{m} = \mathfrak{m}'$.
2) If $\alpha \in \Delta^{im}$, then $\mathbb{N}\alpha \subset \Delta$. Since V_0 is finite dimensional, $\rho_a(\alpha) \neq 0$ and $\mathbb{N}\rho_a(\alpha) \subset \Sigma$, hence $\rho_a(\alpha) \in \Sigma^{im}$. $\square$

Definition 3.2. ([3, 5.2.6]) Suppose that $\Delta^{im} \neq \emptyset$. Let $\alpha, \beta \in \Delta^{im}$.
(i) The imaginary roots α and β are said to be linked if $\mathbb{N}\alpha + \mathbb{N}\beta \subset \Delta$ or $\beta \in \mathbb{Q}^+\alpha$.
(ii) The imaginary roots α and β are said to be linkable if there exists a finite family of imaginary roots $(\beta_i)_{0 \leq i \leq n+1}$ such that $\beta_0 = \alpha$, $\beta_{n+1} = \beta$ and β_i and β_{i+1} are linked for all $i = 0, 1, ..., n$.

Proposition 3.3. *(*[3, 5.2.7]*) Suppose that $\Delta^{im} \neq \emptyset$. Let $\Delta = \overset{m}{\underset{j=1}{\cup}} \Delta_j$ be the decomposition of Δ in indecomposable root systems. Suppose that Δ_1, Δ_2, ..., Δ_r ($r \leq m$) are the indecomposable root subsystems of Δ which are not of finite type. Then to be linkable is an equivalence relation on Δ^{im} and the equivalence classes are the $2r$ sets $\Delta^{im}_{\pm} \cap \Delta_j$, $j = 1, 2, ..., r$.*

Lemma 3.4. *Suppose that $\Delta^{im} \neq \emptyset$, then there exist root bases in Σ and Δ such that $\rho_a(\Delta^{im}_+) \subset \Sigma^{im}_+$.*

Proof. Fix a root basis Π_a for the grading root system Σ. Let $\Delta = \overset{m}{\underset{j=1}{\cup}} \Delta_j$ be, as above, the decomposition of Δ in indecomposable root systems. Denote by $\Pi_j := \Pi \cap \Delta_j$ the root basis of Δ_j, $j = 1, 2, ..., m$. If α, β are two imaginary linkable roots of Δ^{im}_j, then $\rho_a(\alpha)$ and $\rho_a(\beta)$ are also linkable in Σ^{im}. By Proposition 3.3, $\rho_a(\alpha)$ and $\rho_a(\beta)$ are of the same sign. Since α and β are of the same sign in Δ^{im}_j relative to the root basis Π_j, one can, if necessary, change the sign of Π_j so that $\rho_a(\alpha)$ and $\rho_a(\beta)$ are positive imaginary roots of Σ^+ relative to the fixed root basis Π_a. Hence we get a root basis of $\Delta = \overset{m}{\underset{j=1}{\cup}} \Delta_j$ satisfying $\rho_a(\Delta^{im}_+) \subset \Sigma^{im}_+$. $\square$

In the following, we will show that the indecomposable Kac-Moody Lie algebra $\mathfrak{g}$ and the grading subalgebra $\mathfrak{m}$ are of the same type.

Lemma 3.5. *The Kac-Moody Lie algebra* $\mathfrak{g}$ *is of indefinite type if and only if* Δ^{im} *generates the dual space* $(\mathfrak{h}/\mathfrak{c})^*$ *of* $\mathfrak{h}/\mathfrak{c}$.

Proof. Note that the root basis $\Pi = \{\alpha_i, i \in I\}$ induces a basis for the quotient vector space $(\mathfrak{h}/\mathfrak{c})^*$. It follows that the condition $(\Delta^{im} \neq \emptyset)$ implies $(\dim(\mathfrak{h}/\mathfrak{c})^* \geq 2)$. Suppose now that $\mathfrak{g}$ is of indefinite type. Let $\alpha \in \Delta_+^{sim}$ be a positive strictly imaginary root satisfying $\langle \alpha, \check{\alpha_i} \rangle < 0$, $\forall i \in I$; then, $r_i(\alpha) = \alpha - \langle \alpha, \check{\alpha_i} \rangle \alpha_i \in \Delta_+^{im}$ for all $i \in I$. In particular, the vector subspace $\langle \Delta^{im} \rangle$ spanned by Δ^{im} contains Π and hence is equal to $(\mathfrak{h}/\mathfrak{c})^*$. Conversely, if Δ^{im} generates $(\mathfrak{h}/\mathfrak{c})^*$, then Δ^{im} is non-empty and contains at least two linearly independent imaginary roots; hence Δ can not be of finite or affine type. $\square$

Proposition 3.6. *1) If* Δ^{im} *is not empty, then* $\mathfrak{m}$ *is indecomposable.*
2) The Kac-Moody Lie Algebra $\mathfrak{g}$ *and the grading subalgebra* $\mathfrak{m}$ *are of the same type.*
3) Suppose $\mathfrak{g}$ *Lorentzian, then* $\mathfrak{m}$ *is also Lorentzian.*

N.B. We will see below that $\mathfrak{m}$ is always indecomposable (3.11) and symmetrizable (3.17).

Proof. 1) We saw in Lemma 3.4 that $\rho_a(\Delta_+^{im})$ is in a unique linkable equivalence class of Σ_+^{im}. So, if $\Sigma = \Sigma_1 \cup \Sigma_2$ is decomposable, we may assume $\rho_a(\Delta_+^{im}) \subset \Sigma_1^{im}$. But there is $\delta \in \Delta_+^{im}$ such that $\alpha + n\delta \in \Delta_+$ for all $\alpha \in \Delta_+$ and $n \in \mathbb{N}$ [12, 4.3, 5.6 and 6.3]. So $\rho_a(\alpha) + n\rho_a(\delta) \in \Sigma$ for $n >> 0$ and $\rho_a(\alpha) \in \Sigma_1 \cup \{0\}$. As $\rho_a(\Delta \cup \{0\}) = \Sigma \cup \{0\}$, we have $\Sigma_2 = \emptyset$.

2) If $\mathfrak{g}$ is of finite type, then Δ is finite and hence $\Sigma = \rho_a(\Delta) \setminus \{0\}$ is finite.
If $\mathfrak{g}$ is affine, let δ be the lowest positive imaginary root. One can choose a root basis $\Pi_a = \{\gamma_i, i \in \bar{I}\}$ of Σ so that $\bar{\delta} := \rho_a(\delta)$ is a positive imaginary root. Note that $\mathfrak{a}' := \mathfrak{a} \cap \mathfrak{m}' \subset \mathfrak{h}'$; in particular $\bar{\delta}(\mathfrak{a}') = \{0\}$ and $\langle \bar{\delta}, \check{\gamma_i} \rangle = 0$, $\forall i \in \bar{I}$. It follows that $\mathfrak{m}$ is affine (see [12, 4.3]).
Suppose now that $\mathfrak{g}$ is of indefinite type. Thanks to Lemma 3.5, it suffices to prove that Σ^{im} generates $(\mathfrak{a}/\mathfrak{c}_a)^*$, where $\mathfrak{c}_a = \mathfrak{c} \cap \mathfrak{a}$ is the center of $\mathfrak{m}$. The natural homomorphism of vector spaces $\pi : \mathfrak{a} \to \mathfrak{h}/\mathfrak{c}$ induces a monomorphism $\bar{\pi} : \mathfrak{a}/\mathfrak{c}_a \to \mathfrak{h}/\mathfrak{c}$. By duality, the homomorphism $\bar{\pi}^* : (\mathfrak{h}/\mathfrak{c})^* \to (\mathfrak{a}/\mathfrak{c}_a)^*$ is surjective and $\bar{\pi}^*(\Delta^{im}) \subset \Sigma^{im}$ generates $(\mathfrak{a}/\mathfrak{c}_a)^*$.
3) Suppose that $\mathfrak{g}$ is Lorentzian (hence of indefinite type) and let $(.\,,.)$ be an invariant non-degenerate bilinear form on $\mathfrak{g}$. Then, the restriction of $(.\,,.)$ to $\mathfrak{h}_{\mathbb{R}}$ has signature $(+ + \ldots.+, -)$ and any maximal totally isotropic subspace of $\mathfrak{h}_{\mathbb{R}}$ relatively to $(.\,,.)$ is one dimensional. Let $\mathfrak{a}_{\mathbb{R}} := \mathfrak{a} \cap \mathfrak{h}_{\mathbb{R}}$ and let $(.\,,.)_a$ be the restriction of $(.\,,.)$ to $\mathfrak{m}$. As $\mathfrak{m}$ is of indefinite type, dim $(\mathfrak{a}) \geq 2$ and the restriction of $(.\,,.)_a$ to $\mathfrak{a}_{\mathbb{R}}$ is non-null. It follows that the orthogonal subspace $\mathfrak{m}^{\perp}$ of $\mathfrak{m}$ relatively to $(.\,,.)_a$ is a proper ideal of $\mathfrak{m}$. Since $\mathfrak{m}$ is perfect (because $\mathfrak{g}$ is) we deduce that $\mathfrak{m}^{\perp} = \{0\}$ (cf. [12, 1.7]) and the invariant bilinear form $(.\,,.)_a$ is non-degenerate. It follows that $\mathfrak{m}$ is symmetrizable and the bilinear form $(.\,,.)_a$ when restricted to $\mathfrak{a}_{\mathbb{R}}$ is non-degenerate; since $\mathfrak{m}$ is of indefinite type, it can not be positive definite. Hence, the bilinear form $(.\,,.)_a$ has signature $(+ + \ldots.+, -)$ on $\mathfrak{a}_{\mathbb{R}}$ and then the grading subalgebra $\mathfrak{m}$ is Lorentzian. $\square$

Definition 3.7. Let Π_a be a root basis of Σ and let Σ^+ be the corresponding set of positive roots. The root basis is said to be adapted to the root basis Π of Δ if $\rho_a(\Delta^+) \subset \Sigma^+ \cup \{0\}$.

We will see (3.10) that adapted root bases always exist.

Lemma 3.8. *Let Π_a be a root basis of Σ such that $\rho_a(\Delta_+^{im}) \subset \Sigma_+^{im}$ and let X_a be the corresponding positive Tits cone. Then we have $\bar{X}_a \subset \bar{X} \cap \mathfrak{a}$.*

Proof. As $\Delta^{im} \neq \emptyset$, one has $\bar{X} = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \alpha, h\rangle \geq 0, \forall \alpha \in \Delta_+^{im}\}$ (see Remark 1.3). The lemma follows from Lemma 3.4. □

Lemma 3.9. *Suppose that $\Delta^{im} \neq \emptyset$. Let $p \in \bar{X}$ such that $\langle \alpha, p\rangle \in \mathbb{Z}$, $\forall \alpha \in \Delta$, and $\langle \beta, p\rangle > 0$, $\forall \beta \in \Delta_+^{im}$. Then $p \in \overset{\circ}{X}$.*

Proof. The result is clear when Δ is of affine type since $\overset{\circ}{X} = \overset{\circ}{\bar{X}} = \{h \in \mathfrak{h}_{\mathbb{R}}; \langle \delta, h\rangle > 0\}$. Suppose now that Δ is of indefinite type. If one looks to the proof of Proposition 5.8.c) in [12], one can show that an element $p \in \bar{X}$ satisfying the conditions of the lemma lies in X. As Δ_+^{im} is $W-$invariant, we may assume that p lies in the fundamental chamber C. Hence there exists a subset J of I such that $\{\alpha \in \Delta;\ \langle \alpha, p\rangle = 0\} = \Delta_J = \Delta \cap \sum_{j\in J} \mathbb{Z}\alpha_j$. Since $\Delta_J \cap \Delta^{im} = \emptyset$, the root subsystem Δ_J is of finite type and p lies in the finite type facet of type J. Thus $p \in \overset{\circ}{X}$ (see 1.5). □

Theorem 3.10. *There exists a root basis Π_a of Σ which is adapted to the root basis Π of Δ. Moreover, there exists a finite type subset J of I such that $\Delta_J = \{\alpha \in \Delta;\ \rho_a(\alpha) = 0\}$.*

N.B. This is part 1) of Theorem 2.

Proof. Let $\Pi_a = \{\gamma_i, i \in \bar{I}\}$ be a root basis of Σ such that $\rho_a(\Delta_+^{im}) \subset \Sigma_+^{im}$, where $\bar{I}$ is just a set indexing the basis elements. Let $p \in \mathfrak{a}$ such that $\langle \gamma_i, p\rangle = 1$, $\forall i \in \bar{I}$ and let $P = \{\alpha \in \Delta; \langle \alpha, p\rangle \geq 0\}$. If Δ is finite, then P is clearly a parabolic subsystem of Δ and the result is trivial. Suppose now that $\Delta^{im} \neq \emptyset$; then p satisfies the conditions of the Lemma 3.9 and we may assume that p lies in the facet of type J for some subset J of finite type in I. In which case $P = \Delta_J \cup \Delta^+$ is the standard parabolic subsystem of finite type J. Note that, for $\gamma \in \Sigma^+$, one has $\langle \gamma, p\rangle = ht_a(\gamma)$ the height of γ with respect to Π_a. It follows that $\{\alpha \in \Delta; \rho_a(\alpha) = 0\} = \Delta_J$, in particular, $\rho_a(\Delta^+) = \rho_a(P) \subset \Sigma^+ \cup \{0\}$. Hence, the root basis Π_a is adapted to Π. □

Corollary 3.11. Σ *is indecomposable.*

Proof. For $\gamma_1, \gamma_2 \in \Pi_a$, there are $\alpha_1, \alpha_2 \in \Delta_+$ such that $\gamma_i = \rho_a(\alpha_i)$. But γ_i is not a sum in Σ_+, so, up to Δ_J, α_i is not a sum: we may assume $\alpha_i \in \Pi$. As Δ is indecomposable, there is a root $\alpha \in \Delta \cap (\alpha_1 + \alpha_2 + \sum_{\alpha\in\Pi} \mathbb{Z}^+\alpha)$. Now $\rho_a(\alpha) \in (\Sigma \cup \{0\}) \cap (\gamma_1 + \gamma_2 + \sum_{\gamma\in\Pi_a} \mathbb{Z}^+\gamma) \subset \Sigma$ and γ_1, γ_2 have to be in the same connected component of Π_a. □

From now on, we fix a root basis $\Pi_a = \{\gamma_s, s \in \bar{I}\}$, for the grading root system Σ, which is adapted to the root basis $\Pi = \{\alpha_i, i \in I\}$ of Δ (see Theorem 3.10). As before, let $J := \{j \in I\,;\, \rho_a(\alpha_j) = 0\}$ and $I' := I \setminus J$. For $k \in I'$, we denote, as above, by I_k the connected component of $J \cup \{k\}$ containing k, and $J_k := J \cap I_k$.

Proposition 3.12.
1) Let $s \in \bar{I}$, then there exists $k_s \in I'$ such that $\rho_a(\alpha_{k_s}) = \gamma_s$ and any preimage $\alpha \in \Delta$ of γ_s is equal to α_k modulo $\sum_{j\in J_k} \mathbb{Z}\alpha_j$ for some $k \in I'$ satisfying $\rho_a(\alpha_k) = \gamma_s$.

2) Let $k \in I'$ such that $\rho_a(\alpha_k)$ is a real root of Σ. Then $\rho_a(\alpha_k) \in \Pi_a$ is a simple root.

Proof. This result was proved by J. Nervi for affine algebras (see [17, 2.3.10] and the proof of Prop. 2.3.12). The arguments used there are available for general Kac-Moody algebras. □

We introduce the following notations :

$$I'_{re} := \{i \in I' \,;\, \rho_a(\alpha_i) \in \Pi_a\} \,;\;\; I'_{im} := I' \setminus I'_{re},$$
$$I_{re} = \underset{k \in I'_{re}}{\cup} I_k \,;\;\; J_{re} = I_{re} \cap J = \underset{k \in I'_{re}}{\cup} J_k \,;\;\; J^\circ = J \setminus J_{re}$$
$$\Gamma_s := \{i \in I' \,;\, \rho_a(\alpha_i) = \gamma_s\}\,, \forall s \in \bar{I}.$$

Note that J° is not connected to I_{re}.

Remark 3.13.
1) In view of Proposition 3.12, assertion 2), one has $\rho_a(\alpha_k) \in \Sigma_+^{im}$, $\forall k \in I'_{im}$.
2) $I = I_{re} \cup I'_{im} \cup J^\circ$ is a disjoint union.
3) If $I'_{im} = \emptyset$, then $I = I_{re} \cup J^\circ$. Since I is connected (and I_{re} is not connected to J°) we deduce that $J^\circ = \emptyset$, $I = I_{re}$ and $I'_{re} = I' = I \setminus J$.
4) If $I'_{im} \neq \emptyset$, then I_{re} may be non-connected (see the example in §5 below).

Proposition 3.14.
1) Let $k \in I'_{re}$, then I_k is of finite type.
2) Let $s \in \bar{I}$. If $|\Gamma_s| \geq 2$ and $k \neq l \in \Gamma_s$, then $I_k \cup I_l$ is not connected: $\mathfrak{g}(I_k)$ and $\mathfrak{g}(I_l)$ commute and are orthogonal.
3) For all $k \in I'_{re}$, (I_k, J_k) is an irreducible $C-$admissible pair.
4) The derived subalgebra $\mathfrak{m}'$ of the grading algebra $\mathfrak{m}$ is contained in $\mathfrak{g}'(I_{re})$ (as defined in proposition 1.2).

Proof.
1) Suppose that there exists $k \in I'_{re}$ such that I_k is not of finite type; then there exists an imaginary root β_k whose support is the whole I_k. Hence, there exists a positive integer $m_k \in \mathbb{N}$ such that $\rho_a(\beta_k) = m_k\rho(\alpha_k)$ is an imaginary root of Σ. It follows that $\rho_a(\alpha_k)$ is an imaginary root and this contradicts the fact that $k \in I'_{re}$.
2) Let $s \in \bar{I}$ such that $|\Gamma_s| \geq 2$ and let $k \neq l \in \Gamma_s$. Since $V_{n\gamma_s} = \{0\}$ for all integer $n \geq 2$, the same argument used in 1) shows that $I_k \cup I_l$ is not connected, and I_k and I_l are its two connected components. In particular, $[\mathfrak{g}(I_k), \mathfrak{g}(I_l)] = \{0\}$ and $(\mathfrak{g}(I_k), \mathfrak{g}(I_l)) = \{0\}$.
3) Let $k \in I'_{re}$ and let $s \in \bar{I}$ such that $\rho_a(\alpha_k) = \gamma_s$. Let $(\bar{X}_s,\ \bar{H}_s = \gamma_s^\vee,\ \bar{Y}_s)$ be an $\mathfrak{sl}_2-$triple in $\mathfrak{m}$ corresponding to the simple root γ_s. Let V_{γ_s} be the weight space of $\mathfrak{g}$ corresponding to γ_s. In view of Proposition 3.12, assertion 1), one has :

$$V_{\gamma_s} = \underset{l \in \Gamma_s}{\oplus} V_{\gamma_s} \cap \mathfrak{g}(I_l). \tag{3.1}$$

Hence, one can write :

$$\bar{X}_s = \sum_{l \in \Gamma_s} E_l \,;\quad \bar{Y}_s = \sum_{l \in \Gamma_s} F_l\,, \tag{3.2}$$

with $E_l \in V_{\gamma_s} \cap \mathfrak{g}(I_l)$ and $F_l \in V_{-\gamma_s} \cap \mathfrak{g}(I_l)$. It follows from assertion 2) that

$$\bar{H}_s = \gamma_s^\vee = [\bar{X}_s, \bar{Y}_s] = \sum_{l \in \Gamma_s} [E_l, F_l] = \sum_{l \in \Gamma_s} H_l, \tag{3.3}$$

where $H_l := [E_l, F_l] \in \mathfrak{h}_{I_l}$, $\forall l \in \Gamma_s$. Then one has, for $k \in \Gamma_s$,

$$2 = \langle \gamma_s, \check{\gamma_s} \rangle = \langle \alpha_k, \check{\gamma_s} \rangle = \sum_{l \in \Gamma_s} \langle \alpha_k, H_l \rangle = \langle \alpha_k, H_k \rangle ,$$

and for $j \in J_k$,

$$0 = \langle \alpha_j, \check{\gamma_s} \rangle = \sum_{l \in \Gamma_s} \langle \alpha_j, H_l \rangle = \langle \alpha_j, H_k \rangle.$$

In particular, H_k is the unique semi-simple element of $\mathfrak{h}_{I_k}$ satisfying :

$$\langle \alpha_i, H_k \rangle = 2\delta_{i,k}, \forall i \in I_k. \tag{3.4}$$

Hence, (E_k, H_k, F_k) is an $\mathfrak{sl}_2-$triple in the simple Lie algebra $\mathfrak{g}(I_k)$ and since $V_{2\gamma_s} = \{0\}$, (I_k, J_k) is an irreducible $C-$admissible pair for all $k \in \Gamma_s$. The statement 4) follows from the relation (3.2). □

Corollary 3.15. *The pair (I_{re}, J_{re}) is $C-$admissible (in the eventually decomposable sense of Remark 2.15). If $I'_{im} = \emptyset$, then $I_{re} = I$, $J_{re} = J$ and $\mathfrak{g}$ is finitely Δ^J-graded, with grading subalgebra $\mathfrak{g}^J$.*

N.B. We have got part 2) of Theorem 2.

Proof. The first assertion is a consequence of Proposition 3.14. By remark 3.13, when $I'_{im} = \emptyset$, we have $I = I_{re}$; hence, by Theorem 2.14, $\mathfrak{g}$ is finitely Δ^J-graded. □

Definition 3.16. If $I'_{im} \neq \emptyset$, then (I, J) is called a generalized $C-$admissible pair and the gradation of $\mathfrak{g}$ by Σ and $\mathfrak{m}$ is said imaginary.
On the contrary if $I'_{im} = \emptyset$, the gradation is said real.
If $I'_{im} = J = \emptyset$, the Kac-Moody algebra $\mathfrak{g}$ is said to be maximally finitely $\Sigma-$graded.

Corollary 3.17. *The grading subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ is symmetrizable and the restriction to $\mathfrak{m}$ of the invariant bilinear form of $\mathfrak{g}$ is non-degenerate.*

Proof. Let $(.\,,.)_a$ be the restriction to $\mathfrak{m}$ of the invariant bilinear form $(.\,,.)$ of $\mathfrak{g}$. Recall from the proof of Proposition 3.14 that $\check{\gamma_s} = \sum_{k \in \Gamma_s} H_k, \forall s \in \bar{I}$. In particular $(\check{\gamma_s}, \check{\gamma_s})_a = \sum_{k \in \Gamma_s} (H_k, H_k) > 0$. It follows that $(.\,,.)_a$ is a non-degenerate invariant bilinear form on $\mathfrak{m}$ (see §1.4) and that $\mathfrak{m}$ is symmetrizable. □

Corollary 3.18. *Let $\mathfrak{h}^J$ be the orthogonal of $\mathfrak{h}_J$ in $\mathfrak{h}$. For $k \in I'_{im}$, write*

$$\rho_a(\alpha_k) = \sum_{s \in \bar{I}} n_{s,k} \gamma_s.$$

For $s \in \bar{I}$, choose l_s a representative element of Γ_s. Then $\mathfrak{a}/\mathfrak{c}_a$ can be viewed as the subspace of $\mathfrak{h}^J/\mathfrak{c}$ defined by the following relations :

$$\langle \alpha_k, h \rangle = \langle \alpha_{l_s}, h \rangle, \forall k \in \Gamma_s, \forall s \in \bar{I}$$

$$\langle \alpha_k, h \rangle = \sum_{s \in \bar{I}} n_{s,k} \langle \alpha_{l_s}, h \rangle, \forall k \in I'_{im}.$$

Proof. The subspace of $\mathfrak{h}^J/\mathfrak{c}$ defined by the above relations has dimension $|\bar{I}|$ and contains $\mathfrak{a}/\mathfrak{c}_a$ and hence it is equal to $\mathfrak{a}/\mathfrak{c}_a$. □

Proposition 3.19. *Let $(.,.)_a$ be the restriction to $\mathfrak{m}$ of the invariant bilinear form $(.,.)$ of $\mathfrak{g}$.*
1) Let $\mathfrak{a}' = \mathfrak{a} \cap \mathfrak{m}'$ and let $\mathfrak{a}''$ be a supplementary subspace of $\mathfrak{a}'$ in $\mathfrak{a}$ which is totally isotropic relatively to $(.,.)_a$. Then $\mathfrak{a}'' \cap \mathfrak{h}' = \{0\}$.
2) Let $A_{I_{re}}$ be the submatrix of A indexed by I_{re}. Then there exists a subspace $\mathfrak{h}_{I_{re}}$ of $\mathfrak{h}$ containing $\mathfrak{a}$ such that $(\mathfrak{h}_{I_{re}}, \Pi_{I_{re}}, \Pi_{I_{re}}^{\check{}})$ is a realization of $A_{I_{re}}$. In particular, the Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ associated to this realization (in 1.2) contains the grading subalgebra $\mathfrak{m}$.
3) The Kac-Moody algebra $\mathfrak{g}(I_{re})$ is finitely $\Delta(I_{re})^{J_{re}}-$graded and its grading subalgebra is the subalgebra $\mathfrak{g}(I_{re})^{J_{re}}$ associated to the $C-$admissible pair (I_{re}, J_{re}) as in Proposition 2.11.
4) The Kac-Moody algebra $\mathfrak{g}(I_{re})^{J_{re}}$ contains $\mathfrak{m}$.

Proof.
1) Recall that the center $\mathfrak{c}_a$ of $\mathfrak{m}$ is contained in the center $\mathfrak{c}$ of $\mathfrak{g}$. Since $\mathfrak{h}' = \mathfrak{c}^\perp$ and $\mathfrak{c}_a$ is in duality with $\mathfrak{a}''$ relatively to $(.,.)_a$, we deduce that $\mathfrak{a}'' \cap \mathfrak{h}' = \{0\}$.
2) From the proofs of 3.17 and 3.14 we get $\gamma_s^\vee = \sum_{k\in\Gamma_s} H_k \in \sum_{k\in\Gamma_s} \mathfrak{h}_{I_k} = \mathfrak{h}'_{I_{re}}$. So $\mathfrak{c}_a \subset \mathfrak{a}' \subset \mathfrak{h}'_{I_{re}} \subset \mathfrak{h}'$. It follows that $(\mathfrak{h}'_{I_{re}} + \mathfrak{h}^{I_{re}})$ is contained in $\mathfrak{c}_a^\perp$ the orthogonal subspace of $\mathfrak{c}_a$ in $\mathfrak{h}$. Since $\mathfrak{a}'' \cap \mathfrak{c}_a^\perp = \{0\}$, one can choose a supplementary subspace $\mathfrak{h}''_{I_{re}}$ of $(\mathfrak{h}'_{I_{re}} + \mathfrak{h}^{I_{re}})$ containing $\mathfrak{a}''$. Let $\mathfrak{h}_{I_{re}} = \mathfrak{h}'_{I_{re}} \oplus \mathfrak{h}''_{I_{re}}$, then, by Proposition 1.2, $(\mathfrak{h}_{I_{re}}, \Pi_{I_{re}}, \Pi_{I_{re}}^{\check{}})$ is a realization of $A_{I_{re}}$.
3) As in Corollary 3.15, assertion 3) is a simple consequence of Theorem 2.14.
4) The algebra $\mathfrak{a}$ is in $\mathfrak{h}_{I_{re}} \cap \Pi_J^\perp = (\mathfrak{h}_{I_{re}})^{J_{re}}$. By the proof of Proposition 3.14, for $s \in \overline{I}$, $\overline{X}_s$ and $\overline{Y}_s$ are linear combinations of the elements in $\{E_k, F_k \mid k \in \Gamma_s\} \subset \mathfrak{g}(I_{re})^{J_{re}}$. Hence $\mathfrak{g}(I_{re})^{J_{re}}$ contains all generators of $\mathfrak{m}$. □

Lemma 3.20. *Let $\mathfrak{l}$ be a Kac-Moody subalgebra of $\mathfrak{g}$ containing $\mathfrak{m}$. Then $\mathfrak{l}$ is finitely $\Sigma-$graded. In particular, the Kac-Moody subalgebra $\mathfrak{g}(I_{re})$ or $\mathfrak{g}(I_{re})^{J_{re}}$ is finitely $\Sigma-$graded.*

N.B. Proposition 3.19 and Lemma 3.20 finish the proof of Theorem 2.

Proof. Recall that the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{m}$ is $\mathrm{ad}_{\mathfrak{g}}-$diagonalizable. Since $\mathfrak{l}$ is $\mathrm{ad}(\mathfrak{a})-$invariant, one has $\mathfrak{l} = \sum\limits_{\gamma\in\Sigma\cup\{0\}} V_\gamma \cap \mathfrak{l}$. By assumption $\{0\} \neq \mathfrak{m}_\gamma \subset V_\gamma \cap \mathfrak{l}$ for all $\gamma \in \Sigma$. Thus, $\mathfrak{l}$ is finitely $\Sigma-$graded. □

Proposition 3.21. *If $I'_{im} = \emptyset$, then $\mathfrak{g}(I_{re}) = \mathfrak{g}$ and the $C-$admissible subalgebra $\mathfrak{g}^J$ is maximally finitely $\Sigma-$graded, with grading subalgebra $\mathfrak{m}$.*

Proof. This result is due to J. Nervi ([17, 2.5.10]) for the affine case; it follows from the facts that $V_0 \cap \mathfrak{g}^J = \mathfrak{h}^J$ and $\mathfrak{m} \subset \mathfrak{g}^J$ (see Prop. 3.19). □

We now want a precise description of the gradation of $\mathfrak{g}(I_{re})$ by Σ and $\mathfrak{m}$; particularly in the case (already mentioned in Remark 3.13) where $\mathfrak{g}(I_{re})$ (and so $\mathfrak{g}(I_{re})^{J_{re}}$) is decomposable.

Let I_{re}^1, I_{re}^2, ..., I_{re}^q be the connected components of I_{re} and $J_{re}^i := J_{re} \cap I_{re}^i$, $i = 1, 2, ..., q$. Then $\mathfrak{g}(I_{re}) = \bigoplus\limits_{i=1}^{q} \mathfrak{g}(I_{re}^i)$ and hence $\mathfrak{g}(I_{re})^{J_{re}} = \bigoplus\limits_{i=1}^{q} \mathfrak{g}(I_{re}^i)^{J_{re}^i}$ (see Remark 2.15). Retain the notations introduced just before Proposition 3.14 and

those introduced in its proof.
For $s \in \bar{I}$ and $i = 1, 2, ..., q$, let $\Gamma_s^i := \Gamma_s \cap I_{re}^i$. If Γ_s^i is non-empty, put $E_s^i := \sum_{l \in \Gamma_s^i} E_l$; $F_s^i := \sum_{l \in \Gamma_s^i} F_l$ and $H_s^i := \sum_{l \in \Gamma_s^i} H_l$. We take $E_s^i = F_s^i = H_s^i = 0$ if Γ_s^i is empty. Note that $\Gamma_s = \overset{q}{\underset{i=1}{\cup}} \Gamma_s^i$ (disjoint union) and from the proof of the Proposition 3.14 we get the following relations

$$\bar{X}_s = \sum_{i=1}^{q} E_s^i \,; \quad \bar{Y}_s = \sum_{i=1}^{q} F_s^i \,, \forall s \in \bar{I}\,, \tag{3.5}$$

$$\bar{H}_s = \gamma_s^{\vee} = [\bar{X}_s, \bar{Y}_s] = \sum_{i=1}^{q} [E_s^i, F_s^i] = \sum_{i=1}^{q} H_s^i, \forall s \in \bar{I}. \tag{3.6}$$

Lemma 3.22. *Let $s \in \bar{I}$ and $i \in \{1, 2, ..., q\}$ such that $\Gamma_s^i \neq \emptyset$. Then we have*
1) $\Gamma_t^i \neq \emptyset$ for all $t \in \bar{I}$ satisfying $\langle \gamma_t, \gamma_s^{\vee} \rangle < 0$.
2) $\Gamma_t^i \neq \emptyset$, $\forall t \in \bar{I}$.

Proof. To prove 1), suppose $\Gamma_t^i = \emptyset$ for any t satisfying $\langle \gamma_t, \gamma_s^{\vee} \rangle < 0$. Let $k \in \Gamma_s^i$, then $\langle \gamma_s, \gamma_t^{\vee} \rangle = \sum_{\substack{j=1 \\ j \neq i}}^{q} \langle \alpha_k, H_t^j \rangle = 0$, a contradiction since $\langle \gamma_s, \gamma_t^{\vee} \rangle$ must be negative. Thus $\Gamma_s^i \neq \emptyset$ iff $\Gamma_t^i \neq \emptyset$. The second statement follows from the connectedness of $\bar{I}$: For $t \in \bar{I}$, there exists a sequence $s_0 = s, s_1, ..., s_n = t$ in $\bar{I}$ such that s_j is linked to s_{j+1} for all $j = 0, 1, ..., n-1$. By 1) $\Gamma_{s_j}^i$ is, as Γ_s^i, non-empty for all $j = 0, 1, ..., n$. In particular $\Gamma_t^i \neq \emptyset$. □

Lemma 3.23. *$\Gamma_s^i \neq \emptyset$, $\forall s \in \bar{I}$, $\forall i = 1, 2, ..., q$, and $(H_s^i)_{s \in \bar{I}}$ is free for all $i = 1, 2, ..., q$.*

Proof. Recall that $I_{re} = \underset{k \in I'_{re}}{\cup} I_k$, with all the I_k connected. Let $i \in \{1, 2, ..., q\}$ and let $k \in I'_{re}$ such that $I_k \subset I_{re}^i$. Let $s \in \bar{I}$ such that $\rho_a(\alpha_k) = \gamma_s$, then $k \in \Gamma_s^i$ and $\Gamma_s^i \neq \emptyset$. By the Lemma 3.22, $\Gamma_t^i \neq \emptyset$ for all $t \in \bar{I}$. Thus $H_s^i \neq 0$, $\forall s \in \bar{I}$; $\forall i = 1, 2, ..., q$, and the freeness of $(H_s^i)_{s \in \bar{I}}$ follows from that of $(H_k)_{k \in \ I'_{re}}$. □

Proposition 3.24. *For $i = 1, 2, ..., q$, let p_i be the projection of $\mathfrak{g}(I_{re})$ on $\mathfrak{g}(I_{re}^i)$ with kernel $\oplus_{j \neq i} \mathfrak{g}(I_{re}^j)$ and let $\mathfrak{m}_i := p_i(\mathfrak{m})$. Then we have :*
1) $\mathfrak{m}_i$ is a Kac-Moody subalgebra of $\mathfrak{g}(I_{re}^i)^{J_{re}^i}$ isomorphic to $\mathfrak{m}$.
2) The Kac-Moody subalgebra $\mathfrak{g}(I_{re}^i)^{J_{re}^i}$ is maximally finitely Σ_i-graded, where Σ_i is the root system of $\mathfrak{m}_i$ relative to the Cartan subalgebra $\mathfrak{a}_i := p_i(\mathfrak{a})$.

N.B. Note that $\mathfrak{m}$ is contained in $\overset{q}{\underset{i=1}{\oplus}} \mathfrak{m}_i$. In particular, $\overset{q}{\underset{i=1}{\oplus}} \mathfrak{m}_i$ is finitely $\Sigma-$graded. If we identify $\overset{q}{\underset{i=1}{\oplus}} \mathfrak{m}_i$ with $\mathfrak{m}^q$, then the grading subalgebra $\mathfrak{m}$ can be viewed as the diagonal subalgebra $\Delta(\mathfrak{m}^q)$ of $\mathfrak{m}^q$: $\Delta(\mathfrak{m}^q) := \{(X, X, ..., X)\,; X \in \mathfrak{m}\}$.

Proof. For $i \in \{1, 2, ..., q\}$, p_i is a morphism of Lie algebras and $\mathfrak{m}_i := p_i(\mathfrak{m})$ is contained in $\mathfrak{g}(I_{re}^i)^{J_{re}^i}$. For $s \in \bar{I}$, one has $p_i(\gamma_s^{\vee}) = H_s^i$. Thus the restriction of p_i to $\mathfrak{a}' := [\mathfrak{a}, \mathfrak{a}] = \oplus_{s \in \bar{I}} \mathbb{C}\gamma_s^{\vee}$ is injective by Lemma 3.23. Since $\mathfrak{m}$ is indecomposable, p_i

when restricted to $\mathfrak{m}$ is still injective (see [12, 1.7]). Thus $\mathfrak{m}_i = p_i(\mathfrak{m})$ is isomorphic to $\mathfrak{m}$ and we have the following commutative diagram :

$$\begin{tikzcd}
\mathfrak{m} \arrow[r, "\sim", "p_i"'] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{m}_i \arrow[d, hook] \\
\mathfrak{g}(I_{re})^{J_{re}} \arrow[r, two heads, "p_i"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowleft"] & \mathfrak{g}(I^i_{re})^{J^i_{re}} \arrow[d, hook] \\
\mathfrak{g}(I_{re}) \arrow[r, two heads, "p_i"] & \mathfrak{g}(I^i_{re})
\end{tikzcd}$$

For the second assertion, Let $\mathfrak{a}_i := p_i(\mathfrak{a})$ and $\Sigma_i = \Delta(\mathfrak{m}_i, \mathfrak{a}_i)$. When restricted to $\mathfrak{m}$ p_i induces an isomorphism of root systems $\psi_i : \Sigma_i \to \Sigma$ such that

$$\langle \alpha, h\rangle = \langle \psi_i^{-1}(\alpha), p_i(h)\rangle\,, \quad \forall \alpha \in \Sigma\,, \forall h \in \mathfrak{a}.$$

Note that for $\alpha \in \Sigma$ and $X \in \mathfrak{g}(I_{re})$ satisfying $[h, X] = \langle \alpha, h\rangle X$, $\forall h \in \mathfrak{a}$, one has $[h^i, p_i(X)] = \langle \psi_i^{-1}(\alpha), h^i\rangle p_i(X)$, $\forall h^i \in \mathfrak{a}_i$. Since $\mathfrak{g}(I_{re})$ (resp. $\mathfrak{g}(I_{re})^{J_{re}}$) is finitely $\Sigma-$graded and p_i is surjective, the Kac-Moody subalgebra $\mathfrak{g}(I^i_{re})$ (resp. $\mathfrak{g}(I^i_{re})^{J^i_{re}}$) is finitely Σ_i-graded. For $k \in I^i_{re}$, Let $\rho_i(\alpha_k)$ be the restriction of α_k to $\mathfrak{a}_i$. Then $(\rho_i(\alpha_k) = 0) \iff (\rho_a(\alpha_k) = 0) \iff (k \in J^i_{re})$. By Proposition 3.21, $\mathfrak{g}(I^i_{re})^{J^i_{re}}$ is maximally finitely Σ_i-graded. □

Corollary 3.25. *If $\mathfrak{g}$ is Lorentzian then I_{re} is connected.*

Proof. If $\mathfrak{g}$ is Lorentzian, then By Proposition 3.6, the grading subalgebra $\mathfrak{m}$ and hence all the $\mathfrak{m}_i$ $(i = 1, 2, ..., q)$ are also Lorentzian. When restricted to $\overset{q}{\underset{i=1}{\oplus}} \mathfrak{a}_i$ the invariant bilinear form $(.\,,.)$ is still non-degenerate and has signature $(q(r-1), q)$, where r is the common rank of the $\mathfrak{m}_i$, $i = 1, 2, ..., q$. Hence $q = 1$ and I_{re} is connected. □

Proposition 3.26. *If $\mathfrak{g}$ is of finite, affine or hyperbolic type, then any finite gradation is real: $I'_{im} = \emptyset$ and (I, J) is a $C-$admissible pair.*

Proof. The result is trivial if $\mathfrak{g}$ is of finite type. Suppose $I'_{im} \neq \emptyset$ for one of the other cases. If $\mathfrak{g}$ is affine, then I_{re} is of finite type and by Lemma 3.19 , $\mathfrak{m}$ is contained in the finite dimensional semi-simple Lie algebra $\mathfrak{g}(I_{re})$. This contradicts the fact that $\mathfrak{m}$ is, as $\mathfrak{g}$, of affine type (see Proposition 3.6). If $\mathfrak{g}$ is hyperbolic, then it is Lorentzian and perfect (cf. section 1.1). By Lemma 3.20 and Corollary 3.25, $\mathfrak{g}(I_{re})$ is an indecomposable finitely $\Sigma-$graded Kac-Moody subalgebra of $\mathfrak{g}$. As I_{re} is assumed to be a proper connected subset of I, $\mathfrak{g}(I_{re})$ is of finite or affine type, a contradiction since, by Proposition 3.6, $\mathfrak{m}$ must be Lorentzian. Hence $I'_{im} = \emptyset$ in the two last cases. □

Proposition 3.27. *If $\mathfrak{g}$ is hyperbolic, then the grading subalgebra $\mathfrak{m}$ is also hyperbolic.*

Proof. Recall that in this case, $I_{re} = I$ (see Proposition 3.26 and Corollary 3.15). Let $\bar{I}^1$ be a proper subset of $\bar{I}$ and suppose that $\bar{I}^1$ is connected. Let $I^1 = \underset{s\in\bar{I}^1}{\cup}(\underset{k\in\Gamma_s}{\cup} I_k)$. Then, I^1 is a proper subset of I. We may assume that the subalgebra $\mathfrak{m}(\bar{I}^1)$ of $\mathfrak{m}$ is contained in $\mathfrak{g}(I^1)$. Let $\Sigma^1 := \Sigma(\bar{I}^1)$ be the root system of $\mathfrak{m}(\bar{I}^1)$. Then, it is not difficult to check that $\mathfrak{g}(I^1)$ is finitely Σ^1-graded. The argument used in Proposition 3.24 shows that the indecomposable components of $\mathfrak{g}(I^1)$ (which all are of finite or affine type) are finitely Σ^1-graded. By Proposition 3.6, $\mathfrak{m}(\bar{I}^1)$ is of finite or affine type. Hence $\mathfrak{m}$ is hyperbolic. □

Corollary 3.28. *The problem of classification of finite real gradations of $\mathfrak{g}$ comes down first to classify the $C-$admissible pairs (I, J) of $\mathfrak{g}$ and then the maximal finite gradations of the corresponding admissible algebra $\mathfrak{g}^J$. When $\mathfrak{g}$ is of finite, affine or hyperbolic type, we get thus all finite gradations.*

Proof. This follows from Proposition 3.26, Proposition 3.21 and Lemma 1.5. □

4. Maximal gradations

We assume now moreover that $\mathfrak{g}$ is maximally finitely $\Sigma-$graded. We keep the notations in section 3 but we have $J = I'_{im} = \emptyset$. So $\overline{I}$ is a quotient of I, with quotient map ρ defined by $\rho_a(\alpha_k) = \gamma_{\rho(k)}$. For $s \in \overline{I}$, $\Gamma_s = \rho^{-1}(\{s\})$.

Proposition 4.1.
1) If $k \neq l \in I$ and $\rho(k) = \rho(l)$, then there is no link between k and l in the Dynkin diagram of A: $\alpha_k(\alpha_l^\vee) = \alpha_l(\alpha_k^\vee) = 0$ and $(\alpha_k, \alpha_l) = 0$.
2) $\mathfrak{a} \subset \{h \in \mathfrak{h} \mid \alpha_k(h) = \alpha_l(h)$ whenever $\rho(k) = \rho(l)\}$.
3) For good choices of the simple coroots and Chevalley generators $(\alpha_k^\vee, e_k, f_k)_{k\in I}$ in $\mathfrak{g}$ and $(\gamma_s^\vee, \overline{X}_s, \overline{Y}_s)_{s\in\overline{I}}$ in $\mathfrak{m}$, we have $\gamma_s^\vee = \sum_{k\in\Gamma_s} \alpha_k^\vee$, $\overline{X}_s = \sum_{k\in\Gamma_s} e_k$ and $\overline{Y}_s = \sum_{k\in\Gamma_s} f_k$.
4) In particular, for $s, t \in \overline{I}$, we have $\gamma_s(\gamma_t^\vee) = \sum_{k\in\Gamma_t} \alpha_i(\alpha_k^\vee)$ for any $i \in \Gamma_s$.

Proof. Assertions 1) and 2) are proved in 3.14 and 3.18. For $i \in \Gamma_s$, $\gamma_s = \rho_a(\alpha_i)$ is the restriction of α_i to $\mathfrak{a}$; so 4) is a consequence of 3).

For 3) recall the proof of Proposition 3.14. The $\mathfrak{sl}_2-$triple $(\overline{X}_s, \gamma_s^\vee, \overline{Y}_s)$ may be written $\gamma_s^\vee = \sum_{k\in\Gamma_s} H_k$, $\overline{X}_s = \sum_{k\in\Gamma_s} E_k$ and $\overline{Y}_s = \sum_{k\in\Gamma_s} F_k$ where (E_k, H_k, F_k) is an $\mathfrak{sl}_2-$triple in $\mathfrak{g}(I_k)$, with $\alpha_k(H_k) = 2$. But now $J = I'_{im} = \emptyset$, so $I_k = \{k\}$ and $\mathfrak{g}(I_k) = \mathbb{C}e_k \oplus \mathbb{C}\alpha_k^\vee \oplus \mathbb{C}f_k$, hence the result. □

So the grading subalgebra $\mathfrak{m}$ may be entirely described by the quotient map ρ.

We look now to the reciprocal construction.

So $\mathfrak{g}$ is an indecomposable and symmetrizable Kac-Moody algebra associated to a generalized Cartan matrix $A = (a_{i,j})_{i,j\in I}$. We consider a quotient $\overline{I}$ of I with quotient map $\rho : I \to \overline{I}$ and fibers $\Gamma_s = \rho^{-1}(\{s\})$ for $s \in \overline{I}$. We suppose that ρ is an admissible quotient i.e. that it satisfies the following two conditions:

(MG1) If $k \neq l \in I$ and $\rho(k) = \rho(l)$, then $a_{k,l} = \alpha_l(\alpha_k^\vee) = 0$.

(MG2) If $s \neq t \in \overline{I}$, then $\overline{a}_{s,t} := \sum\limits_{i\in\Gamma_s} a_{i,j} = \sum\limits_{i\in\Gamma_s} \alpha_j(\alpha_i^\vee)$ is independent of the choice of $j \in \Gamma_t$.

Proposition 4.2. *The matrix $\overline{A} = (\overline{a}_{s,t})_{s,t\in\overline{I}}$ is an indecomposable generalized Cartan matrix.*

Proof. Let $s \neq t \in \bar{I}$ and let $j \in \Gamma_t$. By (MG1) one has $\bar{a}_{t,t} = \sum_{i\in\Gamma_t} a_{i,j} = a_{j,j} = 2$, and by (MG2) $\overline{a}_{s,t} := \sum_{i\in\Gamma_s} a_{i,j} \in \mathbb{Z}^-$ $(\forall j \in \Gamma_t)$. Moreover, $\overline{a}_{s,t} = 0$ if and only if $a_{i,j} = 0\,(= a_{j,i})$, $\forall (i,j) \in \Gamma_s \times \Gamma_t$. It follows that $\overline{a}_{s,t} = 0$ if and only if $\overline{a}_{t,s} = 0$, and $\bar{A}$ is a generalized Cartan matrix. Since A is indecomposable, $\bar{A}$ is also indecomposable. $\square$

Let $\mathfrak{h}^\Gamma = \{h \in \mathfrak{h} \mid \alpha_k(h) = \alpha_l(h) \text{ whenever } \rho(k) = \rho(l)\}$, $\gamma_s^\vee = \sum_{k\in\Gamma_s} \alpha_k^\vee$ and $\mathfrak{a}' = \oplus_{s\in\overline{I}} \mathbb{C}\gamma_s^\vee \subset \mathfrak{h}^\Gamma$. We may choose a subspace $\mathfrak{a}''$ in $\mathfrak{h}^\Gamma$ such that $\mathfrak{a}'' \cap \mathfrak{a}' = \{0\}$, the restrictions $\overline{\alpha}_i =: \gamma_{\rho(i)}$ to $\mathfrak{a} = \mathfrak{a}' \oplus \mathfrak{a}''$ of the simple roots α_i (corresponding to different $\rho(i) \in \overline{I}$) are linearly independent and $\mathfrak{a}''$ is minimal for these two properties.

Proposition 4.3. $(\mathfrak{a}, \{\gamma_s \mid s \in \overline{I}\}, \{\gamma_s^\vee \mid s \in \overline{I}\})$ *is a realization of* $\overline{A}$.

Proof. Let ℓ be the rank of $\overline{A}$. Note that $\mathfrak{a}$ contains $\mathfrak{a}' = \oplus_{s\in\overline{I}} \mathbb{C}\gamma_s^\vee$; the family $(\gamma_s)_{s\in\bar{I}}$ is free in the dual space $\mathfrak{a}^*$ of $\mathfrak{a}$ and satisfies $\langle \gamma_t, \gamma_s^\vee \rangle = \bar{a}_{s,t}$, $\forall s,t \in \bar{I}$. It follows that $\dim(\mathfrak{a}) \geq 2|\bar{I}| - \ell$ (see [11, 14.1] or [12, Ex. 1.3]). As $\mathfrak{a}$ is minimal, we have $\dim(\mathfrak{a}) = 2|\bar{I}| - \ell$ (see [11, 14.2] for minimal realization). Hence $(\mathfrak{a}, \{\gamma_s \mid s \in \overline{I}\}, \{\gamma_s^\vee \mid s \in \overline{I}\})$ is a (minimal) realization of $\overline{A}$. $\square$

We note $\Delta^\rho = \Sigma \subset \oplus_{s\in\overline{I}} \mathbb{Z}\gamma_s$ the root system associated to this realization.

We define now $\overline{X}_s = \sum_{k\in\Gamma_s} e_k$ and $\overline{Y}_s = \sum_{k\in\Gamma_s} f_k$. Let $\mathfrak{m} = \mathfrak{g}^\rho$ be the Lie subalgebra of $\mathfrak{g}$ generated by $\mathfrak{a}$ and the elements $\overline{X}_s$, $\overline{Y}_s$ for $s \in \overline{I}$.

Proposition 4.4. *The Lie subalgebra* $\mathfrak{m} = \mathfrak{g}^\rho$ *is the Kac-Moody algebra associated to the realization* $(\mathfrak{a}, \{\gamma_s \mid s \in \overline{I}\}, \{\gamma_s^\vee \mid s \in \overline{I}\})$ *of* $\overline{A}$. *Moreover,* $\mathfrak{g}$ *is an integrable* $\mathfrak{g}^\rho-$*module with finite multiplicities.*

Proof. Clearly, the following relations hold in the Lie subalgebra $\mathfrak{g}^\rho$:

$$\begin{array}{lll} [\mathfrak{a}, \mathfrak{a}] = 0, & [\overline{X}_s, \overline{Y}_t] = \delta_{s,t}\gamma_s^\vee & (s,t \in \bar{I}); \\ [a, \overline{X}_s] = \langle \gamma_s, a \rangle \overline{X}_s, & [a, \overline{Y}_s] = -\langle \gamma_s, a \rangle \overline{Y}_s & (a \in \mathfrak{a}, s \in \bar{I}). \end{array}$$

For the Serre's relations, one has :

$$1 - \overline{a}_{s,t} \geq 1 - a_{i,j},\ \forall (i,j) \in \Gamma_s \times \Gamma_t.$$

In particular, one can see, by induction on $|\Gamma_s|$, that :

$$(\mathrm{ad}\overline{X}_s)^{1-\overline{a}_{s,t}}(e_j) = \big(\sum_{i\in\Gamma_s} \mathrm{ad}e_i\big)^{1-\overline{a}_{s,t}}(e_j) = 0,\ \forall j \in \Gamma_t.$$

Hence

$$(\mathrm{ad}\overline{X}_s)^{1-\overline{a}_{s,t}}(\overline{X}_t) = 0,\ \forall s,t \in \overline{I},$$

and in the same way we obtain that :

$$(\mathrm{ad}\overline{Y}_s)^{1-\overline{a}_{s,t}}(\overline{Y}_t) = 0,\ \forall s,t \in \overline{I}.$$

It follows that $\mathfrak{g}^\rho$ is a quotient of the Kac-Moody algebra $\mathfrak{g}(\overline{A})$ associated to $\overline{A}$ and $(\mathfrak{a}, \{\gamma_s \mid s \in \overline{I}\}, \{\gamma_s^\vee \mid s \in \overline{I}\})$ in which the Cartan subalgebra $\mathfrak{a}$ of $\mathfrak{g}(\overline{A})$ is embedded. By [12, 1.7] $\mathfrak{g}^\rho$ is equal to $\mathfrak{g}(\overline{A})$.

It's clear that $\mathfrak{g}$ is an integrable $\mathfrak{g}^\rho-$module with finite dimensional weight spaces relative to the adjoint action of $\mathfrak{a}$, since for $\alpha = \sum_{i\in I} n_i\alpha_i \in \Delta^+$, its restriction $\rho_a(\alpha)$ to $\mathfrak{a}$ is given by

$$\rho_a(\alpha) = \sum_{s\in\bar{I}} \big(\sum_{i\in\Gamma_s} n_i\big)\,\gamma_s \tag{4.1}$$

□

Proposition 4.5. *The Kac-Moody algebra $\mathfrak{g}$ is maximally finitely $\Delta^\rho-$graded with grading subalgebra $\mathfrak{g}^\rho$.*

Proof. As in Theorem 2.14, we will see that $\rho_a(\Delta^+) \subset Q_+^\Gamma := \underset{s\in\bar{I}}{\oplus} \mathbb{Z}^+\gamma_s$ satisfies, as $\Sigma^+ = \Delta_+^\rho$, the following conditions :
(i) $\gamma_s \in \rho_a(\Delta^+) \subset Q_+^\Gamma$, $2\gamma_s \notin \rho_a(\Delta^+)$, $\forall s \in \bar{I}$.
(ii) if $\gamma \in \rho_a(\Delta^+)$, $\gamma \neq \gamma_s$, then the set $\{\gamma + k\gamma_s; k \in \mathbb{Z}\} \cap \rho_a(\Delta^+)$ is a string $\{\gamma - p\gamma_s, ..., \gamma + q\gamma_s\}$, where $p, q \in \mathbb{Z}^+$ and $p - q = \langle \gamma, \gamma_s^\vee \rangle$;
(iii) if $\gamma \in \rho_a(\Delta^+)$, then $\mathrm{supp}(\gamma)$ is connected.
Clearly $\{\gamma_s \mid s \in \bar{I}\} \subset \rho_a(\Delta_+) \subset Q_+^\Gamma$. For $\alpha \in \Delta$ and $s \in \bar{I}$, the condition $\rho_a(\alpha) \in \mathbb{N}\gamma_s$ implies $\alpha \in \Delta(\Gamma_s)^+ = \{\alpha_i;\ i \in \Gamma_s\}$ [see (4.1)]. It follows that $2\gamma_s \notin \rho_a(\Delta_+)$ and (i) is satisfied. By Proposition 4.4, $\mathfrak{g}$ is an integrable $\mathfrak{g}^\rho-$module with finite multiplicities. Hence, the propriety (ii) follows from [12, 3.6]. Let $\alpha \in \Delta_+$ and let $s, t \in \mathrm{supp}(\rho_a(\alpha))$. By (4.1) there exists $(k, l) \in \Gamma_s \times \Gamma_t$ such that $k, l \in \mathrm{supp}(\alpha)$, which is connected. Hence there exist $i_0 = k, i_1, ..., i_{n+1} = l$ such that $\alpha_{i_j} \in \mathrm{supp}(\alpha)$, $j = 0, 1, ..., n+1$, and for $j = 0, 1, ..., n$, i_j and i_{j+1} are linked relative to the generalized Cartan matrix A. In particular, $\rho(i_j) \neq \rho(i_{j+1}) \in \mathrm{supp}(\rho_a(\alpha))$ and they are linked relative to the generalized Cartan matrix $\overline{A}$, $j = 0, 1, ..., n$, with $\rho(i_0) = s$ and $\rho(i_{n+1}) = t$. Hence the connectedness of $\mathrm{supp}(\rho_a(\alpha))$ relative to $\overline{A}$. It follows that $\rho_a(\Delta^+) = \Delta_+^\rho$ and hence $\rho_a(\Delta) = \Delta^\rho$ (see [12, Ex. 5.4]. In particular, $\mathfrak{g}$ is finitely $\Delta^\rho-$graded with $J = \emptyset = I'_{im}$.

□

Corollary 4.6. *The restriction to $\mathfrak{m} = \mathfrak{g}^\rho$ of the invariant bilinear form $(.\,,.)$ of $\mathfrak{g}$ is non-degenerate. In particular, the generalized Cartan matrix $\overline{A}$ is symmetrizable of the same type as A.*

Proof. The first part of the corollary follows from Proposition 4.5 and Corollary 3.17. The second part follows from Proposition 3.6. □

Remark 4.7. The map ρ coincides with the map (also denoted ρ) defined at the beginning of this section using the maximal gradation of Proposition 4.5. Conversely Proposition 4.1 tells that, for a general maximal finite gradation, ρ is admissible and $\mathfrak{m} = \mathfrak{g}^\rho$ for good choices of the Chevalley generators. So we get a good correspondence between maximal gradations and admissible quotient maps.

By Corollary 3.28 the real finite gradations of a Kac-Moody algebra $\mathfrak{g}$ are bijectively associated to pairs of a $C-$admissible pair (I, J) and an admissible quotient map $\rho : I' = I \setminus J \to \bar{I}'$.

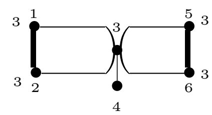
5. An example

The following example shows that imaginary gradations do exist. It shows in particular that, for a generalized $C-$admissible pair (I, J), J° may be non-empty and I_{re} may be non-connected. Moreover, the Kac-Moody algebra $\mathfrak{g}$ may be not graded by the root system of $\mathfrak{g}(I_{re})$.
The imaginary gradations will be studied in a forthcoming paper [7].

Example 5.1. *Consider the Kac Moody algebra $\mathfrak{g}$ corresponding to the indecomposable and symmetric generalized Cartan matrix A :*

$$A = \begin{pmatrix} 2 & -3 & -1 & 0 & 0 & 0 \\ -3 & 2 & -1 & 0 & 0 & 0 \\ -1 & -1 & 2 & -1 & -1 & -1 \\ 0 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -3 \\ 0 & 0 & -1 & 0 & -3 & 2 \end{pmatrix}$$

with the corresponding Dynkin diagram :



Note that $\det(A) = 275$ *and the symmetric submatrix of A indexed by* $\{1, 2, 4, 5, 6\}$ *has signature* $(+++, --)$. *Since* $\det(A) > 0$, *the matrix A should have signature* $(++++, --)$. *Let Σ be the root system associated to the strictly hyperbolic generalized Cartan matrix* $\begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$, *the corresponding Dynkin diagram is the following :*

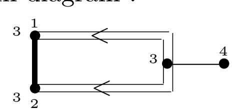
$H_{3,3}$

We will see that $\mathfrak{g}$ is finitely $\Sigma-$graded and describe the corresponding generalized $C-$admissible pair.

1) Let τ be the involutive diagram automorphism of $\mathfrak{g}$ such that $\tau(1) = 5$, $\tau(2) = 6$ and τ fixes the other vertices. Let σ'_n be the normal semi-involution of $\mathfrak{g}$ corresponding to the split real form of $\mathfrak{g}$. Consider the quasi-split real form $\mathfrak{g}^1_{\mathbb{R}}$ associated to the semi-involution $\tau\sigma'_n$ (see [2] or [6]). Then $\mathfrak{t}_{\mathbb{R}} := \mathfrak{h}^{\tau}_{\mathbb{R}}$ is a maximal split toral subalgebra of $\mathfrak{g}^1_{\mathbb{R}}$. The corresponding restricted root system $\Delta' := \Delta(\mathfrak{g}_{\mathbb{R}}, \mathfrak{t}_{\mathbb{R}})$ is reduced and the corresponding generalized Cartan matrix A' is given by :

$$A' = \begin{pmatrix} 2 & -3 & -2 & 0 \\ -3 & 2 & -2 & 0 \\ -1 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{pmatrix}$$

with the corresponding Dynkin diagram :



Following N. Bardy [4, 9] , there exists a split real Kac-Moody subalgebra $\mathfrak{m}^1_{\mathbb{R}}$ of $\mathfrak{g}^1_{\mathbb{R}}$ containing $\mathfrak{t}_{\mathbb{R}}$ such that $\Delta' = \Delta(\mathfrak{m}^1_{\mathbb{R}}, \mathfrak{t}_{\mathbb{R}})$. It follows that $\mathfrak{g}$ is finitely $\Delta'-$graded.

2) Let $\mathfrak{m}^1 := \mathfrak{m}^1_{\mathbb{R}} \otimes \mathbb{C}$ and $\mathfrak{t} := \mathfrak{t}_{\mathbb{R}} \otimes \mathbb{C}$. Denote by $\alpha'_i := \alpha_i/\mathfrak{t}$, $i = 1, 2, 3, 4$. Put $\check{\alpha'_1} = \check{\alpha_1} + \check{\alpha_5}$, $\check{\alpha'_2} = \check{\alpha_2} + \check{\alpha_6}$, $\check{\alpha'_3} = \check{\alpha_3}$ and $\check{\alpha'_4} = \check{\alpha_4}$. Let $I^1 := \{1, 2, 3, 4\}$, then $(\mathfrak{t}, \Pi' = \{\alpha'_i,\ i \in I^1\}, \Pi'^{\vee} = \{\check{\alpha'_i},\ i \in I^1\})$ is a realization of A' which is symmetrizable and Lorentzian.

Let $\mathfrak{m}$ be the Kac-Moody subalgebra of $\mathfrak{m}^1$ corresponding to the submatrix $\bar{A}$ of A' indexed by $\{1, 2\}$. Thus $\bar{A} = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$ is strictly hyperbolic. Let $\mathfrak{a} :=$

$\mathbb{C}\alpha'^{\vee}_1 \oplus \mathbb{C}\alpha'^{\vee}_2$ be the standard Cartan subalgebra of $\mathfrak{m}$ and let $\Sigma = \Delta(\mathfrak{m}, \mathfrak{a})$. For $\alpha' \in \mathfrak{t}^*$, denote by $\rho_1(\alpha')$ the restriction of α' to $\mathfrak{a}$. Put $\gamma_s = \rho_1(\alpha'_s)$, $\gamma_s^{\vee} = \alpha'^{\vee}_s$, $s = 1, 2$. Then $\Pi_a = \{\gamma_1, \gamma_2\}$ is the standard root basis of Σ. One can see easily that $\rho_1(\alpha'_4) = 0$ and $\rho_1(\alpha'_3) = 2(\gamma_1 + \gamma_2)$ is a strictly positive imaginary root of Σ. Now we will show that $\mathfrak{m}^1$ is finitely $\Sigma-$graded.

Let $(.\,,.)_1$ be the normalized invariant bilinear form on $\mathfrak{m}^1$ such that short real roots have length 1 and long real roots have square length 2. Then there exists a positive rational q such that the restriction of $(.\,,.)_1$ to $\mathfrak{t}$ has the matrix B_1 in the basis $\Pi'^{\vee}$, where :

$$B_1 = q \begin{pmatrix} 2 & -3 & -1 & 0 \\ -3 & 2 & -1 & 0 \\ -1 & -1 & 1 & -1/2 \\ 0 & 0 & -1/2 & 1 \end{pmatrix}$$

By duality, the restriction of $(.\,,.)_1$ to $\mathfrak{t}$ induces a non-degenerate symmetric bilinear form on $\mathfrak{t}^*$ (see [12, 2.1]) such that its matrix B'_1 in the basis Π', is the following :

$$B'_1 = q^{-1} \begin{pmatrix} 2 & -3 & -2 & 0 \\ -3 & 2 & -2 & 0 \\ -2 & -2 & 4 & -2 \\ 0 & 0 & -2 & 4 \end{pmatrix}$$

Hence, q equals 2.

Note that for $\alpha' = \sum_{i=1}^{4} n_i \alpha'_i \in \Delta'^{+}$, we have that

$$(\alpha', \alpha')_1 = n_1^2 + n_2^2 + 2n_3^2 + 2n_4^2 - 3n_1n_2 - 2n_1n_3 - 2n_2n_3 - 2n_3n_4. \tag{5.1}$$

We will show that $\rho_1(\Delta'^{+}) = \Sigma^+ \cup \{0\}$. Note that Σ can be identified with $\Delta' \cap (\mathbb{Z}\alpha'_1 + \mathbb{Z}\alpha'_2)$; hence ρ_1 is injective on Σ and $\Sigma^+ \subset \rho_1(\Delta'^{+})$.

Let $(.\,,.)_a$ be the normalized invariant bilinear form on $\mathfrak{m}$ such that all real roots have length 2. Then the restriction of $(.\,,.)_a$ to $\mathfrak{a}$ has the matrix B_a in the basis $\Pi_a^{\vee} = \{\gamma_1^{\vee}, \gamma_2^{\vee}\}$, where :

$$B_a = \begin{pmatrix} 2 & -3 \\ -3 & 2 \end{pmatrix}$$

Since $\bar{A}$ is symmetric, the non-degenerate symmetric bilinear form, on $\mathfrak{a}^*$, induced by the restriction of $(.\,,.)_a$ to $\mathfrak{a}$, has the same matrix B_a in the basis Π_a. In particular, we have that :

$$(\rho_1(\alpha'), \rho_1(\alpha'))_a = 2[(n_1 + 2n_3)^2 + (n_2 + 2n_3)^2 - 3(n_1 + 2n_3)(n_2 + 2n_3)],$$

since $\rho_1(\alpha') = (n_1 + 2n_3)\gamma_1 + (n_1 + 2n_3)\gamma_2$.

Using (5.1), it is not difficult to check that

$$(\rho_1(\alpha'), \rho_1(\alpha'))_a = 2[(\alpha', \alpha')_1 - (n_3 - n_4)^2 - 5n_3^2 - n_4^2] \tag{5.2}$$

Suppose $n_3 = 0$, then, since $\mathrm{supp}(\alpha')$ is connected, we have that $\alpha' = n_1\alpha'_1 + n_2\alpha'_2$ or $\alpha' = \alpha'_4$. Hence $\rho_1(\alpha') = n_1\gamma_1 + n_2\gamma_2 \in \Sigma$ or $\rho_1(\alpha') = 0$.

Suppose $n_3 \neq 0$, then, since $(\alpha', \alpha')_1 \leq 2$, one can see, using (5.2), that

$$(\rho_1(\alpha'), \rho_1(\alpha'))_a < 0.$$

As Σ is hyperbolic and $\rho_1(\alpha') \in \mathbb{N}\gamma_1 + \mathbb{N}\gamma_2$, we deduce that $\rho_1(\alpha')$ is a positive imaginary root of Σ (see [12, 5.10]). It follows that $\rho_1(\Delta'^{+}) = \Sigma^+ \cup \{0\}$.

To see that $\mathfrak{m}^1$ is finitely $\Sigma-$graded, it suffices to prove that, for $\gamma = m_1\gamma_1 + m_2\gamma_2 \in \Sigma^+ \cup \{0\}$, the set $\{\alpha' \in \Delta'^{+}, \rho_1(\alpha') = \gamma\}$ is finite. Note that if $\alpha' = \sum_{i=1}^{4} n_i\alpha'_i \in \Delta'^{+}$ satisfying $\rho_1(\alpha') = \gamma$, then $n_i + 2n_3 = m_i$, $i = 1, 2$. In particular, there are only finitely many possibilities for n_i, $i = 1, 2, 3$. The same argument as the one used in the proof of Proposition 2.13 shows also that there are only finitely many possibilities for n_4.

3) Recall that $\mathfrak{m} \subset \mathfrak{m}^1 \subset \mathfrak{g}$. The fact that $\mathfrak{g}$ is finitely $\Delta'-$graded with grading subalgebra $\mathfrak{m}^1$ and $\mathfrak{m}^1$ is finitely $\Sigma-$graded implies that $\mathfrak{g}$ is finitely $\Sigma-$graded (cf. lemma 1.5). Let $I = \{1, 2, 3, 4, 5, 6\}$, then the root basis Π_a of Σ is adapted to the root basis Π of Δ and we have $I_{re} = \{1, 2, 5, 6\}$ (not connected), $\Gamma_1 = \{1, 5\}$, $\Gamma_2 = \{2, 6\}$, $J = \{4\}$, $J_{re} = \emptyset$, $I'_{im} = \{3\}$ and $J^\circ = J = \{4\}$.
Note that, for this example, $\mathfrak{g}(I_{re})$, which is $\Sigma-$graded, is isomorphic to $\mathfrak{m} \times \mathfrak{m}$. This gradation corresponds to that of the pseudo-complex real form of $\mathfrak{m} \times \mathfrak{m}$ (i.e. the complex Kac-Moody algebra $\mathfrak{m}$ viewed as real Lie algebra) by its restricted reduced root system. Since the pair $(I_3, J_3) = (\{3, 4\}, \{4\})$ is not admissible, it is not possible to build a Kac-Moody algebra $\mathfrak{g}^J$ grading finitely $\mathfrak{g}$ and maximally finitely $\Sigma-$graded.

Acknowledgments We thank the anonymous referee for his/her valuable comments and suggestions.

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Université de Monastir, Département de Mathématiques, Faculté des Sciences, 5019 Monastir. Tunisie.

Université de Lorraine, Institut Élie Cartan de Lorraine, UMR 7502, Vandœuvre lès-Nancy, F-54506, France.

CNRS, Institut Élie Cartan de Lorraine, UMR 7502, Vandœuvre lès-Nancy, F-54506, France.

E-mail address: hechmi.benmessaoud@fsm.rnu.tn Guy.Rousseau@univ-lorraine.fr