SECOND ORDER PERTURBATION THEORY FOR EMBEDDED EIGENVALUES

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ABSTRACT. We study second order perturbation theory for embedded eigenvalues of an abstract class of self-adjoint operators. Using an extension of the Mourre theory, under assumptions on the regularity of bound states with respect to a conjugate operator, we prove upper semicontinuity of the point spectrum and establish the Fermi Golden Rule criterion. Our results apply to massless Pauli-Fierz Hamiltonians for arbitrary coupling.

CONTENTS

1. INTRODUCTION

In this second of a series of papers, we study second order perturbation theory for embedded eigenvalues of an abstract class of self-adjoint operators. Perturbation theory for isolated eigenvalues of finite multiplicity is well-understood, at least if the family of operators under consideration is analytic in the sense of Kato (see [Ka, RS]). The question is more subtle when dealing with unperturbed eigenvalues embedded in the continuous spectrum. A method to tackle this problem, which we shall not develop here, is based on analytic deformation techniques and gives rise to a notion of resonances. It appeared in [AC, BC] and was further extended by many authors in different contexts (let us mention [Si, RS, JP, BFS] among many other contributions). As shown in [AHS], another way of studying the behaviour of

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embedded eigenvalues under perturbation is based on Mourre's commutator method ([Mo]). We shall develop this second approach from an abstract point of view in this paper.

We mainly require two conditions: The first one corresponds to a set of assumptions needed in order to use the Mourre method (see Conditions 1.3 below). We shall work with an extension of the Mourre theory which we call *singular Mourre theory*, and which is closely related to the ones developed in [Sk, MS, GGM1]. Singular Mourre theory refers to the situation where the commutator of the Hamiltonian with the chosen "conjugate operator" is not controlled by the Hamiltonian itself. The *regular Mourre theory*, studied for instance in [Mo, ABG, HüSp, HuSi, Ca, CGH], is a particular case of the theory considered here. A feature of singular Mourre theory is to allow one to derive spectral properties of so-called Pauli-Fierz Hamiltonians. This shall be discussed in Section 2.

Our second set of assumptions concerns the regularity of bound states with respect to a conjugate operator (see Conditions 1.7, 1.9 and 1.10 below). Related questions are discussed in details, in an abstract framework, in the companion paper [FMS] (see also [Ca, CGH]).

Our main concerns are to study upper semicontinuity of point spectrum (Theorem 1.14) and to show that the Fermi Golden Rule criterion (Theorem 1.15) holds. If the Fermi Golden Rule condition is not fulfilled we shall still obtain an expansion to second order of perturbed eigenvalues. Before precisely stating our results and comparing them to the literature, we introduce the abstract framework in which we shall work.

1.1. **Assumptions.** We introduce first our basic conditions, Conditions 1.3, which are related to a set of conditions used in [GGM1]. For a comparison we refer the reader to Remark 1.4 6).

Let $\mathcal{H}$ be a complex Hilbert space. Suppose that H and M are self-adjoint operators on $\mathcal{H}$, with $M \geq 0$, and suppose that a symmetric operator R is given such that $\mathcal{D}(R) \supseteq \mathcal{D}(H)$. Let

$$H' := M + R \quad \text{defined on} \quad \mathcal{D} := \mathcal{D}(M) \cap \mathcal{D}(H). \tag{1.1}$$

Under Condition 1.3 (1), we shall see that $\mathcal{D}$ is dense in $\mathcal{H}$ (see Remark 1.4 2) below). Operators are according to our convention always densely defined. Observe also that we do not impose the condition that H' is closed. To make contact to [GGM1], we note that the operator closure of H' at some points in our exposition will coincide with the operator H' used in Hypothesis (M1) in [GGM1] (see Remark 1.4 6) for a further comment). Let

$$\mathcal{G} := \mathcal{D}(M^{\frac{1}{2}}) \cap \mathcal{D}(|H|^{\frac{1}{2}}), \tag{1.2}$$

equipped with the norm of the intersection topology defined by

$$\|u\|_{\mathcal{G}}^2 := \big\|M^{\frac{1}{2}}u\big\|_{\mathcal{H}}^2 + \big\||H|^{\frac{1}{2}}u\big\|_{\mathcal{H}}^2 + \|u\|_{\mathcal{H}}^2. \tag{1.3}$$

Let A be a closed, maximal symmetric operator on $\mathcal{H}$. In particular, introducing deficiency indices $n_{\mp} = \dim \operatorname{Ker}(A^* \pm \mathrm{i})$, either $n_+ = 0$ or $n_- = 0$. For simplicity we shall assume that $n_+ = 0$ so that A generates a C_0-semigroup of isometries $\{W_t\}_{t\geq 0}$ (if $n_- = 0$ we may mimic the theory explained below with $A \to -A$). At this point we refer to e.g. [Da, Theorem 10.4.4]. We recall that the C_0-semigroup property means (see e.g. [GGM1, Subsection 2.5] for a short general discussion of C_0-semigroups, and [HP, Chapter 10] for an extensive study) that the map $[0,\infty[\ni t \mapsto W_t \in \mathcal{B}(\mathcal{H})$ obeys $W_0 = I$, $W_tW_s = W_{t+s}$ for $t, s \geq 0$, and $\text{w-}\lim_{t\to 0^+} W_t = I$. Here $\mathcal{B}(\mathcal{H})$ denotes the set of bounded operators on $\mathcal{H}$ and w- lim stands for weak limit. We also recall that any C_0-semigroup on a Hilbert space is automatically strongly continuous on $[0,\infty[$, cf. [HP, Theorem 10.6.5]. The operator A is the generator of the C_0-semigroup $\{W_t\}_{t\geq 0}$ meaning that

$$\mathcal{D}(A)=\{u\in\mathcal{H},\ \lim_{t\to 0^+}(\mathrm{i}t)^{-1}(W_tu-u)\text{ exists}\}\text{ and }Au=\lim_{t\to 0^+}(\mathrm{i}t)^{-1}(W_tu-u). \tag{1.4}$$

We write $W_t=e^{\mathrm{i}tA}$.

For any Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, we denote by $\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2)$ the set of bounded operators from $\mathcal{H}_1$ to $\mathcal{H}_2$. We use the notation $\langle B\rangle := (1+B^*B)^{1/2}$ for any closed operator B. Throughout the paper, C_j, $j=1,2,\dots$, will denote positive constants that may differ from one proof to another. Let us recall the following general definition from [GGM1]:

Definition 1.1. Let $\{W_{1,t}\}$, $\{W_{2,t}\}$ be two C_0-semigroups on Hilbert spaces $\mathcal{H}_1$, $\mathcal{H}_2$ with generators A_1, A_2 respectively. A bounded operator $B\in\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2)$ is said to be in $\mathrm{C}^1(A_1;A_2)$ if

$$\|W_{2,t}B-BW_{1,t}\|_{\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2)}\leq Ct,\quad 0\leq t\leq 1, \tag{1.5}$$

for some positive constant C.

We have the following accompanying remarks and definitions.

Remarks 1.2. 1) By [GGM1, Proposition 2.29], $B\in\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2)$ is of class $\mathrm{C}^1(A_1;A_2)$ if and only if the sesquilinear form ${}_2[B,\mathrm{i}A]_1$ defined on $\mathcal{D}(A_2^*)\times\mathcal{D}(A_1)$ by

$$\langle\phi,{}_2[B,\mathrm{i}A]_1\psi\rangle=\mathrm{i}\langle B^*\phi,A_1\psi\rangle-\mathrm{i}\langle A_2^*\phi,B\psi\rangle, \tag{1.6}$$

is bounded relatively to the topology of $\mathcal{H}_2\times\mathcal{H}_1$. The associated bounded operator in $\mathcal{B}(\mathcal{H}_1;\mathcal{H}_2)$ is denoted by $[B,\mathrm{i}A]^0$ and we have

$$[B,\mathrm{i}A]^0=\text{s-}\lim_{t\to 0^+}t^{-1}[BW_{1,t}-W_{2,t}B], \tag{1.7}$$

where s-lim stands for strong limit. We say that B is of class $\mathrm{C}^2(A_1;A_2)$ if and only if $B\in\mathrm{C}^1(A_1;A_2)$ and $[B,\mathrm{i}A]^0\in\mathrm{C}^1(A_1;A_2)$.

2) We recall (see [ABG]) that if A and B are self-adjoint operators on a Hilbert space $\mathcal{H}$, B is said to be in $\mathrm{C}^1(A)$ if there exists $z\in\mathbb{C}\setminus\mathbb{R}$ such that $(B-z)^{-1}\in\mathrm{C}^1(A;A)$ (meaning here that $\mathcal{H}_j=\mathcal{H}$ and $A_j=A$, $j=1,2$). In that case in fact $(B-z)^{-1}\in\mathrm{C}^1(A;A)$ for all $z\in\rho(B)$ ($\rho(B)$ is the resolvent set of B).

3) The standard Mourre class, cf. [Mo], is a subset of $\mathrm{C}^1(A)$ given as follows: Notice that for any $B\in\mathrm{C}^1(A)$ the commutator form $[B,\mathrm{i}A]$ defined on $\mathcal{D}(B)\cap\mathcal{D}(A)$ extends uniquely (by continuity) to a bounded form $[B,\mathrm{i}A]^0$ on $\mathcal{D}(B)$. We shall say that B is *Mourre-*$C^1(A)$ if $[B,\mathrm{i}A]^0$ is a B-bounded operator on $\mathcal{H}$. The subclass of *Mourre-*$C^1(A)$ operators in $\mathrm{C}^1(A)$ is in this paper denoted by $\mathrm{C}^1_{\mathrm{Mo}}(A)$.

Let us now state our first set of conditions which is based on the setting introduced in the beginning of this subsection, in particular the C_0-semigroup of isometries, $W_t=e^{\mathrm{i}tA}$, $t\geq 0$:

Conditions 1.3.

(1) $H\in\mathrm{C}^1_{\mathrm{Mo}}(M)$.

(2) There is an interval $I\subseteq\mathbb{R}$ such that for all $\eta\in I$, there exist $c_0>0$, $C_1\in\mathbb{R}$, $f_\eta\in\mathrm{C}_0^\infty(\mathbb{R})$, $0\leq f_\eta\leq 1$ and $f_\eta=1$ in a neighbourhood of η, and a compact operator K_0 on $\mathcal{H}$ such that, in the sense of quadratic forms on $\mathcal{D}$,

$$H'\geq \mathrm{c}_0I-C_1f_\eta^\perp(H)^2\langle H\rangle-K_0, \tag{1.8}$$

where $f_\eta^\perp(H)=1-f_\eta(H)$.

(3) $\mathcal{G}$ is "*boundedly-stable*" under $\{W_t\}$ and $\{W_t^*\}$ i.e. $W_t\mathcal{G} \subseteq \mathcal{G}$, $W_t^*\mathcal{G} \subseteq \mathcal{G}$, $t > 0$, and for all $\phi \in \mathcal{G}$,

$$\sup_{0<t<1} \|W_t\phi\|_{\mathcal{G}} < \infty, \quad \sup_{0<t<1} \|W_t^*\phi\|_{\mathcal{G}} < \infty. \tag{1.9}$$

Let $A_{\mathcal{G}}$ denote the generator of the C_0-semigroup $W_t|_{\mathcal{G}}$ and let $A_{\mathcal{G}^*}$ denote the generator of the C_0-semigroup given as the extension of W_t to $\mathcal{G}^*$ (see Remark 1.4 1) for justification).

(4) $H \in \mathrm{C}^2(A_{\mathcal{G}}; A_{\mathcal{G}^*})$ (see Remark 1.4 2) for justification of notation), and for all $\phi \in \mathcal{D}$

$$H'\phi = [H, \mathrm{i}A]^0\phi. \tag{1.10}$$

We have several accompanying remarks. In Remarks 1.4 1)– 3) we introduce further notation, give justification of notation and furthermore we state a version of the so-called virial theorem.

Remarks 1.4. 1) Due to the boundedly-stability (1.9), the closed graph theorem and a density argument it follows that $W_t|_{\mathcal{G}}$ belongs to $\mathcal{B}(\mathcal{G})$ and that $W_t|_{\mathcal{G}}$ is a C_0-semigroup, cf. [GGM1, Lemma 2.33]. Arguing similarly we verify that each W_t extends by continuity to a bounded operator on $\mathcal{G}^*$ and that the extensions form a C_0-semigroup in $\mathcal{G}^*$. This justifies the notations $A_{\mathcal{G}}$ and $A_{\mathcal{G}^*}$ in Condition 1.3 (3).

2) It follows from Condition 1.3 (1) that $\mathcal{D}$ is a core for H as well as for M, see e.g. [GG] or [GGM1]. This condition is transcribed from [Sk] and is stronger than [GGM1, Hypothesis (M1)], cf. 6) given below. Another consequence of Condition 1.3 (1) is the following alternative description of the space $\mathcal{G}$: Let G be the Friedrichs extension of the operator $M + \langle H \rangle$ on $\mathcal{D}$. Then $\mathcal{D}(\sqrt{G}) = \mathcal{G}$; this follows from [GGM1, Proposition 3.8]. In Appendix A we give an elementary proof. In particular

$$\mathcal{D} \text{ is dense in } \mathcal{G}. \tag{1.11}$$

Notice that due to (1.11) we can uniquely consider the operators H and H' as being members of $\mathcal{B}(\mathcal{G}; \mathcal{G}^*)$. Whence in particular writing $H' \in \mathcal{B}(\mathcal{G}; \mathcal{G}^*)$ we have the identity (1.10) for all $\phi \in \mathcal{G}$ and we can legitimately introduce the notation

$$H'' := [H', \mathrm{i}A]^0 \in \mathcal{B}(\mathcal{G}; \mathcal{G}^*). \tag{1.12}$$

3) Suppose Conditions 1.3. Then the following identity holds for all $\phi_1 \in \mathcal{D} \cap \mathcal{D}(A^*)$ and $\phi_2 \in \mathcal{D} \cap \mathcal{D}(A)$:

$$\langle \phi_1, (M + R)\phi_2 \rangle = \mathrm{i}\langle H\phi_1, A\phi_2 \rangle - \mathrm{i}\langle A^*\phi_1, H\phi_2 \rangle. \tag{1.13}$$

This is a consequence of (1.7). Another (related) consequence of (1.7) is the following version of the virial theorem: For any eigenstate, $(H - \lambda)\psi = 0$, with $\psi \in \mathcal{D}(M^{1/2})$

$$\langle \psi, (M + R)\psi \rangle := \|M^{1/2}\psi\|^2 + \langle \psi, R\psi \rangle = 0, \tag{1.14}$$

see [GGM1, Proposition 4.2]. Observe that due to (1.11), the assumptions of [GGM1, Proposition 4.2] are indeed satisfied. Notice also that some regularity assumption of H with respect to an operator A is needed for the virial theorem for the pair (H, A) to hold (see [GG]). As a standard corollary of the virial theorem we have under Conditions 1.3 that the number of eigenvalues of H in any compact interval $J \subseteq I$ is finite and that each such eigenvalue has a finite multiplicity (here we assume that the corresponding eigenstates are in $\mathcal{D}(M^{1/2})$). Besides, in Section 3, we will recall a version of the Limiting Absorption Principle (LAP) established in [GGM1] (see Theorem 3.1 of the present paper) which implies that under Conditions 1.3, H has no singular continuous spectrum in I. The fact that H' is a "commutator", cf. (1.10) and (1.13), is also important in the proof of LAP of [GGM1] (this is indeed an integral part of any known proof of LAP in the spirit of Mourre).

4) The conditions of the regular Mourre theory considered for instance in [Mo, ABG, HüSp, HuSi, Ca, CGH] constitute a particular case of Conditions 1.3 assuming that $M = 0$. In [Mo, ABG, HuSi, Ca, CGH], the conjugate operator A is supposed to be self-adjoint, whereas in [HüSp] the weaker assumption that A is the generator of a C_0-semigroup of isometries is required. Notice that in the case where $M = 0$ and A is self-adjoint Condition 1.3 (3) appears replaced by the stronger condition: $\sup_{|t|<1} \|\mathrm{e}^{\mathrm{i}tA}\phi\|_{\mathcal{D}(H)} < \infty$ for any $\phi \in \mathcal{D}(H)$. If $\mathcal{H}$ is separable it follows from [HP, Lemma 10.2.1] that the latter condition is a consequence of the weaker condition that $e^{\mathrm{i}tA}\mathcal{D}(H) \subseteq \mathcal{D}(H)$ for all $t \in \mathbb{R}$. A similar equivalence for semigroups is not known to our knowledge. It should also be noticed that the boundedness of H'' with respect to H is often required in the regular Mourre theory. Condition 1.3 (4) leads to the weaker assumption that $\langle H\rangle^{-1/2}H''\langle H\rangle^{-1/2}$ is bounded.
5) The idea of splitting the formal commutator $\mathrm{i}[H, A]$ into an H-unbounded piece, M, and a H-bounded piece, R, appeared first in [Sk]. As it was shown in [Sk], and later in [GGM2], this extension of the Mourre theory allows one to study spectral properties of N-body systems coupled to bosonic fields (see also [MS] for the use of related assumptions in a different context). This will be discussed more precisely in the next section.
6) We notice that Conditions 1.3 (with $K_0 = 0$ in (2)) are stronger than Hypotheses (M1)–(M5) used in [GGM1]. As mentioned at the beginning of this subsection, the operator H' in [GGM1] is supposed to be closed; it corresponds to the closure of the operator H' considered in this paper (compare Hypothesis (M1) in [GGM1] with Condition 1.3 (1), and see [GGM1, Lemma 2.26]). Therefore, in particular, the results proved in [GGM1] hold under Conditions 1.3, see Remark 3.2 2) for a further discussion.

Throughout the discussion below we impose (mostly tacitly) Conditions 1.3. We introduce the following classes of operators (to be considered as classes of "perturbations"):

Definition 1.5. We say that a symmetric operator V with $\mathcal{D}(V) \supseteq \mathcal{D}(H)$, ϵ-bounded relatively to H, is in $\mathcal{V}_1$ if $V \in \mathrm{C}^1(A_{\mathcal{G}}; A_{\mathcal{G}^*})$ and $V' := [V, \mathrm{i}A]^0$ is given as an H-bounded operator. For any $V \in \mathcal{V}_1$, we set

$$\|V\|_1 := \|V(H - \mathrm{i})^{-1}\| + \|V'(H - \mathrm{i})^{-1}\|. \tag{1.15}$$

It follows from the Kato-Rellich Theorem that for any $V \in \mathcal{V}_1$ the operator $H + V$ is self-adjoint with $\mathcal{D}(H + V) = \mathcal{D}(H)$.

Definition 1.6. We say that $V \in \mathcal{V}_1$ is in $\mathcal{V}_2$ if $V' \in \mathrm{C}^1(A_{\mathcal{G}}; A_{\mathcal{G}^*})$, and we set

$$\|V\|_2 := \|V\|_1 + \|V''\|_{\mathcal{B}(\mathcal{G};\mathcal{G}^*)}, \tag{1.16}$$

where $V'' := [V', \mathrm{i}A]^0$.

Our main assumptions on the unperturbed eigenstates are stated in Condition 1.7 and in its stronger version Condition 1.9.

Condition 1.7. If $\lambda \in I$ is an eigenvalue of H, any eigenstate ψ associated to λ, $H\psi = \lambda\psi$, satisfies $\psi \in \mathcal{D}(A) \cap \mathcal{D}(M)$.

Remark 1.8. Under Condition 1.7 and with ψ given as there, one verifies using (1.7) and the fact that $\mathcal{D}$ is dense in $\mathcal{D}(H)$ that $\psi \in \mathcal{D}(HA) := \{\phi \in \mathcal{D}(A) | \, A\phi \in \mathcal{D}(H)\}$, cf. Remark 1.4 3).

Condition 1.9. If $\lambda \in I$ is an eigenvalue of H, any eigenstate ψ associated to λ, $H\psi = \lambda\psi$, satisfies $\psi \in \mathcal{D}(A^2) \cap \mathcal{D}(M)$.

The (possibly existing) perturbed eigenstates may fulfil the following condition:

Condition 1.10. For any compact interval $J \subseteq I$ there exist $\gamma > 0$ and a subset $\mathcal{B}_{1,\gamma}$ of the ball centered at 0 with radius γ in $\mathcal{V}_1$,

$$\mathcal{B}_{1,\gamma} \subseteq \{V \in \mathcal{V}_1, \|V\|_1 \leq \gamma\}, \tag{1.17}$$

such that $\{0\} \subset \mathcal{B}_{1,\gamma}$, $\mathcal{B}_{1,\gamma}$ is star-shaped and symmetric with respect to 0, and the following holds: There exists $C > 0$ such that, if $V \in \mathcal{B}_{1,\gamma}$ and $(H + V - \lambda)\psi = 0$ with $\lambda \in J$, then $\psi \in \mathcal{D}(A) \cap \mathcal{D}(M)$ and

$$\|A\psi\| \leq C\|\psi\|. \tag{1.18}$$

Observe that Conditions 1.9 and 1.10 are both stronger than Condition 1.7. Condition 1.7 is indeed insufficient for our main theorems to hold and we need to assume either Condition 1.9 or Condition 1.10. As for the application we give in Section 2, we shall verify Condition 1.10 rather than Condition 1.9, see more precisely Proposition 2.4 and Remark 2.5 2).

The following two conditions are needed for our version of the so-called Fermi Golden Rule criterion. The first condition is a technical addition to Conditions 1.3:

Condition 1.11. $\mathcal{D}(M^{1/2}) \cap \mathcal{D}(H) \cap \mathcal{D}(A^*)$ is dense in $\mathcal{D}(A^*)$.

Remarks 1.12.
1) Suppose the following modification of the part of Condition 1.3 (3) concerning the adjoint semigroup: $\mathcal{D}$ is boundedly-stable under $\{W_t^*\}$ i.e. $W_t^*\mathcal{D} \subseteq \mathcal{D}$, $t > 0$, and for all $\phi \in \mathcal{D}$,
$$\sup_{0<t<1} \|W_t^*\phi\|_{\mathcal{D}} < \infty. \tag{1.19}$$
Then $\mathcal{D} \cap \mathcal{D}(A^*)$ is dense in $\mathcal{D}(A^*)$, cf. [GGM1, Remark 2.35]. This statement is of course stronger than Condition 1.11.
2) In our applications Condition 1.11 can be avoided upon changing the definition of $\mathcal{V}_1$. Explicitly this modification is given by imposing in Definition 1.5 the following additional condition (replacing ϵ-boundedness with respect to H): V is $\langle H\rangle^{1/2}$-bounded. (See Remark 5.2 1).)

Our second condition is the so-called Fermi Golden Rule condition.

Condition 1.13. Suppose Conditions 1.7 and 1.11. Suppose $\lambda \in \sigma_{\mathrm{pp}}(H)$ and let P denote the eigenprojection $P = E_H(\{\lambda\})$ and $\bar{P} = I - P$. For given $V \in \mathcal{V}_1$ there exists $c > 0$ such that

$$PV\mathrm{Im}\big((H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}\big)VP \geq cP. \tag{1.20}$$

We shall see in Section 3 that the left-hand-side of (1.20) defines a bounded operator for any $V \in \mathcal{V}_1$ (see Theorem 3.3 and Remark 5.2 1) for details). This point might be surprising for the reader due to the low degree of regularity imposed by Condition 1.7 (for example P may not map into $\mathcal{D}(A^2)$ under the stated conditions, see the end of the next subsection for a further discussion).

1.2. **Main results.** We have the following result on upper semicontinuity of the point spectrum of H, showing, in other words, that the total multiplicity of the perturbed eigenvalues near an unperturbed one, λ, cannot exceed the multiplicity of λ.

Theorem 1.14. *Assume that Conditions 1.3 and Condition 1.10 hold. Let $\lambda \in I$ and $J \subseteq I$ be a compact interval including λ such that $\sigma_{\rm pp}(H) \cap J = \{\lambda\}$. Fix $\gamma > 0$ and $\mathcal{B}_{1,\gamma}$ as in Condition 1.10. There exists $0 < \gamma' \leq \gamma$ such that if $V \in \mathcal{B}_{1,\gamma}$ and $\|V\|_1 \leq \gamma'$, the total multiplicity of the eigenvalues of $H + V$ in J is at most* $\dim \mathrm{Ker}(H - \lambda)$.

Notice that the appearing quantity $\dim \mathrm{Ker}(H - \lambda)$ is finite. This is in fact a consequence of Conditions 1.3 and Condition 1.7, cf. Remark 1.4 3). We remark that Theorem 1.14 is an abstract version of [AHS, Theorem 2.5] where upper semicontinuity of the point spectrum of N-body Schrödinger operators is established. The proof, given in Section 4, is essentially the same.

In the case where H does not have eigenvalues in J, we do not need Condition 1.10 to establish upper semicontinuity of point spectrum. More precisely, we will prove that $\sigma_{\rm pp}(H + \sigma V) \cap J = \emptyset$ for $|\sigma|$ small enough under the condition that $V \in \mathcal{V}_2$ (see Corollary 4.1). If it is only required that $V \in \mathcal{V}_1$, the result still holds true provided we assume in addition that any eigenstate of $H + \sigma V$ belongs to $\mathcal{D}(M^{1/2})$ (see Corollary 4.2).

One might suspect that there is a similar semistability result as the one stated in Theorem 1.14 given upon replacing Condition 1.10 by Condition 1.9 (assuming now smallness of $\|V\|_2$). Although there is a formal argument, Conditions 1.3 are insufficient for a rigorous proof. Nevertheless the analogous assertion is true in the special case where H does not have eigenvalues in the interval J, cf. Corollary 4.1. Notice also that another special case, although treated under additional conditions, is part of Theorem 1.15 stated below.

For any $V \in \mathcal{V}_1$ and $\sigma \in \mathbb{R}$ we set $H_\sigma := H + \sigma V$. A main result of this paper is the following assertion on absence of eigenvalues of H_σ for small non-vanishing $|\sigma|$ and for a V fulfilling (1.20). It will be proven in Section 5.

Theorem 1.15. *Assume that Conditions 1.3, Condition 1.7 and Condition 1.11 hold. Assume that Condition 1.13 holds for some $V \in \mathcal{V}_1$. Let $J \subseteq I$ be any compact interval such that $\sigma_{\rm pp}(H) \cap J = \{\lambda\}$. Suppose one of the following two conditions:*

i) *Condition 1.9 and $V \in \mathcal{V}_2$.*
ii) *Condition 1.10 and $V \in \mathcal{B}_{1,\gamma}$.*

There exists $\sigma_0 > 0$ such that for all $\sigma \in]-\sigma_0, \sigma_0[\setminus \{0\}$,

$$\sigma_{\rm pp}(H_\sigma) \cap J = \emptyset. \tag{1.21}$$

This type of theorem is usually referred to as the Fermi Golden Rule criterion (or in short just Fermi Golden Rule). In the framework of regular Mourre theory (that is in particular if $M = 0$, see Remark 1.4 4) above), if A is self-adjoint, Fermi Golden Rule is well-known. It was first proved in [AHS] for N-body Schrödinger operators, under an assumption of the type $V \in \mathcal{V}_2$ and using exponential bounds for eigenstates (yielding in particular an analogue of Condition 1.9). In [HuSi], Theorem 1.15 is proved in an abstract setting assuming Condition 1.9 and the H-boundedness of V''. In [Ca, CGH], still in the framework of regular Mourre theory and with A self-adjoint, it is shown that an assumption of the type $H \in \mathrm{C}^4(A)$ implies Condition 1.9. A similar result also appears in [GJ] under slightly weaker ("local") assumptions, still requiring, however, the boundedness of four commutators.

Theorem 1.15 improves the previous results for the following two reasons: First, as mentioned above, Conditions 1.3 do not require that A be self-adjoint neither that the formal commutator $\mathrm{i}[H, A]$ be H-bounded, which can be important in applications (see in particular Section 2 on Pauli-Fierz Hamiltonians). Second, we prove that the Fermi Golden Rule criterion also holds under Condition 1.10 and the hypothesis $V \in \mathcal{B}_{1,\gamma}$ (that is under condition ii) of Theorem 1.15), which to our knowledge constitutes a new result even in the framework of regular Mourre theory. Let us emphasize that Condition 1.10 does not contain the assumption that the eigenstates are in the domain of A^2, but only in the domain of A. The price we have to pay lies in the fact that Condition 1.10 involves information on the possibly existing perturbed eigenstates, which in concrete models might (at a first glance) seem rather difficult to obtain.

Nevertheless in a separate paper, [FMS] we provide abstract hypotheses under which Condition 1.10 is indeed satisfied. As a consequence, we obtain that Theorem 1.15 applies for a class of Quantum Field Theory models provided that the Hamiltonian only has two bounded commutators with A (defined in a suitable sense), see Section 2. We emphasize that from an abstract point of view, working with $\mathrm{C}^2(A)$ conditions, in fact verifying Condition 1.10 is *doable* while Condition 1.9 might be *false*, see [FMS, Example 1.4] for a counterexample.

Recently Rasmussen together with one us ([MR]) studied the essential energy-momentum spectrum of the translation invariant massive Nelson Hamiltonian H. In particular the authors construct, for a given total momentum P and non-threshold energy E, a conjugate operator A with respect to which the fiber Hamiltonian $H(P)$ satisfies a Mourre estimate, locally uniformly in E and P. From the point of view of the present paper this model is of interest because $H(P)$ is of class $\mathrm{C}^2(A)$ but (presumably) not of class $\mathrm{C}^3(A)$. This means that, even though the context of [MR] is regular Mourre theory, the improvements of this paper and its companion [FMS] are both essential to conclude anything about the structure of embedded non-threshold eigenvalue bands.

We shall use different methods to prove Theorem 1.15 depending on whether we assume i) or ii). In the first case, we shall obtain an expansion to second order of any possibly existing perturbed eigenvalue near the unperturbed one λ. In the second case, ii), this will also be done under the further hypothesis $\dim \mathrm{Ran}(P) = 1$, but we shall proceed differently if the unperturbed eigenvalue is degenerate. In both cases, a key ingredient of the proof consists in obtaining a "reduced Limiting Absorption Principle" at an eigenvalue (see Theorems 3.3 and 3.4 below).

The paper is organized as follows: In the next section, we consider Pauli-Fierz Hamiltonians which constitute our main example of a model satisfying the abstract conditions stated above. Section 3 concerns reduced Limiting Absorption Principles at an eigenvalue λ of H. In Section 4, we study upper semicontinuity of point spectrum and prove Theorem 1.14. Finally in Section 5, we study second order perturbation theory assuming either Condition 1.9 or Condition 1.10, and we prove Theorem 1.15. In Appendix A we present a simple proof of the technically important statement (1.11).

2. Application to the spectral theory of Pauli-Fierz models

2.1. **Massless Pauli-Fierz Hamiltonians.** The main example we have in mind fitting into the framework of Section 1 consists of an abstract class of Quantum Field Theory models, sometimes called massless Pauli-Fierz models (see for instance [DG, DJ, GGM2, FMS]). The latter describe a "small" quantum system linearly coupled to a massless quantized radiation field. The corresponding Hamiltonians H_v^{PF} acts on the Hilbert space $\mathcal{H}_{\mathrm{PF}} := \mathcal{K}\otimes\Gamma(\mathfrak{h})$, where $\mathcal{K}$ is the Hilbert space for the small quantum system, and $\Gamma(\mathfrak{h})$ is the symmetric Fock space over $\mathfrak{h} := \mathrm{L}^2(\mathbb{R}^d, \mathrm{d}k)$. The latter describes a field of massless scalar bosons and is defined by

$$\Gamma(\mathfrak{h}) := \mathbb{C} \oplus \bigoplus_{n=1}^{+\infty} \otimes_s^n \mathfrak{h}, \tag{2.1}$$

where $\otimes_s^n$ denotes the symmetric nth tensor product of $\mathfrak{h}$. The operator H_v^{PF} depends on the form factor v and is written as

$$H_v^{\mathrm{PF}} := K \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(|k|) + \phi(v), \tag{2.2}$$

where K is a bounded below operator on $\mathcal{K}$ describing the dynamics of the small system, $\mathrm{d}\Gamma(|k|)$ is the second quantization of the operator of multiplication by $|k|$ and $\phi(v) := (a^*(v)+a(v))/\sqrt{2}$. We recall that the second quantization of an operator ω on $\mathfrak{h}$ is given by its restriction to the n-bosons Hilbert space as

$$\mathrm{d}\Gamma(\omega)|_{\mathbb{C}} = 0, \quad \mathrm{d}\Gamma(\omega)|_{\otimes_s^n \mathfrak{h}} = \sum_{j=1}^{n} \mathbb{1}_{\mathfrak{h}} \otimes \cdots \otimes \mathbb{1}_{\mathfrak{h}} \otimes \omega \otimes \mathbb{1}_{\mathfrak{h}} \otimes \cdots \otimes \mathbb{1}_{\mathfrak{h}}, \tag{2.3}$$

where in the sum above, ω acts on the jth component of the tensor product. The form factor v is a linear operator from $\mathcal{K}$ to $\mathcal{K}\otimes\mathfrak{h}$, and $a^*(v)$, $a(v)$ are the usual creation and annihilation operators associated with v (see [BD, GGM2]). For convenience, we assume that

$$K \geq 0. \tag{2.4}$$

The hypotheses we make are slightly stronger than the ones considered in [GGM2]. The first one, Hypothesis (**H0**), is related to the fact that the small system is assumed to be confined:

(H0) $(K+1)^{-1}$ is compact on $\mathcal{K}$.

For any $\gamma > 0$, let $\mathcal{O}_\gamma \subseteq \mathcal{B}(\mathcal{D}(K^\gamma); \mathcal{K}\otimes\mathfrak{h})$ be the set of operators which extend by continuity from $\mathcal{D}(K^\gamma)$ to an element of $\mathcal{B}(\mathcal{K}; \mathcal{D}(K^\gamma)^* \otimes \mathfrak{h})$, that is

$$\begin{aligned} \mathcal{O}_\gamma := \big\{ & v \in \mathcal{B}(\mathcal{D}(K^\gamma); \mathcal{K}\otimes\mathfrak{h}), \\ & \exists C > 0, \forall \psi \in \mathcal{D}(K^\gamma), \big\| [(K+1)^{-\gamma} \otimes \mathbb{1}_{\mathfrak{h}}] v\psi \big\|_{\mathcal{K}\otimes\mathfrak{h}} \leq C \|\psi\|_{\mathcal{K}} \big\}. \end{aligned} \tag{2.5}$$

Let $0 \leq \tau < 1/2$ be fixed. Our first assumption on the form factor is the following:

(I1) v and $[\mathbb{1}_{\mathcal{K}} \otimes |k|^{-1/2}]v$ belong to $\mathcal{O}_\tau$.

It follows from [GGM2, Proposition 4.6] that, if $[\mathbb{1}_{\mathcal{K}} \otimes |k|^{-1/2}]v \in \mathcal{O}_\tau$, then H_v^{PF} is self-adjoint with domain

$$\mathcal{D}(H_v^{\mathrm{PF}}) = \mathcal{D}(H_0^{\mathrm{PF}}) = \mathcal{D}(K) \otimes \Gamma(\mathfrak{h}) \cap \mathcal{K} \otimes \mathcal{D}(\mathrm{d}\Gamma(|k|)). \tag{2.6}$$

We consider the unitary operator

$$T : \mathrm{L}^2(\mathbb{R}^d) \to \mathrm{L}^2(\mathbb{R}^+) \otimes \mathrm{L}^2(S^{d-1}) =: \tilde{\mathfrak{h}} \tag{2.7}$$

defined by $(Tu)(\omega,\theta) = \omega^{(d-1)/2} u(\omega\theta)$. Lifting it to the full Hilbert space $\mathcal{H}_{\mathrm{PF}}$ by setting $\mathcal{T} := \mathbb{1}_{\mathcal{K}} \otimes \Gamma(T)$ (recall that $\Gamma(T)$ is defined by its restriction to the n-bosons Hilbert space as $\Gamma(T)|_{\otimes_n^s \mathfrak{h}} = T \otimes \cdots \otimes T$ for $n \geq 1$, and $\Gamma(T)|_{\mathbb{C}} = \mathbb{1}_{\mathbb{C}}$ for $n = 0$), we get a unitary map

$$\mathcal{T} : \mathcal{H}_{\mathrm{PF}} \to \tilde{\mathcal{H}}_{\mathrm{PF}} := \mathcal{K} \otimes \Gamma(\tilde{\mathfrak{h}}). \tag{2.8}$$

This allows us to write the Hamiltonian in polar coordinates in the following way:

$$\tilde{H}_v^{\mathrm{PF}} := \mathcal{T} H_v^{\mathrm{PF}} \mathcal{T}^{-1} = K \otimes \mathbb{1}_{\Gamma(\tilde{\mathfrak{h}})} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega) + \phi(\tilde{v}), \tag{2.9}$$

on $\tilde{\mathcal{H}}_{\rm PF}$, where

$$\tilde{v} := [\mathbb{1}_{\mathcal{K}} \otimes T]v \tag{2.10}$$

is a linear operator from $\mathcal{K}$ to $\mathcal{K} \otimes \tilde{\mathfrak{h}}$, and $\mathrm{d}\Gamma(\omega)$ denotes the second quantization of the operator of multiplication by $\omega \in \mathbb{R}^+$.

Let us consider a function $d \in \mathrm{C}^\infty((0,\infty))$ satisfying $d'(\omega) < 0$, $|d'(\omega)| \le C\omega^{-1}d(\omega)$ for some positive constant C, $d(\omega) = 1$ if $\omega \ge 1$, and $\lim_{\omega\to 0} d(\omega) = +\infty$ (see Figure 1).

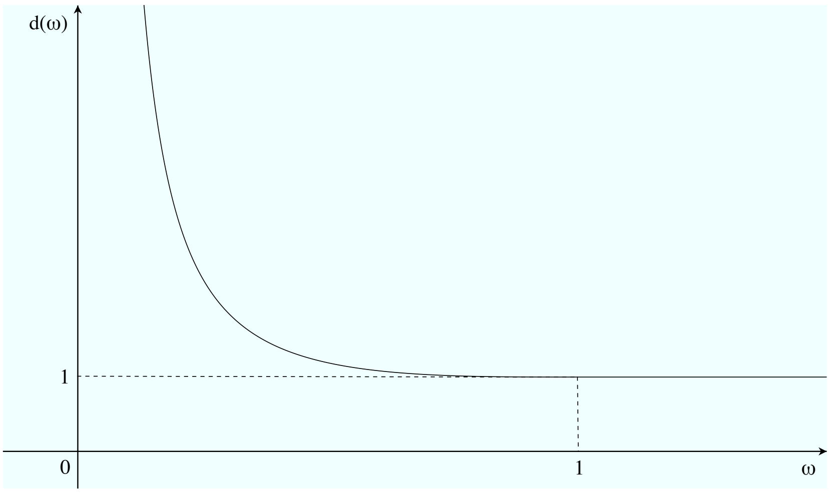

FIGURE 1. **The map** $\omega \mapsto d(\omega)$

Let

$$\tilde{\mathcal{O}}_\tau := [\mathbb{1}_{\mathcal{K}} \otimes T]\mathcal{O}_\tau. \tag{2.11}$$

The following further assumptions on the interaction are made:

(I2) The following holds:

$$\big[\mathbb{1}_{\mathcal{K}} \otimes (1+\omega^{-1/2})\omega^{-1}d(\omega) \otimes \mathbb{1}_{\mathrm{L}^2(S^{d-1})}\big]\tilde{v} \in \tilde{\mathcal{O}}_\tau$$
$$\big[\mathbb{1}_{\mathcal{K}} \otimes (1+\omega^{-1/2})d(\omega)\partial_\omega \otimes \mathbb{1}_{\mathrm{L}^2(S^{d-1})}\big]\tilde{v} \in \tilde{\mathcal{O}}_\tau,$$

(I3) $\big[\mathbb{1}_{\mathcal{K}} \otimes \partial_\omega^2 \otimes \mathbb{1}_{\mathrm{L}^2(S^{d-1})}\big]\tilde{v} \in \mathcal{B}(\mathcal{D}(K^{\frac{1}{2}}); \mathcal{K} \otimes \tilde{\mathfrak{h}})$.

Let us recall the definition of the conjugate operator used in [GGM2]. Let $\chi \in \mathrm{C}_0^\infty([0,\infty))$ be such that $\chi(\omega) = 0$ if $\omega \ge 1$ and $\chi(\omega) = 1$ if $\omega \le 1/2$. For $0 < \delta \le 1/2$, the function $m_\delta \in \mathrm{C}^\infty([0,\infty))$ is defined by

$$m_\delta(\omega) = \chi\Big(\frac{\omega}{\delta}\Big)\, d(\delta) + (1-\chi)\Big(\frac{\omega}{\delta}\Big)\, d(\omega), \tag{2.12}$$

(see Figure 2).

Consider the following operator $\tilde{a}_\delta$ acting on $\tilde{\mathfrak{h}}$:

$$\tilde{a}_\delta := \mathrm{i}m_\delta(\omega)\frac{\partial}{\partial\omega} + \frac{\mathrm{i}}{2}\frac{\mathrm{d}m_\delta}{\mathrm{d}\omega}(\omega), \quad \mathcal{D}(\tilde{a}_\delta) = \mathrm{H}_0^1(\mathbb{R}^+) \otimes \mathrm{L}^2(S^{d-1}), \tag{2.13}$$

where $H_0^1(\mathbb{R}^+)$ denotes the closure of $\mathrm{C}_0^\infty(\mathbb{R}^+)$ in $\mathrm{H}^1(\mathbb{R}^+)$ and $\mathrm{C}_0^\infty(\mathbb{R}^+)$ is the set of smooth compactly supported functions on $\mathbb{R}^+$. Then the operator $\tilde{A}_\delta$ on $\tilde{\mathcal{H}}_{\rm PF}$ is defined by $\tilde{A}_\delta :=$

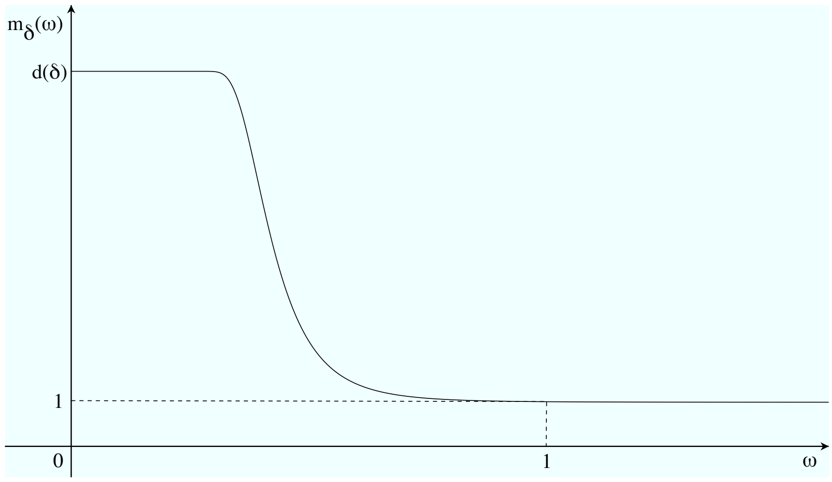

FIGURE 2. **The map** $\omega \mapsto m_\delta(\omega)$

$\mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\tilde{a}_\delta)$. It is proved in [GGM2, Section 6] that $\tilde{A}_\delta$ is closed, densely defined and maximal symmetric.

Let $M_\delta := \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(m_\delta)$ and $R_\delta(\tilde{v}) := -\phi(\mathrm{i}\tilde{a}_\delta \tilde{v})$. Then M_δ is self-adjoint, $M_\delta \geq 0$, and if v satisfies Hypotheses (**I1**) and (**I2**), then, by [GGM2, Lemma 6.4 *i*)], $R_\delta(\tilde{v})$ is symmetric and $\tilde{H}_v^{\mathrm{PF}}$-bounded.

2.2. **Checking the abstract assumptions.** In this subsection, we verify that, on the Hilbert space $\mathcal{H} = \tilde{\mathcal{H}}_{\mathrm{PF}}$, the operators $H = \tilde{H}_v^{\mathrm{PF}}$, $M = M_\delta$, $R = R_\delta(\tilde{v})$, $A = \tilde{A}_\delta$ fulfil Conditions 1.3, 1.10 and 1.11 stated in Section 1 (provided that v satisfies, in particular, the hypotheses stated above). The following lemma shows that Condition 1.3 (1) is satisfied.

Lemma 2.1. *Assume that v satisfies Hypothesis* (**I1**). *Then for all $\delta > 0$,*

$$\tilde{H}_v^{\mathrm{PF}} \in \mathrm{C}^1_{\mathrm{Mo}}(M_\delta). \tag{2.14}$$

Proof. The fact that $\tilde{H}_v^{\mathrm{PF}} \in \mathrm{C}^1(M_\delta)$ follows from [GGM2, Lemma 6.4 *i*)]. Moreover, since m_δ is bounded and $[\omega, m_\delta] = 0$, we have that $[\tilde{H}_v^{\mathrm{PF}}, \mathrm{i}M_\delta]^0 = -\phi(\mathrm{i}m_\delta \tilde{v})$ by [GGM2, Corollary 4.13]. Using again that m_δ is bounded, we then conclude from Hypothesis (**I1**) and [GGM2, Proposition 4.6] that $[\tilde{H}_v^{\mathrm{PF}}, \mathrm{i}M_\delta]^0$ is $\tilde{H}_0^{\mathrm{PF}}$-bounded, and hence $\tilde{H}_v^{\mathrm{PF}}$-bounded (with relative bound 0). □

Lemma 2.1 together with [GGM2, Propositions 6.6, 6.7 and Theorem 7.12] imply:

Proposition 2.2. *Assume Hypothesis* (**H0**) *and that v satisfies Hypotheses* (**I1**), (**I2**) *and* (**I3**). *Then for all $E_0 \in \mathbb{R}$, there exists $\delta_0 > 0$ such that for all $0 < \delta \leq \delta_0$, the operators $H = \tilde{H}_v^{\mathrm{PF}}$, $M = M_\delta$, $R = R_\delta(\tilde{v})$, $A = \tilde{A}_\delta$ fulfil Conditions 1.3 with $I = (-\infty, E_0)$.*

Remark 2.3. We remark that the formulation of the Mourre estimate stated in [GGM2, Theorem 7.12] is not the same as the one considered in Condition 1.3 (2). However, one can verify that the latter is indeed a consequence of [GGM2, Theorem 7.12].

In order to verify Condition 1.10, we need to impose a further condition on v:

(I4) The form $(K\otimes \mathbb{1}_{\tilde{\mathfrak{h}}})\tilde{v}-\tilde{v}K$ extends by continuity from $\mathcal{D}(K\otimes \mathbb{1}_{\tilde{\mathfrak{h}}})\times\mathcal{D}(K)$ to an element of $\tilde{\mathcal{O}}_{\frac{1}{2}}$.

Here $\tilde{\mathcal{O}}_{\frac{1}{2}}$ is defined as $\tilde{\mathcal{O}}_\tau$ (see (2.5) and (2.11)). Notice that, assuming (**I1**), the statement above is meaningful.

We have to identify the set $\mathcal{B}_{1,\gamma}$ used in Condition 1.10. To this end, let us first introduce some definitions. Let $\mathcal{I}_{\mathrm{PF}}(d)$ be defined by:

$$\mathcal{I}_{\mathrm{PF}}(d) := \big\{ v\in\mathcal{L}(\mathcal{K};\mathcal{K}\otimes\mathfrak{h}), v \text{ satisfies } (\mathbf{I1}),(\mathbf{I2}),(\mathbf{I3}),(\mathbf{I4})\big\}. \tag{2.15}$$

Observe that $\mathcal{I}_{\mathrm{PF}}(d)$ can be equipped with a norm, $\|\cdot\|_{\mathrm{PF}}$, matching the four conditions (**I1**), (**I2**) , (**I3**), (**I4**) (see [FMS, Subsection 5.1]).

Let $v\in\mathcal{I}_{\mathrm{PF}}(d)$. Let $W_{\delta,t}$ denote the C_0-semigroup generated by $\tilde{A}_\delta$. We set

$$\mathcal{G}_\delta^{\mathrm{PF}} := \mathcal{D}(|\tilde{H}_v^{\mathrm{PF}}|^{\frac{1}{2}})\cap\mathcal{D}(M_\delta^{\frac{1}{2}}). \tag{2.16}$$

By Proposition 2.2, we have that $H=\tilde{H}_v^{\mathrm{PF}}$, $M=M_\delta$, $A=\tilde{A}_\delta$ fulfil Condition 1.3 (3), and hence $W_{\delta,t}|_{\mathcal{G}_\delta^{\mathrm{PF}}}$ is a C_0-semigroup (see Remark 1.4 1)). Its generator is denoted by $\tilde{A}_{\mathcal{G}_\delta^{\mathrm{PF}}}$. Likewise, the extension of $W_{\delta,t}$ to $(\mathcal{G}_\delta^{\mathrm{PF}})^*$ is a C_0-semigroup whose generator is denoted by $\tilde{A}_{(\mathcal{G}_\delta^{\mathrm{PF}})^*}$.

Let $\mathcal{V}_1^{\mathrm{PF}}$ denote the set of symmetric operators V, ϵ-bounded relatively to $\tilde{H}_v^{\mathrm{PF}}$, such that $V\in\mathrm{C}^1(\tilde{A}_{\mathcal{G}_\delta^{\mathrm{PF}}};\tilde{A}_{(\mathcal{G}_\delta^{\mathrm{PF}})^*})$ and $[V,\mathrm{i}\tilde{A}_\delta]^0$ is $\tilde{H}_v^{\mathrm{PF}}$-bounded. It is equipped with the norm

$$\|V\|_1^{\mathrm{PF}} = \|V(\tilde{H}_v^{\mathrm{PF}}-\mathrm{i})^{-1}\| + \|[V,\mathrm{i}\tilde{A}_\delta]^0(\tilde{H}_v^{\mathrm{PF}}-\mathrm{i})^{-1}\|. \tag{2.17}$$

By [GGM2, Proposition 4.6], if w satisfies Hypothesis (**I1**), then $\phi(\tilde{w})$ is ϵ-bounded relatively to $\tilde{H}_v^{\mathrm{PF}}$, and, by [GGM2, Lemma 6.4 *i)*], if in addition w satisfies Hypothesis (**I2**), then, for any $\delta>0$, $[\phi(\tilde{w}),\mathrm{i}\tilde{A}_\delta]^0=-\phi(\mathrm{i}\tilde{a}_\delta\tilde{w})$ is $\tilde{H}_v^{\mathrm{PF}}$-bounded. Moreover, one can verify that the map

$$\mathcal{I}_{\mathrm{PF}}(d)\ni w\mapsto\phi(\tilde{w})\in\mathcal{V}_1^{\mathrm{PF}} \tag{2.18}$$

is continuous (see [FMS, Lemma 5.8]).

In a separate paper, [FMS], we prove (see [FMS, Theorem 5.2]):

Proposition 2.4. *Assume Hypothesis* (**H0**) *and let* $v\in\mathcal{I}_{\mathrm{PF}}(d)$. *For all* $E_0\in\mathbb{R}$, *there exists* $\delta_0>0$ *such that for all* $0<\delta\le\delta_0$, *the operators* $H=\tilde{H}_v^{\mathrm{PF}}$, $M=M_\delta$, $R=R_\delta(\tilde{v})$, $A=\tilde{A}_\delta$ *fulfil Condition 1.10. Here* $I=(-\infty,E_0)$ *and* $\mathcal{B}_{1,\gamma}$ *is given by*

$$B_{1,\gamma}=\{\phi(\tilde{w}), w\in\mathcal{I}_{\mathrm{PF}}(d),\|w\|_{\mathrm{PF}}\le\tilde{\gamma}\}, \tag{2.19}$$

where $\tilde{\gamma}>0$ *is fixed sufficiently small.*

Remarks 2.5. 1) Since the map (2.18) is continuous, for any $\gamma>0$, the set $\mathcal{B}_{1,\gamma}$ is included in $\{V\in\mathcal{V}_1^{\mathrm{PF}},\|V\|_1^{\mathrm{PF}}\le\gamma\}$ provided that $\tilde{\gamma}$ is chosen small enough. Moreover, $B_{1,\gamma}$ is clearly star-shaped and symmetric with respect to 0. Hence the requirements of Condition 1.10 are satisfied.

2) Under the conditions of Proposition 2.4, we do not expect Condition 1.9 to be satisfied in general. Indeed, the assumption that $v\in\mathcal{I}_{\mathrm{PF}}(d)$ in the statement of Proposition 2.4 allows us to control two commutators of $\tilde{H}_v^{\mathrm{PF}}$ with $\tilde{A}_\delta$. In order to be able to conclude that Condition 1.9 is satisfied using the method of [FMS], one would need to control three commutators of $\tilde{H}_v^{\mathrm{PF}}$ with $\tilde{A}_\delta$ (see [FMS]). This would require a stronger restriction on

the infrared behavior of the form factor v than the one imposed by Hypotheses (**I1**)–(**I2**)–(**I3**).

In order to apply Theorems 1.14 and 1.15, it remains to verify Condition 1.11. Let

$$\mathcal{S} = \mathcal{D}(K) \otimes \Gamma_{\rm fin}(\mathrm{C}_0^\infty(\mathbb{R}^+) \otimes \mathrm{L}^2(S^{d-1})), \tag{2.20}$$

where for $\mathcal{E} \subseteq \mathrm{L}^2(\mathbb{R}^+) \otimes \mathrm{L}^2(S^{d-1})$,

$$\Gamma_{\rm fin}(\mathcal{E}) := \big\{ \Phi = (\Phi^{(0)}, \Phi^{(1)}, \Phi^{(2)}, \dots) \in \Gamma(\mathcal{E}), \exists n_0, \Phi^{(n)} = 0 \text{ for } n \geq n_0 \big\}. \tag{2.21}$$

For any $\delta > 0$, $\mathcal{S}$ is included in $\mathcal{D}(\tilde{H}_v^{\rm PF}) \cap \mathcal{D}(M_\delta) \cap \mathcal{D}(\tilde{A}_\delta)$. Moreover, $\mathcal{S}$ is a core for $\tilde{A}_\delta^*$. Therefore we get:

Proposition 2.6. *Assume that v satisfies Hypothesis* (**I1**). *Then, for all $\delta > 0$, the operators $H = \tilde{H}_v^{\rm PF}$, $M = M_\delta$, $A = \tilde{A}_\delta$ fulfil Condition 1.11.*

Let us finally mention the particular case for which the unperturbed Hamiltonian under consideration is the non-interacting one, $\tilde{H}_0^{\rm PF}$, given by

$$\tilde{H}_0^{\rm PF} := K \otimes \mathbb{1}_{\Gamma(\tilde{\mathfrak{h}})} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\omega). \tag{2.22}$$

In this case, one can choose $M = \mathbb{1}_{\mathcal{K}} \otimes \mathcal{N}$, where $\mathcal{N} := \mathrm{d}\Gamma(\mathbb{1}_{\tilde{\mathfrak{h}}})$ is the number operator, and $A = \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\mathrm{i}\partial_\omega)$. Then one can easily check the following proposition:

Proposition 2.7. *Assume Hypothesis* (**H0**). *Then the operators $H = \tilde{H}_0^{\rm PF}, M = \mathbb{1}_{\mathcal{K}} \otimes \mathcal{N}, R = 0, A = \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(\mathrm{i}\partial_\omega)$ fulfil Conditions 1.3 (with $I = \mathbb{R}$) and Condition 1.9.*

Remark 2.8. The fact that Condition 1.9 is fulfilled under the conditions of Proposition 2.7 is obvious, since the unperturbed eigenstates are of the form $\phi \otimes \Omega$, where ϕ is an eigenstate of K, and Ω denotes the vacuum in $\Gamma(\tilde{\mathfrak{h}})$.

2.3. **Results.** As a consequence of Propositions 2.2, 2.4 and 2.6, applying Theorems 1.14 and 1.15, we obtain:

Theorem 2.9. *Assume Hypothesis* (**H0**). *Let $v_0, v \in \mathcal{I}_{\rm PF}(d)$. Let J be a compact interval such that $\sigma_{\rm pp}(H_{v_0}^{\rm PF}) \cap J = \{\lambda\}$. Let P_{v_0} denote the eigenprojection $P_{v_0} = E_{H_{v_0}^{\rm PF}}(\{\lambda\})$ and $\bar{P}_{v_0} = I - P_{v_0}$. Then the following holds:*

i) *There exists $\sigma_0 > 0$ such that for all $0 \leq |\sigma| \leq \sigma_0$, the total multiplicity of the eigenvalues of $H_{v_0}^{\rm PF} + \sigma\phi(v)$ in J is at most* $\dim \mathrm{Ran}(P_{v_0})$.

ii) *Suppose in addition that*

$$P_{v_0}\phi(v) \mathrm{Im}\big((H_{v_0}^{\rm PF} - \lambda - \mathrm{i}0^+)^{-1}\bar{P}_{v_0}\big)\phi(v)P_{v_0} \geq cP_{v_0}, \tag{2.23}$$

for some $c > 0$. Then there exists $\sigma_0 > 0$ such that for all $0 < |\sigma| \leq \sigma_0$,

$$\sigma_{\rm pp}\big(H_{v_0}^{\rm PF} + \sigma\phi(v)\big) \cap J = \emptyset. \tag{2.24}$$

Remarks 2.10. 1) In view of Propositions 2.2, 2.4 and 2.6, Theorems 1.14 and 1.15 imply Theorem 2.9 with $\tilde{H}_{v_0}^{\rm PF}$ replacing $H_{v_0}^{\rm PF}$ and $\phi(\tilde{v})$ replacing $\phi(v)$. However, using the unitary transformation mapping $\mathcal{H}_{\rm PF}$ to $\tilde{\mathcal{H}}_{\rm PF}$, the statement of Theorem 2.9 clearly follows.

2) In the case where the unperturbed Hamiltonian is the non-interacting one, that is $H^{\rm PF}_{v_0} = H^{\rm PF}_0$ with $H^{\rm PF}_0 = K \otimes \mathbb{1}_{\Gamma(\mathfrak{h})} + \mathbb{1}_{\mathcal{K}} \otimes {\rm d}\Gamma(|k|)$, one can use Proposition 2.7 instead of Proposition 2.4 in order to conclude Theorem 2.9 ii). Indeed, it follows from [GGM2, Proposition 4.11, Lemma 6.2 and proof of Proposition 6.6] that if v satisfies (**I1**)–(**I2**)–(**I3**), then $\phi(\tilde v) \in \mathcal{V}_2$ (in the sense of Definition 1.6). Hence, since Condition 1.9 is satisfied by Proposition 2.7, we can apply Theorem 1.15 with Condition *i)* instead of Condition *ii)*. For a general $v_0 \in \mathcal{I}_{\rm PF}(d)$, however, we have to apply Theorem 1.15 with Condition *i)* (see Remark 2.5 2) above).

The latter result (the absence of eigenvalues of $H^{\rm PF}_0 + \sigma\phi(v)$ for sufficiently small $\sigma \neq 0$ according to Fermi Golden Rule) already appears in [DJ] assuming in particular that $\langle \partial_\omega \rangle^\nu \tilde v \in \mathcal{B}(\mathcal{K}; \mathcal{K} \otimes \tilde{\mathfrak{h}})$ for some $\nu > 1$. More recently the same result was also proven in [Go], still for sufficiently small values of the coupling constant, under the assumptions that $\partial_\omega \tilde v$, $\omega^{-1/2}\partial_\omega \tilde v$ and $\omega^{-\nu}\tilde v$ (for some $\nu > 1$) belong to $\mathcal{B}(\mathcal{K}; \mathcal{K} \otimes \tilde{\mathfrak{h}})$. Besides, in [DJ], upper semicontinuity of the point spectrum of $H^{\rm PF}_0 + \sigma\phi(v)$ (in the sense stated in Theorem 2.9 i)) is obtained for sufficiently small σ, assuming that $\langle \partial_\omega \rangle^\nu \tilde v \in \mathcal{B}(\mathcal{K}; \mathcal{K} \otimes \tilde{\mathfrak{h}})$ for some $\nu > 2$. The main achievement of our paper, as far as massless Pauli-Fierz models are concerned, is to provide a method which allows us to consider $H^{\rm PF}_{v_0}$ as the unperturbed Hamiltonian, for *any* v_0 belonging to $\mathcal{I}_{\rm PF}(d)$.

A model sharing several properties with the one considered in this subsection is the so-called "standard model of non-relativistic QED". For results on spectral theory in this context involving the Mourre method, we refer to [Sk, BFS, BFSS, DJ, FGS].

2.4. **Example: The massless Nelson model.** An example of a model satisfying the hypotheses of Subsection 2.1 is the Nelson model of confined non-relativistic quantum particles interacting with massless scalar bosons. The Hilbert space is given by

$$\mathcal{H}_{\rm N} := {\rm L}^2(\mathbb{R}^{3P}) \otimes \mathcal{F}, \tag{2.25}$$

where $\mathcal{F} := \Gamma({\rm L}^2(\mathbb{R}^3))$ is the symmetric Fock space over ${\rm L}^2(\mathbb{R}^3)$ (see (2.1)). The Nelson Hamiltonian acts on $\mathcal{H}_{\rm N}$ and is defined by

$$H^{\rm N}_\rho := K \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{{\rm L}^2(\mathbb{R}^{3P})} \otimes {\rm d}\Gamma(|k|) + I_\rho(x). \tag{2.26}$$

Here $x = (x_1, \dots, x_P)$, and K is a Schrödinger operator on ${\rm L}^2(\mathbb{R}^{3P})$ describing the dynamics of P non-relativistic particles. We suppose that K is given by

$$K := \sum_{i=1}^{P} \frac{1}{2m_i}\Delta_i + \sum_{i<j} V_{ij}(x_i - x_j) + W(x_1, \dots, x_p), \tag{2.27}$$

where the masses m_i are positive, the confining potential W satisfies

(W0) $W \in {\rm L}^2_{\rm loc}(\mathbb{R}^{3P})$ and there exist positive constants $c_0, c_1 > 0$ and $\alpha > 2$ such that $W(x) \geq c_0|x|^{2\alpha} - c_1$,

and the pair potentials V_{ij} satisfy

(V0) The V_{ij}'s are Δ-bounded with relative bound 0.

Without loss of generality, we can assume that $K \geq 0$. Note that (**W0**) implies that Hypothesis (**H0**) of Subsection 2.1 is satisfied.

The coupling $I_\rho(x)$ in (2.26) is of the form

$$I_\rho(x) := \sum_{i=1}^{P} \Phi_\rho(x_i), \tag{2.28}$$

where, for $y \in \mathbb{R}^3$, $\Phi_\rho(y)$ is the field operator defined by

$$\Phi_\rho(y) := \frac{1}{\sqrt{2}} \int_{\mathbb{R}^3} \big(\rho(k) e^{-\mathrm{i}k\cdot y} a^*(k) + \bar{\rho}(k) e^{\mathrm{i}k\cdot x} a(k)\big) \mathrm{d}k. \tag{2.29}$$

In particular, $I_\rho(x)$ can be written under the form $I_\rho(x) = \phi(\Psi_{\mathrm{N}}(\rho))$, where

$$\Psi_{\mathrm{N}}(\rho) \in \mathcal{B}(\mathrm{L}^2(\mathbb{R}^{3P}); \mathrm{L}^2(\mathbb{R}^{3P}) \otimes \mathrm{L}^2(\mathbb{R}^3)) = \mathcal{B}(\mathrm{L}^2(\mathbb{R}^{3P}); \mathrm{L}^2(\mathbb{R}^3; \mathrm{L}^2(\mathbb{R}^{3P})))$$

is defined by

$$(\Psi_{\mathrm{N}}(\rho)\psi)(k)(x_1, \dots, x_P) = \sum_{j=1}^{P} e^{-\mathrm{i}k\cdot x_j} \rho(k) \psi(x_1, \dots, x_P). \tag{2.30}$$

Hence H_ρ^{N} is a Pauli-Fierz Hamiltonian in the sense of Subsection 2.1, with $\mathcal{K} = \mathrm{L}^2(\mathbb{R}^{3P})$ and $\mathfrak{h} = \mathrm{L}^2(\mathbb{R}^3)$.

For simplicity, we assume that ρ only depends on k through its norm, $|k|$, and, going to polar coordinates, we introduce

$$\tilde{\rho}(\omega) = \omega \rho(\omega, 0, 0), \quad \omega \in \mathbb{R}^+. \tag{2.31}$$

Our set of conditions on $\tilde{\rho}$ is the following:

(ρ1) $\displaystyle\int_0^\infty (1 + \omega^{-1}) |\tilde{\rho}(\omega)|^2 \mathrm{d}\omega < \infty,$

(ρ2) $\displaystyle\int_0^\infty (1 + \omega^{-1}) d(\omega)^2 \left[\omega^{-2} |\tilde{\rho}(\omega)|^2 + \left| \frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\omega}(\omega) \right|^2 \right] \mathrm{d}\omega < \infty,$

(ρ3) $\displaystyle\int_0^\infty \left| \frac{\mathrm{d}^2\tilde{\rho}}{\mathrm{d}\omega^2}(\omega) \right|^2 \mathrm{d}\omega < \infty,$

(ρ4) $\displaystyle\int_0^\infty \omega^4 |\tilde{\rho}(\omega)|^2 \mathrm{d}\omega < \infty,$

where d denotes the function considered in Subsection 2.1. Note that **(ρ1)**–**(ρ2)**–**(ρ3)** are the assumptions made in [GGM2]. The further assumption **(ρ4)** is made in order that Hypothesis **(I4)** of Subsection 2.2 is satisfied. We observe that **(ρ2)** and **(ρ4)** imply **(ρ1)**.

The set of functions ρ satisfying **(ρ1)**–**(ρ2)**–**(ρ3)**–**(ρ4)** is denoted by $\mathcal{I}_{\mathrm{N}}(d)$. The following proposition is proven in [FMS, Subsection 5.2]:

Proposition 2.11. *Let $\rho \in \mathcal{I}_{\mathrm{N}}(d)$. Then $\Psi_{\mathrm{N}}(\rho)$ defined as in* (2.30) *belongs to $\mathcal{I}_{\mathrm{PF}}(d)$.*

An example of ρ, and hence $\tilde{\rho}$, satisfying **(ρ1)**–**(ρ2)**–**(ρ3)**–**(ρ4)** is

$$\rho(k) = e^{-\frac{|k|^2}{2\Lambda^2}} |k|^{-\frac{1}{2}+\epsilon}, \quad \tilde{\rho}(\omega) = e^{-\frac{\omega^2}{2\Lambda^2}} \omega^{\frac{1}{2}+\epsilon}, \tag{2.32}$$

with $0 < \Lambda < \infty$ and $\epsilon > 1$.

From Proposition 2.11 and Theorem 2.9, we obtain:

Theorem 2.12. *Assume that Hypotheses* (**W0**) *and* (**V0**) *hold. Let $\rho_0, \rho \in \mathcal{I}_{\mathrm{N}}(d)$. Let J be a compact interval such that $\sigma_{\mathrm{pp}}(H_{\rho_0}^{\mathrm{N}}) \cap J = \{\lambda\}$. Let P_{ρ_0} denote the eigenprojection $P_{\rho_0} = E_{H_{\rho_0}^{\mathrm{N}}}(\{\lambda\})$ and $\bar{P}_{\rho_0} = I - P_{\rho_0}$. Then the following holds:*

i) *There exists* $\sigma_0 > 0$ *such that for all* $0 \le |\sigma| \le \sigma_0$*, the total multiplicity of the eigenvalues of* $H^{\mathrm{N}}_{\rho_0} + \sigma I_\rho(x)$ *in* J *is at most* $\dim \mathrm{Ran}(P_{\rho_0})$.

ii) *Suppose in addition that*

$$P_{\rho_0} I_\rho(x) \mathrm{Im}\big((H^{\mathrm{N}}_{\rho_0} - \lambda - \mathrm{i}0^+)^{-1} \bar{P}_{\rho_0}\big) I_\rho(x) P_{\rho_0} \ge c P_{\rho_0}, \tag{2.33}$$

for some $c > 0$*. Then there exists* $\sigma_0 > 0$ *such that for all* $0 < |\sigma| \le \sigma_0$,

$$\sigma_{\mathrm{pp}}\big(H^{\mathrm{N}}_{\rho_0} + \sigma I_\rho(x)\big) \cap J = \emptyset. \tag{2.34}$$

In fact, the confinement assumption (**W0**) allows one to make use of a unitary dressing transformation (see e.g. [GGM2, FMS]) in order to "improve" the infrared behavior of the form factor in the Hamiltonian $H^{\mathrm{N}}_{\rho_0}$. More precisely, let **(**ρ**1')** denote the following condition:

(ρ**1')** $\displaystyle\int_0^\infty (1+\omega^{-2})|\tilde{\rho}(\omega)|^2 \mathrm{d}\omega < \infty.$

Assuming that ρ_0 satisfies this condition, the unitary operator

$$U_{\rho_0} := e^{-\mathrm{i}P\Phi_{\mathrm{i}\rho_0/|\cdot|}} \tag{2.35}$$

is well-defined and we can consider the Hamiltonian

$$\begin{aligned} H^{\mathrm{N}'}_{\rho_0} :=& (\mathbb{1}_{\mathcal{K}} \otimes U_{\rho_0}) H^{\mathrm{N}}_{\rho_0} (\mathbb{1}_{\mathcal{K}} \otimes U^*_{\rho_0}) \\ =& K_{\rho_0} \otimes \mathbb{1}_{\mathcal{F}} + \mathbb{1}_{\mathcal{K}} \otimes \mathrm{d}\Gamma(|k|) + I_{\rho_0}(x) - I_{\rho_0}(0), \end{aligned} \tag{2.36}$$

where

$$K_{\rho_0} := K + \frac{P^2}{2} \int_{\mathbb{R}^3} \frac{|\rho_0(k)|^2}{|k|} \mathrm{d}k - P \sum_{j=1}^P \int_{\mathbb{R}^3} \frac{|\rho_0(k)|^2}{|k|} \cos(k \cdot x_j) \mathrm{d}k. \tag{2.37}$$

In the same way as in (2.30), we observe that $I_{\rho_0}(x) - I_{\rho_0}(0) = \phi(\Psi'_{\mathrm{N}}(\rho_0))$, where $\Psi'_{\mathrm{N}}(\rho_0)$ is defined by

$$(\Psi'_{\mathrm{N}}(\rho_0)\psi)(k)(x_1, \dots, x_P) = \sum_{j=1}^P (e^{-\mathrm{i}k\cdot x_j} - 1)\rho_0(k)\psi(x_1, \dots, x_P). \tag{2.38}$$

In particular, $H^{\mathrm{N}'}_{\rho_0}$ is a Pauli-Fierz Hamiltonian in the sense of Subsection 2.1.

We consider the following further conditions:

(ρ**2')** $\displaystyle\int_0^\infty \Big|\frac{\mathrm{d}\tilde{\rho}}{\mathrm{d}\omega}(\omega)\Big|^2 \mathrm{d}\omega < \infty,$

(ρ**3')** $\displaystyle\int_0^\infty (1+\omega^2)^{-1}\omega^2 \Big|\frac{\mathrm{d}^2\tilde{\rho}}{\mathrm{d}\omega^2}(\omega)\Big|^2 \mathrm{d}\omega < \infty,$

and we denote by $\mathcal{I}'_{\mathrm{N}}(d)$ the set of functions ρ satisfying **(**ρ**1')–(**ρ**2')–(**ρ**3')–(**ρ**4)**. In [FMS, Subsection 5.2], we verify that if $\rho_0 \in \mathcal{I}'_{\mathrm{N}}(d)$, then $\Psi'_{\mathrm{N}}(\rho_0)$ defined as in (2.38) belongs to $\mathcal{I}_{\mathrm{PF}}(d)$. Notice that for any $0 < \Lambda < \infty$ and $\epsilon > 0$, the function given in (2.32) belongs to $\mathcal{I}'_{\mathrm{N}}(d)$.

As in the statement of Theorem 2.12, we consider a perturbation of the Hamiltonian $H^{\mathrm{N}}_{\rho_0}$ of the form $\sigma I_\rho(x)$. After the dressing transformation, the perturbation becomes

$$\begin{aligned} \sigma I_{\rho_0,\rho}(x) :=& \sigma(\mathbb{1}_{\mathcal{K}} \otimes U_{\rho_0}) I_\rho(x) (\mathbb{1}_{\mathcal{K}} \otimes U^*_{\rho_0}) \\ =& I_\rho(x) - P \sum_{j=1}^P \mathrm{Re} \int_{\mathbb{R}^3} \frac{\bar{\rho}_0(k)\rho(k)}{|k|} e^{-\mathrm{i}k\cdot x_j} \mathrm{d}k. \end{aligned} \tag{2.39}$$

Notice that $\sigma I_{\rho_0,\rho}(x)$ is *not* a field operator in the sense of Subsection 2.1. Hence it does not belong to the class of perturbations considered in Theorem 2.9. Nevertheless, proceeding in the same way as what we did in Subsection 2.2 to deduce Theorem 2.9 (see in particular [FMS, Theorem 1.2 2)] for the verification of Condition 1.10 in the present context), we obtain:

Theorem 2.13. *Assume that Hypotheses* (**W0**) *and* (**V0**) *are satisfied and let* $\rho_0 \in \mathcal{I}'_{\mathrm{N}}(d)$ *and* $\rho \in \mathcal{I}_{\mathrm{N}}(d)$*. Let* J *be a compact interval such that* $\sigma_{\mathrm{pp}}(H^{\mathrm{N}}_{\rho_0}) \cap J = \{\lambda\}$*. Let* P_{ρ_0} *denote the eigenprojection* $P_{\rho_0} = E_{H^{\mathrm{N}}_{\rho_0}}(\{\lambda\})$ *and* $\bar{P}_{\rho_0} = I - P_{\rho_0}$*. Then the conclusions* i) *and* ii) *of Theorem 2.12 hold.*

Observe that, thanks to the unitary dressing transformation U_{ρ_0}, the Fermi Golden Rule condition (2.33) is equivalent to the following one:

$$P'_{\rho_0} I_{\rho_0,\rho}(x) \mathrm{Im}\left((H^{\mathrm{N}'}_{\rho_0} - \lambda - \mathrm{i}0^+)^{-1} \bar{P}'_{\rho_0}\right) I_{\rho_0,\rho}(x) P'_{\rho_0} \geq c P'_{\rho_0}, \tag{2.40}$$

where

$$P'_{\rho_0} := E_{H^{\mathrm{N}'}_{\rho_0}}(\{\lambda\}). \tag{2.41}$$

Hence the conclusions of Theorem 2.13 for $H^{\mathrm{N}}_{\rho_0}$ follow from the corresponding statements for $H^{\mathrm{N}'}_{\rho_0}$.

In Theorem 2.13, ρ_0 and ρ do not belong to the same class of form factors (as far as the infrared singularity is concerned, ρ_0 is allowed to have a more singular infrared behavior than ρ). This is due to the fact that the unitary transformation U_{ρ_0} is ρ_0-dependent, so that the Hamiltonian obtained after the transformation, $H^{\mathrm{N}'}_{\rho_0}$, does not depend linearly on ρ_0. Thus, a perturbation of the form $H^{\mathrm{N}'}_{\rho_0+\sigma\rho} - H^{\mathrm{N}'}_{\rho_0}$ does not belong to the class of linear perturbations considered in this paper (at least as far as the Fermi Golden Rule criterion is concerned). Nevertheless, since the non-linear terms in σ in the expression of $H^{\mathrm{N}'}_{\rho_0+\sigma\rho} - H^{\mathrm{N}'}_{\rho_0}$ act only on the particle Hilbert space $\mathrm{L}^2(\mathbb{R}^{3P})$ (and hence, in particular, commute with the conjugate operator $\tilde{A}_\delta$ of Subsection 2.1), we expect that the method of this paper can be extended to cover the case where both ρ_0 and ρ belong to $\mathcal{I}'_{\mathrm{N}}(d)$.

3. Reduced Limiting Absorption Principle at an eigenvalue

In this section we prove two different "reduced Limiting Absorption Principles". Assuming Conditions 1.3 and 1.7, we shall prove a Limiting Absorption Principle for the reduced unperturbed Hamiltonian $H\bar{P}$ (where $P = E_H(\{\lambda\})$ and $\bar{P} = I - P$). If the stronger Condition 1.9 is satisfied, we shall obtain a Limiting Absorption Principle for the reduced perturbed Hamiltonian $H + \alpha P + \sigma V$ (for some $\alpha > 0$), provided that $V \in \mathcal{V}_2$ and that σ is sufficiently small.

Various versions of the Limiting Absorption Principle appear in [GGM1]. We only give here the following theorem which is a particular case of the results established in [GGM1], stated in a form useful for our context. Recall the notation $\langle A \rangle = (1 + A^* A)^{1/2} = (1 + |A|^2)^{1/2}$.

Theorem 3.1. *Assume that Conditions 1.3 hold. Suppose* $J \subseteq I$ *is a compact interval such that* $\sigma_{\mathrm{pp}}(H) \cap J = \emptyset$*. Let* $S = \{z \in \mathbb{C}, \mathrm{Re}\, z \in J, 0 < |\mathrm{Im}\, z| \leq 1\}$*. For any* $1/2 < s \leq 1$*,*

$$\sup_{z \in S} \|\langle A \rangle^{-s} (H - z)^{-1} \langle A \rangle^{-s}\| < \infty. \tag{3.1}$$

Moreover the function $S \ni z \to \langle A\rangle^{-s}(H-z)^{-1}\langle A\rangle^{-s} \in \mathcal{B}(\mathcal{H})$ is uniformly Hölder continuous of order $s-1/2$. In particular, the limit

$$\langle A\rangle^{-s}(H-\lambda-\mathrm{i}0^+)^{-1}\langle A\rangle^{-s} := \lim_{\epsilon\downarrow 0}\langle A\rangle^{-s}(H-\lambda-\mathrm{i}\epsilon)^{-1}\langle A\rangle^{-s}, \tag{3.2}$$

exists in the norm topology of $\mathcal{B}(\mathcal{H})$ uniformly in $\lambda \in J$, and the map $J \ni \lambda \to \langle A\rangle^{-s}(H-\lambda-\mathrm{i}0^+)^{-1}\langle A\rangle^{-s} \in \mathcal{B}(\mathcal{H})$ is uniformly Hölder continuous of order $s-1/2$.

Remarks 3.2. 1) Strictly speaking, the Mourre estimate formulated in Condition 1.3 (2) together with [GGM1] yield that, for any $\eta \in J$, there is a neighbourhood I_η such that, for any compact interval $J_\eta \subseteq I_\eta$, the Limiting Absorption Principle (3.1) holds with J_η replacing J. The statement of Theorem 3.1 then follows from the compactness of J and a covering argument (see Step II in the proof of Theorem 3.4 below for the use of the same argument).

2) For $s=1$, the result [GGM1, Theorem 3.3] is stronger in that the bound (3.1) holds in a stronger operator topology (given in terms of the Hilbert spaces $\mathcal{G}$ and $\mathcal{G}^*$). For our purposes (3.1) suffices. A similar remark is due for the bounds (3.3) and (3.25) given below. Besides, [GGM1, Theorem 3.3] does not require that A is maximal symmetric, only the generator of a C_0-semigroup.

We shall now obtain a result similar to Theorem 3.1 for a reduced resolvent.

Theorem 3.3. *Assume that Conditions 1.3 and Condition 1.7 hold. Suppose $J \subseteq I$ is a compact interval such that $\sigma_{\mathrm{pp}}(H)\cap J = \{\lambda\}$. Let P denote the eigenprojection $P = E_H(\{\lambda\})$ and let $\bar P = I - P$. Let $S = \{z \in \mathbb{C}, \operatorname{Re} z \in J, 0 < |\operatorname{Im} z| \le 1\}$. For any $1/2 < s \le 1$,*

$$\sup_{z\in S}\|\langle A\rangle^{-s}(H-z)^{-1}\bar P\langle A\rangle^{-s}\| < \infty. \tag{3.3}$$

Moreover there exists $C>0$ such that for all $z, z' \in S$,

$$\left\|\langle A\rangle^{-s}\left((H-z)^{-1}-(H-z')^{-1}\right)\bar P\langle A\rangle^{-s}\right\| \le C|z-z'|^{s-\frac{1}{2}}. \tag{3.4}$$

In particular, the limit

$$\langle A\rangle^{-s}(H-\lambda-\mathrm{i}0^+)^{-1}\bar P\langle A\rangle^{-s} := \lim_{\epsilon\downarrow 0}\langle A\rangle^{-s}(H-\lambda-\mathrm{i}\epsilon)^{-1}\langle A\rangle^{-s}, \tag{3.5}$$

exists in the norm topology of $\mathcal{B}(\mathcal{H})$ uniformly in $\lambda \in J$, and the map $J \ni \lambda \to \langle A\rangle^{-s}(H-\lambda-\mathrm{i}0^+)^{-1}\bar P\langle A\rangle^{-s} \in \mathcal{B}(\mathcal{H})$ is uniformly Hölder continuous of order $s-1/2$.

Proof. It follows from Conditions 1.3 and Condition 1.7 that $\sigma_{\mathrm{pp}}(H)$ is finite in a neighbourhood of λ. Hence, possibly by considering a bigger compact interval, we can assume without loss of generality that λ is included in the interior of J.

Consider Condition 1.3 (2) with $\eta = \lambda$. Let $J_\lambda \subseteq J$ be a compact neighbourhood of λ such that $f_\lambda = 1$ on a neighbourhood of J_λ. Applying Theorem 3.1 on $[\inf J, \inf J_\lambda]$ and using that $P + \bar P = I$, we obtain that

$$\sup_{z\in\mathbb{C}, \operatorname{Re} z\in[\inf J,\inf J_\lambda], 0<|\operatorname{Im} z|\le 1}\|\langle A\rangle^{-s}(H-z)^{-1}\bar P\langle A\rangle^{-s}\| < \infty, \tag{3.6}$$

and that $z \mapsto \langle A\rangle^{-s}(H-z)^{-1}\bar P\langle A\rangle^{-s}$ is Hölder continuous of order $s-1/2$ on $\{z \in \mathbb{C}, \operatorname{Re} z \in [\inf J, \inf J_\lambda], 0 < |\operatorname{Im} z| \le 1\}$. The same holds with $[\sup J_\lambda, \sup J]$ replacing $[\inf J, \inf J_\lambda]$. Therefore, to conclude the proof, one can verify that it is sufficient to establish the statement of Theorem 3.3 with J replaced by J_λ. We can follow the proof of [GGM1, Theorem 3.3]. We emphasize the differences with [GGM1] and refer the reader to that paper for more details.

We obtain from (1.8) with $\eta = \lambda$ that

$$M + R \geq 2^{-1}c_0 I - C_2 f_\lambda^\perp(H)^2\langle H\rangle - f_\lambda(H) K f_\lambda(H). \tag{3.7}$$

Since $f_\lambda(H)$ goes strongly to P as $\lambda \to 0$, we obtain

$$M + R \geq 3^{-1}c_0 I - C_2 f_\lambda^\perp(H)^2\langle H\rangle - C_3 P, \tag{3.8}$$

which is valid if the support of f_λ is sufficiently close to λ. Applying $\bar{P}$ from the left and from the right in (3.8) yields

$$\bar{P}(M + R)\bar{P} \geq 3^{-1}c_0\bar{P} - C_2\bar{P} f_\lambda^\perp(H)^2\langle H\rangle\bar{P}. \tag{3.9}$$

Next, we can mimic the proof of [GGM1, Theorem 3.3] using (3.9) and the following slightly different constructions: In Subsection 3.4 of [GGM1] the operator H_ϵ (related to the one from the seminal paper [Mo]) is taken as $H_\epsilon = H - \mathrm{i}\epsilon H'$. Notice that here and henceforth we can assume without loss that H' is closed (possibly by taking the closure).

We propose to take

$$\bar{H}_\epsilon := H - \mathrm{i}\epsilon\bar{P}H'\bar{P}, \tag{3.10}$$

with domain $\mathcal{D}(\bar{H}_\epsilon) := \mathcal{D}(H) \cap \mathcal{D}(M) \cap \mathrm{Ran}(\bar{P})$ on the Hilbert space $\bar{\mathcal{H}} := \bar{P}\mathcal{H}$. It follows from the assumption $\mathrm{Ran}(P) \subseteq \mathcal{D}(M)$ that $\bar{H}_\epsilon$ is well-defined and commutes with $\bar{P}$. Similarly, denoting $H_{\bar{P}} := H|_{\mathcal{D}(H)\cap\mathrm{Ran}(\bar{P})}$ and $M_{\bar{P}} := (\bar{P}M\bar{P})|_{\mathcal{D}(M)\cap\mathrm{Ran}(\bar{P})}$, the assumption that $\mathrm{Ran}(P) \subseteq \mathcal{D}(M)$ implies that $H_{\bar{P}}$ and $M_{\bar{P}}$ are self-adjoint. Moreover, since $H \in \mathrm{C}^1_{\mathrm{Mo}}(M)$ by Condition 1.3 (1), one verifies that $H_{\bar{P}} \in \mathrm{C}^1_{\mathrm{Mo}}(M_{\bar{P}})$, and hence in particular that $\mathcal{D}(H_{\bar{P}})\cap\mathcal{D}(M_{\bar{P}})$ is a core for $M_{\bar{P}}$. One also verifies that $\bar{P}H'\bar{P}$ coincides with the closure of $M_{\bar{P}} + R_{\bar{P}}$ defined on $\mathcal{D}(M_{\bar{P}}) \cap \mathcal{D}(H_{\bar{P}})$, where $R_{\bar{P}} := (\bar{P}R\bar{P})|_{\mathcal{D}(H)\cap\mathrm{Ran}(\bar{P})}$. Therefore the assumptions of [GGM1, Theorem 2.25] are satisfied (see [GGM1, Lemma 2.26]), which implies that $\bar{H}_\epsilon$ is closed, densely defined, and $\bar{H}_\epsilon^* = \bar{H}_{-\epsilon}$.

Let $\bar{\mathcal{G}} := \mathcal{G}\cap\mathrm{Ran}(\bar{P})$. By Conditions 1.3 and the fact that $\mathrm{Ran}(P) \subseteq \mathcal{D}(M)$, $\bar{H}_\epsilon$ extends to a bounded operator: $\bar{H}_\epsilon \in \mathcal{B}(\bar{\mathcal{G}};\bar{\mathcal{G}}^*)$. Mimicking [GGM1, Subsection 3.4] (replacing $u \in \mathcal{D}(H_\epsilon)$ in Lemmata 3.9 and 3.10 by $u \in \mathcal{D}(\bar{H}_\epsilon)$, and using (3.9)), one can show that there exists ϵ_0 such that for all $0 < |\epsilon| \leq \epsilon_0$, for all $z = \eta + \mathrm{i}\mu$ with $\eta \in J_\lambda$ and $\epsilon\mu > 0$, $\bar{H}_\epsilon - z$ is invertible with bounded inverse $\bar{R}_\epsilon(z) \in \mathcal{B}(\bar{\mathcal{H}};\mathcal{D}(\bar{H}_\epsilon))$. Furthermore $\bar{R}_\epsilon(z)$ extends to a bounded operator in $\mathcal{B}(\bar{\mathcal{G}}^*;\bar{\mathcal{G}})$ which coincides with the inverse of $(\bar{H}_\epsilon - z) \in \mathcal{B}(\bar{\mathcal{G}};\bar{\mathcal{G}}^*)$, and which satisfies

$$\|\bar{R}_\epsilon(z)\|_{\mathcal{B}(\bar{\mathcal{G}};\bar{\mathcal{G}}^*)} \leq \frac{C}{|\epsilon|}, \tag{3.11a}$$

$$\|\bar{R}_\epsilon(z)v\|_{\bar{\mathcal{G}}} \leq \frac{C}{|\epsilon|^{\frac{1}{2}}}\left(\left|(v, \bar{R}_\epsilon(z)v)\right|^{\frac{1}{2}} + \|v\|_{\bar{\mathcal{H}}}\right) \text{ for all } v \in \bar{\mathcal{H}}, \tag{3.11b}$$

$$\text{s-}\lim_{\epsilon\to 0^\pm} \bar{R}_\epsilon(z) = (H_{\bar{P}} - z)^{-1} \in \mathcal{B}(\bar{\mathcal{H}}), \tag{3.11c}$$

(see [GGM1, Proposition 3.11 and Lemma 3.12]).

Let

$$\rho_\epsilon := \langle\epsilon A\rangle^{s-1}\langle A\rangle^{-s} = (1 + \epsilon^2|A|^2)^{\frac{s-1}{2}}(1 + |A|^2)^{-\frac{s}{2}}, \tag{3.12}$$

with $1/2 < s \leq 1$. Instead of looking at the expectation of the resolvent $R_\epsilon(z) := (H_\epsilon - z)^{-1}$, $\epsilon \neq 0$, we propose to show a differential inequality for the quantity

$$F_\epsilon(z) := \left\langle \rho_\epsilon u, \bar{P}\bar{R}_\epsilon(z)\bar{P}\rho_\epsilon u\right\rangle; \tag{3.13}$$

here $u \in \mathcal{H}$, so that $\rho_\epsilon u \in \mathcal{D}(A) \subseteq \mathcal{D}(A^*)$. Note that the assumption $\mathrm{Ran}(P) \subseteq \mathcal{D}(A)$ implies that $\bar{P}$ leaves $\mathcal{D}(A)$ invariant.

In the same way as in the proof of Theorem 3.3 in [GGM1], one can verify that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon}F_\epsilon(z) = &\left\langle \left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}\rho_\epsilon\right) u, \bar{R}_\epsilon(z)\bar{P}\rho_\epsilon u\right\rangle + \left\langle \rho_\epsilon u, \bar{R}_\epsilon(z)\bar{P}\left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}\rho_\epsilon\right) u\right\rangle \\ &+ \left\langle \bar{R}^*_\epsilon(z)\bar{P}\rho_\epsilon u, A\rho_\epsilon u\right\rangle - \left\langle A\rho_\epsilon u, \bar{R}_\epsilon(z)\bar{P}\rho_\epsilon u\right\rangle \\ &+ \epsilon\left\langle \bar{R}^*_\epsilon(z)\bar{P}\rho_\epsilon u, \left(H'PA - APH' - H''\right)\bar{R}_\epsilon(z)\bar{P}\rho_\epsilon u\right\rangle, \end{aligned} \tag{3.14}$$

where

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\rho_\epsilon := (s-1)\epsilon|A|^2\langle\epsilon A\rangle^{s-3}\langle A\rangle^{-s} = (s-1)\epsilon|A|^2(1+\epsilon^2|A|^2)^{\frac{s-3}{2}}(1+|A|^2)^{-\frac{s}{2}}. \tag{3.15}$$

In particular it follows from the Spectral Theorem that

$$\|\mathrm{d}\rho_\epsilon/\mathrm{d}\epsilon\| \le C|\epsilon|^{s-1} \quad \text{and} \quad \|A\rho_\epsilon\| \le C|\epsilon|^{s-1}. \tag{3.16}$$

Next it follows from Conditions 1.3 and Condition 1.7 that $PA \in \mathcal{B}(\mathcal{G}^*)$, $AP \in \mathcal{B}(\mathcal{G})$, and hence that

$$H'PA - APH' - H'' \in \mathcal{B}(\mathcal{G};\mathcal{G}^*).$$

This implies

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\epsilon}F_\epsilon(z)\right| \le &C_1|\epsilon|^{s-1}\|u\|_{\bar{\mathcal{H}}}\left(\|\bar{R}_\epsilon(z)\bar{P}\rho_\epsilon u\|_{\bar{\mathcal{H}}} + \|\bar{R}^*_\epsilon(z)\bar{P}\rho_\epsilon u\|_{\bar{\mathcal{H}}}\right) \\ &+ C_2|\epsilon|\left(\|\bar{R}_\epsilon(z)\bar{P}\rho_\epsilon u\|_{\bar{\mathcal{G}}}\|\bar{R}^*_\epsilon(z)\bar{P}\rho_\epsilon u\|_{\bar{\mathcal{G}}}\right).\end{aligned} \tag{3.17}$$

By (3.11b) and since $\|\cdot\|_{\bar{\mathcal{H}}} \le \|\cdot\|_{\bar{\mathcal{G}}}$, we obtain

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}\epsilon}F_\epsilon(z)\right| &\le C_3|\epsilon|^{s-1}\|u\|_{\bar{\mathcal{H}}}|\epsilon|^{-\frac{1}{2}}\left(|F_\epsilon(z)|^{\frac{1}{2}} + \|\bar{P}\rho_\epsilon u\|_{\bar{\mathcal{H}}}\right) \\ &\quad + C_4|\epsilon|\Big(|\epsilon|^{-\frac{1}{2}}\left(|F_\epsilon(z)|^{\frac{1}{2}} + \|\bar{P}\rho_\epsilon u\|_{\bar{\mathcal{H}}}\right)\Big)^2 \\ &\le C_5|\epsilon|^{s-\frac{3}{2}}\left(|F_\epsilon(z)| + \|u\|^2_{\bar{\mathcal{H}}}\right),\end{aligned} \tag{3.18}$$

for $0 < |\epsilon| \le \epsilon_0$. Applying Gronwall's lemma, this yields

$$|F_\epsilon(z)| \le C_6\|u\|^2_{\bar{\mathcal{H}}}, \tag{3.19}$$

which combined with (3.11c) gives

$$\sup_{z\in\mathbb{C}, \mathrm{Re}\, z\in J_\lambda, 0<|\mathrm{Im}\, z|\le 1} \|\langle A\rangle^{-s}(H-z)^{-1}\bar{P}\langle A\rangle^{-s}\| < \infty. \tag{3.20}$$

In order to prove the Hölder continuity in z, we use that, for $0 < \epsilon_1 < \epsilon_0$,

$$\begin{aligned}F_0(z) - F_0(z') = &-\int_0^{\epsilon_1}\frac{\mathrm{d}}{\mathrm{d}\epsilon}(F_\epsilon(z) - F_\epsilon(z'))\mathrm{d}\epsilon \\ &-\int_{\epsilon_1}^{\epsilon_0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}(F_\epsilon(z) - F_\epsilon(z'))\mathrm{d}\epsilon + (F_{\epsilon_0}(z) - F_{\epsilon_0}(z')).\end{aligned} \tag{3.21}$$

It follows from (3.18) and (3.19) that

$$\Big|\int_0^{\epsilon_1}\frac{\mathrm{d}}{\mathrm{d}\epsilon}(F_\epsilon(z) - F_\epsilon(z'))\mathrm{d}\epsilon\Big| \le C_7\epsilon_1^{s-\frac{1}{2}}\|u\|^2. \tag{3.22}$$

Moreover, using the first resolvent equation together with (3.14), (3.11a), (3.11b), (3.16) and (3.19), we obtain

$$\Big|\frac{\mathrm{d}}{\mathrm{d}\epsilon}(F_\epsilon(z)-F_\epsilon(z'))\Big| \le C_8|\epsilon|^{s-\frac{5}{2}}|z-z'|\cdot\|u\|^2,$$

which implies

$$\Big|\int_{\epsilon_1}^{\epsilon_0}\frac{\mathrm{d}}{\mathrm{d}\epsilon}(F_\epsilon(z)-F_\epsilon(z'))\mathrm{d}\epsilon\Big| \le C_9\epsilon_1^{s-\frac{3}{2}}|z-z'|\cdot\|u\|^2. \tag{3.23}$$

Finally, the first resolvent equation and (3.11a) give

$$\big|(F_{\epsilon_0}(z)-F_{\epsilon_0}(z'))\big| \le C(\epsilon_0)|z-z'|\cdot\|u\|^2, \tag{3.24}$$

for some positive constant $C(\epsilon_0)$ depending on ϵ_0. Taking $\epsilon_1=|z-z'|$, Equation (3.4) follows from (3.21)–(3.24). □

We have the following stronger result if Condition 1.9 is further assumed.

Theorem 3.4. *Assume that Conditions 1.3 and Condition 1.9 hold. Suppose $J\subseteq I$ is a compact interval such that $\sigma_{\mathrm{pp}}(H)\cap J\subseteq\{\lambda\}$. Let $P=E_H(\{\lambda\})$ and $V\in\mathcal{V}_2$. For $\sigma\in\mathbb{R}$, define $H_\sigma:=H+\sigma V$ and $\bar H_\sigma:=H_\sigma+\alpha_J P$, where $\alpha_J\in\mathbb{R}$ is fixed such that $\alpha_J>\sup J-\inf J$. Let $S=\{z\in\mathbb{C},\mathrm{Re}\,z\in J, 0<|\mathrm{Im}\,z|\le 1\}$. For all $1/2<s\le 1$, there exists $\sigma_0>0$ such that for all $|\sigma|\le\sigma_0$,*

$$\sup_{z\in S}\|\langle A\rangle^{-s}(\bar H_\sigma-z)^{-1}\langle A\rangle^{-s}\|<\infty. \tag{3.25}$$

Moreover there exists $C>0$ such that for all $\sigma,\sigma'\in[-\sigma_0,\sigma_0]$, for all $z,z'\in S$,

$$\big\|\langle A\rangle^{-s}\left((\bar H_\sigma-z)^{-1}-(\bar H_{\sigma'}-z')^{-1}\right)\langle A\rangle^{-s}\big\| \le C\left(|\sigma-\sigma'|^{s-\frac12}+|z-z'|^{s-\frac12}\right). \tag{3.26}$$

Remarks 3.5. 1) In the case $\sigma_{\mathrm{pp}}(H)\cap J=\emptyset$, we have $P=0$ and hence $\bar H_\sigma=H_\sigma$. Of course, Condition 1.9 is not required in this case.

2) The assumption that $\alpha_J>\sup J-\inf J$ implies that $H+\alpha_J P$ does not have eigenvalues in J.

3) Equations (3.25)–(3.26) with $\sigma=\sigma'=0$ yield that

$$\sup_{z\in S}\|\langle A\rangle^{-s}(H-z)^{-1}\bar P\langle A\rangle^{-s}\|<\infty, \tag{3.27}$$

and that $z\mapsto\langle A\rangle^{-s}(H-z)^{-1}\bar P\langle A\rangle^{-s}$ is Hölder continuous of order $s-1/2$ on S. Hence we recover the Limiting Absorption Principles of Theorems 3.1 and 3.3.

Proof of Theorem 3.4 Considering the Mourre estimate, Condition 1.3 (2), for any $\eta\in J$, we denote by $J_\eta\subseteq I$ a compact neighbourhood of η such that $f_\eta=1$ on a neighbourhood of J_η.

Step 1 Let us prove that, for any $\eta\in J$, there exists $\sigma_\eta>0$ such that for all $|\sigma|\le\sigma_\eta$,

$$\sup_{z\in\mathbb{C},\mathrm{Re}\,z\in J_\eta,0<|\mathrm{Im}\,z|\le1}\|\langle A\rangle^{-s}(\bar H_\sigma-z)^{-1}\langle A\rangle^{-s}\|<\infty, \tag{3.28}$$

and that the function $(\sigma,z)\mapsto\langle A\rangle^{-s}(\bar H_\sigma-z)^{-1}\langle A\rangle^{-s}$ is Hölder continuous of order $s-1/2$ in σ and z on $[-\sigma_\eta,\sigma_\eta]\times\{z\in\mathbb{C},\mathrm{Re}\,z\in J_\eta,0<|\mathrm{Im}\,z|\le1\}$.

Let $\bar H:=H+\alpha_J P$. Condition 1.7 implies that $[P,\mathrm{i}A]^0$ extends to a compact operator. Since $HP=\lambda P$ and $H\bar P=\bar H\bar P$, we have

$$f_\eta^\perp(H)^2\langle H\rangle = f_\eta^\perp(\bar H)^2\langle\bar H\rangle+f_\eta^\perp(\lambda)^2\langle\lambda\rangle P-f_\eta^\perp(\lambda+\alpha_J)^2\langle\lambda+\alpha_J\rangle P. \tag{3.29}$$

Using that the second and third terms in the right-hand-side of (3.29) are compact, the Mourre estimate (1.8) yields

$$M + \big(R + \alpha_J[P, \mathrm{i}A]^0\big) \geq c_0 I - C_0 f_\eta^\perp(\bar{H})^2\langle\bar{H}\rangle - K_0', \tag{3.30}$$

where K_0' is compact. Since $\eta \notin \sigma_{\mathrm{pp}}(\bar{H})$ (see Remark 3.5 2)), we can put $K_0' = 0$ provided we choose the function f_η supported in a sufficiently small interval containing η. We get

$$M + \big(R + \alpha_J[P, \mathrm{i}A]^0\big) \geq 2^{-1} c_0 I - C_1 f_\eta^\perp(\bar{H})^2\langle\bar{H}\rangle. \tag{3.31}$$

The estimate (3.31) is stable under perturbation from the class $\mathcal{V}_1$. In particular (and more precisely) there exists $\sigma_\eta > 0$ such that if $|\sigma| \leq \sigma_\eta$, then

$$M + \big(R + \sigma V' + \alpha_J[P, \mathrm{i}A]^0\big) \geq 3^{-1} c_0 I - C_2 f_\eta^\perp(\bar{H}_\sigma)^2\langle\bar{H}_\sigma\rangle. \tag{3.32}$$

Indeed, since $V \in \mathcal{V}_1$, we have that

$$\pm V' \leq C_3\langle H\rangle + C_4 \leq C_5 + C_6 f_\eta^\perp(\bar{H})\langle\bar{H}\rangle f_\eta^\perp(\bar{H}), \tag{3.33}$$

and

$$\begin{aligned} f_\eta^\perp(\bar{H})\langle\bar{H}\rangle f_\eta^\perp(\bar{H}) &\leq C_7 f_\eta^\perp(\bar{H})\langle\bar{H}_\sigma\rangle f_\eta^\perp(\bar{H}) \\ &\leq C_7 f_\eta^\perp(\bar{H}_\sigma)\langle\bar{H}_\sigma\rangle f_\eta^\perp(\bar{H}_\sigma) + C_8|\sigma|. \end{aligned} \tag{3.34}$$

The first inequality in (3.34) follows from elementary interpolation while the second inequality follows, for instance, from the Helffer-Sjöstrand functional calculus.

We set for shortness $H_\sigma' := H' + \sigma V'$, $H_\sigma'' := H'' + \sigma V''$, $P' := [P, \mathrm{i}A]^0$ and $P'' := [P', \mathrm{i}A]^0$. Remark that Conditions 1.3, Condition 1.9 and the assumption $V \in \mathcal{V}_2$ imply that

$$H_\sigma'',\ P'',\ H_\sigma'' + \alpha_J P'' \in \mathcal{B}(\mathcal{G}; \mathcal{G}^*).$$

Note that equation (3.32) can be written

$$H_\sigma' + \alpha_J P' \geq 3^{-1} c_0 I - C_2 f_\eta^\perp(\bar{H}_\sigma)^2\langle\bar{H}_\sigma\rangle. \tag{3.35}$$

We emphasize that the constant C_2 is independent of z and σ.

To prove (3.28), we can proceed as in the proof of Theorem 3.3, using (3.35) instead of (3.9), and replacing $\bar{H}_\epsilon$ and $F_\epsilon(z)$ in (3.10) and (3.13) respectively by

$$\bar{H}_{\sigma,\epsilon} := \bar{H}_\sigma - \mathrm{i}\epsilon(H_\sigma' + \alpha_J P'), \tag{3.36}$$

and

$$F_{\sigma,\epsilon}(z) := \langle \rho_\epsilon u, \bar{R}_{\sigma,\epsilon}(z)\rho_\epsilon u\rangle. \tag{3.37}$$

Here we have set

$$\bar{R}_{\sigma,\epsilon}(z) := (\bar{H}_{\sigma,\epsilon} - z)^{-1} \tag{3.38}$$

and, as before, $\rho_\epsilon = \langle\epsilon A\rangle^{s-1}\langle A\rangle^{-s}$. Notice that, by [GGM1, Theorem 2.25 and Lemma 2.26], $\bar{H}_{\sigma,\epsilon}$ is closed, densely defined and satisfies $\bar{H}_{\sigma,\epsilon}^* = \bar{H}_{\sigma,-\epsilon}$. Moreover, following [GGM1, Subsection 3.4], one can indeed verify that there exists ϵ_0 such that for all $0 < |\epsilon| \leq \epsilon_0$ and

$z = \eta' + \mathrm{i}\mu$ with $\eta' \in J_\eta$ and $\epsilon\mu > 0$, $\bar{H}_{\sigma,\epsilon} - z$ is invertible with bounded inverse $\bar{R}_{\sigma,\epsilon}(z)$ satisfying properties similar to (3.11a)–(3.11c). We can compute:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\epsilon} F_{\sigma,\epsilon}(z) = & \left\langle \left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}\rho_\epsilon\right) u, \bar{R}_{\sigma,\epsilon}(z)\rho_\epsilon u \right\rangle + \left\langle \rho_\epsilon u, \bar{R}_{\sigma,\epsilon}(z) \left(\frac{\mathrm{d}}{\mathrm{d}\epsilon}\rho_\epsilon\right) u \right\rangle \\ & + \left\langle \bar{R}^*_{\sigma,\epsilon}(z)\rho_\epsilon u, A\rho_\epsilon u\right\rangle - \langle A\rho_\epsilon u, \bar{R}_{\sigma,\epsilon}(z)\rho_\epsilon u\rangle \\ & - \epsilon \left\langle \bar{R}^*_{\sigma,\epsilon}(z)\rho_\epsilon u, \left(H''_\sigma + \alpha_J P''\right) \bar{R}_{\sigma,\epsilon}(z)\rho_\epsilon u \right\rangle . \end{aligned} \tag{3.39}$$

We obtain as in (3.18) that

$$\left| \frac{\mathrm{d}}{\mathrm{d}\epsilon} F_{\sigma,\epsilon}(z) \right| \le C_9 |\epsilon|^{s-\frac{3}{2}} \|u\|^2. \tag{3.40}$$

Estimate (3.25) (with J_λ in place of J) and the Hölder continuity in z then follow as in the proof of Theorem 3.3.

It remains to prove the Hölder continuity in σ. We follow again the proof of Theorem 3.3. For $0 < \epsilon_1 < \epsilon_0$, we have

$$\begin{aligned} F_{\sigma,0}(z) - F_{\sigma',0}(z) = & - \int_0^{\epsilon_1} \frac{\mathrm{d}}{\mathrm{d}\epsilon}(F_{\sigma,\epsilon}(z) - F_{\sigma',\epsilon}(z))\mathrm{d}\epsilon \\ & - \int_{\epsilon_1}^{\epsilon_0} \frac{\mathrm{d}}{\mathrm{d}\epsilon}(F_{\sigma,\epsilon}(z) - F_{\sigma',\epsilon}(z))\mathrm{d}\epsilon + (F_{\sigma,\epsilon_0}(z) - F_{\sigma',\epsilon_0}(z)). \end{aligned} \tag{3.41}$$

The first term in the right-hand-side of (3.41) is estimated thanks to (3.40), which gives

$$\Big| \int_0^{\epsilon_1} \frac{\mathrm{d}}{\mathrm{d}\epsilon}(F_{\sigma,\epsilon}(z) - F_{\sigma',\epsilon}(z))\mathrm{d}\epsilon \Big| \le C_{10} \epsilon_1^{s-\frac{1}{2}} \|u\|^2. \tag{3.42}$$

As for the second and third terms on the right-hand-side of (3.41), we use that, by the second resolvent equation,

$$\bar{R}_{\sigma,\epsilon}(z) - \bar{R}_{\sigma',\epsilon}(z) = -(\sigma - \sigma')\bar{R}_{\sigma,\epsilon}(z)(V - \mathrm{i}\epsilon V')\bar{R}_{\sigma',\epsilon}(z).$$

Since V and V' are H-bounded by assumption, this implies in the same way as in the proof of (3.23) and (3.24) that

$$\Big| \int_{\epsilon_1}^{\epsilon_0} \frac{\mathrm{d}}{\mathrm{d}\epsilon}(F_{\sigma,\epsilon}(z) - F_{\sigma',\epsilon}(z))\mathrm{d}\epsilon \Big| \le C_{11} \epsilon_1^{s-\frac{3}{2}} |\sigma - \sigma'| \cdot \|u\|^2, \tag{3.43}$$

and

$$\left| (F_{\sigma,\epsilon_0}(z) - F_{\sigma',\epsilon_0}(z)) \right| \le C(\epsilon_0) |\sigma - \sigma'| \cdot \|u\|^2. \tag{3.44}$$

The Hölder continuity in σ follows from (3.41)–(3.44) by choosing $\epsilon_1 = |\sigma - \sigma'|$.

Step 2 Since J is compact, it follows from Step 1 and a covering argument that there exist $\eta_1, \dots, \eta_l$ (with $l < \infty$) such that $J \subseteq J_{\eta_1} \cup \dots \cup J_{\eta_l}$. Taking $\sigma_0 = \min(\sigma_{\eta_1}, \dots, \sigma_{\eta_l})$, Equation (3.25) and the Hölder continuity in σ follow. The Hölder continuity in z is a straightforward consequence of the fact that

$$\sum_{n=1}^{l} (a_n)^{s-\frac{1}{2}} \le l^{\frac{3}{2}-s} \big(\sum_{n=1}^{l} a_n \big)^{s-\frac{1}{2}}, \tag{3.45}$$

for any sequence of positive numbers $(a_n)_{n=1,\dots,l}$, and $1/2 < s \le 1$. □

4. Upper semicontinuity of point spectrum

In this section we study upper semicontinuity of the point spectrum of H. The main result is Theorem 1.14 proven below.

Let us begin with stating a consequence of Theorem 3.4, which shows that if the unperturbed Hamiltonian does not have eigenvalues in a compact interval, the same holds for the perturbed Hamiltonian (provided that the perturbation V belongs to $\mathcal{V}_2$).

Corollary 4.1. *Assume that Conditions 1.3 hold. Let $J \subseteq I$ be a compact interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \emptyset$. Let $V \in \mathcal{V}_2$. There exists $\sigma_0 > 0$ such that for any $|\sigma| \le \sigma_0$,*

$$\sigma_{\mathrm{pp}}(H + \sigma V) \cap J = \emptyset. \tag{4.1}$$

The statement of Corollary 4.1 remains true under the weaker assumption that $V \in \mathcal{V}_1$, provided that a priori eigenstates of $H+\sigma V$ belong to $\mathcal{D}(M^{1/2})$. This is a consequence of the Mourre estimate established in the proof of Theorem 3.4 (see (3.32)), together with the virial property that $\langle \psi, (H' + \sigma V')\psi \rangle = 0$ which holds for any eigenstate ψ of $H + \sigma V$ satisfying $\psi \in \mathcal{D}(M^{1/2})$ (see Remark 1.4 3)). Hence we have the following:

Corollary 4.2. *Assume that Conditions 1.3 hold. Let $J \subseteq I$ be a compact interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \emptyset$. Let $V \in \mathcal{V}_1$. There exists $\sigma_0 > 0$ such that for any $|\sigma| \le \sigma_0$, the following holds: Suppose that any eigenstate ψ of $H + \sigma V$ associated to an eigenvalue $\lambda \in J$ satisfies $\psi \in \mathcal{D}(M^{1/2})$, then*

$$\sigma_{\mathrm{pp}}(H + \sigma V) \cap J = \emptyset. \tag{4.2}$$

We now turn to the proof of Theorem 1.14. Here we need Condition 1.10 and that $V \in \mathcal{B}_{1,\gamma}$ in addition to Conditions 1.3.

Proof of Theorem 1.14 Let $\lambda \in I$ and $J \subseteq I$ as in the statement of the theorem.
Step 1 Let us prove that, for any $\eta \in J$, there exist $\beta_\eta > 0$ and $\gamma_\eta > 0$ such that, for $\|V\|_1 \le \gamma_\eta$, the total multiplicity of the eigenvalues of $H + V$ in $(\eta - \beta_\eta, \eta + \beta_\eta)$ is at most $\dim \mathrm{Ker}(H - \eta)$.

If η is an eigenvalue, we proceed as in [AHS, Section 2] introducing the (finite rank) eigenprojection, say P, corresponding to this eigenvalue and the auxiliary operator $\bar{H} = H + \alpha_J P$. Here $\alpha_J > \sup J - \inf J$ as in Theorem 3.4. Then in the same way as in (3.32), for $\|V\|_1 \le \gamma_\eta$ with $\gamma_\eta > 0$ small enough, we have that

$$M + \big(R + \alpha_J[P, \mathrm{i}A]^0 + [V, \mathrm{i}A]^0\big) \ge 3^{-1}c_0 I - C_1 f_\eta^\perp(\bar{H} + V)^2 \langle \bar{H} + V \rangle, \tag{4.3}$$

where $f_\eta \in \mathrm{C}_0^\infty(\mathbb{R})$ is such that $0 \le f_\eta \le 1$ and $f_\eta = 1$ in a neighbourhood of η. Let us in the following agree on the convention that $P = 0$ and $\bar{H} = H$ if $\eta \notin \sigma_{\mathrm{pp}}(H)$. Then (4.3) holds no matter whether η is an eigenvalue or not (provided $\|V\|_1$ is sufficiently small and that the support of f_η is chosen sufficiently close to η).

Now, it suffices to follow the proof of [AHS, Theorem 2.5], combining Condition 1.10 and (4.3). More precisely, let m be the multiplicity of η and let us assume that $H + V$ has eigenvalues (η_j), $j = 1, \dots, m_1$, of total multiplicity $m_1 > m$, located in $(\eta - \beta_\eta, \eta + \beta_\eta) \subseteq I$. Let (ψ_j), $j = 1, \dots, m_1$, be an orthonormal set of eigenvectors, ψ_j being associated with η_j. Consider a linear combination $\psi = \sum_j a_j \psi_j$ such that $\|\psi\| = 1$ and $P\psi = 0$. Since $V \in \mathcal{B}_{1,\gamma}$, it follows from Condition 1.10 that $\psi \in \mathcal{D} \cap D(A)$, whence (4.3) together with Remark 1.4 3) yields

$$\begin{aligned}3^{-1}c_0 &\le \left\langle \psi, (M + R + \alpha_J[P, \mathrm{i}A]^0 + [V, \mathrm{i}A]^0)\psi \right\rangle + C_1 \left\| f_\eta^\perp(\bar{H} + V)\langle \bar{H} + V\rangle^{1/2}\psi \right\|^2 \\ &= \mathrm{i} \left\langle (\bar{H} + V - \eta)\psi, A\psi \right\rangle - \mathrm{i} \left\langle A\psi, (\bar{H} + V - \eta)\psi \right\rangle + C_1 \left\| f_\eta^\perp(\bar{H} + V)\langle \bar{H} + V\rangle^{1/2}\psi \right\|^2 \\ &\le \beta_\eta \left(2\|A\psi\| + C_2\beta_\eta \right). \end{aligned} \tag{4.4}$$

In the second inequality, we used that

$$\left\|(\bar{H} + V - \eta)\psi\right\| = \|\sum_j a_j(\eta_j - \eta)\psi_j\| \le \beta_\eta, \tag{4.5}$$

and hence also that that $\left\| f_\eta^\perp(\bar{H} + V)\langle \bar{H} + V\rangle^{1/2}\psi \right\| \le C_3\beta_\eta$ by the Spectral Theorem, where the constant C_3 depends on $\operatorname{supp}(f_\eta)$. By Condition 1.10, we obtain a contradiction provided that β_η is chosen sufficiently small.

Step 2 Let us prove that the total multiplicity of the eigenvalues of $H + V$ in J is at most $\dim \operatorname{Ker}(H - \lambda)$.

It follows from Step 1 that, for any $\eta \in [\inf J, \lambda - \beta_\lambda] \cup [\lambda + \beta_\lambda, \sup J]$, there exist $\beta_\eta > 0$ and $\gamma_\eta > 0$ such that, for $\|V\|_1 \le \gamma_\eta$, $H + V$ does not have eigenvalues in $(\eta - \beta_\eta, \eta + \beta_\eta)$. Since $[\inf J, \lambda - \beta_\lambda] \cup [\lambda + \beta_\lambda, \sup J]$ is compact, it follows from a covering argument that there exist $\eta_1, \ldots, \eta_l$ such that

$$[\inf J, \lambda - \beta_\lambda] \cup [\lambda + \beta_\lambda, \sup J] \subset \bigcup_{j=1}^{l} (\eta_j - \beta_{\eta_j}, \eta_j + \beta_{\eta_j}). \tag{4.6}$$

Hence, for $\|V\|_1 \le \min(\gamma_{\eta_1}, \ldots, \gamma_{\eta_l})$, $H + V$ does not have eigenvalues in $[\inf J, \lambda - \beta_\lambda] \cup [\lambda + \beta_\lambda, \sup J]$. Applying Step 1 again with $\eta = \lambda$, this concludes the proof. □

The next proposition is a consequence of Theorem 1.14. It will be used in Section 5.

Proposition 4.3. *Assume that Conditions 1.3 and Condition 1.10 hold. Suppose $\lambda \in \sigma_{\mathrm{pp}}(H)$ and that $J \subseteq I$ is a compact interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \{\lambda\}$. Let $P = E_H(\{\lambda\})$, $\bar{P} = I - P$ and $P_{V,J} = E_{(H+V)_{\mathrm{pp}}}(J)$ for any $V \in \mathcal{V}_1$ (with sufficiently small norm). Then for any sequence $V^{(n)} \in \mathcal{B}_{1,\gamma}$ such that $\|V^{(n)}\|_1 \to 0$,*

$$\|\bar{P}P_{V^{(n)},J}\| \to 0. \tag{4.7}$$

One of the following two alternatives i) or ii) holds:

- i) *There exists $0 < \gamma' \le \gamma$ such that if $V \in \mathcal{B}_{1,\gamma}$ and $0 \ne \|V\|_1 \le \gamma'$, then the operator $H + V$ does not have eigenvalues in J.*
- ii) *There exists a sequence of operators $V_n \in \mathcal{B}_{1,\gamma}$ with $0 \ne \|V_n\|_1 \to 0$ and a sequence of normalized eigenstates, $(H + V_n - \lambda_n)\psi_n = 0$, with eigenvalues $\lambda_n \to \lambda$, such that for some $\psi_\infty \in \operatorname{Ran}(P)$ we have $\|\psi_n - \psi_\infty\| \to 0$.*

Proof. If (4.7) fails there exist an $\epsilon > 0$, a sequence of elements $V^{(n)} \in \mathcal{B}_{1,\gamma}$ with $0 \ne \|V^{(n)}\|_1 \to 0$, a linear combination of eigenstates of $H + V^{(n)}$, viz. $\psi^{(n)} = \sum_{j \le m(n)} a_j^{(n)} \psi_j^{(n)}$, such that

$$\|\psi^{(n)}\| \le 1 \text{ and } \|\bar{P}\psi^{(n)}\| > \epsilon. \tag{4.8}$$

Here $m(n) \le \dim \operatorname{Ran}(P)$ specifies the dimension of the range of $P_{V^{(n)},J}$.

Due to Theorem 1.14 the corresponding eigenvalues, say $\lambda_j^{(n)}$, concentrate at λ. More precisely

$$\max_{j\le m(n)} |\lambda_j^{(n)} - \lambda| \to 0 \text{ for } n \to \infty. \tag{4.9}$$

In particular we have

$$\max_{j\le m(n)} \|(H-\lambda)\psi_j^{(n)}\| \to 0, \text{ and } \max_{j\le m(n)} \|f_\lambda^\perp(H)\psi_j^{(n)}\| \to 0, \tag{4.10}$$

and therefore also

$$\|(H-\lambda)\psi^{(n)}\| \to 0, \text{ and } \|f_\lambda^\perp(H)\psi^{(n)}\| \to 0. \tag{4.11}$$

Next by the Banach-Alaoglu Theorem [Yo, Theorem 1 on p. 126] we can assume that there exists the weak limit $\psi_\infty := \mathrm{w}-\lim \psi^{(n)}$ (by passing to a subsequence and change notation). From the first identity of (4.11) we learn that $\psi_\infty \in \mathrm{Ran}(P)$. Consequently

$$\mathrm{w}-\lim \bar P\psi^{(n)} = \bar P\psi_\infty = 0. \tag{4.12}$$

Now we apply a similar argument as the one for proving Theorem 1.14 now based on (1.8) rather than (4.3): Looking at the expectation of both sides of (1.8) in the states $\phi_n := \bar P\psi^{(n)}$, using Remark 1.4 3), we obtain

$$c_0\|\phi_n\|^2 \le 2\|(H-\lambda)\phi_n\|\|A\phi_n\| + C\|\langle H\rangle^{1/2} f_\lambda^\perp(H)\phi_n\|^2 + \langle \phi_n, K_0\phi_n\rangle. \tag{4.13}$$

Since K_0 is compact we obtain from (4.12) that $\langle\phi_n, K_0\phi_n\rangle \to 0$. By (1.18), $\|A\phi_n\|$ is uniformly bounded, and therefore we conclude in combination with (4.11) that $\|\phi_n\| \to 0$. This contradicts (4.8).

Let us now prove that either i) of ii) holds. If i) fails indeed there exists a sequence of normalized eigenstates, $(H + V_n - \lambda_n)\psi_n = 0$, with eigenvalues $\lambda_n \to \lambda$ and with $V_n \in \mathcal{B}_{1,\gamma}$, $0 \ne \|V_n\|_1 \to 0$. Due to (4.7) $\|\bar P\psi_n\| \to 0$. By compactness there exists $\psi \in \mathrm{Ran}(P)$ such that along some subsequence $P\psi_{n_k} \to \psi$. Whence

$$\|\psi_{n_k} - \psi\| \le \|\bar P\psi_{n_k}\| + \|P\psi_{n_k} - \psi\| \to 0 \text{ for } k \to \infty, \tag{4.14}$$

and we conclude ii). □

There is a different version of the second part of Proposition 4.3 given by first fixing $V \in \mathcal{B}_{1,\gamma}$ (but otherwise given under the same conditions). Now we look at the eigenvalue problem in I of the family of perturbed Hamiltonians $H_\sigma = H + \sigma V$ with $\sigma \in \mathbb{R}$ and $|\sigma| > 0$ sufficiently small. In this framework there is a similar dichotomy (it can be shown by applying Proposition 4.3 under the same conditions, replacing $\mathcal{B}_{1,\gamma}$ by the subset $\{\sigma V, |\sigma| \le \sigma_0\} \subseteq \mathcal{B}_{1,\gamma}$).

Corollary 4.4. *Assume that Conditions 1.3 and Condition 1.10 hold. Suppose $\lambda \in \sigma_{\mathrm{pp}}(H)$ and that $J \subseteq I$ is a compact interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \{\lambda\}$. Let $P = E_H(\{\lambda\})$ and let $V \in \mathcal{B}_{1,\gamma}$. One of the following two alternatives i) or ii) holds:*

i) *For some sufficiently small $\sigma_0 > 0$ there are no eigenvalues of $H_\sigma := H + \sigma V$ in J for all $\sigma \in]-\sigma_0, \sigma_0[\ \setminus\{0\}$.*
ii) *For some sequence of coupling constants , $0 \ne \sigma_n \to 0$, and some sequence of normalized eigenstates ψ_n, $(H + \sigma_n V - \lambda_n)\psi_n = 0$ with $\lambda_n \to \lambda$, there exists $\psi_\infty \in \mathrm{Ran}(P)$ such that $\|\psi_n - \psi_\infty\| \to 0$.*

5. Second order perturbation theory

In this section we shall study second order perturbation theory. Our main interest is the Fermi Golden Rule, which indeed we shall show is a consequence of having an expansion to second order of any possible existing perturbed eigenvalue near an unperturbed one. This is done in Subsection 5.1 under Conditions 1.3 and Condition 1.10, in the case where the unperturbed eigenvalue is simple. In the degenerate case, this is done in Subsection 5.2 assuming Condition 1.9 rather than Condition 1.10. We do not obtain an expansion to second order of the perturbed eigenvalues assuming Condition 1.10 only. Nevertheless we shall show a similar version of the Fermi Golden Rule in this case also (done in Subsection 5.2).

5.1. **Second order perturbation theory – simple case.**

Theorem 5.1. *Assume that Conditions 1.3, Condition 1.10 and Condition 1.11 hold. Suppose $\lambda \in \sigma_{\rm pp}(H)$ and that $J \subseteq I$ is a compact interval such that $\sigma_{\rm pp}(H) \cap J = \{\lambda\}$. Let $P = E_H(\{\lambda\})$, $\bar P = I - P$. Let $V \in \mathcal{B}_{1,\gamma}$. Suppose*

$$\dim \operatorname{Ran}(P) = 1, \ \textit{viz.}\ P = |\psi\rangle\langle\psi|. \tag{5.1}$$

For all $1/2 < s \le 1$ and $\epsilon > 0$, there exists $\sigma_0 > 0$ such that if $|\sigma| \le \sigma_0$ and $\lambda_\sigma \in J$ is an eigenvalue of H_σ, then

$$\big|\lambda_\sigma - \lambda - \sigma\langle\psi, V\psi\rangle + \sigma^2\langle V\psi, (H - \lambda - \mathrm{i}0^+)^{-1}\bar P V\psi\rangle\big| \le \epsilon\sigma^2, \tag{5.2}$$

and there exists a normalized eigenstate ψ_σ, $H_\sigma\psi_\sigma = \lambda_\sigma\psi_\sigma$, such that

$$\big\|\psi_\sigma - \psi + \sigma(H - \lambda - \mathrm{i}0^+)^{-1}\bar P V\psi\big\|_{\mathcal{D}(\langle A\rangle^s)^*} \le \epsilon|\sigma|. \tag{5.3}$$

Remarks 5.2. 1) It is a consequence of Conditions 1.3, Condition 1.7, Remark 1.8 and Condition 1.11 that

$$\operatorname{Ran}(VP) \subseteq \mathcal{D}(A) \text{ for all } V \in \mathcal{V}_1. \tag{5.4}$$

Notice that we can compute the commutator form $[V, \mathrm{i}A]$ on $\big(\mathcal{D}(M^{1/2}) \cap \mathcal{D}(H) \cap \mathcal{D}(A^*)\big) \times \big(\mathcal{D}(M^{1/2}) \cap \mathcal{D}(H) \cap \mathcal{D}(A)\big)$ by a formula similar to (1.7). Whence this form is given by V', cf. (1.13), which by assumption is an H-bounded operator. In combination with Theorem 3.3 (5.4) implies that indeed the operator

$$PV(H - \lambda - \mathrm{i}0^+)^{-1}\bar P V P \in \mathcal{B}(\mathcal{H}). \tag{5.5}$$

2) Due to Theorem 1.14 there is at most one eigenvalue λ_σ of H_σ near λ, and if it exists it is simple.

Corollary 5.3. *Under the conditions of Theorem 5.1 and the condition*

$$\operatorname{Im}\langle V\psi, (H - \lambda - \mathrm{i}0^+)^{-1}\bar P V\psi\rangle > 0, \tag{5.6}$$

there exists $\sigma_0 > 0$ such that for all $\sigma \in]-\sigma_0, \sigma_0[\, \setminus \{0\}$

$$\sigma_{\rm pp}(H_\sigma) \cap J = \emptyset. \tag{5.7}$$

Proof of Theorem 5.1 Assume by contradiction that (5.2) does not hold. Then there exist $\epsilon > 0$ and a sequence $\sigma_n \to 0$ such that H_{σ_n} has an eigenvalue λ_n in J satisfying, for all n and for some $\psi \in \operatorname{Ran}(P)$, $\|\psi\| = 1$,

$$\big|\lambda_n - \lambda - \sigma_n\langle\psi, V\psi\rangle + \sigma_n^2\langle V\psi, (H - \lambda - \mathrm{i}0^+)^{-1}\bar P V\psi\rangle\big| \ge \epsilon\sigma_n^2. \tag{5.8}$$

Since $\dim \mathrm{Ran}(P) = 1$, (5.8) actually holds for any $\psi \in \mathrm{Ran}(P)$ such that $\|\psi\| = 1$. Let ψ_n be a normalized eigenstate of $H_n := H_{\sigma_n}$ associated to λ_n, $H_n\psi_n = \lambda_n\psi_n$. Arguing as in the proof of Proposition 4.3 we can assume that there exists $\tilde{\psi} \in \mathrm{Ran}(P)$ such that $\|\psi_n - \tilde{\psi}\| \to 0$. Henceforth we set $\psi = \tilde{\psi}$. Let $P_n := E_{H_n}(\{\lambda_n\})$. It follows from the fact that $\dim \mathrm{Ran}(P) = 1$ together with Theorem 1.14 that $\dim \mathrm{Ran}(P_n) = 1$. Hence $P_n = |\psi_n\rangle\langle\psi_n|$. The equation $(H_n - \lambda_n)P_n = 0$ is equivalent to the following system of equations:

$$\begin{cases} P\big(\sigma_n V + \lambda - \lambda_n\big)P_n = 0, \\ \sigma_n \bar{P}VP_n + (\lambda - \lambda_n)\bar{P}P_n + (H - \lambda)\bar{P}P_n = 0. \end{cases} \tag{5.9}$$

Since $\|\psi_n - \psi\| \to 0$, we have $\|\bar{P}P_n\| \to 0$ and $\|PP_n\| \to 1$. Hence the first equation of (5.9) yields

$$\lambda - \lambda_n = O(|\sigma_n|). \tag{5.10}$$

Now, using the second equation of (5.9), we can write, for any $\phi \in \mathcal{H}$ such that $\|\phi\| = 1$, and any $1/2 < s \le 1$,

$$\begin{aligned} \|\bar{P}P_n\phi\|^2 &= \big|\langle \bar{P}P_n\phi, \bar{P}P_n\phi\rangle\big| \\ &= \big|\langle \bar{P}P_n\phi, (H - \lambda - \mathrm{i}0^+)^{-1}\big(\sigma_n\bar{P}VP_n + (\lambda - \lambda_n + \mathrm{i}0^+)\bar{P}P_n\big)\phi\rangle\big| \\ &\le C|\sigma_n| \,\big\|\langle A\rangle^{-s}(H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}\langle A\rangle^{-s}\big\| \times \\ &\qquad \|\langle A\rangle^s\bar{P}P_n\| \left(\|\langle A\rangle\bar{P}P_n\| + \|\langle A\rangle\bar{P}VP_n\|\right). \end{aligned} \tag{5.11}$$

Using Condition 1.10 and the assumption that $V \in \mathcal{B}_{1,\gamma}$, one can prove that $\|\langle A\rangle\bar{P}P_n\|$ and $\|\langle A\rangle\bar{P}VP_n\|$ are uniformly bounded in n. In addition we claim that for $s < 1$, $\|\langle A\rangle^s\bar{P}P_n\| \to 0$ as $n \to \infty$. To prove this, it suffices to use that $\|\langle A\rangle^s(\langle A\rangle + \mathrm{i}k)^{-1}\| \to 0$ as $k \to \infty$, together with $\|\langle A\rangle\bar{P}P_n\|$ being uniformly bounded in n and $\|\bar{P}P_n\| \to 0$ as $n \to \infty$. Therefore by Theorem 3.3,

$$\|\bar{P}P_n\|^2 = o(|\sigma_n|). \tag{5.12}$$

Since $\dim \mathrm{Ran}(P) = \dim \mathrm{Ran}(P_n) = 1$, Equation (5.12) implies

$$\|\bar{P}\psi_n\|^2 = \|\bar{P}_n\psi\|^2 = o(|\sigma_n|), \tag{5.13}$$

and in particular also

$$\|\bar{P}_nP\|^2 = o(|\sigma_n|), \tag{5.14}$$

where we have set $\bar{P}_n = I - P_n$. Taking the expectation of the first equation of (5.9) in the state ψ gives

$$\begin{aligned} \lambda - \lambda_n &= -\sigma_n\langle\psi, V\psi\rangle + (\lambda - \lambda_n)(1 - \|P_n\psi\|^2) - \sigma_n\langle\psi, V(P_n - P)\psi\rangle \\ &= -\sigma_n\langle\psi, V\psi\rangle + \sigma_n\langle\psi, V\bar{P}_n\psi\rangle + o(\sigma_n^2), \end{aligned} \tag{5.15}$$

where we used (5.10) and (5.13) in the second equality. Let us write

$$\bar{P}_n\psi = P\bar{P}_n\psi - \bar{P}P_n\psi. \tag{5.16}$$

Estimate (5.14) yields $\|P\bar{P}_n\psi\| = o(|\sigma_n|)$. Inserting (5.16) and the second equation of (5.9) into (5.15), we obtain

$$\begin{aligned} \lambda - \lambda_n = &- \sigma_n\langle\psi, V\psi\rangle + \sigma_n^2\langle V\psi, (H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}VP_n\psi\rangle \\ &+ \sigma_n(\lambda - \lambda_n)\langle V\psi, (H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}P_n\psi\rangle + o(\sigma_n^2). \end{aligned} \tag{5.17}$$

As above we can use $\lambda - \lambda_n = O(|\sigma_n|)$ together with the fact that $\|\langle A\rangle^s \bar{P}P_n\| \to 0$ for $s < 1$ and Theorem 3.3 to obtain

$$\sigma_n(\lambda - \lambda_n)\langle V\psi, (H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}P_n\psi\rangle = o(\sigma_n^2). \tag{5.18}$$

Finally, it follows from Condition 1.10 and the assumption $V \in \mathcal{B}_{1,\gamma}$ that $\|\langle A\rangle^s V(P_n - P)\psi\| \to 0$ for $s < 1$. This leads to

$$\lambda - \lambda_n = -\sigma_n\langle\psi, V\psi\rangle + \sigma_n^2\langle V\psi, (H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}V\psi\rangle + o(\sigma_n^2), \tag{5.19}$$

which contradicts (5.8), and hence proves (5.2).

It remains to prove (5.3). Assume, again by contradiction, that (5.3) does not hold. Then there exist $\epsilon > 0$ and a sequence $\sigma_n \to 0$ such that $H_n = H_{\sigma_n}$ has an eigenvalue $\lambda_n \in J$ associated to a normalized eigenstate ψ_n satisfying, for any $\psi \in \mathrm{Ran}(P)$, $\|\psi\| = 1$,

$$\left\|\psi_n - \psi + \sigma_n(H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}V\psi\right\|_{(\mathcal{D}(\langle A\rangle^s))^*} \geq \epsilon|\sigma_n|. \tag{5.20}$$

As above we can assume that there exists $\psi \in \mathrm{Ran}(P)$ such that $\|\psi_n - \psi\| \to 0$. Let $\tilde{\psi} := e^{i\theta_n}\psi$, where $\theta_n \in \mathbb{R}$ is defined by the equation $\langle\psi, \psi_n\rangle = e^{i\theta_n}|\langle\psi, \psi_n\rangle|$. Using the second equation of (5.9), we can write

$$\begin{aligned}
\psi_n &= P\psi_n + \bar{P}\psi_n \\
&= \langle\psi, \psi_n\rangle\psi - (\lambda - \lambda_n)(H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}\psi_n - \sigma_n(H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}V\psi_n \\
&= \tilde{\psi} - \sigma_n(H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}V\tilde{\psi} + R_n,
\end{aligned} \tag{5.21}$$

where

$$\begin{aligned}
R_n =& \big(\|P\psi_n\| - 1\big)\tilde{\psi} - (\lambda - \lambda_n)(H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}\psi_n \\
&- \sigma_n(H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}V\big(\psi_n - \tilde{\psi}\big).
\end{aligned} \tag{5.22}$$

By arguments similar to the ones used to prove (5.2), one can see that $\|R_n\|_{D(\langle A\rangle^s)^*} = o(|\sigma_n|)$ for any fixed $1/2 < s < 1$, which contradicts (5.20), and hence proves (5.3). □

5.2. **Fermi Golden Rule criterion – general case.** We begin this section with a result similar to Theorem 5.1 that we shall obtain without requiring an hypothesis of simplicity. Here we need Condition 1.9 rather than Condition 1.10.

Theorem 5.4. *Suppose Conditions 1.3, Condition 1.9 and Condition 1.11. Let $V \in \mathcal{V}_2$. Suppose $\lambda \in \sigma_{\mathrm{pp}}(H)$ and that $J \subseteq I$ is a compact interval such that $\sigma_{\mathrm{pp}}(H) \cap J = \{\lambda\}$. Let $P = E_H(\{\lambda\})$, $\bar{P} = I - P$.*

There exist $C \geq 0$ and $\sigma_0 > 0$ such that if $|\sigma| \leq \sigma_0$ and $\lambda_\sigma \in J$ is an eigenvalue of $H_\sigma = H + \sigma V$, then there exists $\psi \in \mathrm{Ran}(P)$, $\|\psi\| = 1$, such that

$$\big|\lambda_\sigma - \lambda - \sigma\langle\psi, V\psi\rangle + \sigma^2\langle V\psi, (H - \lambda - \mathrm{i}0^+)^{-1}\bar{P}V\psi\rangle\big| \leq C|\sigma|^{5/2}. \tag{5.23}$$

Remarks 5.5. 1) In the simple case, $P = |\psi\rangle\langle\psi|$, (5.23) is stronger than (5.2).

2) We do not have an analogue of (5.3) under the conditions of Theorem 5.4, even if we assume in addition $\dim \mathrm{Ran}(P) = 1$. Similarly, cf. Remark 5.2 2), we do not have upper semicontinuity of point spectrum at λ even if $\dim \mathrm{Ran}(P) = 1$.

Proof of Theorem 5.4 We can argue in a way similar to the proofs of Proposition 5.2 and Lemma 5.3 in [AHS]. For $\sigma = 0$, there is nothing to prove. Let $\sigma \neq 0$.

As in the proof of Theorem 3.4, we set $\bar{H} = H + \alpha_J P$ with $\alpha_J > \sup J - \inf J$, and $\bar{H}_\sigma = \bar{H} + \sigma V$. Assume that $\lambda_\sigma \in \sigma_{\rm pp}(H_\sigma)$ and let ϕ_σ be such that $(H_\sigma - \lambda_\sigma)\phi_\sigma = 0$, $\|\phi_\sigma\| = 1$. Hence

$$(\bar{H}_\sigma - \lambda_\sigma)\phi_\sigma = \alpha_J P\phi_\sigma. \tag{5.24}$$

By Theorem 3.4, $\lambda_\sigma \notin \sigma_{\rm pp}(\bar{H}_\sigma)$, and hence in particular $P\phi_\sigma \neq 0$. Moreover, it follows from (5.24) that, for any $\epsilon > 0$,

$$P\phi_\sigma = \alpha_J P\big(\bar{H}_\sigma - \lambda_\sigma - {\rm i}\epsilon\big)^{-1}P\phi_\sigma - {\rm i}\epsilon\alpha_J P\big(\bar{H}_\sigma - \lambda_\sigma - {\rm i}\epsilon\big)^{-1}\phi_\sigma. \tag{5.25}$$

Letting $\epsilon \to 0$, since $\lambda_\sigma \notin \sigma_{\rm pp}(\bar{H}_\sigma)$, we obtain

$$P\phi_\sigma = \alpha_J P\big(\bar{H}_\sigma - \lambda_\sigma - {\rm i}0^+\big)^{-1}P\phi_\sigma. \tag{5.26}$$

Note that the right-hand-side of (5.26) is well-defined by Theorem 3.4 since, by Condition 1.9, ${\rm Ran}(P) \subseteq \mathcal{D}(A)$.

Let $\beta := \alpha_J + \lambda - \lambda_\sigma$. Hence $P\big(\bar{H} - \lambda_\sigma\big)P = \beta P$. Using twice the second resolvent equation, one easily verifies that, for any $\epsilon > 0$,

$$\begin{aligned} &P\big(\bar{H}_\sigma - \lambda_\sigma - {\rm i}\epsilon\big)^{-1}P \\ &= (\beta - {\rm i}\epsilon)^{-1}P - (\beta - {\rm i}\epsilon)^{-2}\sigma PVP + (\beta - {\rm i}\epsilon)^{-2}\sigma^2 PV\big(\bar{H}_\sigma - \lambda_\sigma - {\rm i}\epsilon\big)^{-1}VP. \end{aligned} \tag{5.27}$$

Letting $\epsilon \to 0$ and using Theorem 3.4 with $s = 1$, this yields

$$\begin{aligned} &P\big(\bar{H}_\sigma - \lambda_\sigma - {\rm i}0^+\big)^{-1}P \\ &= \beta^{-1}P - \beta^{-2}\sigma PVP + \beta^{-2}\sigma^2 PV\big(\bar{H} - \lambda_\sigma - {\rm i}0^+\big)^{-1}VP + R_1, \end{aligned} \tag{5.28}$$

where R_1 is a bounded operator on ${\rm Ran}(P)$ satisfying $\|R_1\| \le C_1|\sigma|^{5/2}$. Note that the right-hand-side of (5.28) is well-defined by Theorem 3.4 and Remark 5.2 2).

Now let $\psi := \|P\phi_\sigma\|^{-1}P\phi_\sigma$. Multiplying (5.28) by $\alpha_J\beta$ and taking the expectation in ψ, we obtain thanks to (5.26):

$$\begin{aligned} \lambda - \lambda_\sigma = &- \alpha_J\beta^{-1}\sigma\langle\psi, V\psi\rangle \\ &+ \alpha_J\beta^{-1}\sigma^2\langle V\psi, \big(\bar{H} - \lambda_\sigma - {\rm i}0^+\big)^{-1}V\psi\rangle + \langle\psi, R_1\psi\rangle. \end{aligned} \tag{5.29}$$

Using again Theorem 3.4 with $s = 1$, this implies

$$\begin{aligned} \lambda - \lambda_\sigma = &- \alpha_J\beta^{-1}\sigma\langle\psi, V\psi\rangle \\ &+ \alpha_J\beta^{-1}\sigma^2\langle V\psi, \big(\bar{H} - \lambda - {\rm i}0^+\big)^{-1}V\psi\rangle + \langle\psi, R_2\psi\rangle, \end{aligned} \tag{5.30}$$

where R_2 is a bounded operator on ${\rm Ran}(P)$ satisfying $\|R_2\| \le C_2|\sigma|^{5/2}$. In particular, $|\lambda - \lambda_\sigma| \le C_3|\sigma|$. We then obtain from (5.26) and (5.28) that

$$\begin{aligned} \frac{\lambda - \lambda_\sigma}{\sigma}\psi &= -\alpha_J\beta^{-1}PVP\psi + \alpha_J\beta^{-1}\sigma PV\big(\bar{H} - \lambda - {\rm i}0^+\big)^{-1}VP\psi + \sigma^{-1}R_2\psi \\ &= (-PVP + \sigma R_3)\psi, \end{aligned} \tag{5.31}$$

where R_3 is an operator on the finite dimensional space ${\rm Ran}(P)$ uniformly bounded in σ. It follows from the usual perturbation theory (see [Ka]) that ψ can be written as $\psi = \psi_1 + \sigma\psi_2$

where ψ_1 is an eigenstate of $-PVP$ and $\psi_2 \in \mathrm{Ran}(P)$. Now, multiplying (5.30) by $\alpha_J^{-1}\beta$ gives

$$\begin{aligned}(\lambda-\lambda_\sigma)\alpha_J^{-1}\beta &= -\sigma\langle\psi, V\psi\rangle + \sigma^2\langle V\psi, (\bar H-\lambda-\mathrm{i}0^+)^{-1}V\psi\rangle + \alpha_J^{-1}\beta\langle\psi, R_2\psi\rangle\\ &= -\sigma\langle\psi, V\psi\rangle + \alpha_J^{-1}\sigma^2\langle V\psi, PV\psi\rangle + \sigma^2\langle V\psi, (H-\lambda-\mathrm{i}0^+)^{-1}\bar P V\psi\rangle\\ &\quad + \alpha_J^{-1}\beta\langle\psi, R_2\psi\rangle. \end{aligned} \tag{5.32}$$

By (5.30), we can write

$$\lambda-\lambda_\sigma = -\sigma\langle\psi, V\psi\rangle + \langle\psi, R_4\psi\rangle, \tag{5.33}$$

with $\|R_4\| \le C_4\sigma^2$, and hence

$$\begin{aligned}(\lambda-\lambda_\sigma)\alpha_J^{-1}\beta &= (\lambda-\lambda_\sigma) + \alpha_J^{-1}(\lambda-\lambda_\sigma)^2\\ &= (\lambda-\lambda_\sigma) + \alpha_J^{-1}\sigma^2\langle\psi, V\psi\rangle^2 + O(|\sigma|^3). \end{aligned} \tag{5.34}$$

Since $\psi = \psi_1 + \sigma\psi_2$ where ψ_1 is an eigenstate of $-PVP$, we have

$$\langle\psi, V\psi\rangle^2 - \|PV\psi\|^2 = O(|\sigma|). \tag{5.35}$$

Therefore,

$$\alpha_J^{-1}\sigma^2\langle V\psi, PV\psi\rangle - \alpha_J^{-1}\sigma^2\langle\psi, V\psi\rangle^2 = O(|\sigma|^3). \tag{5.36}$$

Combining Equations (5.32), (5.34) and (5.36), the statement of the theorem follows. □

We come now to the proof of Theorem 1.15 on the absence of eigenvalues of the perturbed Hamiltonian $H_\sigma = H + \sigma V$, generalizing Corollary 5.3:

Proof of Theorem 1.15 Suppose first that Condition 1.9 holds and that $V \in \mathcal{V}_2$. By Theorem 5.4, there exists $\sigma_0 > 0$ such that if λ_σ is an eigenvalue of H_σ with $|\sigma| \le \sigma_0$, then (5.23) is satisfied. Taking the imaginary part of (5.23) contradicts (1.20).

Suppose now Condition 1.10 and that $V \in \mathcal{B}_{1,\gamma}$. Assume by contradiction that (1.21) is false. Then the second alternative ii) of Corollary 4.4 holds. Hence we consider a sequence of normalized eigenstates $\psi_n \to \psi_\infty \in \mathrm{Ran}(P)$ of a sequence of Hamiltonians $H_n := H_{\sigma_n}$ given in terms of a certain sequence of coupling constants $\sigma_n \to 0,\ \sigma_n \ne 0$. Let $P_n = |\psi_n\rangle\langle\psi_n|$. As in the proof of Theorem 5.1, the equation $(H_n - \lambda_n)P_n = 0$ is equivalent to (5.9). We notice that

$$\mathrm{Im}\big(P_nV\bar P P_n\big) = -\mathrm{Im}\big(P_nVPP_n\big) = \tfrac{\lambda-\lambda_n}{\sigma_n}\mathrm{Im}\big(P_nPP_n\big) = 0, \tag{5.37}$$

due to the first equation of (5.9). Next we apply $P_nV(H-\lambda-\mathrm{i}0^+)^{-1}\bar P$ from the left in the second equation of (5.9), take the imaginary part and use (5.37) yielding

$$\begin{aligned}&\sigma_nP_nV\mathrm{Im}\big((H-\lambda-\mathrm{i}0^+)^{-1}\bar P\big)VP_n\\ &= (\lambda_n-\lambda)\mathrm{Im}\big(P_nV(H-\lambda-\mathrm{i}0^+)^{-1}\bar P P_n\big). \end{aligned} \tag{5.38}$$

Now we take the expectation of (5.38) in the state ψ_∞, use the first equation of (5.9) and divide by σ_n yielding

$$\begin{aligned}&\mathrm{Im}\,\langle(H-\lambda-\mathrm{i}0^+)^{-1}\bar P\rangle_{VP_n\psi_\infty}\\ &= \mathrm{Im}\,\langle P_nV(H-\lambda-\mathrm{i}0^+)^{-1}\bar P P_nV\rangle_{\psi_\infty}. \end{aligned} \tag{5.39}$$

Again, using Condition 1.10, we have that $\|\langle A\rangle^s \bar{P} P_n\| \to 0$ for $1/2 < s < 1$. We then conclude by letting $n \to \infty$ in the above identity, using Theorem 3.3, which yields

$$\operatorname{Im} \langle (H - \lambda - \mathrm{i}0^+)^{-1} \bar{P} \rangle_{V\psi_\infty} = 0. \tag{5.40}$$

Clearly (5.40) contradicts (1.20). □

Appendix A

We give an independent proof of (1.11) under Condition 1.3 (1), in fact we shall give an alternative proof of the fact that

$$\mathcal{D}(\sqrt{G}) = \mathcal{G}, \tag{A.1}$$

cf. Remark 1.4 2). Obviously $\mathcal{D}(\sqrt{G}) \subseteq \mathcal{G}$ and the graph norm of $\sqrt{G}$ is equivalent to the norm on $\mathcal{G}$ (defined by (1.3)). In particular $\mathcal{D}(\sqrt{G})$ is closed in $\mathcal{G}$. Whence (A.1) is in turn a consequence of (1.11). The proof of (1.11) is in two steps.

Step I We shall show that

$$\mathcal{D}(M) \cap \mathcal{D}(|H|^{1/2}) \text{ is dense in } \mathcal{G}. \tag{A.2}$$

We will essentially use [FMS, (3.15)]. Whence, introducing the notation

$$I_n(M) = -\mathrm{i}n(M - \mathrm{i}n)^{-1} \text{ for } n \in \mathbb{N},$$

we have

$$\operatorname{s-}\lim_{n\to\infty} \langle H\rangle^{1/2} I_n(M) \langle H\rangle^{-1/2} = I. \tag{A.3}$$

For completeness let us here give the proof of (A.3) following [FMS]: Due to [Mo, Proposition II.3]

$$\operatorname{s-}\lim_{n\to\infty} \langle H\rangle I_n(M) \langle H\rangle^{-1} = I. \tag{A.4}$$

Introducing

$$B_n^s = \langle H\rangle^s \left(I_n(M) - I\right) \langle H\rangle^{-s};\ \operatorname{Re} s \in [0,1],$$

we observe that the families $\{B_n^1\}$ and $\{B_n^0\}$ are bounded. Whence also $\{B_n^{1/2}\}$ is bounded (by interpolation). Using this fact and the fact that $\|B_n^{1/2}\phi\| \to 0$ for $\phi \in \mathcal{D}(\langle H\rangle^{1/2})$ (due to (A.4)) we obtain (A.3).

Now, to show (A.2), we let $\phi \in \mathcal{G}$ be given and define $\phi_n = I_n(M)\phi$. By (A.3) we have $\phi_n \in \mathcal{D}(M) \cap \mathcal{D}(|H|^{1/2})$ and in fact that $\|\langle H\rangle^{1/2}(\phi - \phi_n)\| \to 0$. Obviously $\|M^{1/2}(\phi - \phi_n)\| \to 0$. We conclude that $\|\phi - \phi_n\|_{\mathcal{G}} \to 0$.

Step II We shall show that

$$\mathcal{D} \text{ is dense in } \mathcal{D}(M) \cap \mathcal{D}(|H|^{1/2}) \subseteq \mathcal{G}. \tag{A.5}$$

Whence let $\phi \in \mathcal{D}(M) \cap \mathcal{D}(|H|^{1/2})$ be given. Define similarly $\phi_n = I_n(H)\phi$. Since $H \in C^1_{\mathrm{Mo}}(M)$ we can compute

$$[M, I_n(H)] = n^{-1} I_n(H) [H, \mathrm{i}M]^0 I_n(H) \in \mathcal{B}(\mathcal{H}),$$

are therefore deduce that

$$\operatorname{s-}\lim_{n\to\infty} [M, I_n(H)] = 0. \tag{A.6}$$

It follows from (A.6) that $\phi_n \in \mathcal{D}$ and that $\|M(\phi - \phi_n)\| \to 0$. Clearly $\|\langle H\rangle^{1/2}(\phi - \phi_n)\| \to 0$. In particular $\|\phi - \phi_n\|_{\mathcal{G}} \to 0$.

Clearly (1.11) follows by combining (A.2) and (A.5). □

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