ALMOST SPLIT K-FORMS OF KAC-MOODY ALGEBRAS

Guy Rousseau

Unité associée au C.N.R.S. n° 750
Département de mathématiques de l'Université Nancy 1
B.P. 239 54506 Vandoeuvre lès Nancy Cedex
FRANCE

ABSTRACT

If $\mathfrak{g}_K$ is an almost split K-form of a Kac-Moody algebra over a field K of characteristic 0, we prove, as in the finite-dimensional case, the conjugacy of the maximal K-split toral subalgebras of $\mathfrak{g}_K$.

§ 1 KAC-MOODY ALGEBRAS:

1.1 Let K be a field of characteristic 0 and $\overline{K}$ its algebraic closure.

Let $\mathfrak{s}$ be a Lie algebra over K . A K-split toral subalgebra of $\mathfrak{s}$ is a subalgebra $\mathfrak{t}$ which is $ad_{\mathfrak{s}}$-diagonalizable .A toral subalgebra is a subalgebra $\mathfrak{t}$ such that $\mathfrak{t}\otimes\overline{K}$ is a $\overline{K}$-split toral subalgebra of $\mathfrak{s}\otimes\overline{K}$;it is a commutative subalgebra of $\mathfrak{s}$ .A maximal toral subalgebra of $\mathfrak{s}$ contains its center $\mathfrak{c}$

A Cartan-toral subalgebra of $\mathfrak{s}$ is a toral subalgebra $\mathfrak{h}$ equal to its normalizer ;it is a maximal toral subalgebra of $\mathfrak{s}$ .If $\mathfrak{h}$ is in a subalgebra $\mathfrak{s}_1$ of $\mathfrak{s}$ then $\mathfrak{h}$ is a Cartan-toral subalgebra of $\mathfrak{s}_1$ .Suppose now that $\mathfrak{s}_1$ is finite dimensional ,then $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{s}_1$ (in the classical sense :[4)]VII),and all Cartan subalgebras or maximal toral subalgebras of $\mathfrak{s}_1$ are Cartan-toral subalgebras ([4)]VII 3.2 and 2.3 prop.10,see [11)] 1.6.2).

In the sequel as in [10)] Cartan subalgebra will mean maximal toral subalgebra ; but all Cartan subalgebras we shall speak of will be Cartan-toral subalgebras .

In the sequel of this paragraph we suppose K= $\overline{K}$ algebraically closed . For the standard facts stated below proofs may be found in [6)] or [10)] sometimes in [7)], [8)]or [11)] . In these references K is often $\mathbb{C}$ but the same proofs give the same results for K.

1.2 Let $\mathfrak{g}$ be a Kac-Moody algebra over K . We shall always suppose $\mathfrak{g}$ indecomposable and infinite dimensional . We have a root space decomposition of $\mathfrak{g}$ with respect to the standard Cartan (toral) subalgebra $\mathfrak{h}$: $\mathfrak{g}=\mathfrak{h}\oplus(\oplus\mathfrak{g}_\alpha)$ where α runs in the root system $\Delta=\Delta(\mathfrak{g},\mathfrak{h})\subset\mathfrak{h}^*-\{0\}$.

The standard basis of Δ is $\Pi=\{\alpha_i / i\in I\}$ .There are corresponding coroots $\alpha_i^\vee$ in $\mathfrak{h}$ such that $a_{ij}=\alpha_j(\alpha_i^\vee)$ are the coefficients of the generalized Cartan matrix associated to $\mathfrak{g}$. The Weyl group W of $(\mathfrak{g},\mathfrak{h})$ is generated by the fundamental reflections r_i defined by $r_i(h)=h-\alpha_i(h)\alpha_i^\vee$ for h in $\mathfrak{h}$. A real root is a root conjugated by W to a root in Π .

1.3 Given a subset J of I we define a subset of Δ by :

$$\Delta(J)=\Delta\cap((\oplus_{i\in J}\mathbb{Z}\alpha_i)\oplus(\oplus_{i\notin J}\mathbb{N}\alpha_i))$$

Then the subalgebra $\mathfrak{p}^+(J)=\mathfrak{p}(J)$ (resp. $\mathfrak{p}^-(J)$) is $\mathfrak{h}\oplus\mathfrak{g}_\alpha$ where α runs in $\Delta(J)$ (resp $-\Delta(J)$) and is called a (positive resp. negative) standard parabolic subalgebra .The subset J and the corresponding parabolics are said of finite type if the matrix of the a_{ij} for i,j in J is a Cartan matrix (of a semi-simple algebra).

We always suppose that the parabolics are proper i.e. $J\neq I$. When $J=\emptyset$ the parabolics $\mathfrak{b}^+=\mathfrak{p}^+(\emptyset)$ and $\mathfrak{b}^-=\mathfrak{p}^-(\emptyset)$ are called standard Borel subalgebras .

1.4 One defines a group G acting on $\mathfrak{g}$ via the adjoint representation $Ad : G\to Aut(\mathfrak{g})$. It is generated by the subgroups U_α for α a real root and $Ad(U_\alpha)=exp(ad(\mathfrak{g}_\alpha))$.

The Cartan subalgebras of $\mathfrak{g}$ are conjugated by G .

A Borel subalgebra of $\mathfrak{g}$ is a maximal completely solvable subalgebra . It is always conjugated by G to $\mathfrak{b}^+$ or $\mathfrak{b}^-$. But $\mathfrak{b}^+$ and $\mathfrak{b}^-$ are not conjugated under G , so there are exactly two conjugacy classes of Borel subalgebras : the positive or negative ones .

A (proper) parabolic subalgebra of $\mathfrak{g}$ is a subalgebra (different from $\mathfrak{g}$) containing a Borel subalgebra ; it is conjugated to a (positive or negative) standard parabolic subalgebra .Its sign is the sign of the Borel subalgebra it contains . Two different standard parabolic subalgebras cannot be conjugated by G .

A linear or semilinear automorphism of $\mathfrak{g}$ is of first kind (resp. second kind) if it transforms a Borel subalgebra into a Borel subalgebra of the same (resp. opposite)sign .

1.5 If $\mathfrak{t}$ is a toral subalgebra we denote $X(\mathfrak{t})$ the sub-$\mathbb{Z}$-module of the dual $\mathfrak{t}^*$ generated by the weights of $\mathfrak{t}$ for $ad_{\mathfrak{g}}$ .As $\mathfrak{t}$ is a subalgebra of some Cartan subalgebra $\mathfrak{h}$, $X(\mathfrak{t})$ is a quotient of $X(\mathfrak{h})$ hence finitely generated . It is also torsionfree so it is a free $\mathbb{Z}$-module .

The group $\tilde{T}=Hom(X(\mathfrak{t}),K^*)$ is a product $(K^*)^n$ .Moreover t in $\tilde{T}$ acts by multiplication by $t(\chi)$ on the eigenspace $\mathfrak{g}_\chi$ of $\mathfrak{t}$, so $\tilde{T}$ is a subgroup of $Aut(\mathfrak{g})$. We then define the subgroup $T=Ad^{-1}(\tilde{T})$ of G .

Let $\mathfrak{t}_r$ be the set of elements in $\mathfrak{g}$ which are diagonalizable with the same eigenspaces as $\mathfrak{t}$ in any tensor power of the adjoint representation . It is clear that $\mathfrak{t}_r$ is a toral subalgebra of $\mathfrak{g}$ containing $\mathfrak{t}$. In particular if $\mathfrak{h}$ is a Cartan subalgebra then $\mathfrak{h}=\mathfrak{h}_r$.

Proposition 1.6 : *a) If* $\mathfrak{t}'$ *is a toral subalgebra of* $\mathfrak{g}$ *contained in* $\mathfrak{t}$ *,then* $\tilde{T}'\subset\tilde{T}$ *and* $T'\subset T$

b) If $\mathfrak{t}$ *is in the Cartan subalgebra* $\mathfrak{h}$ *then* $\mathfrak{t}_r$ *is the subspace of* $\mathfrak{h}$ *defined by equations which are linear combinations of roots with integer coefficients and which are satisfied by* $\mathfrak{t}$ *. In particular* $\mathfrak{t}\subset\mathfrak{t}_r$, $X(\mathfrak{t}_r)=X(\mathfrak{t})$, $\tilde{T}_r=\tilde{T}$ *and* $T_r=T$.

c) $\tilde{T}$ *is the Zariski closure of* $Ad(T)$ *in* $GL(\mathfrak{g})$.

d) The eigenspaces of $\tilde{T}$ *,T and* $\mathfrak{t}$ *are the same .The subspaces of* $\mathfrak{g}$ *stabilized by* $\tilde{T}$ *,T or* $\mathfrak{t}$ *are the same .*

Remark: More precisely c) means that for any finite dimensional subspace V of $\mathfrak{g}$ stabilized by $\tilde{T}$,the group $\tilde{T}$ is the Zariski closure of Ad(T) in GL(V) .

Proof:a) If $\mathfrak{t}'$ is in $\mathfrak{t}$ then $X(\mathfrak{t}')$ is a quotient of $X(\mathfrak{t})$ and $\tilde{T}'\subset\tilde{T}$,$T'\subset T$.

b)As the opposite of a root is a root ,the weights of $\mathfrak{h}$ in tensor powers of ad may be any linear combination of the roots, so b) is clear.

c) Let $\mathfrak{h}$ be a Cartan subalgebra; the group H stabilizes $\mathfrak{h}$ so it is the same as the group of the same name defined in [10] ,cf [11]1.6.5 .From the definition in [10] one can see that Ad(H) contains $Hom(X(\mathfrak{h}),(K^*)^2)$ .If $\mathfrak{t}$ is in $\mathfrak{h}$ then $X(\mathfrak{t})$ is a quotient of $X(\mathfrak{h})$ and $Ad(T)=Ad(H)\cap\tilde{T}$ contains $Hom(X(\mathfrak{t}),(K^*)^2)$ which has clearly for Zariski closure $\tilde{T}=Hom(X(\mathfrak{t}),K^*)$.

d) As $\tilde{T}$ is a product $(K^*)^n$,we may identify $X(\mathfrak{t})$ with the $\mathbb{Z}$-module $X(\tilde{T})$ of algebraïc group homomorphisms from $\tilde{T}$ to K^*.The eigenspaces of $\mathfrak{t}$ and $\tilde{T}$ are thus the same . So d) is clear in view of c) .

Definitions 1.7 : A toral subalgebra $\mathfrak{t}$ satisfying $\mathfrak{t}_r=\mathfrak{t}$ will be called rational .

A subgroup of the form $\tilde{T}$ (resp. T) will be called a rational torus of $Aut(\mathfrak{g})$ (resp.G) , cf [8] 3.2 p. 131.

It is a consequence of 1.6 that there is a one to one correspondence between rational toral subalgebras of $\mathfrak{g}$ and rational torus of G (or $Aut(\mathfrak{g})$).This correspondence preserves inclusion .

If $\mathfrak{t}$ is a toral subalgebra, the $\mathbb{Z}$-dual of $X(\mathfrak{t})$ (resp. the vector space $V(\mathfrak{t})$) is the set of the x in $\mathfrak{t}_r/\mathfrak{c}$ such that $\chi(x)$ is in $\mathbb{Z}$ (resp. $\mathbb{Q}$) for all χ in $X(\mathfrak{t})$.

1.8 Parabolic Subgroups :

The group U^+ (resp. U^-) is the subgroup of G generated by all U_α for α a positive (resp. negative) real root .

The group $B^+=B=HU^+$ (resp $B^-=HU^-$) is the standard positive (resp. negative) Borel subgroup .

For J in I the associated standard positive (resp. negative) parabolic subgroup of G is $P(J)=P^+(J)=B^+W(J)B^+$ (resp. $P^-(J)=B^-W(J)B^-$) where W(J) is the subgroup of W generated by the r_i for i in J . The group P(J) (resp. $P^-(J)$) is the stabilizer of $\mathfrak{p}(J)$ (resp. $\mathfrak{p}^-(J)$) .

A positive or negative parabolic subgroup of G is a conjugate of a positive or negative standard parabolic subgroup .

The applications $\mathfrak{p}(J)\to P(J)$, $\mathfrak{p}^-(J)\to P^-(J)$ extend to a G-equivariant one to one correspondence between parabolic subalgebras and parabolic subgroups (cf [11]1.6.4). This correspondence preserves inclusion .

Remark 1.9 : If the toral subalgebra $\mathfrak{t}$ is in the parabolic subalgebra $\mathfrak{p}$, then $\mathfrak{t}$ and T stabilize $\mathfrak{p}$, so T is in P . If T is in P then T and $\mathfrak{t}$ stabilize $\mathfrak{p}$, so $\mathfrak{t}$ is in $\mathfrak{p}$ as $\mathfrak{p}$ is its own normalizer ([11]1.2.a) . Thus the correspondences between the subalgebras of $\mathfrak{g}$ and the subgroups of G we considered before preserve inclusion . In particular all the results we shall prove for these subalgebras of $\mathfrak{g}$ may be translated for these subgroups of G .

§ 2 K-FORMS OF $\mathfrak{g}$:

2.1 Let $\mathfrak{g}$ be a Kac-Moody algebra (indecomposable and infinite dimensional) over $\overline{K}$.

A K-form of $\mathfrak{g}$ is a Lie algebra $\mathfrak{g}_K$ such that there exists an isomorphism from $\mathfrak{g}$ to $\mathfrak{g}_K\otimes\overline{K}$.

In the sequel (of §2 and §4) we fix such a K-form $\mathfrak{g}_K$ and such an isomorphism . Then the Galois group $\Gamma=Gal(\bar{K}/K)$ acts on $\mathfrak{g}$ and the corresponding group G . We identify $\mathfrak{g}_K$ with the fixed point set $\mathfrak{g}^\Gamma$ and define $G_K=G^\Gamma$.

Definitions 2.2 : If Γ consists of first kind automorphisms we shall say that $\mathfrak{g}_K$ is almost split .Otherwise Γ contains a second kind automorphism and we shall say that $\mathfrak{g}_K$ is almost anisotropic .

If $\mathfrak{g}_K$ is almost anisotropic then no proper parabolic subalgebra of $\mathfrak{g}$ is stable by Γ (i.e. is defined over K). If $\mathfrak{g}_K$ is almost split we shall see (4.3) that Γ stabilizes a proper parabolic subalgebra . So this definition of almost anisotropic agrees with that used by Bruhat and Tits in the finite dimensional case : [5]1.7 .

Almost split is a generalization of split or quasi-split (in this case a Borel subalgebra is defined over K) .

2.3 A way to build all almost split real forms of affine Kac-Moody algebras is given in [12] .Here we shall prove for all almost split K-forms the conjugacy under G_K of maximal K-split toral subalgebras . This will be done in §4 after some more preliminaries in §3 .

If $\mathfrak{g}_K$ is almost anisotropic , then the set of first kind automorphisms in Γ is a normal subgroup of index 2 . So $\mathfrak{g}_K$ becomes almost split over a Galois extension of degree 2 . This shows the importance of almost split K-forms and of the following example .

Example 2.4 [11] : An almost anisotropic (or "almost compact") real form of a symmetrizable Kac-Moody algebra $\mathfrak{g}$ over $\mathbb{C}$ correspond to an anti-involution σ_0 of the second kind of $\mathfrak{g}$. One of these anti-involutions is the so-called Cartan anti-involution ω_0 corresponding to the "compact form" . But if σ_0 is an anti-involution of second kind , there exists a (conjugate of the) Cartan anti-involution ω_0 which commutes with σ_0 hence $\sigma=\sigma_0\omega_0$ is an involution of the first kind : [11] or [8] 3.6 .

This gives bijections between conjugacy classes (under G or Aut($\mathfrak{g}$)) of :

1) first kind involutions of $\mathfrak{g}$.

2) pairs of an almost compact real form $\mathfrak{g}_\mathbb{R}$ and a "maximally compact" Cartan subalgebra of $\mathfrak{g}_\mathbb{R}$.

3) pairs of an almost compact real form $\mathfrak{g}_\mathbb{R}$ and a "maximal compact subalgebra" of $\mathfrak{g}_\mathbb{R}$

The first kind involutions of affine algebras were classified by F.Levstein [9] and J.Bausch [1] , [2] and [3] . Unfortunately this gives perhaps not a classification of almost compact real forms of affine algebras .

§ 3 THE BUILDING OF $\mathfrak{g}$ OVER $\bar{K}$:

In this paragraph we suppose $K=\bar{K}$ algebraically closed .

3.1 The Twin Tits System .

We chose a Cartan subalgebra $\mathfrak{h}$ and a basis of roots .

Let N be the normalizer of H in G , one knows that N/H is isomorphic to the Weyl group W and so is generated by $S=\{ r_i / i\in I \}$.

Then Kac and Peterson proved in [7] 4.2 that (G,N,U^+,U^-,H,S) is a twin (or refined) Tits system . To this twin Tits system are associated two saturated Tits system ([7] 3.1 and cor. 3.4) : (G,HU^+,N,S) and (G,HU^-,N,S) .

We thus have two Bruhat decompositions : $G=U^+NU^+=U^-NU^-$,

but also a Birkhoff decomposition ([7] 3.3) : $G=U^-NU^+$;

(for these three decompositions , if g=u.n.u' the element n is well determined by g) .

3.2 The Abstract Building :

The Tits systems above give two buildings $\mathfrak{B}^+$ and $\mathfrak{B}^-$ that we shall now describe by their sets $\mathfrak{F}^+$ and $\mathfrak{F}^-$ of facets and their sets $\mathfrak{A}^+$ and $\mathfrak{A}^-$ of apartments ; the proofs may be found in [4] exercises of IV § 1 and § 2 .

The facets in $\mathfrak{F}^+$ are in one to one correspondence with the positive proper parabolics . The facets are ordered by the opposite of the inclusion of parabolics ;this order is called the inclusion of facets . The maximal facets called chambers correspond to positive Borel subgroups i.e. conjugates of $B^+=HU^+$.

The facet corresponding to a parabolic subgroup P (resp. parabolic subalgebra $\mathfrak{p}$) is called F(P) (resp. F($\mathfrak{p}$)) , we say F(P) is of finite type iff P is . The parabolic subgroup (resp. subalgebra) corresponding to the facet F is P(F) (resp. $\mathfrak{p}$(F)); when $C=F({}^gB^+_\cdot)$ is a chamber we define $U(C)={}^gU^+$.

The group G acts on $\mathfrak{F}^+$ by $gF(P)=F({}^gP)$ so the stabilizer of a facet F is P(F) .

The standard apartment $A^+(\mathfrak{h})$ of $\mathfrak{B}^+$ is the union of facets corresponding to parabolic subalgebras i.e. the union of transforms by W of facets corresponding to standard parabolics . The set $\mathfrak{A}^+$ of apartments is the set of transforms of $A^+(\mathfrak{h})$ by G . As the Tits system is saturated the stabilizer of $A^+(\mathfrak{h})$ is N , so $\mathcal{A} \cong G/N$. As N is also the normalizer of $\mathfrak{h}$ or H , the apartments are in one to one correspondence with the Cartan subalgebras or Cartan subgroups . By [11] 1.7 The apartment $A^+(\mathfrak{h}')$ corresponding to the Cartan subalgebra $\mathfrak{h}'$ is the union of positive facets F such that $\mathfrak{p}(F)$ contains $\mathfrak{h}'$.

All the preceding definitions may also be given for $\mathcal{B}^-$, replacing each + by a - .

The twin building $\mathcal{B}$ has a set $\mathcal{F}$ of facets which is the disjoint union of $\mathcal{F}^+$ and $\mathcal{F}^-$. The apartments of $\mathcal{B}^+$ and $\mathcal{B}^-$ are in one to one correspondence with the Cartan subalgebras; and we define $A(\mathfrak{h})=A^+(\mathfrak{h})\cup A^-(\mathfrak{h})$ as the apartment of $\mathcal{B}$ corresponding to $\mathfrak{h}$. The set $\mathcal{A}$ of apartments of $\mathcal{B}$ is thus identified with $\mathcal{A}^+$ and $\mathcal{A}^-$. A facet F is in the apartment $A(\mathfrak{h})$ if and only if $\mathfrak{h}$ is in $\mathfrak{p}(F)$.

It is a consequence of Bruhat and Birkhoff decompositions that two facets of $\mathcal{B}$ are contained in a same apartment .

3.3 Geometric Description of $\mathcal{B}$:

In $V(\mathfrak{h})$ we may consider the cone $C=\{x \in V(\mathfrak{h}) \,/\, \alpha_i(x)\geq 0 \;\; \forall i \in I\}$ and the union $TC^+(\mathfrak{h})$ of the wC for w in W . This set is the Tits cone inside $V(\mathfrak{h})$ (the true Tits cone is inside $\mathfrak{h}_{\mathbb{R}}$: [6] 3.12) . For J inside I we define $C_J=\{x \in C \,/\, \alpha_i(x)=0 \;\forall i \in J\}$. As a consequence of [6] 3.12 the conjugates wC_J for w in W and J inside I are the (closed) facets defined inside $TC^+(\mathfrak{h})$ by the walls $Ker(\alpha)$ for α a real root , and the stabilizer (or fixator) in W of C_J is W(J) . So there is a W-equivariant identification between the sets of facets of $A^+(\mathfrak{h})$ and of $TC^+(\mathfrak{h})$ which identify C_J to F(P(J)) . As we want only proper parabolics , we shall remove the point 0 from C , C_J and $TC^+(\mathfrak{h})$.

For J inside I , $J\neq I$, we define $int(C_J)$ as the set of x in C_J such that $\alpha_i(x)>0$ for i not in J . The open facets of $TC^+(\mathfrak{h})$ are the sets $w.int(C_J)$ for w in W , J inside I ,$J\neq I$; they are in one to one correspondence with the (closed) facets and $TC^+(\mathfrak{h})$ is the disjoint union of its open facets . The interior of the Tits cone $int(TC^+(\mathfrak{h}))$ is the union of its open facets of finite type ([8] 2.1 p.118) . If x is in the open facet F , we define P(x)=P(F) ,etc .

Now let $\mathcal{B}^+=G\times TC^+(\mathfrak{h})/\sim$ where $(g,x)\sim(h,y)$ if and only if there exists n in N such that y=nx and $g^{-1}hn$ is in P(x) . There is an obvious action of G on $\mathcal{B}^+$. It is a consequence of Bruhat decomposition that $P(J)\cap N=W(J)H$ which fixes C_J . So if $(1,x)\sim(1,y)$ then x=y and we may identify $TC^+(\mathfrak{h})$ with a subset of $\mathcal{B}^+$ by identifying x and the class of (1,x) .

Also if x , y are in $int(C_J)$ and $(g,x)\sim(h,y)$ then x=y and $g^{-1}h$ is in P(J) . We define a (closed resp. open) facet of $\mathcal{B}^+$ as a set gF where g is in G and F is a (closed resp. open) facet of $TC^+(\mathfrak{h})$, the stabilizer or the (point by point) fixator of this facet is thus the corresponding parabolic ${}^gP(F)$. So there is a G-equivariant identification between $\mathcal{F}^+$ and the set of these (closed resp. open) facets .

The intersection of the sets $N.{}^wB^+=N.{}^wU^+$ for w in W is N ([10] cor.5) ; the stabilizer of $TC^+(\mathfrak{h})$ is thus N . If we define the apartments of $\mathcal{B}^+$ as the sets $g.TC^+(\mathfrak{h})$ for g in G , we have a G-equivariant identification between $\mathcal{A}^+$ and the set of these apartments .

We have given a geometric description of $\mathcal{B}^+$ by gluing the apartments along facets . .

The same things may be done for $\mathcal{B}^-$ (replacing each + by a -) and we define the twin building $\mathcal{B}$ as the disjoint union of $\mathcal{B}^+$ and $\mathcal{B}^-$. The apartment of $\mathcal{B}$ corresponding to the Cartan subalgebra $\mathfrak{h}$ is $A(\mathfrak{h})=TC(\mathfrak{h})=TC^+(\mathfrak{h})\cup TC^-(\mathfrak{h})$, where $TC^-(\mathfrak{h})=-TC^+(\mathfrak{h})$.

Definitions 3.4 :Let Ω be a non empty subset of $\mathcal{B}$ contained in $A(\mathfrak{h})$.

The vectorial extension $ve(\Omega)$ (resp. convex hull $ch(\Omega)$) of Ω in $A(\mathfrak{h})$ is the intersection of $A(\mathfrak{h})=TC(\mathfrak{h})$ with the subvectorspace generated by Ω (resp. the convex hull of Ω) in $V(\mathfrak{h})$.

$\Delta^m(\Omega)$ (resp.$\Delta^u(\Omega)$) is the set of roots α such that $\alpha(\Omega)=0$ (resp.$\alpha(\Omega)\geq 0$ and $\alpha(\Omega)\neq 0$).

The enclosure (resp. vectorial enclosure)of Ω in $A(\mathfrak{h})$ is the subset $cl(\Omega)$ (resp. $vcl(\Omega)$) of $TC(\mathfrak{h})$ defined by $\alpha(x)\geq 0$ for α in $\Delta^m(\Omega)\cup\Delta^u(\Omega)$ (resp. $\alpha(x)=0$ for α in $\Delta^m(\Omega)$) ; it contains the convex hull (resp.vectorial extension) .

$\mathfrak{p}(\Omega)$ (resp. $P(\Omega)$) is the intersection of the $\mathfrak{p}(F)$ (resp. P(F)) for F an open facet of $A(\mathfrak{h})$ which meets Ω .

$\mathfrak{m}(\Omega)=\mathfrak{h}\oplus(\oplus\mathfrak{g}_\alpha)$ where α runs in $\Delta^m(\Omega)$.

$\mathfrak{u}(\Omega)=\oplus\mathfrak{g}_\alpha$ where α runs in $\Delta^u(\Omega)$.

$M(\Omega)$ is the subgroup of G generated by H and the U_α for α a real root in $\Delta^m(\Omega)$.

$U(\Omega)$ is the smallest subgroup of $P(\Omega)$, normalized by $M(\Omega)$ and containing the intersection of the U(C) for C a closed chamber of $A(\mathfrak{h})$ meeting Ω .

Remark : All these things except $P(\Omega)$, $\mathfrak{p}(\Omega)$ (and as we shall see $ch(\Omega)$, $cl(\Omega)$, $U(\Omega)$ and $\mathfrak{u}(\Omega)$) depend not only of Ω but also of $\mathfrak{h}$. They all depend not of Ω but of $ch(\Omega)$ or $cl(\Omega)$ (for $\mathfrak{p}(\Omega)$,$P(\Omega)$ and $U(\Omega)$ this is a consequence of 3.5 below) .

Proposition 3.5 : *a)* $P(\Omega)$ *is the semi-direct product of* $M(\Omega)$ *by* $U(\Omega)$ *.It is the (point by point) fixator of* Ω *and of* $cl(\Omega)$ *; it normalizes* $\mathfrak{p}(\Omega)$ *and* $\mathfrak{u}(\Omega)$.

b) $\mathfrak{u}(\Omega)$ *is an ideal of the algebra* $\mathfrak{p}(\Omega)$ *and* $\mathfrak{p}(\Omega)=\mathfrak{m}(\Omega)\oplus\mathfrak{u}(\Omega)$.

c) The apartments of $\mathfrak{B}$ *containing* Ω *correspond to the Cartan subalgebras of* $\mathfrak{g}$ *contained in* $\mathfrak{p}(\Omega)$ *.The group* $P(\Omega)$ *acts transitively on these sets of apartments or Cartan subalgebras .*

d) $\mathfrak{m}(\Omega)$ *is the commutative direct sum of a Kac-Moody algebra and a subspace of* $\mathfrak{h}$ *; its Cartan subalgebras are the Cartan subalgebras of* $\mathfrak{g}$ *it contains .*

e) $\mathfrak{m}(\Omega)$ *is the centralizer in* $\mathfrak{g}$ *of* Ω *(or* $\mathrm{vcl}(\Omega)$ *)considered as inside* $\mathfrak{h}/\mathfrak{c}$ *hence inside* $\mathfrak{g}/\mathfrak{c}$

f) $M(\Omega)$ *is the fixator of* $\mathrm{ve}(\Omega)$ *or* $\mathrm{vcl}(\Omega)$ *in* $A(\mathfrak{h})$ *. It normalizes* $\mathfrak{m}(\Omega)$ *and its image in* $GL(\mathfrak{m}(\Omega))$ *is that of the Kac-moody group corresponding to* $\mathfrak{m}(\Omega)$ *. In particular* $M(\Omega)$ *is transitive on the Cartan subalgebras of* $\mathfrak{m}(\Omega)$ *and it centralizes* Ω *considered as inside* $\mathfrak{g}/\mathfrak{c}$ *.*

g) $\mathfrak{u}(\Omega)$, $U(\Omega)$, $\mathrm{ch}(\Omega)$ *and* $\mathrm{cl}(\Omega)$ *are well defined by* Ω *.*

h) We have $N.M(\Omega)\cap U(\Omega)=\{1\}$ *and* $U(\Omega)$ *is simply transitive on the vectorial extensions (or vectorial enclosures) of* Ω *(in apartments of* $\mathfrak{B}$ *containing* Ω *) .*

i) If Ω *meets an open facet of finite type , then* $\mathfrak{m}(\Omega)$ *is finite dimensional . If* Ω *meets* $\mathfrak{B}^+$ *and* $\mathfrak{B}^-$ *then* $\mathfrak{u}(\Omega)$ *is finite dimensional and the roots in* $\Delta^u(\Omega)$ *are real .*

j) When Ω *meets* $\mathfrak{B}^+$ *and* $\mathfrak{B}^-$ *or* Ω *is a facet , the Cartan subalgebras of* $\mathfrak{p}(\Omega)$ *are the Cartan subalgebras of* $\mathfrak{g}$ *contained in* $\mathfrak{p}(\Omega)$ *and* $U(\Omega)$ *is the smallest subgroup* $U_1(\Omega)$ *of* $P(\Omega)$ *normalized by* $M(\Omega)$ *and containing* U_α *for* α *in* $\Delta^u(\Omega)$ *.*

Remarks : 1) When Ω meets $\mathfrak{B}^+$ and $\mathfrak{B}^-$, then $U(\Omega)$ is in fact a (non commutative) product of the U_α for α in $\Delta^u(\Omega)$.

2) By a) and c) the intersection of two apartments is convex (= equal to its convex hull) . If x and y are two points in $\mathfrak{B}$, they are in a same apartment (3.2) ; the segment $[x,y]_{\mathfrak{B}}$ is the intersection of the usual segment $[x,y]$ with the Tits cone ; it doesn't depend of the apartment ;it is equal to the usual segment when x and y are of the same sign . A subset of $\mathfrak{B}$ is called $\mathfrak{B}$-convex if , when it contains x and y , it contains $[x,y]_{\mathfrak{B}}$. One should notice that this definition is a little awkward when the subset is not entirely contained in $\mathfrak{B}^+$ or $\mathfrak{B}^-$; e.g. if x and y in $\mathfrak{B}$ are of different signs and not in the interior of the Tits cone , then the set consisting of x and y could be $\mathfrak{B}$-convex .

Proof : It is clear that $P(\Omega)$ is the fixator of Ω and normalizes $\mathfrak{p}(\Omega)$. Results b) and e) are also clear .

To look at $\mathfrak{m}(\Omega)$ and $M(\Omega)$, we may suppose that Ω is equal to its vectorial extension . But $TC^+(\mathfrak{h})$ is convex so if F is a (positive) open facet which meets Ω in a subset of maximal dimension , then F contains an open subset of Ω and $\Delta^m(\Omega)=\Delta^m(F)$. We may suppose $F=\mathrm{int}(C_J)$, in this case $\mathfrak{m}(\Omega)$ is the sum of $\mathfrak{h}$ and the Kac-Moody algebra corresponding to J . We have $M(\Omega)=M(F)=P(F)\cap P(-F)$ ([10] cor. 5a) and $M(\Omega)$ is the

fixator of $F \cup -F$ hence of Ω (=$\mathrm{ve}(\Omega)$) and $\mathrm{vcl}(\Omega)$. The other properties in d) and f) are then clear . Also $M(\Omega)$ normalizes $\mathfrak{u}(\Omega)$.

The first property of i) is now clear , and the second is a consequence of [6] 5.2a and exercise 3.6 . Properties a) , c) and j) when Ω is a facet are proved in [10] cor. 5a and [11] 1.7 .

The first part of c) is an obvious consequence of 3.2 . If $\mathfrak{t}$ is a toral subalgebra of $\mathfrak{p}(\Omega)$, we may consider its image $\mathfrak{t}_1$ in $\mathfrak{m}(\Omega)\equiv\mathfrak{p}(\Omega)/\mathfrak{u}(\Omega)$. It is a toral subalgebra , so by conjugating by $M(\Omega)$ we may suppose $\mathfrak{t}_1$ in the image of $\mathfrak{h}$ i.e. $\mathfrak{t}$ in $\mathfrak{h}\oplus\mathfrak{u}(\Omega)$. If Ω meets $\mathfrak{B}^+$ and $\mathfrak{B}^-$, then we are in a finite dimensional case and , by i) , 1.1 and [4] VII n° 3.2 , $\mathfrak{t}$ is conjugated into $\mathfrak{h}$ by $U_1(\Omega)$. This proves c) and the first part of j) in this case .

Suppose now Ω in $A^+(\mathfrak{h})$ and $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{g}$ contained in $\mathfrak{h}\oplus\mathfrak{u}(\Omega)$. To prove c) , we shall prove that $\mathfrak{t}$ is conjugated to $\mathfrak{h}$ by $U(\Omega)$. The apartment $A_1=A(\mathfrak{t})$ contains all chambers of the apartment $A=A(\mathfrak{h})$ meeting Ω ; so we may suppose Ω is a union of chambers . Let C be a chamber in Ω ; as a) , c) and h) are known for a chamber , we have $A_1=uA$ for a unique u in $U(C)$. We prove now that u is in $U(\Omega)$ which fixes the enclosure of Ω . We know that $A_1\cap A$ contains all minimal galleries between two chambers of $A_1\cap A$ ([4] IV exer. 1.24b) , but the union of these galleries is $\mathrm{cl}(\Omega)$ ([4] IV exer.1.21c) .So we just have to prove that if A_1 and A contain the chamber C_1 ,then u is in $U(C_1)$. But as we saw , a minimal gallery inside A from C to C_1 is also in A_1 , and its transform by u is a gallery in A_1 of the same type ; so it's the same and u fixes C_1 .But $U(C)\cap P(C_1)=U(C)\cap HU(C_1)\subset U(C_1)$ by [10] cor. 5 b,c . So u is in $U(\Omega)$ which proves c) in this case .

From the preceding proofs we obtain $P(\Omega)\subset M(\Omega)U(\Omega)N=U(\Omega)M(\Omega)N$ (where $U(\Omega)$ may be replaced by its subgroup $U_1(\Omega)$ when Ω meets $\mathfrak{B}^+$ and $\mathfrak{B}^-$) ; so $P(\Omega)= U(\Omega)M(\Omega)(N\cap P(\Omega)) =U(\Omega)M(\Omega)$ as $N\cap P(\Omega)\subset M(\Omega)$. So $P(\Omega)$ fixes $\mathrm{cl}(\Omega)$ and normalizes $U(\Omega)$. If F is a facet of maximal dimension in $\mathrm{cl}(\Omega)$, then F contains an open subset of $\mathrm{cl}(\Omega)$ and $\mathrm{vcl}(\Omega)$, so $M(\Omega)\subset M(F)$. It is then clear that $U(\Omega)\subset U(F)$, so $M(\Omega)\cap U(\Omega)=\{1\}$ and $P(\Omega)=M(\Omega)\ltimes U(\Omega)$. Also $\mathfrak{u}(\Omega)$ is the intersection of the $\mathfrak{u}(C)$ for the chambers C meeting Ω , $U(C)$ normalizes $\mathfrak{u}(C)$ and $M(\Omega)$ normalizes $\mathfrak{u}(\Omega)$ so a) follows . When Ω meets $\mathfrak{B}^+$ and $\mathfrak{B}^-$, we proved $P(\Omega)=M(\Omega)U_1(\Omega)$, $U_1(\Omega)\subset U(\Omega)\subset P(\Omega)$ and $U(\Omega)\cap M(\Omega)=\{1\}$, so $U_1(\Omega)=U(\Omega)$ and we proved j) .

If n in N and m in $M(\Omega)$ are such that $nm\in U(\Omega)$, then $n\in N\cap P(\Omega)\subset M(\Omega)$, so $nm\in M(\Omega)\cap U(\Omega)=\{1\}$. Hence $NM(\Omega)\cap U(\Omega)=\{1\}$ and by f) 1 is the only element in $U(\Omega)$ stabilizing $\mathrm{ve}(\Omega)$ or $\mathrm{vcl}(\Omega)$. So h) is a consequence of a) , f) and c) .

By definition $\mathfrak{u}(\Omega)$, $U(\Omega)$, $ch(\Omega)$ and $cl(\Omega)$ depend only of Ω and $\mathfrak{h}$. The group $P(\Omega)$ is transitive on the possible $\mathfrak{h}$, normalizes $\mathfrak{u}(\Omega)$, $U(\Omega)$ and fixes $ch(\Omega)$ and $cl(\Omega)$. So g) is clear .

3.6 Generic Toral Subalgebras :

If $\mathfrak{h}$ is a Cartan subalgebra of $\mathfrak{g}$, a subvectorspace E of $V(\mathfrak{h})$ is called generic if it meets the interior of the Tits cone $TC(\mathfrak{h})$. We then call $E \cap TC(\mathfrak{h})$ a generic subspace of the apartment $A(\mathfrak{h})=TC(\mathfrak{h})$.

A toral subalgebra $\mathfrak{t}$ of $\mathfrak{g}$ is called generic if for a Cartan subalgebra $\mathfrak{h}$ containing $\mathfrak{t}$, the subspace $V(\mathfrak{t})$ of $V(\mathfrak{h})$ is generic . Its Tits cone is then $TC(\mathfrak{t})=V(\mathfrak{t}) \cap TC(\mathfrak{h})$, it contains an open subset of $V(\mathfrak{t})$. It is clear that $\mathfrak{t}$ is generic if and only if $\mathfrak{t}_r$ is generic and then $TC(\mathfrak{t})=TC(\mathfrak{t}_r)$.

As $M(TC(\mathfrak{t}))$ fixes $TC(\mathfrak{t})$, is transitive on the Cartan subalgebras of $\mathfrak{m}(TC(\mathfrak{t}))$ and centralizes $\mathfrak{t}$, we see that the apartments containing $TC(\mathfrak{t})$ correspond to the Cartan subalgebras of $\mathfrak{g}$ containing $\mathfrak{t}$ and the above definitions for $\mathfrak{t}$ don't depend of the choice of $\mathfrak{h}$.

Let Z be the centralizer of $\mathfrak{t}$ in G , then Z acting on $\mathfrak{B}$ stabilizes $TC(\mathfrak{t})$. But as Z acts trivially on $V(\mathfrak{t})$, Z fixes $TC(\mathfrak{t})$. So $M(TC(\mathfrak{t}))$ is the fixator of $TC(\mathfrak{t})$ and the centralizer of $\mathfrak{t}$. We thus obtain a bijection between rational generic toral subalgebras of $\mathfrak{g}$ and generic subspaces of $\mathfrak{B}$.

§ 4 THE DESCENT :

4.1 Let $\mathfrak{g}_K$ be an almost split K-form of a Kac-Moody algebra $\mathfrak{g}$ as in 2.1 and 2.2 . Then the Galois group Γ transforms a parabolic into a parabolic of the same sign , so Γ acts on the sets of facets of $\mathfrak{B}^+$, $\mathfrak{B}^-$ and thus $\mathfrak{B}$, in a way compatible with its actions on $\mathfrak{g}$ and G .

If B is a subset of $\mathfrak{g}$ or G stabilized by Γ , we say B is defined over K and we write B_K for the fixed point set B^Γ . We sometimes say e.g. P is a "K-parabolic" instead of a parabolic defined over K , etc . It is clear that the subvectorspaces (or subalgebras , Cartan subalgebras ,...) of $\mathfrak{g}_K$ are in one to one correspondence with subvectorspaces (or subalgebras , Cartan subalgebras ,...) of $\mathfrak{g}$ defined over K .

Proposition 4.2 : *Let Ω be an union of facets of an apartment of $\mathfrak{B}$, we suppose that Ω is stabilized by Γ and meets $\mathfrak{B}^+$, $\mathfrak{B}^-$ and an open facet of finite type , then there exists an apartment containing Ω which is stable under Γ . If moreover $\mathfrak{t}$ is a toral subalgebra defined over K contained in $\mathfrak{p}(\Omega)$ then one may suppose that the Cartan subalgebra corresponding to this apartment contains $\mathfrak{t}$.*

Consequence : If a toral subalgebra $\mathfrak{t}$ and some parabolic subalgebras $\mathfrak{p}_j$ containing $\mathfrak{t}$ are defined over K and such that these parabolic subalgebras contain the same Cartan subalgebra $\mathfrak{h}$, are not all of the same sign and one of them is of finite type , then one may suppose $\mathfrak{h}$ is defined over K and contains $\mathfrak{t}$.

Proof : By 3.5 $\mathfrak{p}(\Omega)$ is finite dimensional and defined over K ; moreover it contains a Cartan subalgebra of $\mathfrak{g}$. So its Cartan subalgebras in the classical sense are Cartan subalgebras of $\mathfrak{g}$ (1.1 , see also [11] 1.6.2) . But if $\mathfrak{t}$ is a toral subalgebra of $\mathfrak{p}(\Omega)$ defined over K (e.g. $\mathfrak{t}=\{0\}$) , it is contained in a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{p}(\Omega)$ defined over K ([4] VII §2 prop. 3 and prop. 10) . Then $A(\mathfrak{h})$ is stable under Γ and contains Ω .

Proposition 4.3 : *Let $\mathfrak{t}$ be a toral subalgebra of $\mathfrak{g}$ defined over K , then $\mathfrak{t}$ is contained in a Cartan subalgebra $\mathfrak{h}$ defined over K and $\mathfrak{h}$ itself is contained in parabolic subalgebras $\mathfrak{p}^+$ and $\mathfrak{p}^-$ defined over K and of opposite signs .*

Proof : Let $\mathfrak{b}$ be a positive (resp. negative) Borel subalgebra containing $\mathfrak{t}$ and B the corresponding group . Then $\tilde{T}$ normalizes $\mathfrak{b}$ and B . Moreover $\mathfrak{b}$ is defined over a finite extension of K , so the set of transforms of $\mathfrak{b}$ or B by Γ is finite . Hence the set of transforms of B by the semi-direct product $\Gamma \ltimes \tilde{T}$ acting on G is finite . Then $\Gamma \ltimes \tilde{T}$ stabilizes a positive (resp. negative) parabolic subgroup P of G of finite type , as we may apply [8] Theorem $\tilde{1}$ with remarks a and b (see [10] Theorem 2.c) . The corresponding parabolic subalgebra $\mathfrak{p}$ is defined over K ; moreover T stabilizes P , hence T is in P ant $\mathfrak{t}$ in $\mathfrak{p}$ (1.9) .

We have thus parabolics $\mathfrak{p}^+$ and $\mathfrak{p}^-$ and we obtain $\mathfrak{h}$ by applying 4.2 and Birkhoff decomposition (see end of 3.2) .

4.4 Action of Γ on $\mathfrak{B}$:

As a consequence of 4.3 , there exists a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ defined over K . So Γ acts on $A(\mathfrak{h})$ through its linear action on $V(\mathfrak{h})$. As $\mathfrak{B}^+=G \times A(\mathfrak{h})/\sim$, it is easy to see that this action extend to an action of Γ on $\mathfrak{B}^+$ compatible with the actions of Γ on G and of G on $\mathfrak{B}^+$. The same may be done for $\mathfrak{B}^-$, so Γ acts on $\mathfrak{B}$.

We write $\mathfrak{B}_K=\mathfrak{B}^\Gamma$. If F is a facet of $\mathfrak{B}$ stabilized by Γ, then F^Γ is non empty and the intersection of F with a vectorspace . Such subsets are called <u>K-facets</u> . Any point of $\mathfrak{B}_K$ is contained in a K-facet .

A <u>K-chamber</u> (in $\mathfrak{B}_K$) is a maximal K-facet . So every K-facet is in a K-chamber and there is a one to one correspondence between K-chambers and minimal K-parabolic groups (or subalgebras) .

A <u>K-apartment</u> (in $\mathfrak{B}_K$)is a generic subspace D of an apartment of $\mathfrak{B}$ which is (point by point) fixed by Γ and maximal with these properties .

Proposition 4.2 may be extended to the case where Ω is a subset of an apartment (not necessarily an union of facets) satisfying the other assumptions .

Let Ω be a subset of an apartment of $\mathfrak{B}$ which is stabilized by Γ . Then Γ stabilizes $\mathfrak{p}(\Omega)$, $\mathfrak{u}(\Omega)$, $P(\Omega)$ and $U(\Omega)$ and we write $\mathfrak{p}_K(\Omega)$, $\mathfrak{u}_K(\Omega)$, $P_K(\Omega)$ and $U_K(\Omega)$ for the corresponding fixed point sets of Γ . If the Cartan subalgebra $\mathfrak{h}$ in $\mathfrak{p}(\Omega)$ is defined over K then Γ stabilizes $\mathfrak{m}(\Omega)$ and $M(\Omega)$ and we write $\mathfrak{m}_K(\Omega)=\mathfrak{m}(\Omega)^\Gamma$ and $M_K(\Omega)=M(\Omega)^\Gamma$.

4.5 <u>K-Split Toral Subalgebras</u> :

If $\mathfrak{t}$ is a toral subalgebra defined over K of $\mathfrak{g}$, then Γ acts on $X(\mathfrak{t})$ and $\mathfrak{t}$ is K-split if and only if this action is trivial (so $\mathfrak{t}_r$ is then also K-split) . In this case the action of Γ on $V(\mathfrak{t})$ is trivial and if $\mathfrak{h}$ is a Cartan subalgebra defined over K containing $\mathfrak{t}$ then $V(\mathfrak{t})\cap A(\mathfrak{h})\subset A(\mathfrak{h})^\Gamma$.

If $\mathfrak{t}$ is generic then $TC(\mathfrak{t})=V(\mathfrak{t})\cap A(\mathfrak{h})$ contains an open subset of $V(\mathfrak{t})$. So if $TC(\mathfrak{t})$ is (point by point) fixed by Γ then $\mathfrak{t}$ is K-split .

We thus obtain a one to one correspondence between generic subspaces (point by point) fixed by Γ of $\mathfrak{B}$ and K-split rational generic toral subalgebras of $\mathfrak{g}$ (c.f. 3.6) .

<u>Remark</u> : With our definitions $\mathfrak{c}$ is always K-split and contained in any maximal K-split toral subalgebra , even if $\mathfrak{c}_K$ is not contained in the K-vectorspace generated by the coroots : in fact we work in $\mathfrak{g}/\mathfrak{c}$.

<u>Proposition</u> 4.6 : *A maximal K-split toral subalgebra is rational generic .*

<u>Consequence</u> : There is a one to one correspondence between maximal K-split toral subalgebras of $\mathfrak{g}$ and K-apartments of $\mathfrak{B}_K$.

<u>Proof</u> : Let $\mathfrak{t}$ be a maximal K-split toral subalgebra . By 4.5 and 1.6 , we know that $\mathfrak{t}$ is rational . By 4.3 , $\mathfrak{t}$ is in a Cartan subalgebra $\mathfrak{h}$ defined over K such that $A(\mathfrak{h})^\Gamma$ contains facets of finite type . By 4.5 and as $\mathfrak{t}$ is maximal , we then know that $\mathfrak{t}$ is generic .

<u>THEOREM 4.7</u> : *a) The group* G_K *acts transitively on the set of K-apartments and on the set of maximal K-split toral subalgebras .*

b) There exists an integer d such that a K-chamber (resp. a K-apartment) of $\mathfrak{B}_K$ *is a K-facet (resp. a generic subspace of* $\mathfrak{B}$ *fixed by* Γ *) of dimension d . The K-chambers are of finite type .*

c) If D is a K-chamber , then $U_K(D)$ *is simply transitive on the K-apartments containing* D .

d) Two K-facets are contained in the same K-apartment .

<u>Remarks</u> : 1) When $\mathfrak{g}_K$ is K-split (i.e. a Cartan subalgebra is K-split) the conjugacy of maximal K-split toral subalgebras (= K-split Cartan subalgebras) is proved in 10) .

2) The number d is the <u>K-rank</u> of $\mathfrak{g}_K$; it is at least 1 .

3) A K-facet is contained in a K-apartment (d)) .

<u>Proof</u> : By 4.3 there exist positive and negative K-facets of finite type . By 4.2 and 3.2 , d) is true when the K-facets are of opposite sign and one of them of finite type .

Let d be the maximal dimension of a K-chamber and D a K-chamber of this dimension ; suppose e.g. D is positive . For every negative K-facet F of finite type , there exists a K-apartment E containing D and F ; by the maximality of D and the convexity of E , we see that E is of dimension d , D is of finite type and F is contained in a K-facet of dimension d . As we now know that D is of finite type , the same conclusion is also true if we drop the hypothesis F of finite type . So the negative K-chambers are the negative K-facets of dimension d . We may now exchange + and - , and the "chamber-part" of b) is proved . Let B be a generic subspace of $\mathfrak{B}$ fixed by Γ and of dimension d' , it contains two opposite K-facets F_1 and F_2 fixed by Γ and of dimension d' . So we know $d'\leq d$. Let D be a K-chamber containing F_2 and E a K-apartment containing F_1 and D . Clearly E is of dimension d . But $E\supset ch(F_1\cup F_2)=B$, so B is a K-apartment if and only if $d'=d$, and b) is proved .

It is then clear that the K-apartments containing a K-chamber D are the vectorial extensions of D which are stable under Γ . As U(D) acts simply transitively on the vectorial extensions of D (3.5.h) , c) is then clear .

As a consequence of b), d) is now proved when the K-facets are of opposite signs, and when there is only one K-facet. Take D_1 and D_2 two K-chambers of the same sign (positive e.g.). Let A_1 (resp. A_2) be an apartment stabilized by Γ containing D_1 (resp. D_2) and A an apartment containing D_1 and D_2. By 3.5 the convex hull of D_1 and D_2 in A is (point by point) fixed by Γ, and, as D_1 is maximal, D_2 is in the vectorial extension of D_1 in A. If u in $U(D_1)$ is such that $A=uA_1$, then $u^{-1}D_2$ and its opposite D_3 in A_1 ($D_3=-u^{-1}D_2$) are in the vectorial extension of D_1 in A_1. Hence D_3 is fixed by Γ. Now $u=u_1.u_2$ where u_2 fixes $u^{-1}D_2 \cup D_1$ and u_1 fixes $D_3 \cup D_1$ ([10] cor. 5b), and the apartment u_1A_1 contains D_1, D_2 and D_3. Applying 4.2 we obtain the K-apartment wanted for the end of proof of d).

Let A_1 and A_2 be two K-apartments. If D_1 (resp. D_2) is a K-chamber in A_1 (resp. A_2) there exists, by d), a K-apartment A containing D_1 and D_2, and by c) A is conjugated to A_1 (resp. A_2) by an element of $U_K(D_1)$ (resp. $U_K(D_2)$). Hence A_1 and A_2 are conjugated by G_K and a) is proved.

4.8 Final Remark :

Let $\mathfrak{t}$ be a maximal K-split toral subalgebra. Then we may define the set Δ_K of relative roots, i.e. of non-zero weights of $\mathfrak{t}$ in $\mathfrak{g}_K$. If we choose a K-chamber D inside the K-apartment $TC(\mathfrak{t})$, we obtain a system of positive roots Δ_K^+ and a basis Π_K. But it may happen that a fundamental root β (= root in Π_K) is imaginary in the sense that the corresponding facet (of codimension 1 in D) is not of finite type and that $\mathbb{Z}\beta \cap \Delta_K$ is infinite.

This is the case when the K-rank is one. A concrete example may be described in the spirit of [12]: If $\mathfrak{k}$ is a compact real form of a semi-simple Lie algebra $\mathfrak{g}$, then, with classical notations, $(\mathfrak{k}\otimes\mathbb{R}[t,t^{-1}])\oplus\mathbb{R}c\oplus\mathbb{R}d$ is a real form of the affine Lie algebra $(\mathfrak{g}\otimes\mathbb{C}[t,t^{-1}])\oplus\mathbb{C}c\oplus\mathbb{C}d$; its maximal $\mathbb{R}$-split toral subalgebra is $\mathfrak{t}_{\mathbb{R}}=\mathbb{R}c\oplus\mathbb{R}d$.

This problem and the corresponding structure of G_K will be discussed in a forthcoming paper.

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90i:17036 17B67 17B05

Rousseau, Guy (F-NANC)

Almost split K-forms of Kac-Moody algebras.

Infinite-dimensional Lie algebras and groups (*Luminy-Marseille,* 1988), 70–85, *Adv. Ser. Math. Phys.*, 7, *World Sci. Publishing, Teaneck, NJ*, 1989.

Let L be an indecomposable infinite-dimensional Kac-Moody algebra over $\overline{K}$, where K is a field of characteristic 0 and $\overline{K}$ its algebraic closure. A K-form of L is a Lie algebra L_K such that L is isomorphic to $L_K \otimes \overline{K}$. Then the Galois group $\Gamma = \mathrm{Gal}(\overline{K}|K)$ acts on L and the corresponding group G. We identify L_K with the fixed point set L^Γ. The Lie algebra L_K is said to be almost split if Γ consists of linear or semilinear automorphisms that transform a Borel subalgebra of L into a Borel subalgebra of the same sign. In the paper under review the author shows that any two maximal K-split toral subalgebras (subalgebras which are ad L_K-diagonalizable) of L_K are conjugate under $G_K = G^\Gamma$.

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Mohammad Hailat (KWT-KUWA)