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Spherical Hecke algebras for Kac-Moody groups over local fields

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Abstract

We define the spherical Hecke algebra $\mathcal{H}$ for an almost split Kac-Moody group G over a local non-archimedean field. We use the hovel $\mathscr{I}$ associated to this situation, which is the analogue of the Bruhat-Tits building for a reductive group. The stabilizer K of a special point on the standard apartment plays the role of a maximal open compact subgroup. We can define $\mathcal{H}$ as the algebra of K-bi-invariant functions on G with almost finite support. As two points in the hovel are not always in a same apartment, this support has to be in some large subsemigroup G^+ of G. We prove that the structure constants of $\mathcal{H}$ are polynomials in the cardinality of the residue field, with integer coefficients depending on the geometry of the standard apartment. We also prove the Satake isomorphism between $\mathcal{H}$ and the algebra of Weyl invariant elements in some completion of a Laurent polynomial algebra. In particular, $\mathcal{H}$ is always commutative. Actually, our results apply to abstract "locally finite" hovels, so that we can define the spherical algebra with unequal parameters.

Contents

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Introduction

Let G be a connected reductive group over a local non-archimedean field $\mathcal{K}$, and let K be an open compact subgroup. The space $\mathcal{H}$ of complex functions on G, bi-invariant by K and with compact support is an algebra for the natural convolution product. Ichiro Satake [Sat63] studied this algebra $\mathcal{H}$ to define the spherical functions and proved, in particular, that $\mathcal{H}$ is commutative for good choices of K. We know now that one of the good choices for K is the stabilizer of some special vertex for the action of G on its Bruhat-Tits building $\mathscr{I}$, whose structure is explained in [BT72]. Moreover, $\mathcal{H}$, now called the spherical Hecke algebra, may be entirely defined using $\mathscr{I}$; see, e.g., [Par06].

Kac-Moody groups are interesting generalizations of reductive groups, and it is natural to try to define the spherical Hecke algebra in that case. So, let G be a Kac-Moody group. Unfortunately, there is, up to now, no good topology on G and no good compact subgroup, so the "convolution product" has to be defined only by algebraic means. Alexander Braverman and David Kazhdan [BK11] succeeded in defining such a spherical Hecke algebra, when G is split and untwisted affine; see also the survey [BK14] by the same authors. For a well-chosen subgroup K, they define $\mathcal{H}$ as the algebra of K-bi-invariant complex functions with "almost finite" support. There are two new features: the support has to be in a subsemigroup G^+ of G, and it is an infinite union of double classes. Hence, $\mathcal{H}$ is naturally a module over the ring of complex formal power series.

Our idea is to define this spherical Hecke algebra using the hovel associated to the almost split Kac-Moody group G that we built in [GR08], [Rou10] and [Rou12]. This hovel $\mathscr{I}$ is a set with an action of G and a covering by subsets called apartments. They are in one-to-one correspondence with the maximal split subtori, hence permuted transitively by G. Each apartment A is a finite-dimensional real affine space, and its stabilizer N in G acts on it via a generalized affine Weyl group $W = W^v \ltimes Y$ (where $Y \subset \overrightarrow{A}$ is a discrete subgroup of translations), which stabilizes a set $\mathcal{M}$ of affine hyperplanes called walls. So, $\mathscr{I}$ looks much like the Bruhat-Tits building of a reductive group, but $\mathcal{M}$ is not a locally finite system of hyperplanes (as the root system Φ is infinite) and two points in $\mathscr{I}$ are not always in a same apartment. (This is why $\mathscr{I}$ is called a hovel.) There is on $\mathscr{I}$ a G-invariant preorder $\leq$ that induces on each apartment A the preorder given by the Tits cone $\mathcal{T} \subset \overrightarrow{A}$.

Now, we consider the stabilizer K in G of a special point 0 in a chosen standard apartment $\mathbb{A}$. Fix a ring R. The spherical Hecke algebra $\mathcal{H}_R$ is the space of some K-bi-invariant functions on G with values in R. In other words, it is the space $\mathcal{H}_R^{\mathscr{I}}$ of some G-invariant functions on $\mathscr{I}_0 \times \mathscr{I}_0$ where $\mathscr{I}_0 = G/K$ is the orbit of 0 in $\mathscr{I}$. The convolution product is easy to guess from this point of view: $(\varphi*\psi)(x,y) = \sum_{z\in\mathscr{I}_0} \varphi(x,z)\psi(z,y)$ (if this sum means something). As two points x, y in $\mathscr{I}$ are not always in a same apartment (i.e., the Cartan decomposition fails: $G \neq KNK$), we have to consider pairs $(x,y) \in \mathscr{I}_0 \times \mathscr{I}_0$, with $x \leq y$. (This implies that x, y are in a same apartment.) For $\mathcal{H}_R$, this means that the support of $\varphi \in \mathcal{H}_R$ has to be in $K\backslash G^+/K$, where $G^+ = \{g \in G \mid 0 \leq g.0\}$ is a semigroup. In addition, $K\backslash G^+/K$ is in one-to-one correspondence with the subsemigroup $Y^{++} = Y \cap C_f^v$ of Y (where C_f^v is the fundamental Weyl chamber). Now, to get a well-defined convolution product, we have to ask (as in [BK11]) that the support of any $\varphi \in \mathcal{H}_R$ is almost finite: $\mathrm{supp}(\varphi) \subset \bigcup_{i=1}^n (\lambda_i - Q_+^\vee) \cap Y^{++}$, where $\lambda_i \in Y^{++}$ and $Q_+^\vee$ is the subsemigroup of Y generated by the fundamental coroots. Note that $(\lambda - Q_+^\vee) \cap Y^{++}$ is infinite except when G is reductive.

With this definition we are able to prove that $\mathcal{H}_R$ is really an algebra, which generalizes the known spherical Hecke algebras in the finite or affine split case (see Section 2). In Section 3, in the Kac-Moody split case, we describe the hovel $\mathscr{I}$ and give a direct proof that $\mathcal{H}_R$ is commutative.

The structure constants of $\mathcal{H}_R$ are the nonnegative integers $m_{\lambda,\mu}(\nu)$ (for $\lambda, \mu, \nu \in Y^{++}$) such that $c_\lambda * c_\mu = \sum_{\nu\in Y^{++}} m_{\lambda,\mu}(\nu)c_\nu$, where c_λ is the characteristic function of $K\lambda K$. Each chamber (= alcove) in $\mathscr{I}$ has only a finite number of adjacent chambers along a given panel. These numbers are called parameters of $\mathscr{I}$, and they form a finite set $\mathcal{Q}$. In the split case, there is only one parameter q: the number of elements of the residue field κ of $\mathcal{K}$. In Section 4, we show that the structure constants are polynomials in these parameters with integral coefficients depending only on the geometry of the model apartment.

In Section 5, we build an action of $\mathcal{H}_R$ on the module of functions from $\mathbb{A} \cap \mathscr{I}_0$ to R. This gives an injective homomorphism from $\mathcal{H}_R$ into a suitable completion $R[[Y]]$ of the group algebra $R[Y]$; hence $\mathcal{H}_R$ is abelian (5.3). After being modified by a character, this homomorphism gives the Satake isomorphism from $\mathcal{H}_R$ onto the subalgebra $R[[Y]]^{W^v}$ of W^v-invariant elements in $R[[Y]]$. The proof involves a parabolic retraction of $\mathscr{I}$ onto an extended tree inside the hovel.

Actually, this article is written in a more general framework (explained in Section 1): we ask $\mathscr{I}$ to be an abstract ordered hovel (as defined in [Rou11]) and G to be a strongly transitive group of (positive, type-preserving) automorphisms.

The general definition and study of Hecke algebras for split Kac-Moody groups over local fields was also undertaken by Alexander Braverman, David Kazhdan and Manish Patnaik (as we knew from [P10]). A preliminary draft appeared recently [BKP]. Their arguments are algebraic without use of a geometric object as a hovel, and the proofs seem complete (temporarily?) only for the untwisted affine case. In addition to the construction of the spherical Hecke algebra and the Satake isomorphism (as here), they give a formula for spherical functions and they build the Iwahori-Hecke algebra.

One should notice that these authors use, instead of our group K, a smaller group, *a priori* slightly different; see Remark in Section 3.4. This group is also used in [BGKP14] to perform computations on some double cosets in an affine Kac-Moody group over a local non-archimedian field in order to prove an affine Gindikin-Karpelevich formula.

In an article in preparation, we generalize the Iwahori-Hecke algebra to our general framework and investigate its relationship with the spherical Hecke algebra. In the same vein, it should be possible to define the Hecke algebra associated to any type of parahoric subgroups.

1. **General framework**

1.1. *Vectorial data.* We consider a quadruple $(V, W^v, (\alpha_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$, where V is a finite-dimensional real vector space, W^v a subgroup of $\mathrm{GL}(V)$ (the vectorial Weyl group), I a finite set, $(\alpha_i^\vee)_{i\in I}$ a family in V and $(\alpha_i)_{i\in I}$ a free family in the dual V^*. We ask these data to verify the conditions of [Rou11, 1.1]. In particular, the formula $r_i(v) = v - \alpha_i(v)\alpha_i^\vee$ defines a linear involution in V that is an element in W^v and $(W^v, \{r_i \mid i \in I\})$ is a Coxeter system.

To be more concrete, we consider the Kac-Moody case of [Rou11, 1.2]: the matrix $\mathbb{M} = (\alpha_j(\alpha_i^\vee))_{i,j\in I}$ is a generalized Cartan matrix. Then W^v is the Weyl group of the corresponding Kac-Moody Lie algebra $\mathfrak{g}_{\mathbb{M}}$ and the associated real root system is

$$\Phi = \{w(\alpha_i) \mid w \in W^v, i \in I\} \subset Q = \bigoplus_{i\in I} \mathbb{Z}.\alpha_i.$$

We set $\Phi^\pm = \Phi \cap Q^\pm$, where $Q^\pm = \pm(\bigoplus_{i\in I} (\mathbb{Z}_{\geq 0}).\alpha_i)$ and $Q^\vee = (\bigoplus_{i\in I} \mathbb{Z}.\alpha_i^\vee)$, $Q^\vee_\pm = \pm(\bigoplus_{i\in I} (\mathbb{Z}_{\geq 0}).\alpha_i^\vee)$. We have $\Phi = \Phi^+ \cup \Phi^-$ and, for $\alpha = w(\alpha_i) \in \Phi$, $r_\alpha = w.r_i.w^{-1}$ and $r_\alpha(v) = v - \alpha(v)\alpha^\vee$, where $\alpha^\vee = w(\alpha_i^\vee)$ depends only on α.

The set Φ is an (abstract, reduced) real root system in the sense of [MP89], [MP95] or [Bar96]. We shall sometimes also use the set $\Delta = \Phi \cup \Delta^+_{im} \cup \Delta^-_{im}$ of all roots (with $-\Delta^-_{im} = \Delta^+_{im} \subset Q^+$, W^v-stable) defined in [Kac90]. It is an (abstract, reduced) root system in the sense of [Bar96].

The *fundamental positive chamber* is $C^v_f = \{v \in V \mid \alpha_i(v) > 0 \text{ for all } i \in I\}$. Its closure $\overline{C^v_f}$ is the disjoint union of the vectorial faces $F^v(J) = \{v \in V \mid \alpha_i(v) = 0 \text{ for all } i \in J, \alpha_i(v) > 0 \text{ for all } i \in I \setminus J\}$ for $J \subset I$. The positive (resp. negative) vectorial faces are the sets $w.F^v(J)$ (resp. $-w.F^v(J)$) for $w \in W^v$ and $J \subset I$. The set J or the face $w.F^v(J)$ is called *spherical* if the group $W^v(J)$ generated by $\{r_i \mid i \in J\}$ is finite.

The *Tits cone* $\mathcal{T}$ is the (disjoint) union of the positive vectorial faces. It is a W^v-stable convex cone in V.

1.2. *The model apartment.* As in [Rou11, 1.4] the model apartment $\mathbb{A}$ is V considered as an affine space and endowed with a family $\mathcal{M}$ of walls. These walls are affine hyperplanes directed by $\mathrm{Ker}(\alpha)$ for $\alpha \in \Phi$.

We ask this apartment to be *semi-discrete* and the origin 0 to be *special*. This means that these walls are the hyperplanes defined as follows:

$$M(\alpha, k) = \{v \in V \mid \alpha(v) + k = 0\} \qquad \text{for } \alpha \in \Phi \text{ and } k \in \Lambda_\alpha$$

(with $\Lambda_\alpha = k_\alpha.\mathbb{Z}$ a nontrivial discrete subgroup of $\mathbb{R}$). Using the following lemma (i.e., replacing Φ by $\widetilde{\Phi}$) we shall (and will) assume that $\Lambda_\alpha = \mathbb{Z}$ for all $\alpha \in \Phi$.

Lemma 1.3. *For all $\alpha \in \Phi$, we choose $k_\alpha > 0$ and define $\widetilde{\alpha} = \alpha/k_\alpha$, $\widetilde{\alpha}^\vee = k_\alpha.\alpha^\vee$. Then $\widetilde{\Phi} = \{\widetilde{\alpha} \mid \alpha \in \Phi\}$ is the (abstract reduced) real root system (in the sense of* [MP89], [MP95] *or* [Bar96]) *associated to*

$$(V, W^v, (k_{\alpha_i}^{-1}.\alpha_i)_{i\in I}, (k_{\alpha_i}.\alpha_i^\vee)_{i\in I})$$

and hence to the generalized Cartan matrix $\widetilde{\mathbb{M}} = (k_{\alpha_j}^{-1}.\alpha_j(k_{\alpha_i}.\alpha_i^\vee))_{i,j\in I}$. Moreover, with $\widetilde{\Phi}$, the walls are described using the subgroups $\widetilde{\Lambda}_\alpha = \mathbb{Z}$.

Proof. For $\alpha, \beta \in \Phi$, the group W^a contains the translation τ by $k_\alpha.\alpha^\vee$ and $\tau(M(\beta, 0)) = M(\beta, -\beta(k_\alpha.\alpha^\vee))$. So $k_\alpha.\beta(\alpha^\vee) \in \Lambda_\beta$; i.e.,

$$\widetilde{\beta}(\widetilde{\alpha}^\vee) = k_\beta^{-1}.k_\alpha.\beta(\alpha^\vee) \in \mathbb{Z}.$$

Hence $\widetilde{\mathbb{M}} = (k_{\alpha_j}^{-1}.\alpha_j(k_{\alpha_i}.\alpha_i^\vee))_{i,j\in I}$ is a generalized Cartan matrix and the lemma is clear, as $k_{w\alpha} = k_\alpha$. □

For $\alpha = w(\alpha_i) \in \Phi$, $k \in \mathbb{Z}$ and $M = M(\alpha, k)$, the reflection $r_{\alpha,k} = r_M$ with respect to M is the affine involution of $\mathbb{A}$ with fixed points the wall M and associated linear involution r_α. The affine Weyl group W^a is the group generated by the reflections r_M for $M \in \mathcal{M}$; we assume that W^a stabilizes $\mathcal{M}$.

For $\alpha \in \Phi$ and $k \in \mathbb{R}$, $D(\alpha, k) = \{v \in V \mid \alpha(v) + k \geq 0\}$ is a half-space. It is called a *half-apartment* if $k \in \mathbb{Z}$.

The Tits cone $\mathcal{T}$ and its interior $\mathcal{T}^o$ are convex and W^v-stable cones; therefore, we can define two W^v-invariant preorder relations on $\mathbb{A}$:

$$x \leq y \iff y - x \in \mathcal{T} \quad \text{and} \quad x \overset{o}{\leq} y \iff y - x \in \mathcal{T}^o.$$

If W^v has no fixed point in $V \setminus \{0\}$ and no finite factor, then they are orders; but they are not in general.

1.4. *Faces*, *sectors*, *chimneys.* The faces in $\mathbb{A}$ are associated to the above systems of walls and half-apartments (i.e., $D(\alpha,k)=\{v\in\mathbb{A}\mid\alpha(v)+k\geq 0\}$). As in [BT72], they are no longer subsets of $\mathbb{A}$, but filters of subsets of $\mathbb{A}$. For the definition of that notion and its properties, we refer to [BT72] or [GR08].

If F is a subset of $\mathbb{A}$ containing an element x in its closure, the germ of F in x is the filter $\mathrm{germ}_x(F)$ consisting of all subsets of $\mathbb{A}$ that are intersections of F and neighbourhoods of x. In particular, if $x\neq y\in E$, we denote the germ in x of the segment $[x,y]$ (resp. of the interval $]x,y]$) by $[x,y)$ (resp. $]x,y)$).

Given F a filter of subsets of $\mathbb{A}$, its *enclosure* $\mathrm{cl}_{\mathbb{A}}(F)$ is the filter made of the subsets of $\mathbb{A}$ containing an element of F of the shape $\cap_{\alpha\in\Delta}D(\alpha,k_\alpha)$, where $k_\alpha\in\mathbb{Z}\cup\{\infty\}$. (Here, $D(\alpha,\infty)=\mathbb{A}$.)

A *face* F in the apartment $\mathbb{A}$ is associated to a point $x\in\mathbb{A}$ and a vectorial face F^v in V. More precisely, a subset S of $\mathbb{A}$ is an element of the face $F=F(x,F^v)$ if, and only if, it contains an intersection of half-spaces $D(\alpha,k)$ or open half-spaces $D^\circ(\alpha,k)$ (for $\alpha\in\Delta$ and $k\in\mathbb{Z}\sqcup\{\infty\}$) that contains $\Omega\cap(x+F^v)$, where Ω is an open neighborhood of x in $\mathbb{A}$. The enclosure of a face $F=F(x,F^v)$ is its closure: the closed-face $\overline{F}$. It is the enclosure of the local-face in x, $\mathrm{germ}_x(x+F^v)$.

There is an order on the faces: the assertions "F is a face of F'," "F' covers F" and "$F\leq F'$" are by definition equivalent to $F\subset\overline{F'}$. The dimension of a face F is the smallest dimension of an affine space generated by some $S\in F$. The (unique) such affine space E of minimal dimension is the support of F. Any $S\in F$ contains a nonempty open subset of E. A face F is spherical if the direction of its support meets the open Tits cone; then its pointwise stabilizer W_F in W is finite.

Any point $x\in\mathbb{A}$ is contained in a unique face $F(x,V_0)$, which is minimal (but seldom spherical); x is a vertex if, and only if, $F(x,V_0)=\{x\}$.

A *chamber* (or alcove) is a maximal face or, equivalently, a face such that all its elements contain a nonempty open subset of $\mathbb{A}$.

A *panel* is a spherical face maximal among faces that are not chambers or, equivalently, a spherical face of dimension $n-1$. Its support is a wall. So, the set of spherical faces of $\mathbb{A}$ and the Tits cone completely determine the set $\mathcal{M}$ of walls.

A *sector* in $\mathbb{A}$ is a V-translate $\mathfrak{s}=x+C^v$ of a vectorial chamber $C^v=\pm w.C^v_f$ ($w\in W^v$), x is its *base point* and C^v its *direction.* Two sectors have the same direction if, and only if, they are conjugate by V-translation and if, and only if, their intersection contains another sector.

The *sector-germ* of a sector $\mathfrak{s}=x+C^v$ in $\mathbb{A}$ is the filter $\mathfrak{S}$ of subsets of $\mathbb{A}$ consisting of the sets containing a V-translate of $\mathfrak{s}$; it is well determined by the direction C^v. So, the set of translation classes of sectors in $\mathbb{A}$, the set of vectorial chambers in V and the set of sector-germs in $\mathbb{A}$ are in canonical bijection. We denote the sector-germ associated to the negative fundamental vectorial chamber $-C_f^v$ by $\mathfrak{S}_{-\infty}$.

A *sector-face* in $\mathbb{A}$ is a V-translate $\mathfrak{f} = x + F^v$ of a vectorial face $F^v = \pm wF^v(J)$. The sector-face-germ of $\mathfrak{f}$ is the filter $\mathfrak{F}$ of subsets containing a translate $\mathfrak{f}'$ of $\mathfrak{f}$ by an element of F^v (i.e., $\mathfrak{f}' \subset \mathfrak{f}$). If F^v is spherical, then $\mathfrak{f}$ and $\mathfrak{F}$ are also called spherical. The sign of $\mathfrak{f}$ and $\mathfrak{F}$ is the sign of F^v.

A *chimney* in $\mathbb{A}$ is associated to a face $F = F(x, F_0^v)$ and to a vectorial face F^v; it is the filter $\mathfrak{r}(F, F^v) = \mathrm{cl}_{\mathbb{A}}(F + F^v)$. The face F is the basis of the chimney and the vectorial face F^v its direction. A chimney $\mathfrak{r} = \mathfrak{r}(F, F^v)$ is *splayed* if F^v is spherical, and it is *solid* if its support (as a filter, i.e., the smallest affine subspace containing $\mathfrak{r}$) has a finite pointwise stabilizer in W^v. A splayed chimney is therefore solid. The enclosure of a sector-face $\mathfrak{f} = x + F^v$ is a chimney.

A ray δ with origin in x and containing $y \neq x$ (or the interval $]x, y]$, the segment $[x, y]$) is called *preordered* if $x \leq y$ or $y \leq x$ and *generic* if $x \overset{o}{\leq} y$ or $y \overset{o}{\leq} x$. With these new notions, a chimney can be defined as the enclosure of a preordered ray and a preordered segment-germ sharing the same origin. The chimney is splayed if, and only if, the ray is generic.

1.5. *The hovel.* In this section, we recall the definition of an ordered affine hovel given by Guy Rousseau in [Rou11].

An apartment of type $\mathbb{A}$ is a set A endowed with a set $\mathrm{Isom}(\mathbb{A}, A)$ of bijections (called isomorphisms) such that if $f_0 \in \mathrm{Isom}(\mathbb{A}, A)$, then $f \in \mathrm{Isom}(\mathbb{A}, A)$ if, and only if, there exists $w \in W^a$ satisfying $f = f_0 \circ w$. An isomorphism between two apartments $\phi : A \to A'$ is a bijection such that $f \in \mathrm{Isom}(\mathbb{A}, A)$ if, and only if, $\phi \circ f \in \mathrm{Isom}(\mathbb{A}, A')$. As the filters in $\mathbb{A}$ defined in 1.4 above (e.g., faces, sectors, walls, etc.) are permuted by W^a; they are well defined in any apartment of type $\mathbb{A}$.

Definition. An ordered affine hovel of type $\mathbb{A}$ is a set $\mathscr{I}$ endowed with a covering $\mathcal{A}$ of subsets called apartments such that

(MA1) any $A \in \mathcal{A}$ admits a structure of an apartment of type $\mathbb{A}$;

(MA2) if F is a point, a germ of a preordered interval, a generic ray or a solid chimney in an apartment A and if A' is another apartment containing F, then $A \cap A'$ contains the enclosure $\mathrm{cl}_A(F)$ of F and there exists an isomorphism from A onto A' fixing $\mathrm{cl}_A(F)$;

(MA3) if $\mathfrak{R}$ is the germ of a splayed chimney and if F is a face or a germ of a solid chimney, then there exists an apartment that contains $\mathfrak{R}$ and F;

(MA4) if two apartments A, A' contain $\mathfrak{R}$ and F as in (MA3), then their intersection contains $\mathrm{cl}_A(\mathfrak{R} \cup F)$ and there exists an isomorphism from A onto A' fixing $\mathrm{cl}_A(\mathfrak{R} \cup F)$;

(MAO) if x, y are two points contained in two apartments A and A', and if $x \leq_A y$, then the two segments $[x,y]_A$ and $[x,y]_{A'}$ are equal.

We ask here $\mathscr{I}$ to be thick of *finite thickness*: the number of chambers (=alcoves) containing a given panel has to be finite ≥ 3. This number is the same for any panel in a given wall M [Rou11, 2.9]; we denote it by $1 + q_M$.

We assume that $\mathscr{I}$ has a strongly transitive group of automorphisms G. (That is, all isomorphisms involved in the above axioms are induced by elements of G; cf. [Rou12, 4.10].) We choose in $\mathscr{I}$ a fundamental apartment, which we identify with $\mathbb{A}$. As G is strongly transitive, the apartments of $\mathscr{I}$ are the sets $g.\mathbb{A}$ for $g \in G$. The stabilizer N of $\mathbb{A}$ in G induces a group $\nu(N)$ of affine automorphisms of $\mathbb{A}$, which permutes the walls, sectors, sector-faces, etc. and contains the affine Weyl group W^a [Rou12, 4.13.1]. We denote the stabilizer of $0 \in \mathbb{A}$ in G by K.

We ask $\nu(N)$ to be *positive* and *type-preserving* for its action on the vectorial faces. This means that the associated linear map $\overrightarrow{w}$ of any $w \in \nu(N)$ is in W^v. As $\nu(N)$ contains W^a and stabilizes $\mathcal{M}$, we have $\nu(N) = W^v \ltimes Y$, where W^v fixes the origin 0 of $\mathbb{A}$ and Y is a group of translations such that $Q^\vee \subset Y \subset P^\vee = \{v \in V \mid \alpha(v) \in \mathbb{Z} \text{ for all } \alpha \in \Phi\}$.

We ask Y to be *discrete* in V. This is clearly satisfied if Φ generates V^*; i.e., $(\alpha_i)_{i \in I}$ is a basis of V^*.

Examples. The main examples of all the above situation are provided by the hovels of almost split Kac-Moody groups over fields complete for a discrete valuation and with a finite residue field; see [Rou10], [Cha10], [Cha] or [Rou12]. Some details in the split case can be found in Section 3.

Remarks. (a) In the following, we often refer to [GR08], which deals with split Kac-Moody groups and residue fields containing $\mathbb{C}$. But the results cited are easily generalized to our present framework using the above references.

(b) For an almost split Kac-Moody group over a local field $\mathcal{K}$, the set of roots Φ is ${}^{\mathcal{K}}\Phi_{\text{red}} = \{{}^{\mathcal{K}}\alpha \in {}^{\mathcal{K}}\Phi \mid \frac{1}{2}.{}^{\mathcal{K}}\alpha \notin {}^{\mathcal{K}}\Phi\}$, where the relative root system ${}^{\mathcal{K}}\Phi$ describes well the commuting relations between the root subgroups. Unfortunately $\widetilde{\Phi}$ gives a worst description of these relations.

1.6. *Type* 0 *vertices.* The elements of Y, through the identification $Y = N.0$, are called *vertices of type* 0 in $\mathbb{A}$; they are special vertices. We note $Y^+ = Y \cap \mathcal{T}$ and $Y^{++} = Y \cap \overline{C^v_f}$. The type 0 vertices in $\mathscr{I}$ are the points on the orbit $\mathscr{I}_0$ of 0 by G. This set $\mathscr{I}_0$ is often called the affine Grassmannian as it is equal to G/K.

In general, G is not equal to $KYK = KNK$ [GR08, 6.10]; i.e., $\mathscr{I}_0 \neq K.Y$. We know that $\mathscr{I}$ is endowed with a G-invariant preorder $\leq$ that induces the known one on $\mathbb{A}$ [Rou11, 5.9]. We set $\mathscr{I}^+ = \{x \in \mathscr{I} \mid 0 \leq x\}$, $\mathscr{I}_0^+ = \mathscr{I}_0 \cap \mathscr{I}^+$

and $G^+ = \{g \in G \mid 0 \leq g.0\}$; so $\mathscr{I}_0^+ = G^+.0 = G^+/K$. As $\leq$ is a G-invariant preorder, G^+ is a semigroup.

If $x \in \mathscr{I}_0^+$, there is an apartment A containing 0 and x (by definition of $\leq$) and all apartments containing 0 are conjugated to $\mathbb{A}$ by K (axiom (MA2)); so $x \in K.Y^+$ as $\mathscr{I}_0^+ \cap \mathbb{A} = Y^+$. But $\nu(N \cap K) = W^v$ and $Y^+ = W^v.Y^{++}$ (with uniqueness of the element in Y^{++}); so $\mathscr{I}_0^+ = K.Y^{++}$; more precisely $\mathscr{I}_0^+ = G^+/K$ is the disjoint union of the KyK/K for $y \in Y^{++}$. Hence, we have proved that the map $Y^{++} \to K\backslash G^+/K$ is one-to-one and onto.

1.7. *Vectorial distance and $Q^\vee$-order.* For $x \in \mathcal{T}$, we denote by x^{++} the unique element in $\overline{C_f^v}$ conjugated by W^v to x.

Let $\mathscr{I} \times_\leq \mathscr{I} = \{(x,y) \in \mathscr{I} \times \mathscr{I} \mid x \leq y\}$ be the set of increasing pairs in $\mathscr{I}$. Such a pair (x,y) is always in a same apartment $g.\mathbb{A}$; so $g^{-1}y - g^{-1}x \in \mathcal{T}$, and we define the *vectorial distance* $d^v(x,y) \in \overline{C_f^v}$ by $d^v(x,y) = (g^{-1}y - g^{-1}x)^{++}$. It does not depend on the choices we made.

For $(x,y) \in \mathscr{I}_0 \times_\leq \mathscr{I}_0 = \{(x,y) \in \mathscr{I}_0 \times \mathscr{I}_0 \mid x \leq y\}$, the vectorial distance $d^v(x,y)$ takes values in Y^{++}. Actually, as $\mathscr{I}_0 = G.0$, K is the stabilizer of 0 and $\mathscr{I}_0^+ = K.Y^{++}$ (with uniqueness of the element in Y^{++}), the map d^v induces a bijection between the set $\mathscr{I}_0 \times_\leq \mathscr{I}_0/G$ of G-orbits in $\mathscr{I}_0 \times_\leq \mathscr{I}_0$ and Y^{++}.

Any $g \in G^+$ is in $K.d^v(0, g.0).K$.

For $x, y \in \mathbb{A}$, we say that $x \leq_{Q^\vee} y$ (resp. $x \leq_{Q^\vee_{\mathbb{R}}} y$) when $y - x \in Q^\vee_+$ (resp. $y - x \in Q^\vee_{\mathbb{R}+} = \sum_{i \in I} \mathbb{R}_{\geq 0}.\alpha_i^\vee$). Thus we get a preorder, which is an order at least when $(\alpha_i^\vee)_{i \in I}$ is free or $\mathbb{R}_+$-free (i.e., $\sum a_i \alpha_i^\vee = 0, a_i \geq 0 \Rightarrow a_i = 0$ for all i).

1.8. *Paths.* We consider piecewise linear continuous paths $\pi : [0,1] \to \mathbb{A}$ such that each (existing) tangent vector $\pi'(t)$ belongs to an orbit $W^v.\lambda$ for some $\lambda \in \overline{C_f^v}$. Such a path is called a λ-*path*; it is increasing with respect to the preorder relation $\leq$ on $\mathbb{A}$.

For any $t \neq 0$ (resp. $t \neq 1$), we let $\pi'_-(t)$ (resp. $\pi'_+(t)$) denote the derivative of π at t from the left (resp. from the right). Further, we define $w_\pm(t) \in W^v$ to be the smallest element in its $(W^v)_\lambda$-class such that $\pi'_\pm(t) = w_\pm(t).\lambda$ (where $(W^v)_\lambda$ is the stabilizer in W^v of λ). Moreover, we denote the negative (resp. positive) segment-germ of π at t by $\pi_-(t) = \pi(t) - [0,1)\pi'_-(t) = [\pi(t), \pi(t-\varepsilon))$ (resp. $\pi_+(t) = \pi(t) + [0,1)\pi'_+(t) = [\pi(t), \pi(t+\varepsilon))$ (for $\varepsilon > 0$ small).

The reverse path $\overline{\pi}$ defined by $\overline{\pi} = \pi(1-t)$ has symmetric properties; it is a $(-\lambda)$-path.

For any choices of $\lambda \in \overline{C_f^v}$, $\pi_0 \in \mathbb{A}$, $r \in \mathbb{N} \setminus \{0\}$ and sequences $\underline{\tau} = (\tau_1, \tau_2, \ldots, \tau_r)$ of elements in $W^v/(W^v)_\lambda$ and $\underline{a} = (a_0 = 0 < a_1 < a_2 < \cdots < a_r = 1)$ of elements in $\mathbb{R}$, we define a λ-path $\pi = \pi(\lambda, \pi_0, \underline{\tau}, \underline{a})$ by the formula

$$\pi(t) = \pi_0 + \sum_{i=1}^{j-1} (a_i - a_{i-1})\tau_i(\lambda) + (t - a_{j-1})\tau_j(\lambda) \quad \text{for} \quad a_{j-1} \leq t \leq a_j.$$

Any λ-path may be defined in this way (and we may assume $\tau_j \neq \tau_{j+1}$).

Definition ([KM08, 3.27]). A *Hecke path* of shape λ with respect to $-C^v_f$ is a λ-path such that $\pi'_+(t) \leq_{W^v_{\pi(t)}} \pi'_-(t)$ for all $t \in [0,1] \setminus \{0,1\}$, which means that there exists a $W^v_{\pi(t)}$-chain from $\pi'_-(t)$ to $\pi'_+(t)$, i.e., finite sequences ($\xi_0 = \pi'_-(t), \xi_1, \ldots, \xi_s = \pi'_+(t)$) of vectors in V and $(\beta_1, \ldots, \beta_s)$ of real roots such that, for all $i = 1, \ldots, s$,

(i) $r_{\beta_i}(\xi_{i-1}) = \xi_i$.
(ii) $\beta_i(\xi_{i-1}) < 0$.
(iii) $r_{\beta_i} \in W^v_{\pi(t)}$; i.e., $\beta_i(\pi(t)) \in \mathbb{Z}$: $\pi(t)$ is in a wall of direction $\mathrm{Ker}(\beta_i)$.
(iv) Each β_i is positive with respect to $-C^v_f$; i.e., $\beta_i(C^v_f) > 0$.

Remarks. (1) The path is folded at $\pi(t)$ by applying successive reflections along the walls $M(\beta_i, -\beta_i(\pi(t)))$. Moreover, conditions (ii) and (iv) tell us that the path is "positively folded" (cf. [GL05]), i.e., centrifugally folded with respect to the sector-germ $\mathfrak{S}_{-\infty} = \mathrm{germ}_\infty(-C^v_f)$.

(2) Let $\mathfrak{c}_- = \mathrm{germ}_0(-C^v_f)$ be the negative fundamental chamber (= alcove). A *Hecke path* of shape λ with respect to $\mathfrak{c}_-$ [BCGR] is a λ-path in the Tits cone $\mathcal{T}$ satisfying the above conditions except that we replace (iv) by

(iv′) each β_i is positive with respect to $\mathfrak{c}_-$; i.e., $\beta_i(\pi(t) - \mathfrak{c}_-) > 0$.

Then (ii) and (iv′) tell us that the path is centrifugally folded with respect to the center $\mathfrak{c}_-$.

2. **Convolution algebras**

2.1. *Wanted.* We consider the space

$$\widehat{\mathcal{H}}^{\mathscr{I}}_R = \widehat{\mathcal{H}}_R(\mathscr{I}, G) = \{\varphi^{\mathscr{I}} : \mathscr{I}_0 \times_\leq \mathscr{I}_0 \to R \mid \varphi^{\mathscr{I}}(gx, gy) = \varphi^{\mathscr{I}}(x,y) \forall g \in G\}$$

of G-invariant functions on $\mathscr{I}_0 \times_\leq \mathscr{I}_0$ with values in some ring R (essentially $\mathbb{C}$ or $\mathbb{Z}$). We want to make $\widehat{\mathcal{H}}^{\mathscr{I}}_R$ (or some large subspace) an algebra for the following convolution product:

$$(\varphi^{\mathscr{I}} * \psi^{\mathscr{I}})(x,y) = \sum_{x \leq z \leq y} \varphi^{\mathscr{I}}(x,z)\psi^{\mathscr{I}}(z,y).$$

It is clear that this product is associative and R-bilinear if it exists.

Via d^v, $\widehat{\mathcal{H}}^{\mathscr{I}}_R$ is linearly isomorphic to the space

$$\widehat{\mathcal{H}}_R = \{\varphi^G : Y^{++} = K \backslash G^+ / K \to R\},$$

which can be interpreted as the space of K-bi-invariant functions on G^+. The correspondence $\varphi^{\mathscr{I}} \leftrightarrow \varphi^G$ between $\widehat{\mathcal{H}}^{\mathscr{I}}_R$ and $\widehat{\mathcal{H}}_R$ is given by

$$\varphi^G(g) = \varphi^{\mathscr{I}}(0, g.0) \quad \text{and} \quad \varphi^{\mathscr{I}}(x,y) = \varphi^G(d^v(x,y)).$$

In this setting, the convolution product should be

$$(\varphi^G * \psi^G)(g) = \sum_{h \in G^+/K} \varphi^G(h)\psi^G(h^{-1}g),$$

where we consider φ^G and ψ^G trivial on $G\setminus G^+$. In the following, we shall often make no difference between $\varphi^{\mathscr{I}}$ or φ^G and forget the exponents ${}^{\mathscr{I}}$ and ^{G}.

We consider the subspace $\mathcal{H}_R^f$ of functions with finite support in $Y^{++} = K\backslash G^+/K$; its natural basis is $(c_\lambda)_{\lambda\in Y^{++}}$, where c_λ sends λ to 1 and $\mu \neq \lambda$ to 0. Clearly c_0 is a unit for $*$. In $\widehat{\mathcal{H}}_R^{\mathscr{I}}$, $(c_\lambda * c_\mu)^{\mathscr{I}}(x,y)$ is the number of triangles $[x,z,y]$ with $d^v(x,z)=\lambda$ and $d^v(z,y)=\mu$.

As suggested by [BK11] and Lemma 2.4, we also consider the subspace $\mathcal{H}_R$ of $\widehat{\mathcal{H}}_R$ of functions φ with *almost finite* support; i.e., $\mathrm{supp}(\varphi) \subset \cup_{i=1}^n (\lambda_i - Q_+^\vee) \cap Y^{++}$, where $\lambda_i \in Y^{++}$.

2.2. *Retractions onto* Y^+. For all $x \in \mathscr{I}^+$, there is an apartment containing x and $\mathfrak{c}_-$ [Rou11, 5.1], and this apartment is conjugated to $\mathbb{A}$ by an element of K fixing $\mathfrak{c}_-$ (axiom (MA2)). So, by the usual arguments and [Rou11, 5.5] we can define the retraction $\rho_{\mathfrak{c}_-}$ of $\mathscr{I}^+$ into $\mathbb{A}$ with center $\mathfrak{c}_-$; its image is $\rho_{\mathfrak{c}_-}(\mathscr{I}^+) = \mathcal{T} = \mathscr{I}^+ \cap \mathbb{A}$ and $\rho_{\mathfrak{c}_-}(\mathscr{I}_0^+) = Y^+$. There is also the retraction $\rho_{-\infty}$ of $\mathscr{I}$ onto $\mathbb{A}$ with center the sector-germ $\mathfrak{S}_{-\infty}$ [GR08, 4.4].

For $\rho = \rho_{\mathfrak{c}_-}$ or $\rho_{-\infty}$, the image of any segment $[x,y]$ with $(x,y)\in \mathscr{I}\times_\leq \mathscr{I}$ and $d^v(x,y)=\lambda\in\overline{C_f^v}$ is a λ-path [GR08, 4.4]. In particular, $\rho(x)\leq\rho(y)$.

2.3. *Convolution product*. The convolution product in $\widehat{\mathcal{H}}_R$ should be defined (for $y\in Y^{++}$) by

$$(\varphi * \psi)(y) = \sum \varphi(z)\psi(d^v(z,y)),$$

where the sum runs over the $z \in \mathscr{I}_0^+$ such that $0 \leq z \leq y$ and $\varphi(z) = \varphi^{\mathscr{I}}(0,z) = \varphi^G(d^v(0,z))$.

(1) Using $\rho_{\mathfrak{c}_-}$ we have, for $\lambda,\mu,y\in Y^{++}$,

$$(c_\lambda * c_\mu)(y) = \sum_{w\in W^v/(W^v)_\lambda} N_{\mathfrak{c}_-}(\mu, w.\lambda, y),$$

where $N_{\mathfrak{c}_-}(\mu,w.\lambda,y)$ is the number of $z\in\mathscr{I}_0^+$ with $d^v(z,y)=\mu$ and $\rho_{\mathfrak{c}_-}(z) = w.\lambda\in Y^+$. Note that, if $N_{\mathfrak{c}_-}(\mu,w.\lambda,y)>0$, there exists a μ-path from $w.\lambda$ to y, and hence $y\in w.\lambda+Y^+$.

So $c_\lambda * c_\mu$ is the formal sum $c_\lambda * c_\mu = \sum_{\nu\in Y^{++}} m_{\lambda,\mu}(\nu)c_\nu$ where the structure constant $m_{\lambda,\mu}(\nu) = \sum_{w\in W^v/(W^v)_\lambda} N_{\mathfrak{c}_-}(\mu,w.\lambda,\nu)\in\mathbb{Z}_{\geq 0}\cup\{+\infty\}$ is also equal to the number of triangles $[x,z,y]$ with $d^v(x,z)=\lambda$ and $d^v(z,y)=\mu$ for any fixed pair $(x,y)\in\mathscr{I}_0\times_\leq\mathscr{I}_0$ with $d^v(x,y)=\nu$ (e.g., $(x,y)=(0,\nu)$).

(2) Using $\rho_{-\infty}$ we have $m_{\lambda,\mu}(\nu)=\sum_{z'} N_{-\infty}(\mu,z',\nu)$ where the sum runs over the z' in $Y^+(\lambda) = \rho_{-\infty}(\{z\in\mathscr{I}_0^+ \mid d^v(0,z)=\lambda\})$ and $N_{-\infty}(\mu,z',\nu)\in\mathbb{Z}_{\geq 0}\cup\{+\infty\}$ is the number of $z\in\mathscr{I}_0^+$ with $d^v(0,z)=\lambda$, $d^v(z,y)=\mu$ (for any $y\in\mathscr{I}_0^+$ with $d^v(0,y)=\nu$, e.g., $y=\nu$) and $\rho_{-\infty}(z)=z'$. But $\rho_{-\infty}([0,z])$ is a λ-path hence increasing with respect to $\leq$, so $Y^+(\lambda)\subset Y^+$. Moreover,

$\rho_{-\infty}([z,\nu])$ is a μ-path, so z' has to be in $\nu - Y^+$. Hence, z' has to run over the set $Y^+(\lambda) \cap (\nu - Y^+) \subset Y^+ \cap (\nu - Y^+)$.

Actually, the image by $\rho_{-\infty}$ of any segment $[x,y]$ with $(x,y) \in \mathscr{I} \times_{\leq} \mathscr{I}$ and $d^v(x,y) = \lambda \in Y^{++}$ is a Hecke path of shape λ with respect to $-C^v_f$ [GR08, Th. 6.2]. Hence we have the following results:

Lemma 2.4.

(a) *For $\lambda \in Y^{++}$ and $w \in W^v$, $w\lambda \in \lambda - Q^\vee_+$; i.e., $w\lambda \leq_{Q^\vee} \lambda$.*
(b) *Let π be a Hecke path of shape $\lambda \in Y^{++}$ with respect to $-C^v_f$, from $y_0 \in Y$ to $y_1 \in Y$. Then $\lambda = \pi'_+(0)^{++} = \pi'_-(1)^{++}$, $\pi'_+(0) \leq_{Q^\vee} \lambda$, $\pi'_+(0) \leq_{Q^\vee_\mathbb{R}} (y_1 - y_0) \leq_{Q^\vee_\mathbb{R}} \pi'_-(1) \leq_{Q^\vee} \lambda$ and $y_1 - y_0 \leq_{Q^\vee} \lambda$.*
(c) *If moreover, $(\alpha_i^\vee)_{i\in I}$ is free, we may replace above $\leq_{Q^\vee_\mathbb{R}}$ by $\leq_{Q^\vee}$.*
(d) *For $\lambda,\mu,\nu \in Y^{++}$, if $m_{\lambda,\mu}(\nu) > 0$, then $\nu \in \lambda+\mu-Q^\vee_+$, i.e., $\nu \leq_{Q^\vee} \lambda+\mu$.*

Note. By (d) above, if $x \leq z \leq y$ in $\mathscr{I}_0$, then $d^v(x,y) \leq_{Q^\vee} d^v(x,z) + d^v(z,y)$.

Proof. (a) By definition, for $\lambda \in Y$, $w.\lambda \in \lambda + Q^\vee$, hence (a) follows from [Kac90, 3.12d] used in a realization where $(\alpha_i^\vee)_{i\in I}$ is free.

(b) By definition of Hecke paths in 1.8, $\lambda = \pi'_+(0)^{++} = \pi'_-(1)^{++}$. Moreover, for all $t \in [0,1]$, $\lambda = \pi'_-(t)^{++} = \pi'_+(t)^{++}$, and we know how to get $\pi'_+(t)$ from $\pi'_-(t)$ by successive reflections; this proves that $\pi'_+(t) \in \pi'_-(t) + Q^\vee_{\mathbb{R}+}$. By integrating the locally constant function $\pi'(t)$, we get

$$\pi'_+(0) \leq_{Q^\vee_\mathbb{R}} (y_1 - y_0) \leq_{Q^\vee_\mathbb{R}} \pi'_-(1) \leq_{Q^\vee_\mathbb{R}} \lambda.$$

It is proved (but not stated) in [GR08, 5.3.3] that any Hecke path of shape λ starting in $y_0 \in Y$ can be transformed in the path $\pi_\lambda(t) = y_0 + \lambda t$ by applying successively the operators e_{α_i} or $\widetilde{e}_{\alpha_i}$ for $i \in I$; moreover, $e_{\alpha_i}(\pi)(1) = \pi(1) + \alpha_i^\vee$ and $\widetilde{e}_{\alpha_i}(\pi)(1) = \pi(1)$, hence $y_1 - y_0 \leq_{Q^\vee} \lambda$.

(c) By (b), $y_1 - y_0 - \pi'_+(0) \in Q^\vee_{\mathbb{R}+} \cap Q^\vee = Q^\vee_+$, so $\pi'_+(0) \leq_{Q^\vee} (y_1 - y_0)$. The same follows for $y_1 - y_0 \leq_{Q^\vee} \pi'_-(1)$.

(d) If $m_{\lambda,\mu}(\nu) > 0$, we have an Hecke path of shape λ (resp. μ) from 0 to z' (resp. from z' to ν). So (d) follows from (b). □

Proposition 2.5. *Suppose $(\alpha_i^\vee)_{i\in I}$ is free in V. Then for all $\lambda,\mu,\nu \in Y^{++}$, $m_{\lambda,\mu}(\nu)$ is finite.*

Note. Actually, we may replace the condition on freeness of $(\alpha_i^\vee)_{i\in I}$ by the condition on $\mathbb{R}^+$-freeness.

Proof. We have to count the $z \in \mathscr{I}_0^+$ such that $d^v(0,z) = \lambda$ and $d^v(z,\nu) = \mu$. We set $z' = \rho_{-\infty}(z)$. By Lemma 2.4b, $z' \in \lambda - Q^\vee_+$ and $\nu \in z' + \mu - Q^\vee_+$. Hence z' is in $(\lambda - Q^\vee_+) \cap (\nu - \mu + Q^\vee_+)$, which is finite as $(\alpha_i^\vee)_{i\in I}$ is free or $\mathbb{R}^+$-free. So, we fix now z'. By [GR08, Cor. 5.9] there is a finite number of Hecke paths π' of shape μ from z' to ν. So, we fix now π'. And by [GR08, Th. 6.3] (see also

Remark 4.10 and Proposition 4.11) there is a finite number of segments $[z,\nu]$ retracting to π'; hence the number of z is finite. □

THEOREM 2.6. *Suppose $(\alpha_i^\vee)_{i\in I}$ is free or $\mathbb{R}^+$-free, then $\mathcal{H}_R$ is an algebra.*

Proof. We saw that for $\lambda,\mu,\nu\in Y^{++}$, $m_{\lambda,\mu}(\nu)$ is finite; hence $c_\lambda * c_\mu$ is well defined (eventually as an infinite formal sum). Let us consider $\varphi,\psi\in\mathcal{H}_R$: $\mathrm{supp}(\varphi)\subset\cup_{i=1}^m(\lambda_i-Q_+^\vee)$, $\mathrm{supp}(\psi)\subset\cup_{j=1}^n(\mu_j-Q_+^\vee)$. Let $\nu\in Y^{++}$. If $m_{\lambda,\mu}(\nu)>0$ with $\lambda\in\mathrm{supp}(\varphi)$, $\mu\in\mathrm{supp}(\psi)$ (hence $\lambda\in\lambda_i-Q_+^\vee$, $\mu\in\mu_j-Q_+^\vee$ for some i,j), we have $\lambda+\mu\in\nu+Q_+^\vee$ by Lemma 2.4(d). So,

$$\lambda\in(\nu-\mu+Q_+^\vee)\cap(\lambda_i-Q_+^\vee)\subset(\nu-\mu_j+Q_+^\vee)\cap(\lambda_i-Q_+^\vee),$$

which is a finite set. For the same reasons μ is in a finite set, so $\varphi*\psi$ is well defined.

With the above notation, $\nu\in(\lambda+\mu-Q_+^\vee)\subset\cup_{i,j}(\lambda_i+\mu_j-Q_+^\vee)$, so $\varphi*\psi\in\mathcal{H}_R$. □

Definition 2.7. $\mathcal{H}_R=\mathcal{H}_R(\mathscr{I},G)$ is the *spherical Hecke algebra* (with coefficients in R) associated to the hovel $\mathscr{I}$ and its strongly transitive automorphism group G.

Remark. We shall now investigate $\mathcal{H}_R$ and some other possible convolution algebras in $\widehat{\mathcal{H}}_R$ by separating the cases: finite, indefinite and affine.

2.8. *Finite case.* In this case Φ and W^v are finite, $(\alpha_i^\vee)_{i\in I}$ is free, $\mathcal{T}=V$ and the relation $\leq$ is trivial. The hovel $\mathscr{I}=\mathscr{I}^+$ is a locally finite Bruhat-Tits building.

Let ρ be the half-sum of positive roots. As $2\rho\in Q$ and $\rho(\alpha_i^\vee)=1$ for all $i\in I$, we see that an almost finite set in Y^{++} is always finite. So $\mathcal{H}_R$ and $\mathcal{H}_R^f$ are equal.

The algebra $\mathcal{H}_\mathbb{C}$ was already studied by I. Satake in [Sat63]. Its close link with buildings is explained in [Par06]. The algebra $\mathcal{H}_\mathbb{Z}$ is the spherical Hecke ring of [KLM08], where the interpretation of $m_{\lambda,\mu}(\nu)$ as a number of triangles in $\mathscr{I}$ is already given.

Note that $\widehat{\mathcal{H}}_R$ is not an algebra as, e.g., $m_{\lambda,(-w_0)\lambda}(0)\neq 0$ for all $\lambda\in Y^{++}$ (where w_0 is the greatest element in W^v).

2.9. *Indefinite case.*

LEMMA. *Suppose now Φ associated to an indefinite indecomposable generalized Cartan matrix. Then there is an element δ in Δ_{im}^+ (of support I) such that $\delta(\alpha_i^\vee)<0$ for all $i\in I$ and a basis $(\delta_i)_{i\in I}$ of the real vector space $Q_\mathbb{R}$ spanned by Φ such that $\delta_i(\mathcal{T})\geq 0$ for all $i\in I$.*

Proof. Any $\delta\in\Delta_{im}^+$ takes positive values on $\mathcal{T}$ [Kac90, 5.8]. Now, in the indefinite case, there is $\delta\in\Delta_{im}^+\cap(\oplus_{i\in I}\mathbb{R}_{>0}.\alpha_i)$ such that $\delta(\alpha_i^\vee)<0$ for all

$i \in I$ [Kac90, 4.3], and hence $\delta + \alpha_i \in \Delta^+$ for all $i \in I$. Eventually replacing δ by 3δ [Kac90, 5.5], we have $(\delta + \alpha_i)(\alpha_j^\vee) < 0$ for all $i, j \in I$, and hence $\delta + \alpha_i \in \Delta^+_{im}$. The wanted basis is inside $\{\delta\} \cup \{\delta + \alpha_i \mid i \in I\}$. □

The existence of $\delta \in \Delta^+_{im}$ as in the lemma proves that $(\alpha_i^\vee)_{i\in I}$ is $\mathbb{R}^+$-free. So $\mathcal{H}_R$ is an algebra. The following Example 2.10 proves that $\mathcal{H}_R^f$ is in general not a subalgebra.

If $(\alpha_i)_{i\in I}$ generates (i.e., is a basis of) V^*, $\widehat{\mathcal{H}}_R$ is also an algebra (the *formal spherical Hecke algebra*). Let $\nu \in Y^{++}$. We have to prove that there is only a finite number of pairs $(\lambda, \mu) \in (Y^{++})^2$ such that $m_{\lambda,\mu}(\nu) > 0$. Let z' be as in the proof of 2.5. We saw in 2.3 that $z' \in Y^+ \cap (\nu - Y^+) = Y \cap \mathcal{T} \cap (\nu - \mathcal{T})$. By the lemma, $\mathcal{T} \cap (\nu - \mathcal{T})$ is bounded; hence $Y \cap \mathcal{T} \cap (\nu - \mathcal{T})$ is finite. So we may fix z'. Now $\lambda \in z' + Q_+^\vee$, and hence (for δ as in the lemma) $\delta(\lambda) \leq \delta(z')$; as $\alpha_i(\lambda) \in \mathbb{Z}_{\geq 0}$ for all $i \in I$ and $\delta \in \oplus_{i\in I} \mathbb{R}_{>0}.\alpha_i$, this gives only a finite number of possibilities for λ. Similarly, $\mu \in \nu - z' + Q_+^\vee$ has to be in a finite set.

Actually $\widehat{\mathcal{H}}_R$ is often equal to $\mathcal{H}_R$ when $(\alpha_i^\vee)_{i\in I}$ is free and $(\alpha_i)_{i\in I}$ generates V^* (hence the matrix $\mathbb{M} = (\alpha_j(\alpha_i^\vee))$ is invertible); see Example 2.10.

2.10. *An indefinite rank* 2 *example.* Let us consider the Kac-Moody matrix $\mathbb{M} = \left(\begin{smallmatrix} 2 & -3 \\ -3 & 2 \end{smallmatrix}\right)$. A basis of Φ and of V^* is $\{\alpha_1, \alpha_2\}$, and we consider the dual basis $(\varpi_1^\vee, \varpi_2^\vee)$ of V. In this basis, $\alpha_1^\vee = \left(\begin{smallmatrix} 2 \\ -3 \end{smallmatrix}\right)$, $\alpha_2^\vee = \left(\begin{smallmatrix} -3 \\ 2 \end{smallmatrix}\right)$ and the matrices of r_1, r_2, $r_2 r_1$ and $r_1 r_2$ are respectively $\left(\begin{smallmatrix} -1 & 0 \\ 3 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 3 \\ 0 & -1 \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} 8 & 3 \\ -3 & -1 \end{smallmatrix}\right)$ and $M^{-1} = \left(\begin{smallmatrix} -1 & -3 \\ 3 & 8 \end{smallmatrix}\right)$. The eigenvalues of M or M^{-1} are $a_\pm = (7 \pm \sqrt{45})/2$. In any basis diagonalizing M and M^{-1} we see easily that $(r_2 r_1)^n + (r_1 r_2)^n = a_n.\mathrm{Id}_V$, where $a_n = a_+^n + a_-^n$ is in $\mathbb{N}$ and increasing up to infinity ($a_0 = 2$, $a_1 = 7$, $a_2 = 47$, $a_3 = 322, \ldots$).

Consider now $\lambda = \mu = -\alpha_1^\vee - \alpha_2^\vee = \left(\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right)$ in $Y^{++} \subset \mathbb{Z}_{\geq 0}.\varpi_1^\vee \oplus \mathbb{Z}_{\geq 0}.\varpi_2^\vee$. We have $(r_2 r_1)^n.\lambda + (r_1 r_2)^n.\lambda = a_n.\lambda$. This means that

$$m_{\lambda,\lambda}(a_n.\lambda) \geq N_{\mathfrak{c}_-}(\lambda, (r_2 r_1)^n \lambda, a_n.\lambda) \geq 1$$

for all positive n (and the same thing for $N_{-\infty}$). So $c_\lambda * c_\lambda$ is an infinite formal sum. Actually $(-Q_+^\vee) \cap Y^{++} \supset \mathbb{Z}_{\geq 0}.5\varpi_1^\vee \oplus \mathbb{Z}_{\geq 0}.5\varpi_2^\vee$; hence Y^{++} itself is almost finite!

2.11. *An affine rank* 2 *example.* Let us consider the Kac-Moody matrix $\mathbb{M} = \left(\begin{smallmatrix} 2 & -2 \\ -2 & 2 \end{smallmatrix}\right)$. A basis of Φ is $\{\alpha_1, \alpha_2\}$, but we consider a realization V of dimension 3 for which $\{\alpha_1^\vee, \alpha_2^\vee\}$ is free and with basis of V^*, $\{\alpha_o = -\rho, \alpha_1, \alpha_2\}$. More precisely, if $(\varpi_0^\vee, \varpi_1^\vee, \varpi_2^\vee)$ is the dual basis of V, we have $\alpha_1^\vee = \left(\begin{smallmatrix} -1 \\ 2 \\ -2 \end{smallmatrix}\right)$, $\alpha_2^\vee = \left(\begin{smallmatrix} -1 \\ -2 \\ 2 \end{smallmatrix}\right)$ and the matrices of r_1, r_2, $r_1 r_2$ and $r_2 r_1$ are respectively $\left(\begin{smallmatrix} 1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 2 & 1 \end{smallmatrix}\right)$, $\left(\begin{smallmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & -1 \end{smallmatrix}\right)$, $M = \left(\begin{smallmatrix} 1 & 1 & 3 \\ 0 & -1 & -2 \\ 0 & 2 & 3 \end{smallmatrix}\right)$ and $M^{-1} = \left(\begin{smallmatrix} 1 & 3 & 1 \\ 0 & 3 & 2 \\ 0 & -2 & -1 \end{smallmatrix}\right)$. A classical calculus using triangulation tells us that $(r_2r_1)^n + (r_1r_2)^n = \begin{pmatrix} 2 & 4n^2 & 4n^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$. Actually, $c = \alpha_1^\vee + \alpha_2^\vee = -2\varpi_0^\vee \in Q_+^\vee$ is the canonical central element [Kac90, §6.2] and the above calculations are peculiar cases of [Kac90, §6.5].

Now let us consider $\lambda = \mu = \sum_{i=1}^2 a_i\varpi_i^\vee \in Y^{++} \subset \oplus_{i=1}^2 \mathbb{Z}_{\geq 0}\varpi_i^\vee$. We have $(r_2r_1)^n(\lambda) + (r_1r_2)^n(\lambda) = \lambda - 2n^2|\lambda|c$ with $|\lambda| = a_1 + a_2$. This means that $m_{\lambda,\lambda}(\lambda - 2n^2|\lambda|c) \geq N_{\mathfrak{c}_-}(\lambda, (r_2r_1)^n(\lambda), \lambda - 2n^2|\lambda|c) \geq 1$ for all $n \in \mathbb{Z}$ (and the same thing for $N_{-\infty}$). So $c_\lambda * c_\lambda$ is an infinite formal sum.

Moreover, as c is fixed by r_1 and r_2, $(r_2r_1)^n(\lambda + 2n^2|\lambda|c) + (r_1r_2)^n(\lambda) = \lambda$, so $m_{\lambda+2n^2|\lambda|c,\lambda}(\lambda) \geq 1$ for all $n \in \mathbb{Z}$, and $\widehat{\mathcal{H}}_R$ is not an algebra. Remark also that, if we consider the essential quotient $V^e = V/\mathbb{R}c$, the above calculus tells that $m_{\lambda,\lambda}(\lambda) \geq \sum_{n\in\mathbb{Z}} N_{\mathfrak{c}_-}(\lambda, (r_2r_1)^n(\lambda), \lambda)$ is infinite if $|\lambda| > 0$.

2.12. *Affine indecomposable case.* We saw in Example 2.11 that $m_{\lambda,\lambda}(\lambda)$ may be infinite for all $\lambda \in Y^{++}$ when $(\alpha_i^\vee)_{i\in I}$ is not free. So, in this case, $\widehat{\mathcal{H}}_R$ seems to contain no algebra except $R.c_0$. Remark also that $(\alpha_i^\vee)_{i\in I}$ free is equivalent to $(\alpha_i^\vee)_{i\in I}$ $\mathbb{R}^+$-free in the affine indecomposable case as the only possible relation between the $\alpha_i^\vee$ is $c = 0$, where $c = \sum_{i\in I} a_i^\vee.\alpha_i^\vee$ (with $a_i^\vee \in \mathbb{Z}_{>0}$ for all $i \in I$) is the canonical central element.

An almost finite subset in Y^{++} is a finite union of subsets like $Y_\lambda = (\lambda - Q_+^\vee) \cap Y^{++}$. Let δ be the smallest positive imaginary root in Δ. Then $\delta(Q_+^\vee) = 0$ so $Y_\lambda \subset \{y \in Y^{++} \mid \delta(y) = \delta(\lambda)\} = Y'_\lambda$. But $\delta = \sum_{i\in I} a_i.\alpha_i$ with $a_i \in \mathbb{Z}_{>0}$ for all $i \in I$, so the image of Y'_λ in $V^e = V/\mathbb{R}c$ (where $\mathbb{R}c = \cap_{i\in I}\operatorname{Ker}(\alpha_i)$) is finite. It is now clear that Y_λ is a finite union of sets like $\mu - \mathbb{Z}_{\geq 0}.c$ with $\mu \in Y^{++}$. Hence an almost finite subset as defined above is the same as an almost finite union (of double cosets) as defined in [BK11].

The algebra $\mathcal{H}_\mathbb{C}$ is the one introduced by A. Braverman and D. Kazhdan in [BK11]. We gave above a combinatorial proof that it is an algebra, without algebraic geometry.

3. The split Kac-Moody case

3.1. *Situation.* As in [Rou10] or [Rou12], we consider a split Kac-Moody group $\mathfrak{G}$ associated to a root generating system

$$\mathcal{S} = (\mathbb{M}, Y_\mathcal{S}, (\overline{\alpha}_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$$

over a field $\mathcal{K}$ endowed with a discrete valuation ω (with value group $\Lambda = \mathbb{Z}$ and ring of integers $\mathcal{O} = \omega^{-1}([0, +\infty]))$ whose residue field $\kappa = \mathbb{F}_q$ is finite. So, $\mathbb{M} = (a_{i,j})_{i,j\in I}$ is a Kac-Moody matrix, $Y_\mathcal{S}$ a free $\mathbb{Z}$-module, $(\alpha_i^\vee)_{i\in I}$ a family in $Y_\mathcal{S}$, $(\overline{\alpha}_i)_{i\in I}$ a family in the dual $X = Y_\mathcal{S}^*$ of $Y_\mathcal{S}$ and $\overline{\alpha}_j(\alpha_i^\vee) = a_{i,j}$. We denote by W^v the associated Weyl group.

If $(\overline{\alpha}_i)_{i\in I}$ is free in X, we consider $V = V_Y = Y_\mathcal{S} \otimes_\mathbb{Z} \mathbb{R}$ and the quadruple $(V, W^v, (\alpha_i = \overline{\alpha}_i)_{i\in I}, (\alpha_i^\vee)_{i\in I})$. In general, we may define $Q = \mathbb{Z}^I$ with canonical basis $(\alpha_i)_{i\in I}$; then $V = V_Q = \operatorname{Hom}_\mathbb{Z}(Q, \mathbb{R})$ is also in a quadruple as in 1.1.

A third example V^{xl} of choice for V is explained in [Rou12]. We always denote by bar : $Q \to X$ the linear map sending α_i to $\overline{\alpha}_i$.

With these vectorial data we may define what was considered in 1.1 and 1.2. We choose $\Lambda_\alpha = \Lambda = \mathbb{Z}$ for all $\alpha \in \Phi$. Now the hovel $\mathscr{I}$ in 1.5 is as defined in [Rou10] or [Rou12], and the strongly transitive group is $G = \mathfrak{G}(\mathcal{K})$. By [Rou11, 6.11] or [Rou10, 5.16] we have $q_M = q$ for any wall M. When $\mathfrak{G}$ is a split reductive group, $\mathscr{I}$ is its extended Bruhat-Tits building.

3.2. *Generators for G.* The Kac-Moody group $\mathfrak{G}$ contains a split maximal torus $\mathfrak{T}$ with character group X and cocharacter group $Y_\mathcal{S}$. We set $T = \mathfrak{T}(\mathcal{K})$. For each $\alpha \in \Phi \subset Q$, there is a group homomorphism $x_\alpha : \mathcal{K} \to G$ that is one-to-one; its image is the subgroup U_α. Now G is generated by T and the subgroups U_α for $\alpha \in \Phi$, submitted to some relations given by Tits [Tit87], also available in [Rém02] or [Rou10]. We denote the subgroup generated by the subgroups U_α, for $\alpha \in \Phi^\pm$, by $U^\pm$.

Now we shall explain only a few of the relations. For $u \in \mathcal{K}$, $t \in T$ and $\alpha \in \Phi$, one has

(KMT4) $t.x_\alpha(u).t^{-1} = x_\alpha(\overline{\alpha}(t).u)$ (where $\overline{\alpha} = \mathrm{bar}(\alpha)$).

For $u \neq 0$, we note $\widetilde{s}_\alpha(u) = x_\alpha(u).x_{-\alpha}(u^{-1}).x_\alpha(u)$ and $\widetilde{s}_\alpha = \widetilde{s}_\alpha(1)$.

(KMT5) $\widetilde{s}_\alpha(u).t.\widetilde{s}_\alpha(u)^{-1} = r_\alpha(t)$. ($W^v$ acts on V, $Y_\mathcal{S}$, X and hence on T.)

3.3. *Weyl groups.* Actually the stabilizer N of $\mathbb{A} \subset \mathscr{I}$ is the normalizer of $\mathfrak{T}$ in G. The image $\nu(N)$ of N in $\mathrm{Aut}(\mathbb{A})$ is a semi-direct product $\nu(N) = \nu(N_0) \ltimes \nu(T)$ with

- N_0 is the stabilizer of 0 in N and $\nu(N_0)$ is isomorphic to W^v acting linearly on $\mathbb{A} = V$. Actually $\nu(N_0)$ is generated by the elements $\nu(\widetilde{s}_\alpha)$, which act as r_α (for $\alpha \in \Phi$).
- $t \in T$ acts on $\mathbb{A}$ by a translation of vector $\nu(t) \in V$ such that $\overline{\chi}(\nu(t)) = -\omega(\chi(t))$ for any $\overline{\chi} \in X = Y_\mathcal{S}^*$ and $\chi \in X$ or Q that are related by $\overline{\chi} = \chi$ if $V = V_Y$ or $\overline{\chi} = \mathrm{bar}(\chi)$ if $V = V_Q$.

So, $\nu(N) = W^v \ltimes Y$ where Y is closely related to $Y_\mathcal{S} \simeq T/\mathfrak{T}(\mathcal{O})$: as $\Lambda = \omega(\mathcal{K}) = \mathbb{Z}$, they are equal if $V = V_Y$ and if $V = V_Q$, $Y = \mathrm{bar}^*(Y_\mathcal{S})$ is the image of $Y_\mathcal{S}$ by the map $\mathrm{bar}^* : Y_\mathcal{S} \to \mathrm{Hom}_\mathbb{Z}(Q,\mathbb{Z})$ dual to bar.

So, the choice $V = V_Y$ is more pleasant. The choice $V = V_Q$ is made, e.g., in [Cha10], [Cha] or [Rém02], and has good properties in the indefinite case; cf. 2.9. They both coincide when $(\overline{\alpha}_i)_{i \in I}$ is a basis of $X \otimes \mathbb{R} = V_Y^*$. This assumption generalizes semi-simplicity; in particular, the center of $\mathfrak{G}$ is then finite [Rém02, 9.6.2].

3.4. *The group K.* The group $K = G_0$ should be equal to $\mathfrak{G}(\mathcal{O})$ for some integral structure of $\mathfrak{G}$ over $\mathcal{O}$; cf. [GR08, 3.14]. But the appropriate integral structure is difficult to define in general. So, we define K by its generators.

The group N_0 is generated by $T_0 = \mathfrak{T}(\mathcal{O}) = T \cap K$ and the elements $\widetilde{s}_\alpha$ for $\alpha \in \Phi$. (This is clear by 3.3.) The group U_0, generated by the groups $U_{\alpha,0} = x_\alpha(\mathcal{O})$ for $\alpha \in \Phi$, is in K. We set $U_0^{\pm} = U_0 \cap U^{\pm}$. In general, $U_0^{\pm}$ is not generated by the groups $U_{\alpha,0}$ for $\alpha \in \Phi^{\pm}$ [Rou10, 4.12.3a].

It is likely that K may be greater than the group generated by N_0 and U_0 (i.e., by U_0 and T_0). We have to define groups $U_0^{pm+} \supset U_0^+$ and $U_0^{nm-} \supset U_0^-$ as follows. In some formal positive completion $\widehat{G}^+$ of G, we can define the subgroup $U_0^{ma+} = \prod_{\alpha \in \Delta^+} U_{\alpha,0}$ of the subgroup $U^{ma+} = \prod_{\alpha \in \Delta^+} U_\alpha$ of $\widehat{G}^+$, with $U^+ \subset U^{ma+}$ (where $U_{\alpha,0}$ and U_α are suitably defined for α imaginary). Then $U_0^{pm+} = U_0^{ma+} \cap G = U_0^{ma+} \cap U^+$. The group U_0^{nm-} is defined similarly with Δ^- using the group $U_0^{ma-} \subset U^{ma-}$ in some formal negative completion $\widehat{G}^-$ of G.

Now $K = G_0 = U_0^{nm-}.U_0^+.N_0 = U_0^{pm+}.U_0^-.N_0$ (see [Rou10, 4.14, 5.1]).

Remark. Let us denote by K_1 the group used by A. Braverman, D. Kazhdan and M. Patnaik in their definition of the spherical Hecke algebra. With the notation above, K_1 is generated by T_0 and U_0, i.e., by T_0, U_0^+ and U_0^-, hence $K = U_0^{nm-}.K_1 = U_0^{pm+}.K_1$, with $U_0^- \subset U_0^{nm-} \subset U^-$ and $U_0^+ \subset U_0^{pm+} \subset U^+$. But they prove, at least in the untwisted affine case, that $U^- \cap U^+.K_1 \subset K_1$ [BKP, proof of Lemma A.3]; so $U_0^{nm-} \subset U^- \cap K \subset U^- \cap U^+.K_1 \subset K_1$ and $K = K_1$. This result answers positively a question in [Rou12, 5.4], at least for points of type 0 and in the untwisted affine split case.

PROPOSITION 3.5. *There is an involution θ (called Chevalley involution) of the group G such that $\theta(t) = t^{-1}$ for all $t \in T$ and $\theta(x_\alpha(u)) = x_{-\alpha}(u)$ for all $\alpha \in \Phi$ and $u \in \mathcal{K}$. Moreover, K is θ-stable and θ induces the identity on $W^v = N/T$.*

Proof. This involution is well known on the corresponding complex Lie algebra. See [Kac90, 1.3.4], where for the generators e_α, one uses a different convention from ours ($[e_\alpha, e_{-\alpha}] = -\alpha^\vee$ as in [Tit87] or [Rém02]). Hence the proposition follows when κ contains $\mathbb{C}$ or is at least of characteristic 0. But here we have to use the definition of G by generators and relations.

We see in [Rou10, 1.5, 1.7.5] that $\widetilde{s}_\alpha(-u) = \widetilde{s}_\alpha(u)^{-1}$ and $\widetilde{s}_\alpha(u) = \widetilde{s}_{-\alpha}(u^{-1})$. So for the wanted involution θ, we have $\theta(\widetilde{s}_\alpha(u)) = \widetilde{s}_{-\alpha}(u) = \widetilde{s}_\alpha(u^{-1})$. We now have to verify the relations between the $\theta(x_\alpha(u)) = x_{-\alpha}(u)$, $\theta(t) = t^{-1}$ and $\theta(\widetilde{s}_\alpha(u)) = \widetilde{s}_\alpha(u^{-1})$. This is clear for (KMT4) and (KMT5) (as $r_\alpha = r_{-\alpha}$). The three other relations are

(KMT3) $(x_\alpha(u), x_\beta(v)) = \prod x_\gamma(C_{p,q}^{\alpha,\beta}.u^p v^q)$ for $(\alpha, \beta) \in \Phi^2$ prenilpotent and, for the product, $\gamma = p\alpha + q\beta$ runs in $(\mathbb{Z}_{>0}\alpha + \mathbb{Z}_{>0}\beta) \cap \Phi$. But the integers $C_{p,q}^{\alpha,\beta}$ are picked up from the corresponding formula between exponentials in the automorphism group of the corresponding complex Lie algebra. As we know that θ is defined in this Lie algebra, we have $C_{p,q}^{-\alpha,-\beta} = C_{p,q}^{\alpha,\beta}$ and (KMT3) is still true for the images by θ.

(KMT6) $\widetilde{s}_\alpha(u^{-1}) = \widetilde{s}_\alpha.\alpha^\vee(u)$ for α simple and $u \in \mathcal{K} \setminus \{0\}$. This is still true after applying θ as $\theta(\widetilde{s}_\alpha(u^{-1})) = \widetilde{s}_\alpha(u)$ and $(-\alpha)^\vee(u) = \alpha^\vee(u^{-1})$.

(KMT7) $\widetilde{s}_\alpha.x_\beta(u).\widetilde{s}_\alpha^{-1} = x_\gamma(\varepsilon.u)$ if $\gamma = r_\alpha(\beta)$ and $\widetilde{s}_\alpha(e_\beta) = \varepsilon.e_\gamma$ in the Lie algebra (with $\varepsilon = \pm 1$). This is still true after applying θ because $\widetilde{s}_\alpha(e_\beta) = \varepsilon.e_\gamma \Rightarrow \widetilde{s}_\alpha(e_{-\beta}) = \varepsilon.e_{-\gamma}$ (as $r_\alpha(\beta^\vee) = \gamma^\vee$).

So, θ is a well-defined involution of G, $\theta(U_0) = U_0$, $\theta(N_0) = N_0$ and $\theta(U_0^\pm) = U_0^\mp$. But the isomorphism θ of U^+ onto U^- can clearly be extended to an isomorphism θ from U^{ma+} onto U^{ma-} sending U_0^{ma+} onto U_0^{ma-}. So $\theta(U_0^{pm+}) = U_0^{nm-}$ and $\theta(K) = K$. As $\theta(\widetilde{s}_\alpha) = \widetilde{s}_\alpha$, θ induces the identity on $W^v = N/T$. □

THEOREM 3.6. *The algebra $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$ is commutative, when it exists.*

Notation. To be clearer we shall sometimes write $\widehat{\mathcal{H}}_R(\mathfrak{G}, \mathcal{K})$ or $\mathcal{H}_R(\mathfrak{G}, \mathcal{K})$ instead of $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$.

Proof. The formula $\theta^\#(g) = \theta(g^{-1})$ defines an anti-involution ($\theta^\#(gh) = \theta^\#(h).\theta^\#(g)$) of G that induces the identity on T and stabilizes K. In particular, $\theta^\#(G^+) = \theta^\#(KY^{++}K) = G^+$ and $\theta^\#(K\lambda K) = K\lambda K$ for all $\lambda \in Y^{++}$. For $\varphi, \psi \in \widehat{\mathcal{H}}_R$ and $g \in G^+$, one has

$$(\varphi * \psi)(g) = (\varphi * \psi)(\theta^\#(g)) = \sum_{h \in G^+/K} \varphi(h)\psi(h^{-1}\theta^\#(g)).$$

The map $h \mapsto h' = \theta^\#(h^{-1}\theta^\#(g)) = g\theta^\#(h^{-1})$ is one-to-one from G^+/K onto G^+/K. So,

$$\begin{aligned}(\varphi * \psi)(g) &= \sum_{h' \in G^+/K} \varphi(\theta^\#(h'^{-1}g))\psi(\theta^\#(h')) \\ &= \sum_{h' \in G^+/K} \varphi(h'^{-1}g)\psi(h') = (\psi * \varphi)(g).\end{aligned}$$ □

Remarks 3.7. (1) Below, this commutativity will be proved in general, as a consequence of the Satake isomorphism. The above proof generalizes well-known proofs in the reductive case; e.g., for $\mathfrak{G} = \mathrm{GL}_n$, $\theta^\#$ is the transposition.

(2) When $\mathfrak{G}$ is an almost split Kac-Moody group over the field $\mathcal{K}$ (supposed complete or henselian) it splits over a finite Galois extension $\mathcal{L}$, the hovel ${}^{\mathcal{K}}\mathscr{I}$ over $\mathcal{K}$ exists and embeds in the hovel ${}^{\mathcal{L}}\mathscr{I}$ over $\mathcal{L}$ [Rou12, §6]. After eventually enlarging $\mathcal{L}$, one may suppose that 0 is a special point in ${}^{\mathcal{K}}\mathscr{I}$ and ${}^{\mathcal{L}}\mathscr{I}$ — more precisely, in the fundamental apartments ${}^{\mathcal{K}}\mathbb{A} \subset {}^{\mathcal{L}}\mathbb{A} = \mathbb{A}$ associated respectively to a maximal $\mathcal{K}$-split torus ${}_{\mathcal{K}}\mathfrak{S}$ and a $\mathcal{L}$-split maximal torus $\mathfrak{T} \supset {}_{\mathcal{K}}\mathfrak{S}$. If we make a good choice of the homomorphisms $x_\alpha : \mathcal{L} \to \mathfrak{G}(\mathcal{L})$, the associated involution θ of $\mathfrak{G}(\mathcal{L})$ should commute with the action of the Galois group

$\Gamma = \mathrm{Gal}(\mathcal{L}/\mathcal{K})$ and hence induce an involution ${}^{\mathcal{K}}\theta$ and an anti-involution ${}^{\mathcal{K}}\theta^{\#}$ of $\mathfrak{G}(\mathcal{K}) = \mathfrak{G}(\mathcal{L})^{\Gamma}$ such that ${}^{\mathcal{K}}\theta(K) = {}^{\mathcal{K}}\theta^{\#}(K) = K$ and ${}^{\mathcal{K}}\theta^{\#}$ induces the identity in $Y({}_{\mathcal{K}}\mathfrak{S}) = {}_{\mathcal{K}}\mathfrak{S}(K)/{}_{\mathcal{K}}\mathfrak{S}(\mathcal{O})$. The commutativity of $\widehat{\mathcal{H}}_R(\mathfrak{G},\mathcal{K})$ or $\mathcal{H}_R(\mathfrak{G},\mathcal{K})$ would follow.

This strategy works well when $\mathfrak{G}$ is quasi split over $\mathcal{K}$; unfortunately, it seems to fail in the general case.

(3) The commutativity of $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$ is related to the choice of a special vertex for the origin 0. Even in the semi-simple case, other choices may give noncommutative convolution algebras; see [Sat63] and [KR07].

4. **Structure constants**

We come back to the general framework of Section 1. We shall compute the structure constants of $\widehat{\mathcal{H}}_R$ or $\mathcal{H}_R$ by formulas depending on $\mathbb{A}$ and the numbers q_M of 1.5. Note that there is only a finite number of them: as $q_{wM} = q_M$ for all $w \in \nu(N)$ and $wM(\alpha,k) = M(w\alpha,k)$ for all $w \in W^v$, we may suppose $M = M(\alpha_i,k)$ with $i \in I$ and $k \in \mathbb{Z}$. Now $\alpha_i^\vee \in Q^\vee \subset Y$; as $\alpha_i(\alpha_i^\vee) = 2$, the translation by $\alpha_i^\vee$ permutes the walls $M = M(\alpha_i,k)$ (for $k \in \mathbb{Z}$) with two orbits. So, Y has at most two orbits in the set of the constants $q_{M(\alpha_i,k)}$: one containing the $q_i = q_{M(\alpha_i,0)}$ and the other containing the $q_i' = q_{M(\alpha_i,\pm 1)}$. Hence, the number of (possibly) different parameters is at most $2.|I|$. We denote by $\mathcal{Q} = \{q_1,\dots,q_l,q_1' = q_{l+1},\dots,q_l' = q_{2l}\}$ this set of parameters.

4.1. *Centrifugally folded galleries of chambers*. Let x be a point in the standard apartment $\mathbb{A}$. Let Φ_x be the set of all roots α such that $\alpha(x) \in \mathbb{Z}$. It is a closed subsystem of roots. Its associated Weyl group W_x^v is a Coxeter group.

We have twinned buildings $\mathscr{I}_x^+$ (resp. $\mathscr{I}_x^-$) whose elements are segment germs $[x,y) = \mathrm{germ}_x([x,y])$ for $y \in \mathscr{I}$, $y \neq x$, $y \geq x$ (resp. $y \leq x$). We consider their unrestricted structure, so the associated Weyl group is W^v and the chambers (resp. closed chambers) are the local chambers $C = \mathrm{germ}_x(x + C^v)$ (resp. local closed chambers $\overline{C} = \mathrm{germ}_x(x + \overline{C^v})$), where C^v is a vectorial chamber; cf. [GR08, 4.5] or [Rou11, §5]. To $\mathbb{A}$ is associated a twin system of apartments $\mathbb{A}_x = (\mathbb{A}_x^-, \mathbb{A}_x^+)$.

We choose in $\mathbb{A}_x^-$ a negative (local) chamber C_x^- and denote by C_x^+ its opposite in $\mathbb{A}_x^+$. We consider the system of positive roots Φ^+ associated to C_x^+ (i.e., $\Phi^+ = w\Phi_f^+$ if Φ_f^+ is the system Φ^+ defined in 1.1 and $C_x^+ = \mathrm{germ}_x(x + wC_f^v)$). We denote by $(\alpha_i)_{i\in I}$ the corresponding basis of Φ and by $(r_i)_{i\in I}$ the corresponding generators of W^v.

Fix a reduced decomposition of an element $w \in W^v$, $w = r_{i_1}\cdots r_{i_r}$, and let $\mathbf{i} = (i_1,\dots,i_r)$ be the type of the decomposition. Now we consider galleries of (local) chambers $\mathbf{c} = (C_x^-, C_1,\dots,C_r)$ in the apartment $\mathbb{A}_x^-$ starting at C_x^-

and of type $\mathbf{i}$. The set of all these galleries is in bijection with the set $\Gamma(\mathbf{i}) = \{1, r_{i_1}\} \times \cdots \times \{1, r_{i_r}\}$ via the map $(c_1, \ldots, c_r) \mapsto (C_x^-, c_1 C_x^-, \ldots, c_1 \cdots c_r C_x^-)$. Let $\beta_j = -c_1 \cdots c_j(\alpha_{i_j})$; then β_j is the root corresponding to the common limit wall $M_j = M_{\beta_j}$ of $C_{j-1} = c_1 \cdots c_{j-1} C_x^-$ and $C_j = c_1 \cdots c_j C_x^-$ and satisfying $\beta_j(C_j) \geq \beta_j(x)$. (Actually, M_j is a wall $\iff \beta_j \in \Phi_x$.) In the following, we shall identify a sequence $(c_1, \ldots, c_r)$ and the corresponding gallery.

Definition 4.2. Let $\mathfrak{Q}$ be a chamber in $\mathbb{A}_x^+$. A gallery $\mathbf{c} = (c_1, \ldots, c_r) \in \Gamma(\mathbf{i})$ is said to be centrifugally folded with respect to $\mathfrak{Q}$ if $c_j = 1$ implies $\beta_j \in \Phi_x$ and $w_{\mathfrak{Q}}^{-1}\beta_j < 0$, where $w_{\mathfrak{Q}} = w(C_x^+, \mathfrak{Q}) \in W^v$ (i.e., $\mathfrak{Q} = w_{\mathfrak{Q}} C_x^+$). We denote this set of centrifugally folded galleries by $\Gamma_{\mathfrak{Q}}^+(\mathbf{i})$.

PROPOSITION 4.3. *A gallery* $\mathbf{c} = (C_x^-, C_1, \ldots, C_r) \in \Gamma(\mathbf{i})$ *belongs to* $\Gamma_{\mathfrak{Q}}^+(\mathbf{i})$ *if, and only if,* $C_j = C_{j-1}$ *implies that* $M_j = M_{\beta_j}$ *is a wall and separates* $\mathfrak{Q}$ *from* $C_j = C_{j-1}$.

Proof. We saw that M_j is a wall if and only if $\beta_j \in \Phi_x$. We have the following equivalences:

$$
\begin{aligned}
&M_j \text{ separates } \mathfrak{Q} \text{ from } C_j = C_{j-1}\\
&\qquad\iff w_{\mathfrak{Q}}^{-1} M_j \text{ separates } C_x^+ \text{ from } w_{\mathfrak{Q}}^{-1} C_j = w_{\mathfrak{Q}}^{-1} C_{j-1}\\
&\qquad\iff w_{\mathfrak{Q}}^{-1}\beta_j \text{ is a negative root.} \qquad \square
\end{aligned}
$$

The group $\overline{G}_x = G_x / G_{\mathscr{I}_x}$ acts strongly transitively on $\mathscr{I}_x^+$ and $\mathscr{I}_x^-$. For any root $\alpha \in \Phi_x$ with $\alpha(x) = k \in \mathbb{Z}$, the group $\overline{U}_\alpha = U_{\alpha,k}/U_{\alpha,k+1}$ is a finite subgroup of $\overline{G}_x$ of cardinality $q_{x,\alpha} = q_{M(\alpha,-\alpha(x))} \in \mathcal{Q}$. We denote by u_α the elements of this group.

Next, let $\rho_{\mathfrak{Q}} : \mathscr{I}_x \to \mathbb{A}_x$ be the retraction centered at $\mathfrak{Q}$. To a gallery of chambers $\mathbf{c} = (c_1, \ldots, c_r) = (C_x^-, C_1, \ldots, C_r)$ in $\Gamma(\mathbf{i})$, one can associate the set of all galleries of type $\mathbf{i}$ starting at C_x^- in $\mathscr{I}_x^-$ that retract onto $\mathbf{c}$. We denote this set by $\mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$. We denote the set of minimal galleries in $\mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$ by $\mathcal{C}_{\mathfrak{Q}}^m(\mathbf{c})$. Set

$$
g_j = \begin{cases} c_j & \text{if } w_{\mathfrak{Q}}^{-1}\beta_j > 0 \text{ or } \beta_j \notin \Phi_x,\\ u_{c_j(\alpha_{i_j})} c_j & \text{if } w_{\mathfrak{Q}}^{-1}\beta_j < 0 \text{ and } \beta_j \in \Phi_x. \end{cases} \tag{1}
$$

PROPOSITION 4.4. $\mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$ *is the nonempty set of all galleries* $(C_x^- = C'_0, C'_1, \ldots, C'_r)$ *where* $C'_j = g_1 \cdots g_j C_x^-$ *for all* j, *with each* g_j *chosen as in equation* (1) *above. For all* j, *the local chambers* $\mathfrak{Q}$ *and* C'_j *are in the apartment* $g_1 \cdots g_j \mathbb{A}_x$.

The set $\mathcal{C}_{\mathfrak{Q}}^m(\mathbf{c})$ *is empty if, and only if, the gallery* $\mathbf{c}$ *is not centrifugally folded with respect to* $\mathfrak{Q}$. *The gallery* $(C_x^- = C'_0, C'_1, \ldots, C'_r)$ *is minimal if, and only if,* $c_j \neq 1$ *for any* j *with* $w_{\mathfrak{Q}}^{-1}\beta_j > 0$ *or* $\beta_j \notin \Phi_x$ *and* $u_{c_j(\alpha_{i_j})} \neq 1$ *for any* j *with* $c_j = 1$ *and* $w_{\mathfrak{Q}}^{-1}\beta_j < 0$.

Remark. For g_j as in equation (1), we may write $g_j = u_{c_j(\alpha_{i_j})}c_j$ (with $u_{c_j(\alpha_{i_j})} = 1$ if $w_{\mathfrak{Q}}^{-1}\beta_j > 0$ or $\beta_j \notin \Phi_x$). Then in the product $g_1 \cdots g_j$ we may gather the c_k on the right and, as $c_1 \cdots c_k(\alpha_{i_k}) = -\beta_k$, we may write $g_1 \cdots g_j = u_{-\beta_1} \cdots u_{-\beta_j}.c_1 \cdots c_j$. Hence $C'_j := g_1 \cdots g_j C_x^- = u_{-\beta_1} \cdots u_{-\beta_j} C_j$. When $u_{-\beta_k} \neq 1$, we have $\beta_k \in \Phi_x$ and $w_{\mathfrak{Q}}^{-1}\beta_k < 0$; so it is clear that $\rho_{\mathfrak{Q}}(C'_j) = C_j$.

The gallery $(C_x^- = C'_0, C'_1, \ldots, C'_r)$ (of type $\mathbf{i}$) is minimal if, and only if, we may also write (uniquely)

$$\begin{aligned} C'_j &= u_{-\alpha_{i_1}}.u_{r_{i_1}(-\alpha_{i_2})} \cdots u_{r_{i_1}\cdots r_{i_{j-1}}(-\alpha_{i_j})}.r_{i_1} \cdots r_{i_j}(C_x^-) \\ &= h_1 \cdots h_j.r_{i_1} \cdots r_{i_j}(C_x^-) \end{aligned}$$

with

$$h_k = u_{r_{i_1}\cdots r_{i_{k-1}}(-\alpha_{i_k})} \in \overline{U}_{r_{i_1}\cdots r_{i_{k-1}}(-\alpha_{i_k})}$$

(which fixes C_x^-). In particular, $C'_j \in h_1 \cdots h_j \mathbb{A}_x$. But this formula gives no way to know when $\rho_{\mathfrak{Q}}(C'_j) = C_j$. We know only that, when $\beta_k \notin \Phi_x$, i.e., $r_{i_1} \cdots r_{i_{k-1}}(-\alpha_{i_k}) \notin \Phi_x$, we necessarily have $h_k = 1$.

Proof. As the type $\mathbf{i}$ of $(C_x^- = C'_0, C'_1, \ldots, C'_r)$ is the type of a minimal decomposition, this gallery is minimal if, and only if, two consecutive chambers are different. So the last assertion is a consequence of the first ones. We prove these properties for $(C_x^- = C'_0, C'_1, \ldots, C'_j)$ by induction on j. In the following we just write H_j for the common limit hyperplane H_{β_j} of C_{j-1} and C_j of type i_j.

There are five possible relative positions of $\mathfrak{Q}$, C_x^- and C_1 with respect to H_1, and we seek C'_1 with $\rho_{\mathfrak{Q}}(C'_1) = C_1$ and $\overline{C'_1} \supset \overline{C_x^-} \cap H_1$.

(0) $\beta_1 = -c_1\alpha_{i_1} \notin \Phi_x$. Then H_1 is not a wall, each C'_1 with $\overline{C'_1} \supset \overline{C_x^-} \cap H_1$ is equal to C_x^- or $r_{i_1}C_x^-$ and C'_1 or C_x^- are contained in the same apartments. So $C'_1 = C_1 = c_1 C_x^-$; C_1 and $\mathfrak{Q}$ are in $g_1\mathbb{A}_x = \mathbb{A}_x$ with $g_1 = c_1$. When $C'_1 = C_x^-$, we have $c_1 = 1$ and $\mathbf{c}$ is not centrifugally folded.

We suppose now $\beta_1 \in \Phi_x$, so H_1 is a wall.

(1) C_x^- is on the same side of H_1 as $\mathfrak{Q}$ and C_1 is not. Then $c_1 = r_{i_1}$, $\beta_1 = \alpha_{i_1}$, $w_{\mathfrak{Q}}^{-1}\beta_1 < 0$, $C'_1 = g_1 C_x^- = u_{-\alpha_{i_1}} r_{i_1} C_x^- = u_{-\alpha_{i_1}} C_1$. But $u_{-\alpha_{i_1}}$ pointwise stabilizes the half-space bounded by H_1 containing C_x^-; hence $u_{-\alpha_{i_1}}(\mathfrak{Q}) = \mathfrak{Q}$ and C'_1 are in the apartment $g_1\mathbb{A}_x$.

(2) $\mathfrak{Q}$ and $C_x^- = C_1$ are separated by H_1. Then $c_1 = 1$, $\beta_1 = -\alpha_{i_1}$, $w_{\mathfrak{Q}}^{-1}\beta_1 < 0$, $C'_1 = g_1 C_x^- = u_{\alpha_{i_1}} C_x^-$ but $u_{\alpha_{i_1}}$ pointwise stabilizes the half-space bounded by H_1 not containing C_x^-; hence $\mathfrak{Q}$ and C'_1 are in the apartment $g_1\mathbb{A}_x$.

(3) C_1 is on the same side of H_1 as $\mathfrak{Q}$ and C_x^- is not. Then $c_1 = r_{i_1}$, $\beta_1 = \alpha_{i_1}$, $w_{\mathfrak{Q}}^{-1}\beta_1 > 0$ and C'_1 has to be C_1, so $g_1 = c_1 = r_{i_1}$, $w_{\mathfrak{Q}}^{-1}(\alpha_{i_1}) > 0$; moreover, $\mathfrak{Q}$ and $C'_1 = r_{i_1} C_x^- = C_1$ are in the apartment $g_1\mathbb{A}_x$.

(4) $\mathfrak{Q}$ and $C_x^- = C_1$ are on the same side of H_1. Then $c_1 = 1$ and $w_{\mathfrak{Q}}^{-1}\beta_1 > 0$; the gallery $\mathbf{c}$ is not centrifugally folded. So $\rho_{\mathfrak{Q}}(C_1') = C_1$ implies $C_1' = C_x^- = g_1 C_x^-$ with $g_1 = c_1 = 1$ as in (1). But the gallery $(C_x^- = C_0', C_1', \dots, C_j')$ cannot be minimal.

By induction we assume now that the chambers $\mathfrak{Q}$ and $C_{j-1}' = g_1 \cdots g_{j-1} C_x^-$ are in the apartment $A_{j-1} = g_1 \cdots g_{j-1}\mathbb{A}_x$. Again, we have five possible relative positions for $\mathfrak{Q}, C_{j-1}$ and C_j with respect to H_j. We seek C_j' with $\rho_{\mathfrak{Q}}(C_j') = C_j$ and $\overline{C_j'} \supset \overline{C_{j-1}'} \cap g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$.

(0) $\beta_j = -c_1 \cdots c_j \alpha_{i_j} \notin \Phi_x$. Then H_j is not a wall, and each C_j' with $\overline{C_j'} \supset \overline{C_{j-1}'} \cap g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ is equal to $C_{j-1}' = g_1 \cdots g_{j-1} C_x^-$ or $g_1 \cdots g_{j-1} r_{i_j} C_x^-$; moreover, C_j' or C_{j-1}' are contained in the same apartments. Therefore $C_j' = g_1 \cdots g_{j-1} c_j C_x^-$ and $\mathfrak{Q}$ are in $g_1 \cdots g_j \mathbb{A}_x = g_1 \cdots g_{j-1}\mathbb{A}_x$ with $g_j = c_j$. When $C_j' = C_{j-1}'$, we have $c_j = 1$ and $\mathbf{c}$ is not centrifugally folded.

Now we suppose $\beta_j \in \Phi_x$, so H_j is a wall.

(1) C_{j-1} is on the same side of $H_j = c_1 \cdots c_{j-1} H_{\alpha_{i_j}}$ as $\mathfrak{Q}$ and C_j is not. Then $c_j = r_{i_j}$, $\beta_j = c_1 \cdots c_{j-1}\alpha_{i_j}$, $w_{\mathfrak{Q}}^{-1}\beta_j < 0$. Moreover, $\mathfrak{Q}$ and C_{j-1}' are on the same side of $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ in A_{j-1}, and

$$\begin{aligned} C_j' &= g_1 \cdots g_{j-1} u_{-\alpha_{i_j}} r_{i_j} C_x^- \\ &= g_1 \cdots g_{j-1} u_{-\alpha_{i_j}} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C_{j-1}' \\ &= g_1 \cdots g_{j-1} u_{-\alpha_{i_j}} (g_1 \cdots g_{j-1})^{-1} g_1 \cdots g_{j-1} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C_{j-1}', \end{aligned}$$

where $g_1 \cdots g_{j-1} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C_{j-1}'$ is the chamber adjacent to C_j' along $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ in A_{j-1}. Moreover, $g_1 \cdots g_{j-1} u_{-\alpha_{i_j}} (g_1 \cdots g_{j-1})^{-1}$ pointwise stabilizes the half-space bounded by $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ containing C_{j-1}' and $\mathfrak{Q}$. So $\mathfrak{Q}$ and C_j' are in the apartment $g_1 \cdots g_j \mathbb{A}_x$.

(2) $C_{j-1} = C_j$ and $\mathfrak{Q}$ are separated by H_j. Then $c_j = 1$, $\beta_j = -c_1 \cdots c_{j-1}\alpha_{i_j}$, $w_{\mathfrak{Q}}^{-1}\beta_j < 0$. Moreover, C_{j-1}' and $\mathfrak{Q}$ are separated by $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ in A_{j-1}, and $\mathfrak{Q}$ and the chamber

$$g_1 \cdots g_{j-1} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C_{j-1}'$$

are on the same side of this wall. For $u_{\alpha_{i_j}} \neq 1$,

$$C_j' = g_1 \cdots g_{j-1} u_{\alpha_{i_j}} C_x^- = g_1 \cdots g_{j-1} u_{\alpha_{i_j}} (g_1 \cdots g_{j-1})^{-1} C_{j-1}'$$

is a chamber adjacent (or equal) to C_{j-1}' along

$$g_1 \cdots g_{j-1} H_{\alpha_{i_j}} = g_1 \cdots g_{j-1} u_{\alpha_{i_j}} H_{\alpha_{i_j}}$$

in $g_1 \cdots g_j \mathbb{A}_x$ (with $g_j = u_{\alpha_{i_j}}$).

The root-subgroup $g_1 \cdots g_{j-1} U_{\alpha_{i_j}} (g_1 \cdots g_{j-1})^{-1}$ pointwise stabilizes the half-space bounded by $g_1 \cdots g_{j-1} H_{\alpha_{i_j}}$ and containing the chamber

$$g_1 \cdots g_{j-1} r_{i_j} (g_1 \cdots g_{j-1})^{-1} C'_{j-1}.$$

So $\mathfrak{Q}$ and C'_j are in the apartment $g_1 \cdots g_j \mathbb{A}_x$.

(3) C_j is on the same side of $H_j = c_1 \cdots c_{j-1} H_{\alpha_{i_j}}$ as $\mathfrak{Q}$ and C_{j-1} is not. Then $c_j = r_{i_j}$, $\beta_j = c_1 \cdots c_{j-1} \alpha_{i_j}$, $w_{\mathfrak{Q}}^{-1} \beta_j > 0$, and so $C'_j = g_1 \cdots g_{j-1} r_{i_j} C_x^-$. Whence $\mathfrak{Q}$ and C'_j are in the apartment $g_1 \cdots g_j \mathbb{A}_x$.

(4) $C_{j-1} = C_j$ and $\mathfrak{Q}$ are on the same side of $H_j = c_1 \cdots c_{j-1} H_{\alpha_{i_j}}$. Then $c_j = 1$, $\beta_j = -c_1 \cdots c_{j-1} \alpha_{i_j}$ and $w_{\mathfrak{Q}}^{-1} \beta_j > 0$. The gallery $\mathbf{c}$ is not centrifugally folded. So $\rho_{\mathfrak{Q}}(C'_j) = C_j$ implies $C'_j = C'_{j-1} = g_1 \cdots g_j C_x^-$ with $g_j = c_j = 1$ as in (1). But the gallery $(C_x^- = C'_0, C'_1, \ldots, C'_j)$ cannot be minimal. □

Corollary 4.5. *If* $\mathbf{c} \in \Gamma_{\mathfrak{Q}}^+(\mathbf{i})$, *then the number of elements in* $\mathcal{C}_{\mathfrak{Q}}^m(\mathbf{c})$ *is*

$$\sharp \mathcal{C}_{\mathfrak{Q}}^m(\mathbf{c}) = \prod_{k=1}^{t(\mathbf{c})} q_{j_k} \times \prod_{l=1}^{r(\mathbf{c})} (q_{j_l} - 1),$$

where $q_j = q_{x,\beta_j} = q_{x,\alpha_{i_j}} \in \mathcal{Q}$, $t(\mathbf{c}) = \sharp\{j \mid c_j = r_{i_j}, \beta_j \in \Phi_x$ *and* $w_{\mathfrak{Q}}^{-1}\beta_j < 0\}$ *and* $r(\mathbf{c}) = \sharp\{j \mid c_j = 1, \beta_j \in \Phi_x$ *and* $w_{\mathfrak{Q}}^{-1}\beta_j < 0\}$.

Remark. In the case of Section 3, where all parameters are equal to q, $\sharp\mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$ is the number of points, over the field $\mathbb{F}_q$, on a cell in a Bott-Samelson variety (which is defined over $\mathbb{Z}$). And $\mathcal{C}_{\mathfrak{Q}}^m(\mathbf{c})$ is a subset of that cell isomorphic to $\mathbb{G}_a^{t(c)} \times \mathbb{G}_m^{r(c)}$.

4.6. *Galleries and opposite segment germs.* Suppose now $x \in \mathbb{A} \cap \mathscr{I}^+$. Let ξ and η be two segment germs in $\mathbb{A}_x^+$. Let $-\eta$ and $-\xi$ opposite respectively η and ξ in $\mathbb{A}_x^-$. Let $\mathbf{i}$ be the type of a minimal gallery between C_x^- and $C_{-\xi}$, where $C_{-\xi}$ is the negative (local) chamber containing $-\xi$ such that $w(C_x^-, C_{-\xi})$ is of minimal length. Let $\mathfrak{Q}$ be a chamber of $\mathbb{A}_x^+$ containing η. We suppose ξ and η conjugated by W_x^v.

Lemma. *The following conditions are equivalent*:

(i) *there exists an opposite* ζ *to* η *in* $\mathscr{I}_x^-$ *such that* $\rho_{\mathbb{A}_x, C_x^-}(\zeta) = -\xi$;

(ii) *there exists a gallery* $\mathbf{c} \in \Gamma_{\mathfrak{Q}}^+(\mathbf{i})$ *ending in* $-\eta$;

(iii) $\xi \leq_{W_x^v} \eta$ (*in the sense of* 1.8, *with* Φ^+ *defined as in* 4.1 *using* C_x^-).

Moreover, the possible ζ *are in one-to-one correspondence with the disjoint union of the sets* $\mathcal{C}_{\mathfrak{Q}}^m(\mathbf{c})$ *for* $\mathbf{c}$ *in the set* $\Gamma_{\mathfrak{Q}}^+(\mathbf{i}, -\eta)$ *of galleries in* $\Gamma_{\mathfrak{Q}}^+(\mathbf{i})$ *ending in* $-\eta$. *More precisely, if* $\mathbf{m} \in \mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$ *is associated to* $(h_1, \ldots, h_r)$ *as in Remark* 4.4, *then* $\zeta = h_1 \cdots h_r(-\xi)$.

Proof. If $\zeta \in \mathscr{I}_x^-$ opposites η and if $\rho_{\mathbb{A}_x,C_x^-}(\zeta) = -\xi$, then any minimal gallery $\mathbf{m} = (C_x^-, M_1, \dots, M_r \ni \zeta)$ retracts onto a minimal gallery between C_x^- and $C_{-\xi}$. So we can assume as well that $\mathbf{m}$ has type $\mathbf{i} = (i_1, \dots, i_r)$ and then ζ determines $\mathbf{m}$. Now, if we retract $\mathbf{m}$ from $\mathfrak{Q}$, we get a gallery $\mathbf{c} = \rho_{\mathbb{A}_x,\mathfrak{Q}}(\mathbf{m})$ in $\mathbb{A}_x^-$ starting at C_x^-, ending in $-\eta$ and centrifugally folded with respect to $\mathfrak{Q}$.

Reciprocally, let $\mathbf{c} = (C_x^-, C_1, \dots, C_r) \in \Gamma_{\mathfrak{Q}}^+(\mathbf{i})$, such that $-\eta \in C_r$. According to Proposition 4.4 and the remark that follows it, there exists a minimal gallery $\mathbf{m} = (C_x^-, C_1', \dots, C_r')$ in the set $\mathcal{C}_{\mathfrak{Q}}(\mathbf{c})$, and the chambers C_j' can be described by $C_j' = g_1 \cdots g_j C_x^- = h_1 \cdots h_j . r_{i_1} \cdots r_{i_j} C_x^-$ where each h_k fixes C_x^-, hence $\rho_{\mathbb{A}_x,C_x^-}$ restricts on C_j' to the action of $(h_1 \cdots h_j)^{-1}$.

Let $\zeta \in C_r'$ opposite η in any apartment containing those two. The minimality of the gallery $\mathbf{m} = (C_x^-, C_1', \dots, C_r')$ ensures that $\rho_{\mathbb{A}_x,C_x^-}(\zeta) \in C_{-\xi}$; hence $\rho_{\mathbb{A}_x,C_x^-}(\zeta) = -\xi$ as they are both opposite η up to conjugation by W_x^v. So we proved the equivalence (i) $\Longleftrightarrow$ (ii) and the last two assertions.

Now the equivalence (i) $\Longleftrightarrow$ (iii) is proved in [GR08, Prop. 6.1, Th. 6.3]: in this reference we speak of Hecke paths with respect to $-C_f^v$, but the essential part is a local discussion in $\mathscr{I}_x$ (using only C_x^- and the twin building structure of $\mathscr{I}_x^{\pm}$) that gives this equivalence. □

4.7. *Liftings of Hecke paths.* Let π be a λ-path from $z' \in Y^+$ to $y \in Y^+$ entirely contained in the Tits cone $\mathcal{T}$, hence in a finite union of closed sectors $w\overline{C_f^v}$ with $w \in W^v$. By [GR08, 5.2.1], for each $w \in W^v$, there is only a finite number of $s \in]0,1]$ such that the reverse path $\bar{\pi}(t) = \pi(1-t)$ leaves, in $\pi(s)$, a wall positively with respect to $-w\overline{C_f^v}$; i.e., this wall separates $\pi_-(s)$ from $-w\overline{C_f^v}$. Therefore, we are able to define $\ell \in \mathbb{N}$ and $0 < t_1 < t_2 < \cdots < t_\ell \leq 1$ such that the $z_k = \pi(t_k)$, $k \in \{1, \dots, \ell\}$, are the only points in the path where at least one wall containing z_k separates $\pi_-(t_k)$ and the local chamber $\mathfrak{c}_-$ of 1.8(2).

For each $k \in \{1, \dots, \ell\}$, we choose for $C_{z_k}^-$ (as in 4.1) the germ in z_k of the sector of vertex z_k containing $\mathfrak{c}_-$. Let $\mathbf{i}_k$ be a fixed reduced decomposition of the element $w_-(t_k) \in W^v$, and let $\mathfrak{Q}_k$ be a fixed chamber in $\mathscr{I}_{z_k}^+$ containing $\eta_k = \pi_+(t_k)$. We set $-\xi_k = \pi_-(t_k)$. When π is a Hecke path (or a billiard path as in [GR08]), ξ_k and η_k are conjugated by $W_{z_k}^v$.

When π is a Hecke path with respect to $\mathfrak{c}_-$, $\{z_1, \dots, z_\ell\}$ includes all points where the piecewise linear path π is folded and, in the other points, all galleries in $\Gamma_{\mathfrak{Q}_k}^+(\mathbf{i}_k, -\eta_k)$ are unfolded.

Let $S_{\mathfrak{c}_-}(\pi, y)$ be the set of all segments $[z, y]$ such that $\rho_{\mathfrak{c}_-}([z, y]) = \pi$.

THEOREM 4.8. *$S_{\mathfrak{c}_-}(\pi, y)$ is nonempty if, and only if, π is a Hecke path with respect to $\mathfrak{c}_-$. Then, we have a bijection*

$$S_{\mathfrak{c}_-}(\pi, y) \simeq \prod_{k=1}^{\ell} \coprod_{\mathbf{c} \in \Gamma_{\mathfrak{Q}_k}^+(\mathbf{i}_k, -\eta_k)} \mathcal{C}_{\mathfrak{Q}_k}^m(\mathbf{c}).$$

In particular, the number of elements in this set is a polynomial in the numbers $q \in \mathcal{Q}$ *with coefficients in* $\mathbb{Z}$ *depending only on* $\mathbb{A}$.

Note. So the image by $\rho_{\mathfrak{c}_-}$ of a segment in $\mathscr{I}^+$ is a Hecke path with respect to $\mathfrak{c}_-$.

Proof. The restriction of $\rho_{\mathfrak{c}_-}$ to $\mathscr{I}_{z_k}$ is clearly equal to $\rho_{\mathbb{A}_{z_k}, C^-_{z_k}}$; therefore Lemma 4.6 tells us that π is a Hecke path with respect to $\mathfrak{c}_-$ if, and only if, each $\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k, -\eta_k)$ is nonempty.

We set $t_0 = 0$ and $t_{\ell+1} = 1$. We shall build a bijection from $S_{\mathfrak{c}_-}(\pi_{|[t_{n-1},1]}, y)$ onto $\prod_{k=n}^{\ell} \coprod_{\mathbf{c}\in\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)} \mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})$ by decreasing induction on $n \in \{1, \dots, \ell+1\}$. For $n = \ell+1$ and if $t_\ell \neq 1$, no wall cutting $\pi([t_\ell, 1])$ separates $y = \pi(1)$ from $\mathfrak{c}_-$; so a segment s in $\mathscr{I}$ with $s(1) = y$ and $\rho_{\mathfrak{c}_-} \circ s = \pi$ has to coincide with π on $[t_\ell, 1]$.

Suppose now that $s \in S_{\mathfrak{c}_-}(\pi_{|[t_n,1]}, y)$ is determined by a unique element in

$$\prod_{k=n+1}^{\ell} \coprod_{\mathbf{c}\in\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)} \mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})$$

in the following way: For an element $(\mathbf{m}_{n+1}, \mathbf{m}_{n+2}, \dots, \mathbf{m}_\ell)$ in this last set, each $\mathbf{m}_k = (C^-_{z_k}, C^k_1, \dots, C^k_{r_k})$ is the minimal gallery given by a sequence of elements $(h^k_1, \dots, h^k_{r_k}) \in (\overline{G}_{z_k})^{r_k}$, as in the remark after Proposition 4.4 and, for $t \in [t_n, t_{n+1}]$, we have $s(t) = (h^\ell_1 \cdots h^\ell_{r_\ell}) \cdots (h^{n+1}_1 \cdots h^{n+1}_{r_{n+1}})\pi(t)$ where, actually, each h^k_j is a chosen element of $U_{-r_{i_1}\cdots r_{i_{j-1}}(\alpha_{i_j})}$ whose class in $\overline{U}_{-r_{i_1}\cdots r_{i_{j-1}}(\alpha_{i_j})}$ is the h^k_j defined above; in particular, each h^k_j fixes $\mathfrak{c}_-$.

We set $g = (h^\ell_1 \cdots h^\ell_{r_\ell}) \cdots (h^{n+1}_1 \cdots h^{n+1}_{r_{n+1}}) \in G_{\mathfrak{c}_-}$. Then

$$g^{-1}s(t_n) = \pi(t_n) = z_n.$$

If $s \in S_{\mathfrak{c}_-}(\pi_{|[t_{n-1},1]}, y)$ and $s_{|[t_n,1]}$ is as above, then $g^{-1}s_-(t_n)$ is a segment germ in $\mathscr{I}^-_{z_n}$ opposite $g^{-1}s_+(t_n) = \pi_+(t_n) = \eta_n$ and retracting to $\pi_-(t_n)$ by $\rho_{\mathfrak{c}_-}$. By Lemma 4.6 and the above remark, this segment germ determines uniquely a minimal gallery $\mathbf{m}_n \in \mathcal{C}^m_{\mathfrak{Q}_n}(\mathbf{c})$ with $\mathbf{c} \in \Gamma^+_{\mathfrak{Q}_n}(\mathbf{i}_n, -\eta_n)$.

Conversely, such a minimal gallery $\mathbf{m}_n$ determines a segment germ $\zeta \in \mathscr{I}^-_{z_n}$, opposite $\pi_+(t_n) = \eta_n$ such that $\rho_{\mathbb{A}_{z_n}, C^-_{z_n}}(\zeta) = \pi_-(t_n)$. By Lemma 4.6, $\zeta = (h^n_1 \cdots h^n_{r_n})\pi_-(t_n)$ for some well-defined $(h^n_1, \dots, h^n_{r_n}) \in (\overline{G}_{z_n})^{r_n}$. As above we replace each g^n_j by a chosen element of $G_{(z_n \cup \mathfrak{c}_-)}$ whose class in $\overline{G}_{z_n}$ is this g^n_j. As no wall cutting $[z_{n-1}, z_n]$ separates $z_n = \pi(t_n)$ from $\mathfrak{c}_-$, any segment retracting by $\rho_{\mathfrak{c}_-}$ onto $[z_{n-1}, z_n]$ and with $[z_n, x) = \pi_-(t_n)$ (resp. $= \zeta$, $= g\zeta$) is equal to $[z_{n-1}, z_n]$ (resp. $(h^n_1 \cdots h^n_{r_n})[z_{n-1}, z_n]$, $g(h^n_1 \cdots h^n_{r_n})[z_{n-1}, z_n]$). We set $s(t) = (h^\ell_1 \cdots h^\ell_{r_\ell}) \cdots (h^{n+1}_1 \cdots h^{n+1}_{r_{n+1}})(h^n_1 \cdots h^n_{r_n})\pi(t)$ for $t \in [t_{n-1}, t_n]$.

With this inductive definition, s is a λ-path, $s(1) = y$, $\rho_{\mathfrak{c}_-} \circ s = \pi$ and $s_{|[t_{k-1},t_k]}$ is a segment for all $k \in \{1, \dots, \ell+1\}$. Moreover, for $k \in \{1, \dots, \ell\}$, the segment germs $[s(t_k), s(t_{k+1}))$ and $[s(t_k), s(t_{k-1}))$ are opposite. By the following lemma this proves that s itself is a segment. □

LEMMA 4.9. *Let x, y, z be three points in an ordered hovel $\mathscr{I}$, with $x \leq y \leq z$, and suppose the segment germs $[y, z)$, $[y, x)$ opposite in the twin buildings $\mathscr{I}_y$. Then $[x, y] \cup [y, z]$ is the segment $[x, z]$.*

Proof. For any $u \in [y, z]$, we have $x \leq y \leq u \leq z$. Hence x and $[u, y)$ or $[u, z)$ are in a same apartment [Rou11, 5.1]. As $[y, z]$ is compact we deduce that there are points $u_0 = y, u_1, \dots, u_\ell = z$ such that x and $[u_{i-1}, u_i]$ are in a same apartment A_i for $1 \leq i \leq \ell$. Now A_1 contains x and $[y, u_1]$, hence also $[x, y]$ (axiom (MAO) of 1.5). But $[y, x)$ and $[y, u_1) = [y, z)$ are opposite, so $[x, y] \cup [y, u_1] = [x, u_1]$. The lemma follows by induction. □

Remark 4.10. Analogue results can be proven for the retraction $\rho_{-\infty}$ instead of $\rho_{\mathfrak{c}_-}$: for all x, we choose $C_x^- = \text{germ}_x(x - C_f^v)$. For a λ-path π in $\mathbb{A}$ from z' to y, [GR08, 5.2.1] tells that we have a finite number of points $z_k = \pi(t_k)$ where at least a wall is left positively by the path $\bar{\pi}(t) = \pi(1-t)$. As above, we define $\mathbf{i}_k$, $\mathfrak{Q}_k$, η_k and ξ_k. Now $S_{-\infty}(\pi, y)$ is the set of all segments $[z, y]$ such that $\rho_{-\infty}([z, y]) = \pi$.

In [GR08, Ths. 6.2 and 6.3], we have proven that $S_{-\infty}(\pi, y)$ is nonempty if, and only if, π is a Hecke path with respect to $-C_f^v$. Moreover, we have shown that for $\mathscr{I}$ associated to a split Kac-Moody group over $\mathbb{C}((t))$, $S_{-\infty}(\pi, y)$ is isomorphic to a quasi-affine toric complex variety. The arguments above prove that with our choice for $\mathscr{I}$, $S_{-\infty}(\pi, y)$ is finite, with the following precision (which generalizes to the Kac-Moody case some formulae of [GL12]):

PROPOSITION 4.11. *Let π be a Hecke path with respect to $-C_f^v$ from z' to y. Then we have a bijection*

$$S_{-\infty}(\pi, y) \simeq \prod_{k=1}^{\ell} \coprod_{\mathbf{c} \in \Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k, -\eta_k)} \mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})$$

In particular, the number of elements in this set is a polynomial in the numbers $q \in \mathcal{Q}$ with coefficients in $\mathbb{Z}$ depending only on $\mathbb{A}$.

THEOREM 4.12. *Let $\lambda, \mu, \nu \in Y^{++}$, $\mathfrak{c}_-$ be the negative fundamental alcove, and suppose $(\alpha_i^\vee)_{i \in I}$ is $\mathbb{R}^+$-free. Then*

(a) *The number of Hecke paths of shape μ with respect to $\mathfrak{c}_-$ starting in $z' = w\lambda$ (for some $w \in W^v$ fixing 0) and ending in $y = \nu$ is finite.*

(b) *The structure constant* $m_{\lambda,\mu}(\nu)$, *i.e., the number of triangles* $[0, z, \nu]$ *in* $\mathscr{I}$ *with* $d^v(0, z) = \lambda$ *and* $d^v(z, \nu) = \mu$, *is equal to*

$$m_{\lambda,\mu}(\nu) = \sum_{w\in W^v/(W^v)_\lambda} \sum_{\pi} \prod_{k=1}^{\ell_\pi} \sum_{\mathbf{c}\in\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k,-\eta_k)} \sharp\mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c}), \tag{2}$$

where π *runs over the set of Hecke paths of shape* μ *with respect to* $\mathfrak{c}_-$ *from* $w\lambda$ *to* ν *and* ℓ_π, $\Gamma^+_{\mathfrak{Q}_k}(\mathbf{i}_k, -\eta_k)$ *and* $\mathcal{C}^m_{\mathfrak{Q}_k}(\mathbf{c})$ *are defined as above for each such* π.

(c) *In particular, the structure constants of the Hecke algebra* $\mathcal{H}_R$ *are polynomials in the numbers* $q\in\mathcal{Q}$ *with coefficients in* $\mathbb{Z}$ *depending only on* $\mathbb{A}$.

Proof. We saw in 2.3.1 that $m_{\lambda,\mu}(\nu)$ is the number of $z \in \mathscr{I}_0^+$ such that $d^v(0, z) = \lambda$ and $d^v(z, \nu) = \mu$. Such a z determines uniquely a Hecke path $\pi = \rho_{\mathfrak{c}_-}([z, \nu])$ of shape μ with respect to $\mathfrak{c}_-$ from $z' = \rho_{\mathfrak{c}_-}(z)$ to ν. But $d^v(0, z) = \lambda$ and $0 \in \mathfrak{c}_-$, so $d^v(0, z') = \lambda$; i.e., $z' = w\lambda$ with $w \in W^v$. So the formula (2) follows from Theorem 4.8.

We know already that $m_{\lambda,\mu}(\nu)$ is finite (2.5) and $S_{\mathfrak{c}_-}(\pi, y) \neq \emptyset$ (Theorem 4.8); hence (a) is clear. Now (c) follows from Corollary 4.5 □

5. **Satake isomorphism**

In this section, we prove the Satake isomorphism. From now on, we assume that the $\alpha_i^\vee$'s are free.

We denote by U^- the pointwise stabilizer in G of the sector-germ $\mathfrak{S}_{-\infty}$; i.e., any $u \in U^-$ has to pointwise stabilize a sector $x - C^v_f \subset \mathbb{A}$. By definition, for $z \in \mathscr{I}$, $\rho_{-\infty}(z)$ is the only point of the orbit $U^-.z$ in $\mathbb{A}$.

5.1. *The module of functions on the type* 0 *vertices in* $\mathbb{A}$. Let $\mathbb{A}_0 = \nu(N).0 = Y.0$ be the set of vertices of type 0 in $\mathbb{A}$. Note that $\mathbb{A}_0$ can be identified with the set of horocycles of U^- in $\mathscr{I}_0$, i.e., with $\mathscr{I}_0/U^-$, via the retraction $\rho_{-\infty}$. First we consider $\widehat{\mathscr{F}} = \widehat{\mathscr{F}}_R = \mathscr{F}(\mathbb{A}_0, R)$, the set of functions on $\mathbb{A}_0$ with values in the ring R. Equivalently, $\widehat{\mathscr{F}}$ can be identified with the set of U^--invariant functions on $\mathscr{I}_0$.

For $\mu \in Y$, we define $\chi_\mu \in \widehat{\mathscr{F}}$ as the characteristic function of $U^-.\mu$ in $\mathscr{I}_0$ (or $\{\mu\}$ in Y). Then, any $\chi \in \widehat{\mathscr{F}}_R$ may be written $\chi = \sum_{\mu\in Y} a_\mu \chi_\mu$ with $a_\mu \in R$. We set $\mathrm{supp}(\chi) = \{\mu \mid a_\mu \neq 0\}$. Now, let

$$\mathscr{F} = \mathscr{F}_R = \{\chi \in \widehat{\mathscr{F}} \mid \mathrm{supp}(\chi) \subset \cup_{j=1}^n (\mu_j - Q^\vee_+) \text{ for some } \mu_j \in \mathbb{A}_0\}$$

be the set of functions on $\mathscr{I}_0$ with almost finite support.

We define also the following completion of the group algebra $R[Y]$:

$$R[[Y]] = \Big\{ f = \sum_{y \in Y} a_y e^y \mid \mathrm{supp}(f) = \{ y \in Y \mid a_y \neq 0 \} \\ \subset \cup_{j=1}^{n} (\mu_j - Q_+^\vee) \text{ for some } \mu_j \in \mathbb{A}_0 \Big\};$$

it is clearly a commutative algebra (with $e^y.e^z = e^{y+z}$). Actually, it is the Looijenga's coweight algebra; see Section 4.1 in [Loo80].

The formula $(f.\chi)(\mu) = \sum_{y \in Y} a_y \chi(\mu - y)$, for $f = \sum a_y e^y \in R[[Y]]$, $\chi \in \mathscr{F}$ and $\mu \in Y$, defines an element $f.\chi \in \mathscr{F}$; in particular, $e^y.\chi_\mu = \chi_{\mu+y}$. Clearly, the map $R[[Y]] \times \mathscr{F} \to \mathscr{F}$, $(f, \chi) \mapsto f.\chi$ makes $\mathscr{F}$ into a free $R[[Y]]$-module of rank 1, with any χ_μ as basis element.

Definition-Proposition 5.2. The map

$$\begin{array}{ccc} \mathscr{F} \times \mathcal{H} & \to & \mathscr{F} \\ (\chi, \varphi) & \mapsto & \chi * \varphi \end{array}$$

where, for $x \in \mathscr{I}_0$, $(\chi * \varphi)(x) = \sum_{y \in \mathscr{I}_0} \chi(y) \varphi^{\mathscr{I}}(y, x)$, defines a right action of $\mathcal{H}$ on $\mathscr{F}$ that commutes with the actions of $Z = \{ n \in N \mid \nu(n) \in Y \}$ and (more generally) $R[[Y]]$.

Proof. It is relatively clear that $\chi * \varphi$ is a function on $\mathscr{I}_0 / U^-$ and that the map indeed defines an action. Let us check that this action commutes with the one of Z. Let $t \in Z$ and $x \in \mathscr{I}_0$. Then

$$\begin{aligned} (\chi * \varphi)(tx) &= \sum_{y \in \mathscr{I}_0} \chi(y) \varphi^{\mathscr{I}}(y, tx) \\ &= \sum_{y' \in \mathscr{I}_0} \chi(ty') \varphi^{\mathscr{I}}(ty', tx) \qquad (y = ty') \\ &= \sum_{y' \in \mathscr{I}_0} \chi(ty') \varphi^{\mathscr{I}}(y', x) \\ &= ((\chi \circ t) * \varphi)(x). \end{aligned}$$

So, $(\chi \circ t) * \varphi = (\chi * \varphi) \circ t$. For $\nu(t) = \mu \in Y$ and $\chi \in \mathscr{F}$, clearly we have $\chi \circ t = e^{-\mu}.\chi$. As a formal consequence, the right action of $\mathcal{H}$ commutes with the left action of $R[[Y]]$.

The difficult point is to show that the support condition is satisfied. For any $\lambda \in Y^{++}$ and any $\nu \in Y$,

$$\begin{aligned} (\chi_\mu * c_\lambda)(\nu) &= \sum_{y \in \mathscr{I}_0} \chi_\mu(y) c_\lambda^{\mathscr{I}}(y, \nu) \\ &= \sharp\{ y \in \mathscr{I}_0 \mid \rho_{-\infty}(y) = \mu \text{ and } d^v(y, \nu) = \lambda \}. \end{aligned}$$

The latest is also the cardinality of the set of all segments $[y, \nu]$ in $\mathscr{I}$ $(y \leq \nu)$ of "length" λ such that $y \in U^- \cdot \mu$. In addition, since the action of $\mathcal{H}$ commutes with the one of Z, we set $n_\lambda(\nu - \mu) = (\chi_\mu * c_\lambda)(\nu)$. Then $n_\lambda(\nu - \mu) =$

$\sum_\pi \sharp S_{-\infty}(\pi,\nu)$ where the sum runs over the set of Hecke λ-paths with respect to $-C_f^v$ from μ to ν. (See 4.10 for the definition of $S_{-\infty}(\pi,\nu)$.)

Now, Lemma 2.4(b) shows that $n_\lambda(\nu-\mu)\neq 0$ implies $\nu-\mu \leq_{Q_+} \lambda$. Moreover, if $\nu=\lambda+\mu$, then $n_\lambda(\lambda)=1$. Therefore, we get

$$\chi_\mu * c_\lambda = \sum_{\nu\leq_{Q^\vee}\lambda+\mu} n_\lambda(\nu-\mu)\chi_\nu = \chi_{\lambda+\mu} + \sum_{\nu<_{Q^\vee}\lambda+\mu} n_\lambda(\nu-\mu)\chi_\nu. \tag{3}$$

This formula shows that, for any $\varphi\in\mathcal{H}$ with $\mathrm{supp}(\varphi)\subset\cup_{i=1}^n(\lambda_i-Q_+^\vee)$ and any $\chi\in\mathscr{F}$ with $\mathrm{supp}(\chi)\subset\cup_{j=1}^n(\mu_j-Q_+^\vee)$, the support of $\chi*\varphi$ is contained in $\cup_{i,j}(\lambda_i+\mu_j-Q_+^\vee)$. More precisely, for any $\nu\in\cup_{i,j}(\lambda_i+\mu_j-Q_+^\vee)$, there exists a finite number of $\lambda\in\mathrm{supp}(\varphi)$ and $\mu\in\mathrm{supp}(\chi)$ such that $\nu\leq_{Q_+}\lambda+\mu$. Hence, $\chi*\varphi$ is well defined. □

5.3. *The Satake isomorphism.*

5.3.1. *The morphism $\mathcal{S}_*$.* As $\mathscr{F}$ is a free $R[[Y]]$-module of rank 1, we have $\mathrm{End}_{R[[Y]]}(\mathscr{F})=R[[Y]]$. So the right action of $\mathcal{H}$ on the $R[[Y]]$-module $\mathscr{F}$ gives an algebra homomorphism $\mathcal{S}_*:\mathcal{H}\to R[[Y]]$ such that $\chi*\varphi=\mathcal{S}_*(\varphi).\chi$ for any $\varphi\in\mathcal{H}$ and any $\chi\in\mathscr{F}$.

As $e^\nu.\chi_\mu=\chi_{\mu+\nu}$, equation (3) gives

$$\mathcal{S}_*(c_\lambda)=\sum_{\nu\leq_{Q^\vee}\lambda} n_\lambda(\nu)e^\nu = e^\lambda+\sum_{\nu<_{Q^\vee}\lambda} n_\lambda(\nu)e^\nu.$$

We shall modify $\mathcal{S}_*$ by some character to get the Satake isomorphism.

5.3.2. *The module δ.* We define the map $\delta: Q^\vee\to\mathbb{R}_+^*$ by $\sum_{i\in I} a_i\alpha_i^\vee\mapsto\prod_{i\in I}(q_iq_i')^{a_i}$, where $q_i,q_i'\in\mathcal{Q}\subset\mathbb{N}$ are as in the beginning of Section 4. We extend this homomorphism and its square root to Y (as $\mathbb{R}_+^*$ is uniquely divisible). So, we get homomorphisms $\delta,\delta^{1/2}:Y\to\mathbb{R}_+^*$ and $\delta=\delta\circ\nu,\delta^{1/2}=\delta^{1/2}\circ\nu: Z\to\mathbb{R}_+^*$.

We made a choice for δ. But we shall see in Theorem 5.4 that the expected properties depend only on $\delta_{|Q^\vee}$.

In the classical case, where G is a split semi-simple group and $\mathscr{I}$ its Bruhat-Tits building, we have $q_i=q_i'=q$ for any $i\in I$. Hence, if we set $\mu=\sum_{i\in I} a_i\alpha_i^\vee$, then $\delta^{1/2}(\mu)=q^{\sum a_i}=q^{\rho(\mu)}$, where ρ is the half-sum of positive roots.

5.3.3. *The Satake isomorphism.* From now on, we suppose that the algebra R contains the image of $\delta^{1/2}$ in $\mathbb{R}_+^*$. We define

$$\mathcal{S}(c_\lambda)=\sum_{\mu\leq_{Q^\vee}\lambda}\delta^{1/2}(\mu)n_\lambda(\mu)e^\mu=\delta^{1/2}(\lambda)e^\lambda+\sum_{\mu<_{Q^\vee}\lambda}\delta^{1/2}(\mu)n_\lambda(\mu)e^\mu$$

and extend it to formal combinations of the c_λ with almost finite support.

Thus we get an algebra homomorphism $\mathcal{S} : \mathcal{H} \to R[[Y]]$ called the *Satake isomorphism*, as it is one-to-one. For $\varphi = \sum_\lambda a_\lambda c_\lambda \in \mathcal{H}$, we have

$$\mathcal{S}(\varphi) = \sum_\lambda a_\lambda \Big(\delta^{1/2}(\lambda) e^\lambda + \sum_{\mu <_{Q^\vee} \lambda} \delta^{1/2}(\mu) n_\lambda(\mu) e^\mu \Big).$$

If $\varphi \neq 0$ and λ_0 is a maximum element in $\mathrm{supp}(\varphi)$, then λ_0 is also a maximum element in $\mathrm{supp}(\mathcal{S}(\varphi))$ and $\mathcal{S}(\varphi) \neq 0$.

Remarks. (a) So, now we know that $\mathcal{H}$ is commutative.

(b) In the classical case where G is a split semi-simple group, $\mathcal{S}(c_\lambda)$ is defined as an integral over a maximal unipotent subgroup; here we choose U^-. The Haar measure du on U^- is chosen to give volume 1 to $K \cap U^-$ and, for an element t in the torus Z, the formula for changing variables is given by $d(tut^{-1}) = \delta(t)^{-1} du$. So the classical formula for the Satake isomorphism given, e.g., in [Car79, (19) p 146] when $\nu(t) = \mu$, is

$$\begin{aligned}
\mathcal{S}(c_\lambda)(t) &= \delta(t)^{1/2} \int_{U^-} c_\lambda^G(ut) du = \delta(t)^{1/2} \int_{U^-} c_\lambda^{\mathscr{I}}(0, ut.0) du \\
&= \delta(t)^{1/2} \int_{U^-} c_\lambda^{\mathscr{I}}(u^{-1}.0, t.0) du = \delta(t)^{1/2} \sum_{y \in U^-.0} c_\lambda^{\mathscr{I}}(y, \mu) \\
&= \delta(t)^{1/2} \sum_{y \in \mathscr{I}_0} \chi_0(y).c_\lambda^{\mathscr{I}}(y, \mu) = \delta(t)^{1/2} (\chi_0 * c_\lambda)(\mu).
\end{aligned}$$

This is the same formula as ours.

5.3.4. *W^v-invariance.* There is an action of W^v on Y, hence on $R[Y]$ by setting $w.e^\lambda = e^{w\lambda}$ for $w \in W^v$ and $\lambda \in Y$. This action does not extend to $R[[Y]]$, but we define

$$R[[Y]]^{W^v} = \{ f = \sum a_\lambda e^\lambda \in R[[Y]] \mid a_\lambda = a_{w\lambda} \forall \lambda \in Y \; \forall w \in W^v \}.$$

This is a subalgebra of $R[[Y]]$ and actually the image of the Satake isomorphism (see Theorem 5.4).

Remark. Let $C^\vee = \{ \pi \in V^* \mid \alpha_i^\vee(\pi) \geq 0 \text{ for all } i \in I \}$ and $\mathcal{T}^\vee = \cup_{w \in W^v} wC^\vee$ be the fundamental dual chamber and the dual Tits cone in V^*. By definition, for $f \in R[[Y]]$ and $\pi \in C^\vee$, $\pi(\mathrm{supp}(f))$ is bounded above. Hence, for $f \in R[[Y]]^{W^v}$, $\pi(\mathrm{supp}(f))$ is also bounded above for any $\pi \in \mathcal{T}^\vee$. We know that the dual cone of $\overline{\mathcal{T}}^\vee$ is the closed convex hull $\overline{\Gamma}$ of the set $\Delta_+^{\vee im} \cup \{0\}$, where $\Delta_+^{\vee im} \subset Q_+^\vee$ is the set of positive imaginary roots in the dual system of roots $\Delta^\vee$ ([Kac90, 5.8]). So, the only directions along which points in $\mathrm{supp}(f)$ (for $f \in R[[Y]]^{W^v}$) may go to infinity are the directions in $-\overline{\Gamma}$.

THEOREM 5.4. *The Hecke algebra $\mathcal{H}_R$ is isomorphic via $\mathcal{S}$ to the commutative algebra $R[[Y]]^{W^v}$ of Weyl invariant elements in $R[[Y]]$.*

Proof. As $\mathcal{S}(c_\lambda) = \sum_{\mu \leq_{Q^\vee} \lambda} \delta^{1/2}(\mu) n_\lambda(\mu) e^\mu$, we only have to prove that for $w \in W^v$, $\delta^{1/2}(\mu) n_\lambda(\mu) = \delta^{1/2}(w\mu) n_\lambda(w\mu)$ or $n_\lambda(w\mu) = n_\lambda(\mu)\delta^{1/2}(\mu - w\mu)$. It is sufficient to prove this for $w = r_i$ a fundamental reflection, hence to prove that $n_\lambda(r_i\mu) = n_\lambda(\mu)\delta^{1/2}(\mu - r_i\mu) = n_\lambda(\mu)\delta^{1/2}(\alpha_i(\mu)\alpha_i^\vee)$. By the given definition of δ, the wanted formula is

$$n_\lambda(r_i\mu) = n_\lambda(\mu)\Big(\sqrt{q_i q_i'}\Big)^{\alpha_i(\mu)}. \tag{4}$$

The proof of this formula is postponed to the following subsections, starting with 5.5. One can already notice that $\alpha_i(\mu)$ is an integer. We know that any $t \in Z$ with $\nu(t) = \mu$ exchanges the walls $M(\alpha_i, 0)$ and $M(\alpha_i, \alpha_i(\mu))$, hence, if $\alpha_i(\mu)$ is odd, we get that $q_i = q_i'$. So, in any case $\Big(\sqrt{q_i q_i'}\Big)^{|\alpha_i(\mu)|}$ is an integer.

Once formula (4) is proved we know that $\mathcal{S}(\mathcal{H}) \subset R[[Y]]^{W^v}$. For $f = \sum a_\mu e^\mu \in R[[Y]]^{W^v}$ with $\operatorname{supp}(f) \subset \cup_{j=1}^r (\lambda_j - Q_+^\vee)$, we shall build a sequence φ_n in $\mathcal{H}$ such that $\operatorname{supp}(f - \mathcal{S}(\varphi_n)) \subset \cup_{j=1}^r (\lambda_j - Q_{+n}^\vee)$ and $\operatorname{supp}(\varphi_{n+1} - \varphi_n) \subset Y^{++} \cap (\cup_{j=1}^r (\lambda_j - Q_{+n}^\vee))$, where $Q_{+n}^\vee = \{\sum_{i\in I} n_i \alpha_i^\vee \in Q_+^\vee \mid \sum n_i \geq n\}$. Then, the limit φ of this sequence exists in $\mathcal{H}$ and $\mathcal{S}(\varphi) = f$. So, $\mathcal{S}$ is onto.

We build the sequence by induction. We set $\varphi_0 = 0$. If $\varphi_0, \dots, \varphi_n$ are given as above, we set $\{\mu_1, \dots, \mu_s\} = \operatorname{supp}(f - \mathcal{S}(\varphi_n)) \setminus \cup_{j=1}^r (\lambda_j - Q_{+(n+1)}^\vee)$. For any $w \in W^v$, $w\mu_k \in \operatorname{supp}(f - \mathcal{S}(\varphi_n)) \subset \cup_{j=1}^r (\lambda_j - Q_{+n}^\vee)$, so $w\mu_k$ cannot be strictly greater than μ_k for $\leq_{Q^\vee}$; this proves that $\mu_k \in Y^{++}$. So we define $\varphi_{n+1} = \varphi_n - \sum_{k=1}^s a_{\mu_k}(f - \mathcal{S}(\varphi_n))\delta(\mu_k)^{-1/2} c_{\mu_k}$. As $\mathcal{S}(c_\lambda) = \delta^{1/2}(\lambda)e^\lambda + \sum_{\mu <_{Q^\vee} \lambda} \delta^{1/2}(\mu) n_\lambda(\mu) e^\mu$, this φ_{n+1} is suitable. □

Remark. Suppose G is a split Kac-Moody group as in Section 3. Consider the complex Kac-Moody algebra $\mathfrak{g}^\vee$ associated with $G^\vee$, the Langlands dual of G. Let $\mathfrak{h}^\vee = \mathbb{C} \otimes_\mathbb{Z} Y$ be the Cartan subalgebra of $\mathfrak{g}^\vee$. Let $\operatorname{Rep}(\mathfrak{g}^\vee)$ be the category of $\mathfrak{g}^\vee$-modules V such that V is $\mathfrak{h}^\vee$-diagonalizable, the weight spaces V_λ are finite-dimensional and the set $\mathscr{P}(V)$ of weights of V satisfies $\mathscr{P}(V) \subset \cup_{j=1}^r (\lambda_j - Q_+^\vee)$ for some λ_j. One can check that $\operatorname{Rep}(\mathfrak{g}^\vee)$ is stable by tensoring, and hence we can consider its Grothendieck ring $K(\mathfrak{g}^\vee)$. Now, the map $[V] \mapsto \sum_\lambda (\dim V_\lambda) e^\lambda$ is an isomorphism from $K(\mathfrak{g}^\vee)$ onto $\mathbb{C}[[Y]]^{W^v}$. Therefore, by composing it with $\mathcal{S}$, we get an isomorphism between $\mathcal{H}_\mathbb{C}$ and $K(\mathfrak{g}^\vee)$.

5.5. *Extended tree associated to* $(\mathbb{A}, \alpha_i)$. We consider the vectorial panel $-F^v(\{i\})$ in $-\overline{C_f^v}$ and its support the vectorial wall $\operatorname{Ker}(\alpha_i)$. Their respective directions are a panel $\mathfrak{F}_\infty$ in a wall M_∞, in the twin buildings $\mathscr{I}^{\pm\infty}$ at infinity of $\mathscr{I}$ [Rou11, 3.3, 3.4, 3.7].

The germs of the sector-panels in $\mathscr{I}$ of direction $\mathfrak{F}_\infty$ are the points of an (essential) affine building $\mathscr{I}(\mathfrak{F}_\infty)$, which is of rank 1, i.e., a tree [Rou11, 4.6].

The union $\mathscr{I}(M_\infty)$ of the apartments in $\mathscr{I}$ containing a wall of direction M_∞ is an inessential affine building whose essential quotient is $\mathscr{I}(\mathfrak{F}_\infty)$

[Rou11, 4.9]. More precisely, $\mathscr{I}(M_\infty)$ may be identified with the product of the tree $\mathscr{I}(\mathfrak{F}_\infty)$ and an affine space quotient of $\mathbb{A}$.

The canonical apartment of $\mathscr{I}(M_\infty)$ is $\mathbb{A}$ endowed with a smaller set of walls: uniquely the walls of direction $\mathrm{Ker}(\alpha_i)$. As we chose $\mathscr{I}$ semi-discrete (1.2), this is a locally finite set of hyperplanes; hence $\mathscr{I}(M_\infty)$ is discrete and $\mathscr{I}(\mathfrak{F}_\infty)$ a discrete tree (not an $\mathbb{R}$-tree). By [Rou11, 2.9] the valencies of these walls are the same in $\mathscr{I}(M_\infty)$ and in $\mathscr{I}$, i.e., $1 + q_i$ and $1 + q'_i$; hence $\mathscr{I}(\mathfrak{F}_\infty)$ is a semi-homogeneous tree of valencies $1 + q_i$ and $1 + q'_i$. By definition, $0 \in \mathbb{A}$ is in a wall of valence $1 + q_i$.

We asked that the stabilizer N of $\mathbb{A}$ in G be positive and type preserving (1.5), i.e., that it act on $V = \overrightarrow{\mathbb{A}}$ via W^v. So, the stabilizer in W^v of M_∞ is $\{1, r_i\}$, and M_∞ determines in V a supplementary vector subspace of dimension one: $M_\infty^\perp = \mathrm{Ker}(1 + r_i)$. The affine space $\mathbb{A}$ decomposes as the product of the affine space $E = \mathbb{A}/M_\infty^\perp$ with associated vector space $\mathrm{Ker}(\alpha_i)$ and an affine line ($= \mathbb{A}/\mathrm{Ker}(\alpha_i)$). This decomposition is canonical, i.e., invariant by the stabilizer $N(M_\infty)$ of M_∞ in N. As a consequence we get the decomposition $\mathscr{I}(M_\infty) = E \times \mathscr{I}(\mathfrak{F}_\infty)$ that is canonical, i.e., invariant by the stabilizer $G(M_\infty)$ of M_∞ in G. Moreover, $G(M_\infty)$ acts on E by translations only.

Remark. Suppose $\mathfrak{G}$ is an almost split Kac-Moody group over a local field $\mathcal{K}$ and $\mathscr{I}$ its associated hovel as in [Rou12]. Then the stabilizer $G(\mathfrak{F}_\infty)$ of $\mathfrak{F}_\infty$ in G is a parabolic subgroup, endowed with a Levi decomposition $G(\mathfrak{F}_\infty) = G(M_\infty) \ltimes U(\mathfrak{F}_\infty)$ (with $U(\mathfrak{F}_\infty) \subset U^-$) and $\mathscr{I}(M_\infty)$ (resp. $\mathscr{I}(\mathfrak{F}_\infty)$) is the extended (resp. essential) Bruhat-Tits building associated to the reductive group of rank 1 $G(M_\infty)$, embedded in $\mathscr{I}$ [Rou12, 6.12.2]. Any orbit of $U(\mathfrak{F}_\infty)$ in $\mathscr{I}$ meets $\mathscr{I}(M_\infty)$ in one and only one point.

The tree $\mathscr{I}(\mathfrak{F}_\infty)$ is a piece of the polyhedral "compactification" of $\mathscr{I}$ (a true compactification when $\mathfrak{G}$ is reductive). With the notation of [Rou12], $\mathscr{I}(M_\infty)$ (resp. $\mathscr{I}(\mathfrak{F}_\infty)$) is the façade $\mathscr{I}(\mathfrak{G}, \mathcal{K}, \overline{\overline{\mathbb{A}}})_{\mathfrak{F}_\infty}$ (resp. $\mathscr{I}(\mathfrak{G}, \mathcal{K}, \overline{\mathbb{A}}^e)_{\mathfrak{F}_\infty}$).

5.6. *Parabolic retraction.* Let x be a point in $\mathscr{I}$. There is a unique sector-panel $x + \mathfrak{F}_\infty$ of vertex x and direction $\mathfrak{F}_\infty$ [Rou11, 4.7.1]. The germ of this sector-panel is a point in $\mathscr{I}(\mathfrak{F}_\infty)$, the *projection* $\mathrm{pr}_{\mathfrak{F}_\infty}(x)$ of x onto $\mathscr{I}(\mathfrak{F}_\infty)$; cf. [Cha10], [Cha] or [Rou12, 4.3.5] in the Kac-Moody case.

Let A_x be an apartment in $\mathscr{I}$ containing x and $\mathfrak{F}_\infty$; hence $x + \mathfrak{F}_\infty$ and $\mathrm{germ}_\infty(x+\mathfrak{F}_\infty)$. But this germ is in an apartment B_x of $\mathscr{I}(M_\infty)$ (axiom (MA3) applied to $\mathrm{germ}_\infty(x + \mathfrak{F}_\infty)$ and a sector of direction C_f^v) and there exists an isomorphism ψ_x of A_x onto B_x fixing this germ (axiom (MA2)). One writes $\rho(x) = \psi_x(x) \in \mathscr{I}(M_\infty)$. We have thus defined the *retraction* $\rho = \rho_{\mathfrak{F}_\infty, M_\infty}$ *of* $\mathscr{I}$ *onto* $\mathscr{I}(M_\infty)$ *with center* $\mathfrak{F}_\infty$. We shall now verify that $\rho(x)$ does not depend on the choices made.

By definition, $\rho(x)$ is in the hyperplane H_x of B_x of direction M_∞ and containing $germ_\infty(x+\mathfrak{F}_\infty)$; this H_x does not depend on the choice of B_x. Moreover, for two choices $\psi_x : A_x \to B_x$ and $\psi'_x : A'_x \to B_x$, $\psi'_x \circ \psi_x^{-1}$ is the identity on $\mathrm{germ}_\infty(x+\mathfrak{F}_\infty)$ and hence on H_x. It is now clear that $\psi_x(x) = \psi'_x(x)$. Actually $\rho(x)$ may also be defined in the following simple way: there exist $y, z \in (x+\mathfrak{F}_\infty) \cap B_x$ such that y is the middle of $[x,z]$ in A_x. Then $\rho(x)$ is the point of $H_x \subset B_x$ such that y is the middle of $[\rho(x), z]$ in B_x.

Remark. It is possible to prove that the image by ρ of a preordered segment is a polygonal line and, in some generalized sense, a Hecke path.

5.7. *Factorization of* $\rho_{-\infty}$. The panel $\mathfrak{F}_\infty$ is in the closure of the chamber $\mathfrak{C}_{-\infty}$ of $\mathscr{I}^{-\infty}$ associated to $-C^v_f$. So this chamber or the associated sector-germ $\mathfrak{S}_{-\infty}$ determines an end of the tree $\mathscr{I}(\mathfrak{F}_\infty)$ [Rou11, 4.6]; i.e., a sector-germ $\mathfrak{S}'$ in $\mathscr{I}(M_\infty)$: $\mathfrak{S}'$ is one of the two sector-germs in $\mathbb{A}$ (considered as an apartment of $\mathscr{I}(M_\infty)$ with its small set of walls), and each element in $\mathfrak{S}'$ contains a half-apartment of equation $\alpha_i(y) \le k$ with $k \in \mathbb{Z}$. We denote by $\rho'_{-\infty}$ the retraction of $\mathscr{I}(M_\infty)$ onto $\mathbb{A}$ with center $\mathfrak{S}'$.

Lemma. *The retraction* $\rho_{-\infty}$ *factorizes through* ρ: $\rho_{-\infty} = \rho'_{-\infty} \circ \rho$.

Proof. For $x \in \mathscr{I}$, one chooses an apartment A_x containing x and $\mathfrak{C}_{-\infty}$, and hence also containing the sector $x+\mathfrak{C}_{-\infty}$, its sector-germ $\mathfrak{S}_{-\infty}$ and its panel $x+\mathfrak{F}_\infty$. One chooses an apartment B_x of $\mathscr{I}(M_\infty)$ containing $\mathrm{germ}_\infty(x+\mathfrak{F}_\infty)$ and $\mathfrak{S}_{-\infty}$. Hence, A_x and B_x contain both $\mathrm{germ}_\infty(x+\mathfrak{F}_\infty)$ and $\mathfrak{S}_{-\infty}$; by axiom (MA4) there exists an isomorphism ψ_x of A_x onto B_x fixing these two germs. By the definition of the parabolic retraction, in 5.6, $\rho(x) = \psi_x(x)$.

Now the apartments A_x and B_x of $\mathscr{I}(M_\infty)$ contain both $\mathfrak{S}_{-\infty}$ and hence $\mathfrak{S}'$. So there is an isomorphism $\theta : B_x \to \mathbb{A}$ fixing $\mathfrak{S}'$ and hence $\mathfrak{S}_{-\infty}$. As $\rho(x) \in B_x$, one has $\rho'_{-\infty} \circ \rho(x) = \theta(\rho(x)) = \theta \circ \psi_x(x)$, and this is $\rho_{-\infty}(x)$ as $\theta \circ \psi_x : A_x \to \mathbb{A}$ is an isomorphism fixing $\mathfrak{S}_{-\infty}$. □

5.8. *Counting*. We want to prove equation (4): $n_\lambda(r_i\mu) = n_\lambda(\mu)\left(\sqrt{q_i q'_i}\right)^{\alpha_i(\mu)}$ for $\lambda \in Y^{++}$ and $\mu \in Y$, where $n_\lambda(\mu)$ is the number of points $y \in \mathscr{I}_0$ such that $\rho_{-\infty}(y) = -\mu$ and $d^v(y,0) = \lambda$; cf. 5.2. For $z \in \mathscr{I}(M_\infty)$, one writes $p_\lambda(z) \in \mathbb{Z}_{\ge 0} \cup \{\infty\}$ for the number of points $y \in \mathscr{I}_0$ such that $\rho(y) = z$ and $d^v(y,0) = \lambda$. By Lemma 5.7, $n_\lambda(\mu)$ is the sum of $p_\lambda(z)$ for $z \in \mathscr{I}(M_\infty) \cap \mathscr{I}_0$ such that $\rho'_{-\infty}(z) = -\mu$.

Let $M_0 = 0 + M_\infty = \mathrm{Ker}(\alpha_i)$ be the wall in $\mathbb{A}$ of direction M_∞ containing 0. Its pointwise stabilizer $G(M_0)$ ($\subset G(M_\infty)$) acts transitively on the apartments of $\mathscr{I}$ or $\mathscr{I}(M_\infty)$ containing it (by axiom (MA4), as M_0 is the enclosure of two sector-panel germs). Moreover, $G(M_\infty)$ fixes $\mathfrak{F}_\infty$; hence ρ is $G(M_\infty)$-equivariant. As a consequence, the weight function p_λ is constant on the orbits of $G(M_0)$ in $\mathscr{I}(M_\infty) \cap \mathscr{I}_0$. Hence $n_\lambda(\mu) = \sum_\Omega p_\lambda(\Omega) n^\Omega(-\mu)$, where the sum runs over the orbits Ω of $G(M_0)$ in $\mathscr{I}(M_\infty) \cap \mathscr{I}_0$ and $n^\Omega(\nu)$ is the number of points z in the orbit Ω such that $\rho'_{-\infty}(z) = \nu$.

To prove formula (4), it is sufficient to prove that for any orbit Ω as above and any $\nu \in Y$,

$$n^\Omega(r_i\nu) = n^\Omega(\nu)\left(\sqrt{q_i q'_i}\right)^{-\alpha_i(\nu)}.$$

In 5.5 we saw that $G(M_\infty)$ leaves the decomposition $\mathscr{I}(M_\infty) = \mathscr{I}(\mathfrak{F}_\infty) \times E$ invariant and acts on E by translations. But $G(M_0)$ fixes $M_0 \ni 0$, so it acts trivially on E. As $G(M_0)$ is transitive on the apartments containing M_0, an orbit Ω is a set $S_r \times \{e\}$, where S_r is the sphere of radius $r \in \mathbb{Z}_{\geq 0}$ and center 0 in the tree $\mathscr{I}(\mathfrak{F}_\infty)$. The apartment $\mathbb{A}$ (with its small set of walls) is the product $(\mathbb{R}, \mathbb{Z}) \times E$, where α_i is the projection of $\mathbb{A}$ onto the one-dimensional apartment $\mathbb{R}$ with vertex set $\mathbb{Z}$.

So, the above formula and hence formula (4) and Theorem 5.4 are consequences of the following proposition. The fact that $q_i = q'_i$ when $m = \alpha_i(\nu)$ is odd was explained in the proof of 5.4.

5.9. *The tree case.* Let $\mathbb{T}$ be a (discrete) semi-homogeneous tree. Let $\mathbb{A} \simeq \mathbb{R}$ be an apartment in $\mathbb{T}$ whose vertices are identified with $\mathbb{Z}$. The valency of the vertex $s \in \mathbb{Z}$ is $1 + q$ (resp. $1 + q'$) if s is even (resp. odd). Let $-\infty$ be the end of $\mathbb{A}$ corresponding to integers converging towards $-\infty$. Let ρ' be the retraction of $\mathbb{T}$ onto $\mathbb{A}$ with center $-\infty$. For $m \in \mathbb{Z} \subset \mathbb{A}$ and $r \in \mathbb{Z}_{\geq 0}$, we write $n_r(m)$ the number of vertices in the sphere S_r of center 0 and radius r in $\mathbb{T}$ such that $\rho'(z) = m$.

If m is odd, we ask that $q = q'$.

Proposition. *One has* $n_r(m) = n_r(-m)(\sqrt{qq'})^m$.

Remark. This formula is equivalent to the $W^v(\mathbb{T})$-invariance of the image of the Satake isomorphism for the Bruhat-Tits tree $\mathbb{T}$. As this invariance is known, the following proof is not necessary; we give it for the convenience of the reader.

For a Bruhat-Tits tree $\mathscr{I} = \mathbb{T}$, there are two choices for $\mathscr{I}_0$ (and Y): the set of vertices at even distance from 0 or the full set of vertices. In this last case, we have to allow m to be odd, and we see below that the hypothesis $q = q'$ is necessary to get the formula. So, even for classical Bruhat-Tits buildings, to get the good image for the Satake isomorphism, $\mathscr{I}_0$ cannot be any G-stable set of special vertices. (We chose $\mathscr{I}_0$ to be a G-orbit.)

Proof. For $z \in S_r$, let $s_z \in \mathbb{Z}$ be the vertex of $\mathbb{A}$ such that $[0, s_z] = [0, z] \cap \mathbb{A}$. Then $\rho'(z) = s_z + (r - |s_z|) \in \mathbb{Z}$. We can calculate the number $n_r(m)$ of vertices $z \in S_r$ such that $\rho'(z) = m$:

First case: $s_z \geq 0 \iff \rho'(z) = r$. So $n_r(r) = qq'qq' \cdots$ (r factors).

Second case: $-r \leq s_z < 0 \iff \rho'(z) < r$ and then $\rho'(z) = r + 2s_z$, i.e., $s_z = (\rho'(z) - r)/2$. The number $n_r(m)$ is then

$$\begin{array}{ll} 1 & \text{if} \quad m = s_z = -r, \\ (q-1)q'qq' \ \cdots \ (r + s_z = (r+m)/2 \text{ factors}) & \text{if} \quad s_z \in]-r, 0[\ \text{ is even}, \\ (q'-1)qq'q \ \cdots \ (r + s_z = (r+m)/2 \text{ factors}) & \text{if} \quad s_z \in]-r, 0[\ \text{ is odd}. \end{array}$$

It is now easy to compare $n_r(m)$ and $n_r(-m)$. We get the wanted formula, using that $q = q'$ when m is odd. □

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