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Some Forms of Kac–Moody Algebras*

NICOLÁS ANDRUSKIEWITSCH

Universidad Nacional de Córdoba, Córdoba, Argentina

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DEDICADO A LUISA GARIBOTTI

1. INTRODUCTION

Let k be an algebraically closed field of char 0. The (unique up to isomorphism) three-dimensional simple Lie algebra $sl(2, k)$ plays an important role in the theory of the semisimple Lie algebras over k. For example, Chevalley, Harish Chandra, and Serre's construction of the semisimple Lie algebras g over k corresponding to a Dynkin diagram $\mathscr{D}$ can be viewed as "*glueing together*" copies of $sl(2, k)$ in "*$\mathscr{D}$-fashion.*" And, in turn, this was the inspiration for the definition of the Kac–Moody algebras.

Now let us drop the algebraic closedness assumption on k and denote by $\bar{k}$ the algebraic closure of k. Then the three-dimensional simple Lie algebras over k (TDS, for short) are in one-to-one correspondence with the quaternion algebras over k. This paper is concerned with a construction by generators and relations of k-forms of (symmetrizable) Kac–Moody algebras over $\bar{k}$. This construction is inspired by (and depends on) that of Chevalley, Harish Chandra, Serre, Kac, and Moody and can be roughly described as "*glueing together*" suitably choosen TDS over k, *in* "*$\mathscr{D}$-fashion.*" To be more clear, we present two constructions: first, we "*glue*" copies of the same three-dimensional simple Lie algebra (Definition 2) and show that one gets forms of Kac–Moody algebras (Proposition 1). Second, we "*glue*" copies of distinct TDS Lie algebras (Definition 4); but we can not choose the distinct TDS in an arbitrary way. We need some compatibility conditions between the TDS attached to neighbouring vertices: they are described by some scalars s_{ij}. And for simplicity, we restrict our attention in this definition to Dynkin diagrams without cycles. For both definitions, we also make the additional hypothesis that the

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number of strokes connecting two vertices does not exceed 3 (since we have not developed a general formula yet).

It turns out, however, that these constructions do not exhaust all the forms of Kac–Moody algebras. For example, one also needs to "*glue*" quaternion algebras over finite extensions of k: this is the case for the fixed point set of an antilinear finite order automorphism arising from a diagram automorphism. Here we shall only give some indications about this construction; we will postpone its discussion to a future paper.

When $k=\mathbf{R}$, a somewhat different approach is used in [BP], by means of antilinear involutions. Our viewpoint, however, is not only more general, but enables us to construct a symmetric bilinear invariant form. In the real case, this has some interesting consequences: if the matrix is of affine type, Cartan decompositions are available, using, by the way, some strong results of [PK]. (This was also obtained using a slightly different method in [R, KP3]). Let us also add that the classification of all the involutive automorphisms (of the first kind) of affine Kac–Moody algebras is contained in [L] (see also [B]). On the other hand, we are concerned here only with "derived" Kac–Moody algebras associated to a symmetrizable generalized Cartan matrix (cf. [K, 0.3]). In a forthcoming article, we will discuss the general case.

Finally, we can also attach a "Kac–Moody" group to the forms constructed here, in the same vein as in [PK]. This suggests the existence of some "infinite dimensional symmetric spaces." But as there is not a unique way to attach a group to Kac–Moody data in the split case (see [T, GW]) we are not sure yet of what could be the right definition for such spaces.

2. Definitions

Let k be a field of characteristic 0, X, Y, Z a basis of a 3-dimensional k-vector space V. For fixed $a, b \in k^* = k - 0$ we can define a Lie algebra structure, which we shall call $sq(a, b)$ on V by the rule:

$$[X, Y] = 2Z$$

$$[Y, Z] = -2bX$$

$$[Z, X] = -2aY.$$

Let $(d_1, ..., d_n)$ denote the quadratic space (k^n, q), where q is the quadratic form such that $q(\sum_i \lambda_i e_i) = \sum_i d_i \lambda_i^2$. ($\{e_i\}$ is the canonical basis.) In addition let $\langle\!\langle -a, -b \rangle\!\rangle$ denote the quaternion algebra having a basis $\{1, i, j, k\}$ with a multiplicative table

$$i^2 = a, \qquad j^2 = b, \qquad ij = -ji = k.$$

Endowed with the usual norm, it is a quadratic space isomorphic to $(1, -a, -b, ab)$, which is in turn the Pfister 2-form $\langle\langle -a, -b\rangle\rangle$ (hence the notation). Then it is well known that $sq(a, b)$ is isomorphic to $sq(c, d)$ if and only if the quadratic spaces $(-a, -b, ab)$ and $(-c, -d, cd)$ are; moreover $sq(a, b)$ is simple and every 3-dimensional simple Lie over a k algebra arises in this way. In fact, $sq(a, b)$ can be realized as the Lie algebra of the traceless elements of the quaternion algebra $\langle\langle -a, -b\rangle\rangle$; it is the Lie algebra of the group $SQ(a, b)$ of the elements of $\langle\langle -a, -b\rangle\rangle$ having norm equal to one. $sl(2, k)$ is isomorphic to $sq(1, -1)$ and if $k = \mathbf{R}$, $sq(-1, -1)$ is $su(2, \mathbf{R})$.

Now let $A = (a_{ij}) \in \mathbf{Z}^{n \times n}$ be a generalized Cartan matrix, i.e.,

$$a_{ii} = 2$$

$$a_{ij} \leqslant 0 \qquad i \neq j$$

$$a_{ij} = 0 \Rightarrow a_{ji} = 0.$$

As usual, we will say that A is a Cartan matrix if it corresponds to a finite dimensional complex Lie algebra. Let us also recall that an $n \times n$ matrix A is called symmetrizable if there exists a non-degenerate diagonal $n \times n$ matrix D such that DA is symmetric. In the rest of the paper, A will denote a symmetrizable generalized Cartan matrix. We will also assume, for convenience, that $-3 \leqslant a_{ij}$ and that the corresponding Dynkin diagram is connected. Let us recall the definition of a Kac–Moody algebra. (In this paper we will work with the "derived" Kac–Moody algebra; see, for example, [K, GK]):

DEFINITION 1. $g_k(A)$ is the Lie algebra over k with $3n$ generators $\{E_i, F_i, H_i\colon 1 \leqslant i \leqslant n\}$ and defining relations

$$[H_i, H_j] = 0 \tag{1}$$

$$[E_i, F_j] = \delta_{ij} H_j \tag{2}$$

$$[H_i, E_j] = a_{ij} E_j \tag{3}$$

$$[H_i, F_j] = -a_{ij} F_j \tag{4}$$

and for $i \neq j$

$$(\mathrm{ad}\, E_i)^{1 - a_{ij}} E_j = 0 \tag{5}$$

$$(\mathrm{ad}\, F_i)^{1 - a_{ij}} F_j = 0. \tag{6}$$

Let us fix $a, b \in k^*$.

DEFINITION 2. $g_k(A, a, b)$ is the Lie algebra over k with $3n$ generators $\{X_i, Y_i, Z_i\colon 1 \leqslant i \leqslant n\}$ and defining relations

$$[Z_i, Z_j] = 0 \tag{7}$$

$$[X_i, Y_i] = 2Z_i \tag{8}$$

$$[Z_i, X_j] = -a_{ij} a Y_j \tag{9}$$

$$[Y_j, Z_i] = -a_{ij} b X_j \tag{10}$$

and if $i \neq j$

$$[X_i, Y_j] = [Y_i, X_j] \tag{11}$$

$$[X_i, X_j] = -ab^{-1}[Y_i, Y_j] \tag{12}$$

$$(\operatorname{ad} X_i)^{1-a_{ij}} X_j = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ aX_j & \text{if } a_{ij} = -1 \\ 4a(\operatorname{ad} X_i) X_j & \text{if } a_{ij} = -2 \\ 10a(\operatorname{ad} X_i)^2 X_j - 9a^2 X_j & \text{if } a_{ij} = -3 \end{cases} \tag{13}$$

$$(\operatorname{ad} X_i)^{1-a_{ij}} Y_j = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ aY_j & \text{if } a_{ij} = -1 \\ 4a(\operatorname{ad} X_i) Y_j & \text{if } a_{ij} = -2 \\ 10a(\operatorname{ad} X_i)^2 Y_j - 9a^2 Y_j & \text{if } a_{ij} = -3 \end{cases} \tag{14}$$

$$(\operatorname{ad} Y_i)^{1-a_{ij}} X_j = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ bX_j & \text{if } a_{ij} = -1 \\ 4b(\operatorname{ad} Y_i) X_j & \text{if } a_{ij} = -2 \\ 10b(\operatorname{ad} Y_i)^2 X_j - 9b^2 X_j & \text{if } a_{ij} = -3 \end{cases} \tag{15}$$

$$(\operatorname{ad} Y_i)^{1-a_{ij}} Y_j = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ bY_j & \text{if } a_{ij} = -1 \\ 4b(\operatorname{ad} Y_i) Y_j & \text{if } a_{ij} = -1 \\ 10b(\operatorname{ad} Y_i)^2 Y_j - 9b^2 Y_j & \text{if } a_{ij} = -3. \end{cases} \tag{16}$$

We will drop the subscript k whenever it is clear from the context which field we are working in.

PROPOSITION 1. (i) *There is a natural isomorphism* $g_k(A, a, b) \otimes_k k' \simeq g_{k'}(A, a, b)$ *if* k' *is an extension of* k.

(ii) *Let* $t, s \in k^*$. *Then* $X'_i \mapsto tX_i$, $Y'_i \mapsto sY_i$, $Z'_i \mapsto tsZ_i$ *provides an isomorphism between* $g(A, at^2, bs^2)$ (*with generators* X'_i, Y'_i, Z'_i) *and* $g(A, a, b)$.

(iii) $X_i^* \mapsto Y_i$, $Y_i^* \mapsto X_i$, $Z_i^* \mapsto -Z_i$ *provides an isomorphism between* $g(A, a, b)$ *(with generators* X_i^*, Y_i^*, Z_i^**) and* $g(A, b, a)$.

(iv) $X_i \mapsto -X_i$, $Y_i \mapsto Y_i$, $Z_i \mapsto -Z_i$ *provides an automorphism of* $g(A, a, b)$, *called the Cartan involution.*

(v) *Let* μ *be an automorphism of the Dynkin diagram (in other words,* $\mu \in S_n$ *and* $a_{ij} = a_{\mu(i)\,\mu(j)}$*). Then there exists a unique automorphism of* $g(A, a, b)$ *satisfying* $X_i \mapsto X_{\mu(i)}$, $Y_i \mapsto Y_{\mu(i)}$, $Z_i \mapsto Z_{\mu(i)}$.

(vi) $g_k(A, 1, -1)$ *is isomorphic to the Kac–Moody algebra over* k *associated with* A, $g_k(A)$. *In particular, if* A *is a Cartan matrix,* $g(A, a, b)$ *is absolutely simple.*

Proof. (i) to (v) are easy. (vi): The applications

$$\begin{aligned} &g_k(A) \to g_k(A, 1, -1) \\ &H_i \mapsto Z_i \\ &E_i \mapsto \tfrac{1}{2}(X_i - Y_i) \\ &F_i \mapsto \tfrac{1}{2}(X_i + Y_i) \end{aligned}$$

and

$$\begin{aligned} &g_k(A, 1, -1) \to g_k(A) \\ &Z_i \mapsto H_i \\ &X_i \mapsto E_i + F_i \\ &Y_i \mapsto -E_i + F_i \end{aligned}$$

are well defined and inverse of each other. Clearly, we only need to check the well defined part. Let us show first that the relations (1), ..., (6) imply (7), ..., (16). (7), ..., (12) are easy. Let us see (13) and leave to the reader (14), ..., (16), since they are very similar.

$a_{ij} = 0$: obvious.

$a_{ij} = -1$:

$$\begin{aligned} [X_i, [X_i, X_j]] &= [E_i + F_i, [E_i + F_i, E_j + F_j]] \\ &= [E_i + F_i, [E_i, E_j] + [F_i, F_j]] \\ &= \operatorname{ad} E_i^2 E_j + [E_i, [F_i, F_j]] \\ &\quad + [F_i, [E_i, E_j]] + \operatorname{ad} F_i^2 F_j = E_j + F_j = X_j \end{aligned}$$

since

$$\begin{aligned} [E_i, [F_i, F_j]] &= [H_i, F_j] = -a_{ij} F_j \\ [F_i, [E_i, E_j]] &= [-H_i, E_j] = -a_{ij} E_j. \end{aligned}$$

$a_{ij}=-2$:

$$\begin{aligned}\operatorname{ad} X_i^3 X_j &= [E_i+F_i, \operatorname{ad} E_i^2 E_j - a_{ij}(E_j+F_j) + \operatorname{ad} F_i^2 F_j] \\ &= \operatorname{ad} E_i^3 E_j - a_{ij}[E_i, E_j] + [E_i, \operatorname{ad} F_i^2 F_j] \\ &\quad - a_{ij}[F_i, F_j] + [F_i, \operatorname{ad} E_i^2 E_j] + \operatorname{ad} F_i^3 F_j \\ &= -(2+3a_{ij})([E_i, E_j] + [F_i, F_j]) = 4[X_i, X_j]\end{aligned}$$

since

$$\begin{aligned}&[E_i, [F_i, [F_i, F_j]]] \\ &\quad = [H_i, [F_i, F_j]] + [F_i, [H_i, F_j]] = -(2+2a_{ij})[F_i, F_j] \\ &[F_i, [E_i, [E_i, E_j]]] \\ &\quad = [-H_i, [E_i, E_j]] - [E_i, [H_i, E_j]] = -(2+2a_{ij})[E_i, E_j].\end{aligned}$$

$a_{ij}=-3$:

$$\begin{aligned}\operatorname{ad} X_i^4 X_j &= [E_i+F_i, \operatorname{ad} E_i^3 E_j \\ &\quad - (2+3a_{ij})([E_i, E_j] + [F_i, F_j]) + \operatorname{ad} F_i^3 F_j] \\ &= \operatorname{ad} E_i^4 E_j - (2+3a_{ij})[E_i+F_i, [E_i, E_j] + [F_i, F_j]] \\ &\quad + [E_i, \operatorname{ad} F_i^3 F_j] + [F_i, \operatorname{ad} E_i^3 E_j] + \operatorname{ad} F_i^4 F_j \\ &= -(8+6a_{ij})(\operatorname{ad} E_i^2 E_j + \operatorname{ad} F_i^2 F_j + 3E_j + 3F_j) \\ &\quad - a_{ij}(8+6a_{ij} - 2 - 3a_{ij})(E_j+F_j) \\ &= 10 \operatorname{ad} X_i^2 X_j - 9X_j\end{aligned}$$

since

$$\begin{aligned}[E_i, [F_i[F_i, [F_i, F_j]]]] &= [H_i, \operatorname{ad} F_i^2 F_j] + [F_i, (-2-2a_{ij})[F_i, F_j]] \\ &= -(6+3a_{ij}) \operatorname{ad} F_i^2 F_j \\ [F_i, [E_i[E_i, [E_i, E_j]]]] &= [-H_i, \operatorname{ad} E_i^2 E_j] + [E_i, (-2-2a_{ij})[E_i, E_j]] \\ &= -(6+3a_{ij}) \operatorname{ad} E_i^2 E_j.\end{aligned}$$

Conversely, let us show that (7), ..., (16) imply (1), ..., (6). As before, (1), ..., (4) are easy. Let us show (5) and leave (6) to the reader.

$a_{ij}=0$:

$$\tfrac{1}{4}[X_i - Y_i, X_j - Y_j] = \tfrac{1}{2}([X_i, Y_j] - [X_i, Y_j]).$$

$a_{ij}=-1$:

$$\begin{aligned}[E_i, [E_i, E_j]] &= \tfrac{1}{4}[X_i - Y_i, [X_i, X_j] - [X_i, Y_j]] \\ &= \tfrac{1}{4}(\operatorname{ad} X_i^2 X_j - \operatorname{ad} X_i^2 Y_j + \operatorname{ad} Y_i^2 X_j - \operatorname{ad} Y_i^2 Y_j).\end{aligned}$$

$a_{ij} = -2$:

$$\begin{aligned}[E_i, \operatorname{ad} E_i^2 E_j] &= \tfrac{1}{8}[X_i - Y_i, \operatorname{ad} X_i^2 X_j - \operatorname{ad} X_i^2 Y_j + \operatorname{ad} Y_i^2 X_j - \operatorname{ad} Y_i^2 Y_j] \\ &= \tfrac{1}{8}(\operatorname{ad} X_i^3 X_j - \operatorname{ad} X_i^3 Y_j + [X_i, \operatorname{ad} Y_i^2 X_j] - [X_i, \operatorname{ad} Y_i^2 Y_j] \\ &\quad - \operatorname{ad} Y_i^3 X_j + \operatorname{ad} Y_i^3 Y_j - [Y_i, \operatorname{ad} X_i^2 X_j] + [Y_i, \operatorname{ad} X_i^2 Y_j]) \\ &= \tfrac{1}{4}(\operatorname{ad} X_i^3 X_j - \operatorname{ad} X_i^3 Y_j - \operatorname{ad} Y_i^3 X_j + \operatorname{ad} Y_i^3 Y_j)\end{aligned}$$

since

$$\begin{aligned}[X_i, \operatorname{ad} Y_i^2 X_j] &= -4(1 + a_{ij})[X_i, X_j] + \operatorname{ad} Y_i^3 Y_j \\ [X_i, \operatorname{ad} Y_i^2 Y_j] &= -4(1 + a_{ij})[X_i, Y_j] + \operatorname{ad} Y_i^3 X_j \\ [Y_i, \operatorname{ad} X_i^2 X_j] &= 4(1 + a_{ij})[X_i, Y_j] + \operatorname{ad} X_i^3 Y_j \\ [Y_i, \operatorname{ad} X_i^2 Y_j] &= 4(1 + a_{ij})[X_i, X_j] + \operatorname{ad} X_i^3 X_j.\end{aligned}$$

$a_{ij} = -3$:

$$\begin{aligned}[E_i, \operatorname{ad} E_i^3 E_j] &= \tfrac{1}{8}[X_i - Y_i, \operatorname{ad} X_i^3 X_j - \operatorname{ad} X_i^3 Y_j - \operatorname{ad} Y_i^3 X_j + \operatorname{ad} Y_i^3 Y_j] \\ &= \tfrac{1}{4}(\operatorname{ad} X_i^4 X_j - \operatorname{ad} X_i^4 Y_j + \operatorname{ad} Y_i^4 X_j - \operatorname{ad} Y_i^4 Y_j) \\ &\quad + \tfrac{14}{8}(\operatorname{ad} X_i^2 Y_j - \operatorname{ad} X_i^2 X_j - \operatorname{ad} Y_i^2 Y_j + \operatorname{ad} Y_i^2 X_j)\end{aligned}$$

since

$$\begin{aligned}[X_i, \operatorname{ad} Y_i^3 X_j] &= 2[Z_i, \operatorname{ad} Y_i^2 X_j] + [Y_i, [X_i, \operatorname{ad} Y_i^2 X_j]] \\ &\quad - 4 \operatorname{ad} X_i^2 Y_j - (8 + 6a_{ij}) \operatorname{ad} Y_i^2 Y_j + \operatorname{ad} Y_i^4 Y_j \\ [X_i, \operatorname{ad} Y_i^3 Y_j] &= 2[Z_i, \operatorname{ad} Y_i^2 Y_j] + [Y_i, [X_i, \operatorname{ad} Y_i^2 Y_j]] \\ &\quad - 4 \operatorname{ad} X_i^2 X_j - (8 + 6a_{ij}) \operatorname{ad} Y_i^2 X_j + \operatorname{ad} Y_i^4 X_j \\ [Y_i, \operatorname{ad} X_i^3 X_j] &= -2[Z_i, \operatorname{ad} X_i^2 X_j] + [X_i, [Y_i, \operatorname{ad} X_i^2 X_j]] \\ &\quad + 4 \operatorname{ad} Y_i^2 Y_j + (8 + 6a_{ij}) \operatorname{ad} X_i^2 Y_j + \operatorname{ad} X_i^4 Y_j \\ [Y_i, \operatorname{ad} X_i^3 Y_j] &= -2[Z_i, \operatorname{ad} X_i^2 Y_j] + [X_i, [Y_i, \operatorname{ad} X_i^2 Y_j]] \\ &\quad + 4 \operatorname{ad} Y_i^2 X_j + (8 + 6a_{ij}) \operatorname{ad} X_i^2 X_j + \operatorname{ad} X_i^4 X_j.\end{aligned}$$

We also need to use the following consequence of relations (8) to (12):

$$\begin{aligned}(\operatorname{ad} X_i)^2 X_j &= -2a_{ij} X_j + (\operatorname{ad} Y_i)^2 X_j \\ (\operatorname{ad} Y_i)^2 Y_j &= 2a_{ij} Y_j + (\operatorname{ad} X_i)^2 Y_j.\end{aligned}$$

Q.E.D.

Now let us recall from [PK]:

THEOREM 1. *Two maximal* ad-*diagonalizable subalgebras of a Kac–Moody algebra are conjugate.*

This suggests the following definition:

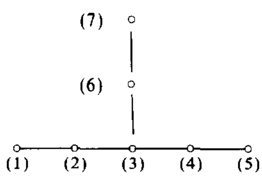
DEFINITION 3. Let G be an arbitrary Lie algebra over k, H a subalgebra of G. H is a Cartan subalgebra if it is maximal in the set of abelian subalgebras consisting of ad-locally finite semisimple elements.

Let $\mathbf{h} = \mathbf{h}_k$ be the k-span of $\{Z_i: 1 \leqslant i \leqslant n\}$. If $Z = \sum_{i=1}^{n} \lambda_i Z_i \in \mathbf{h}$, then we define as usual $\alpha_j \in \mathbf{h}^*$ by $\alpha_j(Z) = \sum_{i=1}^{n} \lambda_i a_{ij}$. We get from Proposition 1:

COROLLARY 1. $\mathbf{h}_k$ *is a Cartan subalgebra of* $g_k(A, a, b)$. *Moreover, the center of* $g_k(A, a, b)$ *is* $\{Z \in h_k: \alpha_j(Z) = 0\ \forall j\}$.

Proof. The first assertion is clear; for the second, $\{Z \in h_k: \alpha_j(Z) = 0\ \forall j\}$ is evidently contained in the center and one gets the equality tensoring with $\bar{k}$ and using [K, 1.6]. Q.E.D.

Now let us assume that the Dynkin diagram of A has no cycles and fix a "sequential" ordering of the simple roots from left to right; for example, for $e_6^{(1)}$ we may choose it as follows:



For each k: $1 \leqslant k \leqslant n$, let us denote by I'_k the (unique) subset of $\{1, ..., n\}$ given by the vertices of the path from 1 to k; $I_k = I'_k - \{k\}$; if $h \in I_k$, then $S_k(h) \in I'_k$ ($S(h)$ if no confusion arises) is preceded by h.

Now let us fix elements of k^* a_i $(1 \leqslant i \leqslant n)$, b, s_{ij} (for $i < j$ such that $a_{ij} \neq 0$). Let us set

$$b_1 = b$$

$$b_j = a_i^{-1} b_i a_j s_{ij}^2 \qquad (\text{if } i < j,\ a_{ij} \neq 0)$$

$$s_{ii} = 1$$

$$s_{ji} = s_{ij}^{-1} \qquad (\text{if } i < j,\ a_{ij} \neq 0)$$

$$s_{1j} = \prod_{h \in I_j} s_{h, S(h)}, \qquad s_{ij} = s_{1i}^{-1} s_{1j}.$$

Let us observe that $\forall i, j, r$:

$$b_j = b_i a_i^{-1} a_j s_{ij}^2$$
$$s_{ij} = s_{ir} s_{rj}.$$

Remark 1. One can allow cycles in the Dynkin diagram by imposing the following compatibility condition on the data (a_i, s_{ij}, b): for general i, j s_{ij} can be defined as above and this does not depend on the path.

DEFINITION 4. $g_k(A, a_i, s_{ij}, b)$ is the Lie algebra over k given by $3n$ generators $\{X_i, Y_i, Z_i\colon 1 \leqslant i \leqslant n\}$ and relations

$$[Z_i, Z_j] = 0 \tag{17}$$
$$[X_i, Y_i] = 2Z_i \tag{18}$$
$$[Z_i, X_j] = -a_i s_{ij}^{-1} a_{ij} Y_j \tag{19}$$
$$[Y_j, Z_i] = -b_i s_{ij} a_{ij} X_j \tag{20}$$

and if $i \neq j$

$$[X_i, Y_j] = s_{ij} [Y_i, X_j] \tag{21}$$
$$[X_i, X_j] = -a_i b_i^{-1} s_{ij}^{-1} [Y_i, Y_j] \tag{22}$$
$$(\text{ad } X_i)^{1-a_{ij}} X_j = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ a_i X_j & \text{if } a_{ij} = -1 \\ 4a_i (\text{ad } X_i) X_j & \text{if } a_{ij} = -2 \\ 10 a_i (\text{ad } X_i)^2 X_j - 9a_i^2 X_j & \text{if } a_{ij} = -3 \end{cases} \tag{23}$$
$$(\text{ad } X_i)^{1-a_{ij}} Y_j = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ a_i Y_j & \text{if } a_{ij} = -1 \\ 4a_i (\text{ad } X_i) Y_j & \text{if } a_{ij} = -2 \\ 10 a_i (\text{ad } X_i)^2 Y_j - 9a_i^2 Y_j & \text{if } a_{ij} = -3 \end{cases} \tag{24}$$
$$(\text{ad } Y_i)^{1-a_{ij}} X_j = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ b_i X_j & \text{if } a_{ij} = -1 \\ 4b_i (\text{ad } Y_i) X_j & \text{if } a_{ij} = -2 \\ 10 b_i (\text{ad } Y_i)^2 X_j - 9b_i^2 X_j & \text{if } a_{ij} = -3 \end{cases} \tag{25}$$
$$(\text{ad } Y_i)^{1-a_{ij}} Y_j = \begin{cases} 0 & \text{if } a_{ij} = 0 \\ b_i Y_j & \text{if } a_{ij} = -1 \\ 4b_i (\text{ad } Y_i) Y_j & \text{if } a_{ij} = -2 \\ 10 b_i (\text{ad } Y_i)^2 Y_j - 9b_i^2 Y_j & \text{if } a_{ij} = -3. \end{cases} \tag{26}$$

PROPOSITION 2. (i) *There is a natural isomorphism* $g_k(A, a_i, s_{ij}, b)\otimes_k k' \simeq g_{k'}(A, a_i, s_{ij}, b)$ *if* k' *is an extension of* k.

(ii) *Let* $a\in k^*$ *and set* $a_i=a$, $s_{ij}=1$ $(1\leqslant i\leqslant n \quad a_{ij}\neq 0)$. *Then* $g(A, a_i, s_{ij}, b)$ *is isomorphic to* $g(A, a, b)$.

(iii) *Let* γ, λ_i, $v_{ij}\in k^*$ $(1\leqslant i\leqslant n,\ i<j$ *and* $a_{ij}\neq 0)$. *Then* $g(A, a_i\lambda_i^2, s_{ij}v_{ij}, b\gamma^2)$ *is isomorphic to* $g(A, a_i, s_{ij}, b)$.

(iv) *Let* $c, d\in k^*$ *and let us assume that there exist* λ_i, $\gamma\in k^*$ *satisfying*:

$$\lambda_i^2=ca_i^{-1}$$
$$\gamma^2=db_1^{-1}.$$

Then $g(A, a_i, s_{ij}, b)$ *is isomorphic to* $g(A, c, d)$ (*cf. Definition* 2).

(v) *If* k *is algebraically closed then* $g(A, a_i, s_{ij}, b)$ *is isomorphic to* $g(A)$ (*cf. Definition* 1).

(vi) *If* A *is a Cartan matrix then* $g_k(A, a_i, s_{ij}, b)$ *is absolutely simple.*

Proof. (i) and (ii) are clear.

(iii) Let us denote

$$a_i'=a_i\lambda_i^2 \qquad (1\leqslant i\leqslant n) \qquad b_1'=b_1\gamma^2$$

and for $j>i$, $a_{ij}\neq 0$

$$v_{ji}=v_{ij}^{-1} \qquad v_{1j}=\prod_{h\in I_j} v_{h,S(h)} \qquad v_{ij}=v_{1i}^{-1}v_{1j}$$
$$b_j'=a_i'^{-1}b_i'a_j's_{ij}'^2 \qquad s_{ij}'=s_{ij}v_{ij}.$$

We have:

$$b_j'=\gamma^2\lambda_1^{-2}\lambda_j^2v_{1j}^2b_j.$$

We will show that

$$X_j'\mapsto\lambda_jX_j$$
$$Y_j'\mapsto\gamma\lambda_1^{-1}\lambda_jv_{1j}Y_j$$
$$Z_j'\mapsto\gamma\lambda_1^{-1}\lambda_j^2v_{1j}Z_j$$

gives an isomorphism from $g(A, a_i\lambda_i^2, s_{ij}v_{ij}, b\gamma^2)$ (with generators X_i', Y_i', Z_i' and relations (17′), ..., (26′)) onto $g(A, a_i, s_{ij}, b)$.

We can reduce ourselves to show that the isomorphism is well defined, i.e., that the images of X_j', Y_j', Z_j' satisfy relations (17′), ..., (26′). Relations

(17′) and (18′) are obvious. For (19′), ..., (26′) it suffices to treat the case $a_{ij} \neq 0$, which is a straightforward computation, using the following formula:

$$v_{ij} = v_{ir} v_{rj} \qquad \forall i, r, j. \tag{*}$$

Finally, (iv) follows from (ii), (iii) putting $v_{ij} = cs_{ij}^{-1}$; (v) and (vi) are consequences of (iv) and Proposition 1. Q.E.D.

Exactly as after Proposition 1, let us define $\mathbf{h} = \mathbf{h}_k$ as the k-span of $\{Z_i : 1 \leqslant i \leqslant n\}$; and if $Z = \sum_{i=1}^{n} \lambda_i Z_i \in \mathbf{h}$, then $\alpha_j \in \mathbf{h}^*$ by $\alpha_j(Z) = \sum_{i=1}^{n} \lambda_i a_{ij}$. Thus we have:

COROLLARY 2. *$\mathbf{h}_k$ is a Cartan subalgebra of $g_k(A, a_i, s_{ij}, b)$. Moreover, if A is affine the center is one-dimensional.*

Remark 2. Let us define $g_k^{\mathrm{loc}}(A, a_i, s_{ij}, b) = g^{\mathrm{loc}}$ as the linear subspace of $g_k(A, a_i, s_{ij}, b)$ spanned by $\{X_i, Y_i, Z_i\}$; the proposition shows that it is $3n$-dimensional.

The Lie algebra L freely generated by $\{X_i, Y_i, Z_i\}$ has an $\mathbf{N}_0$-graded structure given by $\deg(X_i) = \deg(Y_i) = 1$, $\deg(Z_i) = 0$. Let us consider the ascending filtration on L given by

$$L_m = \{u \in L : \deg(u) \leqslant m\}$$

and let $g_m = \pi(L_m)$, where $\pi: L \to g(A, a_i, s_{ij}, b) = g$ is the canonical projection. Thus $(g_m)_{m \in \mathbf{N}_0}$ is an ascending filtration of g. Moreover, $[g_m, g_n] = g_{m+n}$ and $g_1 = g^{\mathrm{loc}}$. Now, if $u \in g$, let us put

$$v(u) = \inf\{m : u \in g_m\}.$$

Let us also remark, though it is obvious, that the introduced filtration is compatible with the isomorphisms given by Propositions 1 and 2. On the other hand, let us consider the principal gradation of $g(A, 1, -1) = g(A)$ (cf. [K, 1.5]) denoted $(g_j(1))_{j \in \mathbf{Z}}$. In this case we have

$$g_m = \bigoplus_{-m \leqslant j \leqslant m} g_j(1).$$

3. EXAMPLES

In this section, we will study some forms of Kac–Moody algebras over quadratic extensions of k. So let $q \in k^* - k^{*2}$, $k(\sqrt{q})$ a quadratic extension of k having a square root of q, $\eta \in \mathrm{Gal}(k(\sqrt{q})/k) - \mathrm{id}$. We begin with a well-known lemma:

LEMMA 1. *Let T be an order 2 automorphism of $g_k(A, a_i, s_{ij}, b)$. Then there exists a unique antilinear involution J of $g_{k(\sqrt{q})}(A, a_i, s_{ij}, b)$ such that $x \mapsto T(x)$ $\forall x \in g_k(A, a_i, s_{ij}, b)$ (with the canonical inclusion given by Prop. 1).*

Proof. J is antilinear if for $\lambda \in k(\sqrt{q})$, $v \in g_{k(\sqrt{q})}(A, a_i, s_{ij}, b)$, $J(\lambda v) = \eta(\lambda)\, J(v)$. We define $J(u + \sqrt{q}\, w) = T(u) - \sqrt{q}\, T(w)$, for all $u, w \in g_k(A, a_i, s_{ij}, b)$. Clearly, J satisfies the required conditions. Q.E.D.

(I) $g_{\mathbf{R}}(A, -1, -1)$ is the compact form of $g_{\mathbf{C}}(A)$ (cf. [K, 2.7]). More generally,

PROPOSITION 3. *There exists a unique involutive antilinear automorphism J of $g_{k(\sqrt{q})}(A, 1, b)$ (with generators $\{X_i, Y_i, Z_i\}$) such that $X_i \mapsto -X_i$, $Y_i \mapsto Y_i$, $Z_i \mapsto -Z_i$. Moreover $g_k(A, q, b)$ (with generators $\{X_i', Y_i', Z_i'\}$) is isomorphic to the fixed point set of J.*

Proof. Lemma 1 and Proposition 1 (iv) give the existence of J. Now $X_i' \mapsto \sqrt{q}\, X_i$, $Y_i' \mapsto Y_i$, $Z_i' \mapsto \sqrt{q}\, Z_i$ induces a morphism from $g_k(A, q, b)$ into the fixed point set of J and it is not too difficult to show that this is an isomorphism. (Indeed, it carries the center onto the center; use [K, 1.7].) Q.E.D.

(II) Let us fix h, $1 \leqslant h \leqslant n$. There exists an involution φ_h of $g_k(A, a_i, s_{ij}, b)$ given by

$$\varphi_h(X_i) = X_i, \qquad \varphi_h(Y_i) = Y_i \qquad \text{if} \quad i \neq h$$

$$\varphi_h(X_h) = -X_h, \qquad \varphi_h(Y_h) = -Y_h$$

$$\varphi_h(Z_i) = Z_i \qquad \forall i.$$

From Lemma 1 follows the existence of an antilinear involution ψ_h of $g_{k(\sqrt{q})}(A, a_i, s_{ij}, b)$ ψ_h which, restricted to $g_k(A, a_i, s_{ij}, b)$ is φ_h. We shall give a presentation of g_0, the k-form of ψ_h-fixed points.

Let us consider $g_k(A, a_i', s_{ij}, b')$ (with generators X_i', Y_i', Z_i') where

$$a_i' = a_i \quad (i \neq h) \qquad a_i' = q a_i \quad (i = h)$$

$$b' = b \quad (1 \neq h) \qquad b' = qb \quad (1 = h)$$

and consider $X_i' \to X_i$, $Y_i' \to Y_i$, (for $i \neq h$), $Z_i' \to Z_i$, $X_h' \to \sqrt{q}\, X_h$, $Y_h' \to \sqrt{q}\, Y_h$.

This assignment gives rise to a morphism from $g_k(A, a_i', s_{ij}, b')$ into g_0, and it is possible to see that it is an isomorphism.

As a more concrete example, if A is the Cartan matrix of type A_n, $g_{\mathbf{R}}(A, a_i, s_{ij}, -1) = su(p, n-p)$ if

$$a_i = -1 \quad (i \neq p), \qquad a_p = 1, \qquad s_{ij} = 1.$$

(III) In this example, we will assume that A has a diagram automorphism ν of order 2. Let J_ν be the unique involutive antilinear automorphism of $g_{k(\sqrt{q})}(A, a, b)$ (with generators $\{X_i, Y_i, Z_i\}$) such that $X_i \mapsto X_{\nu(i)}$, $Y_i \mapsto Y_{\nu(i)}$, $Z_i \mapsto Z_{\nu(i)}$. We shall outline a presentation by generators and relations of the fixed point set of J_ν which we shall denote by g_0.

Let us first give some easy facts about TDS over finite extensions of k. Let $f \supset k$ be a finite extension of degree m, $a, b \in k^*$. We shall regard the "special quaternion" algebra over f $sq_f(a, b)$ as a (simple, $3m$-dimensional) Lie algebra over k which we shall denote $sq_{f|k}(a, b)$. Let us assume for simplicity that $f = k(\sqrt[m]{q})$ $(q \in k^* - k^{*m})$ is a cyclic extension of k having an m-root of q. Then $sq_{k(\sqrt[m]{q})|k}(a, b)$ is generated by k-vector spaces V_i: $i = 0, ..., m-1$ where $V_0 = sq_k(a, b)$ is a subalgebra, φ_j: $V_0 \to V_j$ $(1 \leqslant j \leqslant m-1)$ is an isomorphism of vector spaces, and the bracket is given by

$$[\varphi_j(u), \varphi_k(v)] = q^s \varphi_r[u, v]$$

if $j + k = ms + r$, $0 \leqslant r \leqslant m-1$, and $u, v \in V_0$. Thus $sq_{k(\sqrt[m]{q})|k}(a, b)$ has a natural $\mathbf{Z}_m$-graded structure.

Let $(|)$ be the invariant bilinear form on $V_0 = sq_k(a, b)$ given by

$$(Z \mid Z) = -ab$$

$$(X \mid X) = a$$

$$(Y \mid Y) = b.$$

It can be extended to an invariant bilinear form on $sq_{k(\sqrt[m]{q})|k}(a, b)$ if for $u, v \in V_0$ one puts

$$(\varphi_j u \mid \varphi_k(v)) = 0 \qquad \text{if} \quad j + k \neq m$$

$$(\varphi_j(u) \mid \varphi_{m-j}(v)) = q(u \mid v).$$

Now g_0 has a presentation by generators and relations which can be roughly described as attaching TDS over k to the vertices i such that $\nu(i) = i$ or $a_{i\nu(i)} \neq 0$ and TDS over $k(\sqrt{q})$ to the ν orbit of i if $\nu(i) \neq i$ and $a_{i\nu(i)} = 0$. We will also have an ascending filtration as above.

4. The Invariant Bilinear Form

Let us recall that $D = (d_1, ..., d_n)$ is a diagonal matrix such that DA is symmetric. Let us define a symmetric bilinear form $(|)_0$ on $g_k^{\mathrm{loc}}(A, a_i, s_{ij}, b) = g^{\mathrm{loc}}$ as follows:

$$(Z_i|Z_j)_0 = -\tfrac{1}{2}s_{ij}^{-1}a_ib_ja_{ij}d_j^{-1}$$
$$(X_i|X_j)_0 = \delta_{ij}d_i^{-1}a_i$$
$$(Y_i|Y_j)_0 = \delta_{ij}d_i^{-1}b_i$$
$$(Z_i|X_j)_0 = (Z_i|Y_j)_0 = (X_i|Y_j)_0 = 0.$$

THEOREM 2. *There exists a unique symmetric bilinear form* $(|)$ $((|)_k$ *if necessary) on* $g_k(A, a_i, s_{ij}, b)$ *satisfying*

(i) $(|)$ *is invariant, i.e.,* $([u, v]|w) = (u|[v, w])$;

(ii) $(|)|_{g^{\mathrm{loc}} \times g^{\mathrm{loc}}} = (|)_0$.

Moreover we have

(iii) $\nu(u) < \nu(v) \Rightarrow (u|v) = 0$.

Proof. First at all, let us observe that the isomorphisms given by Propositions 1 and 2 preserve $(|)_0$. (For Proposition 2(iii) use $(*)$.) Moreover, $(|)_0$ is invariant, i.e., satisfies (i) whenever u, v, w, $[u, v]$, $[v, w] \in g^{\mathrm{loc}}$. Indeed, we only need to show that

$$(X_i|[Z_j, Y_i])_0 = ([X_i, Z_j]|Y_i)_0 = (Z_j|[Y_i, X_i])_0.$$

But

$$(X_i|[Z_j, Y_i])_0 = a_ib_js_{ji}a_{ji}d_i^{-1}$$
$$([X_i, Z_j]|Y_i)_0 = a_jb_is_{ji}^{-1}a_{ji}d_i^{-1}$$
$$(Z_j|[Y_i, X_i])_0 = a_jb_is_{ji}^{-1}a_{ji}d_i^{-1}.$$

Now for $g_k(A, 1, -1)$ the theorem is just [K, Th. 2.2]. Thus we only need to prove:

$$(|)_{k|g_k \times g_k} \subseteq k.$$

We can do this on g_m by induction on m; for $m = 1$ it is clear and for the inductive step we use (i), (iii). Q.E.D.

Remark 3. The preceding theorem should be true for the forms considered in Example (III). On the other hand, if g is any Lie algebra, $(|)$ a symmetric bilinear invariant form on g, $x \in g$ is locally nilpotent, and $T = \exp x$, then $(u|u) = (T(u)|T(v))$ $\forall u, v \in g$. Furthermore, going back to the form given by the theorem, it is preserved by the isomorphisms in Examples (I), (II). Finally, if A is of affine type, the radical coincides with the center.

As a first application of the existence of this bilinear form, we shall begin to study Cartan decompositions in real forms of Kac–Moody algebras. In what follows $g_{\mathbf{R}}$ is a real form of a complex Kac–Moody algebra $g(A)$ (constructed as in Def. 4 or Remark 3) where A has no cycles (automatically, it is symmetrizable) and $a_{ij} \leqslant -3$. Moreover, $k(A)$ denotes a compact form of $g(A)$ constructed as above; we shall call compact form every image of $k(A)$ by an automorphism of $g(A)$ which preserves the bilinear form. We shall extend some well-known results (see [H]) to this setup. Let us remark once more that by definition both $g_{\mathbf{R}}$ and $k(A)$ have Cartan subalgebras which coincide after complexification. The proof is quite similar to the classical case but we will include it for the sake of completeness. (This was also done in [KP3, R]).

THEOREM 3. *Let us assume that A is of finite or affine type. Let σ and τ denote the conjugations of $g(A)$ with respect to $g_{\mathbf{R}}$ and $k(A)$, respectively. Then there exists an automorphism φ of $g(A)$ such that the compact real form $\varphi(k(A))$ is σ-invariant.*

Proof. Let $\omega(|)$ (cf. also [K, 2.7]) be the Hermitian form

$$\omega(u \mid v) = -(u \mid \tau(v))$$

which is positive semidefinite [K, 11.7]. Let $\theta = \sigma\tau$; it is an automorphism of $g(A)$. We have

$$(\theta(u) \mid v) = (u \mid \theta^{-1}(v))$$

because (thanks to Theorem 2)

$$(\sigma(u) \mid v) = \overline{(u \mid \sigma(v))}$$

$$(\tau(u) \mid v) = \overline{(u \mid \tau(v))}.$$

Hence

$$\omega(\theta(u) \mid v) = \omega(u \mid \theta(v)).$$

As both σ, τ preserve the above introduced filtration, θ also does. Thus, there exists a basis $\mathbf{B}$ of $g(A)$ with respect to which θ^2 acts "diagonally" by positive real numbers $\{\lambda_i\}_{i \in \mathbf{B}}$. Let us denote by P^t the "diagonal" linear transformation (in the same basis) represented by $\{\lambda_i^t\}$ $(t \in \mathbf{R})$. Each P^t is an automorphism of $g(A)$, preserves the invariant bilinear form, and $\tau P^t = P^{-t}\tau$. We are done by choosing $\varphi = P^{1/4}$. Q.E.D.

DEFINITION 5. A decomposition

$$g_{\mathbf{R}} = k_0 \oplus p_0$$

is called a Cartan decomposition if there exists a compact real form k of $g(A)$ such that

$$\sigma(k) \subseteq k$$
$$k_0 = g_{\mathbf{R}} \cap k$$
$$p_0 = g_{\mathbf{R}} \cap (\sqrt{-1}\, k).$$

Remark 4. It follows from Theorem 3 that if A is affine (or finite, in which case is well known) then $g_{\mathbf{R}}$ has a Cartan decomposition. It is clear that $k = k_0 + \sqrt{-1}\, p_0$. Moreover

$$(X \mid X) \leqslant 0, \qquad \forall X \in k_0$$
$$(X \mid X) \geqslant 0, \qquad \forall X \in p_0$$

and one of the inequalities must be strict (since the centers of g, $g_{\mathbf{R}}$, and k are one-dimensional) (use [K, 11.7]). Moreover

$$s_0 \colon X + Y \mapsto X - Y, \qquad \forall X \in k_0,\ Y \in p_0$$

is an automorphism of g_0 which preserves $(\mid)$. And this condition will characterize a Cartan decomposition whenever one knows that an arbitrary $g_{\mathbf{R}}$ with a negative semidefinite form is conjugated to $k(A)$ (cf. Remark 3).

5. Kac–Moody Groups

Let us now recall the definition of Kac–Moody groups ([PK], see also [G, KP1, KP2, MT]). First, a $g(A)$-module V is called integrable if the following two properties hold:

(i) $V = \bigoplus_{\lambda \in \mathbf{h}^*} V_\lambda$, where $V_\lambda = \{v \in V / h \cdot v = \lambda(h) v \ \forall h \in \mathbf{h}\}$,

(ii) E_i and F_i are locally nilpotent for all $i \in I$.

Remark 5. Let g be a Lie algebra, $s \subseteq g$ a subalgebra, V a g-module. We will say that V is s-locally finite if every $v \in V$ lies in a finite dimensional s-submodule. Clearly, V is integrable if and only if is s_i-locally finite $\forall i$ where $s_i = kE_i + kH_i + kF_i$ (cf. [K, 3.6]).

Definition 6. The Kac–Moody group $G(A)$ attached to A is G^*/N^* where G^* is the free product of the additive groups g_α ($\alpha \in \Delta^{re}$) and

$$N^* = \bigcap_{(V,\pi)\ \text{integrable}} \operatorname{Ker} \pi^*.$$

Here $\pi^*\colon G^* \to GL(V)$ is the representation defined by

$$\pi^*(x) = \exp \pi(x) = \sum_{n \geq 0} 1/n!\, \pi(x)^n \qquad (x \in g_\alpha,\ \alpha \in \Delta^{re}).$$

Remark 6. There exist isomorphisms of groups between:

(i) $G(A)$.

(ii) G^{**}/N^{**}, where G^{**} is the free product of the additive groups kE_i, kF_i $(1 \leq i \leq n)$, $N^{**} = \bigcap_{(V,\pi)\text{ integrable}} \operatorname{Ker} \pi^{**}$, and π^{**} as above.

(iii) G^{***}/N^{***} where G^{***} is the free product of n copies of $SL(2,k)$ (namely, the i-copy S_i has Lie algebra s_i) and

$$N^{***} = \bigcap_{(V,\pi)\text{ integrable}} \operatorname{Ker} \pi^{***}.$$

Here π^{***} is defined by the representations $S_i \to GL(V)$ obtained from the restriction

$$s_i \to gl(V).$$

(Let us also denote by abuse of notation $\pi^*\colon G^*/N^* \to GL(V)$, etc., the corresponding representation.) In fact, one has a canonical inclusion $i\colon G^{**} \to G^*$ such that $\pi^{**} = \pi^* \circ i$. Thus $i^{-1}(N^*) = N^{**}$ and one has a monomorphism $G^{**}/N^{**} \to G^*/N^*$, which is also an epimorphism by [KP2].

On the other hand, there exists a unique homomorphism (see [PK])

$$\varphi_i\colon SL(2,k) \to G^{**}/N^{**}$$

such that

$$\varphi_i\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \quad \left(\text{resp., } \varphi_i\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix}\right)$$

is the canonical $kE_i \to G^{**}/N^{**}$ (resp., $kF_i \to G^{**}/N^{**}$). Thus we have $\varphi\colon G^{***} \to G^{**}/N^{**}$ and $\pi^{***} = \pi^{**} \circ \varphi$; and hence a morphism $G^{***}/N^{***} \to G^{**}/N^{**}$. But we also have $G^{**} \to G^{***}$ given by $kE_i \to \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right)$, $kF_i \to \left(\begin{smallmatrix} 1 & 0 \\ t & 1 \end{smallmatrix}\right)$ (in the i-copy).

Now let $g = g(A, a_i, s_{ij}, b)$ and set $s_i = kX_i + kY_i + kZ_i$; s_i is isomorphic to $sq(a_i, b_i)$. The preceding remarks motivate the following definitions:

DEFINITION 7. A g-module (V, π) is integrable if it is s_i-locally finite $\forall i$: $1 \leq i \leq n$.

A standard argument shows that if (V, π) is integrable, then it is a direct sum of simple finite dimensional s_i-submodules, $\forall i$.

DEFINITION 8. $G(A, a_i, s_{ij}, b)$ is G^*/N^*, where G^* is the free product of the quaternion groups $SQ(a_i, b_i)$ and

$$N^* = \bigcap_{(V,\pi)\text{ integrable}} \operatorname{Ker} \pi^*.$$

Here $\pi^*: G^* \to GL(V)$ is defined by the representations $SQ(a_i, b_i) \to GL(V)$ obtained from $s_i \to gl(V)$.

APPENDIX

By Nicolás Andruskiewitsch and Guy Rousseau

FAMAF, Universidad Nacional de Córdoba and Département de mathématiques, Université Nancy I

Let us remark first that $g_k(A, a_i, s_{ij}, b) = g_k$ splits after tensoring with a quadratic extension. Indeed, let us assume that $g_k \not\simeq g_k(A)$; then for some l we have $-a_l b_l \notin k^2$ (see the proof of the lemma below!); but $-a_j b_j = -a_l b_l (a_j a_l^{-1} s_{lj})^2$. Let us denote by $\sqrt{-a_l b_l}$ a fixed root of the polynomial $T^2 + a_l b_l$ in $\bar{k}$, $k' = k(\sqrt{-a_l b_l})$, $\sqrt{-a_j b_j} = \sqrt{-a_l b_l}\, a_j a_l^{-1} s_{lj} \in k'$. Let us observe that

$$\sqrt{-a_j b_j} = \sqrt{-a_i b_i}\, a_j a_i^{-1} s_{ij} \qquad \forall i, j.$$

LEMMA. *$g_k \otimes k'$ is isomorphic to $g_{k'}(A)$ via*

$$H_i \mapsto \frac{1}{\sqrt{-a_i b_i}} Z_i$$

$$E_i \mapsto \frac{1}{2}\left(X_i - \frac{a_i}{\sqrt{-a_i b_i}} Y_i\right)$$

$$F_i \mapsto \frac{1}{2}\left(\frac{1}{a_i} X_i + \frac{1}{\sqrt{-a_i b_i}} Y_i\right).$$

Proof. The above assignment gives rise to a morphism from the free Lie algebra in the variables $\{E_i, F_i, H_i\}$ onto $g_k \otimes k'$. It is easy to see that its kernel contains the ideal generated by the relations (1), ..., (4). Now the other relations are also satisfied and the quotient map is an isomorphism as follows from Proposition 2, since after tensoring with $\bar{k}$ we know that a certain multiple of the left hand side of (5) (or (6)) vanishes and that the quotient map is an isomorphism. Q.E.D.

Now let $\mathbf{h}$ be the (n-dimensional) span of $Z_1, ..., Z_n$. Let $\lambda_i \in \mathbf{h}^*$ be defined by the rule

$$\lambda_i(Z_j) = a_{ij}$$

and let $\alpha_i \in (\mathbf{h} \otimes k')^*$ be given by $\alpha_i = \sqrt{-a_i b_i}\, \lambda_i$. Clearly, $\{\alpha_i\}$ is a set of simple roots of the root system Δ of $g_{k'}(A)$ with respect to the Cartan subalgebra $\mathbf{h} \otimes k'$.

On the other hand, let $\theta: g_{k'}(A) \to g_{k'}(A)$ be the antilinear involution whose fixed point set is g_k, via the above identification. Clearly

$$\theta(\alpha_i) = -\alpha_i.$$

Let us recall that the antilinear Cartan involution τ is defined by

$$E_i \mapsto -F_i, \qquad F_i \mapsto -E_i, \qquad H_i \mapsto -H_i.$$

Therefore $\theta\tau$ is a linear involution of $g_{k'}(A)$ and fixes point by point the Cartan subalgebra $\mathbf{h} \otimes k'$.

Now we shall geeralize Example (II). Let us fix $J \subset \{h: 1 \leqslant h \leqslant n\}$. There exists an involution φ_J of $g_k(A, a_i, s_{ij}, b)$ given by

$$\varphi_J(X_i) = X_i, \qquad \varphi_J(Y_i) = Y_i \qquad \text{if} \quad i \notin J$$

$$\varphi_J(X_i) = -X_i, \qquad \varphi_J(Y_i) = -Y_i \qquad \text{if} \quad i \in J$$

$$\varphi_J(Z_i) = Z_i \qquad \forall i.$$

From Lemma 1 follows the existence of an antilinear involution ψ_J of $g_{k(\sqrt{q})}(A, a_i, s_{ij}, b)$ ψ_J which, restricted to $g_k(A, a_i, s_{ij}, b)$ is φ_J. We shall give a presentation of g_0, the k-form of ψ_J-fixed points.

Let us consider $g_k(A, a_i^J, s_{ij}, b_i^J)$ (with generators X_i', Y_i', Z_i') where

$$a_i^J = a_i \ (i \notin J), \qquad a_i^J = q a_i \ (i \in J)$$

$$b_i^J = b_i \ (i \notin J), \qquad b_i^J = q b_i \ (i \in J)$$

and consider

$$X_i' \mapsto X_i \qquad Y_i' \mapsto Y_i \ (i \notin J)$$

$$X_i' \mapsto \sqrt{q}\, X_i \qquad Y_i' \mapsto \sqrt{q}\, Y_i \ (i \in J)$$

$$Z_i' \mapsto Z_i \qquad \forall i.$$

This assignment gives rise to a morphism $g_k(A, a_i^J, s_{ij}, b_i^J)$ into g_0 which, it is possible to see, is an isomorphism.

Now let us recall [KP3, Prop. 3.7]:

PROPOSITION. *Let σ be an antilinear involution of the second kind of $g_{\mathbf{C}}(A)$. Then, for some $v \in \operatorname{Aut} A$, σ can be conjugated to an antilinear involution of the following form*:

$$\sigma(E_i) = F_{vi}, \qquad \sigma(F_i) = E_{vi} \qquad \text{if} \quad vi \neq i;$$
$$\sigma(E_i) = \pm F_i, \qquad \sigma(F_i) = \pm E_i \qquad \text{if} \quad vi = i.$$

A form of the second kind will be called *inner* if the corresponding antilinear involution is conjugate to one as in the proposition, with $v = 1$. Notice that g_0 is an inner form of the second kind.

Let us also recall the following terminology (see [R]): A real form is called *almost compact* (resp., *almost split*) if the corresponding antilinear involution is of the second (resp., first) kind.

It follows from the above that:

THEOREM. *If $k = \mathbf{R}$, and $g_{\mathbf{R}}(A, a_i, s_{ij}, b_i)$ is not isomorphic to $g_{\mathbf{R}}(A)$ then $g_{\mathbf{R}}(A, a_i, s_{ij}, b_i)$ is an inner almost compact form of $g_{\mathbf{C}}(A)$; any inner almost compact form arises in this way.*

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