To Jürg Fröhlich whose vision and ideas shaped
the non-relativistic quantum electrodynamics

ON RAYLEIGH SCATTERING IN NON-RELATIVISTIC QUANTUM ELECTRODYNAMICS

JÉRÉMY FAUPIN AND ISRAEL MICHAEL SIGAL

Abstract. We consider a particle system coupled to the quantized electromagnetic or phonon field. Assuming that the coupling is small enough and that Fermi's Golden Rule is satisfied, we prove asymptotic completeness for Rayleigh scattering on the states for which the expectation of either the photon/phonon number operator or an operator testing the photon/phonon infrared behaviour is uniformly bounded on corresponding dense sets. By extending a recent result of De Roeck and Kupiainen in a straightforward way, we show that the second of these conditions is satisfied for the spin-boson model.

1. Introduction

In this paper we study the long-time dynamics of a non-relativistic particle system coupled to the quantized electromagnetic or phonon field. For energies below the ionization threshold, we prove asymptotic completeness (for Rayleigh scattering) on the states for which the expectation of the photon number or an operator testing the photon infrared behaviour is bounded uniformly in time. In this introduction we formulate the model, the problem, the results and the outline of the proof.

Standard model of non-relativistic quantum electrodynamics. First, we consider the standard model of non-relativistic quantum electrodynamics in which particles are minimally coupled to the quantized electromagnetic field. The state space for this model is given by $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}$, where $\mathcal{H}_p$ is the particle state space, say, $L^2(\mathbb{R}^{3n})$, or a subspace thereof, and $\mathcal{F}$ is the bosonic Fock space, $\mathcal{F} \equiv \Gamma(\mathfrak{h}) := \mathbb{C} \oplus_{n=1}^{\infty} \otimes_s^n \mathfrak{h}$, based on the one-photon space $\mathfrak{h} := L^2(\mathbb{R}^3, \mathbb{C}^2)$ ($\otimes_s^n$ stands for the symmetrized tensor product of n factors, $\mathbb{C}^2$ accounts for the photon polarization). Its dynamics is generated by the hamiltonian

$$H := \sum_{j=1}^{n} \frac{1}{2m_j}\big(-i\nabla_{x_j} - \kappa_j A_\xi(x_j)\big)^2 + U(x) + H_f. \tag{1.1}$$

Here, m_j and x_j, $j = 1, \dots, n$, are the ('bare') particle masses and the particle positions, $U(x)$, $x = (x_1, \dots, x_n)$, is the total potential affecting the particles, and κ_j are coupling constants related to the particle charges. Moreover, $A_\xi := \check{\xi} * A$, where ξ is an *ultraviolet cut-off* satisfying e.g. $|\partial^m \xi(k)| \lesssim \langle k\rangle^{-3}$, $|m| = 0, \dots, 3$, and $A(y)$ is the *quantized vector potential* in the Coulomb gauge ($\operatorname{div} A(y) = 0$), describing the quantized electromagnetic field and given by

$$A_\xi(y) = \sum_{\lambda=1,2} \int \frac{\xi(k)dk}{\sqrt{2\omega(k)}} \varepsilon_\lambda(k)\big(e^{ik\cdot y} a_\lambda(k) + e^{-ik\cdot y} a^*_\lambda(k)\big). \tag{1.2}$$

Here, $\omega(k) = |k|$ denotes the photon dispersion relation (k is the photon wave vector), λ is the polarization, and $a_\lambda(k)$ and $a^*_\lambda(k)$ are photon annihilation and creation operators acting on the Fock space $\mathcal{F}$ (see Supplement II for the definition). In (1.2) and in what follows, the integrals without indication of the domain of integration are taken over entire $\mathbb{R}^3$.

The operator H_f in (1.1) is the quantum hamiltonian of the quantized electromagnetic field, describing the dynamics of the latter, given by $H_f = \mathrm{d}\Gamma(\omega)$, where $\mathrm{d}\Gamma(\tau)$ denotes the lifting of a one-photon operator τ to the photon Fock space, $\mathrm{d}\Gamma(\tau)|_{\mathbb{C}} = 0$ for $n = 0$ and, for $n \geq 1$,

$$\mathrm{d}\Gamma(\tau)_{|\otimes_s^n \mathfrak{h}} = \sum_{j=1}^{n} \underbrace{1 \otimes \cdots \otimes 1}_{j-1} \otimes \tau \otimes \underbrace{1 \otimes \cdots \otimes 1}_{n-j}. \tag{1.3}$$

(See Supplement II for the expression of $\mathrm{d}\Gamma(\tau)$ in terms of $a_\lambda(k)$ and $a^*_\lambda(k)$.)

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We assume that $U(x) \in L^2_{\rm loc}(\mathbb{R}^{3n})$ and is either confining or relatively bounded with relative bound 0 w.r.t. $-\Delta_x$, so that the particle hamiltonian $H_p := -\sum_{j=1}^n \frac{1}{2m_j}\Delta_{x_j} + U(x)$, and therefore the total hamiltonian H, are self-adjoint.

This model goes back to the early days of quantum mechanics (it appears in the review [23] as a well-known model and is elaborated in an important way in [56]); its rigorous analysis was pioneered in [24, 25] (see [59, 65] for extensive references).

Phonon hamiltonian. We also consider the standard phonon model of solid state physics (see e.g. [48]). The state space for it is given by $\mathcal{H} := \mathcal{H}_p \otimes \mathcal{F}$, where $\mathcal{H}_p$ is the particle state space and $\mathcal{F} \equiv \Gamma(\mathfrak{h}) = \mathbb{C} \oplus_{n=1}^\infty \otimes_s^n \mathfrak{h}$ is the bosonic Fock space based on the one-phonon space $\mathfrak{h} := L^2(\mathbb{R}^3, \mathbb{C})$. Its dynamics is generated by the hamiltonian

$$H := H_p + H_f + I(g), \tag{1.4}$$

acting on $\mathcal{H}$, where H_p is a self-adjoint particle system Hamiltonian, acting on $\mathcal{H}_p$, and $H_f = \mathrm{d}\Gamma(\omega)$ is the phonon hamiltonian acting on $\mathcal{F}$, where $\omega = \omega(k)$ is the phonon dispersion law (k is the phonon wave vector). For *acoustic phonons*, $\omega(k) \asymp |k|$ for small $|k|$ and $c \le \omega(k) \le c^{-1}$, for some $c > 0$, away from 0, while for *optical phonons*, $c \le \omega(k) \le c^{-1}$, for some $c > 0$, for all k. To fix ideas, we consider below only the most difficult case $\omega(k) = |k|$.

The operator $I(g)$ acts on $\mathcal{H}$ and represents an interaction energy, labeled by a coupling family $g(k)$ of operators acting on the particle space $\mathcal{H}_p$. In the simplest case of linear coupling (the dipole approximation in QED or the phonon models), $I(g)$ is given by

$$I(g) := \int (g^*(k) \otimes a(k) + g(k) \otimes a^*(k)) dk, \tag{1.5}$$

where $a^*(k)$ and $a(k)$ are the phonon creation and annihilation operators acting on $\mathcal{F}$, and $g(k)$ is a family of operators on $\mathcal{H}_p$ (coupling operators), for which we assume the following condition

$$\|\eta_1 \eta_2^{|\alpha|} \partial^\alpha g(k)\|_{\mathcal{H}_p} \lesssim |k|^{\mu - |\alpha|} \langle k \rangle^{-2-\mu}, \quad |\alpha| \le 2, \tag{1.6}$$

where η_1 and η_2 are bounded, positive operators with unbounded inverses, the specific form of which depends on the models considered and will be given below. Moreover we assume that there is $\Sigma > \inf \sigma(H_p)$ such that the following estimate holds

$$\|\eta_2^{-n} \eta_1^{-m} \eta_2^{-n} f(H)\| \lesssim 1, \quad 0 \le n, m \le 2, \tag{1.7}$$

for any $f \in \mathrm{C}_0^\infty((-\infty, \Sigma))$.

A primary example for the particle system to have in mind is an electron in a vacuum or in a solid in an external potential V. In this case, $H_p = \epsilon(p) + V(x)$, $p := -i\nabla_x$, with $\epsilon(p)$ being the standard non-relativistic kinetic energy, $\epsilon(p) = \frac{1}{2m}|p|^2 \equiv -\frac{1}{2m}\Delta_x$ (the Nelson model), or the electron dispersion law in a crystal lattice (a standard model in solid state physics), acting on $\mathcal{H}_p = L^2(\mathbb{R}^3)$. The coupling family is given by $g(k) = |k|^\mu \xi(k) e^{ikx}$, where $\xi(k)$ is the ultraviolet cut-off, satisfying e.g. $|\partial^m \xi(k)| \lesssim \langle k \rangle^{-2-\mu}$, $m = 0, \dots, 3$ (and therefore $g(k)$ satisfies (1.6), with $\eta_1 = \mathbf{1}$ and $\eta_2 = \langle x \rangle^{-1}$ with $\langle x \rangle = (1 + |x|^2)^{1/2}$). For phonons, $\mu = 1/2$, and for the Nelson model, $\mu \ge -1/2$. To have a self-adjoint operator H we assume that V is a Kato potential and that $\mu \ge -1/2$. This can be easily upgraded to an $N-$body system (e.g. an atom or a molecule, see e.g. [40, 59]). A key fact here is that for the particle models discussed above (both for the non-relativistic QED and phonon models), there is a spectral point $\Sigma \in \sigma(H) \cup \{\infty\}$, called the *ionization threshold*, s.t. below Σ, the particle system is well localized:

$$\|\langle p \rangle^2 e^{\delta |x|} f(H)\| \lesssim 1, \tag{1.8}$$

for any $0 \le \delta < \mathrm{dist}(\mathrm{supp}\, f, \Sigma)$ and any $f \in \mathrm{C}_0^\infty((-\infty, \Sigma))$. In other words, states decay exponentially in the particle coordinates x ([37, 6, 7]). Hence (1.7) holds with $\eta_1 = \langle p \rangle^{-1}$ and $\eta_2 = \langle x \rangle^{-1}$. To guarantee that $\Sigma > \inf \sigma(H_p) \ge \inf \sigma(H)$, we assume that the potentials $U(x)$ or $V(x)$ are such that the particle hamiltonian H_p has discrete eigenvalues below the essential spectrum ([37, 6, 7]). Furthermore, Σ, for which (1.8) is true, is given by $\Sigma := \lim_{R \to \infty} \inf_{\varphi \in D_R} \langle \varphi, H\varphi \rangle$, where the infimum is taken over $D_R = \{\varphi \in \mathcal{D}(H) |\ \varphi(x) = 0 \text{ if } |x| < R, \|\varphi\| = 1\}$ (see [37]; Σ is close to $\inf \sigma_{\rm ess}(H_p)$).

For the coupling function g, we introduce the norm

$$\langle g \rangle := \sum_{|\alpha| \le 2} \|\eta_1 \eta_2^{|\alpha|} \partial^\alpha g\|_{L^2(\mathbb{R}^3, \mathcal{H}_p)}. \tag{1.9}$$

Spin-boson model. Another example fitting into our framework, and one of the simplest one, is the spin-boson model describing an idealized two-level atom, with state space $\mathcal{H}_p = \mathbb{C}^2$ and hamiltonian $H_p = \varepsilon\sigma^3$, where $\sigma^1, \sigma^2, \sigma^3$ are the usual 2×2 Pauli matrices, and $\varepsilon > 0$ is an atomic energy, interacting with the massless bosonic field. This model is a rather special case of (1.4)–(1.5). The total hamiltonian is given by (1.4)–(1.5), with the coupling family given by $g(k) = |k|^\mu \xi(k)\sigma^+$, $\sigma^\pm = \frac{1}{2}(\sigma^1 \mp i\sigma^2)$. For the spin-boson model, we can take $\Sigma = \infty$.

Problem. In all above cases, the hamiltonian H is self-adjoint and generates the dynamics through the Schrödinger equation,

$$i\partial_t \psi_t = H\psi_t. \tag{1.10}$$

As initial conditions, ψ_0, we consider states below the ionization threshold Σ, i.e. ψ_0 in the range of the spectral projection $E_{(-\infty,\Sigma)}(H)$. In other words, we are interested in processes, like emission and absorption of radiation, or scattering of photons on an electron bound by an external potential (created e.g. by an infinitely heavy nucleus or impurity of a crystal lattice), in which the particle system (say, an atom or a molecule) is not being ionized. One of the the key problems here is understanding asymptotic behaviour of the evolution (1.10), with the corresponding statement called asymptotic completeness. To formulate it, we denote by Φ_j and E_j the eigenfunctions and the corresponding eigenvalues of the hamiltonian H, below Σ, i.e. $E_j < \Sigma$. Then *Asymptotic completeness* on the interval $(-\infty, \Sigma)$ states that, for any $\psi_0 \in \mathrm{Ran}\ E_{(-\infty,\Sigma)}(H)$, and any $\epsilon > 0$, there are photon wave functions $f_{j\epsilon} \in \mathcal{F}$, with a finite number of photons, s.t. the solution, $\psi_t = e^{-itH}\psi_0$, of the Schrödinger equation, (1.10), satisfies

$$\limsup_{t\to\infty} \|e^{-itH}\psi_0 - \sum_j e^{-iE_j t}\Phi_j \otimes_s e^{-iH_f t} f_{j\epsilon}\| \le \epsilon. \tag{1.11}$$

(It will be shown in the text that $\Phi_j \otimes_s f_{j\epsilon}$ is well-defined, at least for the ground state ($j = 0$).) In other words, for any $\epsilon > 0$ and with probability $\ge 1 - \epsilon$, the Schrödinger evolution ψ_t approaches asymptotically a superposition of states in which the particle system with a photon cloud bound to it is in one of its bound states Φ_j, with additional photons (or possibly none) escaping to infinity with the velocity of light.

The reason for $\epsilon > 0$ in (1.11) is that for the state $\Phi_j \otimes_s f$ to be well defined, as one would expect, one would have to have a very tight control on the number of photons in f, i.e. the number of photons escaping the particle system. (See the remark at the end of Subsection 5.4 for a more technical explanation.) For massive bosons $\epsilon > 0$ can be dropped (set to zero), as the number of photons can be bound by the energy cut-off.[1]

Results. Now we formulate our results. We consider both the minimal coupling model (1.1) and the phonon model (1.4) with the linear interaction (1.5) and the coupling operators $g(k)$ satisfying (1.6) with $\mu > -1/2$.

We begin with giving the precise definition of asymptotic completeness. We define the space $\mathcal{H}_{\mathrm{fin}} := \mathcal{H}_p \otimes \mathcal{F}_{\mathrm{fin}} \otimes \mathcal{F}_{\mathrm{fin}}$, where $\mathcal{F}_{\mathrm{fin}} \equiv \mathcal{F}_{\mathrm{fin}}(\mathfrak{h})$ is the subspace of $\mathcal{F}$ consisting of vectors $\Psi = (\psi_n)_{n=0}^\infty \in \mathcal{F}$ such that $\psi_n = 0$, for all but finitely many n, and the (*scattering*) map $I : \mathcal{H}_{\mathrm{fin}} \to \mathcal{H}$ as the extension by linearity of the map (see [43, 19, 27])

$$I : \Phi \otimes \prod_1^n a^*(h_i)\Omega \to \prod_1^n a^*(h_i)\Phi, \tag{1.12}$$

for any $\Phi \in \mathcal{H}_p \otimes \mathcal{F}_{\mathrm{fin}}$ and for any $h_1, \dots h_n \in \mathfrak{h}$. Here $a^\#(h)$ are the creation and annihilation operators evaluated on a function h, see Supplement II. Another useful representation of I is

$$I : \Phi \otimes f \to \begin{pmatrix} p+q \\ p \end{pmatrix}^{1/2} \Phi \otimes_s f, \tag{1.13}$$

for any $\Phi \in \mathcal{H}_p \otimes (\otimes_s^p \mathfrak{h})$ and $f \in \otimes_s^q \mathfrak{h}$. (We call I the *Hübner-Spohn scattering map.*) As already clear from (1.12), the operator I is unbounded.

Now, it is known (see [7, 38]) that the operator H has *a unique ground state* (denoted here as Φ_{gs}). Let E_{gs} be the ground state energy and $E_{\mathrm{gs}} < a < \Sigma$ be such that the hamiltonian H has no eigenvalues in the

[1] For a discussion of scattering of massless bosons in QFT see [11].

interval $(E_{\rm gs}, a]$. We say that asymptotic completeness holds on the interval $\Delta = [E_{\rm gs}, a]$, if, for every $\epsilon > 0$ and $\phi_0 \in \mathrm{Ran}\chi_\Delta(H)$, there is $\phi_{0\epsilon} \in \mathcal{F}_{\rm fin}$ s.t.

$$\limsup_{t\to\infty} \|e^{-iHt}\phi_0 - I(e^{-iE_{\rm gs}t}P_{\rm gs} \otimes e^{-iH_f t}\chi_{\Delta'}(H_f))\phi_{0\epsilon}\| = \mathcal{O}(\epsilon), \tag{1.14}$$

where $\Delta' = [0, a - E_{\rm gs}]$ and $P_{\rm gs}$ is the orthogonal projection onto $\Phi_{\rm gs}$.

Generically (e.g. under the Fermi Golden Rule condition), H has no eigenvalues in the interval $(E_{\rm gs}, a]$, where $a < \Sigma$ can be taken arbitrarily close to Σ, depending on the coupling constant and on whether the particle system has an infinite number of eigenvalues accumulating to its ionization threshold (see [8, 30, 34]). *We assume* that this is exactly the case:

$$\textit{Fermi's Golden Rule } ([6, 7]) \text{ holds for all excited eigenvalues} \le a \text{ of } H_p. \tag{1.15}$$

Assumption (1.15) means that for every excited eigenvalue $e_j \le a$ of H_p, we have

$$\Pi_j W \mathrm{Im}((H_0 - e_j - i0^+)^{-1}\bar\Pi_j)W\Pi_j \ge c_j\Pi_j, \quad c_j > 0, \tag{1.16}$$

where $H_0 := H_p + H_f$ (for either model), $W := H - H_0$, Π_j denotes the projection onto the eigenspace of H_0 associated to e_j and $\bar\Pi_j := \mathbf{1} - \Pi_j$. In fact, there is an explicit representation of (1.16). Since it differs slightly for different models, we present it for the phonon one, assuming for simplicity that the eigenvalue e_j is simple:

$$\int \langle \phi_j, g^*(k)\mathrm{Im}(H_p + \omega(k) - e_j - i0^+)^{-1} g(k)\phi_j\rangle dk > 0, \tag{1.17}$$

where ϕ_j is an eigenfunction of H_p corresponding to the eigenvalue e_j and the inner product is in the space $\mathcal{H}_p$.

It is clear from (1.17) that Fermi's Golden Rule holds generally, with a very few exceptions. Treatment of the (exceptional) situation when excited embedded eigenvalues do occur requires, within our approach, proving a delicate estimate $\|P_\Omega f(H)\| \lesssim \langle g\rangle$, where P_Ω denotes the projection onto $\mathcal{H}_p \otimes \Omega$ (where $\Omega := 1 \oplus 0 \oplus \dots$ is the vacuum in $\mathcal{F}$) and $f \in \mathrm{C}_0^\infty((E_{\rm gs}, \Sigma) \setminus \sigma_{\rm pp}(H))$, uniformly in $\mathrm{dist}(\mathrm{supp}\, f, \sigma_{\rm pp}(H))$.

Let $N := \mathrm{d}\Gamma(\mathbf{1})$ be the photon (or phonon) number operator and $N_\rho := \mathrm{d}\Gamma(\omega^{-\rho})$ be the photon (or phonon) low momentum number operator. In what follows we let ψ_t denote the Schrödinger evolution, $\psi_t = e^{-itH}\psi_0$, i.e. the solution of the Schrödinger equation (1.10), with an initial condition ψ_0, satisfying $\psi_0 = f(H)\psi_0$, with $f \in \mathrm{C}_0^\infty((-\infty, \Sigma))$. We have

Theorem 1.1 (Asymptotic Completeness)**.** *Consider the hamiltonian* (1.1) *with the coupling constants κ_j sufficiently small, or the hamiltonian* (1.4)–(1.5) *satisfying* (1.6) *with $\mu > 0$,* (1.7) *and $\langle g\rangle \ll 1$. Assume* (1.15) *and suppose that either*

$$\|N^{\frac12}\psi_t\| \lesssim \|N^{\frac12}\psi_0\| + \|\psi_0\|, \tag{1.18}$$

for any $\psi_0 \in f(H)D(N^{1/2})$, with $f \in \mathrm{C}_0^\infty((E_{\rm gs}, \Sigma))$, uniformly in $t \in [0, \infty)$, or

$$\|N_1^{\frac12}\psi_t\| \lesssim 1, \tag{1.19}$$

uniformly in $t \in [0, \infty)$, for any $\psi_0 \in \mathcal{D}$, where $\mathcal{D}$ is such that $\mathcal{D} \cap D(\mathrm{d}\Gamma(\omega^{-1/2}\langle y\rangle\omega^{-1/2})^{\frac12})$ is dense in $\mathrm{Ran}\, E_{(-\infty,\Sigma)}(H)$*. Then asymptotic completeness holds on $[E_{\rm gs}, a]$.*

Assumption (1.18) can be replaced by the slightly weaker hypothesis that there exist $1/2 \le \delta_1 \le \delta_2$ such that for any $\psi_0 \in f(H)D(N^{\delta_2})$, with $f \in \mathrm{C}_0^\infty((E_{\rm gs}, \Sigma))$, $\|N^{\delta_1}\psi_t\| \lesssim \|N^{\delta_2}\psi_0\| + \|\psi_0\|$, uniformly in $t \in [0, \infty)$.

The advantage of Assumption (1.19) is that the uniform bound on $N_1 = \mathrm{d}\Gamma(\omega^{-1})$ is required to hold only for an *arbitrary* dense set of initial states and, as a result, can be verified for the massless spin-boson model by modifying slightly the proof of [14] (see the discussion below). Hence asymptotic completeness in this case holds with no implicit conditions.

As we see from the results above, the uniform bounds, (1.18) or (1.19), on the number of photons (or phonons) emerge as the remaining stumbling blocks to proving asymptotic completeness without qualifications. The difficulty in proving these bounds for massless fields is due to the same infrared problem which pervades this field and which was successfully tackled in other central issues, such as the theory of ground states and resonances (see [5, 59] for reviews), the local decay and the maximal velocity bound.

For massive bosons (e.g. optical phonons), the inequality (1.18) (as well as (2.4), with $\nu_0 = 0$) is easily proven and the proof below simplifies considerably as well. In this case, the result is unconditional. It was first proven in [19] for models with confined particles, and in [27] for Rayleigh scattering.

As was mentioned above, for the spin-boson model, a uniform bound, $\langle \psi_t, e^{\delta N}\psi_t\rangle \le C(\psi_0) < \infty,\ \delta > 0$, on the number of photons, on a dense set of ψ_0's, was recently proven in the remarkable paper [14].

To verify (1.19) for the spin-boson model, with $\mu > 0$, we proceed precisely in the same way as in [14], but using a stronger condition on the decay of correlation functions,

$$\int_0^\infty dt\,(1+t)^\alpha |h(t)| < \infty, \quad \text{with} \quad h(t) := \int_{\mathbb{R}^3} dk\, e^{-it|k|}(1+|k|^{-1})|g(k)|^2, \tag{1.20}$$

for some $\alpha \ge 1$, instead of Assumption A of [14], and bounding the observable $(1+\kappa N_{1/2})^2$ instead of $e^{\kappa N}$. Assumption C of [14] on initial states has to be replaced in the same manner. Assuming that our condition (1.19) on the coupling function g is satisfied with $\mu > 0$ (and $\eta = 1$), we see that (1.20) holds with $\alpha = 1+2\mu$.

The form of the observable $e^{\kappa N}$ enters [14] through the estimate $\|K_{u,v}\|_\diamond \le C|h(u-v)|$ of the operator $K_{u,v}$ defined in [14, (3.4)] and the standard estimate [14, (4.36)]. Both extend readily to our case (the former with $h(t)$ given in (1.20)). Moreover, [14, (4.36)] is used in the proof that pressure vanishes – Eq. (4.39) in [14] – and the latter also follows from our Proposition A.1 (We can also use the observable $e^{-\mathrm{d}\Gamma(\lambda \ln \omega)}$ – equal to $\Gamma(\omega^{-\lambda})$, see (1.24) below for the definition of $\Gamma(\chi)$ – and analyticity – rather than perturbation – in λ.)

Earlier results. Considerable progress has been made in understanding the asymptotic dynamics of non-relativistic particle systems coupled to quantized electromagnetic or phonon field. The local decay property was proven in [7, 8, 9, 12, 30, 31, 33, 34], by the combination of the renormalization group and positive commutator methods. The maximal velocity estimate was proven in [10].

As mentioned above, an important breakthrough was achieved recently in [14], where the authors proved relaxation to the ground state and uniform bounds on the number of emitted massless bosons in the spin-boson model. (Importance of both questions was emphasized earlier by Jürg Fröhlich.)

In quantum field theory, asymptotic completeness was proven for (a small perturbation of) a solvable model involving a harmonic oscillator (see [3, 64]), and for models involving massive boson fields, in [19] for confined systems, in [27] below the ionization threshold for non-confined systems, and in [28] for Compton scattering.

Moreover, the remarkable paper [35] obtained some important results for massless bosons (the Nelson model) in confined potentials (see below for a more detailed discussion). Motivated by the many-body quantum scattering, [19, 27, 28, 29, 35] defined the main notions of scattering theory on Fock spaces, such as wave operators, asymptotic completeness and propagation estimates.

Comparison with [35]. The paper [35] treats the Nelson model (1.4)–(1.5), with abstract conditions on the coupling function g (allowing a coupling function of the form $g(k) = |k|^\mu \xi(k) e^{ikx}$ where $\xi(k)$ is the ultraviolet cut-off, with various conditions on μ depending on the results involved), and with $V(x)$ growing at infinity as $V(x) \ge c_0|x|^{2\alpha} - c_1$, $c_0 > 0$, $\alpha > 0$. In this case, in particular, the ionization threshold Σ is equal to ∞.

We reproduce the main results of [35] (Theorems 12.4, 12.5 and 13.3), which are coached in different terms than ours and present another important view of the subject. Let $f, f_0 \in \mathrm{C}^\infty(\mathbb{R})$ such that $0 \le f, f_0 \le 1$, $f' \ge 0$, $f = 0$ for $s \le \alpha_0$, $f = 1$ for $s \ge \alpha_1$, $f_0' \le 0$, $f_0 = 1$ for $s \le \alpha_1$, $f_0 = 0$ for $s \ge \alpha_2$, with $0 < \alpha_0 < \alpha_1 < \alpha_2$. Let $P_c^+ := \inf_{c<c'} \hat{P}_{c'}^+$, with $\hat{P}_{c'}^+ := \text{s-lim}_{\epsilon\to 0}\, \epsilon^{-1}\hat{R}_c^+(\epsilon^{-1})$, $\hat{R}_c^+(\epsilon^{-1}) := \text{s-lim}_{t\to\infty}\, e^{itH}(B_{ct}+\lambda)^{-1}e^{-itH}$, $B_{ct} := \mathrm{d}\Gamma(b_{ct})$, $b_{ct} := f(\frac{|y|-ct}{t^\rho})$ and $\Gamma_{c'}^+(f_0) := \text{s-lim}_{t\to\infty}\, e^{itH}\Gamma(f_{0,c',t})e^{-itH}$, where $f_{0,c',t} := f_0(\frac{|y|-c't}{t^\rho})$. Then Proposition 12.2 and Theorem 12.3 of [35] state that the operators P_c^+ exist provided $\rho > \frac{1}{\mu+1}$, are independent of the choice of f, and are orthogonal projections commuting with H. Furthermore, let $\mathcal{K}^+ := \{\Phi \in \mathcal{H} : a_\pm(h)\Phi = 0, \forall h \in \mathfrak{h}\}$ (called in [35] the set of asymptotic vacua), where (formally) $a_\pm(h) := \text{s-lim}_{t\to\pm\infty}\, e^{itH}a(e^{-it\omega}h)e^{-itH}$ and $\mathcal{H}_c^+ := \operatorname{Ran} P_c^+$ (the spaces containing states with only a finite number of photons in the region $\{|y| \ge c't\}$ as $t \to \infty$, for all $c' > c$). Assuming $\alpha > 1$ and $\mu > 0$, Theorems 12.4 and 12.5 state that the operator $\Gamma_{c'}^+(f_0)$ exists and is equal to the orthogonal projection on the space $\mathcal{K}_c^+ := \mathcal{K}^+ \cap \mathcal{H}_c^+$, provided $0 < c < c' < 1$ and $\rho > \frac{1}{\mu+1}$. (The latter property is called in [35] geometric asymptotic completeness.) Assuming in addition that the Mourre estimate $\mathbf{1}_\Delta(H)[H, iB]\mathbf{1}_\Delta(H) \ge c_0\mathbf{1}_\Delta(H) + R$ holds on an open interval $\Delta \subset \mathbb{R}$, with the conjugate operator $B := \mathrm{d}\Gamma(b)$, $b = \frac{1}{2}(k\cdot y + y\cdot k)$,

$c_0 > 0$ and R a compact operator on $\mathcal{H}$, then for $0 < c < c(\Delta, c_0)$, one has $\mathbf{1}_\Delta(H)\mathcal{K}_c^+ = \mathbf{1}_\Delta(H)\mathcal{H}_{\rm pp}$, where $\mathcal{H}_{\rm pp}$ is the pure point spectrum eigenspace of H. (Combining results of [7, 8, 30] one can probably prove a Mourre estimate, with B as conjugate operator, in any spectral interval above $E_{\rm gs}$ and below Σ and for the coupling function g given by $g(k) = |k|^\mu \xi(k) e^{ikx}$, with $\mu \geq 1/2$.)

Our approach is similar to the one of [35] in as much as it also originates in ideas of the quantum many-body scattering theory. At this the similarities end.

Approach and organization of the paper. In this paper, as in earlier works, we use the method of propagation observables, originating in the many body scattering theory ([61, 62, 45, 36, 67, 16], see [18, 44] for a textbook exposition and a more recent review). It was extended to the non-relativistic quantum electrodynamics in [19, 35, 26, 27, 28, 29] and to the $P(\varphi)_2$ quantum field theory, in [20] and was used in [10] to prove the *maximal velocity estimate*, which states that, for any $c' > 1$,

$$\big\|\mathrm{d}\Gamma\big(\chi_{\frac{|y|}{c't}\geq 1}\big)^{\frac12}\psi_t\big\| \lesssim t^{-\gamma}\big\|(\mathrm{d}\Gamma(\langle y\rangle)+1)^{\frac12}\psi_0\big\|, \tag{1.21}$$

with $\gamma < \min(\frac12(1-\frac{1}{c'}), \frac{1}{10})$ for (1.1), and $\gamma < \min(\frac{\mu}{2}(\frac{c'-1}{2c'-1}), \frac{1}{2+\mu})$ for (1.4)–(1.6) with $\mu > 0$. We formalize the method of propagation observables in the next section.

We mention that the observables $\mathrm{d}\Gamma(\mathbf{1}_\Omega(y))$ can be interpreted as giving the number of photons in Borel sets $\Omega \subset \mathbb{R}^3$. They are closely related to those used in [27, 35, 50] (and discussed earlier in [52] and [1]) and are consistent with a theoretical description of the detection of photons (usually via the photoelectric effect, see e.g. [53]). The quantity $\langle\psi, \Gamma(\mathbf{1}_\Omega(y))\psi\rangle$ is interpreted as the probability that the photons are in the set Ω in the state ψ. This said, we should mention that the subject of photon localization is still far from being settled. For more discussion see [22].

In Sections 3 and 4, we prove our key propagation estimates – minimal photon escape velocity estimates. These estimates are formulated in terms of the self-adjoint operators b_ϵ defined as $b_\epsilon := \frac12(v(k)\cdot y + y\cdot v(k))$, where $v(k) := \frac{k}{\omega+\epsilon}$, for $\epsilon = t^{-\kappa}$, with some $\kappa > 0$. Since the vector field $v(k)$ is Lipschitz continuous and therefore generates a global flow, the operator b_ϵ is self-adjoint. Our minimal photon escape velocity estimate are of the form

$$\int_1^\infty dt\ t^{-\alpha'}\big\|\mathrm{d}\Gamma(\chi_{\frac{b_\epsilon}{ct^\alpha}=1})^{\frac12}\psi_t\big\|^2 \lesssim \|(N_1+1)^{\frac12}\psi_0\|^2, \tag{1.22}$$

for some α' and α satisfying $0 < \alpha \leq \alpha' \leq 1$, and

$$\big\|\Gamma(\chi_{\frac{b_\epsilon}{ct^\alpha}\leq 1})^{\frac12}\psi_t\big\|^2 \lesssim t^{-\delta}\big(\|(\mathrm{d}\Gamma(\langle y\rangle)+1)^{\frac12}\psi_0\|^2 + \|(\mathrm{d}\Gamma(b)+1)\psi_0\|^2\big), \tag{1.23}$$

for some $\alpha \leq 1$ and $\delta > 0$, where $b = \frac12(k\cdot y + y\cdot k)$ and $\Gamma(\chi)$ is the lifting of a one-photon operator χ (e.g. a smoothed out characteristic function of y) to the photon Fock space, defined by

$$\Gamma(\chi) = \oplus_{n=0}^\infty(\otimes^n\chi), \tag{1.24}$$

(so that $\Gamma(e^b) = e^{\mathrm{d}\Gamma(b)}$).

Once the minimal velocity estimates are proven, the first step in the proof of asymptotic completeness is to decouple the photons in the expanding ball $\{b_\epsilon \leq ct^\alpha\}$ from those inside $\{b_\epsilon \geq ct^\alpha\}$. To this end we use the second quantization, $\Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}\oplus\mathfrak{h})$ of a partition of unity $j : h \to j_0 h \oplus j_\infty h$ on the one-photon space, $j : \mathfrak{h} \to \mathfrak{h}\oplus\mathfrak{h}$, with j_0 localizing a photon to a region $\{b_\epsilon \leq ct^\alpha\}$, and j_∞, to $\{b_\epsilon \geq ct^\alpha\}$, and satisfying $j_0^2 + j_\infty^2 = \mathbf{1}$. Defining the adjoint map $j^* : h_0 \oplus h_\infty \to j_0^* h_0 + j_\infty^* h_\infty$, so that $j^* j = j_0^2 + j_\infty^2 = \mathbf{1}$, and using $\Gamma(j)^*\Gamma(j) = \Gamma(j^*j)$, we see that $\Gamma(j)^*\Gamma(j) = \mathbf{1}$.

The partition $\Gamma(j)$ is further refined as ([19, 27]) $\check\Gamma(j) := U\Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})$, where $U : \Gamma(\mathfrak{h}\oplus\mathfrak{h}) \to \Gamma(\mathfrak{h})\otimes\Gamma(\mathfrak{h})$ is the unitary map defined through the relations $U\Omega = \Omega\otimes\Omega$, $Ua^*(h) = [a^*(h_1)\otimes\mathbf{1} + \mathbf{1}\otimes a^*(h_2)]U$, for any $h = (h_1, h_2) \in \mathfrak{h}\oplus\mathfrak{h}$, and is then lifted from the Fock space $\mathcal{F} = \Gamma(\mathfrak{h})$ to the full state space $\mathcal{H} = \mathcal{H}_p\otimes\mathcal{F}$. As above, $\check\Gamma(j)^*\check\Gamma(j) = \mathbf{1}$. (We call $\check\Gamma(j)$ the *Dereziński-Gérard partition of unity.*) Using $\check\Gamma(j)$, we define the Deift-Simon wave operators ([15, 60, 19, 27]),

$$W_\pm := \underset{t\to\pm\infty}{\text{s-lim}}\, e^{i\hat Ht}\check\Gamma(j)e^{-iHt}, \tag{1.25}$$

where $\hat H := H\otimes\mathbf{1} + \mathbf{1}\otimes H_f$, on the auxiliary space $\hat{\mathcal{H}} := \mathcal{H}\otimes\mathcal{F}$. The first minimal velocity estimate for b_ϵ implies that these operators exist (see Subsection 5.2). The existence of the Deift-Simon wave operators implies that

$$\psi_t = \check{\Gamma}(j)^* e^{-i\hat{H}t} e^{i\hat{H}t} \check{\Gamma}(j) e^{-iHt}\psi_0 = \check{\Gamma}(j)^* e^{-i\hat{H}t}\phi_0 + o_t(1), \tag{1.26}$$

where $\phi_0 := W_+\psi_0$. Since $e^{-i\hat{H}t} = e^{-iHt} \otimes e^{-iH_f t}$, we see that the first term on the r.h.s. describes the photons in the expanding ball $\{b_\epsilon \le ct^\alpha\}$ decoupled from those inside $\{b_\epsilon \ge ct^\alpha\}$.

Next, let $\Delta = [E_{\rm gs}, a] \subset \mathbb{R}$, where $a < \Sigma$, and $\Delta' = [0, a - E_{\rm gs}]$. The existence of W_+ implies the property $W_+\chi_\Delta(H) = \chi_\Delta(\hat{H})W_+$, which gives $\phi_0 = \chi_\Delta(\hat{H})\phi_0$ if $\psi_0 \in \mathrm{Ran}(\chi_\Delta(H))$. The latter relation together with $\chi_\Delta(\hat{H}) = (\chi_\Delta(H) \otimes \chi_{\Delta'}(H_f))\chi_\Delta(\hat{H})$ implies $\phi_0 = \big(\chi_\Delta(H) \otimes \chi_{\Delta'}(H_f)\big)\phi_0$. Next, we use that for all $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$, such that

$$\big\|(\chi_\Delta(H) \otimes \mathbf{1})\phi_0 - (\chi_{\Delta_\epsilon}(H) \otimes \mathbf{1})\phi_0 - (P_{\rm gs} \otimes \mathbf{1})\phi_0\big\| \le \epsilon, \tag{1.27}$$

where $\Delta_\epsilon = [E_{\rm gs} + \delta, a]$ and $P_{\rm gs}$ is the orthogonal projection onto the ground state of H. Applying this equation and the relations $e^{-i\hat{H}t} = e^{-iHt} \otimes e^{-iH_f t}$ and $e^{-iHt}P_{\rm gs} = e^{-iE_{\rm gs}t}P_{\rm gs}$ to (1.26) gives, after some manipulations with energy cut-offs,

$$\psi_t = \check{\Gamma}(j)^*\big(e^{-iE_{\rm gs}t}P_{\rm gs} \otimes e^{-iH_f t}\chi_{\Delta'}(H_f)\big)\phi_0 + \check{\Gamma}(j)^*\phi_t + \mathcal{O}(\epsilon) + o_t(1), \tag{1.28}$$

where $\phi_t = \big(e^{-iHt}\chi_{\Delta_\epsilon}(H) \otimes e^{-iH_f t}\chi_{\Delta'}(H_f)\big)\phi_0$. Now, let $(\tilde{j}_0, \tilde{j}_\infty)$ be localized similarly to (j_0, j_∞) and satisfy $j_0\tilde{j}_0 = j_0$, $j_\infty\tilde{j}_\infty = j_\infty$. Then, as shown below, the adjoint $\check{\Gamma}(j)^*$ to the operator $\check{\Gamma}(j)$ can be represented as $\check{\Gamma}(j)^* = \check{\Gamma}(j)^*\big(\Gamma(\tilde{j}_0) \otimes \Gamma(\tilde{j}_\infty)\big)$. Using this equation in (1.26) and using that $\big(\Gamma(\tilde{j}_0) \otimes \mathbf{1}\big)\phi_t \to 0$, as $t \to \infty$, by the second minimal velocity estimate for b_ϵ, we see that the second term on the r.h.s. of (1.28) vanishes, as $t \to \infty$.

To conclude the proof of asymptotic completeness, we pass from the operator $\check{\Gamma}(j)^*$ to the (*scattering*) map I defined in (1.12)–(1.13). To this end we use the formula $\check{\Gamma}(j)^* = I\Gamma(j_0^*) \otimes \Gamma(j_\infty^*)$, for any operator $j : h \to j_0 h \oplus j_\infty h$, and some elementary estimates in order to remove $\Gamma(j_0^*) \otimes \Gamma(j_\infty^*)$.

Remark. At the expense of slightly lengthier computations, but gaining simpler technicalities, one can also modify b_ϵ to make it bounded, by multiplying it with the cut-off function $\chi_{\frac{|y|}{c't} \le 1}$ with $c' > 1$, such that the maximal velocity estimate (1.21) holds, or use the smooth vector field $v(k) = \frac{k}{\sqrt{\omega^2+\epsilon^2}}$, instead of $v(k) = \frac{k}{\omega+\epsilon}$.

To simplify the exposition, in Sections 2–5, we consider hamiltonians of the form (1.4)–(1.5), with the coupling operators $g(k)$ satisfying (1.6), where η_1 and η_2 obey (1.7). In Section 6, we extend the results to a general class of hamiltonians that are introduced in the next paragraph. In Section 7, we show that the minimal coupling model (1.1) can be mapped unitarily to a hamiltonian from this class, and we deduce Theorem 1.1 for this model.

A general class of hamiltonians. The QED hamiltonian (1.1) can be written in the form (1.4), with $I(g)$ being quadratic in the creation and annihilation operators $a_\lambda^\#(k)$, and the coupling functions satisfying estimates of the form (1.6) with $\mu = -1/2$, $\eta_1 = \langle p\rangle^{-1}$ or $\mathbf{1}$, and $\eta_2 = \langle x\rangle^{-1}$. This infrared behaviour is too singular for our techniques. However, we show in Subsection 7.1 that under the generalized Pauli-Fierz transform of [58], (1.1) is unitary equivalent to an operator of the form described below, whose infrared behaviour is considerably better. We introduce the class of hamiltonians of the form

$$\widetilde{H} = H_p + H_f + \tilde{I}(g), \tag{1.29}$$

where $H_p := -\Delta + V(x)$, and $H_f = \mathrm{d}\Gamma(\omega)$ are the same as before, but the interaction operator, $\tilde{I}(g)$, is of a more general form

$$\tilde{I}(g) := \sum_{ij} \iint d\underline{k}_{(i)} d\underline{k}'_{(j)} g_{ij}(\underline{k}_{(i)}, \underline{k}'_{(j)}) \otimes a^*(\underline{k}_{(i)}) a(\underline{k}'_{(j)}). \tag{1.30}$$

Here the summation in i, j ranges over the set $i, j \ge 0, 1 \le i + j \le 2$, $\underline{k}_{(p)} := (\underline{k}_1, \dots, \underline{k}_p)$, $\underline{k}_j := (k_j, \lambda_j)$, $\int d\underline{k}_{(p)} := \prod_1^p \sum_{\lambda_j} \int dk_j$, $a^\#(\underline{k}_{(p)}) := \prod_1^p a^\#(\underline{k}_j)$ if $p \ge 1$ and $= \mathbf{1}$, if $p = 0$, $a^\#(\underline{k}_j) := a^\#_{\lambda_j}(k_j)$, and $g := (g_{ij})$. We suppose that the coupling operators, $g_{ij} = g_{ij}(\underline{k}_{(i)}, \underline{k}_{(j)})$ satisfy

$$g_{ij}(\underline{k}_{(i)}, \underline{k}'_{(j)}) = g_{ji}^*(\underline{k}'_{(j)}, \underline{k}_{(i)}), \tag{1.31}$$

and

$$\|\eta_1^{2-i-j}\eta_2^{|\alpha|}\partial^\alpha g_{ij}(\underline{k}_{(i+j)})\|_{\mathcal{H}_p} \lesssim \sum_{m=1}^{i+j}\prod_{\ell=1}^{i+j}(|k_\ell|^\mu\langle k_\ell\rangle^{-2-\mu})|k_m|^{-|\alpha|}, \tag{1.32}$$

where $\mu > -1/2$ and, as above, η_1 and η_2 are estimating operators (unbounded, positive operators with bounded inverses) on the particle space $\mathcal{H}_p$ such that there exists $\Sigma > \inf\sigma(H_p)$ so that (1.7) holds.

We define the norm $\langle g\rangle := \sum_{1\le i+j\le 2}\sum_{|\alpha|\le 2}\|\eta_1^{2-i-j}\eta_2^{|\alpha|}\partial^\alpha g_{ij}\|$ of the vector coupling operators $g := (g_{ij})$, extending the norms of the scalar coupling operators g, introduced above. It is easy to extend Theorem 1.1 to the hamiltonians of the form (1.29)–(1.32) satisfying (1.7):

Theorem 1.2. *Theorem 1.1 still holds if we replace hamiltonians of the form* (1.1) *or* (1.4)–(1.6) *with hamiltonians of the form* (1.29)–(1.32)*, with* (1.7)*.*

As mentioned above, Theorem 1.2 is proven in Section 6.

Finally, a low momentum bound of [10] and some standard technical statements are given in Appendices A, B, C and D. The paper is essentially self-contained. In order to make it more accessible to non-experts, we included Supplement I giving standard definitions, proof of the existence and properties of the wave operators, and Supplement II defining and discussing the creation and annihilation operators (see also [21, 17]).

Notations. For functions A and B, we will use the notation $A \lesssim B$ signifying that $A \le CB$ for some absolute (numerical) constant $0 < C < \infty$. The symbol E_Δ stands for the characteristic function of a set Δ, while $\chi_{\cdot\le 1}$ denotes a smoothed out characteristic function of the interval $(-\infty, 1]$, that is it is in $\mathrm{C}^\infty(\mathbb{R})$, non-increasing, equal to 1 if $x \le 1/2$ and equal to 0 if $x \ge 1$. Moreover, $\chi_{\cdot\ge 1} := \mathbf{1} - \chi_{\cdot\le 1}$ and $\chi_{\cdot=1}$ stands for the derivative of $\chi_{\cdot\ge 1}$. Given a self-adjoint operator a and a real number α, we write $\chi_{a\le\alpha} := \chi_{\frac{a}{\alpha}\le 1}$, and likewise for $\chi_{a\ge\alpha}$. Finally, $D(A)$ denotes the domain of an operator A, $\langle x\rangle := (1+|x|^2)^{1/2}$, $\mathcal{O}(\epsilon)$ denotes an operator bounded by $C\epsilon$, $o_t(1)$ denotes a real number tending to 0 as $t\to\infty$, and $C(\epsilon)o_t(1)$ denotes a real number (depending on ϵ and t) which goes to 0 as $t\to\infty$ for any fixed ϵ.

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2. Method of propagation observables

Many steps of our proof use the method of propagation observables which we formalize in what follows. Let $\psi_t = e^{-itH}\psi_0$, where H is a hamiltonian of the form (1.4)–(1.5), with the coupling operator $g(k)$ satisfying (1.6) and (1.7). The method reduces propagation estimates for our system say of the form

$$\int_0^\infty dt\,\big\|G_t^{\frac12}\psi_t\big\|^2 \lesssim \|\psi_0\|_\#^2, \tag{2.1}$$

for some norm $\|\cdot\|_\# \ge \|\cdot\|$, to differential inequalities for certain families ϕ_t of positive, one-photon operators on the one-photon space $L^2(\mathbb{R}^3)$.

We introduce some notation and definitions. For $A \ge -C$, we denote $\|\psi_0\|_A := \|(A+C+1)^{\frac12}\psi_0\|$. Recall the notations $N_\rho = \mathrm{d}\Gamma(\omega^{-\rho})$ and let

$$\Upsilon_\rho = \big\{\psi_0 \in f(H)D(N_\rho^{\frac12}),\ \text{for some } f\in \mathrm{C}_0^\infty((-\infty,\Sigma))\big\}. \tag{2.2}$$

Notice that, since $N_{-1}f(H) = H_f f(H)$ is bounded as follows from the bound

$$\|\psi_t\|_{H_f} \lesssim \|\psi_0\|_H, \tag{2.3}$$

one easily verifies that $\Upsilon_\rho \subset \Upsilon_{\rho'}$ for $\rho \ge \rho' \ge -1$.

We define $\nu_\rho \ge 0$ as the smallest real number satisfying the inequality

$$\langle\psi_t, N_\rho\psi_t\rangle \lesssim t^{\nu_\rho}\|\psi_0\|_\rho^2, \tag{2.4}$$

for any $\psi_0 \in \operatorname{Ran} E_{(-\infty,\Sigma)}(H)$, where $\|\psi\|_\rho^2 := \|\psi\|_{N_\rho}^2$. It was shown in [10] (see (A.1) of Appendix A) that, for any $-1 \le \rho \le 1$, the inequality (2.4) is satisfied with

$$\nu_\rho \le \frac{1+\rho}{2+\mu}, \tag{2.5}$$

where μ is defined by (1.6) (this generalizes an earlier result due to [35]). Also, (2.3) implies that (2.4) holds for $\rho = -1$ with $\nu_{-1} = 0$. Let

$$d\phi_t := \partial_t \phi_t + i[\omega, \phi_t].$$

We isolate the following useful class of families of positive, one-photon operators:

Definition 2.1. A family of positive operators ϕ_t on $L^2(\mathbb{R}^3)$ will be called a *one-photon weak propagation observable*, if it has the following properties

- there are $\delta \ge 0$ and a family p_t of non-negative operators, such that
$$\|\omega^{\delta/2}\phi_t\omega^{\delta/2}\| \lesssim \langle t\rangle^{-\nu_\delta} \quad \text{and} \quad d\phi_t \ge p_t + \sum_{\text{finite}} \text{rem}_i, \tag{2.6}$$
where rem_i are one-photon operators satisfying
$$\|\omega^{\rho_i/2}\, \text{rem}_i\, \omega^{\rho_i/2}\| \lesssim \langle t\rangle^{-\lambda_i}, \tag{2.7}$$
for some ρ_i and λ_i, s.t. $\lambda_i > 1 + \nu_{\rho_i}$,
- for some $\lambda' > 1 + \nu_\delta$ and with η_1, η_2 satisfying (1.7),
$$\Big(\int \|\eta_1\eta_2^2(\phi_t g)(k)\|^2_{\mathcal{H}_p}\omega(k)^\delta dk\Big)^{\frac{1}{2}} \lesssim \langle t\rangle^{-\lambda'}. \tag{2.8}$$
(Here ϕ_t acts on g as a function of k.)

Similarly, a family of operators ϕ_t on $L^2(\mathbb{R}^3)$ will be called a *one-photon strong propagation observable*, if

$$d\phi_t \le -p_t + \sum_{\text{finite}} \text{rem}_i, \tag{2.9}$$

with $p_t \ge 0$, rem_i are one-photon operators satisfying (2.7) for some $\lambda_i > 1 + \nu_{\rho_i}$, and (2.8) holds for some $\lambda' > 1 + \nu_\delta$.

The following proposition reduces proving inequalities of the type of (2.1) to showing that ϕ_t is a one-photon weak or strong propagation observable, i.e. to *one-photon estimates* of $d\phi_t$ and $\phi_t g$.

Proposition 2.2. *If ϕ_t is a one-photon weak (resp. strong) propagation observable, then we have either the weak propagation estimate,* (2.1)*, or the strong propagation estimate,*

$$\langle\psi_t, \Phi_t\psi_t\rangle + \int_0^\infty dt\, \big\|G_t^{\frac{1}{2}}\psi_t\big\|^2 \lesssim \|\psi_0\|_\#^2, \tag{2.10}$$

with the norm $\|\psi_0\|_\#^2 := \|\psi_0\|_\diamond^2 + \|\psi_0\|_*^2$, where $\Phi_t := \mathrm{d}\Gamma(\phi_t)$, $G_t := \mathrm{d}\Gamma(p_t)$, $\|\psi_0\|_* := \|\psi_0\|_\delta$ and $\|\psi_0\|_\diamond := \sum \|\psi_0\|_{\rho_i}$, on the subspace $\Upsilon_{\max(\delta,\rho_i)}$.

Before proceeding to the proof we present some useful definitions. Consider families Φ_t of operators on $\mathcal{H}$ and introduce the Heisenberg derivative

$$D\Phi_t := \partial_t\Phi_t + i\big[H, \Phi_t\big],$$

with the property

$$\partial_t\langle\psi_t, \Phi_t\psi_t\rangle = \langle\psi_t, D\Phi_t\psi_t\rangle. \tag{2.11}$$

Definition 2.3. A family of self-adjoint operators Φ_t on a subspace $\mathcal{H}_1 \subset \mathcal{H}$ will be called a (second quantized) *weak propagation observable*, if for all $\psi_0 \in \mathcal{H}_1$, it has the following properties

- $\sup_t\langle\psi_t, \Phi_t\psi_t\rangle \lesssim \|\psi_0\|_*^2$;
- $D\Phi_t \ge G_t + \text{Rem}$, where $G_t \ge 0$ and $\int_0^\infty dt\, |\langle\psi_t, \text{Rem}\,\psi_t\rangle| \lesssim \|\psi_0\|_\diamond^2$,

for some norms $\|\psi_0\|_*$, $\|\cdot\|_\diamond \ge \|\cdot\|$. Similarly, a family of operators Φ_t will be called a *strong propagation observable*, if it has the following properties

- Φ_t is a family of non-negative operators;

- $D\Phi_t \leq -G_t + \text{Rem}$, where $G_t \geq 0$ and $\int_0^\infty dt\, |\langle \psi_t, \text{Rem}\, \psi_t\rangle| \lesssim \|\psi_0\|_\#^2$,

for some norm $\|\cdot\|_\# \geq \|\cdot\|$.

If Φ_t is a weak propagation observable, then integrating the corresponding differential inequality sandwiched by ψ_t's and using the estimate on $\langle \psi_t, \Phi_t \psi_t\rangle$ and on the remainder Rem, we obtain the (weak propagation) estimate (2.1), with $\|\psi_0\|_\#^2 := \|\psi_0\|_\diamond^2 + \|\psi_0\|_*^2$. If Φ_t is a strong propagation observable, then the same procedure leads to the (strong propagation) estimate (2.10).

Proof of Proposition 2.2. Let $\Phi_t := \mathrm{d}\Gamma(\phi_t)$. To prove the above statement we use the relations (see Supplement II)

$$D_0 \mathrm{d}\Gamma(\phi_t) = \mathrm{d}\Gamma(d\phi_t), \qquad i[I(g), \mathrm{d}\Gamma(\phi_t)] = -I(i\phi_t g), \tag{2.12}$$

where D_0 is the free Heisenberg derivative,

$$D_0 \Phi_t := \partial_t \Phi_t + i[H_0, \Phi_t],$$

valid for any family of one-particle operators ϕ_t, to compute

$$D\Phi_t = \mathrm{d}\Gamma(d\phi_t) - I(i\phi_t g). \tag{2.13}$$

Denote $\langle A\rangle_\psi := \langle \psi, A\psi\rangle$. Applying the Cauchy-Schwarz inequality, we find the following version of a standard estimate

$$|\langle I(g)\rangle_\psi| \leq 2\Big(\int \|\eta_1 \eta_2^2 g(k)\|_{\mathcal{H}_p}^2 \omega(k)^\delta d^3k\Big)^{\frac{1}{2}} \|\eta_1^{-1}\eta_2^{-2}\psi\| \|\psi\|_\delta. \tag{2.14}$$

Using that $\psi_t = f_1(H)\psi_t$, with $f_1 \in \mathrm{C}_0^\infty((-\infty, \Sigma))$, $f_1 f = f$, and using (1.7), we find $\|\eta_1^{-1}\eta_2^{-2}\psi_t\| \lesssim \|\psi_t\|$. Taking this into account, we see that the equations (2.14), (2.8) and (2.3) yield

$$|\langle I(i\phi_t g)\rangle_{\psi_t}| \lesssim \langle t\rangle^{-\lambda' + \nu_\delta} \|\psi_0\|_\delta^2. \tag{2.15}$$

Next, using (2.7), we find $\pm \mathrm{rem}_i \leq \|\omega^{\rho_i/2}\, \mathrm{rem}_i\, \omega^{\rho_i/2}\| \omega^{\rho_i} \lesssim \langle t\rangle^{-\lambda_i}\omega^{-\rho_i}$. This gives $\pm\mathrm{d}\Gamma(\mathrm{rem_i}) \lesssim \langle t\rangle^{-\lambda_i}\mathrm{d}\Gamma(\omega^{-\rho_i})$, which, due to the bound (2.4), leads to the estimate

$$\big|\langle \mathrm{d}\Gamma(\mathrm{rem_i})\rangle_{\psi_t}\big| \lesssim \langle t\rangle^{-\lambda_i + \nu_{\rho_i}} \|\psi_0\|_{\rho_i}^2. \tag{2.16}$$

Let $G_t := \mathrm{d}\Gamma(p_t)$ and $\mathrm{Rem} := \sum_{\text{finite}} \mathrm{d}\Gamma(\mathrm{rem_i}) - I(i\phi_t g)$. We have $G_t \geq 0$, and, by (2.15) and (2.16),

$$\int_0^\infty dt\, \big|\langle \psi_t, \mathrm{Rem}\, \psi_t\rangle\big| \lesssim \|\psi_0\|_\diamond^2, \tag{2.17}$$

with $\|\psi_0\|_\#^2 := \|\psi_0\|_\diamond^2 + \|\psi_0\|_*^2$, $\|\psi_0\|_* := \|\psi_0\|_\delta$, $\|\psi_0\|_\diamond := \sum \|\psi_0\|_{\rho_i}$.

In the strong case, (2.9) and (2.13) imply

$$D\Phi_t \leq -G_t + \mathrm{Rem}, \tag{2.18}$$

and hence by (2.17), Φ_t is a strong propagation observable.

In the weak case, (2.6) and (2.13) imply

$$D\Phi_t \geq G_t + \mathrm{Rem}. \tag{2.19}$$

Since $\phi_t \leq \|\omega^{\delta/2}\phi_t\omega^{\delta/2}\|\omega^{-\delta} \lesssim \langle t\rangle^{-\nu_\delta}\omega^{-\delta}$, we have $\mathrm{d}\Gamma(\phi_t) \lesssim \langle t\rangle^{-\nu_\delta}\mathrm{d}\Gamma(\omega^{-\delta})$. Using this estimate and using again the bound (2.4), we obtain

$$\langle \psi_t, \Phi_t\psi_t\rangle \lesssim \langle t\rangle^{-\nu_\delta}\langle \mathrm{d}\Gamma(\omega^{-\delta})\rangle_{\psi_t} \lesssim \|\psi_0\|_\delta^2. \tag{2.20}$$

Estimates (2.17) and (2.20) show that Φ_t is a weak propagation observable. □

Remarks.

1) Proposition 2.2 reduces a proof of propagation estimates for the dynamics (1.10) to estimates involving the *one-photon* datum (ω, g) (an 'effective one-photon system'), parameterizing the hamiltonian (1.4). (The remaining datum H_p does not enter our analysis explicitly, but through the bound states of H_p which lead to the localization in the particle variables, (1.7)).

2) The condition on the remainder in (2.6) can be weakened to $\mathrm{rem} = \mathrm{rem}' + \mathrm{rem}''$, with rem' and rem'' satisfying (2.7) and

$$|\mathrm{rem}''| \lesssim \chi_{|y| \geq c't}, \tag{2.21}$$

for c' as in (1.21), respectively. Moreover, (2.7) can be further weakened to

$$\int_0^\infty dt\,|\langle\psi_t, \mathrm{d}\Gamma(\mathrm{rem}_i)\psi_t\rangle| < \infty. \tag{2.22}$$

3. The first propagation estimate

Let $\nu_\delta \geq 0$ be the same as in (2.4) and recall the operator b_ϵ defined in the introduction. We write it as

$$b_\epsilon := \frac{1}{2}(\theta_\epsilon \nabla\omega \cdot y + y \cdot \nabla\omega\,\theta_\epsilon), \quad \text{where} \quad \theta_\epsilon := \frac{\omega}{\omega_\epsilon},\ \omega_\epsilon := \omega + \epsilon,\ \epsilon = t^{-\kappa}. \tag{3.1}$$

We prove the following two results.

Theorem 3.1. *Consider hamiltonians of the form* (1.4)–(1.5) *with the coupling operators satisfying* (1.6) *with* $\mu > -1/2$ *and* (1.7)*. Let* $\nu_1 - \nu_0 < \kappa < 1$*. If either* $\alpha < 1$*, or* $\alpha = 1$ *and* $c < 1$*, and*

$$\alpha > \max((3/2+\mu)^{-1}, (1+\kappa)/2, 1-\kappa+\nu_1-\nu_0), \tag{3.2}$$

then for any initial condition $\psi_0 \in \Upsilon_1$*, the Schrödinger evolution,* ψ_t*, satisfies, for any* $a > 1$*, the following estimates*

$$\int_1^\infty dt\ t^{-\alpha-a\nu_0}\left\|\mathrm{d}\Gamma(\chi_{\frac{b_\epsilon}{ct^\alpha}=1})^{\frac12}\psi_t\right\|^2 \lesssim \|\psi_0\|_1^2. \tag{3.3}$$

If $\nu_0 = 0$*,* $\mu > 0$*,* α *satisfies* (3.2) *and* $\alpha < \frac{1}{\bar c}$*, with* $\bar c > 1$*, then, with the notation* $\chi \equiv \chi_{(\frac{|y|}{\bar c t})^2 \leq 1}$*,*

$$\int_1^\infty dt\ t^{-\alpha}\left\|\mathrm{d}\Gamma(\theta_\epsilon^{\frac12}\chi\chi_{\frac{b_\epsilon}{ct^\alpha}=1}\chi\theta_\epsilon^{\frac12})^{\frac12}\psi_t\right\|^2 \lesssim \|\psi_0\|_0^2. \tag{3.4}$$

Proof. We will use the method of propagation observables outlined in Section 2. We consider the one-parameter family of one-photon operators

$$\phi_t := t^{-a\nu_0}\chi_\alpha, \quad \chi_\alpha \equiv \chi_{v\geq 1}, \quad v := \frac{b_\epsilon}{ct^\alpha}, \tag{3.5}$$

where $a > 1$. To show that ϕ_t is a weak one-photon propagation observable, we obtain differential inequalities for ϕ_t. Recall that $d\phi_t = \partial_t\phi_t + i[\omega, \phi_t]$. To compute $d\phi_t$, we use the expansion

$$d\phi_t = t^{-a\nu_0}(dv)\chi_\alpha' + \sum_{i=1}^2 \mathrm{rem}_i, \tag{3.6}$$

$$\mathrm{rem}_1 := t^{-a\nu_0}[d\chi_\alpha - (dv)\chi_\alpha'], \quad \mathrm{rem}_2 := -a\nu_0 t^{-1}\phi_t. \tag{3.7}$$

Using the definitions in (3.1), we compute

$$dv = \frac{1}{ct^\alpha}\big(\theta_\epsilon - \frac{\alpha b_\epsilon}{t} + \partial_t b_\epsilon\big). \tag{3.8}$$

Next, we have $\partial_t b_\epsilon = \frac{\kappa}{2t^{1+\kappa}}(\omega_\epsilon^{-1}\theta_\epsilon\nabla\omega\cdot y +$ h.c.$)$ on $D(b_\epsilon)$, which, due to the relation $\frac12(\omega_\epsilon^{-1}\theta_\epsilon\nabla\omega\cdot y +$ h.c.$) = \omega_\epsilon^{-1/2}b_\epsilon\omega_\epsilon^{-1/2}$, becomes

$$\partial_t b_\epsilon = \frac{\kappa}{t^{1+\kappa}}\omega_\epsilon^{-1/2}b_\epsilon\omega_\epsilon^{-1/2}. \tag{3.9}$$

Using that (see Lemma B.1 of Appendix B)

$$\omega_\epsilon^{-1/2}b_\epsilon\omega_\epsilon^{-1/2}\chi_\alpha' = \omega_\epsilon^{-1/2}b_\epsilon\chi_\alpha'\omega_\epsilon^{-1/2} + \mathcal{O}(t^{\frac32\kappa}),$$

and that $b_\epsilon \geq 0$ on $\operatorname{supp}\chi_\alpha'$, we obtain

$$\partial_t b_\epsilon\chi_\alpha' \geq -\frac{\mathrm{const}}{t^{1-\kappa/2}}. \tag{3.10}$$

The relations (3.6)–(3.10), together with $\frac{b_\epsilon}{ct^\alpha}\chi_\alpha' \leq \chi_\alpha'$, imply

$$d\phi_t \geq t^{-a\nu_0}\big(\frac{\theta_\epsilon}{ct^\alpha} - \frac{\alpha}{t}\big)\chi_\alpha' + \sum_{i=1}^3 \mathrm{rem}_i, \tag{3.11}$$

where rem_1 and rem_2 are given in (3.7) and

$$\mathrm{rem}_3 = \mathcal{O}(t^{-1-\alpha+\frac{\kappa}{2}-a\nu_0}). \tag{3.12}$$

This, together with $\theta_\epsilon = 1 - \frac{t^{-\kappa}}{\omega_\epsilon}$ and $\omega_\epsilon^{-1}\chi'_\alpha = \omega_\epsilon^{-1/2}\chi'_\alpha\omega_\epsilon^{-1/2} + \mathcal{O}(t^{-\alpha+\frac{3}{2}\kappa})$ (see again Lemma B.1 of Appendix B), implies

$$d\phi_t \geq t^{-a\nu_0}\big(\frac{1}{ct^\alpha} - \frac{\alpha}{t}\big)\chi'_\alpha + \sum_{i=1}^{5}\mathrm{rem}_i, \tag{3.13}$$

$$\mathrm{rem}_4 := \frac{1}{ct^{\alpha+\kappa+a\nu_0}}\omega_\epsilon^{-1/2}\chi'_\alpha\omega_\epsilon^{-1/2}, \quad \mathrm{rem}_5 = \mathcal{O}(t^{-2\alpha+\frac{\kappa}{2}-a\nu_0}). \tag{3.14}$$

We have $\|\phi_t\| \leq t^{-a\nu_0}$ and therefore the first estimate in (2.6) holds with $\delta = 0$. If either $\alpha < 1$ (and t sufficiently large), or $\alpha = 1$ and $c < 1$, then $p_t := t^{-a\nu_0}(\frac{1}{ct^\alpha} - \frac{\alpha}{t})\chi'_\alpha$ is non-negative, which implies the second estimate in (2.6). Thus (2.6) holds. By the definition (3.6) and Corollary B.3 of Appendix B for $i = 1$, and by an explicit form for $i = 2, 3, 4, 5$, we have the estimates

$$\|\omega^{\rho_i/2}\,\mathrm{rem}_i\,\omega^{\rho_i/2}\| \lesssim t^{-\lambda_i}, \tag{3.15}$$

$i = 1, 2, 3, 4, 5$, with $\rho_1 = \rho_2 = \rho_3 = \rho_5 = 0$, $\rho_4 = 1$, $\lambda_1 = 2\alpha - \kappa + a\nu_0$, $\lambda_2 = 1 + a\nu_0$, $\lambda_3 = 1 + \alpha - \kappa/2 + a\nu_0$, $\lambda_4 = \alpha + \kappa + a\nu_0$, and $\lambda_5 = 2\alpha - \kappa/2 + a\nu_0$. We remark here that the $i = 2$ term is absent if $\nu_0 = 0$. The relation (3.15) implies (2.7) with $\rho = \rho_i$ and $\lambda = \lambda_i$ provided $\lambda_i > 1 + \nu_{\rho_i}$.

Finally, in the same way as [10, Lemma 3.1], one shows (by replacing $|y|$ with b_ϵ in that lemma) that, under (1.6) for some $\mu \geq -\frac{1}{2}$,

$$\big\|\eta_1\eta_2^2\chi_{\frac{b_\epsilon}{ct^\alpha}\geq 1}g(k)\big\|_{L^2(\mathbb{R}^3;\mathcal{H}_p)} \lesssim t^{-\tau}, \qquad \tau < (\frac{3}{2} + \mu)\alpha, \tag{3.16}$$

which implies (2.8) with $\lambda' < a\nu_0 + (\frac{3}{2} + \mu)\alpha$. Hence ϕ_t is a weak one-photon propagation observable, provided $2\alpha > 1 + \kappa + \nu_0 - a\nu_0$, $\alpha - \kappa/2 > \nu_0 - a\nu_0$, $\alpha + \kappa > 1 + \nu_1 - a\nu_0$, and $(\frac{3}{2} + \mu)\alpha > 1$. Therefore, by Proposition 2.2, we have, under the conditions on the parameters above,

$$\int_1^\infty dt\; t^{-\alpha-a\nu_0}\|\mathrm{d}\Gamma(\chi'_\alpha)^{\frac{1}{2}}\psi_t\|^2 \lesssim \|\psi_0\|_1^2. \tag{3.17}$$

This, due to the definition of χ'_α, implies the estimate (3.3).

We now prove (3.4). We use again the notation $\chi_\alpha \equiv \chi_{v\geq 1}$, where $v := \frac{b_\epsilon}{ct^\alpha}$, and we denote $w := (\frac{|y|}{\bar{c}t})^2$. We consider the one-parameter family of one-photon operators

$$\phi_t := \chi\chi_\alpha\chi, \tag{3.18}$$

and show that ϕ_t is a weak one-photon propagation observable. We have $\|\phi_t\| \leq 1$ and therefore, due to the assumption $\nu_0 = 0$, the first estimate in (2.6) holds with $\delta = 0$. Now, we show the second estimate in (2.6). To compute $d\phi_t$, we use the expansion

$$d\phi_t = \chi(dv)\chi'_\alpha\chi + \chi'(dw)\chi_\alpha\chi + \chi\chi_\alpha(dw)\chi' + \sum_{i=1,2}\mathrm{rem}_i, \tag{3.19}$$

where

$$\mathrm{rem}_1 := \chi(d\chi_\alpha - (dv)\chi'_\alpha)\chi, \quad \mathrm{rem}_2 := (d\chi - (dw)\chi')\chi_\alpha\chi + \text{h.c.}. \tag{3.20}$$

As in (3.8)–(3.10), we have

$$\chi(dv)\chi'_\alpha\chi \geq \frac{1}{ct^\alpha}\chi(\theta_\epsilon - \frac{\alpha b_\epsilon}{t})\chi'_\alpha\chi + \mathrm{rem}_3, \tag{3.21}$$

where $\mathrm{rem}_3 = \mathcal{O}(t^{-1-\alpha+\kappa/2})$. We consider the term $-(\alpha b_\epsilon)/(ct^{\alpha+1})$ in (3.21). By Lemma B.1 of Appendix B, we have

$$\chi b_\epsilon\chi'_\alpha\chi = \chi(\chi'_\alpha)^{\frac{1}{2}}b_\epsilon(\chi'_\alpha)^{\frac{1}{2}}\chi = (\chi'_\alpha)^{\frac{1}{2}}\chi b_\epsilon\chi(\chi'_\alpha)^{\frac{1}{2}} + \mathcal{O}(t^\kappa).$$

Next, we recall (3.1) and observe that b_ϵ can be rewritten as $b_\epsilon = \theta_\epsilon^{1/2}b_0\theta_\epsilon^{1/2}$, with $b_0 := \frac{1}{2}(\nabla\omega\cdot y + \text{h.c.})$. Note that b_0 is not a self-adjoint operator, only maximal symmetric. Nevertheless, using Hardy's inequality, one easily verifies that b_0 is well-defined on $D(|y|)$ and that $b_0\langle y\rangle^{-1}$ and $\langle y\rangle^{-1}b_0$ extend to bounded operators. Thus, using again Lemma B.1 of Appendix B, we deduce that

$$(\chi'_\alpha)^{\frac12}\chi b_\epsilon\chi(\chi'_\alpha)^{\frac12} = (\chi'_\alpha)^{\frac12}\theta_\epsilon^{\frac12}\chi b_0\chi\theta_\epsilon^{\frac12}(\chi'_\alpha)^{\frac12} + \mathcal{O}(t^{\frac12+\frac{\kappa}{2}}).$$

The maximal velocity cut-off gives $\chi b_0\chi \le \bar{c}t\chi^2$ and hence, commuting again χ through $\theta_\epsilon^{1/2}$ and $(\chi'_\alpha)^{1/2}$, using Lemma B.1, we obtain

$$-\chi\frac{b_\epsilon}{t}\chi'_\alpha\chi \ge -\bar{c}\chi\theta_\epsilon^{\frac12}\chi'_\alpha\theta_\epsilon^{\frac12}\chi + \mathcal{O}(\frac{1}{t^{\frac12-\frac{\kappa}{2}}}). \tag{3.22}$$

Another application of Lemma B.1 for the term $\theta_\epsilon/(ct^\alpha)$ in (3.21) gives

$$\chi\theta_\epsilon\chi'_\alpha\chi = \chi\theta_\epsilon^{\frac12}\chi'_\alpha\theta_\epsilon^{\frac12}\chi + \mathcal{O}(\frac{1}{t^{\alpha-\kappa}}). \tag{3.23}$$

Since $\alpha > (1+\kappa)/2$, we deduce from (3.22) and (3.23) that

$$\begin{aligned}\chi\big(\theta_\epsilon - \frac{\alpha b_\epsilon}{t}\big)\chi'_\alpha\chi &\ge (1-\alpha\bar{c})\chi\theta_\epsilon^{\frac12}\chi'_\alpha\theta_\epsilon^{\frac12}\chi + \mathcal{O}(\frac{1}{t^{\frac12-\frac{\kappa}{2}}})\\ &\ge (1-\alpha\bar{c})\theta_\epsilon^{\frac12}\chi\chi'_\alpha\chi\theta_\epsilon^{\frac12} + \mathcal{O}(\frac{1}{t^{\frac12-\frac{\kappa}{2}}}),\end{aligned} \tag{3.24}$$

where in the last inequality we used again Lemma B.1 and the fact that $\alpha > (1+\kappa)/2$.

Next, we address the second and third terms on the r.h.s. of (3.19). We compute

$$dw = 2\big(\frac{b_0}{(\bar{c}t)^2} - \frac{w}{t}\big).$$

By Lemma B.1 of Appendix B and the observation that dw enters (3.19) in combination with $(-\chi')^{\frac12}$, which is bounded, we have

$$\chi'(dw)\chi_\alpha\chi + \chi\chi_\alpha(dw)\chi' = -2(\chi_\alpha)^{\frac12}(-\chi'\chi)^{\frac12}(dw)(-\chi'\chi)^{\frac12}(\chi_\alpha)^{\frac12} + \mathcal{O}(\frac{1}{t^{1+\alpha-\kappa}}). \tag{3.25}$$

Using that $dw \le (\frac{1}{\bar{c}} - 1)\frac{1}{t}$ on the support of χ' and that $\chi' \le 0$ and $\bar{c} > 1$, we obtain

$$(-\chi'\chi)^{\frac12}(dw)(-\chi'\chi)^{\frac12} \ge (1-\frac{1}{\bar{c}})\frac{1}{t}(-\chi'\chi). \tag{3.26}$$

The relations (3.19), (3.21), (3.24), (3.25) and (3.26) imply

$$d\phi_t \ge p_t + \tilde{p}_t - \sum_{i=1,2,3,4}\mathrm{rem}_i, \tag{3.27}$$

where $\mathrm{rem}_4 = \mathcal{O}(\frac{1}{t^{\alpha+1/2-\kappa/2}})$ and

$$p_t := \frac{1-\alpha\bar{c}}{ct^\alpha}\theta_\epsilon^{\frac12}\chi\chi'_\alpha\chi\theta_\epsilon^{\frac12}, \tag{3.28}$$

$$\tilde{p}_t := (1-\frac{1}{\bar{c}})\frac{1}{t}\chi_\alpha^{\frac12}(-\chi')\chi\chi_\alpha^{\frac12}. \tag{3.29}$$

The terms p_t and $\tilde{p}_t$ are non-negative, provided $\alpha < 1/\bar{c}$ and $\bar{c} > 1$. This implies the second estimate in (2.6). Next, we claim the estimates

$$\|\mathrm{rem}_i\| \lesssim t^{-\lambda}, \tag{3.30}$$

$i = 1,2,3,4$, with $\lambda = 1/2 + \alpha - \kappa/2$. Indeed, the definition (3.20) and Corollary B.3 of Appendix B imply (3.30) for $i = 1$ since $1/2+\alpha-\kappa/2 < 2\alpha-\kappa$. The estimate for $i = 3,4$ are obvious. To estimate rem_2, we write

$$(d\chi - (dw)\chi')\chi_\alpha\chi = (d\chi - (dw)\chi')v\tilde{\chi}_\alpha\chi = \frac{1}{ct^\alpha}(d\chi - (dw)\chi')\tilde{\chi}_\alpha b_\epsilon\chi,$$

where $\tilde{\chi}_\alpha = v^{-1}\chi_\alpha$ and, recall, $v = \frac{b_\epsilon}{ct^\alpha}$. Using that $b_\epsilon = \theta_\epsilon b_0 + i\epsilon\omega_\epsilon^{-2}$ and that, by Lemma B.4 of Appendix B,

$$\|d\chi - (dw)\chi'\| \lesssim t^{-1},$$

gives

$$(d\chi-(dw)\chi')\chi_\alpha\chi=\frac{1}{ct^\alpha}(d\chi-(dw)\chi')\tilde\chi_\alpha\theta_\epsilon b_0\chi+\mathcal{O}(\frac{1}{t^{1+\alpha-\kappa}}). \tag{3.31}$$

Using in addition Lemma B.1 of Appendix B and the estimate $b_0\chi=\mathcal{O}(t)$, this yields

$$(d\chi-(dw)\chi')\chi_\alpha\chi=\frac{1}{ct^\alpha}(d\chi-(dw)\chi')\omega\tilde\chi_\alpha\omega_\epsilon^{-1}b_0\chi+\mathcal{O}(\frac{1}{t^{2\alpha-\kappa}}). \tag{3.32}$$

By Lemma B.4, we also have

$$\|(d\chi-(dw)\chi')\omega\|\lesssim t^{-2}.$$

Combining this with (3.32) and the estimates $\omega_\epsilon^{-1}=\mathcal{O}(t^\kappa)$ and $b_0\chi=\mathcal{O}(t)$, we obtain

$$(d\chi-(dw)\chi')\chi_\alpha\chi=\mathcal{O}(\frac{1}{t^{2\alpha-\kappa}}), \tag{3.33}$$

and hence, since $1/2+\alpha-\kappa/2<2\alpha-\kappa$, the estimate for $i=2$ follows.

The relation (3.30) implies (2.7) with $\lambda=1/2+\alpha-\kappa/2$, for rem $=$ rem$_i$, provided $1/2+\alpha-\kappa/2>1$. Finally, as above, (2.8) holds with $\lambda'<a\nu_0+(\frac{3}{2}+\mu)\alpha$ by (3.16). This yields (3.4). □

Remark. The estimate (3.3) is sharp if $\nu_0=0$. Assuming this and taking $\nu_1\le(3/2+\mu)^{-1}$ (see (A.7) of Appendix A), the conditions on α in Theorems 3.1 and 4.1 become $\alpha>\frac{5}{6}+\frac{1}{6(3/2+\mu)}$, and $(3/2+\mu)^{-1}<\alpha<1$, respectively.

4. The second propagation estimate

Recall the norm $\langle g\rangle=\sum_{|\alpha|\le2}\|\eta_1\eta_2^{|\alpha|}\partial^\alpha g\|_{L^2(\mathbb{R}^3,\mathcal{H}_p)}$ for the coupling function g and the notation $\langle A\rangle_\psi=\langle\psi,A\psi\rangle$. We will use the following set of initial conditions.

$$\Upsilon_\#:=\{\psi_0\in f(H)\big(D(\mathrm{d}\Gamma(\langle y\rangle))\cap D(\mathrm{d}\Gamma(b)^2)\big),\text{ for some } f\in C_0^\infty((E_{\rm gs},a])\},$$

where $b=\frac{1}{2}(k\cdot y+y\cdot k)$ and $a<\Sigma$ is given by Assumption (1.15).

Theorem 4.1. *Consider hamiltonians of the form* (1.4)–(1.5) *with the coupling operators satisfying* (1.6) *with $\mu>-1/2$ and* (1.7)*. Assume that* (1.15) *holds. Let $\langle g\rangle$ be sufficiently small, $\nu_1<\kappa<1$, and $0<\alpha<1$. Let $\psi_0\in\Upsilon_\#$. Then the Schrödinger evolution, ψ_t, satisfies the estimate*

$$\big\|\Gamma(\chi_{\frac{b_\epsilon}{ct^\alpha}\le1})^{\frac12}\psi_t\big\|^2\lesssim t^{-\delta}\big(\|\psi_0\|^2_{\mathrm{d}\Gamma(\langle y\rangle)}+\|\psi_0\|^2_{\mathrm{d}\Gamma(b)^2}\big), \tag{4.1}$$

for $0\le\delta<\min(\kappa-\nu_1,1-\kappa,1-\alpha-\nu_0)$ and any $c>0$, where, recall, $b=\frac12(k\cdot y+y\cdot k)$.

We define $B_\epsilon:=\mathrm{d}\Gamma(b_\epsilon)$ and $B_{\epsilon,t}:=B_\epsilon/(ct)$. As in [10, Proposition B.3 and Remark B.4], one verifies that $\Upsilon_\#\subset D(\mathrm{d}\Gamma(\langle y\rangle))\subset D(B_\epsilon)$. The proof of Theorem 4.1 is based on the following result (cf. [61, 45]).

Proposition 4.2. *Under the conditions of Theorem 4.1, the evolution $\psi_t=e^{-iHt}\psi_0$ obeys*

$$\big\|\chi_{B_{\epsilon,t}\le1}\psi_t\big\|^2\lesssim t^{-\delta'}\big(\|\psi_0\|^2_{\mathrm{d}\Gamma(\langle y\rangle)}+\|\psi_0\|^2_{\mathrm{d}\Gamma(b)^2}\big), \tag{4.2}$$

for any $0<c<(1-C\langle g\rangle)/(1+\kappa)$, where $\delta':=\min\big(\frac{1-C\langle g\rangle}{c}-1-\kappa,1-\kappa,\kappa-\nu_1\big)$.

Remark. The constant C is independent of $\gamma_0:=\mathrm{dist}(E_{\rm gs},\operatorname{supp}f)$ (but the implicit constant appearing in the right hand side of (4.2) does depend on γ_0).

Proof. Let $\epsilon>0$ be a constant. Let $\rho<\min\big(\frac{1-C\langle g\rangle}{c}-1,1\big)$ where $C>0$ is a positive constant defined below (see (4.10)). Consider the propagation observable

$$\Phi_t:=-t^\rho\varphi\big(B_{\epsilon,t}\big),$$

where $\varphi\big(B_{\epsilon,t}\big):=\big(B_{\epsilon,t}-2\big)\chi_{B_{\epsilon,t}\le1}$. Note that $\varphi\le0$, but $\varphi'\ge0$. Let $\varphi'=\varphi_1^2$. We use the notations $\varphi:=\varphi(B_{\epsilon,t})$, $\chi:=\chi_{B_{\epsilon,t}\le1}\equiv\chi(B_{\epsilon,t})$, and likewise for φ', φ_1 and χ'. The relations below are understood in the sense of quadratic forms on $\Upsilon_\#$. The IMS formula gives

$$D\Phi_t=M+R, \tag{4.3}$$

where $M := -t^{\rho}\varphi_1(DB_{\epsilon,t})\varphi_1 - \rho t^{-1+\rho}\varphi$ and

$$R := \frac{1}{ct^{1-\rho}}[[B_1,\varphi_1],\varphi_1] + t^{\rho}\big([H,\varphi] - \frac{1}{2ct}(\varphi' B_1 + B_1\varphi')\big), \tag{4.4}$$

where $B_1 := i[H, B_\epsilon]$. First, we compute the main term, M, in (4.3). We leave out a standard proof of $f(H) \in C^1(B_\epsilon)$ (see e.g. [30, Theorem 8]) and standard domain questions such as that $\Upsilon_\# \subset D(B_\epsilon)$. We have

$$DB_{\epsilon,t} = \frac{1}{ct}DB_\epsilon - \frac{1}{t}B_{\epsilon,t}. \tag{4.5}$$

Since, by (II.3) of Supplement II, $i[H_f, B_\epsilon] = N_\epsilon$, where $N_\epsilon := \mathrm{d}\Gamma(\theta_\epsilon)$, we have

$$DB_\epsilon = N_\epsilon + I_1, \tag{4.6}$$

where $I_1 := i[I(g), B_\epsilon] = -I(ib_\epsilon g)$ (see (II.5) of Supplement II). To estimate the operator N_ϵ from below, we use that $\theta_\epsilon = 1 - \frac{\epsilon}{\omega_\epsilon}$, to obtain

$$N_\epsilon = N - \epsilon\mathrm{d}\Gamma(\omega_\epsilon^{-1}). \tag{4.7}$$

Next, Lemma C.2 of Appendix C and the bound (2.4) show that

$$\big\langle\varphi_1\mathrm{d}\Gamma(\omega_\epsilon^{-1})\varphi_1\big\rangle_{\psi_t} \lesssim t^{\nu_1}\|\psi_0\|_1^2 + t^{-1+\nu_0}\epsilon^{-2}\|\psi_0\|_0^2. \tag{4.8}$$

Define the first estimating operator $E_1 := N + \eta_2^{-1}\eta_1^{-2}\eta_2^{-1} + \mathbf{1}$. By (1.6), the condition $\mu > -1/2$ and (2.14) (with $\delta = 0$), we have

$$\|\eta_1\eta_2 I_1(N+\mathbf{1})^{-1/2}\| \lesssim \|\eta_1\eta_2 b_\epsilon g\| \lesssim \langle g\rangle, \tag{4.9}$$

and hence,

$$I_1 \geq -C\langle g\rangle E_1. \tag{4.10}$$

Combining this with the definition of M, (1.7), (4.5), (4.6), (4.7) and (4.8), we obtain

$$\langle M\rangle_{\psi_t} \leq -\frac{1}{ct^{1-\rho}}\big\langle\varphi_1\big((1-C\langle g\rangle)N - t^{-1}B_\epsilon - C\langle g\rangle\big)\varphi_1 + c\rho\varphi\big\rangle_{\psi_t} + \frac{C}{t^{1-\rho}}\big(\epsilon t^{\nu_1}\|\psi_0\|_1^2 + t^{-1+\nu_0}\epsilon^{-1}\|\psi_0\|_0^2\big). \tag{4.11}$$

Let $\Omega := 1 \oplus 0 \oplus \dots$ be the vacuum in $\mathcal{F}$ and P_Ω be the orthogonal projection on the subspace $\mathcal{H}_p \otimes \Omega$, $P_\Omega\Psi := \langle\Omega,\Psi\rangle_{\mathcal{F}} \otimes \Omega$. We now use the relation $\varphi_1 P_\Omega = P_\Omega$, together with the estimate

$$\big\|P_\Omega e^{-itH}f(H)u\big\| \lesssim t^{-s}\|\langle B\rangle u\|, \quad s < 1/2,$$

proven in Lemma 4.3 below, to obtain

$$\langle\varphi_1 P_\Omega\varphi_1\rangle_{\psi_t} = \langle P_\Omega\rangle_{\psi_t} \lesssim t^{-2s}\|\langle B\rangle\psi_0\|^2 \lesssim t^{-2s}\|\psi_0\|_{B^2}^2. \tag{4.12}$$

Combining this with $N \geq \mathbf{1} - P_\Omega$ and (4.11), we obtain

$$\langle M\rangle_{\psi_t} \leq -\frac{1}{ct^{1-\rho}}\big\langle\varphi_1[1 - t^{-1}B_\epsilon - C\langle g\rangle]\varphi_1 + c\rho\varphi\big\rangle_{\psi_t} + \frac{C}{t^{1-\rho}}\big(\epsilon t^{\nu_1}\|\psi_0\|_1^2 + t^{-1+\nu_0}\epsilon^{-1}\|\psi_0\|_0^2 + t^{-2s}\|\psi_0\|_{B^2}^2\big). \tag{4.13}$$

Now, recalling the definition $\varphi\big(B_{\epsilon,t}\big) := \big(B_{\epsilon,t} - 2\big)\chi_{B_{\epsilon,t}\leq 1}$, we compute

$$\begin{aligned} B_{\epsilon,t}\varphi' + \rho(-\varphi) &= B_{\epsilon,t}\big(\chi + (B_{\epsilon,t}-2)\chi'\big) - \rho(B_{\epsilon,t}-2)\chi \\ &= \big((1-\rho)B_{\epsilon,t} + 2\rho\big)\chi + B_{\epsilon,t}(B_{\epsilon,t}-2)\chi'. \end{aligned} \tag{4.14}$$

Next, using that $B_{\epsilon,t}\chi \leq \chi$, $B_{\epsilon,t}(B_{\epsilon,t}-2)\chi' \leq (B_{\epsilon,t}-2)\chi'$, we find furthermore

$$B_{\epsilon,t}\varphi' + \rho(-\varphi) \leq (1+\rho)\chi + (B_{\epsilon,t}-2)\chi' = \rho\chi + \varphi' \leq (1+\rho)\varphi'. \tag{4.15}$$

This, together with (4.13), and notation $\varphi_1^2 = \varphi'$ and $\sigma := 1 - O(\langle g\rangle)$, gives

$$\langle M\rangle_{\psi_t} \leq -\big[\frac{\sigma}{c} - 1 - \rho\big]\frac{1}{t^{1-\rho}}\langle\varphi'\rangle_{\psi_t} + \frac{C}{t^{1-\rho}}\big(\epsilon t^{\nu_1}\|\psi_0\|_1^2 + t^{-1+\nu_0}\epsilon^{-1}\|\psi_0\|_0^2 + t^{-2s}\|\psi_0\|_{B^2}^2\big). \tag{4.16}$$

Next, we introduce the second estimating operator $E_2 := N + \eta^{-2} + \mathbf{1}$, with $\eta^2 := \eta_2^2\eta_1^2\eta_2^2$, and show that the remainder, R, defined in (4.4) satisfies

$$R \leq Ct^{-2}\epsilon^{-1}E_2. \tag{4.17}$$

To prove (4.17), it suffices to show that

$$\big\|E_2^{-\frac12}RE_2^{-\frac12}\big\| \lesssim t^{-2}\epsilon^{-1}. \tag{4.18}$$

Proceeding as in the proof of Lemma B.2 of Appendix B, using the Helffer-Sjöstrand formula (B.1), one verifies that

$$\big\|E_2^{-\frac12}RE_2^{-\frac12}\big\| \lesssim t^{-2}\big\|E_2^{-\frac12}B_2E_2^{-\frac12}\big\|, \tag{4.19}$$

where $B_2 := [B_\epsilon,[B_\epsilon,H]]$. Now, writing $B_2 = [B_\epsilon,[B_\epsilon,H_f]] + I_2$, where $I_2 := [B_\epsilon,[B_\epsilon,I(g)]]$, and using the elementary computations (II.3) and (II.5) of Supplement II, we find $[B_\epsilon,[B_\epsilon,H_f]] = \mathrm{d}\Gamma(\epsilon\theta_\epsilon\omega_\epsilon^{-2})$ and $I_2 = I(b_\epsilon^2 g)$. The estimate $\epsilon\theta_\epsilon\omega_\epsilon^{-2} \leq \epsilon^{-1}$ implies

$$\big\|(\mathbf{1}+N)^{-\frac12}\mathrm{d}\Gamma(\epsilon\theta_\epsilon\omega_\epsilon^{-2})(\mathbf{1}+N)^{-\frac12}\big\| \lesssim \epsilon^{-1}. \tag{4.20}$$

Moreover, (1.6), the condition $\mu > -1/2$ and (2.14) (with $\delta = 0$) yield

$$\|\eta_1\eta_2^2 I_2(\mathbf{1}+N)^{-\frac12}\| \lesssim \|\eta_1\eta_2^2 b_\epsilon^2 g\| \lesssim \epsilon^{-1}\langle g\rangle, \tag{4.21}$$

and hence

$$\big\|E_2^{-\frac12}I_2E_2^{-\frac12}\big\| \lesssim \epsilon^{-1}\langle g\rangle. \tag{4.22}$$

Thus, we obtain

$$\big\|E_2^{-\frac12}B_2E_2^{-\frac12}\big\| \lesssim \epsilon^{-1}, \tag{4.23}$$

which together with (4.19) implies (4.18). Together with Equations (4.3) and (4.16) and the fact that $\|\eta_1^{-1}\eta_2^{-2}f(H)\| \lesssim 1$, this implies

$$\begin{aligned}\langle D\Phi_t\rangle_{\psi_t} &\leq -\big(\frac{\sigma}{c}-1-\rho\big)t^{-1+\rho}\langle\varphi'\rangle_{\psi_t}\\ &\quad + C\big(\epsilon t^{\nu_1+\rho-1}\|\psi_0\|_1^2 + t^{-2+\nu_0+\rho}\epsilon^{-1}\|\psi_0\|_0^2 + t^{-1+\rho-2s}\|\psi_0\|_{B^2}^2\big).\end{aligned} \tag{4.24}$$

Thus, choosing s such that $2s-\rho > 0$, (4.24), together with the observation $\Phi_t \geq t^\rho\chi_{B_{\epsilon,t}\leq 1}$, the conditions $\frac{\sigma}{c} - 1 > \rho$, $\rho < 1 \leq 2-\nu_0$, Hardy's inequality $\|\psi_0\|_1 \lesssim \|\psi_0\|_{\mathrm{d}\Gamma(\langle y\rangle)}$ and the trivial inequality $\|\psi_0\|_0 \leq \|\psi_0\|_{\mathrm{d}\Gamma(\langle y\rangle)}$, implies that

$$\begin{aligned}t^\rho\langle\chi\rangle_{\psi_t} &\leq \langle\Phi_t\rangle_{\psi_t} = \langle\Phi_t\rangle_{\psi_t}|_{t=0} + \int_0^t\langle D\Phi_s\rangle_{\psi_s}ds\\ &\leq \langle -B_\epsilon\chi_{B_\epsilon\leq 0}\rangle_{\psi_0} + C(\epsilon^{-1}+\epsilon t^{\rho+\nu_1}+1)(\|\psi_0\|^2_{\mathrm{d}\Gamma(\langle y\rangle)} + \|\psi_0\|^2_{B^2}).\end{aligned}$$

Using $\langle -B_\epsilon\chi_{B_\epsilon\leq 0}\rangle_{\psi_0} \lesssim \|\psi_0\|^2_{\mathrm{d}\Gamma(\langle y\rangle)}$, and choosing $\epsilon = t^{-\kappa}$, we find

$$\langle\chi\rangle_{\psi_t} \leq C(t^{-\rho+\kappa} + t^{\nu_1-\kappa} + t^{-\rho})(\|\psi_0\|^2_{\mathrm{d}\Gamma(\langle y\rangle)} + \|\psi_0\|^2_{B^2}),$$

which in turn gives (4.2). □

We now prove an estimate used in the proof of Proposition 4.2.

Lemma 4.3. *Assume* (1.6) *with* $\mu > -1/2$, *(1.7), (1.15),* $\langle g\rangle$ *sufficiently small and* $f \in \mathrm{C}_0^\infty((E_{\mathrm{gs}},\Sigma))$. *Then*

$$\big\|P_\Omega e^{-itH}f(H)u\big\| \lesssim t^{-s}\|\langle B\rangle u\|, \quad s < 1/2. \tag{4.25}$$

Proof. We use the local decay properties established in [31] and [8]. Let $c_j := (e_j+e_{j+1})/2$ and $\delta_j := e_{j+1}-e_j$. We decompose the support of f into different regions, writing

$$f(H) = f(H)\chi_{H\leq c_0} + \sum_{\text{finite}} f(H)\chi_j(H), \tag{4.26}$$

where $\chi_j(H)$ denotes a smoothed out characteristic function of the interval $[c_j-\delta_j/4, c_{j+1}+\delta_{j+1}/4]$. Using $P_\Omega = P_\Omega\langle B\rangle$, and [31], we obtain

$$\big\|P_\Omega e^{-itH}f(H)\chi_{H\leq c_0}u\big\| = \big\|\langle B\rangle^{-1}e^{-itH}f(H)\chi_{H\leq c_0}u\big\| \lesssim t^{-s}\big\|\langle B\rangle u\big\|, \tag{4.27}$$

for $s < 1/2$.

To estimate $\|P_\Omega e^{-itH} f(H)\chi_j(H)u\|$, we let $\tilde{\chi}_j(H) := f(H)\chi_j(H)$. In [8], assuming (1.15), a conjugate operator $\tilde{B}_j$ is constructed in such a way that the commutators $[\tilde{\chi}_j(H), \tilde{B}_j]$ and $[[\tilde{\chi}_j(H), \tilde{B}_j], \tilde{B}_j]$ are bounded. Moreover, the Mourre estimate

$$\tilde{\chi}_j(H)[H, i\tilde{B}_j]\tilde{\chi}_j(H) \geq m_0 \tilde{\chi}_j(H)^2,$$

holds for some positive constant m_0. By an abstract result of [45], this implies

$$\left\| \langle \tilde{B}_j \rangle^{-s} e^{-itH} \tilde{\chi}_j(H) \langle \tilde{B}_j \rangle^{-s} \right\| \lesssim t^{-s},$$

for $s < 1$. Since the operator $\tilde{B}_j$ is of the form $\tilde{B}_j = B + M_j$, where M_j is a bounded operator, it then follows that

$$\left\| \langle B \rangle^{-s} e^{-itH} \tilde{\chi}_j(H) \langle B \rangle^{-s} \right\| \lesssim t^{-s},$$

and hence, using again that $P_\Omega \langle B \rangle = P_\Omega$, we obtain

$$\left\| P_\Omega e^{-itH} \tilde{\chi}_j(H)u \right\| = \left\| \langle B \rangle^{-1} e^{-itH} \tilde{\chi}_j(H)u \right\| \lesssim t^{-s} \left\| \langle B \rangle u \right\|. \tag{4.28}$$

Equations (4.26), (4.27) and (4.28) give (4.25). □

Proof of Theorem 4.1. Since $N = \mathrm{d}\Gamma(\mathbf{1})$ and $B_\epsilon = \mathrm{d}\Gamma(b_\epsilon)$ commute, we have

$$\begin{aligned} \Gamma(\chi_{\frac{b_\epsilon}{ct^\alpha} \leq 1}) &\leq \chi_{B_\epsilon \leq cNt^\alpha} = \chi_{B_\epsilon \leq cNt^\alpha}(\chi_{N \leq c't^\gamma} + \chi_{N \geq c't^\gamma}) \\ &\leq \chi_{B_\epsilon \leq c''t^\nu} + \chi_{N \geq c't^\gamma}, \end{aligned} \tag{4.29}$$

where $\nu := \alpha + \gamma$ and $c'' := cc'$. We choose $c' \ll 1/c$, so that $0 < c'' \ll 1$. Next, we have

$$\begin{aligned} \|\chi_{N \geq c't^\gamma} \psi_t\| &\leq (c')^{-\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} \ \|\chi_{N \geq c't^\gamma} N^{\frac{1}{2}} \psi_t\| \\ &\leq (c')^{-\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} \ \|N^{\frac{1}{2}} \psi_t\|, \end{aligned}$$

which, together with (2.4) (with $\rho = 0$), implies

$$\|\chi_{N \geq c't^\gamma} \psi_t\| \ \lesssim t^{-\frac{\gamma}{2} + \frac{\nu_0}{2}} \ \|\psi_0\|_0. \tag{4.30}$$

The inequality (4.29) with $\nu = 1$, Proposition 4.2 and the inequality (4.30) (with $\gamma = 1 - \alpha$) imply the estimate (4.1). □

Remarks. 1) The estimate (4.1) states that, as $t \to \infty$, with probability $\to 1$, either all photons are attached to the particle system in the combined ground state, or at least one photon departs the particle system with the distance growing at least as $\mathcal{O}(t^\alpha)$. ((4.1) for $\mu \geq 1/2$, some $\alpha > 0$ and $\psi_0 \in E_\Delta(H)$, with $\Delta \subset (E_{\mathrm{gs}}, e_1 - \mathcal{O}(\langle g \rangle))$ and e_1 the first excited eigenvalue of H_p, can be derived directly from [9, 10].)

2) With some more work, one can remove the assumption (1.15) and relax the condition on ψ_0 in Theorem 4.1 to the natural one: $\psi_0 \in P_\Sigma(D(\mathrm{d}\Gamma(\langle y \rangle)) \cap D(\mathrm{d}\Gamma(b)^2))$, where P_Σ is the spectral projection onto the orthogonal complement of the eigenfunctions of H with the eigenvalues in the interval $(-\infty, \Sigma)$.

5. Proof of Theorem 1.1

5.1. **Partition of unity.** We begin with a construction of a partition of unity which separates photons close to the particle system from those departing it. Following [19, 27] (cf. the many-body scattering construction), it is defined by first constructing a partition of unity (j_0, j_∞), $j_0^2 + j_\infty^2 = \mathbf{1}$, on the one-photon space, $\mathfrak{h} = L^2(\mathbb{R}^3)$, with j_0 localizing a photon to a region near the particle system (the origin) and j_∞ near infinity, and then associating with it the map $j : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$, given by $j : h \to j_0 h \oplus j_\infty h$. After that we lift the map j to the Fock space $\mathcal{F} = \Gamma(\mathfrak{h})$ by using $\Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h} \oplus \mathfrak{h})$ (defined in (1.24)). Next, we consider the adjoint map $j^* : h_0 \oplus h_\infty \to j_0^* h_0 + j_\infty^* h_\infty$, which we also lift to the Fock space $\mathcal{F} := \Gamma(\mathfrak{h})$ by using $\Gamma(j^*) : \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \to \Gamma(\mathfrak{h})$. By definition, the operator $\Gamma(j)$ has the following properties

$$\Gamma(j)^* = \Gamma(j^*), \quad \Gamma(\tilde{j})\Gamma(j) = \Gamma(\tilde{j}j). \tag{5.1}$$

Since $j^* j = j_0^2 + j_\infty^2 = \mathbf{1}$, this implies the relation $\Gamma(j)^*\Gamma(j) = \mathbf{1}$, which is what we mean by a partition of unity of the Fock space $\mathcal{F} := \Gamma(\mathfrak{h})$.

We refine this construction further by defining the unitary map $U : \Gamma(\mathfrak{h} \oplus \mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h})$, through the relations

$$U\Omega = \Omega \otimes \Omega, \qquad Ua^*(h) = [a^*(h_1) \otimes \mathbf{1} + \mathbf{1} \otimes a^*(h_2)]U, \tag{5.2}$$

for any $h = (h_1, h_2) \in \mathfrak{h} \oplus \mathfrak{h}$, and introducing the operators **(the Dereziński - Gérard partition of unity)**

$$\check{\Gamma}(j) := U\Gamma(j) : \Gamma(\mathfrak{h}) \to \Gamma(\mathfrak{h}) \otimes \Gamma(\mathfrak{h}). \tag{5.3}$$

We lift $\Gamma(j)$, as well as $\check{\Gamma}(j)$, from the Fock space $\mathcal{F} = \Gamma(\mathfrak{h})$ to the full state space $\mathcal{H} = \mathcal{H}_p \otimes \mathcal{F}$, so that e.g. $\check{\Gamma}(j) : \mathcal{H} \to \mathcal{H} \otimes \Gamma(\mathfrak{h})$. Now, the partition of unity relation on $\mathcal{H}$ becomes $\check{\Gamma}(j)^*\check{\Gamma}(j) = \mathbf{1}$ (in particular, $\check{\Gamma}(j)$ is an isometry).

Finally, we specify j_0 to be the operator $\chi_{v\le 1}$ and define j_∞ by the relation $j_0^2 + j_\infty^2 = \mathbf{1}$ (hence j_∞ is of the form $\chi_{v\ge 1}$), with $v = \frac{b_\epsilon}{ct^\alpha}$, b_ϵ is defined in the introduction, $\epsilon = t^{-\kappa}$, and the parameters α and κ satisfy $1 - \mu/(6+3\mu) < \alpha < 1$ and $1 + \nu_1 - \alpha < \kappa < \frac{1}{2}(5\alpha - 3)$. Since $j_0 \to 1$ and $j_\infty \to 0$, strongly, as $t \to \infty$, we have the following useful property of $\check{\Gamma}(j)$:

$$\check{\Gamma}(j) \to \mathbf{1} \otimes P_\Omega, \qquad \text{strongly as } t \to \infty, \tag{5.4}$$

where, recall, P_Ω denotes the projection onto the Fock vaccum. This property is easy to verify on product states.

5.2. **Deift-Simon wave operators.** We define the auxiliary space $\hat{\mathcal{H}} := \mathcal{H} \otimes \mathcal{F}$, which will serve as our repository of asymptotic dynamics, which is governed by the hamiltonian $\hat{H} := H \otimes \mathbf{1} + \mathbf{1} \otimes H_f$ on $\hat{\mathcal{H}}$. With the partition of unity $\check{\Gamma}(j)$, we associate the Deift-Simon wave operators,

$$W_\pm := \underset{t\to\pm\infty}{\text{s-lim}}\, W(t), \qquad \text{where} \qquad W(t) := e^{i\hat{H}t}\check{\Gamma}(j)e^{-iHt}, \tag{5.5}$$

which map the original dynamics, e^{-iHt}, into auxiliary one, $e^{-i\hat{H}t}$ (to be further refined later). Recall that P_{gs} denotes the orthogonal projection onto the ground state subspace of H. Our goal is to prove

Theorem 5.1. *Assume* (1.6) *with* $\mu > 0$, (1.7), *and that one of the implicit conditions of Theorem 1.1 is satisfied. Then the Deift-Simon wave operators exist on* $\operatorname{Ran} E_{(-\infty,\Sigma)}(H)$ *and satisfy*

$$W_\pm P_{\text{gs}} = P_{\text{gs}} \otimes P_\Omega, \tag{5.6}$$

and, for any smooth, bounded function f,

$$W_\pm f(H) = f(\hat{H})W_\pm. \tag{5.7}$$

Proof. We begin with the the following lemma

Lemma 5.2. *Assume* (1.6) *with* $\mu > 0$ *and* (1.7). *For any* $f \in \mathrm{C}_0^\infty((-\infty,\Sigma))$ *and* $\psi_0 \in f(H)D(N_1^{1/2})$,

$$\left\|(\check{\Gamma}(j)f(H) - f(\hat{H})\check{\Gamma}(j))\psi_t\right\| \lesssim t^{-\alpha+\frac{1}{2+\mu}}\|\psi_0\|_1. \tag{5.8}$$

Proof. We compute, using the Helffer-Sjöstrand formula (see (B.1) of Appendix B) for $f(H)$ and $f(\hat{H})$,

$$\check{\Gamma}(j)f(H)\psi_t - f(\hat{H})\check{\Gamma}(j)\psi_t = R,$$

where

$$R := \frac{1}{\pi}\int \partial_{\bar z}\tilde{f}(z)(\hat{H} - z)^{-1}(\hat{H}\check{\Gamma}(j) - \check{\Gamma}(j)H)(H - z)^{-1}\psi_t \,\mathrm{dRe}\, z\, \mathrm{dIm}\, z. \tag{5.9}$$

Using $(H_p \otimes \mathbf{1} \otimes \mathbf{1})(\mathbf{1} \otimes \check{\Gamma}(j)) = (\mathbf{1} \otimes \check{\Gamma}(j))(H_p \otimes \mathbf{1})$, we decompose $\hat{H}\check{\Gamma}(j) - \check{\Gamma}(j)H = G_0 + G_1$, where

$$G_0 = \hat{H}_f\check{\Gamma}(j) - \check{\Gamma}(j)H_f, \tag{5.10}$$

with $\hat{H}_f = H_f \otimes \mathbf{1} + \mathbf{1} \otimes H_f$, and

$$G_1 := (I(g) \otimes \mathbf{1})\check{\Gamma}(j) - \check{\Gamma}(j)I(g). \tag{5.11}$$

We consider G_0. A straightforward computation gives $\Gamma(j)\mathrm{d}\Gamma(c) = \mathrm{d}\Gamma(\underline{c})\Gamma(j) + \mathrm{d}\Gamma(j, jc - \underline{c}j)$, where $\underline{c} = \mathrm{diag}(c,c) :\ \mathfrak{h} \oplus \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ and

$$\mathrm{d}\Gamma(a,c)|_{\otimes_s^n \mathfrak{h}} = \sum_{j=1}^n \underbrace{a \otimes \cdots \otimes a}_{j-1} \otimes c \otimes \underbrace{a \otimes \cdots \otimes a}_{n-j}. \tag{5.12}$$

It follows from this relation and the equalities $U\mathrm{d}\Gamma(\underline{c}) = (\mathrm{d}\Gamma(c)\otimes\mathbf{1}+\mathbf{1}\otimes\mathrm{d}\Gamma(c))U$ that ([19, 27])

$$\check{\Gamma}(j)\mathrm{d}\Gamma(c) = (\mathrm{d}\Gamma(c)\otimes\mathbf{1}+\mathbf{1}\otimes\mathrm{d}\Gamma(c))\check{\Gamma}(j)+\mathrm{d}\check{\Gamma}(j, jc-\underline{c}j), \tag{5.13}$$

where $\mathrm{d}\check{\Gamma}(a,c) := U\mathrm{d}\Gamma(a,c)$. We have $\underline{\omega}j - j\omega = ([\omega, j_0],[\omega, j_\infty])$, and, by Corollary B.3 of Appendix B,

$$[\omega, j_\#] = \frac{\theta_\epsilon}{ct^\alpha} j'_\# + r, \tag{5.14}$$

where $j_\#$ stands for j_0 or j_∞, $j'_\#$ is the derivative of $j_\#$ as a function of $v = \frac{b_\epsilon}{ct^\alpha}$, and r satisfies $\|r\| \lesssim t^{-2\alpha+\kappa}$. Since $\theta_\epsilon \le 1$ and since $\kappa < \alpha$, we deduce that $[\omega, j_\#] = \mathcal{O}(t^{-\alpha})$. This gives $G_0 = -\mathrm{d}\check{\Gamma}(j, j\omega-\underline{\omega}j) = \mathrm{d}\check{\Gamma}(j,\mathcal{O}(t^{-\alpha}))$. Let $\hat{N} := N\otimes\mathbf{1}+\mathbf{1}\otimes N$ be the number operator on $\hat{\mathcal{H}}$. (5.13) with $c=\mathbf{1}$ implies $(\hat{N}+\mathbf{1})^{-1/2}G_0 = G_0(N+\mathbf{1})^{-1/2}$. By (C.6) of Appendix C, we then obtain that

$$\|G_0(N+\mathbf{1})^{-1}\| = \|(\hat{N}+\mathbf{1})^{-\frac{1}{2}}G_0(N+\mathbf{1})^{-\frac{1}{2}}\| \lesssim t^{-\alpha}.$$

Using the easy property that $H\in C^1(N)$ (see e.g. [10, Lemma A.6]), we have $\|(N+\mathbf{1})(H-z)^{-1}(N+\mathbf{1})^{-1}\| \lesssim |\mathrm{Im}\, z|^{-2}$, and hence

$$\|G_0(H-z)^{-1}\psi_t\| \lesssim t^{-\alpha}|\mathrm{Im} z|^{-2}\|(N+\mathbf{1})\psi_t\|. \tag{5.15}$$

Applying Corollary A.3 of Appendix A, we obtain

$$\|G_0(H-z)^{-1}\psi_t\| \lesssim t^{-\alpha+\frac{1}{2+\mu}}|\mathrm{Im} z|^{-2}\|\psi_0\|_1. \tag{5.16}$$

Now, we address G_1. We use the definition $\check{\Gamma}(j) = U\Gamma(j)$ to obtain $\check{\Gamma}(j)a^\#(h) = Ua^\#(jh)\Gamma(j)$, where $a^\#$ stands for a or a^*. Then using (5.2), and $j_0^*j_0 + j_\infty^* j_\infty = 1$, we derive

$$\check{\Gamma}(j)a^\#(h) = (a^\#(j_0h)\otimes\mathbf{1}+\mathbf{1}\otimes a^\#(j_\infty h))\check{\Gamma}(j). \tag{5.17}$$

This implies

$$\check{\Gamma}(j)I(g) = (I(j_0 g)\otimes\mathbf{1}+\mathbf{1}\otimes I(j_\infty g))\check{\Gamma}(j). \tag{5.18}$$

The equation (5.18) gives

$$G_1 = (I((1-j_0)g)\otimes\mathbf{1}-\mathbf{1}\otimes I(j_\infty g))\check{\Gamma}(j). \tag{5.19}$$

Due to the inequality (3.16), we have

$$\|\eta_1\eta_2^2 j_\infty g\|_{L^2} \lesssim t^{-\lambda}, \quad \|\eta_1\eta_2^2(1-j_0)g\|_{L^2} \lesssim t^{-\lambda}, \tag{5.20}$$

with $\lambda < (\mu+\frac{3}{2})\alpha$. Using this, we have in addition

$$\|G_1(N+\eta^{-2}+\mathbf{1})^{-1}\| \lesssim t^{-\lambda}, \tag{5.21}$$

where $\eta^2 := \eta_2^2\eta_1^2\eta_2^2$. Hence, using (1.7) and, as above, that $\|(N+\mathbf{1})(H-z)^{-1}(N+\mathbf{1})^{-1}\| \lesssim |\mathrm{Im}\, z|^{-2}$, we obtain

$$\|G_1(H-z)^{-1}\psi_t\| \lesssim t^{-\lambda+\frac{1}{2+\mu}}|\mathrm{Im} z|^{-2}\|\psi_0\|_1. \tag{5.22}$$

From (5.9), (5.16), (5.22), the properties of the almost analytic extension $\tilde{f}$ and the estimate $\|(H-z)^{-1}\| \lesssim |\mathrm{Im} z|^{-1}$, we conclude that (5.8) holds. □

We want to show that the family $W(t) := e^{i\hat{H}t}\check{\Gamma}(j)e^{-iHt}$ form a strong Cauchy sequence as $t\to\infty$. Let $\psi_0 \in f(H)D(N_1^{1/2})$, $f\in \mathrm{C}_0^\infty((-\infty,\Sigma))$ and $f_1 \in \mathrm{C}_0^\infty((-\infty,\Sigma))$ be such that $f_1 f = f$. Lemma 5.2 implies that

$$W(t)\psi_0 = \widetilde{W}(t)\psi_0 + \mathcal{O}(t^{-\alpha+\frac{1}{2+\mu}})\|\psi_0\|_1, \tag{5.23}$$

where

$$\widetilde{W}(t) := e^{i\hat{H}t}f_1(\hat{H})\check{\Gamma}(j)e^{-iHt}f_1(H).$$

Hence, since our conditions on α imply $\alpha > 1/(2+\mu)$, it suffices to show that $\widetilde{W}(t)$ form a strong Cauchy sequence as $t\to\infty$.

First suppose Assumption (1.18) of Theorem 1.1. We define $\chi_m := \chi_{\hat N\le m}$ and $\overline{\chi}_m := \chi_{\hat N\ge m}$, so that $\chi_m + \overline{\chi}_m = \mathbf{1}$. First, we show that, for any $\psi_0 \in D(N^{1/2})$,

$$\sup_t \|\overline{\chi}_m \widetilde{W}(t)\psi_0\| \lesssim m^{-\frac12}\|\psi_0\|_0. \tag{5.24}$$

Indeed, by Assumption (1.18),

$$\|\hat N^{\frac12} e^{i\hat H t} f_1(\hat H)\check\Gamma(j)\psi_s\| \lesssim \|\hat N^{\frac12}\check\Gamma(j)\psi_s\| + \|\check\Gamma(j)\psi_s\|. \tag{5.25}$$

The boundedness of $\check\Gamma(j)$ and the definition $\psi_t := e^{-iHt}\psi_0$ imply $\|\check\Gamma(j)\psi_t\| \le \|\psi_0\|$. Equation (5.13) with $c = \mathbf{1}$ implies $\hat N^{\frac12}\check\Gamma(j) = \check\Gamma(j)N^{\frac12}$. The latter relation, boundedness of $\check\Gamma(j)$ and Assumption (1.18) give

$$\|\hat N^{\frac12}\check\Gamma(j)\psi_s\| = \|\check\Gamma(j)N^{\frac12}\psi_s\| \lesssim \|\psi_0\|_0,$$

and therefore, by (5.25), $\|\hat N^{\frac12} e^{i\hat H t} f_1(\hat H)\check\Gamma(j)\psi_s\| \lesssim \|\psi_0\|_0$. Since this is true uniformly in t, s, it implies $\|\hat N^{\frac12}\widetilde{W}(t)\psi_0\| \lesssim \|\psi_0\|_0$, which yields (5.24). Equation (5.24) implies that

$$\sup_{t,t'} \|\overline{\chi}_m(\widetilde{W}(t') - \widetilde{W}(t))\psi_0\| \lesssim m^{-\frac12}\|\psi_0\|_0. \tag{5.26}$$

Now we show that, for any $m > 0$ and for any $\psi_0 \in D(\mathrm{d}\Gamma(\langle y\rangle)^{\frac12}) \cap \operatorname{Ran} E_{(-\infty,\Sigma)}(H)$,

$$\|\chi_m(\widetilde{W}(t') - \widetilde{W}(t))\psi_0\| \to 0, \tag{5.27}$$

as $t, t' \to \infty$. This together with (5.26) implies that $\widetilde{W}(t)$ form a strong Cauchy sequence. We write

$$\widetilde{W}(t') - \widetilde{W}(t) = \int_t^{t'} ds\, \partial_s \widetilde{W}(s), \tag{5.28}$$

and compute $\partial_t \widetilde{W}(t) = e^{i\hat H t} f_1(\hat H) G e^{-iHt} f_1(H)$, where $G := i(\hat H\check\Gamma(j) - \check\Gamma(j)H) + \partial_t\check\Gamma(j)$. We write $G = \tilde G_0 + iG_1$, where

$$\tilde G_0 := iG_0 + \partial_t\check\Gamma(j),$$

and G_0 and G_1 are defined in (5.10)–(5.11). We consider $\tilde G_0$. Using the notation $\underline{d}j := i(\underline{\omega} j - j\omega) + \partial_t j$, with $\underline{\omega} = \operatorname{diag}(\omega,\omega)$, and (5.13), we compute readily

$$\tilde G_0 = U\mathrm{d}\Gamma(\underline{j}, \underline{d}j) = \mathrm{d}\check\Gamma(\underline{j}, \underline{d}j). \tag{5.29}$$

Write $j' = (j_0', j_\infty')$, where j_0', j_∞' are the derivatives of j_0, j_∞ as functions of $v = \frac{b_\epsilon}{ct^\alpha}$. We first find a convenient decomposition of $\underline{d}j$. Using $\underline{d}jf = (dj_0 f, dj_\infty f)$, with $dc_t = i[\omega, c_t] + \partial_t c_t$, (3.8) and Corollary B.3 of Appendix B, we compute

$$\underline{d}j = (j_0', j_\infty')\Big(\frac{\theta_\epsilon}{ct^\alpha} - \frac{\alpha b_\epsilon}{ct^{\alpha+1}}\Big) + \mathcal{O}(t^{-2\alpha+\kappa}). \tag{5.30}$$

We insert the maximal velocity partition of unity $\chi_{w\le1} + \chi_{w\ge1} = \mathbf{1}$, with $w := (\frac{|y|}{\bar c t})^2$ and $\bar c > 1$, into this formula and use the notation $\chi \equiv \chi_{w\le1}$ and the relation $vj_\#' = \mathcal{O}(1)j_\#'$, valid due to the localization of $j_\#'$, to obtain

$$\underline{d}j = \frac{1}{ct^\alpha}\theta_\epsilon^{1/2}\chi(j_0', j_\infty')\chi\theta_\epsilon^{1/2} + \mathrm{rem}_t, \tag{5.31}$$

$$\mathrm{rem}_t = \mathcal{O}(t^{-1})\chi(j_0', j_\infty')\chi + \mathcal{O}(t^{-2\alpha+\kappa}) + \mathcal{O}(t^{-\alpha})\chi_{w\ge1}. \tag{5.32}$$

These relations give

$$\tilde G_0 = G_0' + \mathrm{Rem}_t, \tag{5.33}$$

where $G_0' := \frac{1}{ct^\alpha}U\mathrm{d}\Gamma(j, \underline{c}_t)$, with $\underline{c}_t = (c_0, c_\infty) := (\theta_\epsilon^{1/2}\chi j_0'\chi\theta_\epsilon^{1/2}, \theta_\epsilon^{1/2}\chi j_\infty'\chi\theta_\epsilon^{1/2})$, and

$$\mathrm{Rem}_t := \tilde G_0 - G_0' = U\mathrm{d}\Gamma(j, \mathrm{rem}_t).$$

Next, we write

$$A := \sup_{\|\hat\phi_0\|=1}\Big|\int_t^{t'} ds\langle\hat\phi_s, G_0\psi_s\rangle\Big|,$$

where $\hat{\phi}_s := e^{-i\hat{H}s} f_1(\hat{H})\chi_m\hat{\phi}_0$. By (C.5) of Appendix C, G_0' satisfies

$$|\langle\hat{\phi}, G_0'\psi\rangle| \le \frac{1}{ct^\alpha}\big(\|\mathrm{d}\Gamma(|c_0|)^{\frac12}\otimes\mathbf{1}\hat{\phi}\|\,\|\mathrm{d}\Gamma(|c_0|)^{\frac12}\psi\| + \|\mathbf{1}\otimes\mathrm{d}\Gamma(|c_\infty|)^{\frac12}\hat{\phi}\|\,\|\mathrm{d}\Gamma(|c_\infty|)^{\frac12}\psi\|\big). \tag{5.34}$$

By the Cauchy-Schwarz inequality, (5.34) implies

$$\int_t^{t'} ds|\langle\hat{\phi}_s, G_0'\psi_s\rangle| \lesssim \Big(\int_t^{t'} ds\, s^{-\alpha}\|\mathrm{d}\Gamma(|c_0|)^{\frac12}\otimes\mathbf{1}\hat{\phi}_s\|^2\Big)^{\frac12}\Big(\int_t^{t'} ds\, s^{-\alpha}\|\mathrm{d}\Gamma(|c_0|)^{\frac12}\psi_s\|^2\Big)^{\frac12} + \Big(\int_t^{t'} ds\, s^{-\alpha}\|\mathbf{1}\otimes\mathrm{d}\Gamma(|c_\infty|)^{\frac12}\hat{\phi}_s\|^2\Big)^{\frac12}\Big(\int_t^{t'} ds\, s^{-\alpha}\|\mathrm{d}\Gamma(|c_\infty|)^{\frac12}\psi_s\|^2\Big)^{\frac12}.$$

Since $|c_0|$, $|c_\infty|$ are of the form $\theta_\epsilon^{1/2}\chi\chi_{\frac{b_\epsilon}{ct^\alpha}=1}\chi\theta_\epsilon^{1/2}$, the minimal velocity estimate (3.4) implies

$$\int_1^\infty ds\, s^{-\alpha}\|\widehat{\mathrm{d}\Gamma}_\#(|c|)^{\frac12}\hat{\phi}_s\|^2 \lesssim \|\chi_m\hat{\phi}_0\|_0^2 \lesssim m\|\hat{\phi}_0\|^2,$$

where $\widehat{\mathrm{d}\Gamma}_\#(|c|)^{\frac12}$ stands for $\mathrm{d}\Gamma(|c_0|)^{\frac12}\otimes\mathbf{1}$ or $\mathbf{1}\otimes\mathrm{d}\Gamma(|c_\infty|)^{\frac12}$, and

$$\int_1^\infty ds\, s^{-\alpha}\|\mathrm{d}\Gamma_\#(|c|)^{\frac12}\psi_s\|^2 \lesssim \|\psi_0\|_0^2,$$

with $\mathrm{d}\Gamma_\#(|c|)^{\frac12} = \mathrm{d}\Gamma(|c_0|)^{\frac12}$ or $\mathrm{d}\Gamma(|c_\infty|)^{\frac12}$, provided that $\alpha < 1/\bar{c}$. The last three relations give

$$\sup_{\|\hat{\phi}_0\|=1}\Big|\int_t^{t'} ds\,\langle\hat{\phi}_s, G_0'\psi_s\rangle\Big| \to 0, \qquad t, t' \to \infty. \tag{5.35}$$

Likewise, applying (C.6) of Appendix C first with $c_1 = \mathbf{1}$ and $c_2 = \mathbf{1}$, next with $c_1 = \mathbf{1}$ and $c_2 = \chi_{w\ge1}$, where recall $w = (\frac{|y|}{\bar{c}t})^2$, and then applying (C.5) with $c_0 = \chi j_0'\chi$ and $c_\infty = \chi j_\infty'\chi$, we see that Rem_t satisfies

$$|\langle\hat{\phi}, \mathrm{Rem}_t\psi\rangle| \lesssim \|\hat{N}^{\frac12}\hat{\phi}\|\Big(t^{-2\alpha+\kappa}\|N^{\frac12}\psi\| + t^{-1}\|\mathrm{d}\Gamma(\chi j_0'\chi)^{\frac12}\psi\| + t^{-1}\|\mathrm{d}\Gamma(\chi j_\infty'\chi)^{\frac12}\psi\| + t^{-\alpha}\|\mathrm{d}\Gamma(\chi_{w\ge1}^2)^{\frac12}\psi\|\Big). \tag{5.36}$$

Now, using (5.36) and the Cauchy-Schwarz inequality, we obtain

$$\int_t^{t'} ds\,|\langle\hat{\phi}_s, \mathrm{Rem}_s\psi_s\rangle| \le \Big(\int_t^{t'} ds\, s^{-\tau}\|\hat{N}^{\frac12}\hat{\phi}_s\|^2\Big)^{\frac12}\Big\{\Big(\int_t^{t'} ds\, s^{-2(2\alpha-\kappa)+\tau}\|N^{\frac12}\psi_s\|^2\Big)^{\frac12} + \Big(\int_t^{t'} ds\, s^{-2+\tau}\|\mathrm{d}\Gamma(\chi j_0'\chi)^{\frac12}\psi_s\|^2\Big)^{\frac12} + \Big(\int_t^{t'} ds\, s^{-2+\tau}\|\mathrm{d}\Gamma(\chi j_\infty'\chi)^{\frac12}\psi_s\|^2\Big)^{\frac12} + \Big(\int_t^{t'} ds\, s^{-2\alpha+\tau}\|\mathrm{d}\Gamma(\chi_{w\ge1}^2)^{\frac12}\psi_s\|^2\Big)^{\frac12}\Big\}. \tag{5.37}$$

Let $\tau > 1$ and $\alpha = 2-\tau$. Then by the estimate (3.3), we have

$$\int_1^\infty ds\, s^{-2+\tau}\|\mathrm{d}\Gamma(\chi j_\infty'\chi)^{\frac12}\psi_s\|^2 \lesssim \|\psi_0\|_1^2,$$

and by the maximal velocity estimate (1.21), we have

$$\int_1^\infty ds\, s^{-2\alpha+\tau}\|\mathrm{d}\Gamma(\chi_{w\ge1}^2)^{\frac12}\psi_s\|^2 \lesssim \|\psi_0\|_{\mathrm{d}\Gamma(\langle y\rangle)},$$

provided that $\alpha > 1-2\gamma/3$, where, recall, $\gamma < \frac{\mu}{2}\min(\frac{\bar{c}-1}{2\bar{c}-1}, \frac{1}{2+\mu})$. One verifies that $\bar{c} > 1$ can be chosen such that this condition is satisfied and $\alpha < 1/\bar{c}$. Moreover, Assumption (1.18) implies

$$\int_1^\infty ds\, s^{-2(2\alpha-\kappa)+\tau}\|N^{\frac12}\psi_s\|^2 \lesssim \|\psi_0\|_0,$$

provided that $5\alpha > 3 + 2\kappa$. This and the fact that, by Assumption (1.18), the first integral on the r.h.s. of (5.37) converge yield

$$\sup_{\|\hat{\phi}_0\|=1} \Big| \int_t^{t'} ds \, \langle \hat{\phi}_s, \mathrm{Rem}_s \psi_s \rangle \Big| \to 0, \quad t, t' \to \infty. \tag{5.38}$$

Equations (5.35) and (5.38) imply that

$$A = \Big\| \int_t^{t'} ds \, \chi_m f_1(\hat{H}) e^{i\hat{H}s} \tilde{G}_0 \psi_s \Big\| \to 0, \quad t, t' \to \infty. \tag{5.39}$$

Now we turn to G_1. The equations (5.19), (5.20), (2.14) (with $\delta = 0$), (1.7) and $\hat{N}^{1/2}\check{\Gamma}(j) = \check{\Gamma}(j) N^{1/2}$ imply that

$$\| f(\hat{H}) G_1 (N + \mathbf{1})^{-\frac{1}{2}} \| \lesssim t^{-\lambda}, \tag{5.40}$$

for $\lambda < (\mu + \frac{3}{2})\alpha$. Together with Assumption (1.18), this implies that $\| f(\hat{H}) G_1 \psi_t \| \lesssim t^{-\lambda} \| \psi_0 \|_0$, and hence

$$\Big\| \int_t^{t'} ds \, f(\hat{H}) e^{i\hat{H}s} G_1 \psi_s \Big\| \to 0, \qquad t, t' \to \infty,$$

provided that $\alpha > (\mu + \frac{3}{2})^{-1}$. This together with (5.39) gives (5.27) which, as was mentioned above, together with (5.26) shows that $\widetilde{W}(t)$ is a Cauchy sequence as $t \to \infty$. Hence by (5.23) $W(t)$ is a strong Cauchy sequence. This implies the existence of W_+. The proof of the existence of W_- is the same.

The proof of the existence of $W_\pm$ under the assumption (1.19) of Theorem 1.1 is similar, except that we do not need to introduce the cutoff χ_m. We use instead a variant of the weighted propagation estimates of Theorem 3.1. For reader's convenience we give this proof in Appendix E.

Finally, (5.6) follows from (5.4) and the relation $W_\pm \Phi_{\rm gs} = e^{i(\hat{H} - E_{\rm gs})t} \check{\Gamma}(j) \Phi_{\rm gs}$. To prove (5.7) we notice that, by (5.5), we have $W_\pm e^{i\hat{H}s} = \text{s-lim}\, e^{i\hat{H}t} \check{\Gamma}(j) e^{-iH(t+s)} = \text{s-lim}\, e^{i\hat{H}(t'-s)} \check{\Gamma}(j) e^{-iHt'} = e^{i\hat{H}s} W_\pm$, which implies (5.7). ☐

5.3. **Scattering map.** We discuss properties of the Hübner-Spohn scattering map, I, defined in the introduction. We begin with the definition

$$\mathfrak{h}_0 := \Big\{ h \in L^2(\mathbb{R}^3), \int dk \, (1 + \omega(k)^{-1}) |h(k)|^2 < \infty \Big\}. \tag{5.41}$$

Properties of the operator I used below are recorded in the following

Lemma 5.3 ([19, 27, 35])**.** *For any operator $j : h \to j_0 h \oplus j_\infty h$ and $n \in \mathbb{N}$, the following relations hold*

$$\check{\Gamma}(j)^* = I \Gamma(j_0^*) \otimes \Gamma(j_\infty^*), \tag{5.42}$$

$$D((H + i)^{-n/2}) \otimes (\otimes_s^n \mathfrak{h}_0) \subset D(I). \tag{5.43}$$

Proof. Since the operators involved act only on the photonic degrees of freedom, we ignore the particle one. For $g, h \in \mathfrak{h}$, we define embeddings $i_0 g := (g, 0) \in \mathfrak{h} \oplus \mathfrak{h}$ and $i_\infty h := (0, h) \in \mathfrak{h} \oplus \mathfrak{h}$. By the definition of U (see (5.2)), we have the relations $U^* a^*(g) \otimes \mathbf{1} = a^*(i_0 g) U^*$, and $U^* \mathbf{1} \otimes a^*(h) = a^*(i_\infty h) U^*$. Hence, using in addition $U^* \Omega \otimes \Omega = \Omega$, we obtain

$$U^* \prod_1^m a^*(g_i) \Omega \otimes \prod_1^n a^*(h_i) \Omega = \prod_1^m a^*(i_0 g_i) \prod_1^n a^*(i_\infty h_i) \Omega.$$

By the definition of $\Gamma(j)$ and the relations $j^* i_0 g = j_0^* g$ and $j^* i_\infty h = j_\infty^* h$, this gives

$$\Gamma(j)^* U^* \prod_1^m a^*(g_i) \Omega \otimes \prod_1^n a^*(h_i) \Omega = \prod_1^n a^*(j_\infty^* g_i) \prod_1^m a^*(j_0^* h_i) \Omega. \tag{5.44}$$

Now, by the definition of $\check{\Gamma}(j)$ (see (5.2)), we have $\check{\Gamma}(j)^* = \Gamma(j)^* U^*$. On the other hand by (1.12), the r.h.s. is $I \Gamma(j_0^*) \otimes \Gamma(j_\infty^*) \prod_1^m a^*(g_i) \Omega \otimes \prod_1^n a^*(h_i) \Omega$. This proves (5.42).

To prove (5.43), we use the following elementary properties ([27, 35]):

$$\text{The operator} \quad H_f^n (H + i)^{-n} \quad \text{is bounded} \quad \forall n \in \mathbb{N}, \tag{5.45}$$

and, for any $h_1, \cdots h_n \in \mathfrak{h}_0$, where $\mathfrak{h}_0$ is defined in (5.41),

$$\|a^*(h_1)\cdots a^*(h_n)(H_f+1)^{-n/2}\| \le C_n\|h_1\|_\omega \cdots \|h_n\|_\omega, \tag{5.46}$$

where $\|h\|_\omega := \int dk\,(1+\omega(k)^{-1})|h(k)|^2$. The previous two estimates and the representation (1.12) imply that for any $\Phi \in D((H+i)^{-n/2})$ and $h_1, \cdots, h_n \in \mathfrak{h}_0$, we have $\|I\Phi \otimes \prod_1^n a^*(h_i)\Omega\| \le C_n\|h_1\|_\omega \cdots \|h_n\|_\omega \|(H+i)^{n/2}\Phi\| < \infty$. This gives the second statement of the lemma. □

5.4. **Asymptotic completeness.** Below, the symbol $C(\epsilon)o_t(1)$ stands for a real function of ϵ and t such that, for any fixed ϵ, $|C(\epsilon)o_t(1)| \to 0$ as $t \to \infty$, and we denote by $\chi_\Omega(\lambda)$ a smoothed out characteristic function of a set Ω. In this section we prove the following result.

Theorem 5.4. *Assume the conditions of Theorem 1.1 for hamiltonians of the form* (1.4)–(1.5)*. Then the asymptotic completeness (in the sense of the definition* (1.14)*) holds on the interval* $\Delta = [E_{\rm gs}, a]$*, where* $a < \Sigma$ *is given by* (1.15)*.*

Proof. Let α and κ be fixed such that the conditions of Theorems 3.1, 4.1 and 5.1 hold. Let $(j_0, j_\infty) = (\chi_{v\le 1}, \chi_{v\ge 1})$ be the partition of unity defined in Subsection 5.1, where $v = \frac{b_\epsilon}{ct^\alpha}$. Since $j_0^2 + j_\infty^2 = \mathbf{1}$, the operator $\check\Gamma(j)$ is, as mentioned above, an isometry. Using the relation $\Gamma(j)^*\Gamma(j) = \mathbf{1}$, the boundedness of $\check\Gamma(j)^*$, and the existence of W_+, we obtain

$$\psi_t = \check\Gamma(j)^* e^{-i\hat H t} e^{i\hat H t}\check\Gamma(j)e^{-iHt}\psi_0 = \check\Gamma(j)^* e^{-i\hat H t}\phi_0 + o_t(1), \tag{5.47}$$

where $\phi_0 := W_+\psi_0$. Next, using the property $W_+\chi_\Delta(H) = \chi_\Delta(\hat H)W_+$, which gives $W_+\psi_0 = \chi_\Delta(\hat H)W_+\psi_0$, and $\chi_\Delta(\hat H) = (\chi_\Delta(H)\otimes\chi_{\Delta'}(H_f))\chi_\Delta(\hat H)$, and again using $\chi_\Delta(\hat H)W_+\psi_0 = W_+\psi_0 = \phi_0$, we obtain

$$\phi_0 = \big(\chi_\Delta(H)\otimes\chi_{\Delta'}(H_f)\big)\phi_0. \tag{5.48}$$

For all $\epsilon' > 0$, there is $\delta' = \delta'(\epsilon') > 0$, such that

$$\big\|(\chi_\Delta(H)\otimes\mathbf{1})\phi_0 - (\chi_{\Delta_{\epsilon'}}(H)\otimes\mathbf{1})\phi_0 - (P_{\rm gs}\otimes\mathbf{1})\phi_0\big\| \le \epsilon', \tag{5.49}$$

with $\Delta_{\epsilon'} = [E_{\rm gs}+\delta', a]$. The last two relations give

$$\phi_0 = \big((\chi_{\Delta_{\epsilon'}}(H) + P_{\rm gs})\otimes\chi_{\Delta'}(H_f)\big)\phi_0 + \mathcal{O}(\epsilon'). \tag{5.50}$$

For any vector space $\mathcal{V}\subset\mathfrak{h}$, we let $\mathcal{F}_{\rm fin}(\mathcal{V})$ denote the subspace of $\mathcal{F}$ consisting of vectors $\Psi = (\psi_n)_{n=0}^\infty \in \mathcal{F}$ such that $\psi_n = 0$, for all but finitely many n and $\psi_n \in \otimes_s^n\mathcal{V}$ for all n. Let $\phi_{0,\epsilon'} \in \mathcal{F}_{\rm fin}(D(\langle y\rangle))\otimes\mathcal{F}_{\rm fin}(\mathfrak{h}_0)$ be such that $\|\phi_0 - \phi_{0\epsilon'}\| \le \epsilon'$. (We require that the 'first components' of $\phi_{0\epsilon'}$ are in $\mathcal{F}_{\rm fin}(D(\langle y\rangle))$ in order to apply the minimal velocity estimate below, and that the 'second components' are in $\mathcal{F}_{\rm fin}(\mathfrak{h}_0)$ in order that $(P_{\rm gs}\otimes\mathbf{1})\phi_{0\epsilon'}$ is in $D(I)$). This together with (5.47) and (5.50) gives

$$\psi_t = \check\Gamma(j)^* e^{-i\hat H t}((\chi_{\Delta_{\epsilon'}}(H) + P_{\rm gs})\otimes\chi_{\Delta'}(H_f))\phi_{0\epsilon'} + \mathcal{O}(\epsilon') + o_t(1). \tag{5.51}$$

Furthermore, let $(\tilde j_0, \tilde j_\infty)$ be of the form $\tilde j_0 = \tilde\chi_{v\le 1}$, $\tilde j_\infty = \tilde\chi_{v\ge 1}$ where $\tilde\chi$, has the same properties as χ, and satisfy $j_0\tilde j_0 = j_0$, $j_\infty\tilde j_\infty = j_\infty$. Then, by (5.42), the adjoint $\check\Gamma(j)^*$ to the operator $\check\Gamma(j)$ can be represented as

$$\check\Gamma(j)^* = \check\Gamma(j)^*\big(\Gamma(\tilde j_0)\otimes\Gamma(\tilde j_\infty)\big). \tag{5.52}$$

Using this equation in (5.51), together with the relations $e^{-i\hat H t} = e^{-iHt}\otimes e^{-iH_f t}$ and $e^{-iHt}P_{\rm gs} = e^{-iE_{\rm gs}t}P_{\rm gs}$, gives

$$\psi_t = \check\Gamma(j)^*\psi_{t\epsilon'} + A + B + C + \mathcal{O}(\epsilon') + o_t(1), \tag{5.53}$$

where

$$\psi_{t\epsilon'} := \big(e^{-iE_{\rm gs}t}P_{\rm gs}\otimes e^{-iH_f t}\chi_{\Delta'}(H_f)\big)\phi_{0\epsilon'}, \tag{5.54}$$

$$A := \check\Gamma(j)^*\big(\Gamma(\tilde j_0)e^{-iHt}\chi_{\Delta_{\epsilon'}}(H)\otimes\Gamma(\tilde j_\infty)e^{-iH_f t}\chi_{\Delta'}(H_f)\big)\phi_{0\epsilon'}, \tag{5.55}$$

$$B := \check\Gamma(j)^*\big((\Gamma(\tilde j_0)-\mathbf{1})e^{-iE_{\rm gs}t}P_{\rm gs}\otimes\Gamma(\tilde j_\infty)e^{-iH_f t}\chi_{\Delta'}(H_f)\big)\phi_{0\epsilon'}, \tag{5.56}$$

$$C := \check\Gamma(j)^*\big(e^{-iE_{\rm gs}t}P_{\rm gs}\otimes(\Gamma(\tilde j_\infty)-\mathbf{1})e^{-iH_f t}\chi_{\Delta'}(H_f)\big)\phi_{0\epsilon'}. \tag{5.57}$$

Since $\Gamma(j)^*$ is bounded, the minimal velocity estimate, (4.1), gives (here we use that the first components of $\phi_{0\epsilon'}$ are in $D(\mathrm{d}\Gamma(\langle y\rangle)))$

$$\|A\| \leq \big\|(\Gamma(\tilde{j}_0)e^{-iHt}\chi_{\Delta_{\epsilon'}}(H)\otimes \mathbf{1})\phi_{0\epsilon'}\big\| = C(\epsilon')o_t(1).$$

Now we consider the term given by B. We begin with

$$\big\|B\| \leq \big\|(\Gamma(\tilde{j}_0)-\mathbf{1})P_{\mathrm{gs}}\big\|. \tag{5.58}$$

Since $0 \leq \tilde{j}_0 \leq 1$, we have that $0 \leq \mathbf{1}-\Gamma(\tilde{j}_0) \leq \mathbf{1}$. Using this, the relations $\mathbf{1}-\Gamma(\tilde{j}_0) \leq \mathrm{d}\Gamma(\tilde{\chi}_{v\geq 1}) \leq t^{-2\alpha}\mathrm{d}\Gamma(b_\epsilon^2)$, we obtain the bound

$$\big\|(\Gamma(\tilde{j}_0)-\mathbf{1})u\|^2 \leq \|(\mathbf{1}-\Gamma(\tilde{j}_0))^{\frac{1}{2}}u\|^2 \leq t^{-2\alpha}\|\mathrm{d}\Gamma(b_\epsilon^2)^{\frac{1}{2}}u\|^2. \tag{5.59}$$

Using the pull-through formula, one verifies that $\mathrm{d}\Gamma(b_\epsilon^2)^{\frac{1}{2}}P_{\mathrm{gs}}$ is bounded and that $\|\mathrm{d}\Gamma(b_\epsilon^2)^{\frac{1}{2}}P_{\mathrm{gs}}\| = \mathcal{O}(t^\kappa)$ (see Appendix D, Lemma D.1). Hence, since $\kappa < \alpha$, the above estimates yield

$$\big\|B\big\| = o_t(1). \tag{5.60}$$

Next, using $\Gamma(j_\infty)e^{-iH_f t} = e^{-iH_f t}\Gamma(e^{i\omega t}j_\infty e^{-i\omega t})$ and $e^{i\omega t}b_\epsilon e^{-i\omega t} = b_\epsilon + \theta_\epsilon t$, it is not difficult to verify (see Appendix C, Lemma C.4) that

$$\big\|C\big\| \leq \big\|\mathbf{1}\otimes(\Gamma(e^{i\omega t}\tilde{j}_\infty e^{-i\omega t})-\mathbf{1})\phi_{0\epsilon'}\big\| \to 0,$$

as $t\to\infty$, $\forall\epsilon'>0$, and hence we obtain

$$\big\|C\big\| = C(\epsilon')o_t(1). \tag{5.61}$$

Inserting the previous estimates into (5.53) shows that

$$\psi_t = \check{\Gamma}(j)^*\psi_{t\epsilon'} + \mathcal{O}(\epsilon') + C(\epsilon')o_t(1). \tag{5.62}$$

Next, we want to pass from $\check{\Gamma}(j)^*$ to I using the formula (5.42). To this end we use estimates of the type (5.60) and (5.61) in order to remove the term $\Gamma(j_0)\otimes\Gamma(j_\infty)$. Hence, we need to bound I, for instance by introducing a cutoff in N. Let $\chi_m := \chi_{N\leq m}$ and $\bar{\chi}_m := \mathbf{1}-\chi_m$ and write $\check{\Gamma}(j)^*\psi_{t\epsilon'} = \chi_m\check{\Gamma}(j)^*\psi_{t\epsilon'} + \bar{\chi}_m\check{\Gamma}(j)^*\psi_{t\epsilon'}$. Using that $N^{1/2}\check{\Gamma}(j)^* = \check{\Gamma}(j)^*\hat{N}^{1/2}$ and that by Lemma D.1 of Appendix D (see also [7, 38]), $\mathrm{Ran}\,P_{\mathrm{gs}} \subset D(N^{1/2})$, and therefore $\psi_{t\epsilon'} \in D(\hat{N}^{1/2})$, we estimate

$$\|\bar{\chi}_m\check{\Gamma}(j)^*\psi_{t\epsilon'}\| \lesssim m^{-\frac{1}{2}}\|\hat{N}^{\frac{1}{2}}\psi_{t\epsilon'}\| = m^{-\frac{1}{2}}C(\epsilon').$$

Now, we can use (5.42) to obtain

$$\psi_t = \chi_m I\big(\Gamma(j_0)\otimes\Gamma(j_\infty)\big)\psi_{t\epsilon'} + \mathcal{O}(\epsilon') + C(\epsilon')o_t(1) + C(\epsilon')o_m(1). \tag{5.63}$$

Using $\|\chi_m I\| \leq 2^{m/2}$ together with estimates of the type (5.60) and (5.61), we find (here we need the cutoff χ_m)

$$\psi_t = \chi_m I\psi_{t\epsilon'} + \mathcal{O}(\epsilon') + C(\epsilon',m)o_t(1) + C(\epsilon')o_m(1). \tag{5.64}$$

Since $\phi_{0\epsilon'} \in \mathcal{H}\otimes\mathcal{F}_{\mathrm{fin}}(\mathfrak{h}_0)$, we can write $\psi_{t\epsilon'}$ as $\psi_{t\epsilon'} = \Phi_{\mathrm{gs}}\otimes f_{t\epsilon'}$, with $f_{t\epsilon'} \in \mathcal{F}_{\mathrm{fin}}(\mathfrak{h}_0)$, and therefore $\psi_{t\epsilon'} \in D(I)$ (here we need that $f_{\epsilon'}$ is in $\mathcal{F}_{\mathrm{fin}}(\mathfrak{h}_0)$). Hence $\chi_m I\psi_{t\epsilon'} = I\psi_{t\epsilon'} + C(\epsilon')o_m(1)$. Combining this with (5.64) and remembering (5.54), we obtain

$$\begin{aligned}\psi_t =& I(e^{-iE_{\mathrm{gs}}t}P_{\mathrm{gs}}\otimes e^{-iH_f t}\chi_{\Delta'}(H_f))\phi_{0\epsilon'} \\ &+ \mathcal{O}(\epsilon') + C(\epsilon',m)o_t(1) + C(\epsilon')o_m(1).\end{aligned} \tag{5.65}$$

Letting $t\to\infty$, next $m\to\infty$, the equation (1.14) follows. □

Remark. The reason for ϵ' in the statement of the theorem is we do not know whether $(P_{\mathrm{gs}}\otimes 1)W_+\psi_0 \in D(I)$. Indeed, if the latter were true, then the relations (5.65), (5.50) and $\|\phi_0 - \phi_{0\epsilon'}\| \leq \epsilon'$, where $\phi_0 := W_+\psi_0$, would give

$$\begin{aligned}\psi_t =& I(e^{-iE_{\mathrm{gs}}t}P_{\mathrm{gs}}\otimes e^{-iH_f t}\chi_{\Delta'}(H_f))\phi_0 \\ &+ \mathcal{O}(\epsilon') + C(\epsilon',m)o_t(1) + C(\epsilon')o_m(1),\end{aligned} \tag{5.66}$$

which, after letting $t\to\infty$, next $m\to\infty$ and then $\epsilon'\to 0$, gives

$$\lim_{t\to\infty}\|\psi_t - I(e^{-iE_{\mathrm{gs}}t}P_{\mathrm{gs}}\otimes e^{-iH_f t}\chi_{\Delta'}(H_f))W_+\psi_0\| = 0. \tag{5.67}$$

6. Proof of Theorem 1.2: the model (1.29)–(1.32)

In this section we extend the results of Sections 3–5 to hamiltonians of the form (1.29)–(1.32), with the operators η_j, $j = 1, 2$, satisfying (1.7), and prove Theorem 1.2. First, to extend the results of Section 2 to the present case, we replace the assumption (2.8) by the assumptions

$$\begin{cases} \Big(\int \|\eta_1 \eta_2^2 (\tilde{\phi}_t g)_{ij}(k)\|_{\mathcal{H}_p}^2 \omega(k)^\delta dk \Big)^{\frac{1}{2}} \lesssim \langle t \rangle^{-\lambda'}, \ i + j = 1, \\ \Big(\int \|\eta_2^2 (\tilde{\phi}_t g)_{ij}(k_1, k_2)\|_{\mathcal{H}_p}^2 \prod_{\ell=1,2} (1 + \omega(k_\ell)^{-\frac{1}{2}} + \omega(k_\ell)^\delta) dk_\ell \Big)^{\frac{1}{2}} \lesssim \langle t \rangle^{-\lambda'}, \ i + j = 2, \end{cases} \tag{6.1}$$

where λ' is the same as in (2.8) and, for any one-particle operator ϕ acting on $\mathfrak{h}$, we define $(\tilde{\phi} g)_{ij} := \phi g_{ij}$, for $i + j = 1$, and $(\tilde{\phi} g)_{2,0} = (\tilde{\phi} g)_{0,2}^* := (\phi \otimes \mathbf{1} + \mathbf{1} \otimes \phi) g_{2,0}$, $(\tilde{\phi} g)_{1,1} := (\phi \otimes \mathbf{1} - \mathbf{1} \otimes \phi) g_{1,1}$. Then we replace the second relation in (2.12) by the relation (see Supplement II)

$$i[\tilde{I}(g), \mathrm{d}\Gamma(\phi_t)] = -\tilde{I}(i\tilde{\phi} g), \tag{6.2}$$

which is valid for any one-particle operator ϕ, and replace the estimate (2.14) by the estimate

$$\begin{aligned} |\langle \tilde{I}(g) \rangle_\psi| \leq & \sum_{i+j=1} \Big(\int \|\eta_1 \eta_2^2 g_{ij}(k)\|_{\mathcal{H}_p}^2 \omega(k)^\delta dk \Big)^{\frac{1}{2}} \|\eta_1^{-1} \eta_2^{-2} \psi\| \|\psi\|_\delta \\ & + \sum_{i+j=2} \Big(\int \|\eta_2^2 g_{ij}(k_1, k_2)\|_{\mathcal{H}_p}^2 \prod_{\ell=1,2} (1 + \omega(k_\ell)^{-1} + \omega(k_\ell)^\delta) dk_\ell \Big)^{\frac{1}{2}} (\|\eta_2^{-4} \psi\| + \|\psi\|_{-1}) \|\psi\|_\delta, \end{aligned} \tag{6.3}$$

which, as in (2.14), implies, together with (6.1) and (1.7),

$$|\langle \tilde{I}(i\tilde{\phi}_t g) \rangle_{\psi_t}| \lesssim t^{-\lambda' + \nu_\delta} \|\psi_0\|_\delta^2, \tag{6.4}$$

for any $\psi_0 \in \Upsilon_\delta$, where Υ_δ is defined in (2.2). This completes the extension of the results of Section 2, and therefore of Section 3, to hamiltonians of the form (1.29)–(1.32).

To extend the results of Section 4, we have to extend the estimates (4.10) and (4.22) for $I_1 = i[I(g), B_\epsilon]$ and $I_2 = [B_\epsilon, [B_\epsilon, I(g)]]$ and the estimate (4.18) for the remainder, R, defined in (4.4), to the interactions of the form (1.30)–(1.32). Using that $\tilde{I}_1 := i[\tilde{I}(g), B_\epsilon] = \tilde{I}(i\tilde{b}_\epsilon g)$ and $\tilde{I}_2 := [B_\epsilon, [B_\epsilon, \tilde{I}(g)]] = \tilde{I}(\tilde{b}_\epsilon^2 g)$, where $\tilde{b}_\epsilon$ is defined by the same rules as $\tilde{\phi}$ after (6.1), and using (6.3), we obtain

$$\tilde{I}_1 \geq -C \langle g \rangle \tilde{E}_1, \tag{6.5}$$

and

$$\|\tilde{E}_2^{-\frac{1}{2}} \tilde{I}_2 \tilde{E}_2^{-\frac{1}{2}}\| \lesssim \epsilon^{-1} \langle g \rangle, \tag{6.6}$$

where, recall, $\langle g \rangle := \sum_{1 \leq i+j \leq 2} \sum_{|\alpha| \leq 2} \|\eta_1^{2-i-j} \eta_2^{|\alpha|} \partial^\alpha g_{ij}\|$ are the norm of the vector coupling operators $g := (g_{ij})$, defined in the introduction after (1.32), and $\tilde{E}_1 := N + \eta_2^{-1} \eta_1^{-2} \eta_2^{-1} + \eta_2^{-8} + \mathbf{1}$, and $\tilde{E}_2 := N + H_f + \eta_2^{-2} \eta_1^{-2} \eta_2^{-2} + \eta_2^{-8} + \mathbf{1}$ are new estimating operators. This extends (4.10) and (4.22). Let $\tilde{R}$ be defined by (4.4), with B_1 and H replaced by $\tilde{B}_1 := i[\tilde{H}, B_\epsilon]$ and $\tilde{H}$. By (4.19), with R and $B_2 = [B_\epsilon, [B_\epsilon, H]]$ replaced by $\tilde{R}$ and $\tilde{B}_2 := [B_\epsilon, [B_\epsilon, \tilde{H}]]$, and (6.6), we obtain the extension of (4.18) to the interactions of the form (1.30)–(1.32):

$$\|\tilde{E}_2^{-\frac{1}{2}} \tilde{R} \tilde{E}_2^{-\frac{1}{2}}\| \lesssim t^{-2} \epsilon^{-1}. \tag{6.7}$$

To extend the results of Section 5 to hamiltonians of the form (1.29)–(1.32), we have to prove estimates of the type (5.21) and (5.40) for the operator

$$\tilde{G}_1 := (\tilde{I}(g) \otimes \mathbf{1}) \check{\Gamma}(j) - \check{\Gamma}(j) \tilde{I}(g), \tag{6.8}$$

which replaces G_1 defined in (5.11). To this end, we first extend the relations (5.18), (5.19) to the interactions of the form (1.30). First, we use

$$\check{\Gamma}(j) a^\#(h) = \hat{a}^\#(h) \check{\Gamma}(j), \tag{6.9}$$

where $\hat{a}^{\#}(h) := a^{\#}(j_0 h) \otimes \mathbf{1} + \mathbf{1} \otimes a^{\#}(j_\infty h)$, with $a^{\#}$ standing for a or a^*. This together with (6.8) and the notation $\tilde{a}^{\#}_\lambda(k) := a^{\#}_\lambda(k) \otimes \mathbf{1} - \hat{a}^{\#}_\lambda(k) = (1-j_0)a^{\#}_\lambda(k) \otimes \mathbf{1} - \mathbf{1} \otimes j_\infty a^{\#}_\lambda(k)$ gives

$$\tilde{G}_1 = I_{\#}(g)\check{\Gamma}(j), \tag{6.10}$$

where

$$I_{\#}(g) = \sum_\lambda \int dk \, \big(g_{01}(k) \otimes \tilde{a}_\lambda(k) + \text{h.c.} \big) \tag{6.11}$$

$$+ \sum_{\lambda_1,\lambda_2} \int dk_1 dk_2 \, \big(g_{02}(k_1,k_2) \otimes (\tilde{a}_{\lambda_1}(k_1)\hat{a}_{\lambda_2}(k_2) + \hat{a}_{\lambda_1}(k_1)\tilde{a}_{\lambda_2}(k_2) + \tilde{a}_{\lambda_1}(k_1)\tilde{a}_{\lambda_2}(k_2)) + \text{h.c.} \big) \tag{6.12}$$

$$+ \sum_{\lambda_1,\lambda_2} \int dk_1 dk_2 \, g_{11}(k_1,k_2) \otimes (\tilde{a}^*_{\lambda_1}(k_1)\hat{a}_{\lambda_2}(k_2) + \hat{a}^*_{\lambda_1}(k_1)\tilde{a}_{\lambda_2}(k_2) + \tilde{a}^*_{\lambda_1}(k_1)\tilde{a}_{\lambda_2}(k_2)). \tag{6.13}$$

Here the notation $g_{01}(k) \otimes \tilde{a}_\lambda(k)$ should be read as $\big((1-j_0)g_{01}\big)(k)(a_\lambda(k) \otimes \mathbf{1}) - (j_\infty g_{01})(k)(\mathbf{1} \otimes a_\lambda(k))$, and likewise in the second and third lines. Using this and (3.16), we have in addition

$$\|(\hat{H}_f + 1)^{-\frac{1}{2}} \tilde{G}_1 (N + \tilde{\eta}^{-2} + \mathbf{1})^{-1}\| \lesssim t^{-\lambda}, \tag{6.14}$$

with $\tilde{\eta}^2 := \eta_2^2(\mathbf{1} + \eta_1^2)\eta_2^2$, recall, $\hat{H}_f = H_f \otimes \mathbf{1} + \mathbf{1} \otimes H_f$, and

$$\|f(\hat{H})\tilde{G}_1(N + \mathbf{1})^{-\frac{1}{2}}\| \lesssim t^{-\lambda}. \tag{6.15}$$

This extends the proof of the existence and properties of the Deift-Simon wave operators (see Theorem 5.1) to the interactions of the form (1.30)–(1.32). The remainder of the proof goes the same way as the proof of Theorem 5.1.

7. Proof of Theorem 1.1 for the QED model

7.1. **Generalized Pauli–Fierz transformation.** We consider the QED hamiltonian defined in (1.1)–(1.2). The coupling function $g^{\text{qed}}_y(k,\lambda) := |k|^{-1/2}\xi(k)\varepsilon_\lambda(k)e^{ik\cdot y}$ in this hamiltonian is more singular in the infrared than can be handled by our techniques ($\mu > 0$). To go around this problem we use the (unitary) generalized Pauli–Fierz transformation (see [58])

$$H \longrightarrow \tilde{H} := e^{-i\sum_{j=1}^n \kappa_j \Phi(q_{x_j})} H e^{i\sum_{j=1}^n \kappa_j \Phi(q_{x_j})}, \tag{7.1}$$

where $\Phi(h)$ is the operator-valued field, $\Phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h))$, and the function $q_y(k,\lambda)$ is defined below, to pass to the new unitarily equivalent hamiltonian $\tilde{H}$. To define $q_y(k,\lambda)$, let $\varphi \in \mathrm{C}^\infty(\mathbb{R};\mathbb{R})$ be a non-decreasing function such that $\varphi(r) = r$ if $|r| \le 1/2$ and $|\varphi(r)| = 1$ if $|r| \ge 1$. For $0 < \nu < 1/2$, we define

$$q_y(k,\lambda) := \frac{\xi(k)}{|k|^{\frac{1}{2}+\nu}} \varphi(|k|^\nu \varepsilon_\lambda(k) \cdot y). \tag{7.2}$$

We note that the definition of $\Phi(h)$ gives $A(y) = \Phi(g^{\text{qed}}_y)$. Using (II.7) and (II.8) of Supplement II, we compute

$$\tilde{H} = \sum_{j=1}^n \frac{1}{2m_j}\big(-i\nabla_{x_j} - \kappa_j \tilde{A}(x_j)\big)^2 + E(x) + H_f + V(x), \tag{7.3}$$

where, recall, $x = (x_1, \dots, x_n)$, and

$$\begin{cases} \tilde{A}(y) := \Phi(\tilde{g}_y), \quad \tilde{g}_y(k,\lambda) := g^{\text{qed}}_y(k,\lambda) - \nabla_x q_y(k,\lambda), \\ E(x) := -\sum_{j=1}^n \kappa_j \Phi(e_{x_j}), \quad e_y(k,\lambda) := i|k|q_y(k,\lambda), \\ V(x) := U(x) + \frac{1}{2}\sum_{\lambda=1,2}\sum_{j=1}^n \kappa_j^2 \int_{\mathbb{R}^3} |k||q_{x_j}(k,\lambda)|^2 dk. \end{cases} \tag{7.4}$$

The operator $\tilde{H}$ is self-adjoint with domain $D(\tilde{H}) = D(H) = D(p^2 + H_f)$ (see [41, 42]).

Now, the coupling functions (form factors) $\tilde{g}_y(k,\lambda)$ and $e_y(k,\lambda)$ in the transformed hamiltonian, $\tilde{H}$, satisfy the estimates that are better behaved in the infrared ([10]):

$$|\partial_k^m \tilde{g}_y(k,\lambda)| \lesssim \langle k\rangle^{-3}|k|^{\frac{1}{2}-|m|}\langle y\rangle^{\frac{1}{\nu}+|m|}, \tag{7.5}$$

$$|\partial_k^m e_y(k,\lambda)| \lesssim \langle k\rangle^{-3}|k|^{\frac{1}{2}-|m|}\langle y\rangle^{1+|m|}. \tag{7.6}$$

We see that the new hamiltonian (7.3) is of the form

$$\widetilde{H} = H_p + H_f + \tilde{I}(g), \tag{7.7}$$

with $H_p := -\sum_{j=1}^n \frac{1}{2m_j}\Delta_{x_j} + V(x)$, and $\tilde{I}(g) := -\sum_{j=1}^n \kappa_j(p_j\cdot\widetilde{A}(x_j) + \widetilde{A}(x_j)\cdot p_j - \kappa_j\widetilde{A}(x_j)^2) + E(x)$. We see that the latter operator is of the form (1.30)–(1.32), with $\eta_1 = \langle p\rangle^{-1}$, $\eta_2 = \langle x\rangle^{-1-1/\nu}$, $\mu = 1/2$, $|\alpha| \le 2$, and $1 \le i+j \le 2$, where $p := (p_1,\dots,p_n)$, and therefore the hamiltonian (1.1) satisfy the bound (1.8) and is of the class described in the introduction.

7.2. **Proof of Theorem 1.1.** We present the parts of the proof of Theorem 1.1 for the hamiltonian (1.1) which differ from that for the hamiltonian (1.4), with the interaction (1.5). To begin with, we note that, in Section 6, we have shown the statements of Theorems 3.1 and 4.1 for hamiltonians of the form (1.29)–(1.32), with the operators η_j, $j = 1,2$, satisfying (1.7), and therefore for the operator (7.3). To translate Theorems 3.1 and 4.1 from $\tilde{H}$, given by (7.3), to the QED hamilonian (1.1), we use the following estimates ([10])

$$\Big\|\mathrm{d}\Gamma(\chi_1(v))^{\frac{1}{2}}\psi\Big\|^2 \lesssim \Big\langle \mathcal{U}\psi, \mathrm{d}\Gamma(\chi_1(v))\mathcal{U}\psi\Big\rangle + t^{-\alpha d}\|\psi\|^2, \tag{7.8}$$

$$\Big\|\Gamma(\chi_2(v))^{\frac{1}{2}}\psi\Big\|^2 \lesssim \Big\langle \mathcal{U}\psi, \Gamma(\chi_2(v))\mathcal{U}\psi\Big\rangle + t^{-\alpha d}\|\psi\|^2, \tag{7.9}$$

where $\mathcal{U} := e^{-i\sum_{j=1}^n \kappa_j\Phi(q_{x_j})}$ and , recall, $v := \frac{b_\epsilon}{ct^\alpha}$, valid for any functions $\chi_1(v)$ and $\chi_2(v)$ supported in $\{|v|\le\varepsilon\}$ and $\{|v|\ge\varepsilon\}$, respectively, for some $\varepsilon > 0$, for any $\psi \in f(H)D(N^{1/2})$, with $f \in \mathrm{C}_0^\infty((-\infty,\Sigma))$, and for $0 \le d < 1/2$. (7.8) follows from estimates of Section 2 of [10] and (7.9) can be obtained similarly (see (II.8) and (II.9)). Using these estimates for $\psi_t = e^{-itH}\psi_0$, with an initial condition ψ_0 in either Υ_1 or Υ_2, together with $\mathcal{U}e^{-itH}\psi_0 = e^{-it\tilde{H}}\mathcal{U}\psi_0$, and applying Theorems 3.1 and 4.1 for $\tilde{H}$ to the first terms on the r.h.s., we see that, to obtain Theorems 3.1 and 4.1 for the hamiltonian (1.1), we need, in addition, the estimates

$$\Big\langle \psi, \mathcal{U}^* N_1\mathcal{U}\psi\Big\rangle \lesssim \langle\psi, (N_1+\mathbf{1})\psi\rangle, \tag{7.10}$$

$$\Big\langle \psi, \mathcal{U}^*\mathrm{d}\Gamma(\langle y\rangle)\mathcal{U}\psi\Big\rangle \lesssim \langle\psi, (\mathrm{d}\Gamma(\langle y\rangle) + \langle x\rangle^2)\psi\rangle, \tag{7.11}$$

$$\|\mathcal{U}^*\mathrm{d}\Gamma(b)\mathcal{U}\psi\| \lesssim \|(\mathrm{d}\Gamma(b) + \langle x\rangle^2)\psi\|, \tag{7.12}$$

where, recall, $N_1 = \mathrm{d}\Gamma(\omega^{-1})$ and $b = \frac{1}{2}(k\cdot y + y\cdot k)$.

Let $q_x := \sum_{j=1}^n \kappa_j q_{x_j}$ so that $\mathcal{U} := e^{-i\Phi(q_x)}$. To prove (7.10), we see that, by (II.8), we have

$$\mathcal{U}^* N_1 \mathcal{U} = e^{i\Phi(q_x)}\mathrm{d}\Gamma(\omega^{-1})e^{-i\Phi(q_x)} = N_1 - \Phi(i\omega^{-1}q_x) + \frac{1}{2}\|\omega^{-1/2}q_x\|_{\mathfrak{h}}^2. \tag{7.13}$$

(Since $\omega^{-1}q_x \notin \mathfrak{h}$, the field operator $\Phi(i\omega^{-1}q_x)$ is not well-defined and therefore this formula should be modified by introducing, for instance, an infrared cutoff parameter σ into q_x. One then removes it at the end of the estimates. Since such a procedure is standard, we omit it here.) This relation, together with

$$|\langle\psi, \Phi(i\omega^{-1}q_x)\psi\rangle| \lesssim \Big(\int \omega^{-3-2\nu+\varepsilon}\langle k\rangle^{-6}dk\Big)^{\frac{1}{2}}\|\mathrm{d}\Gamma(\omega^{-\varepsilon})^{\frac{1}{2}}\psi\|\|\psi\|, \tag{7.14}$$

for any $\varepsilon > 0$, which follows from the bounds of Lemma I.1 of Supplement I, and

$$\|\omega^{-\frac{1}{2}}q_x\|_{\mathfrak{h}} \lesssim \|\omega^{-1-\nu}\langle k\rangle^{-3}\|_{\mathfrak{h}}, \tag{7.15}$$

implies (7.10).

To prove (7.11) and (7.12), we proceed similarly, using, instead of (7.14) and (7.15), the estimates

$$|\langle\psi,\Phi(i\langle y\rangle q_x)\psi\rangle| \lesssim \Big(\int \omega^{-2-2\nu}\langle k\rangle^{-6}dk\Big)^{\frac12}\big\|\mathrm{d}\Gamma(\omega^{-1})^{\frac12}\psi\big\|\|\langle x\rangle\psi\| \\ \lesssim \Big(\int \omega^{-2-2\nu}\langle k\rangle^{-6}dk\Big)^{\frac12}\big\|\mathrm{d}\Gamma(\langle y\rangle)^{\frac12}\psi\big\|\|\langle x\rangle\psi\|, \tag{7.16}$$

and

$$\|\langle y\rangle^{\frac12}q_x\|_{\mathfrak{h}} \lesssim \langle x\rangle^{\frac12}\|\omega^{-1-\nu}\langle k\rangle^{-3}\|_{\mathfrak{h}}, \tag{7.17}$$

and

$$\|\Phi(ibq_x)\psi\| \lesssim \Big(\int \omega^{-2-2\nu}\langle k\rangle^{-6}dk\Big)^{\frac12}\|\langle x\rangle(H_f+1)^{\frac12}\psi\|, \tag{7.18}$$

and

$$\langle q_x, bq_x\rangle_{\mathfrak{h}} \lesssim \langle x\rangle\|\omega^{-\frac12-\nu}\langle k\rangle^{-3}\|^2_{\mathfrak{h}}. \tag{7.19}$$

Next, the existence and the properties of the Deift-Simon wave operators on $\mathrm{Ran}_{(-\infty,\Sigma)}(H)$

$$W_\pm := \operatorname*{s-lim}_{t\to\pm\infty} W(t), \quad \text{with} \quad W(t) := e^{it\hat H}\check\Gamma(j)e^{-itH}, \tag{7.20}$$

where $\hat H = H\otimes\mathbf{1}+\mathbf{1}\otimes H_f$ and the operators $\check\Gamma$ and $j=(j_0,j_\infty)$ are defined in Subsection 5.1, are equivalent to the existence and the properties of the modified Deift-Simon wave operators

$$W_\pm^{(\mathrm{mod})} := \operatorname*{s-lim}_{t\to\pm\infty}\big(e^{-i\Phi(q_x)}\otimes\mathbf{1}\big)e^{it\hat H}\check\Gamma(j)e^{-itH}e^{i\Phi(q_x)}, \tag{7.21}$$

on $\mathrm{Ran}_{(-\infty,\Sigma)}(\tilde H)$ (where $\tilde H = e^{-i\Phi(q_x)}He^{i\Phi(q_x)}$ is given in (1.29)).

To prove the existence of $W_\pm^{(\mathrm{mod})}$, we observe that, due to (6.9), we have $\check\Gamma(j)\Phi(h) = \hat\Phi(h)\check\Gamma(j)$, where

$$\hat\Phi(h) := \Phi(j_0h)\otimes\mathbf{1}+\mathbf{1}\otimes\Phi(j_\infty h), \tag{7.22}$$

which in turn implies that

$$\check\Gamma(j)e^{i\Phi(h)} = e^{i\hat\Phi(h)}\check\Gamma(j). \tag{7.23}$$

Therefore

$$\big(e^{-i\Phi(q_x)}\otimes\mathbf{1}\big)e^{it\hat H}\check\Gamma(j)e^{-itH}e^{i\Phi(q_x)} = \big(e^{-i\Phi(q_x)}\otimes\mathbf{1}\big)e^{it\hat H}e^{i\hat\Phi(q_x)}\check\Gamma(j)e^{-it\tilde H} \\ = e^{it\hat H^{(\mathrm{mod})}}\check\Gamma(j)e^{-it\tilde H}+\mathrm{Rem}_t, \tag{7.24}$$

where $\hat H^{(\mathrm{mod})} := \tilde H\otimes\mathbf{1}+\mathbf{1}\otimes H_f$ and

$$\mathrm{Rem}_t := \big(e^{-i\Phi(q_x)}\otimes\mathbf{1}\big)e^{it\hat H}\big(e^{i\hat\Phi(q_x)}-e^{i\Phi(q_x)}\otimes\mathbf{1}\big)\check\Gamma(j)e^{-it\tilde H}.$$

We claim that

$$\operatorname*{s-lim}_{t\to\pm\infty}\mathrm{Rem}_t = 0. \tag{7.25}$$

Indeed, let $R := \hat\Phi(q_x)-\Phi(q_x)\otimes\mathbf{1} = \Phi((j_0-1)q_x)\otimes\mathbf{1}+\mathbf{1}\otimes\Phi(j_\infty q_x)$ and $\hat N := N\otimes\mathbf{1}+\mathbf{1}\otimes N$. Using (7.2), Lemma II.1 of Supplement II and (3.16), we obtain

$$\big\|R(\hat N+1)^{-\frac12}\big\| \lesssim \|(j_0-1)q_x\|_{\mathfrak{h}}+\|j_\infty q_x\|_{\mathfrak{h}} \lesssim t^{-\alpha\tau}\langle x\rangle^{1+\tau},$$

for any $\tau<1$. From this estimate and the relation $e^{i\hat\Phi(q_x)}-e^{i\Phi(q_x)}\otimes\mathbf{1} = -i\int_0^1 ds e^{(1-s)i\hat\Phi(q_x)}R(e^{si\Phi(q_x)}\otimes\mathbf{1})$, it is not difficult to deduce that

$$\big\|\big(e^{i\hat\Phi(q_x)}-e^{i\Phi(q_x)}\otimes\mathbf{1}\big)(\hat N+\langle x\rangle^{2+2\tau}+\mathbf{1})^{-1}\big\| \lesssim t^{-\alpha\tau}.$$

Furthermore, we have $(\hat N+\langle x\rangle^{2+2\tau}+\mathbf{1})\check\Gamma(j) = \check\Gamma(j)(N+\langle x\rangle^{2+2\tau}+\mathbf{1})$, and, as in Corollary A.3 of Appendix A, with $\mu=1/2$, one can verify that $\|Ne^{-it\tilde H}\psi_0\| \lesssim t^{2/5}\|\psi_0\|_1$ for any $\psi_0\in f(\tilde H)D(N_1^{1/2})$, $f\in\mathrm{C}_0^\infty((-\infty,\Sigma))$. Using in addition that $\|\langle x\rangle^{2+2\tau}f(\tilde H)\|<\infty$, it follows that Rem_t strongly converges to 0 on $\mathrm{Ran}_{(-\infty,\Sigma)}(\tilde H)$ provided that $\alpha\tau>2/5$.

The equations (7.20), (7.24) and (7.25) imply

$$W_{\pm}^{(\mathrm{mod})} = \underset{t\to\pm\infty}{\text{s-lim}}\, e^{it\hat{H}^{(\mathrm{mod})}}\check{\Gamma}(j)e^{-it\tilde{H}}. \tag{7.26}$$

The proof of the existence and properties of the Deift-Simon wave operators (7.26) is a special case of the corresponding proof for the hamiltonian (1.29)–(1.30) (see Section 6).

Finally, we comment on the proof of Theorem 5.4 for the hamiltonian (1.1) in the QED case. It goes in the same way as in Section 5, until the point where we have to show that $\|\mathrm{d}\Gamma(b_\epsilon^2)^{1/2}P_{\mathrm{gs}}\| = \mathcal{O}(t^\kappa)$ in the present case. This estimate can be proven by using the generalized Pauli-Fierz transformation (7.1) together with (II.9), to obtain

$$\left\|\mathrm{d}\Gamma(b_\epsilon^2)^{\frac{1}{2}}\Phi_{\mathrm{gs}}\right\|^2 = \left\langle\tilde{\Phi}_{\mathrm{gs}}, \left(\mathrm{d}\Gamma(b_\epsilon^2) - \Phi(ib_\epsilon^2 q_x) + \frac{1}{2}\langle b_\epsilon^2 q_x, q_x\rangle_{\mathfrak{h}}\right)\tilde{\Phi}_{\mathrm{gs}}\right\rangle, \tag{7.27}$$

where $\tilde{\Phi}_{\mathrm{gs}} := \mathcal{U}\Phi_{\mathrm{gs}}$. Using Lemma I.1 of Supplement I, (1.8) and the fact that $\tilde{\Phi}_{\mathrm{gs}} \in D(N^{1/2})$, we can estimate the second term of the r.h.s. of (7.27) as

$$\left|\left\langle\tilde{\Phi}_{\mathrm{gs}}, \Phi(ib_\epsilon^2 q_x)\tilde{\Phi}_{\mathrm{gs}}\right\rangle\right| \le \left\|\langle x\rangle^3\tilde{\Phi}_{\mathrm{gs}}\right\|\left\|\langle x\rangle^{-3}\Phi(ib_\epsilon^2 q_x)(N+1)^{-\frac{1}{2}}\right\|\|(N+1)^{\frac{1}{2}}\tilde{\Phi}_{\mathrm{gs}}\| \lesssim t^{2\kappa}.$$

Likewise, $|\langle\tilde{\Phi}_{\mathrm{gs}}, \langle b_\epsilon^2 q_x, q_x\rangle_{\mathfrak{h}}\tilde{\Phi}_{\mathrm{gs}}\rangle| \lesssim t^{2\kappa}$. To estimate the first term of the r.h.s. of (7.27), we apply the standard pull-through formula, which gives

$$a_\lambda(k)\tilde{\Phi}_{\mathrm{gs}} = \sum_{j=1}^{n}\frac{\kappa_j}{2m_j}\left(\tilde{H} - E_{\mathrm{gs}} + |k|\right)^{-1}\left((-i\nabla_{x_j} - \kappa_j\tilde{A}(x_j))\cdot\tilde{g}_{x_j}(k,\lambda) - 2m_j e_{x_j}(k,\lambda)\right)\tilde{\Phi}_{\mathrm{gs}}.$$

We then easily conclude that $\|\mathrm{d}\Gamma(b_\epsilon^2)^{1/2}\tilde{\Phi}_{\mathrm{gs}}\| = \mathcal{O}(t^\kappa)$ in the same way as in Lemma D.1 of Appendix D.

Appendix A. Photon # and low momentum estimate

For simplicity, consider hamiltonians of the form (1.4)–(1.5), with the coupling operators $g(k)$ satisfying (1.6) and (1.7) with $\mu > -1/2$. The extension to hamiltonians of the form (1.29)–(1.30) is done along the lines of Section 6. Recall the notations $\langle A\rangle_\psi = \langle\psi, A\psi\rangle$, $N_\rho = \mathrm{d}\Gamma(\omega^{-\rho})$ and $\Upsilon_\rho = \{\psi_0 \in f(H)D(N_\rho^{1/2}),$ for some $f \in \mathrm{C}_0^\infty((-\infty,\Sigma))\}$. The idea of the proof of the following estimate follows [35] and [10].

Proposition A.1. *Let $\rho \in [-1,1]$. For any $\psi_0 \in \Upsilon_\rho$,*

$$\langle N_\rho\rangle_{\psi_t} \lesssim t^{\nu_\rho}\|\psi_0\|_\rho^2, \quad \nu_\rho = \frac{1+\rho}{2+\mu}. \tag{A.1}$$

Proof. Decompose $N_\rho = K_1 + K_2$, where

$$K_1 := \mathrm{d}\Gamma(\omega^{-\rho}\chi_{t^\alpha\omega\le 1}) \qquad \text{and} \qquad K_2 := \mathrm{d}\Gamma(\omega^{-\rho}\chi_{t^\alpha\omega\ge 1}).$$

Then, by (2.3),

$$\langle K_2\rangle_\psi \le \langle\mathrm{d}\Gamma(t^{\alpha(1+\rho)}\omega\chi_{t^\alpha\omega\ge 1})\rangle_{\psi_t} \le t^{\alpha(1+\rho)}\langle H_f\rangle_{\psi_t} \lesssim t^{\alpha(1+\rho)}\|\psi_0\|. \tag{A.2}$$

On the other hand, we have by (2.13),

$$DK_1 = \mathrm{d}\Gamma(\alpha\omega^{1+\rho}t^{\alpha-1}\chi'_{t^\alpha\omega\le 1}) - I(i\omega^{-\rho}\chi_{t^\alpha\omega\le 1}g). \tag{A.3}$$

Since $\|\eta_1 g(k)\|_{\mathcal{H}_p} \lesssim |k|^\mu\langle k\rangle^{-2-\mu}$ (see (1.6)), we obtain

$$\int dk\,\omega(k)^{-2\rho}\chi_{t^\alpha\omega\le 1}\|g(k)\|_{\mathcal{H}_p}^2(\omega(k)^{-1}+1) \lesssim t^{-2(1+\mu-\rho)\alpha}. \tag{A.4}$$

This together with (2.14) and (2.3) gives

$$|\langle I(i\omega^{-\rho}\chi_{t^\alpha\omega\le 1}g)\rangle_{\psi_t}| \lesssim t^{-(1+\mu-\rho)\alpha}\|\psi_0\|^2. \tag{A.5}$$

Hence, by (A.3), since $\partial_t\langle K_1\rangle_{\psi_t} = \langle DK_1\rangle_{\psi_t}$, $\chi'_{t^\alpha\omega\le 1} \le 0$, we obtain

$$\partial_t\langle K_1\rangle_{\psi_t} \lesssim t^{-(1+\mu-\rho)\alpha}\|\psi_0\|^2,$$

and therefore

$$\langle K_1\rangle_{\psi_t} \le Ct^{\nu'}\|\psi_0\|^2 + \langle N_\rho\rangle_{\psi_0}, \tag{A.6}$$

where $\nu' = 1 - (1+\mu-\rho)\alpha$, if $(1+\mu-\rho)\alpha < 1$ and $\nu' = 0$, if $(1+\mu-\rho)\alpha > 1$. Estimates (A.6) and (A.2) with $\alpha = \frac{1}{2+\mu}$, if $\rho > -1$, give (A.1). The case $\rho = -1$ follows from (2.3). □

Remark. A minor modification of the proof above give the following bound for $\rho > 0$ and $\nu'_\rho := \frac{\rho}{\frac{3}{2}+\mu}$,

$$\langle N_\rho \rangle_{\psi_t} \lesssim t^{\nu'_\rho} \big(\|\psi_t\|_N^2 + \|\psi_0\|^2 \big) + \langle N_\rho \rangle_{\psi_0}. \tag{A.7}$$

Corollary A.2. *For any $\psi_0 \in \Upsilon_\rho$, $\gamma \geq 0$ and $c > 0$,*

$$\|\chi_{N_\rho \geq ct^\gamma} \psi_t\|^2 \lesssim t^{-\frac{\gamma}{2} + \frac{1+\rho}{2(2+\mu)}} \|\psi_0\|^2 + t^{-\frac{\gamma}{2}} \langle N_\rho \rangle_{\psi_0}. \tag{A.8}$$

Proof. We have

$$\|\chi_{N_\rho \geq ct^\gamma} \psi_t\| \leq c^{-\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} \ \|\chi_{N_\rho \geq ct^\gamma} N_\rho^{\frac12} \psi_t\| \leq c^{-\frac{\gamma}{2}} t^{-\frac{\gamma}{2}} \ \|N_\rho^{\frac12} \psi_t\|.$$

Now applying (A.1) we arrive at (A.8). □

Corollary A.3. *Let $\psi_0 \in \Upsilon_1$. Then $\psi_0 \in D(N)$ and*

$$\langle N^2 \rangle_{\psi_t} \lesssim t^{\frac{2}{2+\mu}} \|\psi_0\|_1^2. \tag{A.9}$$

Proof. By the Cauchy-Schwarz inequality, we have $N^2 \leq \mathrm{d}\Gamma(\omega)\mathrm{d}\Gamma(\omega^{-1}) = H_f N_1$, and hence

$$\begin{aligned} \langle N^2 \rangle_{\psi_t} &\leq \langle N_1^{\frac12} H_f N_1^{\frac12} \rangle_{\psi_t} \\ &= \langle N_1^{\frac12} H_f (H - E_{\mathrm{gs}} + 1)^{-1} N_1^{\frac12} (H - E_{\mathrm{gs}} + 1) \rangle_{\psi_t} \\ &\quad + \langle N_1^{\frac12} H_f [N_1^{\frac12}, (H - E_{\mathrm{gs}} + 1)^{-1}] (H - E_{\mathrm{gs}} + 1) \rangle_{\psi_t}. \end{aligned}$$

Under the assumption (1.6) with $\mu > 0$, one verifies that $H_f[N_1^{\frac12}, (H - E_{\mathrm{gs}} + 1)^{-1}]$ is bounded. Since $H_f(H - E_{\mathrm{gs}} + 1)^{-1}$ is also bounded, we obtain

$$\langle N^2 \rangle_{\psi_t} \lesssim \|N_1^{\frac12} \psi_t\| \big(\|N_1^{\frac12} (H - E_{\mathrm{gs}} + 1)\psi_t\| + \|(H - E_{\mathrm{gs}} + 1)\psi_t\| \big). \tag{A.10}$$

Applying Proposition A.1 gives

$$\|N_1^{\frac12} \psi_t\| \lesssim t^{\frac{1}{2+\mu}} \|\psi_0\| + \|N_1^{\frac12} \psi_0\|, \tag{A.11}$$

and

$$\begin{aligned} \|N_1^{\frac12} (H - E_{\mathrm{gs}} + 1)\psi_t\| &\lesssim t^{\frac{1}{2+\mu}} \|\psi_0\| + \|N_1^{\frac12} (H - E_{\mathrm{gs}} + 1)\psi_0\| \\ &\lesssim t^{\frac{1}{2+\mu}} \|\psi_0\| + \|N_1^{\frac12} \psi_0\|, \end{aligned} \tag{A.12}$$

where we used in the last inequality that $N_1^{\frac12} \tilde f(H)(N_1 + \mathbf{1})^{-\frac12}$ is bounded for any $\tilde f \in \mathrm{C}_0^\infty(\mathbb{R}^3)$. Combining (A.10), (A.11) and (A.12), we obtain

$$\langle N^2 \rangle_{\psi_t} \lesssim t^{\frac{2}{2+\mu}} (\|N_1^{\frac12} \psi_0\|^2 + \|\psi_0\|^2). \tag{A.13}$$

Hence (A.9) is proven. □

Appendix B. One-particle commutator estimates

In this appendix, we estimate some localization terms and commutators appearing in Section 3. We begin with recalling the Helffer-Sjöstrand formula that will be used several times. Let f be a smooth function satisfying the estimates $|\partial_s^n f(s)| \leq C_n \langle s \rangle^{\rho - n}$ for all $n \geq 0$, with $\rho < 0$. We consider an almost analytic extension $\tilde f$ of f, which means that $\tilde f$ is a C^∞ function on $\mathbb{C}$ such that $\tilde f|_{\mathbb{R}} = f$,

$$\operatorname{supp} \tilde f \subset \big\{ z \in \mathbb{C}, \ |\operatorname{Im} z| \leq C \langle \operatorname{Re} z \rangle \big\},$$

$|\tilde f(z)| \leq C \langle \operatorname{Re} z \rangle^\rho$ and, for all $n \in \mathbb{N}$,

$$\Big| \frac{\partial \tilde f}{\partial \bar z}(z) \Big| \leq C_n \langle \operatorname{Re} z \rangle^{\rho - 1 - n} |\operatorname{Im} z|^n.$$

Moreover, if f is compactly supported, we can assume that this is also the case for $\tilde{f}$. Given a self-adjoint operator A, the Helffer–Sjöstrand formula (see e.g. [18, 44]) allows one to express $f(A)$ as

$$f(A) = \frac{1}{\pi}\int \frac{\partial \tilde{f}(z)}{\partial \bar{z}}(A-z)^{-1}\,\mathrm{d}\mathrm{Re}\,z\,\mathrm{d}\mathrm{Im}\,z. \tag{B.1}$$

Now recall that $b_\epsilon = \frac{1}{2}(\theta_\epsilon \nabla\omega\cdot y + \text{ h.c.})$, where $\theta_\epsilon = \frac{\omega}{\omega_\epsilon}$, $\omega_\epsilon = \omega+\epsilon$, $\epsilon = t^{-\kappa}$, with $\kappa \geq 0$. We have the relations

$$i[\omega, b_\epsilon] = \theta_\epsilon, \quad i[\omega, y^2] = \frac{1}{2}(\nabla\omega\cdot y + y\cdot\nabla\omega), \tag{B.2}$$

and, using in particular Hardy's inequality, one can verify the estimate

$$\big\|[y^2, b_\epsilon]\langle y\rangle^{-2}\big\| = \mathcal{O}(t^\kappa). \tag{B.3}$$

The following lemma gathers several commutator estimates used in the main text. It is a straightforward consequence of the Helffer-Sjöstrand formula together with (B.2), (B.3), and Hardy's inequality. We do not detail the proof.

Lemma B.1. *Let $h, \tilde{h}$ be smooth function satisfying the estimates $\big|\partial_s^n h(s)\big| \leq \mathrm{C}_n\langle s\rangle^{-n}$ for $n\geq 0$ and likewise for $\tilde{h}$. Let $w_\alpha = (|y|/c_1 t^\alpha)^2$, $v_\beta = b_\epsilon/(c_2 t^\beta)$, with $0<\alpha,\beta\leq 1$. The following estimates hold*

$$[h(w_\alpha),\omega] = \mathcal{O}(t^{-\alpha}), \qquad [\tilde{h}(v_\beta),\omega] = \mathcal{O}(t^{-\beta}),$$
$$[h(w_\alpha),\theta_\epsilon^{\frac{1}{2}}] = \mathcal{O}(t^{\frac{1}{2}\kappa-\frac{1}{2}\alpha}), \qquad \langle y\rangle[h(w_\alpha),\theta_\epsilon^{\frac{1}{2}}] = \mathcal{O}(t^{\frac{1}{2}\kappa+\frac{1}{2}\alpha}),$$
$$[\tilde{h}(v_\beta),\omega_\epsilon^{-\frac{1}{2}}] = \mathcal{O}(t^{\frac{3}{2}\kappa-\beta}), \qquad b_\epsilon[\tilde{h}(v_\beta),\omega_\epsilon^{-\frac{1}{2}}] = \mathcal{O}(t^{\frac{3}{2}\kappa}), \qquad [\tilde{h}(v_\beta),\theta_\epsilon^{\frac{1}{2}}] = \mathcal{O}(t^{\kappa-\beta}),$$
$$[h(w_\alpha),b_\epsilon] = \mathcal{O}(t^\kappa), \qquad [h(w_\alpha),\tilde{h}(v_\beta)] = \mathcal{O}(t^{-\beta+\kappa}), \qquad b_\epsilon[h(w_\alpha),\tilde{h}(v_\beta)] = \mathcal{O}(t^\kappa).$$

Now we prove the following abstract result.

Lemma B.2. *Let h be a smooth function satisfying the estimates $\big|\partial_s^n h(s)\big| \leq \mathrm{C}_n\langle s\rangle^{-n}$ for $n\geq 0$. Assume an operator v is s.t. the commutators $[v,\omega]$ and $[v,[v,\omega]]$ are bounded, and for some z in $\mathbb{C}\setminus\mathbb{R}$, $(v-z)^{-1}$ preserves $D(\omega)$. Then the operator $r := [h(v),\omega] - [v,\omega]h'(v)$ is bounded as*

$$\|r\| \lesssim \big\|[v,[v,\omega]]\big\|. \tag{B.4}$$

Proof. We would like to use the Helffer–Sjöstrand formula (B.1) for h. Since h might not decay at infinity, we cannot directly express $h(v)$ by this formula. Therefore, we approximate $h(v)$ as follows. Consider $\varphi \in \mathrm{C}_0^\infty(\mathbb{R};[0,1])$ equal to 1 near 0 and $\varphi_R(\cdot) = \varphi(\cdot/R)$ for $R>0$. Let $\widetilde{h}$ be an almost analytic extensions of h such that $\widetilde{h}|_{\mathbb{R}} = h$,

$$\operatorname{supp}\widetilde{h} \subset \big\{z\in\mathbb{C};\ |\operatorname{Im} z| \leq \mathrm{C}\langle\operatorname{Re} z\rangle\big\}, \tag{B.5}$$

$|\widetilde{h}(z)| \leq \mathrm{C}$ and, for all $n\in\mathbb{N}$,

$$\Big|\partial_{\bar{z}}\widetilde{h}(z)\Big| \leq \mathrm{C}_n\langle\operatorname{Re} z\rangle^{\rho-1-n}|\operatorname{Im} z|^n. \tag{B.6}$$

Similarly let $\widetilde{\varphi}\in\mathrm{C}_0^\infty(\mathbb{C})$ be an almost analytic extension of φ satisfying these estimates. As a quadratic form on $D(\omega)$, we have

$$\big[h(v),\omega\big] = \operatorname*{s-lim}_{R\to\infty}\big[(\varphi_R h)(v),\omega\big]. \tag{B.7}$$

Since $(v-z)^{-1}$ preserves $D(\omega)$ for some z in the resolvent set of v (and hence for any such z, see [2, Lemma 6.2.1]), we can compute, using the Helffer–Sjöstrand representation (see (B.1)) for $(\varphi_R h)(v)$,

$$\begin{aligned}\big[(\varphi_R h)(v),\omega\big] &= \frac{1}{\pi}\int \partial_{\bar{z}}(\widetilde{\varphi}_R\widetilde{h})(z)\big[(v-z)^{-1},\omega\big]\,\mathrm{d}\mathrm{Re}\,z\,\mathrm{d}\mathrm{Im}\,z\\ &= -\frac{1}{\pi}\int \partial_{\bar{z}}(\widetilde{\varphi}_R\widetilde{h})(z)(v-z)^{-1}[v,\omega](v-z)^{-1}\,\mathrm{d}\mathrm{Re}\,z\,\mathrm{d}\mathrm{Im}\,z\\ &= [v,\omega](\varphi_R h)'(v) + r_R,\end{aligned} \tag{B.8}$$

as a quadratic form on $D(\omega)$, where

$$r_R = -\frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)\big[(v-z)^{-1},[v,\omega]\big](v-z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z$$
$$= \frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)(v-z)^{-1}[v,[v,\omega]](v-z)^{-2}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z. \tag{B.9}$$

Now, using $(v-z)^{-1} = \mathcal{O}\big(|\operatorname{Im} z|^{-1}\big)$, we obtain that

$$\big\|(v-z)^{-1}[v,[v,\omega]](v-z)^{-2}\big\| \lesssim |\operatorname{Im} z|^{-3}\big\|[v,[v,\omega]]\big\|. \tag{B.10}$$

Besides, for all $n\in\mathbb{N}$,

$$|\partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)| \le \mathrm{C}_n\langle \operatorname{Re} z\rangle^{\rho-1-n}|\operatorname{Im} z|^n, \tag{B.11}$$

where $\mathrm{C}_n>0$ is independent of $R\ge 1$. Using (B.9) together with (B.10), we see that there exists $\mathrm{C}>0$ such that $\|r_R\| \le \mathrm{C}\big\|[v,[v,\omega]]\big\|$, for all $R\ge 1$. Finally, since $(\varphi_R h)'(v)$ converges strongly to $h'(v)$, the lemma follows from (B.8) and the previous estimate. □

We want apply the lemma above to the *time-dependent* self-adjoint operator $v := \frac{b_\epsilon}{ct^\alpha}$.

Corollary B.3. *Let h be a smooth function satisfying the estimates $\big|\partial_s^n h(s)\big| \le \mathrm{C}_n\langle s\rangle^{-n}$ for $n\ge 0$ and let $v := \frac{b_\epsilon}{ct^\alpha}$, where $c>0$, $\epsilon = t^{-\kappa}$, with $0\le\kappa\le\beta\le 1$. Then the operator $r := dh(v)-(dv)h'(v)$ is bounded as*

$$\|r\| \lesssim t^{-\lambda},\ \lambda := 2\alpha-\kappa. \tag{B.12}$$

Proof. Observe that

$$dh(v)-(dv)h'(v) = [h(v),i\omega]-[v,i\omega]h'(v)+\partial_t h(v)-(\partial_t v)h'(v).$$

It is not difficult to verify that $(v-z)^{-1}$ preserves $D(\omega)$ for any $z\in\mathbb{C}\setminus\mathbb{R}$. Hence it follows from the computations

$$[v,i\omega] = t^{-\alpha}\theta_\epsilon, \qquad [v,[v,i\omega]] = t^{-2\alpha}\theta_\epsilon\omega_\epsilon^{-2}\epsilon, \tag{B.13}$$

that we can apply Lemma B.2. The estimate

$$[v,[v,\omega]] = \mathcal{O}(\omega_\epsilon^{-1}t^{-2\alpha}) = \mathcal{O}\big(t^{-2\alpha+\kappa}\big) \tag{B.14}$$

then gives

$$\|[h(v),i\omega]-[v,i\omega]h'(v)\| \lesssim t^{-2\alpha+\kappa}.$$

It remains to estimate $\|\partial_t h(v)-(\partial_t v)h'(v)\|$. It is not difficult to verify that $D(b_\epsilon)$ is independent of t. Using the notations of the proof of Lemma B.2 and the fact that $\partial_t h(v) = \text{s-lim}_{R\to\infty}\partial_t(\varphi_R h)(v)$, we compute

$$\partial_t(\varphi_R h)(v) = \frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)\partial_t(v-z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z$$
$$= -\frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)(v-z)^{-1}(\partial_t v)(v-z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z$$
$$= (\partial_t v)(\varphi_R h)'(v)+r'_R,$$

where

$$r'_R = -\frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)\big[(v-z)^{-1},\partial_t v\big](v-z)^{-1}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z$$
$$= \frac{1}{\pi}\int \partial_{\bar z}(\widetilde{\varphi}_R\widetilde{h})(z)(v-z)^{-1}[v,\partial_t v](v-z)^{-2}\,\mathrm{dRe}\,z\,\mathrm{dIm}\,z. \tag{B.15}$$

Now using $\partial_t v = -\frac{\alpha b_\epsilon}{ct^{\alpha+1}}+\frac{1}{ct^\alpha}\partial_t b_\epsilon$ together with (3.9), we estimate

$$[v,\partial_t v] = \mathcal{O}(t^{-1-2\alpha+\kappa})b_\epsilon+\mathcal{O}(t^{-1-2\alpha+2\kappa}).$$

From this, the properties of $\tilde\varphi$, $\tilde h$, and $\kappa\le\beta$, we deduce that $\|r'_R\| \lesssim t^{-1-\alpha+\kappa} \lesssim t^{-2\alpha+\kappa}$ uniformly in $R\ge 1$. This concludes the proof of the corollary. □

The following lemma is taken from [10]. Its proof is similar to the proof of Lemma B.2

Lemma B.4. *Let h be a smooth function satisfying the estimates $\left|\partial_s^n h(s)\right| \le \mathrm{C}_n \langle s \rangle^{-n}$ for $n \ge 0$ and $0 \le \delta \le 1$. Let $w_\alpha = (|y|/ct^\alpha)^2$ with $0 < \alpha \le 1$. We have*

$$\big[h(w_\alpha), i\omega\big] = \frac{1}{ct^\alpha} h'(w_\alpha)\big(\frac{y}{ct^\alpha}\cdot\nabla\omega + \nabla\omega\cdot\frac{y}{ct^\alpha}\big) + \mathrm{rem},$$

with

$$\big\|\omega^{\frac{\delta}{2}}\,\mathrm{rem}\,\omega^{\frac{\delta}{2}}\big\| \lesssim t^{-\alpha(1+\delta)}.$$

Appendix C. Estimates of $\mathrm{d}\Gamma$, $\mathrm{d}\check{\Gamma}$ and Γ

In this appendix we prove technical statements about $\mathrm{d}\Gamma$, $\mathrm{d}\check{\Gamma}$ and Γ, used in the main text. Most of the results we present here are close to known ones. We begin with the following standard result, which was used implicitly at several places.

Lemma C.1. *Let a, b be two self-adjoint operators on $\mathfrak{h}$ with $b \ge 0$, $D(b) \subset D(a)$ and $\|a\varphi\| \le \|b\varphi\|$ for all $\varphi \in D(b)$. Then $D(\mathrm{d}\Gamma(b)) \subset D(\mathrm{d}\Gamma(a))$ and $\|\mathrm{d}\Gamma(a)\Phi\| \le \|\mathrm{d}\Gamma(b)\Phi\|$ for all $\Phi \in D(\mathrm{d}\Gamma(b))$.*

Next, we have the following lemma which was used in the proof of Proposition 4.2. We recall the notations $B_\epsilon = \mathrm{d}\Gamma(b_\epsilon)$ and $B_{\epsilon,t} = \frac{B_\epsilon}{ct}$.

Lemma C.2. *Let $f \in \mathrm{C}_0^\infty(\mathbb{R}^3)$. Then*

$$\big\|\mathrm{d}\Gamma(\omega_\epsilon^{-1})^{\frac12} f(B_{\epsilon,t})(\mathbf{1} + \mathrm{d}\Gamma(\omega^{-1}) + t^{-1}\epsilon^{-2}N)^{-\frac12}\big\| \lesssim 1, \tag{C.1}$$

uniformly w.r.t. $\epsilon > 0$ and $t > 0$.

Proof. By interpolation, if suffices to prove that

$$\big\|\mathrm{d}\Gamma(\omega_\epsilon^{-1}) f(B_{\epsilon,t})(\mathbf{1} + \mathrm{d}\Gamma(\omega^{-1}) + t^{-1}\epsilon^{-2}\langle N\rangle)\big\| \lesssim 1. \tag{C.2}$$

To this end, we write

$$\mathrm{d}\Gamma(\omega_\epsilon^{-1}) f(B_{\epsilon,t}) = f(B_{\epsilon,t})\mathrm{d}\Gamma(\omega_\epsilon^{-1}) + [\mathrm{d}\Gamma(\omega_\epsilon^{-1}), f(B_\epsilon, t)].$$

Since $\|f(B_{\epsilon,t})\| \lesssim 1$ and $\mathrm{d}\Gamma(\omega_\epsilon^{-1})^2 \le \mathrm{d}\Gamma(\omega^{-1})^2$, the first term is bounded as

$$\big\|f(B_{\epsilon,t})\mathrm{d}\Gamma(\omega_\epsilon^{-1})(\mathbf{1} + \mathrm{d}\Gamma(\omega^{-1}))\big\| \lesssim 1. \tag{C.3}$$

To estimate the second term, we write as above, using the Helffer-Sjöstrand formula,

$$f(B_{\epsilon,t}) = \frac{1}{\pi}\int \partial_{\bar z}\widetilde{f}(z)(B_{\epsilon,t} - z)^{-1}\,\mathrm{d}\mathrm{Re}\,z\,\mathrm{d}\mathrm{Im}\,z,$$

where $\widetilde{f}$ denotes an almost analytic extension of f. This gives

$$[\mathrm{d}\Gamma(\omega_\epsilon^{-1}), f(B_{\epsilon,t})] = \frac{1}{\pi}\int \partial_{\bar z}\widetilde{f}(z)(B_{\epsilon,t} - z)^{-1}[B_{\epsilon,t}, \mathrm{d}\Gamma(\omega_\epsilon^{-1})](B_{\epsilon,t} - z)^{-1}\,\mathrm{d}\mathrm{Re}\,z\,\mathrm{d}\mathrm{Im}\,z, \tag{C.4}$$

with

$$[B_{\epsilon,t}, \mathrm{d}\Gamma(\omega_\epsilon^{-1})] = (ct)^{-1}\mathrm{d}\Gamma(\theta_\epsilon\omega_\epsilon^{-2}).$$

Since $\|\mathrm{d}\Gamma(\theta_\epsilon\omega_\epsilon^{-2})\langle N\rangle^{-1}\| \lesssim \epsilon^{-2}$, and since $B_{\epsilon,t}$ commutes with N, we obtain that

$$\|(B_{\epsilon,t} - z)^{-1}[B_{\epsilon,t}, \mathrm{d}\Gamma(\omega_\epsilon^{-1})](B_{\epsilon,t} - z)^{-1}\langle N\rangle^{-1}\| \lesssim t^{-1}\epsilon^{-2}|\mathrm{Im} z|^{-2},$$

Hence the formula (C.4) shows that

$$\|[\mathrm{d}\Gamma(\omega_\epsilon^{-1}), f(B_{\epsilon,t})]\langle N\rangle^{-1}\| \lesssim t^{-1}\epsilon^{-2},$$

which, together with (C.3), imples (C.2) and hence (C.1) by interpolation. □

We recall that, given two operators a, c on $\mathfrak{h}$, the operator $\mathrm{d}\Gamma(a, c)$ was defined in (5.12), and $\mathrm{d}\check{\Gamma}(a, c) := U\mathrm{d}\Gamma(a, c)$.

Lemma C.3. *Let $j = (j_0, j_\infty)$ and $c = (c_0, c_\infty)$, where $j_0, j_\infty, c_0, c_\infty$ are operators on $\mathfrak{h}$. Furthermore, assume that $j_0^* j_0 + j_\infty^* j_\infty \leq 1$. Then we have the relation*

$$|\langle \hat{\phi}, \mathrm{d}\check{\Gamma}(j,c)\psi\rangle| \leq \|\mathrm{d}\Gamma(|c_0|)^{\frac{1}{2}} \otimes \mathbf{1}\hat{\phi}\| \|\mathrm{d}\Gamma(|c_0|)^{\frac{1}{2}}\psi\| + \|\mathbf{1} \otimes \mathrm{d}\Gamma(|c_\infty|)^{\frac{1}{2}}\hat{\phi}\| \|\mathrm{d}\Gamma(|c_\infty|)^{\frac{1}{2}}\psi\|. \tag{C.5}$$

Likewise, with $c_1 : \mathfrak{h} \to \mathfrak{h} \oplus \mathfrak{h}$ and $c_2 : \mathfrak{h} \to \mathfrak{h}$, we have

$$|\langle u, \mathrm{d}\Gamma(j, c_1 c_2)v\rangle| \leq \|\mathrm{d}\Gamma(c_1 c_1^*)^{\frac{1}{2}} u\| \|\mathrm{d}\Gamma(c_2^* c_2)^{\frac{1}{2}} v\|. \tag{C.6}$$

Proof. Let $\tilde{\phi} = U^* \hat{\phi}$ and for an operator b on $\mathfrak{h}$ define operators $i_0 b := \mathrm{diag}(b, 0)$ and $i_\infty b := \mathrm{diag}(0, b)$ on $\mathfrak{h} \oplus \mathfrak{h}$. Since $U^* \mathrm{d}\Gamma(|c_0|)^{\frac{1}{2}} \otimes \mathbf{1} U = \mathrm{d}\Gamma(i_0|c_0|)^{\frac{1}{2}}$ and $U^* \mathbf{1} \otimes \mathrm{d}\Gamma(|c_\infty|)^{\frac{1}{2}} U = \mathrm{d}\Gamma(i_\infty|c_\infty|)^{\frac{1}{2}}$, the statement of the lemma is equivalent to

$$|\langle \tilde{\phi}, \mathrm{d}\Gamma(j,c)\psi\rangle| \leq \|\mathrm{d}\Gamma(i_0|c_0|)^{\frac{1}{2}}\tilde{\phi}\| \|\mathrm{d}\Gamma(|c_0|)^{\frac{1}{2}}\psi\| + \|\mathrm{d}\Gamma(i_\infty|c_\infty|)^{\frac{1}{2}}\tilde{\phi}\| \|\mathrm{d}\Gamma(|c_\infty|)^{\frac{1}{2}}\psi\|. \tag{C.7}$$

We decompose $\mathrm{d}\Gamma(j,c) = \mathrm{d}\Gamma(j, i_0 c_0) + \mathrm{d}\Gamma(j, i_\infty c_\infty)$ and estimate each term separately. We have, using that $\|j\| \leq 1$,

$$|\langle \tilde{\phi}, \mathrm{d}\Gamma(j, i_0 c_0)\psi\rangle| \leq \sum_{l=1}^{n} |\langle |i_0 c_0|_l^{\frac{1}{2}}\tilde{\phi}, |i_0 c_0|_l^{\frac{1}{2}}\psi\rangle|,$$

where $|i_0 c_0|_l := \mathbf{1} \otimes \cdots \otimes \mathbf{1} \otimes i_0|c_0| \otimes \mathbf{1} \otimes \cdots \otimes \mathbf{1}$, with the operator $|i_0 c_0|$ appearing in the l^{th} component of the tensor product. By the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned} |\langle \tilde{\phi}, \mathrm{d}\Gamma(j, i_0 c_0)\psi\rangle| &\leq \sum_{l=1}^{n} \||i_0 c_0|_l^{\frac{1}{2}}\tilde{\phi}\| \||i_0 c_0|_l^{\frac{1}{2}}\psi\| \leq \Big(\sum_{l=1}^{n} \||i_0 c_0|_l^{\frac{1}{2}}\tilde{\phi}\|^2\Big)^{\frac{1}{2}} \Big(\sum_{l=1}^{n} \||i_0 c_0|_l^{\frac{1}{2}}\psi\|^2\Big)^{\frac{1}{2}} \\ &= \|\mathrm{d}\Gamma(|i_0 c_0|)^{\frac{1}{2}}\tilde{\phi}\| \|\mathrm{d}\Gamma(|i_0 c_0|)^{\frac{1}{2}}\psi\|. \end{aligned}$$

Since $\|\mathrm{d}\Gamma(|i_0 c_0|)^{\frac{1}{2}}\psi\|_{\mathcal{F}(\mathfrak{h}\oplus\mathfrak{h})} = \|\mathrm{d}\Gamma(|c_0|)^{\frac{1}{2}}\psi\|_{\mathcal{F}(\mathfrak{h})}$, we obtain the first term in the r.h.s. of (C.7). The second one is obtained exactly in the same way. (C.6) can be proven in a similar manner. □

In the following lemma, as in the main text, the operator j_∞ on $\mathrm{L}^2(\mathbb{R}^3)$ is of the form $j_\infty = \chi_{\frac{b_\epsilon}{ct^\alpha} \geq 1}$, where, recall, $b_\epsilon = \frac{1}{2}(v_\epsilon(k) \cdot y + \text{ h.c.})$, where $v_\epsilon(k) = \theta_\epsilon \nabla\omega$, $\theta_\epsilon = \frac{\omega}{\omega+\epsilon}$, and $\epsilon = t^{-\kappa}$, $\kappa > 0$.

Lemma C.4. *Assume $\alpha + \kappa > 1$. Let $u \in \mathcal{F}$. Then $\big\|(\Gamma(j_\infty) - \mathbf{1})e^{-iH_f t}u\big\| \to 0$, as $t \to \infty$.*

Proof. Assume that $u \in D(\mathrm{d}\Gamma(\langle y\rangle))$. Using unitarity of $e^{-iH_f t}$ and the fact that $e^{-iH_f t} = \Gamma(e^{-i\omega t})$, we obtain

$$\big\|(\Gamma(j_\infty) - \mathbf{1})e^{-iH_f t}u\big\| = \big\|(\Gamma(e^{i\omega t} j_\infty e^{-i\omega t}) - \mathbf{1})u\big\| \leq \big\|\mathrm{d}\Gamma(e^{i\omega t}\bar{j}_\infty e^{-i\omega t})u\big\|, \tag{C.8}$$

where $\bar{j}_\infty = \mathbf{1} - j_\infty$. Using the identity $e^{it\omega} b_\epsilon e^{-it\omega} = b_\epsilon + \theta_\epsilon t$, one shows that

$$e^{it\omega}\chi_{\frac{b_\epsilon}{ct^\alpha} \leq 1} e^{-it\omega} = \chi_{\frac{b_\epsilon + \theta_\epsilon t}{ct^\alpha} \leq 1}.$$

Since $\alpha + \kappa > 1$, we have, by the Helffer-Sjöstrand formula, $\chi_{\frac{b_\epsilon + \theta_\epsilon t}{ct^\alpha} \leq 1} = \chi_{\frac{b_\epsilon + t}{ct^\alpha} \leq 1} + \mathcal{O}(t^{-(\alpha+\kappa-1)})$. Due to $\frac{-2b_\epsilon}{t} \geq 1$ on $\mathrm{supp}\, \chi_{\frac{b_\epsilon + t}{ct^\alpha} \leq 1}$ for t sufficiently large, we have

$$\|\chi_{\frac{b_\epsilon + t}{ct^\alpha} \leq 1}\phi\| \leq \Big\|\frac{-2b_\epsilon}{t}\chi_{\frac{b_\epsilon + t}{ct^\alpha} \leq 1}\phi\Big\| \leq \Big\|\frac{2\langle y\rangle}{t}\phi\Big\|,$$

and therefore

$$\Big\|\mathrm{d}\Gamma\big(\chi_{\frac{b_\epsilon + \theta_\epsilon t}{ct^\alpha} \leq 1}\big)u\Big\| \leq \frac{2}{t}\big\|\mathrm{d}\Gamma\big(\langle y\rangle\big)u\big\|.$$

Together with (C.8), this shows that $\big\|(\Gamma(j_\infty) - \mathbf{1})e^{-iH_f t}u\big\| \to 0$, for $u \in D\big(\mathrm{d}\Gamma(\langle y\rangle)\big)$. Since $D\big(\mathrm{d}\Gamma(\langle y\rangle)\big)$ is dense in $\mathcal{F}$, this concludes the proof. □

Appendix D. Estimates of $P_{\rm gs}$

Lemma D.1. *Assume* (1.6) *with* $\mu > -1/2$ *and* (1.7). *Then* $\mathrm{Ran}(P_{\rm gs}) \subset D(N^{\frac12}) \cap D(\mathrm{d}\Gamma(b_\epsilon^2)^{\frac12})$, *in other words, the operators* $N^{\frac12} P_{\rm gs}$ *and* $\mathrm{d}\Gamma(b_\epsilon^2)^{\frac12} P_{\rm gs}$ *are bounded. Moreover, we have* $\|\mathrm{d}\Gamma(b_\epsilon^2)^{\frac12} P_{\rm gs}\| = \mathcal{O}(t^\kappa)$.

Proof. Let $\Phi_{\rm gs} \in \mathrm{Ran}(P_{\rm gs})$. The well-known pull-through formula gives

$$a(k)\Phi_{\rm gs} = -(H - E_{\rm gs} + |k|)^{-1} g(k) \Phi_{\rm gs}. \tag{D.1}$$

Since $\|(H - E_{\rm gs} + |k|)^{-1}\| \le |k|^{-1}$, one easily deduces that

$$\int_{\mathbb{R}^3} \|a(k)\Phi_{\rm gs}\|^2 dk \le \Big(\int_{\mathbb{R}^3} \frac{\|\eta_1 g(k)\|^2_{\mathcal{H}_p}}{|k|^2} dk \Big) \|\eta_1^{-1} \Phi_{\rm gs}\|^2 \lesssim \|\Phi_{\rm gs}\|^2,$$

for any $\mu > -1/2$, where we used (1.6) and (1.7) in the last inequality. This implies that $N^{\frac12} P_{\rm gs}$ is bounded.

To estimate $\|\mathrm{d}\Gamma(b_\epsilon^2)^{\frac12} P_{\rm gs}\|$, we decompose

$$b_\epsilon = \frac{i}{|k| + t^{-\kappa}} k \cdot \nabla_k + \frac{3i|k|}{2(|k| + t^{-\kappa})} - \frac{i|k|}{2(|k|+t^{-\kappa})^2}.$$

Using again that $\|(H - E_{\rm gs} + |k|)^{-1}\| \le |k|^{-1}$, we obtain

$$\Big\| \frac{3i|k|}{2(|k|+t^{-\kappa})} (H - E_{\rm gs} + |k|)^{-1} g(k) \Phi_{\rm gs} \Big\| \lesssim |k|^{-1} \|\eta_1 g(k)\|_{\mathcal{H}_p} \|\eta_1^{-1} \Phi_{\rm gs}\|, \tag{D.2}$$

and

$$\Big\| \frac{i|k|}{2(|k|+t^{-\kappa})^2} (H - E_{\rm gs} + |k|)^{-1} g(k) \Phi_{\rm gs} \Big\| \lesssim t^\kappa |k|^{-1} \|\eta_1 g(k)\|_{\mathcal{H}_p} \|\eta_1^{-1} \Phi_{\rm gs}\|. \tag{D.3}$$

Moreover, we have

$$\|\nabla_k (H - E_{\rm gs} + |k|)^{-1}\| \lesssim \|(H - E_{\rm gs} + |k|)^{-2}\| \le |k|^{-2},$$

and hence

$$\begin{aligned} &\Big\| \frac{i}{|k|+t^{-\kappa}} k \cdot \nabla_k (H - E_{\rm gs} + |k|)^{-1} g(k) \Phi_{\rm gs} \Big\| \\ &\lesssim \big(t^\kappa |k|^{-1} \|\eta_1 g(k)\|_{\mathcal{H}_p} + t^\kappa \|\eta_1 \eta_2 \nabla_k g(k)\|_{\mathcal{H}_p} \big) \|\eta_1^{-1} \eta_2^{-1} \Phi_{\rm gs}\|. \end{aligned} \tag{D.4}$$

Estimates (D.2), (D.3), (D.4) together with (D.1) and (1.6)–(1.7) imply that

$$\|\mathrm{d}\Gamma(b_\epsilon^2)^{\frac12} \Phi_{\rm gs}\|^2 = \int_{\mathbb{R}^3} \Big\| \Big(\frac{i}{|k|+t^{-\kappa}} k \cdot \nabla_k + \frac{3i|k|}{2(|k|+t^{-\kappa})} - \frac{i|k|}{2(|k|+t^{-\kappa})^2} \Big) a(k) \Phi_{\rm gs} \Big\|^2 dk \lesssim t^{2\kappa} \|\Phi_{\rm gs}\|^2,$$

for any $\mu > -1/2$. This shows that $\|\mathrm{d}\Gamma(b_\epsilon^2)^{\frac12} P_{\rm gs}\| = \mathcal{O}(t^\kappa)$. □

Appendix E. The proof of the existence of W_+ under assumption (1.19)

Let $\rho_\nu := \chi \theta_\epsilon^{1/2} \omega^{\nu/2}$ and recall that $\chi \equiv \chi_{w \le 1}$, with $w = (\frac{|y|}{\bar c t})^2$, and $v = \frac{b_\epsilon}{c t^\alpha}$. We begin with the following weighted propagation estimates, which are a straightforward extensions of the estimates of Theorem 3.1:

$$\int_1^\infty dt\, t^{-\beta} \big\| \mathrm{d}\Gamma(\rho_1^* \chi_{v=1} \rho_1)^{\frac12} \psi_t \big\|^2 \lesssim \|\psi_0\|^2, \tag{E.1}$$

for μ and α as in Theorem 3.1 and any $\psi_0 \in \mathcal{H}$, and, if in addition assumption (1.19) of Theorem 1.1 holds,

$$\int_1^\infty dt\, t^{-\alpha} \big\| \mathrm{d}\Gamma(\omega^{-1/2} \chi_{v=1} \omega^{-1/2})^{\frac12} \psi_t \big\|^2 \lesssim C(\psi_0), \tag{E.2}$$

and

$$\int_1^\infty dt\, t^{-\alpha} \big\| \mathrm{d}\Gamma(\rho_{-1}^* \chi_{v=1} \rho_{-1})^{\frac12} \psi_t \big\|^2 \lesssim C(\psi_0). \tag{E.3}$$

for any $\psi_0 \in \mathcal{D}$. Likewise, under assumption (1.19) of Theorem 1.1, the proof of the maximal velocity estimate (1.21) of [10] can easily be extended to the following weighted maximal velocity estimate:

$$\big\| \mathrm{d}\Gamma\big(\omega^{-1/2} \chi_{w \ge 1} \omega^{-1/2}\big)^{\frac12} \psi_t \big\| \lesssim t^{-\gamma} \big(\big\| (\mathrm{d}\Gamma(\omega^{-1/2} \langle y \rangle \omega^{-1/2}) + \mathbf{1})^{\frac12} \psi_0 \big\| + C(\psi_0) \big), \tag{E.4}$$

for any $\bar{c} > 1$, $\gamma < \min(\frac{\mu}{2}\frac{\bar{c}-1}{2\bar{c}-1}, \frac{1}{2})$ and $\psi_0 \in \mathcal{D} \cap D(\mathrm{d}\Gamma(\omega^{-1/2}\langle y\rangle\omega^{-1/2})^{\frac{1}{2}})$.

We only mention that to obtain for instance (E.2), we estimate the interaction term using (2.14) with $\delta = -1/2$ together with the inequality (3.16) and the assumption (1.19).

Now, let $\psi_0 \in \mathcal{D} \cap D(\mathrm{d}\Gamma(\omega^{-1/2}\langle y\rangle\omega^{-1/2})^{\frac{1}{2}})$. We decompose $(\widetilde{W}(t') - \widetilde{W}(t))\psi_0$ as in Equations (5.28)–(5.32). Using the commutator estimates of Appendix B and Hardy's inequality, we verify that

$$\rho^*_{-1}(j'_0, j'_\infty)\rho_1 = \theta_\epsilon^{1/2}\chi(j'_0, j'_\infty)\chi\theta_\epsilon^{1/2} + \mathcal{O}(t^{-\alpha+(1+\kappa)/2}),$$

and likewise for the remainder terms rem_t. Hence Equations (5.31)–(5.32) can be transformed into

$$\underline{d}j = \frac{1}{ct^\alpha}\rho^*_1(j'_0, j'_\infty)\rho_{-1} + \omega^{1/2}\mathrm{rem}'_t\,\omega^{-1/2} \tag{E.5}$$

$$\mathrm{rem}'_t = \mathrm{rem}_t + \mathcal{O}(t^{-2\alpha+(1+\kappa)/2}), \tag{E.6}$$

where rem_t is given in (5.32). These relations give

$$G_0 = \widetilde{G}'_0 + \mathrm{Rem}'_t, \tag{E.7}$$

where $\widetilde{G}'_0 := \frac{1}{ct^\alpha}U\mathrm{d}\Gamma(j, \underline{\widetilde{c}}_t)$, with $\underline{\widetilde{c}}_t = (\widetilde{c}_0, \widetilde{c}_\infty) := (\rho^*_1 j'_0\rho_{-1}, \rho^*_1 j'_\infty\rho_{-1})$, and

$$\mathrm{Rem}'_t := G_0 - \widetilde{G}'_0 = U\mathrm{d}\Gamma(j, \mathrm{rem}'_t).$$

Next, we consider, as above, $\widetilde{A} = \sup_{\|\hat\phi_0\|=1}|\int_t^{t'} ds\langle\hat\phi_s, G_0\psi_s\rangle|$, where $\hat\phi_s = e^{-i\hat{H}s}f(\hat{H})\hat\phi_0$. Let

$$a_0 = \rho^*_1|j'_0|^{1/2}, \quad b_0 = |j'_0|^{1/2}\rho_{-1},$$
$$a_\infty = \rho^*_1|j'_\infty|^{1/2}, \quad b_\infty = |j'_\infty|^{1/2}\rho_{-1}.$$

We have $\widetilde{c}_0 = -a_0b_0$, $\widetilde{c}_\infty = a_\infty b_\infty$. Exactly as for (C.5), one can show that, if $c = (a_0b_0, a_\infty b_\infty)$, where $a_0, b_0, a_\infty, b_\infty$ are operators on $\mathfrak{h}$, then

$$\begin{aligned}|\langle\hat\phi, \mathrm{d}\check\Gamma(j,c)\psi\rangle| \le\ &\|\mathrm{d}\Gamma(a_0a_0^*)^{\frac{1}{2}}\otimes\mathbf{1}\hat\phi\|\|\mathrm{d}\Gamma(b_0^*b_0)^{\frac{1}{2}}\psi\| \\ &+ \|\mathbf{1}\otimes\mathrm{d}\Gamma(a_\infty a_\infty^*)^{\frac{1}{2}}\hat\phi\|\|\mathrm{d}\Gamma(b_\infty^*b_\infty)^{\frac{1}{2}}\psi\|.\end{aligned} \tag{E.8}$$

Hence $\widetilde{G}'_0$ satisfies

$$\begin{aligned}|\langle\hat\phi, \widetilde{G}'_0\psi\rangle| \le \frac{1}{ct^\alpha}\big(&\|\mathrm{d}\Gamma(a_0a_0^*)^{\frac{1}{2}}\otimes\mathbf{1}\hat\phi\|\|\mathrm{d}\Gamma(b_0^*b_0)^{\frac{1}{2}}\psi\| \\ &+ \|\mathbf{1}\otimes\mathrm{d}\Gamma(a_\infty a_\infty^*)^{\frac{1}{2}}\hat\phi\|\|\mathrm{d}\Gamma(b_\infty^*b_\infty)^{\frac{1}{2}}\psi\|\big).\end{aligned} \tag{E.9}$$

By the Cauchy-Schwarz inequality, (E.9) implies

$$\begin{aligned}\int_t^{t'} ds\,|\langle\hat\phi_s, \widetilde{G}'_0\psi_s\rangle| \lesssim\ &\Big(\int_t^{t'} ds\, s^{-\alpha}\|\mathrm{d}\Gamma(a_0a_0^*)^{\frac{1}{2}}\otimes\mathbf{1}\hat\phi_s\|^2\Big)^{\frac{1}{2}}\Big(\int_t^{t'} ds\, s^{-\alpha}\|\mathrm{d}\Gamma(b_0^*b_0)^{\frac{1}{2}}\psi_s\|^2\Big)^{\frac{1}{2}} \\ &+ \Big(\int_t^{t'} ds\, s^{-\alpha}\|\mathbf{1}\otimes\mathrm{d}\Gamma(a_\infty a_\infty^*)^{\frac{1}{2}}\hat\phi_s\|^2\Big)^{\frac{1}{2}}\Big(\int_t^{t'} ds\, s^{-\alpha}\|\mathrm{d}\Gamma(b_\infty^*b_\infty)^{\frac{1}{2}}\psi_s\|^2\Big)^{\frac{1}{2}}.\end{aligned}$$

Since $a_0a_0^*$ and $a_\infty a_\infty^*$ are of the form $\rho^*_1\chi_{b_\epsilon=ct^\alpha}\rho_1$, the weighted minimal velocity estimate (E.3) implies

$$\int_1^\infty ds\, s^{-\alpha}\big\|\widehat{\mathrm{d}\Gamma}(c_{\#1}c_{\#1}^*)^{\frac{1}{2}}\hat\phi_s\big\|^2 \lesssim \|\hat\phi_0\|^2,$$

where $\widehat{\mathrm{d}\Gamma}(c_{\#1}c_{\#1}^*)^{\frac{1}{2}}$ stands for $\mathrm{d}\Gamma(a_0a_0^*)^{\frac{1}{2}}\otimes\mathbf{1}$ or $\mathbf{1}\otimes\mathrm{d}\Gamma(a_\infty a_\infty^*)^{\frac{1}{2}}$. Likewise, since $b_0^*b_0$ and $b_\infty^*b_\infty$ are of the form $\rho^*_{-1}\chi_{b_\epsilon=ct^\alpha}\rho_{-1}$, the weighted minimal velocity estimate (E.1) implies

$$\int_1^\infty ds\, s^{-\alpha}\big\|\mathrm{d}\Gamma(c_{\#2}^*c_{\#2})^{\frac{1}{2}}\psi_s\big\|^2 \lesssim C(\psi_0),$$

with $c_{\#2} = b_0$ or b_∞. The last three relations give

$$\sup_{\|\hat\phi_0\|=1}\Big|\int_t^{t'} ds\,\langle\hat\phi_s, \widetilde{G}'_0\psi_s\rangle\Big| \to 0, \qquad t, t' \to \infty. \tag{E.10}$$

Applying likewise Lemma C.3 of Appendix C, one verifies that Rem'_t satisfies

$$|\langle\hat\phi, \mathrm{Rem}'_t\psi\rangle| \lesssim \|\hat\phi\|\Big(t^{-2\alpha+(1+\kappa)/2}\|\mathrm{d}\Gamma(\omega^{-1})^{\frac12}\psi\| + t^{-1}\|\mathrm{d}\Gamma(\omega^{-1/2}\chi j'_\infty\chi\omega^{-1/2})^{\frac12}\psi\| + t^{-\alpha}\|\mathrm{d}\Gamma(\omega^{-1/2}\chi^2_{w\geq 1}\omega^{-1/2})^{\frac12}\psi\|\Big).$$

Using (1.19), the weighted minimal velocity estimate (E.2) and the weighted maximal velocity estimate (E.4), we conclude as above that

$$\sup_{\|\hat\phi_0\|=1}\Big|\int_t^{t'} ds\langle\hat\phi_s, \mathrm{Rem}'_s\psi_s\rangle\Big| \to 0, \qquad t, t' \to \infty. \tag{E.11}$$

Equations (E.10) and (E.11) then imply

$$\widetilde{A} = \Big\|\int_t^{t'} ds f(\hat H)e^{i\hat H s}G_0\psi_s\Big\| \to 0, \qquad t, t' \to \infty. \tag{E.12}$$

The estimate of G_1 is the same as above, which shows that $\widetilde{W}(t)$, and hence $W(t)$, are strong Cauchy sequences. Thus the limit W_+ exists.

Supplement I. The wave operators

In this supplement we briefly review the definition and properties of the wave operator Ω_+, and establish its relation with W_+ in Theorem I.2 below. For simplicity we consider again hamiltonians of the form (1.4)–(1.5). Let $\mathcal{H}_b \equiv \mathcal{H}_{\mathrm{pp}}(H)\cap E_{(-\infty,\Sigma)}(H)$ be the space spanned by the eigenfunctions of H with the eigenvalues in the interval $(-\infty,\Sigma)$. Define $\tilde{\mathfrak{h}}_0 := \{h \in L^2(\mathbb{R}^3), \int |h|^2(|k|^{-1}+|k|^2)dk < \infty\}$. The wave operator Ω_+ on the space $\mathcal{H}_b\otimes\mathcal{F}_{\mathrm{fin}}(\tilde{\mathfrak{h}}_0)$, is defined by the formula

$$\Omega_+ := \operatorname*{s-lim}_{t\to\infty} e^{itH}I(e^{-itH}\otimes e^{-itH_f}). \tag{I.1}$$

As in [19, 26, 27, 39], it is easy to show

Theorem I.1. *Assume* (1.6) *with* $\mu > -1/2$ *and* (1.7)*. The wave operator* Ω_+ *exists on* $\mathcal{H}_b\otimes\mathcal{F}_{\mathrm{fin}}(\tilde{\mathfrak{h}}_0)$ *and extends to an isometric map,* $\Omega_+ : \mathcal{H}_{\mathrm{as}} \to \mathcal{H}$*, on the space of asymptotic states,* $\mathcal{H}_{\mathrm{as}} := \mathcal{H}_b\otimes\mathcal{F}$.

Proof. Let $h_t(k) := e^{-it|k|}h(k)$. For $h \in D(\omega^{-1/2})$ s.t. $\partial^\alpha h \in D(\omega^{|\alpha|-1/2})$, $|\alpha| \leq 2$, we define the asymptotic creation and annihilation operators by (see [19, 26, 27, 35, 39])

$$a^\#_\pm(h)\Phi := \lim_{t\to\pm\infty} e^{itH}a^\#(h_t)e^{-itH}\Phi,$$

for any $\Phi \in D(|H|^{1/2})\bigcap \mathrm{Ran}\,E_{(-\infty,\Sigma)}(H)$. Here $a^\#$ stands for a or a^*. To show that $a^\#_\pm(h)$ exist (see [26, 39]), we define $a^\#_t(h) := e^{itH}a^\#(h_t)e^{-itH}$ and compute $a^\#_{t'}(h) - a^\#_t(h) = \int_t^{t'} ds\partial_s a^\#_s(h)$ and $\partial_s a^\#_s(h) = ie^{iHt}Ge^{-iHt}$, where $G := [H, a^\#(h_s)] - a^\#(\omega h_t) = \langle g, h_t\rangle_{L^2(dk)}$ for $a^\# = a^*$ and $-\langle h_t, g\rangle_{L^2(dk)}$ for $a^\# = a$. Thus the proof of existence reduces to showing that one-photon terms of the form $\langle \eta g, h_t\rangle$ are integrable in t. By (1.6), we have $\|\langle\eta g, h_t\rangle_{L^2(dk)}\|_{\mathcal{H}_p} \lesssim (1+t)^{-1-\varepsilon}$, with $0<\varepsilon<\mu\ +1$, which is integrable. Moreover, as in [26, 39], one can show that $a^\#_\pm(h)$ satisfy the canonical commutation relations, the relations $a_\pm(h)\Psi = 0$, and

$$\lim_{t\to\pm\infty} e^{itH}a^\#(h_{1,t})\cdots a^\#(h_{n,t})e^{-itH}\Phi = a^\#_\pm(h_1)\cdots a^\#_\pm(h_n)\Phi, \tag{I.2}$$

for any $\Psi \in \mathcal{H}_b$, $h, h_1, \cdots, h_n \in \tilde{\mathfrak{h}}_0$, and any $\Phi \in E_{(-\infty,\Sigma)}(H)$. If we define the wave operator Ω_+ on $\mathcal{H}_{\mathrm{fin}}$ by

$$\Omega_+(\Phi\otimes a^*(h_1)\cdots a^*(h_n)\Omega) := a^*_+(h_1)\cdots a^*_+(h_n)\Phi, \tag{I.3}$$

using the canonical commutation relations, one sees that Ω_+ extends to an isometric map $\Omega_+ : \mathcal{H}^+_{\mathrm{as}} \to \mathcal{H}$. Moreover, using the relation $e^{it\hat H}(\Phi\otimes a^\#(h_1)\cdots a^\#(h_n)\Omega) = (e^{itH}\Phi_{\mathrm{gs}})\otimes(a^\#(h_{1,t})\cdots a^\#(h_{n,t})\Omega)$, together with (I.1) and (I.2), we identify the definition (I.3) with (I.1). ☐

Recall that $P_{\rm gs}$ denotes the orthogonal projection onto the ground state subspace of H. Let $\bar P_{\rm gs} := \mathbf{1} - P_{\rm gs}$ and $\bar P_\Omega := \mathbf{1} - P_\Omega$, where, recall, P_Ω is the projection onto the vacuum sector in $\mathcal{F}$. Theorem 5.4 and its proof imply the following result.

Theorem I.2. *Under the conditions of Theorem 5.4, we have on* $\mathrm{Ran}\,\chi_\Delta(H)$

$$\Omega_+(P_{\rm gs}\otimes \bar P_\Omega)W_+\bar P_{\rm gs} + \Omega_+(P_{\rm gs}\otimes P_\Omega)W_+P_{\rm gs} = \mathbf{1}. \tag{I.4}$$

Proof. Let $\psi_0 \in \mathrm{Ran}\,\chi_\Delta(H)$. For every $\epsilon''>0$ there is $\delta'' = \delta(\epsilon'')>0$, s.t.

$$\|\psi_0 - \psi_{0\epsilon''} - P_{\rm gs}\psi_0\| \le \epsilon'', \tag{I.5}$$

where $\psi_{0\epsilon''} = \chi_{\Delta_{\epsilon''}}(H)\psi_0$, with $\Delta_{\epsilon'} = [E_{\rm gs}+\delta, a]$. Proceeding as in the proof of Theorem 5.4 with $\psi_{0\epsilon''}$ instead of ψ_0, we arrive at (see (5.65))

$$\psi_{0\epsilon''} = e^{-iHt} I(e^{-iE_{\rm gs}t}P_{\rm gs}\otimes e^{-iH_f t}\chi_{(0,a-E_{\rm gs}]}(H_f))\phi_{0\epsilon'} + \mathcal{O}(\epsilon') + C(\epsilon',m)o_t(1) + C(\epsilon')o_m(1), \tag{I.6}$$

where we choose $\phi_{0\epsilon'}$ such that $\phi_{0,\epsilon'} \in D(\mathrm{d}\Gamma(\langle y\rangle))\otimes \mathcal{F}_{\rm fin}(\tilde{\mathfrak{h}}_0)$ and $\|W_+\psi_{0\epsilon''} - \phi_{0\epsilon'}\| \le \epsilon'$. Now using Theorem I.1, we let $t\to\infty$, next $m\to\infty$ to obtain

$$\psi_{0\epsilon''} = \Omega_+(P_{\rm gs}\otimes \chi_{(0,a-E_{\rm gs}]}(H_f))\phi_{0\epsilon'} + \mathcal{O}(\epsilon'). \tag{I.7}$$

Since Ω_+ is isometric, hence bounded, we can let $\epsilon'\to 0$, which gives

$$\psi_{0\epsilon''} = \Omega_+(P_{\rm gs}\otimes \chi_{(0,a-E_{\rm gs}]}(H_f))W_+\psi_{0\epsilon''} = \Omega_+(P_{\rm gs}\otimes \bar P_\Omega)W_+\bar P_{\rm gs}\psi_{0\epsilon''}. \tag{I.8}$$

Here we used that $\chi_{(0,a-E_{\rm gs}]}(H_f) = \bar P_\Omega \chi_{(0,a-E_{\rm gs}]}(H_f)$, together with $\chi_{(0,a-E_{\rm gs}]}(H_f)W_+\psi_{0\epsilon''} = W_+\psi_{0\epsilon''}$ and $\psi_{0\epsilon''} = \bar P_{\rm gs}\psi_{0\epsilon''}$. Introducing (I.8) into (I.5) and letting $\epsilon''\to 0$, we obtain

$$\psi_0 = \Omega_+(P_{\rm gs}\otimes \bar P_\Omega)W_+\bar P_{\rm gs}\psi_0 + P_{\rm gs}\psi_0,$$

that is

$$\Omega_+(P_{\rm gs}\otimes \bar P_\Omega)W_+\bar P_{\rm gs} + P_{\rm gs} = \mathbf{1}.$$

Since, by (5.6) and (I.3), we have $\Omega_+(P_{\rm gs}\otimes P_\Omega)W_+P_{\rm gs} = P_{\rm gs}$, this implies (I.4). □

Supplement II. Creation and annihilation operators on Fock spaces

Recall that the propagation speed of the light and the Planck constant divided by 2π are set equal to 1. Recall also that the one-particle space is $\mathfrak{h} := \mathrm{L}^2(\mathbb{R}^3;\mathbb{C})$, for phonons, and $\mathfrak{h} := \mathrm{L}^2(\mathbb{R}^3;\mathbb{C}^2)$, for photons. In both cases we use the momentum representation and write functions from this space as $u(k)$ and $u(k,\lambda)$, respectively, where $k\in\mathbb{R}^3$ is the wave vector or momentum of the photon and $\lambda\in\{-1,+1\}$ is its polarization.

With each function $f\in\mathfrak{h}$, one associates *creation* and *annihilation operators* $a(f)$ and $a^*(f)$ defined, for $u\in\otimes_s^n\mathfrak{h}$, as

$$a^*(f) : u \to \sqrt{n+1}f\otimes_s u \quad \text{ and } \quad a(f): u\to \sqrt{n}\langle f,u\rangle_{\mathfrak{h}}, \tag{II.1}$$

with $\langle f,u\rangle_{\mathfrak{h}} := \int \overline{f(k)}u(k,k_1,\dots,k_{n-1})\,\mathrm{d}k$, for phonons, and $\langle f,u\rangle_{\mathfrak{h}} := \sum_{\lambda=1,2}\int \mathrm{d}k\overline{f(k,\lambda)}u_n(k,\lambda,k_1,\lambda_1,\dots,k_{n-1},\lambda_{n-1})$, for photons. They are unbounded, densely defined operators of $\Gamma(\mathfrak{h})$, adjoint of each other (with respect to the natural scalar product in $\mathcal{F}$) and satisfy the *canonical commutation relations* (CCR):

$$\left[a^\#(f),a^\#(g)\right] = 0, \qquad \left[a(f),a^*(g)\right] = \langle f,g\rangle,$$

where $a^\# = a$ or a^*. Since $a(f)$ is anti-linear and $a^*(f)$ is linear in f, we write formally

$$a(f) = \int \overline{f(k)}a(k)\,dk, \qquad a^*(f) = \int f(k)a^*(k)\,dk,$$

for phonons, and

$$a(f) = \sum_{\lambda=1,2}\int \overline{f(k,\lambda)}a_\lambda(k)\,dk, \qquad a^*(f) = \sum_{\lambda=1,2}\int f(k,\lambda)a^*_\lambda(k)dk,$$

for photons. Here $a(k)$ and $a^*(k)$ and $a_\lambda(k)$ and $a^*_\lambda(k)$ are unbounded, operator-valued distributions, which obey (again formally) the *canonical commutation relations* (CCR):

$$\big[a^\#(k), a^\#(k')\big] = 0, \qquad \big[a(k), a^*(k')\big] = \delta(k-k'),$$
$$\big[a^\#_\lambda(k), a^\#_{\lambda'}(k')\big] = 0, \qquad \big[a_\lambda(k), a^*_{\lambda'}(k')\big] = \delta_{\lambda,\lambda'}\delta(k-k'),$$

where $a^\# = a$ or a^* and $a^\#_\lambda = a_\lambda$ or a^*_λ.

Given an operator τ acting on the one-particle space $\mathfrak{h}$, the operator $\mathrm{d}\Gamma(\tau)$ (the second quantization of τ) defined on the Fock space $\mathcal{F}$ by (1.3), can be written (formally) as $\mathrm{d}\Gamma(\tau) := \int dk\, a^*(k)\tau a(k)$, for phonons, and $\mathrm{d}\Gamma(\tau) := \sum_{\lambda=1,2}\int dk\, a^*_\lambda(k)\tau a_\lambda(k)$, for photons. Here the operator τ acts on the k-variable. The precise meaning of the latter expression is (1.3). In particular, one can rewrite the quantum Hamiltonian H_f in terms of the creation and annihilation operators, a and a^*, as

$$H_f = \sum_{\lambda=1,2}\int dk\, a^*_\lambda(k)\omega(k)a_\lambda(k) \tag{I.2}$$

for photons, and similarly for phonons.

The relations below are valid for both phonon and photon operators. Commutators of two $\mathrm{d}\Gamma$ operators reduces to commutators of the one-particle operators:

$$[\mathrm{d}\Gamma(\tau), \mathrm{d}\Gamma(\tau')] = \mathrm{d}\Gamma([\tau,\tau']). \tag{II.3}$$

Let τ be a one-photon self-adjoint operator. The following commutation relations involving the field operator $\Phi(f) = \frac{1}{\sqrt{2}}(a^*(f) + a(f))$ can be readily derived from the definitions of the operators involved:

$$[\Phi(f), \Phi(g)] = i\,\mathrm{Im}\langle f, g\rangle_{\mathfrak{h}}, \tag{II.4}$$
$$[\Phi(f), \mathrm{d}\Gamma(\tau)] = i\Phi(i\tau f), \tag{II.5}$$
$$[\Gamma(\tau), \Phi(f)] = \Gamma(\tau)a((1-\tau)f) - a^*((1-\tau)f)\Gamma(\tau). \tag{II.6}$$

Exponentiating these relations, we obtain

$$e^{i\Phi(f)}\Phi(g)e^{-i\Phi(f)} = \Phi(g) - \mathrm{Im}\langle f, g\rangle_{\mathfrak{h}}, \tag{II.7}$$
$$e^{i\Phi(f)}\mathrm{d}\Gamma(\tau)e^{-i\Phi(f)} = \mathrm{d}\Gamma(\tau) - \Phi(i\tau f) + \frac{1}{2}\mathrm{Re}\langle \omega f, f\rangle_{\mathfrak{h}} \tag{II.8}$$
$$e^{i\Phi(f)}\Gamma(\tau)e^{-i\Phi(f)} = \Gamma(\tau) + \int_0^1 ds\, e^{is\Phi(f)}(\Gamma(\tau)a((1-\tau)f) - a^*((1-\tau)f)\Gamma(\tau))e^{-si\Phi(f)}. \tag{II.9}$$

Finally, we have the following standard estimates for annihilation and creation operators $a(f)$ and $a^*(f)$, whose proof can be found, for instance, in [7], [34, Section 3], [40]:

Lemma II.1. *For any $f \in \mathfrak{h}$ such that $\omega^{-\rho/2}f \in \mathfrak{h}$, the operators $a^\#(f)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1/2}$, where $a^\#(f)$ stands for $a^*(f)$ or $a(f)$, extend to bounded operators on $\mathcal{H}$ satisfying*

$$\big\|a(f)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-\frac{1}{2}}\big\| \le \|\omega^{-\rho/2}f\|_{\mathfrak{h}},$$
$$\big\|a^*(f)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-\frac{1}{2}}\big\| \le \|\omega^{-\rho/2}f\|_{\mathfrak{h}} + \|f\|_{\mathfrak{h}}.$$

If, in addition, $g \in \mathfrak{h}$ is such that $\omega^{-\rho/2}g \in \mathfrak{h}$, the operators $a^\#(f)a^\#(g)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1}$ extend to bounded operators on $\mathcal{H}$ satisfying

$$\big\|a(f)a(g)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1}\big\| \le \|\omega^{-\rho/2}f\|_{\mathfrak{h}}\|\omega^{-\rho/2}g\|_{\mathfrak{h}},$$
$$\big\|a^*(f)a(g)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1}\big\| \le \big(\|\omega^{-\rho/2}f\|_{\mathfrak{h}} + \|f\|_{\mathfrak{h}}\big)\|\omega^{-\rho/2}g\|_{\mathfrak{h}},$$
$$\big\|a^*(f)a^*(g)(\mathrm{d}\Gamma(\omega^\rho)+1)^{-1}\big\| \le \big(\|\omega^{-\rho/2}f\|_{\mathfrak{h}} + \|f\|_{\mathfrak{h}}\big)\big(\|\omega^{-\rho/2}g\|_{\mathfrak{h}} + \|g\|_{\mathfrak{h}}\big).$$

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(J. Faupin) Institut de Mathématiques de Bordeaux, UMR-CNRS 5251, Université de Bordeaux 1, 33405 Talence Cedex, France

E-mail address: `jeremy.faupin@math.u-bordeaux1.fr`

(I. M. Sigal) Department of Mathematics, University of Toronto, Toronto, ON M5S 2E4, Canada

E-mail address: `im.sigal@utoronto.ca`