To Herbert Spohn, scientist and friend

MINIMAL PHOTON VELOCITY BOUNDS IN NON-RELATIVISTIC QUANTUM ELECTRODYNAMICS

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Abstract. We consider non-relativistic quantum particle systems, such as atoms and molecules, coupled to the quantized electromagnetic field. We prove several photon velocity bounds for total energies below the ionization threshold. We also consider phonons coupled to such particle systems and prove velocity bounds for them as well.

1. Introduction

In this paper we study the long-time dynamics of a non-relativistic particle system coupled to the quantized electromagnetic or phonon field. For energies below the ionization threshold, we prove several lower bounds on the growth of the distance of the escaping photons to the particle system.

Standard model of non-relativistic quantum electrodynamics. First, we consider the standard model of non-relativistic quantum electrodynamics in which particles are minimally coupled to the quantized electromagnetic field. The state space for this model is given by $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}$, where $\mathcal{H}_p$ is the particle state space, say, $L^2(\mathbb{R}^{3n})$, or a subspace thereof, and $\mathcal{F}$ is the bosonic Fock space, $\mathcal{F} \equiv \Gamma(\mathfrak{h}) := \mathbb{C} \oplus_{n=1}^{\infty} \otimes_s^n \mathfrak{h}$, based on the one-photon space $\mathfrak{h} := L^2(\mathbb{R}^3, \mathbb{C}^2)$ ($\otimes_s^n$ stands for the symmetrized tensor product of n factors, $\mathbb{C}^2$ accounts for the photon polarization). Its dynamics is generated by the hamiltonian

$$H = \sum_{j=1}^{n} \frac{1}{2m_j} \big(-i\nabla_{x_j} - \kappa_j A_\xi(x_j) \big)^2 + U(x) + H_f. \tag{1.1}$$

Here, m_j and x_j, $j = 1, \dots, n$, are the ('bare') particle masses and the particle positions, $U(x)$, $x = (x_1, \dots, x_n)$, is the total potential affecting the particles, and κ_j are coupling constants related to the particle charges. Moreover, $A_\xi := \check{\xi} * A$, where ξ is an *ultraviolet cut-off* satisfying e.g. $|\partial^m \xi(k)| \lesssim \langle k \rangle^{-3}$, $|m| = 0, \dots, 3$, and $A(y)$ is the *quantized vector potential* in the Coulomb gauge ($\operatorname{div} A(y) = 0$), describing the quantized electromagnetic field and given by

$$A_\xi(y) = \sum_{\lambda=1,2} \int \frac{dk}{\sqrt{2\omega(k)}} \xi(k) \varepsilon_\lambda(k) \big(e^{ik\cdot y} a_\lambda(k) + e^{-ik\cdot y} a_\lambda^*(k) \big). \tag{1.2}$$

(Here and in what follows, the integrals without indication of the domain of integration are taken over entire $\mathbb{R}^3$.) In (1.2), $\omega(k) = |k|$ denotes the photon dispersion relation (k is the photon wave vector), λ is the polarization, and $a_\lambda(k)$ and $a_\lambda^*(k)$ are photon annihilation and creation operators acting on the Fock space $\mathcal{F}$ (see Supplement I for the definition). The operator H_f is the quantum hamiltonian of the quantized electromagnetic field, describing the dynamics of the latter, given by $H_f = \mathrm{d}\Gamma(\omega)$, where $\mathrm{d}\Gamma(\tau)$ denotes the lifting of a one-photon operator τ to the photon Fock space, $\mathrm{d}\Gamma(\tau)|_{\mathbb{C}} = 0$ for $n = 0$ and, for $n \geq 1$,

$$\mathrm{d}\Gamma(\tau)_{|\otimes_s^n \mathfrak{h}} = \sum_{j=1}^{n} \underbrace{1 \otimes \cdots \otimes 1}_{j-1} \otimes \tau \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j}. \tag{1.3}$$

(See Supplement I for definitions related to the creation and annihilation operators and for the expression of $\mathrm{d}\Gamma(\tau)$ in terms of these operators.)

We assume that $U(x) \in L^2_{\mathrm{loc}}(\mathbb{R}^{3n})$ and is either confining or relatively bounded with relative bound 0 w.r.t. $-\Delta_x$, so that the particle hamiltonian $H_p := -\sum_{j=1}^{n} \frac{1}{2m_j} \Delta_{x_j} + U(x)$, and therefore the total hamiltonian H, are self-adjoint.

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This model goes back to the early days of quantum mechanics (it appears in the review [22] as a well-known model and is elaborated in an important way in [53]) (see [56, 62] for extensive references).

Phonon hamiltonian. Next, we consider the standard phonon model of the solid state physics (see e.g. [45]). The state space for it is given by $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}$, where $\mathcal{H}_p$ is the particle state space and $\mathcal{F} \equiv \Gamma(\mathfrak{h}) = \mathbb{C} \oplus_{n=1}^{\infty} \otimes_s^n \mathfrak{h}$ is the bosonic Fock space based on the one-phonon space $\mathfrak{h} := L^2(\mathbb{R}^3, \mathbb{C})$. Its dynamics is generated by the hamiltonian

$$H := H_p + H_f + I(g), \tag{1.4}$$

acting on $\mathcal{H}$, where H_p is a self-adjoint particle system hamiltonian, acting on $\mathcal{H}_p$, and $H_f = \mathrm{d}\Gamma(\omega)$ is the phonon hamiltonian acting on $\mathcal{F}$, where $\omega = \omega(k)$ is the phonon dispersion law (k is the phonon wave vector). For *acoustic phonons*, $\omega(k) \asymp |k|$ for small $|k|$ and $c \leq \omega(k) \leq c^{-1}$, for some $c > 0$, away from 0, while for *optical phonons*, $c \leq \omega(k) \leq c^{-1}$, for some $c > 0$, for all k. To fix ideas, we consider below only the most difficult case $\omega(k) = |k|$.

The operator $I(g)$ acts on $\mathcal{H}$ and represents an interaction energy, labeled by a coupling family $g(k)$ of operators acting on the particle space $\mathcal{H}_p$. In the simplest case of linear coupling (the dipole approximation in QED or the phonon models), $I(g)$ is given by

$$I(g) := \int (g^*(k) \otimes a(k) + g(k) \otimes a^*(k)) dk, \tag{1.5}$$

where $a^*(k)$ and $a(k)$ are the phonon creation and annihilation operators acting on $\mathcal{F}$, and $g(k)$ is a family of operators on $\mathcal{H}_p$ (coupling operators), for which we assume the following condition: there are bounded, positive operators, η_1 and η_2, with unbounded inverses, s.t.

$$\|\eta_1 \eta_2^{|\alpha|} \partial^\alpha g(k)\|_{\mathcal{L}(\mathcal{H}_p)} \lesssim |k|^{\mu - |\alpha|} \langle k \rangle^{-2-\mu}, \quad |\alpha| \leq 2, \tag{1.6}$$

where the norm is taken in the Banach space, $\mathcal{L}(\mathcal{H}_p)$, of bounded operators, and for some $\Sigma > \inf \sigma(H_p)$ the following estimate holds

$$\|\eta_2^{-n} \eta_1^{-m} \eta_2^{-n} f(H)\| \lesssim 1, \quad 0 \leq n, m \leq 2, \tag{1.7}$$

for any $f \in \mathrm{C}_0^\infty((-\infty, \Sigma))$. The specific form of η_1 and η_2 depends on the models considered and will be given below.

A primary example for the particle system to have in mind is an electron in a vacuum or in a solid in an external potential V. In this case, $H_p = \epsilon(p) + V(x)$, $p := -i\nabla_x$, with $\epsilon(p)$ being the standard non-relativistic kinetic energy, $\epsilon(p) = \frac{1}{2m}|p|^2 \equiv -\frac{1}{2m}\Delta_x$ (the Nelson model), or the electron dispersion law in a crystal lattice (a standard model in solid state physics), acting on $\mathcal{H}_p = L^2(\mathbb{R}^3)$. The coupling family is given by $g(k) = |k|^\mu \xi(k) e^{ikx}$, where $\xi(k)$ is the ultraviolet cut-off, satisfying e.g. $|\partial^m \xi(k)| \lesssim \langle k \rangle^{-2-\mu}$, $m = 0, \dots, 3$ (and therefore $g(k)$ satisfies (1.6), with $\eta_1 = \mathbf{1}$ and $\eta_2 = \langle x \rangle^{-1}$ with $\langle x \rangle = (1 + |x|^2)^{1/2}$). For phonons, $\mu = 1/2$, and for the Nelson model, $\mu \geq -1/2$. To have a self-adjoint operator H we assume that V is a Kato potential and that $\mu \geq -1/2$. This can be easily upgraded to an $N-$body system (e.g. an atom or a molecule, see e.g. [37, 56]). A key fact here is that for the particle models discussed above (both for non-relativistic QED and for phonon models), there is a spectral point $\Sigma \in \sigma(H) \cup \{\infty\}$, called the *ionization threshold*, s.t. below Σ, the particle system is well localized:

$$\|\langle p \rangle^2 e^{\delta |x|} f(H)\| \lesssim 1, \tag{1.8}$$

for any $0 \leq \delta < \mathrm{dist}(\mathrm{supp}\, f, \Sigma)$ and any $f \in \mathrm{C}_0^\infty((-\infty, \Sigma))$. In other words, states decay exponentially in the particle coordinates x ([34, 6, 7]). Hence (1.7) holds with $\eta_1 = \langle p \rangle^{-1}$ and $\eta_2 = \langle x \rangle^{-1}$. To guarantee that $\Sigma > \inf \sigma(H_p) \geq \inf \sigma(H)$, we assume that the potentials $U(x)$ or $V(x)$ are such that the particle hamiltonian H_p has discrete eigenvalues below the essential spectrum ([34, 6, 7]). Furthermore, Σ, for which (1.8) is true, is given by $\Sigma := \lim_{R \to \infty} \inf_{\varphi \in D_R} \langle \varphi, H\varphi \rangle$, where the infimum is taken over $D_R = \{\varphi \in \mathcal{D}(H) |\ \varphi(x) = 0 \text{ if } |x| < R, \|\varphi\| = 1\}$ (see [34]; Σ is close to $\inf \sigma_{\mathrm{ess}}(H_p)$).

Problem. In all above cases, the hamiltonian H is self-adjoint and generates the dynamics through the Schrödinger equation,

$$i\partial_t \psi_t = H\psi_t. \tag{1.9}$$

As initial conditions, ψ_0, we consider states below the ionization threshold Σ, i.e. ψ_0 in the range of the spectral projection $E_{(-\infty,\Sigma)}(H)$. In other words, we are interested in processes, like emission and absorption of radiation, or scattering of photons on an electron bound by an external potential (created e.g. by an infinitely heavy nucleus or impurity of a crystal lattice), in which the particle system (say, an atom or a molecule) is not being ionized.

Denote by Φ_j and E_j the eigenfunctions and the corresponding eigenvalues of the hamiltonian H, below Σ, i.e. $E_j < \Sigma$. The following are the key characteristics of the evolution of (1.9), in progressive order the depth of information they provide:

- *Local decay* stating that some photons are bound to the particle system while others (if any) escape to infinity, i.e. the probability that they occupy any bounded region of the physical space tends to zero, as $t \to \infty$.
- *Minimal photon velocity bound* with speed c stating that, as $t \to \infty$, with probability $\to 1$, the photons are either bound to the particle system or depart from it with the distance $\geq c't$, for any $c' < c$.
  Similarly, if the probability that at least one photon is at the distance $\geq c''t$, $c'' > c$, from the particle system vanishes, as $t \to \infty$, we say that the evolution satisfies the *maximal photon velocity bound* with speed c.
- *Asymptotic completeness* on the interval $(-\infty, \Sigma)$ stating that, for any $\psi_0 \in \text{Ran } E_{(-\infty,\Sigma)}(H)$, and any $\epsilon > 0$, there are photon wave functions $f_{j\epsilon} \in \mathcal{F}$, with a finite number of photons, s.t. the solution, $\psi_t = e^{-itH}\psi_0$, of the Schrödinger equation, (1.9), satisfies
$$\limsup_{t\to\infty} \left\| e^{-itH}\psi_0 - \sum_j e^{-iE_j t}\Phi_j \otimes_s e^{-iH_f t} f_{j\epsilon} \right\| \leq \epsilon. \tag{1.10}$$
  (One can verify that $\Phi_j \otimes_s f_{j\epsilon}$ is well-defined, at least for the ground state ($j = 0$).) In other words, for any $\epsilon > 0$ and with probability $\geq 1 - \epsilon$, the Schrödinger evolution ψ_t approaches asymptotically a superposition of states in which the particle system with a photon cloud bound to it is in one of its bound states Φ_j, with additional photons (or possibly none) escaping to infinity with the velocity of light.

The reason for $\epsilon > 0$ in (1.10) is that for the state $\Phi_j \otimes_s f_j$ to be well defined, as one would expect, one would have to have a very tight control on the number of photons in f_j, i.e. the number of photons escaping the particle system. (See the remark at the end of Subsection 5.4 of [21] for a more technical explanation.) For massive bosons $\epsilon > 0$ can be dropped (set to zero), as the number of photons can be bound by the energy cut-off.[1]

We define the photon velocity in terms of its space-time (and sometimes phase-space-time) localization. In a quantum theory this is formulated in terms of quantum localization observables and related to quantum probabilities. We describe the photon position by the operator $y := i\nabla_k$ on $L^2(\mathbb{R}^3)$, canonically conjugate to the photon momentum k. To test the photon localization, we use the observables $d\Gamma(\mathbf{1}_S(y))$, where $\mathbf{1}_S(y)$ denotes the characteristic function of a subset S of $\mathbb{R}^3$. We also use the localization observables $\Gamma(\mathbf{1}_S(y))$, where $\Gamma(\chi)$ is the lifting of a one-photon operator χ (e.g. a smoothed out characteristic function of y) to the photon Fock space, defined by
$$\Gamma(\chi) = \oplus_{n=0}^{\infty}(\otimes^n \chi), \tag{1.11}$$
(so that $\Gamma(e^b) = e^{d\Gamma(b)}$), and then to the space of the total system. Let also $T_h = \Gamma(\tau_h)$, with $\tau_h : f(y) \to f(h^{-1}y)$, where $h \in$ group of rigid motions of $\mathbb{R}^3$. The observables $d\Gamma(\mathbf{1}_S(y))$ and $\Gamma(\mathbf{1}_S(y))$ have the following natural properties:

- $d\Gamma(\mathbf{1}_{S_1\cup S_2}(y)) = d\Gamma(\mathbf{1}_{S_1}(y)) + d\Gamma(\mathbf{1}_{S_2}(y))$ and $\Gamma(\mathbf{1}_{S_1}(y))\Gamma(\mathbf{1}_{S_2}(y)) = P_\Omega$, for S_1 and S_2 disjoint, where P_Ω denotes the projection onto the vacuum sector,
- $T_u X_S(y) T_u^{-1} = X_{u^{-1}S}(y)$, where $X_S(y)$ stands for either $d\Gamma(\mathbf{1}_S(y))$ or $\Gamma(\mathbf{1}_S(y))$.

The observables $d\Gamma(\mathbf{1}_S(y))$ can be interpreted as giving the number of photons in Borel sets $S \subset \mathbb{R}^3$. They are closely related to those used in [24, 32, 47] (and discussed earlier in [49] and [1]) and are consistent with a theoretical description of the detection of photons (usually via the photoelectric effect, see e.g. [50]). The quantity $\langle\psi, \Gamma(\mathbf{1}_S(y))\psi\rangle$ is interpreted as the probability that the photons are in the set S in the state ψ. This said, we should mention that the subject of photon localization is still far from being settled.[2]

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[1] For a discussion of scattering of massless bosons in QFT see [11].

[2] The issue of localizability of photons is a tricky one and has been intensely discussed in the literature since the 1930 and 1932 papers by Landau and Peierls [46] and Pauli [52] (see also a review in [44]). A set of axioms for localization observables was proposed by Newton and Wigner [51] and Wightman [63] and further generalized by Jauch and Piron [43]. Observables

The fact that for photons the observables we use depend on the choice of polarization vector fields, $\varepsilon_\lambda(k)$, $\lambda = 1, 2$,[3] is not an impediment here as our results imply analogous results for e.g. localization observables of Mandel [49] and of Amrein and Jauch and Piron [1, 43]: $d\Gamma(f_S^{\mathrm{man}})$ and $d\Gamma(f_S^{\mathrm{ajp}})$, where $f_S^{\mathrm{man}} := P^\perp \mathbf{1}_S(y) P^\perp$ and $f_S^{\mathrm{ajp}} := \mathbf{1}_S(y) \cap P^\perp$, respectively, acting in the Fock space based on the space $\mathfrak{h} = \mathrm{L}^2_{\mathrm{transv}}(\mathbb{R}^3; \mathbb{C}^3) := \{f \in \mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^3) : k \cdot f(k) = 0\}$ instead of $\mathfrak{h} = \mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$. Here $P^\perp : f(k) \to f(k) - |k|^{-2} k\ k \cdot f(k)$ is the orthogonal projection on the transverse vector fields and, for two orthogonal projections P_1 and P_2, the symbol $P_1 \cap P_2$ stand for the orthogonal projection on the largest subspace contained in $\mathrm{Ran}\, P_1$ and $\mathrm{Ran}\, P_2$.

In what follows, we denote by $\chi_S(v)$ a smoothed out characteristic function of the set S, which is defined precisely at the end of the introduction. (For instance, $\chi_{x=1}$ stands for a $\mathrm{C}^\infty(\mathbb{R})$, function, which is $= 1$ if $|x-1| \le 1/10$ and $= 0$ if $|x-1| \ge 1/9$. For a self-adjoint operator A, $\chi_{A=1}$ is defined by the spectral theory.) We say that the system obeys the *minimal photon velocity bound* if the Schrödinger evolution, $\psi_t = e^{-itH}\psi_0$, obeys the estimates

$$\int_1^\infty dt\ t^{-\alpha'} \left\| d\Gamma(\chi_{\frac{|y|}{ct^\alpha}=1})^{\frac12} \psi_t \right\|^2 \lesssim \|\psi_0\|_0^2, \tag{1.12}$$

for some norm $\|\psi_0\|_0$, some $0 < \alpha' \le 1$, and for any $\alpha > 0$ and $c > 0$ such that either $\alpha < 1$, or $\alpha = 1$ and $c < 1$. In other words there are no photons which either diffuse or propagate with speed < 1. The *maximal velocity estimate*, as proven in [10], states that, for any $c' > 1$,

$$\left\| d\Gamma\big(\chi_{\frac{|y|}{c't} \ge 1}\big)^{\frac12} \psi_t \right\| \lesssim t^{-\gamma} \left\| (d\Gamma(\langle y \rangle) + 1)^{\frac12} \psi_0 \right\|, \tag{1.13}$$

with $\gamma < \min(\frac12(1 - \frac{1}{c'}), \frac{1}{10})$ for (1.1), and $\gamma < \min(\frac{\mu}{2}(\frac{c'-1}{2c'-1}), \frac{1}{2+\mu})$ for (1.4)–(1.6) with $\mu > 0$.

Results. Now we formulate our results. We consider both the minimal coupling model (1.1) and the linear coupling model (1.4) with the linear interaction (1.5) and the coupling operators $g(k)$ satisfying (1.6) with $\mu > -1/2$.

It is known (see [7, 35]) that the operator H has *a unique ground state* (denoted here as Φ_{gs}) and that generically (e.g. under the Fermi Golden Rule condition), H has no eigenvalues in the interval $(E_{\mathrm{gs}}, a]$, where $a < \Sigma$ can be taken arbitrarily close to Σ, depending on the coupling constant and on whether the particle system has an infinite number of eigenvalues accumulating to its ionization threshold (see [8, 27, 31]). *We assume* that this is exactly the case:

$$\textit{Fermi's Golden Rule } ([6, 7]) \text{ holds for all excited eigenvalues} \le a \text{ of } H_p. \tag{1.14}$$

Assumption (1.14) means that for every excited eigenvalue $e_j \le a$ of H_p, we have

$$\Pi_j W \mathrm{Im}((H_0 - e_j - i0^+)^{-1} \bar\Pi_j) W \Pi_j \ge c_j \Pi_j, \quad c_j > 0, \tag{1.15}$$

where $H_0 := H_p + H_f$ (for either model), $W := H - H_0$, Π_j denotes the projection onto the eigenspace of H_0 associated to e_j and $\bar\Pi_j := \mathbf{1} - \Pi_j$. In fact, there is an explicit representation of (1.15). Since it differs slightly for different models, we present it for the phonon one, assuming for simplicity that the eigenvalue e_j is simple:

$$\int \langle \phi_j, g^*(k) \mathrm{Im}(H_p + \omega(k) - e_j - i0^+)^{-1} g(k) \phi_j \rangle dk > 0, \tag{1.16}$$

where ϕ_j is an eigenfunction of H_p corresponding to the eigenvalue e_j and the inner product is in the space $\mathcal{H}_p$. It is clear from (1.16) that Fermi's Golden Rule holds generally, with a very few exceptions.

Let $N := d\Gamma(\mathbf{1})$ be the photon (or phonon) number operator and $N_\rho := d\Gamma(\omega^{-\rho})$ be the photon (or phonon) low momentum number operator. In what follows we let ψ_t denote the Schrödinger evolution, $\psi_t = e^{-itH}\psi_0$, i.e. the solution of the Schrödinger equation (1.9), with an initial condition ψ_0, satisfying $\psi_0 = f(H)\psi_0$, with $f \in \mathrm{C}_0^\infty((-\infty, \Sigma))$. More precisely, we will consider the following sets of initial conditions

$$\Upsilon_\rho := \big\{ \psi_0 \in f(H) D(N_\rho)^{\frac12}),\ \text{for some } f \in \mathrm{C}_0^\infty((-\infty, \Sigma)) \big\},$$

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describing localization of massless particles, satisfying the Jauch-Piron version of the Wightman axioms, were constructed by Amrein in [1].

[3]Since polarization vector fields are not smooth, using them to reduce the results from one set of localization observables to another would limit the possible time decay. However, these vector fields can be avoided by using the approach of [48].

and

$$\Upsilon_{\#} := \{\psi_0 \in f(H)\big(D(d\Gamma(\langle y\rangle)) \cap D(d\Gamma(b)^2)\big),\ \text{for some } f \in C_0^\infty((E_{gs}, a])\},$$

where $b := \frac{1}{2}(k\cdot y + y\cdot k)$ and $a < \Sigma$ is given by Assumption (1.14).

For $A \geq -C$, we denote $\|\psi_0\|_A := \|(A+C+1)^{\frac{1}{2}}\psi_0\|$. We define $\nu_\rho \geq 0$ as the smallest real number satisfying the inequality

$$\langle \psi_t, N_\rho \psi_t\rangle \lesssim t^{\nu_\rho}\|\psi_0\|_\rho^2, \tag{1.17}$$

for any $\psi_0 \in \operatorname{Ran} E_{(-\infty,\Sigma)}(H)$, where $\|\psi\|_\rho^2 := \|\psi\|_{N_\rho}^2$. It was shown in [10] (see (A.1) of Appendix A) that, for any $-1 \leq \rho \leq 1$, the inequality (1.17) is satisfied with

$$\nu_\rho \leq \frac{1+\rho}{2+\mu} \tag{1.18}$$

(this generalizes an earlier result due to [32]). Also, the bound

$$\|\psi_t\|_{H_f} \lesssim \|\psi_0\|_H \tag{1.19}$$

shows that (1.17) holds for $\rho = -1$ with $\nu_{-1} = 0$. With ν_ρ defined by (1.17), we prove the following two results.

Theorem 1.1 (Minimal photon velocity bound)**.** *Consider the hamiltonian* (1.1)*, or the hamiltonian* (1.4)–(1.5) *satisfying* (1.6) *with* $\mu > -1/2$ *and* (1.7)*. Let either* $\alpha = 1$ *and* $c < 1$ *or*

$$\max\Big(\frac{1}{6}(5+\nu_1-\nu_0), \frac{1}{2}+\frac{1}{3+2\mu}\Big) < \alpha < 1, \tag{1.20}$$

where $\mu = 1/2$ *for* (1.1)*. Then for any initial condition* $\psi_0 \in \Upsilon_1$*, the Schrödinger evolution,* ψ_t*, satisfies, for any* $a > 1$*, the following estimate*

$$\int_1^\infty dt\ t^{-\alpha - a\nu_0}\|d\Gamma(\chi_{\frac{|y|}{ct^\alpha}=1})^{\frac{1}{2}}\psi_t\|^2 \lesssim \|\psi_0\|_1^2. \tag{1.21}$$

For the coupling function g, we introduce the norm $\langle g\rangle := \sum_{|\alpha|\leq 2}\|\eta_1\eta_2^{|\alpha|}\partial^\alpha g\|_{L^2(\mathbb{R}^3,\mathcal{H}_p)}$. We have

Theorem 1.2 (Weak minimal photon escape velocity estimate)**.** *Consider the hamiltonian* (1.1) *with the coupling constants* κ_j *sufficiently small, or the hamiltonian* (1.4)–(1.5) *satisfying* (1.6) *with* $\mu > -1/2$, (1.7) *and* $\langle g\rangle \ll 1$*. Assume* (1.14)*,* $\nu_0 + \nu_1 < \alpha < 1 - \nu_0$ *and* $c > 0$*. Then for any initial condition* $\psi_0 \in \Upsilon_{\#}$*, the Schrödinger evolution,* ψ_t*, satisfies the estimate*

$$\big\|\Gamma(\chi_{\frac{|y|}{ct^\alpha}\leq 1})\psi_t\big\| \lesssim t^{-\gamma}\big(\|\psi_0\|_{d\Gamma(\langle y\rangle)} + \|\psi_0\|_{d\Gamma(b)^2}\big), \tag{1.22}$$

where $\gamma < \frac{1}{2}\min(1-\alpha-\nu_0, \frac{1}{2}(\alpha-\nu_0-\nu_1))$*.*

Remarks.

1) The estimate (1.21) is sharp if $\nu_0 = 0$. Assuming this and taking $\nu_1 \leq (3/2+\mu)^{-1}$ (see (A.7) of Appendix A), the conditions on α in Theorems 1.1 and 1.2 become $\alpha > \frac{5}{6}+\frac{1}{6(3/2+\mu)}$, and $(3/2+\mu)^{-1} < \alpha < 1$, respectively.

2) The estimate (1.22) states that, as $t \to \infty$, with probability $\to 1$, either all photons are attached to the particle system in the combined ground state, or at least one photon departs the particle system with the distance growing at least as $\mathcal{O}(t^\alpha)$. (Remember that the set $\Upsilon_{\#}$ excludes the ground state and the exited states below Σ are excluded by the condition (1.14). Note that (1.22) for $\mu \geq 1/2$, some $\alpha > 0$ and $\psi_0 \in E_\Delta(H)$, with $\Delta \subset (E_{gs}, e_1 - \mathcal{O}(\langle g\rangle))$ and e_1 the first excited eigenvalue of H_p, can be derived directly from [9, 10].)

3) With some more work, one can remove the assumption (1.14) and relax the condition on ψ_0 in Theorem 1.2 to the natural one: $\psi_0 \in P_\Sigma D(d\Gamma(\langle y\rangle))$, where P_Σ is the spectral projection onto the orthogonal complement of the eigenfunctions of H with corresponding eigenvalues in the interval $(-\infty, \Sigma)$.

4) For the spin-boson model, a uniform bound, $\langle \psi_t, e^{\delta N}\psi_t\rangle \leq C(\psi_0) < \infty$, $\delta > 0$, on the number of photons, on a dense set of ψ_0's, without controlling the dependence of the constant $C(\psi_0)$ on this dense set, was recently proven in [14]. See [21] for a discussion of such bounds.

5) Some key estimates used in the proof Theorem 1.1 were derived in the proof of asymptotic completeness for Rayleigh scattering in [21].

Remarks about the literature. Considerable progress has been made in understanding the asymptotic dynamics of non-relativistic particle systems coupled to quantized electromagnetic or phonon field. The local decay property was proven in [7, 8, 9, 12, 27, 28, 30, 31], by the combination of the renormalization group and positive commutator methods. The maximal velocity estimate was proven in [10].

An important breakthrough was achieved recently in [14], where the authors proved relaxation to the ground state and uniform bounds on the number of emitted massless bosons in the spin-boson model.

For models involving massive bosons fields, some minimal velocity estimates are proven in [18]. For massless bosons, Theorems 1.1 and 1.2 seem to be new.

Asymptotic completeness was proven for (a small perturbation of) a solvable model involving a harmonic oscillator (see [3, 61]), and for models involving massive boson fields ([18, 24, 25, 26]). Moreover, [32] obtained some important results for massless bosons (the Nelson model) in confined potentials.

In [21], we have proven asymptotic completeness for Rayleigh scattering on the states for which the expectation of either the photon/phonon number operator, N, or an operator, N_1, testing the photon/phonon infrared behaviour is uniformly bounded on corresponding dense sets. By extending the result of [14] in a straightforward way, we have shown that the second of these conditions is satisfied for the spin-boson model.

[21] contains also minimal velocity-type estimates (see the paragraph describing our approach), which play an important role in the present paper.

Approach and organization of the paper. We prove Theorems 1.1 and 1.2 in Sections 2 and 3, respectively. The proofs begin with estimates involving the operator $b_\epsilon := \frac{1}{2}(v(k)\cdot y + y\cdot v(k))$, where $v(k) := \frac{k}{\omega+\epsilon}$, for $\epsilon = t^{-\kappa}$, with some $\kappa > 0$. The latter estimates were derived in the proof of asymptotic completeness for Rayleigh scattering in [21]. We reproduce the first of these estimates here for reader's convenience. Then we use them to derive the estimates of Theorems 1.1 and 1.2.

Unlike the operator $b_0 := \frac{1}{2}(\frac{k}{\omega}\cdot y + y \cdot \frac{k}{\omega})$, used in [32], the operator b_ϵ is self-adjoint. This follows from the fact that the vector field $v(k)$ is Lipschitz continuous and therefore generates a global flow. Using b_ϵ avoids some technicalities, as compared to the other operator. At the expense of slightly lengthier computations but gaining simpler technicalities, one can also modify b_ϵ to make it bounded, by multiplying it with the cut-off function $\chi_{\frac{|y|}{c' t}\leq 1}$ with $c' > 1$, such that the maximal velocity estimate (1.13) holds, or use the smooth vector field $v(k) = \frac{k}{\sqrt{\omega^2+\epsilon^2}}$, instead of $v(k) = \frac{k}{\omega+\epsilon}$.)

As in earlier works, to prove the above estimates, we use the method of propagation observables, originating in the many body scattering theory ([58, 59, 42, 33, 64, 15], see [17, 41] for a textbook exposition and a more recent review). It was extended to the non-relativistic quantum electrodynamics in [18, 32, 23, 24, 25, 26] and to the $P(\varphi)_2$ quantum field theory, in [19] and was used in [10] to prove the maximal velocity estimate (1.13). We formalize the method of propagation observables in Appendix B.

To simplify the exposition, in Sections 2–3, we consider hamiltonians of the form (1.4)–(1.5), with the coupling operators $g(k)$ satisfying (1.6), where η_1 and η_2 obey (1.7). In Section 4, we extend the results to a general class of hamiltonians that are introduced in the next paragraph. In Section 5, we show that the minimal coupling model (1.1) can be mapped unitarily to a hamiltonian from this class, and we deduce Theorems 1.1 and 1.2 for this model. Finally, a low momentum bound of [10] and some standard technical statements are given in Appendices A, and C. The paper is essentially self-contained. In order to make it more accessible to non-experts, we included Supplement I defining and discussing the creation and annihilation operators (see also [20, 16]).

General class of hamiltonians. The QED hamiltonian (1.1) can be written in the form (1.4), with $I(g)$ being quadratic in the creation and annihilation operators $a^\#_\lambda(k)$, and the coupling functions satisfying estimates of the form (1.6) with $\mu = -1/2$, $\eta_1 = \langle p\rangle^{-1}$ or $\mathbf{1}$, and $\eta_2 = \langle x\rangle^{-1}$. This infrared behaviour is too singular for our techniques. However, we show in Section 5 that under the generalized Pauli-Fierz transform of [55], (1.1) is unitary equivalent to an operator of the form described below, whose infrared behaviour is considerably better. We introduce the class of hamiltonians of the form

$$\widetilde{H} = H_p + H_f + \tilde{I}(g), \tag{1.23}$$

with $H_p := -\Delta + V(x)$, $H_f = \mathrm{d}\Gamma(\omega)$ and

$$\tilde{I}(g) := \sum_{ij} \iint d\underline{k}_{(i)} d\underline{k}'_{(j)} g_{ij}(\underline{k}_{(i)}, \underline{k}'_{(j)}) \otimes a^*(\underline{k}_{(i)}) a(\underline{k}'_{(j)}), \tag{1.24}$$

where the summation in i, j ranges over the set $i, j \geq 0, 1 \leq i + j \leq 2$, $\underline{k}_{(p)} := (\underline{k}_1, \ldots, \underline{k}_p)$, $\underline{k}_j := (k_j, \lambda_j)$, $\int d\underline{k}_{(p)} := \prod_1^p \sum_{\lambda_j} \int dk_j$, $a^{\#}(\underline{k}_{(p)}) := \prod_1^p a^{\#}(\underline{k}_j)$ if $p \geq 1$ and $= \mathbf{1}$, if $p = 0$, $a^{\#}(\underline{k}_j) := a^{\#}_{\lambda_j}(k_j)$, and $g := (g_{ij})$. We suppose that the coupling operators, $g_{ij} = g_{ij}(\underline{k}_{(i)}, \underline{k}_{(j)})$ obey

$$g_{ij}(\underline{k}_{(i)}, \underline{k}'_{(j)}) = g^*_{ji}(\underline{k}'_{(j)}, \underline{k}_{(i)}), \tag{1.25}$$

and satisfy the estimates

$$\|\eta_1^{2-i-j}\eta_2^{|\alpha|}\partial^\alpha g_{ij}(\underline{k}_{(i+j)})\|_{\mathcal{H}_p} \lesssim \sum_{m=1}^{i+j}\prod_{\ell=1}^{i+j}(|k_\ell|^\mu \langle k_\ell\rangle^{-2-\mu})|k_m|^{-|\alpha|}, \tag{1.26}$$

where $\mu > -1/2$, and, as above, η_1, η_2 are estimating operators (unbounded, positive operators with bounded inverses) on the particle space $\mathcal{H}_p$ satisfying (1.7) (for some $\Sigma > \inf \sigma(H_p)$). We define the norm $\langle g\rangle := \sum_{1\leq i+j\leq 2}\sum_{|\alpha|\leq 2}\|\eta_1^{2-i-j}\eta_2^{|\alpha|}\partial^\alpha g_{ij}\|$ of the vector coupling operators $g := (g_{ij})$, extending the norms of the scalar coupling operators g, introduced above. It is easy to extend Theorems 1.1 and 1.2 to the hamiltonians of the form (1.23)–(1.26) satisfying (1.7):

Theorem 1.3. *Theorems 1.1 and 1.2 hold for hamiltonians of the form* (1.23)–(1.24)*, satisfying* (1.25)–(1.26) *and* (1.7)*.*

As mentioned above, Theorem 1.3 is proven in Section 4.

Notations. For functions A and B, we will use the notation $A \lesssim B$ signifying that $A \leq CB$ for some absolute (numerical) constant $0 < C < \infty$. The symbol E_Δ stands for the characteristic function of a set Δ, while $\chi_{\cdot\leq 1}$ denotes a smoothed out characteristic function of the interval $(-\infty, 1]$, that is it is in $\mathrm{C}^\infty(\mathbb{R})$, is non-increasing, and $= 1$ if $x \leq 1/2$ and $= 0$ if $x \geq 1$. Moreover, $\chi_{\cdot\geq 1} := \mathbf{1} - \chi_{\cdot\leq 1}$ and $\chi_{\cdot=1}$ stands for the derivative of $\chi_{\cdot\geq 1}$. Given a self-adjoint operator a and a real number α, we write $\chi_{a\leq\alpha} := \chi_{\frac{a}{\alpha}\leq 1}$, and likewise for $\chi_{a\geq\alpha}$. Finally, $D(A)$ denotes the domain of an operator A, $\langle x\rangle := (1 + |x|^2)^{1/2}$, $\mathcal{O}(\epsilon)$ denotes an operator bounded by $C\epsilon$.

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2. The first propagation estimate

In this section we to prove the minimal velocity estimates of Theorem 1.1 for hamiltonians of the form (1.4)–(1.5), with the coupling operators $g(k)$ satisfying (1.6) and (1.7). We begin with a technical result proving these estimates for the operator b_ϵ, which is defined in the introduction, instead of $|y|$. Let $\nu_\rho \geq 0$ be the same as in (1.17). We write the operator b_ϵ as

$$b_\epsilon = \frac{1}{2}(\theta_\epsilon\nabla\omega\cdot y + y\cdot\nabla\omega\,\theta_\epsilon), \qquad \text{where} \qquad \theta_\epsilon := \frac{\omega}{\omega_\epsilon},\ \omega_\epsilon := \omega + \epsilon,\ \epsilon = t^{-\kappa}. \tag{2.1}$$

Theorem 2.1. *Consider hamiltonians of the form* (1.4)–(1.5) *with the coupling operators satisfying* (1.6) *with* $\mu > -1/2$ *and* (1.7)*. Let* $\nu_1 - \nu_0 < \kappa < 1$*. If either* $\alpha < 1$*, or* $\alpha = 1$ *and* $c < 1$*, and*

$$\alpha > \max((3/2+\mu)^{-1}, (1+\kappa)/2, 1-\kappa+\nu_1-\nu_0), \tag{2.2}$$

then for any initial condition $\psi_0 \in \Upsilon_1$*, the Schrödinger evolution,* ψ_t*, satisfies, for any* $a > 1$*, the following estimates*

$$\int_1^\infty dt\, t^{-\alpha-a\nu_0}\left\|\mathrm{d}\Gamma(\chi_{\frac{b_\epsilon}{ct^\alpha}=1})^{\frac{1}{2}}\psi_t\right\|^2 \lesssim \|\psi_0\|_1^2. \tag{2.3}$$

Proof. We will use the method of propagation observables outlined in Appendix B. We consider the one-parameter family of one-photon operators

$$\phi_t := t^{-a\nu_0}\chi_\alpha, \quad \chi_\alpha \equiv \chi_{v\geq 1}, \quad v := \frac{b_\epsilon}{ct^\alpha}, \tag{2.4}$$

where $a > 1$. To show that ϕ_t is a weak one-photon propagation observable, we obtain differential inequalities for ϕ_t. Recall that $d\phi_t = \partial_t \phi_t + i[\omega, \phi_t]$. To compute $d\phi_t$, we use the expansion

$$d\phi_t = t^{-a\nu_0}(dv)\chi'_\alpha + \sum_{i=1}^{2} \mathrm{rem}_i, \tag{2.5}$$

$$\mathrm{rem}_1 := t^{-a\nu_0}[d\chi_\alpha - (dv)\chi'_\alpha], \quad \mathrm{rem}_2 := -a\nu_0 t^{-1}\phi_t. \tag{2.6}$$

Using the definitions in (2.1), we compute

$$dv = \frac{1}{ct^\alpha}\big(\theta_\epsilon - \frac{\alpha b_\epsilon}{t} + \partial_t b_\epsilon\big). \tag{2.7}$$

Next, we have $\partial_t b_\epsilon = \frac{\kappa}{2t^{1+\kappa}}(\omega_\epsilon^{-1}\theta_\epsilon \nabla\omega\cdot y + \text{ h.c.})$ on $D(b_\epsilon)$, which, due to the relation $\frac{1}{2}(\omega_\epsilon^{-1}\theta_\epsilon\nabla\omega\cdot y + \text{ h.c.}) = \omega_\epsilon^{-1/2} b_\epsilon \omega_\epsilon^{-1/2}$, becomes

$$\partial_t b_\epsilon = \frac{\kappa}{t^{1+\kappa}}\omega_\epsilon^{-1/2} b_\epsilon \omega_\epsilon^{-1/2}. \tag{2.8}$$

Using that (see Lemma C.1 of Appendix C)

$$\omega_\epsilon^{-1/2} b_\epsilon \omega_\epsilon^{-1/2}\chi'_\alpha = \omega_\epsilon^{-1/2} b_\epsilon \chi'_\alpha \omega_\epsilon^{-1/2} + \mathcal{O}(t^{\frac{3}{2}\kappa}),$$

and that $b_\epsilon \geq 0$ on $\operatorname{supp}\chi'_\alpha$, we obtain

$$\partial_t b_\epsilon \chi'_\alpha \geq -\frac{\text{const}}{t^{1-\kappa/2}}. \tag{2.9}$$

The relations (2.5)–(2.9), together with $\frac{b_\epsilon}{ct^\alpha}\chi'_\alpha \leq \chi'_\alpha$, imply

$$d\phi_t \geq t^{-a\nu_0}\big(\frac{\theta_\epsilon}{ct^\alpha} - \frac{\alpha}{t}\big)\chi'_\alpha + \sum_{i=1}^{3}\mathrm{rem}_i, \tag{2.10}$$

where rem_1 and rem_2 are given in (2.6) and

$$\mathrm{rem}_3 = \mathcal{O}(t^{-1-\alpha+\frac{\kappa}{2}-a\nu_0}). \tag{2.11}$$

This, together with $\theta_\epsilon = 1 - \frac{t^{-\kappa}}{\omega_\epsilon}$ and $\omega_\epsilon^{-1}\chi'_\alpha = \omega_\epsilon^{-1/2}\chi'_\alpha\omega_\epsilon^{-1/2} + \mathcal{O}(t^{-\alpha+\frac{3}{2}\kappa})$ (see again Lemma C.1 of Appendix C), implies

$$d\phi_t \geq t^{-a\nu_0}\big(\frac{1}{ct^\alpha} - \frac{\alpha}{t}\big)\chi'_\alpha + \sum_{i=1}^{5}\mathrm{rem}_i, \tag{2.12}$$

$$\mathrm{rem}_4 := \frac{1}{ct^{\alpha+\kappa+a\nu_0}}\omega_\epsilon^{-1/2}\chi'_\alpha\omega_\epsilon^{-1/2}, \quad \mathrm{rem}_5 = \mathcal{O}(t^{-2\alpha+\frac{\kappa}{2}-a\nu_0}). \tag{2.13}$$

We have $\|\phi_t\| \leq t^{-a\nu_0}$ and therefore the first estimate in (B.2) holds with $\delta = 0$. If either $\alpha < 1$ (and t sufficiently large), or $\alpha = 1$ and $c < 1$, then $p_t := t^{-a\nu_0}(\frac{1}{ct^\alpha} - \frac{\alpha}{t})\chi'_\alpha$ is non-negative, which implies the second estimate in (B.2). Thus (B.2) holds. By the definition (2.5) and Corollary C.3 of Appendix C for $i = 1$, and by an explicit form for $i = 2, 3, 4, 5$, we have the estimates

$$\|\omega^{\rho_i/2}\,\mathrm{rem}_i\,\omega^{\rho_i/2}\| \lesssim t^{-\lambda_i}, \tag{2.14}$$

$i = 1, 2, 3, 4, 5$, with $\rho_1 = \rho_2 = \rho_3 = \rho_5 = 0$, $\rho_4 = 1$, $\lambda_1 = 2\alpha - \kappa + a\nu_0$, $\lambda_2 = 1 + a\nu_0$, $\lambda_3 = 1 + \alpha - \kappa/2 + a\nu_0$, $\lambda_4 = \alpha + \kappa + a\nu_0$, and $\lambda_5 = 2\alpha - \kappa/2 + a\nu_0$. We remark here that the $i = 2$ term is absent if $\nu_0 = 0$. The relation (2.14) implies (B.3) with $\rho = \rho_i$ and $\lambda = \lambda_i$ provided $\lambda_i > 1 + \nu_{\rho_i}$.

Finally, in the same way as [10, Lemma 3.1], one shows (by replacing $|y|$ with b_ϵ in that lemma) that, under (1.6) for some $\mu \geq -\frac{1}{2}$,

$$\big\|\eta_1\eta_2^2\chi_{\frac{b_\epsilon}{ct^\alpha}\geq 1}g(k)\big\|_{L^2(\mathbb{R}^3;\mathcal{H}_p)} \lesssim t^{-\tau}, \qquad \tau < (\frac{3}{2}+\mu)\alpha, \tag{2.15}$$

which implies (B.4) with $\lambda' < a\nu_0 + (\frac{3}{2}+\mu)\alpha$. Hence ϕ_t is a weak one-photon propagation observable, provided $2\alpha > 1 + \kappa + \nu_0 - a\nu_0$, $\alpha - \kappa/2 > \nu_0 - a\nu_0$, $\alpha + \kappa > 1 + \nu_1 - a\nu_0$, and $(\frac{3}{2}+\mu)\alpha > 1$. Therefore, by

Proposition B.2, we have, under the conditions on the parameters above,

$$\int_1^\infty dt\ t^{-\alpha-a\nu_0}\|\mathrm{d}\Gamma(\chi'_\alpha)^{\frac{1}{2}}\psi_t\|^2 \lesssim \|\psi_0\|_1^2. \tag{2.16}$$

This, due to the definition of χ'_α, implies the estimate (2.3). □

Proof of Theorem 1.1 for hamiltonians of the form (1.4)–(1.5). To prove (1.21), we use several iterations of Proposition B.4. We consider the one-parameter family of one-photon operators

$$\phi_t := t^{-a\nu_0}\chi_{w_\alpha\geq 1},$$

with $w_\alpha := \big(\frac{|y|}{c' t^\alpha}\big)^2$, $a > 1$, and $\nu_\rho \geq 0$, the same as in (1.17). We use the notation $\tilde\chi_\alpha \equiv \chi_{w_\alpha\geq 1}$. As in (2.5)–(2.6), we use the expansion

$$d\phi_t = t^{-a\nu_0}(dw_\alpha)\tilde\chi'_\alpha + \sum_{i=1}^{2}\mathrm{rem}_i, \tag{2.17}$$

$$\mathrm{rem}_1 := t^{-a\nu_0}[d\tilde\chi_\alpha - (dv)\tilde\chi'_\alpha], \quad \mathrm{rem}_2 := -a\nu_0 t^{-1}\phi_t. \tag{2.18}$$

We compute

$$dw_\alpha = \frac{2b_0}{(c't^\alpha)^2} - \frac{2\alpha w_\alpha}{t}, \tag{2.19}$$

where $b_0 = \frac{1}{2}(\nabla\omega\cdot y +\ \text{h.c.})$. Note that b_0 is not a self-adjoint operator, only maximal symmetric. Nevertheless, using Hardy's inequality, one easily verifies that b_0 is well-defined on $D(|y|)$ and that $b_0\langle y\rangle^{-1}$ and $\langle y\rangle^{-1}b_0$ extend to bounded operators. We write $b_0 = b_\epsilon + \epsilon\frac{1}{2}(\frac{1}{\omega_\epsilon}\nabla\omega\cdot y +\ \text{h.c.})$, where, recall, $\omega_\epsilon = \omega+\epsilon$, $\epsilon = t^{-\kappa}$. We choose $\kappa > 0$ satisfying

$$4\alpha - 3 > \kappa > 2 - 2\alpha + \nu_1 - \nu_0. \tag{2.20}$$

Using the notation $v = \frac{b_\epsilon}{ct^\alpha}$ and the partition of unity $\chi_{v\geq 1} + \chi_{v\leq 1} = \mathbf{1}$, we find $b_\epsilon \geq ct^\alpha + (b_\epsilon - ct^\alpha)\chi_{v\leq 1}$. Commutator estimates of the type considered in Appendix C (see Lemma C.5) give $\chi_{\frac{b_\epsilon}{\tilde c t^\alpha}\leq -1}(\tilde\chi'_\alpha)^{1/2} = \mathcal{O}(t^{-\alpha+\kappa})$ for $\tilde c > 2c'$, which, together with $b_\epsilon(\tilde\chi'_\alpha)^{1/2} = \mathcal{O}(t^\alpha)$, yields

$$(\tilde\chi'_\alpha)^{1/2} b_\epsilon \chi_{v\leq 1}(\tilde\chi'_\alpha)^{1/2} \geq -\tilde c t^\alpha(\tilde\chi'_\alpha)^{1/2}\chi_{v\leq 1}(\tilde\chi'_\alpha)^{1/2} - Ct^\kappa\tilde\chi'_\alpha.$$

This estimate, together with Lemma C.1 of Appendix C and $w_\alpha \leq 1$ on $\operatorname{supp}\tilde\chi_\alpha$, give $d\phi_t \geq p_t - \tilde p_t + \mathrm{rem}$, where

$$p_t := \frac{2}{t^{a\nu_0}}\Big(\frac{c}{(c')^2 t^\alpha} - \frac{\alpha}{t}\Big)\tilde\chi'_\alpha,$$

$$\tilde p_t := \frac{2(\tilde c + c)}{c'^2 t^{\alpha+a\nu_0}}(\tilde\chi'_\alpha)^{1/2}\chi_{v\leq 1}(\tilde\chi'_\alpha)^{1/2},$$

and $\mathrm{rem} = \sum_{i=1}^4 \mathrm{rem}_i$, with rem_1 and rem_2 given by (2.18),

$$\mathrm{rem}_3 := \frac{c}{(c't^\alpha)^2 t^{\kappa+a\nu_0}}\big(\frac{1}{\omega_\epsilon}\nabla\omega\cdot y +\ \text{h.c.}\big)\tilde\chi'_\alpha,$$

and $\mathrm{rem}_4 = \mathcal{O}(t^{-2\alpha+\kappa-a\nu_0})$. If $\alpha = 1$, then we choose $c > (c')^2$ so that $p_t \geq 0$.

As in the proof of Theorem 2.1, we deduce that the remainders rem_i, $i = 1,2,3,4$, satisfy the estimates (2.14), $i = 1,2,3,4$, with $\rho_1 = \rho_3 = 1$, $\rho_2 = \rho_4 = 0$, $\lambda_1 = 2\alpha + a\nu_0$, $\lambda_2 = 1 + a\nu_0$, $\lambda_3 = \alpha+\kappa+a\nu_0$ and $\lambda_4 = 2\alpha-\kappa+a\nu_0$. In particular, the estimate for $i = 1$ follows from Lemma C.4. Since $2\alpha > \alpha+\kappa > 1+\nu_1-a\nu_0$ and $2\alpha-\kappa > 1$, the remainder $\mathrm{rem} = \sum_{i=1}^4 \mathrm{rem}_i$ gives an integrable term. (Note that $\mathrm{rem}_2 = 0$, if $\nu_0 = 0$.)

Now, we estimate the contribution of $\tilde p_t$. Let $\gamma = 2\alpha - 1 \leq \alpha$ and decompose $\tilde p_t = p_{t1} + p_{t2}$, where

$$p_{t1} := \frac{c''}{t^{\alpha+a\nu_0}}(\tilde\chi'_\alpha)^{1/2}\chi_{c_1 t^\gamma \leq b_\epsilon \leq ct^\alpha}(\tilde\chi'_\alpha)^{1/2},$$

$$p_{t2} := \frac{c''}{t^{\alpha+a\nu_0}}(\tilde\chi'_\alpha)^{1/2}\bar\chi_\gamma(\tilde\chi'_\alpha)^{1/2},$$

with $\chi_{c_1 t^\gamma \le b_\epsilon \le ct^\alpha} \equiv \chi_\gamma \chi_{v\le 1}$, $\chi_\gamma \equiv \chi_{\frac{b_\epsilon}{c_1 t^\gamma}\ge 1}$, $\bar\chi_\gamma \equiv \chi_{\frac{b_\epsilon}{c_1 t^\gamma}\le 1}$, where $c_1 < 1$ if $\gamma = 1$ and $c_1 < \alpha(c')^2$ if $\gamma < 1$, and $c'' := 2(\tilde c + c)/(c')^2$. First, we estimate the contribution of p_{t1}. Since $[(\tilde\chi'_\alpha)^{1/2}, (\chi_{c_1 t^\gamma \le b_\epsilon \le ct^\alpha})^{1/2}] = \mathcal{O}(t^{-\gamma+\kappa})$ (see Lemma C.1 of Appendix C) and since $\alpha + \gamma - \kappa > 1$, it suffices to estimate the contribution of $c'' t^{-\alpha - a\nu_0} \chi_{c_1 t^\gamma \le b_\epsilon \le ct^\alpha}$. To this end, we use the propagation observable

$$\phi_{t1} := t^{-a\nu_0} h_\alpha \chi_\gamma, \tag{2.21}$$

where $h_\alpha \equiv h(\frac{b_\epsilon}{ct^\alpha})$, $h(\lambda) := \int_\lambda^\infty ds \chi_{s\le 1}$. As in (2.9), we have

$$h_\alpha \partial_t b_\epsilon \chi'_\gamma \le \frac{\text{const}}{t^{1-\kappa/2}}, \quad h'_\alpha \partial_t b_\epsilon \chi_\gamma \ge -\frac{\text{const}}{t^{1-\kappa/2}}. \tag{2.22}$$

Using this together with (2.5)–(2.7), we compute

$$d\phi_{t1} \le \frac{1}{ct^{\alpha+a\nu_0}}(\theta_\epsilon - \frac{\alpha b_\epsilon}{t}) h'_\alpha \chi_\gamma + \frac{1}{c_1 t^{\gamma + a\nu_0}} h_\alpha \chi'_\gamma (\theta_\epsilon - \frac{\gamma b_\epsilon}{t}) + \sum_{i=1}^3 \text{rem}'_i,$$

where rem'_1 is a sum of two terms of the form of rem_1 given in (2.5)–(2.6), with χ_α replaced by h_α, in one, and by χ_γ, in the other, $\text{rem}'_2 := \mathcal{O}(t^{-1-\gamma+\kappa/2-a\nu_0})$, and $\text{rem}'_3 := -a\nu_0 t^{-1}\phi_{t1}$. We estimate

$$\theta_\epsilon - \frac{\alpha b_\epsilon}{t} \ge 1 - \frac{1}{\omega_\epsilon t^\kappa} - \frac{\alpha c}{t^{1-\alpha}}$$

on $\operatorname{supp} h'_\alpha$ and

$$\theta_\epsilon - \frac{\gamma b_\epsilon}{t} \le 1 - \frac{1}{\omega_\epsilon t^\kappa} - \frac{\gamma c_1}{2t^{1-\gamma}}$$

on $\operatorname{supp} \chi'_\gamma$. Using $h'_\alpha \le 0$, $\chi'_\gamma \ge 0$, $h_\alpha \le 1 - \frac{b_\epsilon}{ct^\alpha}$ and $\frac{b_\epsilon}{ct^\alpha} = \mathcal{O}(t^{-\alpha+\gamma})$ on $\operatorname{supp}\chi'_\gamma$, this gives

$$d\phi_{t1} \le -p'_{t1} + \tilde p_{t1} + \text{rem}',$$

with $\text{rem}' := \sum_{i=1}^4 \text{rem}'_i$, $\text{rem}'_4 := \omega^{-1/2}\mathcal{O}(t^{-\alpha-\kappa-a\nu_0})\omega^{-1/2}$, and

$$p'_{t1} := t^{-a\nu_0}(1 - \frac{\alpha}{t}) h'_\alpha \chi_\gamma, \qquad \tilde p_{t1} := \frac{1}{c_1 t^{\gamma+a\nu_0}} \chi'_\gamma.$$

By (2.3), since $\gamma > \max((3/2+\mu)^{-1}, (1+\kappa)/2, 1-\kappa+\nu_1-\nu_0)$, the term $\tilde p_{t1}$ gives an integrable contribution. We deduce as above that the remainders rem'_i, $i = 1,2,3,4$, satisfy the estimates (2.14), $i = 1,2,3,4$, with $\rho_1 = \rho_2 = \rho_3 = 0$, $\rho_4 = 1$, $\lambda_1 = 2\gamma - \kappa + a\nu_0$, $\lambda_2 = 1 + \gamma - \kappa/2 + a\nu_0$, $\lambda_3 = 1 + a\nu_0$, and $\lambda_4 = \alpha + \kappa + a\nu_0$. Since $2\gamma - \kappa > 1$, $\gamma > \kappa/2$, and $\alpha + \kappa > 1 + \nu_1 - a\nu_0$, the remainder $\text{rem}' = \sum_i \text{rem}'_i$ is integrable. Finally, (B.4) with $\lambda' < a\nu_0 + (\frac32 + \mu)\gamma$ holds by the inequality (2.15). Hence, ϕ_{t1} is a strong one-photon propagation observable and therefore we have the estimate

$$\int_1^\infty dt\, \|\mathrm{d}\Gamma(p_{t1})^{\frac12}\psi_t\|^2 \lesssim \int_1^\infty dt\, \|\mathrm{d}\Gamma(p'_{t1})^{\frac12}\psi_t\|^2 \lesssim \|\psi_0\|_1^2. \tag{2.23}$$

(In fact, by multiplying the observable (2.21) by t^δ for an appropriate $\delta > 0$, we can obtain a stronger estimate.)

Now, we consider p_{t2}. Recall the notations $\tilde\chi_\alpha \equiv \chi_{w_\alpha \ge 1}$, $w_\alpha = \big(\frac{|y|}{c' t^\alpha}\big)^2$, and let $h_\gamma \equiv h(v_\gamma)$, with $h(\lambda) = \int_\lambda^\infty ds\, \chi_{s\le 1}$ and $v_\gamma = \frac{b_\epsilon}{c_1 t^\gamma}$. We use the propagation observable

$$\phi_{t2} := t^{-a\nu_0}(\tilde\chi_\alpha h_\gamma + h_\gamma \tilde\chi_\alpha). \tag{2.24}$$

Using (2.7), (2.8), (2.19), $b = b_\epsilon + \epsilon\frac12(\frac{1}{\omega_\epsilon}\nabla\omega\cdot y + \text{ h.c.})$, $b_\epsilon \le c_1 t^\gamma$ on $\operatorname{supp}\chi_{v_\gamma\le 1}$, $\gamma = 2\alpha - 1$ and $[(\tilde\chi'_\alpha)^{1/2}, h_\gamma] = \mathcal{O}(t^{-\gamma+\kappa})$ (see Lemma C.1 of Appendix C), we compute

$$d\phi_{t2} \le t^{-a\nu_0}\Big(\big(\frac{c_1}{(c')^2} - \alpha\big)\frac{2}{t}(\tilde\chi'_\alpha)^{1/2} h_\gamma (\tilde\chi'_\alpha)^{1/2} + \tilde\chi_\alpha h'_\gamma (dv_\gamma) + (dv_\gamma) h'_\gamma \tilde\chi_\alpha \Big) + \sum_{i=1}^4 \text{rem}''_i,$$

where $dv_\gamma = \frac{\theta_\epsilon}{c_1 t^\gamma} - \frac{\gamma b_\epsilon}{c_1 t^{\gamma+1}}$, rem''_1 is a term of the form of rem_1 given in (2.6), with χ_α replaced by $\tilde\chi_\alpha$, likewise, rem''_2 is a term of the form of rem_1 given in (2.6), with χ_α replaced by h_γ, $\text{rem}''_3 = \mathcal{O}(t^{-1-\gamma+\kappa/2-a\nu_0})$ and $\text{rem}''_4 := -a\nu_0 t^{-1}\phi_{t2}$. To estimate dv_γ, we use that $\tilde\chi'_\alpha \ge 0$, $h'_\gamma \le 0$, $\theta_\epsilon = 1 - t^{-\kappa}\omega_\epsilon^{-1}$, $v_\gamma h'_\gamma \le h'_\gamma$, and

$$\tilde\chi_\alpha h'_\gamma (dv_\gamma) + (dv_\gamma) h'_\gamma \tilde\chi_\alpha = -\tilde\chi_\alpha^{1/2}(-h'_\gamma)^{1/2}(dv_\gamma)(-h'_\gamma)^{1/2}\tilde\chi_\alpha^{1/2} + \mathcal{O}(t^{-\gamma+\kappa})$$

(see again Lemma C.1 of Appendix C), to obtain

$$d\phi_{t2} \le -p'_{t2} + \mathrm{rem}'',$$

with $\mathrm{rem}'' := \sum_{i=1}^{6} \mathrm{rem}''_i$, $\mathrm{rem}''_5 = \mathcal{O}(t^{-2\gamma+\kappa-a\nu_0})$, $\mathrm{rem}''_6 = \omega^{-1/2}\mathcal{O}(t^{-\gamma-\kappa-a\nu_0})\omega^{-1/2}$ and (at least for t sufficiently large)

$$p'_{t2} := t^{-a\nu_0}\Big[-(\frac{2c_1}{(c')^2} - 2\alpha)\frac{1}{t}(\tilde\chi'_\alpha)^{1/2} h_\gamma (\tilde\chi'_\alpha)^{1/2} + (1 - \frac{\gamma c_1}{t^{1-\gamma}})\frac{1}{c_1 t^\gamma}\tilde\chi_\alpha^{1/2} h'_\gamma \tilde\chi_\alpha^{1/2}\Big].$$

Since $\frac{c_1}{(c')^2} < \alpha$ and either $\gamma < 1$, or $\gamma = 1$ and $c_1 < 1$, and $\tilde\chi'_\alpha \ge 0$ and $h'_\gamma \le 0$, both terms in the square braces on the r.h.s. are non-positive. We deduce as above that the remainders rem''_i, $i = 1, \dots, 6$, satisfy the estimates (2.14), $i = 1, \dots, 6$, with $\rho_1 = \rho_6 = 1$, $\rho_2 = \rho_3 = \rho_4 = \rho_5 = 0$, $\lambda_1 = 2\alpha + a\nu_0$, $\lambda_2 = \lambda_5 = 2\gamma - \kappa + a\nu_\delta$, $\lambda_3 = 1 + \gamma - \kappa/2 + a\nu_0$, $\lambda_4 = 1 + a\nu_0$, $\lambda_6 = \gamma + \kappa + a\nu_0$. Since $2\alpha > \gamma + \kappa > 1 + \nu_1 - a\nu_0$, $2\gamma - \kappa > 1$ and $\gamma > \kappa/2$, the condition (B.3) is satisfied. Moreover, (B.4) with $\lambda' < a\nu_0 + (\frac{3}{2} + \mu)\alpha$ holds by (2.15). Therefore ϕ_{t2} is a strong one-photon propagation observable and we have the estimate

$$\int_1^\infty dt\, \|\mathrm{d}\Gamma(p_{t2})^{\frac12}\psi_t\|^2 \lesssim \int_1^\infty dt\, \|\mathrm{d}\Gamma(p'_{t2})^{\frac12}\psi_t\|^2 \lesssim \|\psi_0\|_1^2. \tag{2.25}$$

(Again, by multiplying the observable (2.24) by t^δ for an appropriate $\delta > 0$, we can obtain a stronger estimate.)

Since $\tilde p_t = p_{t1} + p_{t2}$, estimates (2.23) and (2.25) imply the estimate

$$\int_1^\infty dt\, \|\mathrm{d}\Gamma(p_t)^{\frac12}\psi_t\|^2 \lesssim \|\psi_0\|_1^2, \tag{2.26}$$

which due to $\tilde\chi'_\alpha \equiv \chi_{w_\alpha=1}$, implies the estimate (1.21). □

3. The second propagation estimate

In this section we to prove the minimal velocity estimates of Theorem 1.2 for hamiltonians of the form (1.4)–(1.5), with the coupling operators $g(k)$ satisfying (1.6) and (1.7). Recall the norm $\langle g\rangle = \sum_{|\alpha|\le 2} \|\eta_1\eta_2^{|\alpha|}\partial^\alpha g\|_{L^2(\mathbb{R}^3,\mathcal{H}_p)}$ for the coupling function g and the notation $\langle A\rangle_\psi = \langle\psi, A\psi\rangle$. We begin with a technical result, proven in [21], which proves these estimates for the operator b_ϵ, which is defined in the introduction, instead of $|y|$.

Theorem 3.1. *Consider hamiltonians of the form* (1.4)–(1.5) *with the coupling operators satisfying* (1.6) *with* $\mu > -1/2$ *and* (1.7). *Assume that* (1.14) *holds. Let* $\langle g\rangle$ *be sufficiently small,* $\nu_1 < \kappa < 1$, *and* $0 < \alpha < 1$. *Let* $\psi_0 \in \Upsilon_\#$. *Then the Schrödinger evolution,* ψ_t, *satisfies the estimate*

$$\big\|\Gamma(\chi_{\frac{b_\epsilon}{ct^\alpha}\le 1})^{\frac12}\psi_t\big\|^2 \lesssim t^{-\delta}\big(\|\psi_0\|^2_{\mathrm{d}\Gamma(\langle y\rangle)} + \|\psi_0\|^2_{\mathrm{d}\Gamma(b)^2}\big), \tag{3.1}$$

for $0 \le \delta < \min(\kappa - \nu_1, 1 - \kappa, 1 - \alpha - \nu_0)$ *and any* $c > 0$, *where, recall,* $b = \frac12(k\cdot y + y\cdot k)$.

Proof of Theorem 1.2 for hamiltonians of the form (1.4)–(1.5). Recall the notations $v = \frac{b_\epsilon}{ct^\alpha}$ and $w_\alpha = \big(\frac{|y|}{c't^\alpha}\big)^2$. To prove (1.22), we begin with the following estimate, proven in the localization lemma C.5 of Appendix C:

$$\chi_{v\ge 1}\chi_{w_\alpha\le 1} = \mathcal{O}(t^{-(\alpha-\kappa)}), \tag{3.2}$$

for $\epsilon = t^{-\kappa}$, $\kappa < \alpha$, and $c' < c/2$. Now, let $\chi^2_{v\le 1} + \chi^2_{v\ge 1} = \mathbf{1}$ and write

$$\chi^2_{w_\alpha\le 1} = \chi_{v\le 1}\chi^2_{w_\alpha\le 1}\chi_{v\le 1} + R \le \chi^2_{v\le 1} + R, \tag{3.3}$$

where $R := \chi_{v\le 1}\chi^2_{w_\alpha\le 1}\chi_{v\ge 1} + \chi_{v\ge 1}\chi^2_{w_\alpha\le 1}\chi_{v\le 1} + \chi_{v\ge 1}\chi^2_{w_\alpha\le 1}\chi_{v\ge 1}$. The estimates (3.2) and (3.3) give

$$\chi^2_{w_\alpha\le 1} \le \chi^2_{v\le 1} + \mathcal{O}(t^{-(\alpha-\kappa)}), \tag{3.4}$$

which in turn implies

$$\big\|\Gamma(\chi_{w_\alpha\le 1})^{\frac12}\psi\big\| \lesssim \big\|\Gamma(\chi_{v\le 1})^{\frac12}\psi\big\| + Ct^{-(\alpha-\kappa)/2}\big\|(N+\mathbf{1})^{\frac12}\psi\big\|. \tag{3.5}$$

Choosing $\kappa = (\alpha + \nu_1 - \nu_0)/2$, the estimate (3.5), together with (3.1), yields (1.22). □

4. Proof of Theorem 1.3: the model (1.23)–(1.24)

In this section we extend the results of Sections 2–3 to hamiltonians of the form (1.23)–(1.24), satisfying (1.25) –(1.26) and (1.7), and prove Theorem 1.3. First, to extend the results of Appendix B to the present case, we replace the assumption (B.4) by the assumptions

$$\begin{cases} \Big(\displaystyle\int \|\eta_1\eta_2^2(\tilde\phi_t g)_{ij}(k)\|^2_{\mathcal{H}_p}\omega(k)^\delta dk\Big)^{\frac12} \lesssim \langle t\rangle^{-\lambda'},\ i+j=1, \\ \Big(\displaystyle\int \|\eta_2^2(\tilde\phi_t g)_{ij}(k_1,k_2)\|^2_{\mathcal{H}_p}\prod_{\ell=1,2}(1+\omega(k_\ell)^{-\frac12}+\omega(k_\ell)^\delta)dk_\ell\Big)^{\frac12} \lesssim \langle t\rangle^{-\lambda'},\ i+j=2, \end{cases} \tag{4.1}$$

where λ' is the same as in (B.4) and, for any one-particle operator ϕ acting on $\mathfrak{h}$, we define $(\tilde\phi g)_{ij} := \phi g_{ij}$, for $i+j=1$, and $(\tilde\phi g)_{2,0} = (\tilde\phi g)^*_{0,2} := (\phi\otimes\mathbf{1}+\mathbf{1}\otimes\phi)g_{2,0}$, $(\tilde\phi g)_{1,1} := (\phi\otimes\mathbf{1}-\mathbf{1}\otimes\phi)g_{1,1}$. Then we replace the second relation in (B.9) by the relation (see Supplement I)

$$i[\tilde I(g),\mathrm{d}\Gamma(\phi_t)] = -\tilde I(i\tilde\phi g), \tag{4.2}$$

which is valid for any one-particle operator ϕ, and replace the estimate (B.11) by the estimate

$$\begin{aligned} |\langle\tilde I(g)\rangle_\psi| \le{}& \sum_{i+j=1}\Big(\int \|\eta_1\eta_2^2 g_{ij}(k)\|^2_{\mathcal{H}_p}\omega(k)^\delta dk\Big)^{\frac12}\|\eta_1^{-1}\eta_2^{-2}\psi\|\|\psi\|_\delta \\ &+\sum_{i+j=2}\Big(\int \|\eta_2^2 g_{ij}(k_1,k_2)\|^2_{\mathcal{H}_p}\prod_{\ell=1,2}(1+\omega(k_\ell)^{-1}+\omega(k_\ell)^\delta)dk_\ell\Big)^{\frac12}(\|\eta_2^{-4}\psi\|+\|\psi\|_{-1})\|\psi\|_\delta, \end{aligned} \tag{4.3}$$

which, as in (B.11), implies, together with (4.1) and (1.7),

$$|\langle\tilde I(i\tilde\phi_t g)\rangle_{\psi_t}| \lesssim t^{-\lambda'+\nu_\delta}\|\psi_0\|^2_\delta, \tag{4.4}$$

for any $\psi_0\in\Upsilon_\delta$, where Υ_δ is defined in (B.6). This completes the extension of the results of Section B, and therefore of Section 2, to hamiltonians of the form (1.23)–(1.24).

5. Proof of Theorems 1.1–1.2 for the QED model

5.1. **Generalized Pauli–Fierz transformation.** We consider the QED hamiltonian defined in (1.1)–(1.2). The coupling function $g^{\rm qed}_y(k,\lambda) := |k|^{-1/2}\xi(k)\varepsilon_\lambda(k)e^{ik\cdot y}$ in this hamiltonian is more singular in the infrared than can be handled by our techniques ($\mu>0$). To go around this problem we use the (unitary) generalized Pauli–Fierz transformation (see [55])

$$H \longrightarrow \tilde H := e^{-i\sum_{j=1}^n\kappa_j\Phi(q_{x_j})}He^{i\sum_{j=1}^n\kappa_j\Phi(q_{x_j})}, \tag{5.1}$$

where $\Phi(h)$ is the operator-valued field, $\Phi(h) := \frac{1}{\sqrt2}(a^*(h)+a(h))$, and the function $q_y(k,\lambda)$ is defined below, to pass to the new unitarily equivalent hamiltonian $\tilde H$.

To define $q_y(k,\lambda)$, let $\varphi\in \mathrm{C}^\infty(\mathbb{R};\mathbb{R})$ be a non-decreasing function such that $\varphi(r)=r$ if $|r|\le 1/2$ and $|\varphi(r)|=1$ if $|r|\ge 1$. For $0<\nu<1/2$, we define

$$q_y(k,\lambda) := \frac{\xi(k)}{|k|^{\frac12+\nu}}\varphi(|k|^\nu\varepsilon_\lambda(k)\cdot y). \tag{5.2}$$

We note that the definition of $\Phi(h)$ gives $A(y)=\Phi(g^{\rm qed}_y)$. Using (I.7) and (I.8) of Supplement I, we compute

$$\tilde H = \sum_{j=1}^n\frac{1}{2m_j}\big(-i\nabla_{x_j}-\kappa_j\tilde A(x_j)\big)^2 + E(x)+H_f+V(x), \tag{5.3}$$

where, recall, $x=(x_1,\dots,x_n)$, and

$$\begin{cases} \tilde A(y) := \Phi(\tilde g_y), \quad \tilde g_y(k,\lambda) := g^{\rm qed}_y(k,\lambda)-\nabla_x q_y(k,\lambda), \\ E(x) := -\sum_{j=1}^n\kappa_j\Phi(e_{x_j}), \quad e_y(k,\lambda) := i|k|q_y(k,\lambda), \\ V(x) := U(x)+\frac12\sum_{\lambda=1,2}\sum_{j=1}^n\kappa_j^2\int_{\mathbb{R}^3}|k||q_{x_j}(k,\lambda)|^2dk. \end{cases} \tag{5.4}$$

The operator $\tilde H$ is self-adjoint with domain $D(\tilde H)=D(H)=D(p^2+H_f)$ (see [38, 39]).

Now, the coupling functions (form factors) $\tilde{g}_x(k,\lambda)$ and $e_x(k,\lambda)$ in the transformed hamiltonian, $\tilde{H}$, satisfy the estimates that are better behaved in the infrared ([10]):

$$|\partial_k^m \tilde{g}_y(k,\lambda)| \lesssim \langle k\rangle^{-3}|k|^{\frac{1}{2}-|m|}\langle x\rangle^{\frac{1}{\nu}+|m|}, \tag{5.5}$$

$$|\partial_k^m e_y(k,\lambda)| \lesssim \langle k\rangle^{-3}|k|^{\frac{1}{2}-|m|}\langle x\rangle^{1+|m|}. \tag{5.6}$$

We see that the new hamiltonian (5.3) is of the form

$$\widetilde{H} = H_p + H_f + \tilde{I}(g), \tag{5.7}$$

with $H_p := -\sum_{j=1}^n \frac{1}{2m_j}\Delta_{x_j} + V(x)$, and $\tilde{I}(g) := -\sum_{j=1}^n \kappa_j(p_j\cdot\widetilde{A}(x_j) + \widetilde{A}(x_j)\cdot p_j - \kappa_j\widetilde{A}(x_j)^2) + E(x)$. We see that the latter operator is of the form (1.24)–(1.26), with $\eta_1 = \langle p\rangle^{-1}$, $\eta_2 = \langle x\rangle^{-1-1/\nu}$, $\mu = 1/2$, $|\alpha| \le 2$, and $1 \le i+j \le 2$, where $p := (p_1,\dots,p_n)$, and therefore the hamiltonian (1.1) satisfy the bound (1.8) and is of the class described in the introduction.

5.2. **Proof of Theorems 1.1–1.2 for the QED model.** We have shown the statements of Theorems 1.1 and 1.2 for hamiltonians of the form (1.23)–(1.26), with the operators η_j, $j = 1,2$, satisfying (1.7), and therefore for the operator (5.3). To translate Theorems 1.1 and 1.2 from $\tilde{H}$, given by (5.3), to the QED hamilonian (1.1), we use the following estimates ([10])

$$\left\|\mathrm{d}\Gamma(\chi_1(w))^{\frac{1}{2}}\psi\right\|^2 \lesssim \left\langle \mathcal{U}\psi, \mathrm{d}\Gamma(\chi_1(w))\mathcal{U}\psi\right\rangle + t^{-\alpha d}\|\psi\|^2, \tag{5.8}$$

$$\left\|\Gamma(\chi_2(w))^{\frac{1}{2}}\psi\right\|^2 \lesssim \left\langle \mathcal{U}\psi, \Gamma(\chi_2(w))\mathcal{U}\psi\right\rangle + t^{-\alpha d}\|\psi\|^2, \tag{5.9}$$

where $\mathcal{U} := e^{-i\sum_{j=1}^n \kappa_j\Phi(q_{x_j})}$ and $w := \frac{y}{ct^\alpha}$, valid for any functions $\chi_1(w)$ and $\chi_2(w)$ supported in $\{|w| \le \epsilon\}$ and $\{|w| \ge \epsilon\}$, respectively, for some $\epsilon > 0$, for any $\psi \in f(H)D(N^{1/2})$, with $f \in \mathrm{C}_0^\infty((-\infty,\Sigma))$, and for $0 \le d < 1/2$. (5.8) follows from estimates of Section 2 of [10] and (5.9) can be obtained similarly (see (I.8) and (I.9)). Using these estimates for $\psi_t = e^{-itH}\psi_0$, with an initial condition ψ_0 in either Υ_1 or $\Upsilon_\#$, together with $\mathcal{U}e^{-itH}\psi_0 = e^{-it\tilde{H}}\mathcal{U}\psi_0$, and applying Theorems 1.1 and 1.2 for $\tilde{H}$ to the first terms on the r.h.s., we see that, to obtain Theorems 1.1 and 1.2 for the hamiltonian (1.1), we need, in addition, the estimates

$$\left\langle \psi, \mathcal{U}^* N_1 \mathcal{U}\psi\right\rangle \lesssim \langle\psi, (N_1 + \mathbf{1})\psi\rangle, \tag{5.10}$$

$$\left\langle \psi, \mathcal{U}^*\mathrm{d}\Gamma(\langle y\rangle)\mathcal{U}\psi\right\rangle \lesssim \langle\psi, (\mathrm{d}\Gamma(\langle y\rangle) + \langle x\rangle^2)\psi\rangle, \tag{5.11}$$

$$\|\mathcal{U}^*\mathrm{d}\Gamma(b)\mathcal{U}\psi\| \lesssim \|(\mathrm{d}\Gamma(b) + \langle x\rangle^2)\psi\|, \tag{5.12}$$

where, recall, $N_1 = \mathrm{d}\Gamma(\omega^{-1})$ and $b = \frac{1}{2}(k\cdot y + y\cdot k)$.

Let $q_x := \sum_{j=1}^n \kappa_j q_{x_j}$ so that $\mathcal{U} := e^{-i\Phi(q_x)}$. To prove (5.10), we see that, by (I.8) of Supplement I, we have

$$\mathcal{U}^* N_1 \mathcal{U} = e^{i\Phi(q_x)}\mathrm{d}\Gamma(\omega^{-1})e^{-i\Phi(q_x)} = N_1 - \Phi(i\omega^{-1}q_x) + \frac{1}{2}\|\omega^{-1/2}q_x\|_{\mathfrak{h}}^2. \tag{5.13}$$

(Since $\omega^{-1}q_x \notin \mathfrak{h}$, the field operator $\Phi(i\omega^{-1}q_x)$ is not well-defined and therefore this formula should be modified by introducing, for instance, an infrared cutoff parameter σ into q_x. One then removes it at the end of the estimates. Since such a procedure is standard, we omit it here.) This relation, together with

$$|\langle\psi, \Phi(i\omega^{-1}q_x)\psi\rangle| \lesssim \Big(\int \omega^{-3-2\nu+\varepsilon}\langle k\rangle^{-6}dk\Big)^{\frac{1}{2}}\|\mathrm{d}\Gamma(\omega^{-\varepsilon})^{\frac{1}{2}}\psi\|\|\psi\|, \tag{5.14}$$

for any $\varepsilon > 0$, which follows from the bounds of Lemma I.1 of Supplement I, and

$$\|\omega^{-\frac{1}{2}}q_x\|_{\mathfrak{h}} \lesssim \|\omega^{-1-\nu}\langle k\rangle^{-3}\|_{\mathfrak{h}}, \tag{5.15}$$

implies (5.10).

To prove (5.11) and (5.12), we proceed similarly, using, instead of (5.14) and (5.15), the estimates

$$\begin{aligned}|\langle\psi, \Phi(i\langle y\rangle q_x)\psi\rangle| &\lesssim \Big(\int \omega^{-2-2\nu}\langle k\rangle^{-6}dk\Big)^{\frac{1}{2}}\|\mathrm{d}\Gamma(\omega^{-1})^{\frac{1}{2}}\psi\|\|\langle x\rangle\psi\| \\ &\lesssim \Big(\int \omega^{-2-2\nu}\langle k\rangle^{-6}dk\Big)^{\frac{1}{2}}\|\mathrm{d}\Gamma(\langle y\rangle)^{\frac{1}{2}}\psi\|\|\langle x\rangle\psi\|,\end{aligned} \tag{5.16}$$

and

$$\|\langle y\rangle^{\frac{1}{2}} q_x\|_{\mathfrak{h}} \lesssim \langle x\rangle^{\frac{1}{2}} \|\omega^{-1-\nu}\langle k\rangle^{-3}\|_{\mathfrak{h}}, \tag{5.17}$$

and

$$\|\Phi(ibq_x)\psi\| \lesssim \Big(\int \omega^{-2-2\nu}\langle k\rangle^{-6} dk\Big)^{\frac{1}{2}} \|\langle x\rangle (H_f+1)^{\frac{1}{2}}\psi\|, \tag{5.18}$$

and

$$\langle q_x, bq_x\rangle_{\mathfrak{h}} \lesssim \langle x\rangle \|\omega^{-\frac{1}{2}-\nu}\langle k\rangle^{-3}\|^2_{\mathfrak{h}}. \tag{5.19}$$

Appendix A. Photon # and low momentum estimate

For simplicity, consider hamiltonians of the form (1.4)–(1.5), with the coupling operators $g(k)$ satisfying (1.6) and (1.7) with $\mu > -1/2$. The extension to hamiltonians of the form (1.23)–(1.24) is done along the lines of Section 4. Recall the notations $\langle A\rangle_\psi = \langle \psi, A\psi\rangle$, $N_\rho = \mathrm{d}\Gamma(\omega^{-\rho})$ and $\Upsilon_\rho = \{\psi_0 \in f(H)D(N_\rho^{1/2}),\ \text{for some } f \in \mathrm{C}_0^\infty((-\infty,\Sigma))\}$. The idea of the proof of the following estimate follows [32] and [10].

Proposition A.1. *Let $\rho \in [-1,1]$. For any $\psi_0 \in \Upsilon_\rho$,*

$$\langle N_\rho\rangle_{\psi_t} \lesssim t^{\nu_\rho}\|\psi_0\|^2_\rho, \quad \nu_\rho = \frac{1+\rho}{2+\mu}. \tag{A.1}$$

Proof. Decompose $N_\rho = K_1 + K_2$, where

$$K_1 := \mathrm{d}\Gamma(\omega^{-\rho}\chi_{t^\alpha\omega\le 1}) \qquad \text{and} \qquad K_2 := \mathrm{d}\Gamma(\omega^{-\rho}\chi_{t^\alpha\omega\ge 1}).$$

Then, by (1.19),

$$\langle K_2\rangle_\psi \le \langle \mathrm{d}\Gamma(t^{\alpha(1+\rho)}\omega\chi_{t^\alpha\omega\ge 1})\rangle_{\psi_t} \le t^{\alpha(1+\rho)}\langle H_f\rangle_{\psi_t} \lesssim t^{\alpha(1+\rho)}\|\psi_0\|. \tag{A.2}$$

On the other hand, we have by (B.10),

$$DK_1 = \mathrm{d}\Gamma(\alpha\omega^{1+\rho}t^{\alpha-1}\chi'_{t^\alpha\omega\le 1}) - I(i\omega^{-\rho}\chi_{t^\alpha\omega\le 1}g). \tag{A.3}$$

Since $\|\eta_1 g(k)\|_{\mathcal{H}_p} \lesssim |k|^\mu\langle k\rangle^{-2-\mu}$ (see (1.6)), we obtain

$$\int dk\, \omega(k)^{-2\rho}\chi_{t^\alpha\omega\le 1}\|g(k)\|^2_{\mathcal{H}_p}(\omega(k)^{-1}+1) \lesssim t^{-2(1+\mu-\rho)\alpha}. \tag{A.4}$$

This together with (B.11) and (1.19) gives

$$|\langle I(i\omega^{-\rho}\chi_{t^\alpha\omega\le 1}g)\rangle_{\psi_t}| \lesssim t^{-(1+\mu-\rho)\alpha}\|\psi_0\|^2. \tag{A.5}$$

Hence, by (A.3), since $\partial_t\langle K_1\rangle_{\psi_t} = \langle DK_1\rangle_{\psi_t}$, $\chi'_{t^\alpha\omega\le 1} \le 0$, we obtain

$$\partial_t\langle K_1\rangle_{\psi_t} \lesssim t^{-(1+\mu-\rho)\alpha}\|\psi_0\|^2,$$

and therefore

$$\langle K_1\rangle_{\psi_t} \le Ct^{\nu'}\|\psi_0\|^2 + \langle N_\rho\rangle_{\psi_0}, \tag{A.6}$$

where $\nu' = 1-(1+\mu-\rho)\alpha$, if $(1+\mu-\rho)\alpha < 1$ and $\nu' = 0$, if $(1+\mu-\rho)\alpha > 1$. Estimates (A.6) and (A.2) with $\alpha = \frac{1}{2+\mu}$, if $\rho > -1$, give (A.1). The case $\rho = -1$ follows from (1.19). □

Remark. A minor modification of the proof above give the following bound for $\rho > 0$ and $\nu'_\rho := \frac{\rho}{\frac{3}{2}+\mu}$,

$$\langle N_\rho\rangle_{\psi_t} \lesssim t^{\nu'_\rho}\big(\|\psi_t\|^2_N + \|\psi_0\|^2\big) + \langle N_\rho\rangle_{\psi_0}. \tag{A.7}$$

Corollary A.2. *For any $\psi_0 \in \Upsilon_\rho$, $\gamma \ge 0$ and $c > 0$,*

$$\|\chi_{N_\rho\ge ct^\gamma}\psi_t\|^2 \lesssim t^{-\frac{\gamma}{2}+\frac{1+\rho}{2(2+\mu)}}\|\psi_0\|^2 + t^{-\frac{\gamma}{2}}\langle N_\rho\rangle_{\psi_0}. \tag{A.8}$$

Proof. We have

$$\|\chi_{N_\rho\ge ct^\gamma}\psi_t\| \le c^{-\frac{\gamma}{2}}t^{-\frac{\gamma}{2}}\ \|\chi_{N_\rho\ge ct^\gamma}K_\rho^{\frac{1}{2}}\psi_t\| \le c^{-\frac{\gamma}{2}}t^{-\frac{\gamma}{2}}\ \|N_\rho^{\frac{1}{2}}\psi_t\|$$

Now applying (A.1) we arrive at (A.8). □

Corollary A.3. *Let* $\psi_0 \in \Upsilon_1$. *Then* $\psi_0 \in D(N)$ *and*

$$\langle N^2 \rangle_{\psi_t} \lesssim t^{\frac{2}{2+\mu}} \|\psi_0\|_1^2. \tag{A.9}$$

Proof. By the Cauchy-Schwarz inequality, we have $N^2 \leq \mathrm{d}\Gamma(\omega)\mathrm{d}\Gamma(\omega^{-1}) = H_f N_1$, and hence

$$\begin{aligned}\langle N^2 \rangle_{\psi_t} &\leq \langle N_1^{\frac{1}{2}} H_f N_1^{\frac{1}{2}} \rangle_{\psi_t} \\ &= \langle N_1^{\frac{1}{2}} H_f (H - E_{\mathrm{gs}} + 1)^{-1} N_1^{\frac{1}{2}} (H - E_{\mathrm{gs}} + 1) \rangle_{\psi_t} \\ &\quad + \langle N_1^{\frac{1}{2}} H_f [N_1^{\frac{1}{2}}, (H - E_{\mathrm{gs}} + 1)^{-1}] (H - E_{\mathrm{gs}} + 1) \rangle_{\psi_t}.\end{aligned}$$

Under the assumption (1.6) with $\mu > 0$, one verifies that $H_f[N_1^{\frac{1}{2}}, (H - E_{\mathrm{gs}} + 1)^{-1}]$ is bounded. Since $H_f(H - E_{\mathrm{gs}} + 1)^{-1}$ is also bounded, we obtain

$$\langle N^2 \rangle_{\psi_t} \lesssim \|N_1^{\frac{1}{2}} \psi_t\| \big(\|N_1^{\frac{1}{2}} (H - E_{\mathrm{gs}} + 1)\psi_t\| + \|(H - E_{\mathrm{gs}} + 1)\psi_t\| \big). \tag{A.10}$$

Applying Proposition A.1 gives

$$\|N_1^{\frac{1}{2}} \psi_t\| \lesssim t^{\frac{1}{2+\mu}} \|\psi_0\| + \|N_1^{\frac{1}{2}} \psi_0\|, \tag{A.11}$$

and

$$\begin{aligned}\|N_1^{\frac{1}{2}} (H - E_{\mathrm{gs}} + 1)\psi_t\| &\lesssim t^{\frac{1}{2+\mu}} \|\psi_0\| + \|N_1^{\frac{1}{2}} (H - E_{\mathrm{gs}} + 1)\psi_0\| \\ &\lesssim t^{\frac{1}{2+\mu}} \|\psi_0\| + \|N_1^{\frac{1}{2}} \psi_0\|,\end{aligned} \tag{A.12}$$

where we used in the last inequality that $N_1^{\frac{1}{2}} \tilde{f}(H)(N_1 + \mathbf{1})^{-\frac{1}{2}}$ is bounded for any $\tilde{f} \in \mathrm{C}_0^\infty(\mathbb{R}^3)$. Combining (A.10), (A.11) and (A.12), we obtain

$$\langle N^2 \rangle_{\psi_t} \lesssim t^{\frac{2}{2+\mu}} (\|N_1^{\frac{1}{2}} \psi_0\|^2 + \|\psi_0\|^2). \tag{A.13}$$

Hence (A.9) is proven. □

Appendix B. Method of propagation observables

Many steps of our proof the minimal velocity estimates use the method of propagation observables which we formalize in what follows. Let $\psi_t = e^{-itH}\psi_0$, where H is a hamiltonian of the form (1.4)–(1.5), with the coupling operator $g(k)$ satisfying (1.6). The method reduces propagation estimates for our system say of the form

$$\int_0^\infty dt \left\| G_t^{\frac{1}{2}} \psi_t \right\|^2 \lesssim \|\psi_0\|_\#^2, \tag{B.1}$$

for some norm $\|\cdot\|_\# \geq \|\cdot\|$, to differential inequalities for certain families ϕ_t of positive, one-photon operators on the one-photon space $L^2(\mathbb{R}^3)$. Let

$$d\phi_t := \partial_t \phi_t + i[\omega, \phi_t],$$

and let $\nu_\rho \geq 0$ be determined by the estimate (1.17). We isolate the following useful class of families of positive, one-photon operators:

Definition B.1. A family of positive operators ϕ_t on $L^2(\mathbb{R}^3)$ will be called a *one-photon weak propagation observable*, if it has the following properties

- there are $\delta \geq 0$ and a family p_t of non-negative operators, such that

$$\|\omega^{\delta/2} \phi_t \omega^{\delta/2}\| \lesssim \langle t \rangle^{-\nu_\delta} \quad \text{and} \quad d\phi_t \geq p_t + \sum_{\text{finite}} \mathrm{rem}_i, \tag{B.2}$$

  where rem_i are one-photon operators satisfying

$$\|\omega^{\rho_i/2}\, \mathrm{rem}_i\, \omega^{\rho_i/2}\| \lesssim \langle t \rangle^{-\lambda_i}, \tag{B.3}$$

  for some ρ_i and λ_i, s.t. $\lambda_i > 1 + \nu_{\rho_i}$,

- for some $\lambda' > 1 + \nu_\delta$ and with η_1, η_2 satisfying (1.7),

$$\Big(\int \|\eta_1 \eta_2^2 (\phi_t g)(k)\|^2_{\mathcal{H}_p} \omega(k)^\delta dk \Big)^{\frac{1}{2}} \lesssim \langle t \rangle^{-\lambda'}. \tag{B.4}$$

(Here ϕ_t acts on g as a function of k.)

Similarly, a family of operators ϕ_t on $L^2(\mathbb{R}^3)$ will be called a *one-photon strong propagation observable*, if

$$d\phi_t \le -p_t + \sum_{\text{finite}} \mathrm{rem}_i, \tag{B.5}$$

with $p_t \ge 0$, rem_i are one-photon operators satisfying (B.3) for some $\lambda_i > 1 + \nu_{\rho_i}$, and (B.4) holds for some $\lambda' > 1 + \nu_\delta$.

Recall the notations $N_\rho = \mathrm{d}\Gamma(\omega^{-\rho})$ and

$$\Upsilon_\rho = \big\{ \psi_0 \in f(H) D(N_\rho^{\frac{1}{2}}), \text{ for some } f \in \mathrm{C}_0^\infty((-\infty, \Sigma)) \big\}. \tag{B.6}$$

Notice that, since $N_{-1} f(H) = H_f f(H)$ is bounded, one easily verifies that $\Upsilon_\rho \subset \Upsilon_{\rho'}$ for $\rho \ge \rho' \ge -1$. The following proposition reduces proving inequalities of the type of (B.1) to showing that ϕ_t is a one-photon weak or strong propagation observable, i.e. to *one-photon estimates* of $d\phi_t$ and $\phi_t g$.

Proposition B.2. *If ϕ_t is a one-photon weak (resp. strong) propagation observable, then we have either the weak propagation estimate,* (B.1)*, or the strong propagation estimate,*

$$\langle \psi_t, \Phi_t \psi_t \rangle + \int_0^\infty dt \, \big\| G_t^{\frac{1}{2}} \psi_t \big\|^2 \lesssim \|\psi_0\|^2_\#, \tag{B.7}$$

with the norm $\|\psi_0\|^2_\# := \|\psi_0\|^2_\diamond + \|\psi_0\|^2_*$, where $\Phi_t := \mathrm{d}\Gamma(\phi_t)$, $G_t := \mathrm{d}\Gamma(p_t)$, $\|\psi_0\|_* := \|\psi_0\|_\delta$ and $\|\psi_0\|_\diamond := \sum \|\psi_0\|_{\rho_i}$, on the subspace $\Upsilon_{\max(\delta, \rho_i)}$.

Before proceeding to the proof we present some useful definitions. Consider families Φ_t of operators on $\mathcal{H}$ and introduce the Heisenberg derivative

$$D\Phi_t := \partial_t \Phi_t + i\big[H, \Phi_t\big],$$

with the property

$$\partial_t \langle \psi_t, \Phi_t \psi_t \rangle = \langle \psi_t, D\Phi_t \psi_t \rangle. \tag{B.8}$$

Definition B.3. A family of self-adjoint operators Φ_t on a subspace $\mathcal{H}_1 \subset \mathcal{H}$ will be called a (second quantized) *weak propagation observable*, if for all $\psi_0 \in \mathcal{H}_1$, it has the following properties

- $\sup_t \langle \psi_t, \Phi_t \psi_t \rangle \lesssim \|\psi_0\|^2_*$;
- $D\Phi_t \ge G_t + \mathrm{Rem}$, where $G_t \ge 0$ and $\int_0^\infty dt \, |\langle \psi_t, \mathrm{Rem}\, \psi_t \rangle| \lesssim \|\psi_0\|^2_\diamond$,

for some norms $\|\psi_0\|_*$, $\|\cdot\|_\diamond \ge \|\cdot\|$. Similarly, a family of self-adjoint operators Φ_t will be called a *strong propagation observable*, if it has the following properties

- Φ_t is a family of non-negative operators;
- $D\Phi_t \le -G_t + \mathrm{Rem}$, where $G_t \ge 0$ and $\int_0^\infty dt \, |\langle \psi_t, \mathrm{Rem}\, \psi_t \rangle| \lesssim \|\psi_0\|^2_\#$,

for some norm $\|\cdot\|_\# \ge \|\cdot\|$.

If Φ_t is a weak propagation observable, then integrating the corresponding differential inequality sandwiched by ψ_t's and using the estimate on $\langle \psi_t, \Phi_t \psi_t \rangle$ and on the remainder Rem, we obtain the (weak propagation) estimate (B.1), with $\|\psi_0\|^2_\# := \|\psi_0\|^2_\diamond + \|\psi_0\|^2_*$. If Φ_t is a strong propagation observable, then the same procedure leads to the (strong propagation) estimate (B.7).

Proof. Proof of Proposition B.2. Let $\Phi_t := \mathrm{d}\Gamma(\phi_t)$. To prove the above statement we use the relations (see Supplement I)

$$D_0 \mathrm{d}\Gamma(\phi_t) = \mathrm{d}\Gamma(d\phi_t), \qquad i[I(g), \mathrm{d}\Gamma(\phi_t)] = -I(i\phi_t g), \tag{B.9}$$

where D_0 is the free Heisenberg derivative,

$$D_0 \Phi_t := \partial_t \Phi_t + i[H_0, \Phi_t],$$

valid for any family of one-particle operators ϕ_t, to compute

$$D\Phi_t = \mathrm{d}\Gamma(d\phi_t) - I(i\phi_t g). \tag{B.10}$$

Denote $\langle A\rangle_\psi := \langle\psi, A\psi\rangle$. Applying the Cauchy-Schwarz inequality, we find the following version of a standard estimate

$$|\langle I(g)\rangle_\psi| \le 2\Big(\int \|\eta_1\eta_2^2 g(k)\|^2_{\mathcal{H}_p}\omega(k)^\delta d^3k\Big)^{\frac12}\|\eta_1^{-1}\eta_2^{-2}\psi\|\|\psi\|_\delta. \tag{B.11}$$

Using that $\psi_t = f_1(H)\psi_t$, with $f_1 \in \mathrm{C}_0^\infty((-\infty,\Sigma))$, $f_1 f = f$, and using (1.7), we find $\|\eta_1^{-1}\eta_2^{-2}\psi_t\| \lesssim \|\psi_t\|$. Taking this into account, we see that the equations (B.11), (B.4) and (1.19) yield

$$|\langle I(i\phi_t g)\rangle_{\psi_t}| \lesssim \langle t\rangle^{-\lambda'+\nu_\delta}\|\psi_0\|^2_\delta. \tag{B.12}$$

Next, using (B.3), we find $\pm\mathrm{rem}_i \le \|\omega^{\rho_i/2}\,\mathrm{rem}_i\,\omega^{\rho_i/2}\|\omega^{\rho_i} \lesssim \langle t\rangle^{-\lambda_i}\omega^{-\rho_i}$. This gives $\pm\mathrm{d}\Gamma(\mathrm{rem_i}) \lesssim \langle t\rangle^{-\lambda_i}\mathrm{d}\Gamma(\omega^{-\rho_i})$, which, due to the bound (1.17), leads to the estimate

$$\big|\langle \mathrm{d}\Gamma(\mathrm{rem_i})\rangle_{\psi_t}\big| \lesssim \langle t\rangle^{-\lambda_i+\nu_{\rho_i}}\|\psi_0\|^2_{\rho_i}. \tag{B.13}$$

Let $G_t := \mathrm{d}\Gamma(p_t)$ and $\mathrm{Rem} := \sum_{\text{finite}}\mathrm{d}\Gamma(\mathrm{rem_i}) - I(i\phi_t g)$. We have $G_t \ge 0$, and, by (B.12) and (B.13),

$$\int_0^\infty dt\,\big|\langle\psi_t, \mathrm{Rem}\,\psi_t\rangle\big| \lesssim \|\psi_0\|^2_\diamond, \tag{B.14}$$

with $\|\psi_0\|^2_\# := \|\psi_0\|^2_\diamond + \|\psi_0\|^2_*$, $\|\psi_0\|_* := \|\psi_0\|_\delta$, $\|\psi_0\|_\diamond := \sum\|\psi_0\|_{\rho_i}$.

In the strong case, (B.5) and (B.10) imply

$$D\Phi_t \le -G_t + \mathrm{Rem}, \tag{B.15}$$

and hence by (B.14), Φ_t is a strong propagation observable.

In the weak case, (B.2) and (B.10) imply

$$D\Phi_t \ge G_t + \mathrm{Rem}. \tag{B.16}$$

Since $\phi_t \le \|\omega^{\delta/2}\phi_t\omega^{\delta/2}\|\omega^{-\delta} \lesssim \langle t\rangle^{-\nu_\delta}\omega^{-\delta}$, we have $\mathrm{d}\Gamma(\phi_t) \lesssim \langle t\rangle^{-\nu_\delta}\mathrm{d}\Gamma(\omega^{-\delta})$. Using this estimate and using again the bound (1.17), we obtain

$$\langle\psi_t, \Phi_t\psi_t\rangle \lesssim \langle t\rangle^{-\nu_\delta}\langle\mathrm{d}\Gamma(\omega^{-\delta})\rangle_{\psi_t} \lesssim \|\psi_0\|^2_\delta. \tag{B.17}$$

Estimates (B.14) and (B.17) show that Φ_t is a weak propagation observable. □

To prove Theorem 1.1, in Section 2, we also used the following proposition.

Proposition B.4. *Let ϕ_t be a one-photon family satisfying*

- *either, for some $\delta \ge 0$,*

$$\|\omega^{\delta/2}\phi_t\omega^{\delta/2}\| \lesssim \langle t\rangle^{-\nu_\delta} \text{ and } \quad d\phi_t \ge p_t - d\tilde\phi_t + \mathrm{rem}, \tag{B.18}$$

  or

$$d\phi_t \le -p_t + d\tilde\phi_t + \sum_{\text{finite}}\mathrm{rem_i}, \tag{B.19}$$

  where $p_t \ge 0$, rem_i are one-photon operators satisfying (B.3)*, and $\tilde\phi_t$ is a weak propagation observable,*
- (B.4) *holds.*

Then, depending on whether (B.18) *or* (B.19) *is satisfied, $\Phi_t := \mathrm{d}\Gamma(\phi_t)$ is a weak, or strong, propagation observable, on the subspace $\Upsilon_{\max(\delta,\rho_i)}$, and therefore we have either the weak or strong propagation estimates,* (B.1) *or* (B.7)*, on this subspace.*

Proof. Given Proposition B.2 and its proof, the only term we have to control is $\mathrm{d}\Gamma(d\tilde\phi_t)$. Using that $\tilde\phi_t$ is a weak propagation observable and using (B.8), (B.10) and (B.12) for $\tilde\Phi_t := \mathrm{d}\Gamma(\tilde\phi_t)$, we obtain

$$\Big|\int_0^\infty dt\,\langle\mathrm{d}\Gamma(d\tilde\phi_t)\rangle_{\psi_t}\Big| \lesssim \|\psi_0\|^2_\#, \tag{B.20}$$

with $\|\psi_0\|^2_\# := \|\psi_0\|^2_\diamond + \|\psi_0\|^2_*$, $\|\psi_0\|_* := \|\psi_0\|_\delta$, $\|\psi_0\|_\diamond := \sum\|\psi_0\|_{\rho_i}$, which leads to the desired estimates. □

Remarks.

1) Proposition B.2 reduces a proof of propagation estimates for the dynamics (1.9) to estimates involving the *one-photon* datum (ω, g) (an 'effective one-photon system'), parameterizing the hamiltonian (1.4). (The remaining datum H_p does not enter our analysis explicitly, but through the bound states of H_p which lead to the localization in the particle variables, (1.7)).

2) The condition on the remainder in (B.2) can be weakened to $\mathrm{rem} = \mathrm{rem}' + \mathrm{rem}''$, with rem' and rem'' satisfying (B.3) and

$$|\mathrm{rem}''| \lesssim \chi_{|y|\geq c't}, \tag{B.21}$$

for c' as in (1.13), respectively. Moreover, (B.3) can be further weakened to

$$\int_0^\infty dt\, |\langle \psi_t, \mathrm{d}\Gamma(\mathrm{rem}_i)\psi_t\rangle| < \infty. \tag{B.22}$$

3) An iterated form of Proposition B.4 is used to prove Theorem 1.1.

Appendix C. One-particle commutator estimates

In this appendix, we estimate some localization terms and commutators appearing in Section 2. We begin with recalling the Helffer-Sjöstrand formula that will be used several times. Let f be a smooth function satisfying the estimates $|\partial_s^n f(s)| \leq C_n \langle s\rangle^{\rho-n}$ for all $n \geq 0$, with $\rho < 0$. We consider an almost analytic extension $\tilde f$ of f, which means that $\tilde f$ is a C^∞ function on $\mathbb{C}$ such that $\tilde f|_{\mathbb{R}} = f$,

$$\mathrm{supp}\, \tilde f \subset \big\{ z \in \mathbb{C},\ |\operatorname{Im} z| \leq C\langle \operatorname{Re} z\rangle \big\},$$

$|\tilde f(z)| \leq C\langle \operatorname{Re} z\rangle^\rho$ and, for all $n \in \mathbb{N}$,

$$\Big| \frac{\partial \tilde f}{\partial \bar z}(z) \Big| \leq C_n \langle \operatorname{Re} z\rangle^{\rho-1-n} |\operatorname{Im} z|^n.$$

Moreover, if f is compactly supported, we can assume that this is also the case for $\tilde f$. Given a self-adjoint operator A, the Helffer–Sjöstrand formula (see e.g. [17, 41]) allows one to express $f(A)$ as

$$f(A) = \frac{1}{\pi} \int \frac{\partial \tilde f(z)}{\partial \bar z} (A - z)^{-1}\, \mathrm{dRe}\, z\, \mathrm{dIm}\, z. \tag{C.1}$$

Now recall that $b_\epsilon := \frac{1}{2}(\theta_\epsilon \nabla\omega \cdot y + \text{ h.c.})$, where $\theta_\epsilon = \frac{\omega}{\omega_\epsilon}$, $\omega_\epsilon := \omega + \epsilon$, $\epsilon = t^{-\kappa}$, with $\kappa \geq 0$. We have the relations

$$i[\omega, b_\epsilon] = \theta_\epsilon, \quad i[\omega, y^2] = \frac{1}{2}(\nabla\omega \cdot y + y \cdot \nabla\omega), \tag{C.2}$$

and, using in particular Hardy's inequality, one can verify the estimate

$$\big\| [y^2, b_\epsilon]\langle y\rangle^{-2} \big\| = \mathcal{O}(t^\kappa). \tag{C.3}$$

The following lemma is a straightforward consequence of the Helffer-Sjöstrand formula together with (C.2) and (C.3). We do not detail the proof.

Lemma C.1. *Let $h, \tilde h$ be smooth function satisfying the estimates $\big|\partial_s^n h(s)\big| \leq \mathrm{C}_n \langle s\rangle^{-n}$ for $n \geq 0$ and likewise for $\tilde h$. Let $w_\alpha = (|y|/c_1 t^\alpha)^2$, $v_\beta = b_\epsilon/(c_2 t^\beta)$, with $0 < \alpha, \beta \leq 1$. The following estimates hold*

$$[h(w_\alpha), \omega] = \mathcal{O}(t^{-\alpha}), \qquad [\tilde h(v_\beta), \omega] = \mathcal{O}(t^{-\beta}),$$

$$[h(w_\alpha), \theta_\epsilon^{\frac{1}{2}}] = \mathcal{O}(t^{\frac{1}{2}\kappa - \frac{1}{2}\alpha}), \qquad \langle y\rangle [h(w_\alpha), \theta_\epsilon^{\frac{1}{2}}] = \mathcal{O}(t^{\frac{1}{2}\kappa + \frac{1}{2}\alpha}),$$

$$[\tilde h(v_\beta), \omega_\epsilon^{-\frac{1}{2}}] = \mathcal{O}(t^{\frac{3}{2}\kappa - \beta}), \qquad b_\epsilon[\tilde h(v_\beta), \omega_\epsilon^{-\frac{1}{2}}] = \mathcal{O}(t^{\frac{3}{2}\kappa}), \qquad [\tilde h(v_\beta), \theta_\epsilon^{\frac{1}{2}}] = \mathcal{O}(t^{\kappa - \beta}),$$

$$[h(w_\alpha), b_\epsilon] = \mathcal{O}(t^\kappa), \qquad [h(w_\alpha), \tilde h(v_\beta)] = \mathcal{O}(t^{-\beta+\kappa}), \qquad b_\epsilon [h(w_\alpha), \tilde h(v_\beta)] = \mathcal{O}(t^\kappa).$$

Now we prove the following abstract result.

Lemma C.2. *Let h be a smooth function satisfying the estimates $\left|\partial_s^n h(s)\right| \le \mathrm{C}_n \langle s\rangle^{-n}$ for $n \ge 0$. Assume an operator v is s.t. the commutators $[v,\omega]$ and $[v,[v,\omega]]$ are bounded, and for some z in $\mathbb{C}\setminus\mathbb{R}$, $(v-z)^{-1}$ preserves $D(\omega)$. Then the operator $r := [h(v),\omega] - [v,\omega]h'(v)$ is bounded as*

$$\|r\| \lesssim \big\|[v,[v,\omega]]\big\|. \tag{C.4}$$

Proof. We would like to use the Helffer–Sjöstrand formula (C.1) for h. Since h might not decay at infinity, we cannot directly express $h(v)$ by this formula. Therefore, we approximate $h(v)$ as follows. Consider $\varphi \in \mathrm{C}_0^\infty(\mathbb{R};[0,1])$ equal to 1 near 0 and $\varphi_R(\cdot) = \varphi(\cdot/R)$ for $R > 0$. Let $\widetilde{h}$ be an almost analytic extensions of h such that $\widetilde{h}|_{\mathbb{R}} = h$,

$$\operatorname{supp}\widetilde{h} \subset \big\{z\in\mathbb{C};\ |\operatorname{Im} z| \le \mathrm{C}\langle \operatorname{Re} z\rangle\big\}, \tag{C.5}$$

$|\widetilde{h}(z)| \le \mathrm{C}$ and, for all $n\in\mathbb{N}$,

$$\Big|\partial_{\bar z}\widetilde{h}(z)\Big| \le \mathrm{C}_n \langle \operatorname{Re} z\rangle^{\rho-1-n} |\operatorname{Im} z|^n. \tag{C.6}$$

Similarly let $\widetilde{\varphi} \in \mathrm{C}_0^\infty(\mathbb{C})$ be an almost analytic extension of φ satisfying these estimates. As a quadratic form on $D(\omega)$, we have

$$\big[h(v),\omega\big] = \operatorname*{s-lim}_{R\to\infty} \big[(\varphi_R h)(v),\omega\big]. \tag{C.7}$$

Since $(v-z)^{-1}$ preserves $D(\omega)$ for some z in the resolvent set of v (and hence for any such z, see [2, Lemma 6.2.1]), we can compute, using the Helffer–Sjöstrand representation (see (C.1)) for $(\varphi_R h)(v)$,

$$\begin{aligned}\big[(\varphi_R h)(v),\omega\big] &= \frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)\big[(v-z)^{-1},\omega\big]\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z\\ &= -\frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)(v-z)^{-1}[v,\omega](v-z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z\\ &= [v,\omega](\varphi_R h)'(v) + r_R,\end{aligned} \tag{C.8}$$

as a quadratic form on $D(\omega)$, where

$$\begin{aligned} r_R &= -\frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)\big[(v-z)^{-1},[v,\omega]\big](v-z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z\\ &= \frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)(v-z)^{-1}[v,[v,\omega]](v-z)^{-2}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z.\end{aligned} \tag{C.9}$$

Now, using $(v-z)^{-1} = \mathcal{O}\big(|\operatorname{Im} z|^{-1}\big)$, we obtain that

$$\big\|(v-z)^{-1}[v,[v,\omega]](v-z)^{-2}\big\| \lesssim |\operatorname{Im} z|^{-3}\big\|[v,[v,\omega]]\big\|. \tag{C.10}$$

Besides, for all $n\in\mathbb{N}$,

$$|\partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)| \le \mathrm{C}_n\langle \operatorname{Re} z\rangle^{\rho-1-n}|\operatorname{Im} z|^n, \tag{C.11}$$

where $\mathrm{C}_n > 0$ is independent of $R \ge 1$. Using (C.9) together with (C.10), we see that there exists $\mathrm{C} > 0$ such that $\|r_R\| \le \mathrm{C}\big\|[v,[v,\omega]]\big\|$, for all $R \ge 1$. Finally, since $(\varphi_R h)'(v)$ converges strongly to $h'(v)$, the lemma follows from (C.8) and the previous estimate. □

We want apply the lemma above to the *time-dependent* self-adjoint operator $v = \frac{b_\epsilon}{ct^\alpha}$.

Corollary C.3. *Let h be a smooth function satisfying the estimates $\left|\partial_s^n h(s)\right| \le \mathrm{C}_n\langle s\rangle^{-n}$ for $n\ge 0$ and let $v := \frac{b_\epsilon}{ct^\alpha}$, where $c > 0$, $\epsilon = t^{-\kappa}$, with $0 \le \kappa \le \beta \le 1$. Then the operator $r := dh(v) - (dv)h'(v)$ is bounded as*

$$\|r\| \lesssim t^{-\lambda},\ \ \lambda := 2\alpha - \kappa. \tag{C.12}$$

Proof. Observe that

$$dh(v) - (dv)h'(v) = [h(v), i\omega] - [v, i\omega]h'(v) + \partial_t h(v) - (\partial_t v)h'(v).$$

It is not difficult to verify that $(v-z)^{-1}$ preserves $D(\omega)$ for any $z \in \mathbb{C}\setminus\mathbb{R}$. Hence it follows from the computations

$$[v, i\omega] = t^{-\alpha}\theta_\epsilon, \qquad [v,[v,i\omega]] = t^{-2\alpha}\theta_\epsilon\omega_\epsilon^{-2}\epsilon, \tag{C.13}$$

that we can apply Lemma C.2. The estimate

$$[v,[v,\omega]] = \mathcal{O}(\omega_\epsilon^{-1}t^{-2\alpha}) = \mathcal{O}\big(t^{-2\alpha+\kappa}\big) \tag{C.14}$$

then gives

$$\|[h(v), i\omega] - [v, i\omega]h'(v)\| \lesssim t^{-2\alpha+\kappa}.$$

It remains to estimate $\|\partial_t h(v) - (\partial_t v)h'(v)\|$. It is not difficult to verify that $D(b_\epsilon)$ is independent of t. Using the notations of the proof of Lemma C.2 and the fact that $\partial_t h(v) = \text{s-lim}_{R\to\infty} \partial_t(\varphi_R h)(v)$, we compute

$$\begin{aligned}\partial_t(\varphi_R h)(v) &= \frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)\partial_t(v-z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z\\ &= -\frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)(v-z)^{-1}(\partial_t v)(v-z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z\\ &= (\partial_t v)(\varphi_R h)'(v) + r'_R,\end{aligned}$$

where

$$\begin{aligned}r'_R &= -\frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)\big[(v-z)^{-1}, \partial_t v\big](v-z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z\\ &= \frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)(v-z)^{-1}[v, \partial_t v](v-z)^{-2}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z.\end{aligned} \tag{C.15}$$

Now using $\partial_t v = -\frac{\alpha b_\epsilon}{ct^{\alpha+1}} + \frac{1}{ct^\alpha}\partial_t b_\epsilon$ together with (2.8), we estimate

$$[v, \partial_t v] = \mathcal{O}(t^{-1-2\alpha+\kappa})b_\epsilon + \mathcal{O}(t^{-1-2\alpha+2\kappa}).$$

From this, the properties of $\tilde\varphi$, $\tilde h$, and $\kappa \le \beta$, we deduce that $\|r'_R\| \lesssim t^{-1-\alpha+\kappa} \lesssim t^{-2\alpha+\kappa}$ uniformly in $R \ge 1$. This concludes the proof of the corollary. □

The following lemma is taken from [10]. Its proof is similar to the proof of Lemma C.2

Lemma C.4. *Let h be a smooth function satisfying the estimates $\left|\partial_s^n h(s)\right| \le \mathrm{C}_n\langle s\rangle^{-n}$ for $n \ge 0$ and $0 \le \delta \le 1$. Let $w_\alpha = (|y|/ct^\alpha)^2$ with $0 < \alpha \le 1$. We have*

$$\big[h(w_\alpha), i\omega\big] = \frac{1}{ct^\alpha}h'(w_\alpha)\big(\frac{y}{ct^\alpha}\cdot\nabla\omega + \nabla\omega\cdot\frac{y}{ct^\alpha}\big) + \mathrm{rem},$$

with

$$\big\|\omega^{\frac{\delta}{2}}\,\mathrm{rem}\,\omega^{\frac{\delta}{2}}\big\| \lesssim t^{-\alpha(1+\delta)}.$$

Now we prove a localization lemma. Let $v_\alpha := \frac{b_\epsilon}{c't^\alpha}$, $w_\alpha := (|y|/ct^\alpha)^2$.

Lemma C.5. *Let $\kappa < \alpha$. We have, for $c < c'/2$,*

$$\chi_{v_\alpha\ge1}\chi_{w_\alpha\le1} = \mathcal{O}(t^{-(\alpha-\kappa)}). \tag{C.16}$$

Proof. We omit the subindex α in w_α and v_α write $w \equiv w_\alpha$ and $v \equiv v_\alpha$. Observe that by the definition of χ (see Introduction) and the condition $c < c'/2$, we have $\chi_{|y|\ge c't^\alpha}\chi_{|y|\le ct^\alpha} = 0$. Let $c < \bar c < c'/2$ and let $\tilde\chi_{|y|\le\bar ct}$ be such that $\chi_{|y|\le ct}\tilde\chi_{|y|\le\bar ct} = \chi_{|y|\le ct}$ and $\chi_{|y|\ge c't}\tilde\chi_{|y|\le\bar ct} = 0$. Define $\bar b_\epsilon := \tilde\chi_{|y|\le\bar ct^\alpha}b_\epsilon\tilde\chi_{|y|\le\bar ct^\alpha}$. It follows from the expression of b_ϵ that $|\langle u, b_\epsilon u\rangle| \le \|u\|\||y|u\|$, and hence we deduce that $|\langle u, \bar b_\epsilon u\rangle| \le \bar ct^\alpha\|u\|^2$. This gives $\chi_{\bar b_\epsilon\ge c't^\alpha} = 0$. Using this, we write

$$\chi_{b_\epsilon\ge c't^\alpha}\chi_{|y|\le ct^\alpha} = (\chi_{b_\epsilon\ge c't^\alpha} - \chi_{\bar b_\epsilon\ge c't^\alpha})\chi_{|y|\le ct^\alpha}. \tag{C.17}$$

Let $\bar v := \frac{\bar b_\epsilon}{c't^\alpha}$. Denote $g(v) := \chi_{v\ge1}$ and $g(\bar v) := \chi_{\bar v\ge1}$. We will use the construction and notations of the proof of Lemma C.2. Using the Helffer-Sjöstrand formula for $(\varphi_R g)(c)$, we write

$$\begin{aligned}(\varphi_R g)(v) - (\varphi_R g)(\bar v) &= \frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi_R g})(z)\big[(v-z)^{-1} - (\bar v - z)^{-1}\big]\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z\\ &= -\frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{g})(z)(v-z)^{-1}(v-\bar v)(\bar v - z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z.\end{aligned} \tag{C.18}$$

Now we show that $(v-\bar v)(\bar v - z)^{-1}\chi_{|y|\le ct^\alpha} = \mathcal{O}(t^{-(\alpha-\kappa)}|\operatorname{Im} z|^{-2})$. We have

$$v - \bar v = (1 - \tilde\chi_{|y|\le\bar ct^\alpha})\frac{b_\epsilon}{c't^\alpha} + \tilde\chi_{|y|\le\bar ct^\alpha}\frac{b_\epsilon}{c't^\alpha}(1 - \tilde\chi_{|y|\le\bar ct^\alpha}),$$

and we observe that, by Lemma C.1,

$$[(1 - \tilde\chi_{|y|\le\bar ct^\alpha}), b_\epsilon] = \mathcal{O}(t^\kappa). \tag{C.19}$$

Thus

$$v-\bar{v}=(1+\tilde{\chi}_{|y|\le \bar{c}t^\alpha})\frac{b_\epsilon}{c't^\alpha}(1-\tilde{\chi}_{|y|\le \bar{c}t^\alpha})+\mathcal{O}(t^{-(\alpha-\kappa)}),$$

Moreover, we can write

$$\begin{aligned}(1-\tilde{\chi}_{|y|\le \bar{c}t^\alpha})(\bar{v}-z)^{-1}\chi_{|y|\le ct^\alpha}&=\big[(1-\tilde{\chi}_{|y|\le \bar{c}t^\alpha}),(\bar{v}-z)^{-1}\big]\chi_{|y|\le ct^\alpha}\\&=-(\bar{v}-z)^{-1}\big[(1-\tilde{\chi}_{|y|\le \bar{c}t^\alpha}),\frac{b_\epsilon}{ct^\alpha}\big](\bar{v}-z)^{-1}\chi_{|y|\le ct^\alpha}\\&=\mathcal{O}(t^{-(\alpha-\kappa)}|\operatorname{Im} z|^{-2}),\end{aligned}$$

where we used (C.19) to obtain the last estimate. This implies the statement of the lemma. □

Remark. The estimate (C.16) can be improved to $\chi_{v_\alpha\ge 1}\chi_{w_\alpha\le 1}=\mathcal{O}(t^{-m(\alpha-\kappa)})$, for any $m>0$, if we replace $\omega_\epsilon:=\omega+\epsilon$ in the definition of b_ϵ by the smooth function $\omega_\epsilon:=\sqrt{\omega^2+\epsilon^2}$.

Supplement I. Creation and annihilation operators on Fock spaces

Recall that the propagation speed of the light and the Planck constant divided by 2π are set equal to 1. Recall also that the one-particle space is $\mathfrak{h}:=\mathrm{L}^2(\mathbb{R}^3;\mathbb{C})$, for phonons, and $\mathfrak{h}:=\mathrm{L}^2(\mathbb{R}^3;\mathbb{C}^2)$, for photons. In both cases we use the momentum representation and write functions from this space as $u(k)$ and $u(k,\lambda)$, respectively, where $k\in\mathbb{R}^3$ is the wave vector or momentum of the photon and $\lambda\in\{-1,+1\}$ is its polarization.

With each function $f\in\mathfrak{h}$, one associates *creation* and *annihilation operators* $a(f)$ and $a^*(f)$ defined, for $u\in\otimes_s^n\mathfrak{h}$, as

$$a^*(f):u\to\sqrt{n+1}f\otimes_s u\quad\text{ and }\quad a(f):u\to\sqrt{n}\langle f,u\rangle_{\mathfrak{h}},\tag{I.1}$$

with $\langle f,u\rangle_{\mathfrak{h}}:=\int\overline{f(k)}u(k,k_1,\dots,k_{n-1})\,\mathrm{d}k$, for phonons, and $\langle f,u\rangle_{\mathfrak{h}}:=\sum_{\lambda=1,2}\int\mathrm{d}k\overline{f(k,\lambda)}u_n(k,\lambda,k_1,\lambda_1,\dots,k_{n-1},\lambda_{n-1})$, for photons. They are unbounded, densely defined operators of $\Gamma(\mathfrak{h})$, adjoint of each other (with respect to the natural scalar product in $\mathcal{F}$) and satisfy the *canonical commutation relations* (CCR):

$$\big[a^\#(f),a^\#(g)\big]=0,\qquad\big[a(f),a^*(g)\big]=\langle f,g\rangle,$$

where $a^\#=a$ or a^*. Since $a(f)$ is anti-linear and $a^*(f)$ is linear in f, we write formally

$$a(f)=\int\overline{f(k)}a(k)\,dk,\qquad a^*(f)=\int f(k)a^*(k)\,dk,$$

for phonons, and

$$a(f)=\sum_{\lambda=1,2}\int\overline{f(k,\lambda)}a_\lambda(k)\,dk,\qquad a^*(f)=\sum_{\lambda=1,2}\int f(k,\lambda)a_\lambda^*(k)dk,$$

for photons. Here $a(k)$ and $a^*(k)$ and $a_\lambda(k)$ and $a_\lambda^*(k)$ are unbounded, operator-valued distributions, which obey (again formally) the *canonical commutation relations* (CCR):

$$\begin{aligned}&\big[a^\#(k),a^\#(k')\big]=0,\qquad\big[a(k),a^*(k')\big]=\delta(k-k'),\\&\big[a_\lambda^\#(k),a_{\lambda'}^\#(k')\big]=0,\qquad\big[a_\lambda(k),a_{\lambda'}^*(k')\big]=\delta_{\lambda,\lambda'}\delta(k-k'),\end{aligned}$$

where $a^\#=a$ or a^* and $a_\lambda^\#=a_\lambda$ or a_λ^*.

Given an operator τ acting on the one-particle space $\mathfrak{h}$, the operator $\mathrm{d}\Gamma(\tau)$ (the second quantization of τ) defined on the Fock space $\mathcal{F}$ by (1.3), can be written (formally) as $\mathrm{d}\Gamma(\tau):=\int dk\,a^*(k)\tau a(k)$, for phonons, and $\mathrm{d}\Gamma(\tau):=\sum_{\lambda=1,2}\int dk\,a_\lambda^*(k)\tau a_\lambda(k)$, for photons. Here the operator τ acts on the k-variable. The precise meaning of the latter expression is (1.3). In particular, one can rewrite the quantum Hamiltonian H_f in terms of the creation and annihilation operators, a and a^*, as

$$H_f=\sum_{\lambda=1,2}\int dk\,a_\lambda^*(k)\omega(k)a_\lambda(k)\tag{I.2}$$

for photons, and similarly for phonons.

The relations below are valid for both phonon and photon operators. Commutators of two $\mathrm{d}\Gamma$ operators reduces to commutators of the one-particle operators:

$$[\mathrm{d}\Gamma(\tau),\mathrm{d}\Gamma(\tau')]=\mathrm{d}\Gamma([\tau,\tau']).\tag{I.3}$$

Let τ be a one-photon self-adjoint operator. The following commutation relations involving the field operator $\Phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(f))$ can be readily derived from the definitions of the operators involved:

$$[\Phi(f), \Phi(g)] = i\,\mathrm{Im}\langle f, g\rangle_{\mathfrak{h}}, \tag{I.4}$$

$$[\Phi(f), \mathrm{d}\Gamma(\tau)] = i\Phi(i\tau f), \tag{I.5}$$

$$[\Gamma(\tau), \Phi(f)] = \Gamma(\tau)a((1-\tau)f) - a^*((1-\tau)f)\Gamma(\tau). \tag{I.6}$$

Exponentiating these relations, we obtain

$$e^{i\Phi(f)}\Phi(g)e^{-i\Phi(f)} = \Phi(g) - \mathrm{Im}\langle f, g\rangle_{\mathfrak{h}}, \tag{I.7}$$

$$e^{i\Phi(f)}\mathrm{d}\Gamma(\tau)e^{-i\Phi(f)} = \mathrm{d}\Gamma(\tau) - \Phi(i\tau f) + \frac{1}{2}\,\mathrm{Re}\langle \omega f, f\rangle_{\mathfrak{h}} \tag{I.8}$$

$$e^{i\Phi(f)}\Gamma(\tau)e^{-i\Phi(f)} = \Gamma(\tau) + \int_0^1 ds\, e^{is\Phi(f)}(\Gamma(\tau)a((1-\tau)f) - a^*((1-\tau)f)\Gamma(\tau))e^{-si\Phi(f)}. \tag{I.9}$$

Finally, we have the following standard estimates for annihilation and creation operators $a(f)$ and $a^*(f)$, whose proof can be found, for instance, in [7], [31, Section 3], [37]:

Lemma I.1. *For any $f \in \mathfrak{h}$ such that $\omega^{-\rho/2}f \in \mathfrak{h}$, the operators $a^{\#}(f)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1/2}$, where $a^{\#}(f)$ stands for $a^*(f)$ or $a(f)$, extend to bounded operators on $\mathcal{H}$ satisfying*

$$\big\| a(f)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-\frac{1}{2}}\big\| \leq \|\omega^{-\rho/2}f\|_{\mathfrak{h}},$$

$$\big\| a^*(f)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-\frac{1}{2}}\big\| \leq \|\omega^{-\rho/2}f\|_{\mathfrak{h}} + \|f\|_{\mathfrak{h}}.$$

If, in addition, $g \in \mathfrak{h}$ is such that $\omega^{-\rho/2}g \in \mathfrak{h}$, the operators $a^{\#}(f)a^{\#}(g)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1}$ extend to bounded operators on $\mathcal{H}$ satisfying

$$\big\| a(f)a(g)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1}\big\| \leq \|\omega^{-\rho/2}f\|_{\mathfrak{h}}\|\omega^{-\rho/2}g\|_{\mathfrak{h}},$$

$$\big\| a^*(f)a(g)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1}\big\| \leq \big(\|\omega^{-\rho/2}f\|_{\mathfrak{h}} + \|f\|_{\mathfrak{h}}\big)\|\omega^{-\rho/2}g\|_{\mathfrak{h}},$$

$$\big\| a^*(f)a^*(g)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1}\big\| \leq \big(\|\omega^{-\rho/2}f\|_{\mathfrak{h}} + \|f\|_{\mathfrak{h}}\big)\big(\|\omega^{-\rho/2}g\|_{\mathfrak{h}} + \|g\|_{\mathfrak{h}}\big).$$

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