ANALYTICITY OF THE SELF-ENERGY IN TOTAL MOMENTUM OF AN ATOM COUPLED TO THE QUANTIZED RADIATION FIELD

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Abstract. We study a neutral atom with a non-vanishing electric dipole moment coupled to the quantized electromagnetic field. For a sufficiently small dipole moment and small momentum, $\vec{p}$, the one-particle (self-) energy of an atom is proven to be a real-analytic function of its momentum. The main ingredient of our proof is a suitable form of the Feshbach-Schur spectral renormalization group.

1. Introduction

1.1. **Description of the model and statement of the main result.** In this paper, we consider models of a neutral atom with a non-zero electric dipole moment coupled to the quantized electromagnetic field. We are interested in analyzing properties of the self-energy – or dispersion law – as a function of the momentum $\vec{p}$ of the atom. When the coupling of the atom to the electromagnetic field is turned off, its dispersion law is taken to be $\vec{p}^2/2m$, where m is its bare mass. Our purpose is to study the radiative corrections to this law (self-energy) when the atom is coupled to the electromagnetic field. We will prove real analyticity of the self-energy in the momentum $\vec{p}$, for $|\vec{p}| < m$, provided the dipole moment of the atom is sufficiently small; (the speed of light and Planck's constant are set to 1, throughout this paper). Our result has interesting applications to the study of Compton scattering of atoms and of the effective dynamics of an atom when it moves through an external potential. These matters, as well as the study of atomic resonances for atoms of finite total mass, are left for the subject of a forthcoming work.

1.1.1. *The Hamiltonian.* We study the simplest Hamiltonian describing a freely moving neutral atom coupled to the electromagnetic field via an electric dipole moment. We consider the atom as a two-level system coupled to the electromagnetic field via an interaction Hamiltonian

$$H_I := -\vec{d} \cdot \vec{E}, \tag{1.1}$$

where $\vec{d}$ is the dipole moment of the atom and $\vec{E}$ the quantized electric field. For a two-level system, $\vec{d}$ can be expanded in the basis $(\sigma_x, \sigma_y, \sigma_z)$ of Pauli matrices. The quantization of the electromagnetic field in the Coulomb gauge is accomplished within the usual second quantization formalism. Readers not familiar with it are encouraged to consult [9] and the references given therein.

The Hilbert space of the system is the tensor product

$$\mathcal{H} = \mathcal{H}_{at} \otimes \mathcal{H}_f,$$

where

$$\mathcal{H}_{at} := L^2(\mathbb{R}^3) \otimes \mathbb{C}^2 \qquad \text{and} \qquad \mathcal{H}_f := \mathcal{F}_+(L^2(\underline{\mathbb{R}}^3))$$

are the atomic and the field Hilbert spaces, with $\mathcal{F}_+(L^2(\underline{\mathbb{R}}^3))$ the symmetric Fock space over $L^2(\underline{\mathbb{R}}^3)$. Here and in what follows, we use the shorthand

$$\underline{\mathbb{R}}^3 := \mathbb{R}^3 \times \{1,2\} = \left\{ \underline{k} := (\vec{k}, \lambda) \in \mathbb{R}^3 \times \{1,2\} \right\}, \tag{1.2}$$

where λ is the polarization index of the field. To shorten notations, we also set $\underline{\mathbb{R}}^{3n} := (\underline{\mathbb{R}}^3)^n$, and, for $A \subset \mathbb{R}^3$,

$$\underline{A} := A \times \{1,2\}, \qquad \int_{\underline{A}} d\underline{k} := \sum_{\lambda=1,2} \int_A d\vec{k}. \tag{1.3}$$

Notations (1.2) and (1.3) are used throughout this paper.

The dynamics of the system is given by the Hamiltonian

$$H = H_{at} + H_f + \lambda_0 H_I, \tag{1.4}$$

where

$$H_{at} := -\frac{\Delta}{2m} \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \begin{pmatrix} \omega_0 & 0 \\ 0 & 0 \end{pmatrix} \otimes \mathbb{1} \tag{1.5}$$

is the free atomic Hamiltonian,

$$H_f = \mathbb{1} \otimes \mathbb{1} \otimes \int_{\underline{\mathbb{R}}^3} |\vec{k}| \ a^*(\underline{k}) a(\underline{k}) d\underline{k} \tag{1.6}$$

is the Hamiltonian of the free electromagnetic field, and

$$H_I = i \int_{\underline{B}_1} |\vec{k}|^{\frac{1}{2}} \left(e^{i\vec{k}\cdot\vec{x}} \otimes \vec{\epsilon}(\underline{k}) \cdot \vec{\sigma} \otimes a(\underline{k}) - h.c. \right) d\underline{k} \tag{1.7}$$

is the interaction Hamiltonian; $\lambda_0 \geq 0$, in (1.4), is the coupling constant, $m > 0$ and $\omega_0 > 0$, in (1.5), are the mass of the atom and the energy of the internal excited state of the atom, respectively, and

$$a(\underline{k}) := a_\lambda(\vec{k}), \qquad a^*(\underline{k}) := a^*_\lambda(\vec{k}), \qquad \vec{\epsilon}(\underline{k}) := \vec{\epsilon}_\lambda(\vec{k}), \tag{1.8}$$

are the annihilation and creation operators on $\mathcal{H}_f$ and the polarization vectors of the electromagnetic field in the Coulomb gauge. Furthermore, we denoted by B_1 in (1.7) the closed ball in $\mathbb{R}^3$ centered at 0 with radius 1, by $\vec{\sigma}$ the vector of Pauli matrices and by $\vec{x}$ the position operator of the atom.

Remarks.

- *The Hamiltonian H in* (1.4) *can be derived from the Hamiltonian of a localized neutral system of charges interacting with the quantized electromagnetic field, under the assumption that the typical size of the system of charges is very small in comparison to the typical wavelength of the radiation field. This approximation is called the "dipole" approximation. We refer the reader to* [14] *for details concerning the unitary transformation (the Göppert-Mayer transformation) that brings the Hamiltonian of the system of charges in interaction with the electromagnetic field to an electric dipole Hamiltonian similar to the one in* (1.4).
- *Standard estimates show that H_I is $H_f^{1/2}$ relatively bounded. Therefore, by Kato's Theorem, H is self-adjoint on the domain of $H_{at} + H_f$. We also notice that the interaction term H_I in* (1.7) *behaves well in the infrared and that our model has no infrared catastrophe. This feature simplifies our analysis, as compared to the analysis of charged particles.*

- *To ease notations, the ultraviolet cut-off imposed in the interaction Hamiltonian H_I has been chosen to be equal to 1. We point out that our result and our proofs hold for an arbitrary ultraviolet cut-off.*
- *The Hamiltonian H has the very important property to be translation invariant. Indeed, an easy calculation shows that each component of the total momentum operator*

$$\vec{P}_{tot} := -i\vec{\nabla}_x \otimes \mathbb{1} \otimes \mathbb{1} + \mathbb{1} \otimes \mathbb{1} \otimes \int_{\underline{\mathbb{R}}^3} \vec{k}\ a^*(\underline{k})a(\underline{k})d\underline{k}, \tag{1.9}$$

commutes with H.

The property of translation invariance implies the existence of a unitary map $U : \mathcal{H} \to \tilde{\mathcal{H}}$, where

$$\tilde{\mathcal{H}} := \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}_{\vec{p}}\ d\vec{p}. \tag{1.10}$$

More precisely, U is the generalized Fourier transform, defined, for any $\psi \in \mathcal{H}$ decaying sufficiently rapidly at infinity, by,

$$(U\psi)(\vec{p}) = \frac{1}{(2\pi)^{3/2}} \int_{\mathbb{R}^3} e^{-i(\vec{p}-\vec{P}_f)\cdot\vec{y}}\psi(\vec{y})d\vec{y}; \tag{1.11}$$

The Hilbert spaces $\mathcal{H}_{\vec{p}}$ are isomorphic to $\mathbb{C}^2 \otimes \mathcal{F}_+(L^2(\underline{\mathbb{R}}^3))$. We rewrite the Hamiltonian H in the representation (1.10). We introduce the notations

$$b(\underline{k}) := Ue^{i\vec{k}\cdot\vec{x}}a(\underline{k})U^{-1}, \qquad b^*(\underline{k}) := Ue^{-i\vec{k}\cdot\vec{x}}a^*(\underline{k})U^{-1}. \tag{1.12}$$

The operator-valued distributions $b(\underline{k})$, $b^*(\underline{k}')$ satisfy the canonical commutation relations $[b(\underline{k}), b^*(\underline{k}')] = \delta_{\lambda,\lambda'}\delta(\vec{k}-\vec{k}')$ and $[b^\sharp(\underline{k}), b^\sharp(\underline{k}')] = 0$. Furthermore,

$$\left(\left(U\frac{-\Delta}{2m}U^{-1}\right)\psi\right)(\vec{p}) = \frac{\left(\vec{p}-\vec{P}_f\right)^2}{2m}\psi(\vec{p}).$$

It follows that the Hamiltonian H can be decomposed as a direct integral of fiber Hamiltonians, $H(\vec{p})$, with

$$\begin{aligned} H(\vec{p}) =& \frac{\left(\vec{p}-\vec{P}_f\right)^2}{2m} + \omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ &+ i\lambda_0 \int_{\underline{B}_1} |\vec{k}|^{\frac{1}{2}}\vec{\epsilon}(\underline{k})\cdot\vec{\sigma}\left(b(\underline{k}) - b^*(\underline{k})\right)d\underline{k} + \int_{\underline{\mathbb{R}}^3} |\vec{k}|b^*(\underline{k})b(\underline{k})d\underline{k}. \end{aligned} \tag{1.13}$$

Here and in what follows, we suppressed the tensor product notation, $\mathbb{1}\otimes\ldots$. Using standard estimates and the Kato-Rellich theorem, one can prove that the fiber Hamiltonians $H(\vec{p})$ are self-adjoint on the dense domain $D := D(H_f + P_f^2)$ of $\mathcal{H}_{\vec{p}}$, and bounded from below.

The question to know whether

$$E(\vec{p}) := \inf\left(\sigma(H(\vec{p}))\right) \tag{1.14}$$

is an eigenvalue of $H(\vec{p})$ and whether it has some regularity property with respect to the total momentum $\vec{p}$ has been already addressed in the literature for different models [1, 2, 5, 10, 11, 12, 16, 23, 26, 31, 33]. In particular, for the Nelson model, it has been shown in [19] that $E(\vec{p})$ is a non-degenerate eigenvalue if and only if an infrared regularization is imposed. In [16, 33], using iterative perturbation theory and a multiscale analysis, it has been proven that $\vec{p} \mapsto E(\vec{p})$ is twice differentiable near $\vec{0}$. Without imposing any infrared regularization to the form factor in the Nelson model, [1] has established using in particular a cluster expansion that $E(\vec{p})$ is a real analytic function of $\vec{p}$ and the coupling constant. For charged particles in non-relativistic QED, similar results have been obtained in [5, 10, 11, 12, 23, 26]: It is known that $E(\vec{p})$ is an eigenvalue of $H(\vec{p})$ if and only if $\vec{p} = \vec{0}$ [10, 11, 26] (unless and infrared regularization is imposed) and that $\vec{p} \mapsto E(\vec{p})$ is twice differentiable near $\vec{0}$. The latter property has been proven using an application of the spectral renormalization group (see [10]) and by iterative perturbation theory [12, 23]. For results concerning models describing moving atoms or ions coupled to the electromagnetic fields, we refer to [2, 21, 26, 31]

For the Hamiltonian model (1.4) studied in this paper, we can prove:

Theorem 1.1. *Let $0 < \nu < m$. There exists a constant $\lambda_c(\nu) > 0$ such that, for any coupling constant $\lambda_0 \geq 0$ satisfying $\lambda_0 < \lambda_c(\nu)$, the map $\vec{p} \mapsto E(\vec{p})$ and its associated eigenprojection $\vec{p} \mapsto \Pi(\vec{p})$ are real analytic on $B_\nu = \{\vec{p} \in \mathbb{R}^3, |\vec{p}| < \nu\}$.*

Remarks.

- *The restriction to momenta $|\vec{p}| < m$ is crucial for our result to hold. Indeed, for $|\vec{p}| > m$, we expect $E(\vec{p})$ to dissolve in the continuum. This feature is the mathematical relic of the Cherenkov radiation emitted by the particle when it travels faster than the velocity of the light in the medium; see e.g.* [15].
- *The absence of infrared divergences in our model implies that the one-particle states (or fiber ground states) $\psi(\vec{p})$ stay in $\mathcal{H}_{\vec{p}}$ for all $|\vec{p}| < m$. This result would not hold for a particle with a non-zero net charge, because the infrared catastrophe would force the eigenstate to leave the Fock space, for all $\vec{p} \neq \vec{0}$. We refer the reader to* [19] *for an extensive discussion of this problem.*
- *The critical constant $\lambda_c(\nu)$ decays like a power law in $m - \nu$ when ν approaches m.*
- *We observe that, by rotation invariance, $E(\vec{p})$ only depends on the norm of $\vec{p}$. Theorem 1.1 shows that the map $|\vec{p}| \mapsto E(|\vec{p}|)$ is real analytic on the interval $[0, \mu)$. The regularity of $E(|\vec{p}|)$ with respect to $|\vec{p}|$ is an important physical property, that allows one, in particular, to define the renormalized mass of the dipole by the formula $m_{\text{eff}} = (\partial^2_{|\vec{p}|} E(|\vec{p}|))^{-1}$. In a companion paper* [18]*, we study the effective dynamics of the dipole placed in a slowly varying external potential; We justify that the renormalized mass and the kinetic mass of the dipole coincide. The results of Theorem 1.1 are also expected to be useful in the framework of scattering theory, as we plan to consider in future work; (see also* [21, 34]*).*

1.2. **Outline of the strategy.** Our analysis is based on the operator theoretic renormalization group method introduced in [7], [6] and [4]. For a concise review, the reader is referred to [22]. This method, based on an iteration of the smooth Feshbach-Schur map, has been useful in the study of the existence of ground states and resonances in models of matter coupled to a quantized field; Some results on the analyticity of ground states and their associated eigenvalues have been obtained recently with this method, but the models that have been investigated in the literature so far, (as far as the question of analyticity with respect to a parameter is concerned) all deal with fixed or confined atoms; see [25, 27, 28]. For instance, it has been shown in [28] that the ground state of the spin boson model is analytic in the coupling constant λ_0.

In this paper, we investigate the analyticity of the dispersion law with respect to the total momentum $\vec{p}$. The novelty of our result lies in the fact that $\vec{p}$ appears in the marginal term $\vec{p} \cdot \vec{P}_f$ in the initial Hamiltonian $H(\vec{p})$, and not in the perturbation. The perturbation contracts under the iteration of the renormalization map, whereas the size of the marginal term does not change much. The control of the evolution of the marginal terms under the renormalization flow is therefore an important issue in our analysis. A similar issue appears in [5, 10], but our approach to deal with it differs from that in [5, 10] and is slightly simpler in some respects. In particular, we consider a different Banach space of effective Hamiltonians, and we apply the Feshbach-Schur map at each step of the renormalization procedure in a different way. Besides, [10] shows that $\vec{p} \mapsto E(\vec{p})$ is twice differentiable near $\vec{0}$, while our main result establishes the real analyticity of this map.

Next, we describe the different steps of our analysis and give an overview of the way the spectral renormalization group works in the present setup.

1.2.1. *Complexification of the total momentum.* We start by fixing a vector $\vec{p}^*$ in $\mathbb{R}^3$ of length smaller than m. We set $2\mu = (m - |\vec{p}^*|)/m$, and consider the open set

$$U[\vec{p}^*] := \{\vec{p} \in \mathbb{C}^3 \mid |\vec{p} - \vec{p}^*| < \mu m\},$$

centered at $\vec{p}^*$. For any $\vec{p} \in U[\vec{p}^*]$, we subtract the constant term $\vec{p}^2/2m$ in (1.13), and consider the operator

$$H(\vec{p}) := \frac{\vec{P}_f^2}{2m} - \frac{\vec{p}}{m} \cdot \vec{P}_f + \omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_0 H_I + H_f,$$

where

$$H_I := i \int_{\underline{B}_1} |\vec{k}|^{\frac{1}{2}} \vec{\epsilon}(\underline{k}) \cdot \vec{\sigma} \, (b(\underline{k}) - b^*(\underline{k})) \, d\underline{k}, \quad H_f := \int_{\underline{\mathbb{R}}^3} |\vec{k}| b^*(\underline{k}) b(\underline{k}) d\underline{k}. \tag{1.15}$$

The subtraction of $\vec{p}^2/2m$ leads to a shift in the spectrum of the original Hamiltonian by $\vec{p}^2/2m$. For $\vec{p} \in U[\vec{p}^*] \cap (\mathbb{C}^3 \setminus \mathbb{R}^3)$, the operator $H(\vec{p})$ is not self-adjoint, anymore, but it is closed on D. We show that the spectral renormalization group method can be used to investigate the spectrum of the operator $H(\vec{p})$ near the origin of the complex plane. We will find an eigenvalue of $H(\vec{p})$ in this neighborhood of the origin, for any $\vec{p} \in U[\vec{p}^*]$. This eigenvalue turns out to be equal to $\inf(\sigma(H(\vec{p})))$, for $\vec{p} \in U[\vec{p}^*] \cap \mathbb{R}^3$; see Subsection 4.2.2.

1.2.2. *The first decimation step and the Feshbach-Schur map.* The purpose of the Feshbach-Schur map introduced in [6, 7], is to construct a new operator – called “effective Hamiltonian” – that acts on a subspace of “low-energy photons” of $\mathcal{H}$, with the property that the spectrum and the eigenstates of $H(\vec{p})$ near the origin can be uniquely reconstructed from the study of the kernel of this effective Hamiltonian. In this paper, we use the smooth version of the Feshbach-Schur map that has been developed in [4]. The precise relation between the spectrum, resolvent and eigenvectors of a closed operator and its Feshbach-Schur transform are recalled in Subsection 2.1. We restrict the values of z to the complex open disc

$$D_{\mu/2} := \{z \in \mathbb{C} \mid |z| < \frac{\mu}{2}\}. \tag{1.16}$$

The effective Hamiltonian constructed from $H(\vec{p}) - z\mathbb{1}$, using the Feshbach-Schur map, can be cast into the form

$$H^{(0)}(\vec{p}, z) = \sum_{m+n \geq 0} W_{m,n}^{(0)}(\vec{p}, z), \tag{1.17}$$

where the operators $W_{m,n}^{(0)}(\vec{p}, z)$, $m + n \geq 0$, are called "Wick monomials". The operator $H^{(0)}(\vec{p}, z)$ acts on the subspace $\mathcal{H}_{\text{red}} := \mathbb{1}_{H_f \leq 1} \mathcal{H}_f$ of $\mathcal{H}_f$. The Wick monomials are bounded operators on $\mathcal{H}_{\text{red}}$. They are associated to a sequence

$$\underline{w}^{(0)} := (w_{m,n}^{(0)})_{m+n \geq 0}$$

of bounded measurable functions

$$w^{(0)}_{m,n} : U[\vec{p}^*] \times D_{\mu/2} \times \mathbb{R}_+ \times \mathbb{R}^3 \times \underline{\mathbb{R}}^m \times \underline{\mathbb{R}}^n \to \mathbb{C}$$

that are $\mathcal{C}^1$ in their third and fourth arguments, and symmetric in $\underline{\mathbb{R}}^m$ and $\underline{\mathbb{R}}^n$. The bounded operators $W^{(0)}_{m,n}(\vec{p}, z)$, $m + n \geq 0$, are defined in the sense of quadratic forms by

$$W^{(0)}_{m,n}(\vec{p}, z) := \mathbb{1}_{H_f \leq 1} \int_{\underline{B}_1^{m+n}} \left(\prod_{i=1}^{m} b^*(\underline{k}_i) \right) w^{(0)}_{m,n}(\vec{p}, z, H_f, \vec{P}_f, \underline{k}_1, ..., \underline{k}_m, \underline{\tilde{k}}_1, ..., \underline{\tilde{k}}_n) \left(\prod_{j=1}^{n} b(\underline{\tilde{k}}_j) \right) \prod_{i=1}^{m} d\underline{k}_i \prod_{j=1}^{n} d\underline{\tilde{k}}_j \mathbb{1}_{H_f \leq 1}, \tag{1.18}$$

where $\mathbb{1}_{H_f \leq 1}$ projects on the subspace of photons with energy smaller than one. The functions $w^{(0)}_{m,n}$ are called "kernels" in the literature. They satisfy additional properties that are stated in Section 2.

1.2.3. *Functional calculus.* Before continuing the outline of our proof, we explain how the operators $w^{(0)}_{m,n}(\vec{p}, z, H_f, \vec{P}_f, \underline{k}_1, ..., \underline{k}_m, \underline{\tilde{k}}_1, ..., \underline{\tilde{k}}_n)$ and $w^{(0)}_{0,0}(\vec{p}, z, H_f, \vec{P}_f)$ in (1.18) are defined. Let $f : \mathbb{R} \times \mathbb{R}^3 \to \mathbb{C}$ be a measurable function. Any element $\Psi \in \mathcal{F}_+(L^2(\underline{\mathbb{R}}^3))$ can be written as a sequence of totally symmetric functions $(\psi^{(n)})_{n \geq 0}$ of momenta, with $\psi^{(n)} \in \mathrm{L}^2_s(\underline{\mathbb{R}}^{3n})$. We set

$$f(H_f, \vec{P}_f)\Psi := (f^{(n)}\psi^{(n)})_{n \geq 0}, \tag{1.19}$$

where

$$f^{(0)}\psi^{(0)} := f(0, \vec{0})\psi^{(0)}, \tag{1.20}$$

and, for $n \geq 1$, $(\underline{k_1}, \dots, \underline{k_n})$ in $\underline{\mathbb{R}}^{3n}$,

$$(f^{(n)}\psi^{(n)})(\underline{k_1}, \dots, \underline{k_n}) := f(|\vec{k}_1| + \cdots + |\vec{k}_n|, \vec{k_1} + \cdots + \vec{k_n}) \ \psi^{(n)}(\underline{k_1}, \dots, \underline{k_n}). \tag{1.21}$$

(1.19) defines an (unbounded) operator on $\mathcal{F}_+(L^2(\underline{\mathbb{R}}^3))$ with domain

$$D(f(H_f, \vec{P}_f)) := \{\Psi \in \mathcal{F}_+(L^2(\underline{\mathbb{R}}^3)), \|f(H_f, \vec{P}_f)\Psi\| < \infty\}.$$

Assuming that f is essentially bounded on the subset

$$\mathcal{R} := \{(r, \vec{l}) \in \mathbb{R} \times \mathbb{R}^3 \mid |\vec{l}| \leq r\} \subset \mathbb{R}^4, \tag{1.22}$$

it is easy to check that $f(H_f, \vec{P}_f)$ is bounded with

$$\|f(H_f, \vec{P}_f)\| \leq \text{ess sup}\,\{|f(r, \vec{l})|, (r, \vec{l}) \in \mathcal{R}\}. \tag{1.23}$$

1.2.4. *The renormalization map and its iteration.* Under certain conditions on the norm of the operators $W^{(0)}_{m,n}(\vec{p}, z)$, it is possible to apply to $H^{(0)}(\vec{p}, z)$ a renormalization transformation $\mathcal{R}_\rho$ that leads to a new effective Hamiltonian $H^{(1)}(\vec{p}, z)$ acting on $\mathcal{H}_{\text{red}}$. $H^{(1)}(\vec{p}, z)$ can also be cast into the form

$$H^{(1)}(\vec{p}, z) = \sum_{m+n \geq 0} W^{(1)}_{m,n}(\vec{p}, z), \tag{1.24}$$

where the operators $W^{(1)}_{m,n}(\vec{p}, z)$, $m+n \geq 0$, are Wick monomials associated to a sequence $\underline{w}^{(1)}$ of kernels. The renormalization map $\mathcal{R}_\rho$, constructed in Section 2, is "isospectral" in the same sense as the Feshbach-Schur map discussed above. The map $\mathcal{R}_\rho$ removes the photon degrees of freedom of energy higher than ρ. Under certain assumptions for $H^{(0)}(\vec{p}, z)$, the map $\mathcal{R}_\rho$ can be iterated indefinitely. We then get a sequence of operators $H^{(N)}(\vec{p}, z) := (\mathcal{R}^N_\rho H^{(0)})(\vec{p}, z)$ on

$\mathcal{H}_{\mathrm{red}}$. We prove in Subsection 4.2.2 that the sequence of operators $(W_{m,n}^{(N)}(\vec{p},z))_{N\in\mathbb{N}}$, $m+n\geq 1$, tends to zero in norm when N tends to infinity and that $W_{0,0}^{(N)}(\vec{p},0) = w_{0,0}^{(N)}(\vec{p},0,H_f,\vec{P}_f)$ tends to an element $\alpha(\vec{p})H_f + \vec{\beta}(\vec{p})\cdot\vec{P}_f$ of the fixed-points manifold $\mathcal{M}_{fp} := \mathbb{C}H_f + \mathbb{C}P_{f,x} + \mathbb{C}P_{f,y} + \mathbb{C}P_{f,z}$ of the flow $\mathcal{R}_\rho^N$. This marginal operator has a non-zero eigenvector, the vacuum, and, using the isospectral character of the renormalization map, we can reconstruct the one-particle state of momentum $\vec{p}$ and its associated eigenvalue, for any $\vec{p}\in U[\vec{p}^*]$. This procedure is explained in Subsection 4.2.2. One of the key points of the proof concerns the control of $W_{0,0}^{(N)}(\vec{p},z) = w_{0,0}^{(N)}(\vec{p},z,H_f,\vec{P}_f)$. In particular, to iterate the renormalization map, we need to show that the restriction of $w_{0,0}^{(N)}(\vec{p},z,H_f,\vec{P}_f)$ to the range of $\mathbb{1}_{H_f\geq 3\rho/4}$ is bounded invertible, for all $N\in\mathbb{N}$ and for sufficiently small values of $|z|$. For N large enough, $w_{0,0}^{(N)}(\vec{p},0,H_f,\vec{P}_f)$ is close to $\alpha(\vec{p})H_f + \vec{\beta}(\vec{p})\cdot\vec{P}_f$, and it is important to check that $\alpha(\vec{p})$ stays close to 1 and that $\vec{\beta}(\vec{p})$ stays close to $-\vec{p}/m$ if we want $w_{0,0}^{(N)}(\vec{p},z,H_f,\vec{P}_f)$ to be invertible on the range of $\mathbb{1}_{H_f\geq 3\rho/4}$. For momenta $\vec{p}$ in the open set $U[\vec{p}^*]$, we are able to control the size of the deviations $|\alpha(\vec{p})-1|$ and $|\vec{\beta}(\vec{p})+\vec{p}/m|$ by a fine tuning of the coupling constant λ_0. This fine tuning depends on the parameter $\mu = (m-|\vec{p}^*|)/2m$.

1.2.5. *Renormalization preserves analyticity.* A key ingredient of the proof of Theorem 1.1 is the fact that the renormalization map preserves analyticity. If $(\vec{p},z)\mapsto H^{(0)}(\vec{p},z)$ is analytic on $U[\vec{p}^*]\times D_{\mu/2}$, we can show inductively that the operator-valued functions $(\vec{p},z)\mapsto H^{(N)}(\vec{p},z)$ are analytic on $U[\vec{p}^*]\times D_{\mu/2}$, for all $N\geq 1$. We prove this result in Subsection 4.1. The analyticity of the eigenvalues and the eigenprojections is established in Section 5.

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2. Spectral analysis tools

In this section, we explain in details the spectral tools sketched in Subsection 1.2. We begin with the "smooth" isospectral decimation method introduced in [4] and further improved in [24]. This method maps a closed operator acting on a Hilbert space $\mathcal{H}$ to a new operator acting on a subspace of $\mathcal{H}$. We refer the reader to [4, 24] for proofs.

2.1. **The Feshbach-Schur map.**

2.1.1. *Heuristic derivation of the Feshbach-Schur map.* Let $\mathcal{H}$ be a separable Hilbert space and let H be a self-adjoint operator on $\mathcal{H}$. We assume that H has an eigenvalue E, with associated eigenvector Ψ. Let P be an orthogonal projection with the property that $P\Psi\neq 0$. Introducing P and $P^\perp := 1-P$ on both sides of the identity $H\Psi = E\Psi$, we obtain that

$$(PHP + PHP^\perp)\Psi = EP\Psi, \tag{2.1}$$

$$(P^\perp HP + P^\perp HP^\perp)\Psi = EP^\perp\Psi. \tag{2.2}$$

Now we assume that $P^\perp(H-E)P^\perp$ is bounded invertible on $P^\perp\mathcal{H}$. In this case, we deduce from (2.2) that

$$P^\perp\Psi = -(P^\perp(H-E)P^\perp)^{-1}P^\perp HP\Psi. \tag{2.3}$$

Reporting into (2.1),

$$P(H-E)P\Psi - PH(P^\perp(H-E)P^\perp)^{-1}P^\perp HP\Psi = 0. \tag{2.4}$$

$F_P(H) := PHP - PH(P^\perp HP^\perp)^{-1}P^\perp HP$ is called the Feshbach-Schur map. The operator $F_P(H)$ is defined on $P\mathcal{H}$ and has the remarkable property that $\Psi \in \ker(F_P(H-E))$. There is a one-to-one correspondence between $\mathrm{Ker}(F_P(H-E))$ and $\mathrm{Ker}(H-E)$. In particular, $F_P(H-E)P\phi = 0$ if and only if $(H-E)(P - P^\perp(P^\perp(H-E)P^\perp)^{-1}P^\perp HP)\phi = 0$. We can therefore reconstruct the eigenvectors of H with eigenvalue E from the kernel of $F_P(H-E)$.

2.1.2. *The smooth Feshbach-Schur map.* The smooth Feshbach-Schur map is a generalization of the discrete Feshbach-Schur map discussed above. We generalize the construction to closed operators H and to bounded operators χ and $\overline{\chi}$ that share some common properties with the orthogonal projection P above. Let χ and $\overline{\chi}$ be two commuting and non zero bounded operators on $\mathcal{H}$ satisfying the identity $\chi^2 + \overline{\chi}^2 = \mathbb{1}$.

Definition 2.1 (Feshbach pair). *Let H and T be two closed operators defined on the same domain $D \subset \mathcal{H}$. We define $W = H - T : D \to \mathcal{H}$ and introduce the operators*

$$H_\chi := T + \chi W \chi, \qquad H_{\overline{\chi}} := T + \overline{\chi} W \overline{\chi}. \tag{2.5}$$

We say that (H,T) is a Feshbach pair for χ if it satisfies the following properties:
(a) $\chi T \subset T\chi$, $\overline{\chi} T \subset T\overline{\chi}$,
(b) T and $H_{\overline{\chi}}$ are bounded invertible from $D \cap \mathrm{Ran}(\overline{\chi})$ to $\mathrm{Ran}(\overline{\chi})$,
(c) $\overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi : D \to \mathcal{H}$ is bounded.

If (H,T) is a Feshbach pair for χ, we can define the Feshbach map $F_\chi(H,T) : D \to \mathcal{H}$ by

$$F_\chi(H,T) := H_\chi - \chi W \overline{\chi}(H_{\overline{\chi}})^{-1}\overline{\chi} W \chi. \tag{2.6}$$

The most important feature of the map $(H,T) \mapsto F_\chi(H,T)$ is its isospectrality, meant in the following sense:

Theorem 2.2 ([4],[24]). *Let (H,T) be a Feshbach pair for χ with associated Feshbach map $F_\chi(H,T)$.*
(1) $H : D \to \mathcal{H}$ is bounded invertible iff $F_\chi(H,T) : \mathrm{Ran}(\chi) \cap D \to \mathrm{Ran}(\chi)$ is bounded invertible. Furthermore, if we define the auxiliary operators

$$Q_\chi := \chi - \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} W \chi, \qquad Q_\chi^\sharp := \chi - \chi W \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi}, \tag{2.7}$$

then

$$H^{-1} = \overline{\chi} H_{\overline{\chi}}^{-1} \overline{\chi} + Q_\chi F_\chi^{-1}(H,T) Q_\chi^\sharp, \qquad F_\chi^{-1}(H,T) = \chi H^{-1} \chi + \overline{\chi} T^{-1} \overline{\chi}. \tag{2.8}$$

(2) The restrictions $\chi : \mathrm{Ker}(H) \to \mathrm{Ker}(F_\chi(H,T))$ and $Q_\chi : \mathrm{Ker}(F_\chi(H,T)) \to \mathrm{Ker}(H)$ are linear isomorphisms and inverse to each other.

We reformulate the results (1) and (2) in order to highlight the isospectral property of the Feshbach-Schur map: (1) implies that 0 is in the resolvent set of H if and only if it is in the resolvent set of $F_\chi(H,T)$. (2) means that the kernels of H and $F_\chi(H,T)$ have the same dimension and that the eigenvectors of H and $F_\chi(H,T)$ corresponding to the eigenvalue 0 are in one-to-one correspondence: If ψ satisfies $H\psi = 0$, then $\phi := \chi\psi$ solves $F_\chi(H,T)\phi = 0$. If $\phi \in \mathrm{Ran}(\chi)$ satisfies $F_\chi(H,T)\phi = 0$, then $\psi := Q_\chi \phi$ solves $H\psi = 0$.

Remarks.

- *As we wish to study the spectrum of H, we consider the operator $H - z\mathbb{1} = T + W - z\mathbb{1}$ instead of H. If $H\psi = z\psi$, then $F_\chi(H - z\mathbb{1}, T - z\mathbb{1})\chi\psi = 0$. Furthermore, if $F_\chi(H - z\mathbb{1}, T - z\mathbb{1})\phi = 0$, then $HQ_\chi\phi = zQ_\chi\phi$.*
- *In the following, we choose for χ an operator of the type $\chi(H_f)$, where H_f is the free field Hamiltonian operator defined in* (1.15)*, and χ is a positive smooth function with compact support included in the interval* $[0, 1]$*. The operator $\chi(H_f)$ is defined by the functional calculus for self-adjoint operators. The smoothness of χ is useful in our setting because we will need to take (and to bound) derivatives of χ.*

2.2. **Wick monomials and their kernels.** We already outlined in Subsection 1.2 that the operator constructed from $H(\vec{p}) - z\mathbb{1}$ with the help of the smooth Feshbach-Schur map can be cast into the form

$$H^{(0)}(\vec{p}, z) = \sum_{m+n\geq 0} W^{(0)}_{m,n}(\vec{p}, z), \tag{2.9}$$

where the operators $W^{(0)}_{m,n}(\vec{p}, z)$, $m + n \geq 0$, are defined in the sense of quadratic forms by

$$W^{(0)}_{m,n}(\vec{p}, z) := \mathbb{1}_{H_f\leq 1} \int_{\underline{B}_1^{m+n}} \left(\prod_{i=1}^{m} b^*(\underline{k}_i)\right) w^{(0)}_{m,n}(\vec{p}, z, H_f, \vec{P}_f, \underline{k}_1, ..., \underline{k}_m, \underline{\tilde{k}}_1, ..., \underline{\tilde{k}}_n) \left(\prod_{j=1}^{n} b(\underline{\tilde{k}}_j)\right) \prod_{i=1}^{m} d\underline{k}_i \prod_{j=1}^{n} d\underline{\tilde{k}}_j \mathbb{1}_{H_f\leq 1}. \tag{2.10}$$

The calculation leading to (2.9) is displayed explicitly in Appendix C. For the time being, we focus on the properties of the kernels $w^{(0)}_{m,n}$, for $m+n \geq 0$. Lemma 4.1 below shows that these properties are preserved under the renormalization map if the initial sequence of kernels $\underline{w}^{(0)}$ lies in a suitably small polydisc $\mathcal{B}(\gamma, \delta, \varepsilon)$.

2.2.1. *Notation.* It is a little difficult to come up with concise formulae involving the kernels $w_{m,n}$, because these kernels depend on many arguments. This is the reason why we introduce the following shorthand notations that are used throughout our text:

$$\underline{k}^{(m)} := (\underline{k}_1, \dots, \underline{k}_m) \in \underline{\mathbb{R}}^{3m}, \quad \underline{\tilde{k}}^{(n)} := (\underline{\tilde{k}}_1, \dots, \underline{\tilde{k}}_n) \in \underline{\mathbb{R}}^{3n},$$

$$\underline{K}^{(m,n)} := (\underline{k}^{(m)}, \underline{\tilde{k}}^{(n)}), \quad \mathrm{d}\underline{K}^{(m,n)} := \prod_{i=1}^{m} \mathrm{d}\underline{k}_i \prod_{j=1}^{n} \mathrm{d}\underline{\tilde{k}}_j,$$

$$|\underline{K}^{(m,n)}| := |\underline{k}^{(m)}|\, |\underline{\tilde{k}}^{(n)}|, \quad |\underline{k}^{(m)}| := \prod_{i=1}^{m} |\vec{k}_i|, \quad |\underline{\tilde{k}}^{(n)}| := \prod_{j=1}^{n} |\vec{\tilde{k}}_j|,$$

$$\Sigma[\underline{k}^{(m)}] := \sum_{i=1}^{m} |\vec{k}_i|, \quad \Sigma[\underline{\tilde{k}}^{(n)}] := \sum_{j=1}^{n} |\vec{\tilde{k}}_j|, \quad \vec{\Sigma}[\underline{k}^{(m)}] := \sum_{i=1}^{m} \vec{k}_i, \quad \vec{\Sigma}[\underline{\tilde{k}}^{(n)}] := \sum_{j=1}^{n} \vec{\tilde{k}}_j,$$

$$b^*(\underline{k}^{(m)}) := \prod_{i=1}^{m} b^*_{\lambda_i}(\vec{k}_i), \quad b(\underline{\tilde{k}}^{(n)}) := \prod_{j=1}^{n} b_{\lambda_j}(\vec{\tilde{k}}_j).$$

For $\rho \in \mathbb{C}$, we set

$$\rho\underline{k}^{(m)} := (\rho\vec{k}_1, \lambda_1, \dots, \rho\vec{k}_m, \lambda_m), \quad \rho\underline{K}^{(m,n)} := (\rho\underline{k}^{(m)}, \rho\underline{\tilde{k}}^{(n)}).$$

We remind the reader that $B_1 = \{\vec{k} \in \mathbb{R}^3, |\vec{k}| \leq 1\}$ and we introduce the set

$$\mathcal{B} := \{(r, \vec{l}) \in [0,1] \times B_1, |\vec{l}| \leq r\}. \tag{2.11}$$

2.2.2. *Kernels.* In Section 3, we will show that, for all $\vec{p} \in U[\vec{p}^*] = \{\vec{p} \in \mathbb{C}^3 \mid |\vec{p} - \vec{p}^*| < \mu m\}$ and $z \in D_{\mu/2} = \{z \in \mathbb{C}, |z| < \mu/2\}$, the kernels $w^{(0)}_{m,n}$, $m + n \geq 0$, defined in (2.9)–(2.10) belong to certain Banach spaces of functions that we introduce in the following definition.

Definition 2.3 (The Banach spaces $\mathcal{W}^\sharp_{0,0}$ and $\mathcal{W}^\sharp_{m,n}$)**.**

- *The set of functions $w_{0,0} \in \mathrm{C}^1(\mathcal{B}; \mathbb{C})$ equipped with the norm*

$$\|w_{0,0}\|^\sharp := |w_{0,0}(0, \vec{0})| + \|\partial_r w_{0,0}\|_\infty + \sum_{i=1}^{3} \|\partial_{l_i} w_{0,0}\|_\infty \tag{2.12}$$

is denoted by $\mathcal{W}^\sharp_{0,0}$; $\mathcal{W}^\sharp_{0,0}$ defines a Banach space.

- *For $m + n \geq 1$, the set of functions $w_{m,n} : \mathcal{B} \times \underline{B}_1^m \times \underline{B}_1^n \to \mathbb{C}$ that are measurable on $\underline{B}_1^m \times \underline{B}_1^n$, totally symmetric on $\underline{B}_1^m$ and $\underline{B}_1^n$, of class $\mathrm{C}^1(\mathcal{B})$ for almost every $\underline{K}^{(m,n)} \in \underline{B}_1^m \times \underline{B}_1^n$, and that obey the norm bound*

$$\|w_{m,n}\|^\sharp := \|w_{m,n}\|_{\frac{1}{2}} + \|\partial_r w_{m,n}\|_{\frac{1}{2}} + \sum_{i=1}^{3} \|\partial_{l_i} w_{m,n}\|_{\frac{1}{2}} < \infty, \tag{2.13}$$

where

$$\|w_{m,n}\|_{\frac{1}{2}} := \sup_{(r,\vec{l}) \in \mathcal{B}} \operatorname*{ess\,sup}_{\underline{K}^{(m,n)} \in \underline{B}_1^{m+n}} \left| w_{m,n}(r, \vec{l}, \underline{K}^{(m,n)}) \right| |\underline{K}^{(m,n)}|^{-1/2}, \tag{2.14}$$

defines a Banach space that we denote by $\mathcal{W}^\sharp_{m,n}$.

The choice of the exponent in the factor $|\underline{K}^{(m,n)}|^{-1/2}$ in (2.14) is related to the infrared behavior of the model we consider, and insures an optimal rate of convergence to 0 of the renormalized kernels $w^{(N)}_{m,n}$, $m + n \geq 1$, obtained by the renormalization procedure. More precisely, if the form factor $|k|^{1/2}$ in the interaction Hamiltonian H_I in (1.15) is replaced by $|k|^{-1/2+\varepsilon}$, our method would work in the same way, provided that $|\underline{K}^{(m,n)}|^{-1/2}$ in (2.14) is replaced by $|\underline{K}^{(m,n)}|^{1/2-\varepsilon}$. Furthermore, we will prove below estimates of the form $\|w^{(N)}_{m,n}\|^\sharp = \mathcal{O}(\rho^{N+1})$ where $\rho < 1$ is a scale parameter. Replacing $|\underline{K}^{(m,n)}|^{-1/2}$ in (2.14) by $|\underline{K}^{(m,n)}|^{1/2-\varepsilon}$ with $0 < \varepsilon \leq 1$ would lead to estimates of the form $\|w^{(N)}_{m,n}\|^\sharp = \mathcal{O}(\rho^{\varepsilon(N+1)})$.

To a kernel $w_{0,0} \in \mathcal{W}^\sharp_{0,0}$, we can associate the bounded operator $w_{0,0}(H_f, \vec{P}_f)\mathbb{1}_{H_f \leq 1}$ defined by the functional calculus of Subsection 1.2. The choice of the norm $\| \cdot \|^\sharp$ will allow us to express the fact that the "free" effective Hamiltonian $w^{(0)}_{0,0}(\vec{p}, z, H_f, \vec{P}_f)$ in (2.9) is close to the operator $H_f - m^{-1}\vec{p} \cdot \vec{P}_f - z$, in the sense that

$$\|w^{(0)}_{0,0}(\vec{p}, z, r, \vec{l}) - (r - m^{-1}\vec{p} \cdot \vec{l} - z)\|^\sharp$$

tends to 0 as the coupling constant $\lambda_0 \to 0$ (see Section 3). We remark that another possible choice for $\|w_{0,0}\|^\sharp$ would be given by the equivalent norm

$$\|w_{0,0}\|_\infty + \|\partial_r w_{0,0}\|_\infty + \sum_{i=1}^{3} \|\partial_{l_i} w_{0,0}\|_\infty.$$

To a kernel $w_{m,n} \in \mathcal{W}^\sharp_{m,n}$, we associate the Wick monomial

$$W_{m,n}(w_{m,n}) := \mathbb{1}_{H_f \le 1} \int_{\underline{B}_1^{m+n}} b^*(\underline{k}^{(m)}) w_{m,n}[H_f, \vec{P}_f, \underline{K}^{(m,n)}] b(\underline{\tilde{k}}^{(n)}) d\underline{K}^{(m,n)} \mathbb{1}_{H_f \le 1}, \tag{2.15}$$

where, for a.e. $\underline{K}^{(m,n)} \in \underline{B}_1^{m+n}$, $w_{m,n}[H_f, \vec{P}_f, \underline{K}^{(m,n)}]$ is defined by the functional calculus of Subsection 1.2. The expression in (2.15) defines a quadratic form on the set of vectors in $\mathcal{H}_f$ with finitely many particles. The following lemma shows that it extends to a bounded quadratic form on $\mathcal{H}_{\mathrm{red}}$. The associated bounded operator will be denoted by the same symbol.

Lemma 2.4. *Let $m + n \ge 1$. For all $w_{m,n} \in \mathcal{W}^\sharp_{m,n}$, the quadratic form defined in* (2.15) *extends to a bounded quadratic form with norm satisfying*

$$\|W_{m,n}(w_{m,n})\| \le (m!n!)^{-\frac{1}{2}} (8\pi)^{(m+n)/2} \|w_{m,n}\|_{\frac{1}{2}}. \tag{2.16}$$

Lemma 2.4 asserts that we can control the norm of Wick monomials by controlling the norm of their associated kernels. We remark that we considered a L^∞-norm in (2.14) while a L^2-norm is used in [4]. Moreover the operators $w_{m,n}[H_f, \vec{P}_f, \underline{K}^{(m,n)}]$ in (2.15) depend both on H_f and $\vec{P}_f$ while the corresponding operators considered in [4] only depend on H_f. Nevertheless, the proof of Lemma 2.4 is a straightforward adaptation of the one of [4, Theorems 3.1] (the proof is in fact slightly easier with the L^∞-norm defined in (2.14) than with the L^2-norm used in [4]).

2.2.3. *Hamiltonians associated with a sequence of kernels.* We want to bound the series of Wick monomials in (2.9). This amounts to assume that the kernels $w_{m,n}$ satisfy some summability properties. We introduce the Banach space

$$\mathcal{W}^\sharp := \bigoplus_{m+n \ge 0} \mathcal{W}^\sharp_{m,n}, \tag{2.17}$$

equipped with the norm

$$\|\underline{w}\|^\sharp_\xi := \sum_{m+n \ge 0} \xi^{-(m+n)} \|w_{m,n}\|^\sharp, \tag{2.18}$$

where ξ is a fixed positive number smaller than one, and $\underline{w} = (w_{m,n})_{m+n \ge 0}$. Let $\mathcal{H}_{\mathrm{red}}$ be the subspace of photons of energy smaller than 1, i.e.

$$\mathcal{H}_{\mathrm{red}} := \mathbb{1}_{H_f \le 1} \mathcal{H}_f. \tag{2.19}$$

In order to associate with any element of $\mathcal{W}^\sharp$ a bounded operator (similar to the one in (2.9)), we introduce a map $H : \mathcal{W}^\sharp \to \mathcal{B}(\mathcal{H}_{\mathrm{red}})$.

Definition 2.5 (The linear map $H(\underline{w})$)**.** *For all $\underline{w} \in \mathcal{W}^\sharp$, we set*

$$H(\underline{w}) := \sum_{m+n \ge 0} W_{m,n}(\underline{w}) = W_{0,0}(\underline{w}) + W_{\ge 1}(\underline{w}), \tag{2.20}$$

where

$$W_{0,0}(\underline{w}) := w_{0,0}[H_f, \vec{P}_f] \mathbb{1}_{H_f \le 1}, \quad W_{\ge 1}(\underline{w}) := \sum_{m+n \ge 1} W_{m,n}(\underline{w}), \tag{2.21}$$

and, for $m+n \geq 1$, $W_{m,n}(\underline{w}) := W_{m,n}(w_{m,n})$ *is the Wick monomial defined in* (2.15).

The following lemma shows that $H(\underline{w})$ defines a bounded operator on $\mathcal{H}_{\rm red}$ for all $\underline{w} \in \mathcal{W}^\sharp$. It is a direct consequence of Lemma 2.4.

Lemma 2.6. *For all* $0 < \xi < 1/\sqrt{8\pi}$, *the linear map* $H : \mathcal{W}^\sharp \to \mathcal{B}(\mathcal{H}_{\rm red})$ *defined by* (2.20) *satisfies,*

$$\| H(\underline{w}) \| \leq \| \underline{w} \|_\xi^\sharp, \tag{2.22}$$

for all $\underline{w} \in \mathcal{W}^\sharp$.

Remarks. *As in* [4, Theorem 3.3] *(see also* [28, Theorem 5.4]*), we can verify that the map* H *is injective provided that the kernels* $w_{m,n}$ *are restricted to the set*

$$\mathcal{B}_{m,n} := \{ (r, \vec{l}, \underline{K}^{(m,n)}) \in \mathcal{B} \times \underline{B}_1^m \times \underline{B}_1^n, r \leq 1 - \max\left(\Sigma[\underline{k}^{(m)}], \Sigma[\underline{k}^{(n)}] \right) \Big\}.$$

The proof is similar to that of Lemma 4.2 given in Appendix D.

Next, we take into account the fact that the operators considered in the following depend on the total momentum $\vec{p}$ and the spectral parameter z. We remind the reader that $U[\vec{p}^*]$ is the set of complex momenta $\vec{p}$ such that $|\vec{p} - \vec{p}^*| < \mu m$, where $\mu = (m - |\vec{p}^*|)/2m$, and that $D_{\mu/2}$ is the complex open disc of radius $\mu/2$ around the origin. We will assume that $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$.

The kernels $w_{m,n}(\vec{p}, z, r, \vec{l}, \underline{K}^{(m,n)})$ can be viewed as functions

$$U[\vec{p}^*] \times D_{\mu/2} \ni (\vec{p}, z) \mapsto w_{m,n}(\vec{p}, z) \in \mathcal{W}_{m,n}^\sharp.$$

The set of functions from $U[\vec{p}^*] \times D_{\mu/2}$ to $\mathcal{W}_{m,n}^\sharp$ is denoted by $\mathcal{W}_{m,n}$. Likewise, the set of functions from $U[\vec{p}^*] \times D_{\mu/2}$ to $\mathcal{W}^\sharp$ is denoted by $\mathcal{W}$.

Following [22], we introduce a notation that proves to be convenient for our analysis: We denote the set of functions $U[\vec{p}^*] \times D_{\mu/2} \ni (\vec{p}, z) \mapsto H(\underline{w}(\vec{p}, z)) \in \mathcal{B}(\mathcal{H}_{\rm red})$ by $\mathcal{W}_{\rm op}$.

2.3. **The renormalization map.** We define the renormalization map that we shall use in the sequel. Our construction is similar to the one in [4]. The main difference, as already appears in Definition 2.3, is that the kernels we consider depend on the variables $(r, \vec{l})$ corresponding to the operators $(H_f, \vec{P}_f)$, while the kernels in [4] only depend on r.

Let $0 < \rho < 1$ be a fixed scale parameter. The renormalization map maps any operator $H(\underline{w}) \in \mathcal{B}(\mathcal{H}_{\rm red})$ defined in (2.20), to a new operator $\mathcal{R}_\rho H(\underline{w}) \in \mathcal{B}(\mathcal{H}_{\rm red})$. The spectrum of $\mathcal{R}_\rho H(\underline{w})$ may be easier to analyze than the spectrum of $H(\underline{w})$, because $\mathcal{R}_\rho$ eliminates the degrees of freedom of energy bigger than ρ. Thanks to the isospectrality of $\mathcal{R}_\rho$, we can reconstruct some spectral properties of the initial operator $H(\underline{w})$. The renormalization map is a composition of three distinct transformations:

- An analytic transformation, E_ρ, of the spectral parameter z,
- An application of the Feshbach-Schur map,
- A scale transformation S_ρ.

In what follows, we give an overview of each of these transformations. The $(\vec{p}, z)$ dependence of the kernels is kept explicit when needed.

2.3.1. *The scale transformation S_ρ.* For $\rho > 0$, we define the unitary map Γ_ρ on $\mathcal{H}_f$ by

$$\Gamma_\rho \, \Omega := \Omega,$$
$$\big(\Gamma_\rho \, \Psi\big)^{(p)}(\underline{k}^{(p)}) := \rho^{\frac{3p}{2}} \Psi^{(p)}(\rho \underline{k}^{(p)}).$$

We recall the following scaling properties that can be verified easily, using the definitions of the operators involved:

$$\Gamma_\rho H_f \Gamma_\rho^* = \rho H_f, \qquad \Gamma_\rho \vec{P}_f \Gamma_\rho^* = \rho \vec{P}_f. \tag{2.23}$$

This implies that $\Gamma_\rho : \mathrm{Ran}(\mathbb{1}_{H_f \le \rho}) \to \mathrm{Ran}(\mathbb{1}_{H_f \le 1})$. Recalling the definition $\mathcal{H}_{\mathrm{red}} = \mathbb{1}_{H_f \le 1} \mathcal{H}_f$, we define the map $S_\rho : \mathcal{B}(\mathcal{H}_{\mathrm{red}}) \to \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ by

$$S_\rho(A) := \rho^{-1} \Gamma_\rho A \Gamma_\rho^*,$$

for any bounded operator A on $\mathcal{H}_{\mathrm{red}}$. It follows from (2.23) that

$$S_\rho(H_f) = H_f, \qquad S_\rho(\vec{P}_f) = \vec{P}_f, \qquad S_\rho(f(H_f, \vec{P}_f)) = \rho^{-1} f(\rho H_f, \rho \vec{P}_f),$$

for any measurable function f. Furthermore, one can verify that, for any $m, n \in \mathbb{N} \cup \{0\}$,

$$S_\rho(W_{m,n}(\underline{w})) = W_{m,n}(s_\rho(\underline{w})),$$

with

$$s_\rho(\underline{w})_{m,n}(H_f, \vec{P}_f, \underline{K}^{(m,n)}) := \rho^{\frac{3}{2}(m+n)-1} w_{m,n}(\rho H_f, \rho \vec{P}_f, \rho \underline{K}^{(m,n)}). \tag{2.24}$$

The definition (2.14) of the norm $\| \cdot \|_{\frac{1}{2}}$ implies that

$$\begin{aligned}
\|s_\rho(\underline{w})_{m,n}\|_{\frac{1}{2}} &= \rho^{\frac{3}{2}(m+n)-1} \sup_{(r,\vec{l}) \in \mathcal{B}} \ \operatorname*{ess\,sup}_{\underline{K}^{(m,n)} \in \underline{B}_1^{m+n}} \Big| w_{m,n}(\rho r, \rho \vec{l}, \rho \underline{K}^{(m,n)}) \Big| \, |\underline{K}^{(m,n)}|^{-1/2} \\
&= \rho^{2(m+n)-1} \sup_{(r,\vec{l}) \in \mathcal{B}} \ \operatorname*{ess\,sup}_{\underline{K}^{(m,n)} \in \underline{B}_1^{m+n}} \Big| w_{m,n}(\rho r, \rho \vec{l}, \rho \underline{K}^{(m,n)}) \Big| \, |\rho \underline{K}^{(m,n)}|^{-1/2} \\
&\le \rho^{2(m+n)-1} \|w_{m,n}\|_{\frac{1}{2}},
\end{aligned}$$

and likewise that

$$\|\partial_r s_\rho(\underline{w})_{m,n}\|_{\frac{1}{2}} \le \rho^{2(m+n)} \|\partial_r w_{m,n}\|_{\frac{1}{2}},$$
$$\|\partial_{l_i} s_\rho(\underline{w})_{m,n}\|_{\frac{1}{2}} \le \rho^{2(m+n)} \|\partial_{l_i} w_{m,n}\|_{\frac{1}{2}}.$$

As $\|s_\rho(\underline{w})_{m,n}\|_{\frac{1}{2}}$ controls the norm of the operators $S_\rho(W_{m,n}(\underline{w}))$, these estimates show that the operators $W_{m,n}(\underline{w})$ contract under rescaling. Operators possessing this property are called *irrelevant.* In contrast, $\mathbb{1}$ expands under S_ρ by a factor ρ^{-1} and is called a *relevant* operator. Terms linear in H_f and $\vec{P}_f$ stay unchanged and are called *marginal* operators.

2.3.2. *Transformation of the spectral parameter E_ρ.* The operator $H(\underline{w})$, introduced in Definition (2.5), can be canonically decomposed into the sum of three bounded operators on $\mathcal{H}_{\mathrm{red}}$:

- A marginal operator $w_{0,0}(H_f, \vec{P}_f) - w_{0,0}(0, \vec{0})\mathbb{1}$,
- A relevant operator $w_{0,0}(0, \vec{0})\mathbb{1}$,
- An irrelevant operator $W_{\ge 1}(\underline{w})$.

We introduce a polydisc, $\mathcal{B}(\gamma,\delta,\varepsilon)$, contained in the space of operator-valued functions $\mathcal{W}_{\rm op}$ defined above (the elements of $\mathcal{W}_{\rm op}$ are the maps $U[\vec{p}^*]\times D_{\mu/2}\ni(\vec{p},z)\mapsto H(\underline{w}(\vec{p},z))$).

Definition 2.7 (The polydisc $\mathcal{B}(\gamma,\delta,\varepsilon)$)**.** *Let $\gamma,\delta,\varepsilon>0$. The polydisc $\mathcal{B}(\gamma,\delta,\varepsilon)\subset\mathcal{W}_{\rm op}$ is defined by*

$$\begin{aligned}\mathcal{B}(\gamma,\delta,\varepsilon):=\Big\{H(\underline{w}(\cdot,\cdot))\in\mathcal{W}_{\rm op},\\ &\sup_{(\vec{p},z)\in U[\vec{p}^*]\times D_{\mu/2}}\Big\|w_{0,0}(\vec{p},z,r,\vec{l})-w_{0,0}(\vec{p},z,0,\vec{0})-(r-m^{-1}\vec{p}\cdot\vec{l})\Big\|^\sharp\le\gamma,\\ &\sup_{(\vec{p},z)\in U[\vec{p}^*]\times D_{\mu/2}}\Big|w_{0,0}(\vec{p},z,0,\vec{0})+z\Big|\le\delta,\\ &\sup_{(\vec{p},z)\in U[\vec{p}^*]\times D_{\mu/2}}\Big\|\underline{w}(\vec{p},z)\Big\|^\sharp_{\xi,\ge1}\le\varepsilon\Big\},\end{aligned}$$

where

$$\|\underline{w}\|^\sharp_{\xi,\ge1}:=\sum_{m+n\ge1}\xi^{-(m+n)}\|w_{m,n}\|^\sharp,\qquad\xi<\frac{1}{\sqrt{8\pi}}.\tag{2.25}$$

An element of $\mathcal{B}(\gamma,\delta,\varepsilon)$ is close to the operator-valued functions $(\vec{p},z)\mapsto H_f-m^{-1}\vec{p}\cdot\vec{P}_f-z$, in the sense that

- The marginal part $w_{0,0}(\vec{p},z,r,\vec{l})-w_{0,0}(\vec{p},z,0,\vec{0})$ is at a distance $\le\gamma$ of $r-m^{-1}\vec{p}\cdot\vec{l}$ (for the norm $\|\cdot\|^\sharp$), uniformly in $(\vec{p},z)$,
- The relevant part $w_{0,0}(\vec{p},z,0,\vec{0})$ is at a distance $\le\delta$ of $-z$, uniformly in $(\vec{p},z)$,
- The irrelevant part $(w_{m,n}(\vec{p},z))_{m+n\ge1}$ is smaller than ε for the norm $\|\cdot\|^\sharp_\xi$, uniformly in $(\vec{p},z)$.

The instability of the identity operator $\mathbb{1}$ under S_ρ forces us to fine-tune the choice of the spectral parameter. For $w_{0,0}\in\mathcal{W}_{0,0}$, we introduce the set

$$\mathcal{U}[w_{0,0}]:=\Big\{(\vec{p},z)\in U[\vec{p}^*]\times D_{\mu/2},\ |w_{0,0}(\vec{p},z,0,\vec{0})|<\frac{\mu\rho}{2}\Big\}.$$

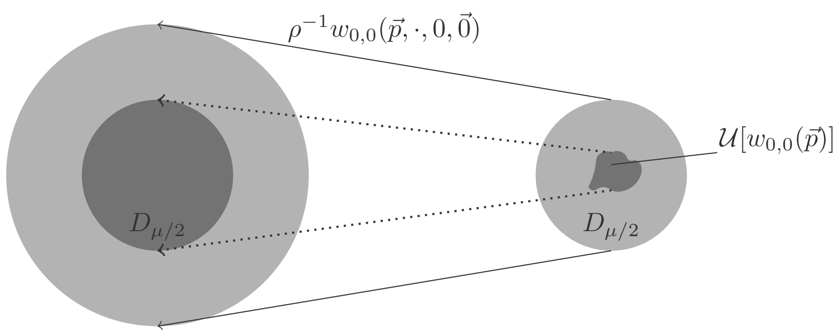
If $(\vec{p},z)\mapsto w_{0,0}(\vec{p},z,0,\vec{0})$ is continuous, $\mathcal{U}[w_{0,0}]$ is an open subset of $\mathbb{C}^4$. Setting

$$\mathcal{U}[w_{0,0}(\vec{p})]:=\Big\{z\in D_{\mu/2},\ |w_{0,0}(\vec{p},z,0,\vec{0})|<\frac{\mu\rho}{2}\Big\},$$

we have that

$$\mathcal{U}[w_{0,0}]=\{(\vec{p},z)\in U[\vec{p}^*]\times D_{\mu/2}\mid z\in\mathcal{U}[w_{0,0}(\vec{p})]\}.$$

The following picture illustrates how the values of the spectral parameter z are restricted.



Definition 2.8 (The rescaling map E_ρ). *Let $w_{0,0} \in \mathcal{W}_{0,0}$. The rescaling map $E_\rho : \mathcal{U}[w_{0,0}] \to U[\vec{p}^*] \times D_{\mu/2}$, is defined by*

$$E_\rho(\vec{p}, z) := (\vec{p}, -\rho^{-1} w_{0,0}(\vec{p}, z, 0, \vec{0})). \tag{2.26}$$

Note that we only rescale z, and do not change the value of $\vec{p}$. The following lemma enables us to control the unstable manifold by rescaling the spectral parameter z before carrying out the scale transformation S_ρ.

Lemma 2.9. *Let $0 < \rho < 1/2$, $0 < \xi < 1/\sqrt{8\pi}$, $\gamma > 0$, $0 < \delta \ll \rho\mu$ and $\varepsilon > 0$. Let $H(\underline{w}(\cdot,\cdot)) \in \mathcal{B}(\gamma, \delta, \varepsilon)$ and assume that the map $(\vec{p}, z) \mapsto w_{0,0}(\vec{p}, z, 0, \vec{0}) \in \mathbb{C}$ is analytic on $U[\vec{p}^*] \times D_{\mu/2}$. Then $\mathcal{U}[w_{0,0}] \neq \emptyset$ and the map $E_\rho : \mathcal{U}[w_{0,0}] \to U[\vec{p}^*] \times D_{\mu/2}$ is biholomorphic. Its inverse is denoted by E_ρ^{-1}.*

Proof. The proof is similar to [4, Lemma 3.4] and [28, Lemma 6.1], except that the models treated in these references describe non moving atoms for which the map E_ρ only depends on z. We give a detailed proof in our case, since E_ρ also depends on the total momentum $\vec{p}$.

Step 1. First, we show that

$$U[\vec{p}^*] \times D_{\mu\rho/2-\delta} \subset \mathcal{U}[w_{0,0}] \subset U[\vec{p}^*] \times D_{\mu\rho/2+\delta}. \tag{2.27}$$

Let $(\vec{p}, z) \in \mathcal{U}[w_{0,0}]$. Since $H(\underline{w}(\cdot,\cdot)) \in \mathcal{B}(\gamma, \delta, \varepsilon)$, we have that

$$|z| \leq |z + w_{0,0}(\vec{p}, z, 0, \vec{0})| + |w_{0,0}(\vec{p}, z, 0, \vec{0})| < \mu\rho/2 + \delta,$$

and hence the second inclusion in (2.27) follows.

Let $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu\rho/2-\delta}$. Using again that $H(\underline{w}(\cdot,\cdot)) \in \mathcal{B}(\gamma, \delta, \varepsilon)$ gives

$$|w_{0,0}(\vec{p}, z, 0, \vec{0})| \leq |w_{0,0}(\vec{p}, z, 0, \vec{0}) + z| + |z| < \mu\rho/2,$$

and hence the first inclusion in (2.27) is proven. This shows that $\mathcal{U}[w_{0,0}] \neq \emptyset$, because $\delta \ll \rho\mu$ and, therefore, $D_{\mu\rho/2-\delta} \neq \emptyset$.

Step 2. We prove that $E_\rho : \mathcal{U}[w_{0,0}] \to E_\rho(\mathcal{U}[w_{0,0}])$ is biholomorphic. To this end, by the inverse function theorem, it suffices to show that $\det dE_\rho(\vec{p}, z) \neq 0$ for any $(\vec{p}, z) \in \mathcal{U}[w_{0,0}]$. It follows directly from (2.26) that

$$\det dE_\rho(\vec{p}, z) = -\rho^{-1} \partial_z w_{0,0}(\vec{p}, z, 0, \vec{0}).$$

Now, let $(\vec{p}, z) \in \mathcal{U}[w_{0,0}]$. By Step 1, $|z| < \mu\rho/2 + \delta$, and hence, with $\mathcal{C} := \{z' \in \mathbb{C}, |z'| = \mu/3\}$, Cauchy's formula gives

$$\Big|\partial_z\Big(w_{0,0}(\vec{p}, z, 0, \vec{0}) + z\Big)\Big| \leq \frac{1}{2\pi}\Big|\int_{\mathcal{C}} \frac{w_{0,0}(\vec{p}, z', 0, \vec{0}) + z'}{(z' - z)^2} dz'\Big| < \frac{\delta\mu}{(\frac{\mu}{3}(1 - 3\rho/2) - \delta)^2},$$

where in the last inequality we used that $H(\underline{w}(\cdot,\cdot)) \in \mathcal{B}(\gamma, \delta, \varepsilon)$. For $\rho < 1/2$ and $\delta \ll \rho\mu$, the right-hand side is strictly smaller than 1, which implies that $\partial_z w_{0,0}(\vec{p}, z, 0, \vec{0}) \neq 0$.

Step 3. We prove that $E_\rho(\mathcal{U}[w_{0,0}]) = U[\vec{p}^*] \times D_{\mu/2}$. Let $(\vec{p}_0, u_0) \in U[\vec{p}^*] \times D_{\mu/2}$. For any z with $|z| = \mu/2 - \beta$ and $0 < \beta \ll \mu$, we have that $|z - \rho u_0| \geq (1 - \rho)\mu/2 - \beta > \delta$, provided that δ is small enough. This implies that

$$|(-w_{0,0}(\vec{p}_0, z, 0, \vec{0}) - \rho u_0) - (z - \rho u_0)| \leq \delta < |z - \rho u_0|.$$

By Rouché's Theorem, the number of zeros of $z \mapsto -w_{0,0}(\vec{p}_0, z, 0, \vec{0}) - \rho u_0$ in $D_{\mu/2-\beta}$ is equal to the number of zeros of $z \mapsto z - \rho u_0$, which is equal to 1 if β is small enough. Since $\beta > 0$

is arbitrary, there is one (and only one) z_0 in $D_{\mu/2}$ such that $-\rho^{-1}w_{0,0}(\vec{p}_0, z_0, 0, \vec{0}) = u_0$. In other words, $(\vec{p}_0, u_0) = E_\rho(\vec{p}_0, z_0)$. □

2.3.3. *The smooth Feshbach-Schur map.* We choose a smooth function $\chi : \mathbb{R} \to [0,1]$ with compact support included in $[0,1]$, having the property that $\chi(x) = 1$ for any $x \in [0, 3/4]$. For any $\rho > 0$, we introduce the smooth function $\chi_\rho : \mathbb{R} \to [0,1]$, defined by the rescaling

$$\chi_\rho(r) := \chi\left(\frac{r}{\rho}\right). \tag{2.28}$$

We set

$$\overline{\chi}_\rho(r) := \sqrt{1 - \chi_\rho^2(r)}.$$

The functional calculus for self-adjoint operators allows us to construct the operators $\chi_\rho(H_f)$ and $\overline{\chi}_\rho(H_f)$ on the space $\mathcal{H}_f$. The next lemma shows that the pair $(H(\underline{w}(\vec{p}, z)), W_{0,0}(\underline{w}(\vec{p}, z)))$ is a Feshbach pair for $\chi_\rho(H_f)$ for suitable values of the parameters $\rho, \xi, \gamma, \delta, \epsilon$.

Lemma 2.10. *Let $0 < \rho < 1/2$, $0 < \xi < 1/\sqrt{8\pi}$, $0 < \gamma \ll \mu$, $\delta > 0$ and $0 < \varepsilon \ll \rho\mu$. For all $H(\underline{w}(\cdot,\cdot)) \in \mathcal{B}(\gamma, \delta, \varepsilon)$ and $(\vec{p}, z) \in \mathcal{U}[w_{0,0}]$,*

$$\Big(H(\underline{w}(\vec{p}, z)), W_{0,0}(\underline{w}(\vec{p}, z))\Big),$$

is a Feshbach pair for $\chi_\rho(H_f)$.

Proof. The proof consists in checking the conditions (a), (b) and (c) of Definition 2.1. Condition (a) is trivial. To prove that $W_{0,0}(\underline{w}(\vec{p}, z))$ is bounded invertible on $\operatorname{Ran} \overline{\chi}_\rho(H_f)$, we use the functional calculus of Paragraph 1.2.3. It is sufficient to show that there exists a constant $C > 0$ such that $w_{0,0}(\vec{p}, z, r, \vec{l}) \geq C > 0$ for all $r \in [\frac{3\rho}{4}, 1]$, $|\vec{l}| \leq r$, and $(\vec{p}, z) \in \mathcal{U}[w_{0,0}]$. To shorten the length of the formulas, we set $t(\vec{p}, z, r, \vec{l}) = w_{0,0}(\vec{p}, z, r, \vec{l}) - w_{0,0}(\vec{p}, z, 0, \vec{0})$. One has

$$|w_{0,0}(\vec{p}, z, r, \vec{l})| \geq r - |\frac{\vec{p}}{m} \cdot \vec{l}| - \Big|t(\vec{p}, z, r, \vec{l}) - \Big(r - \frac{\vec{p}}{m} \cdot \vec{l}\Big)\Big| - |w_{0,0}(\vec{p}, z, 0, \vec{0})|. \tag{2.29}$$

But

$$\begin{aligned} &t(\vec{p}, z, r, \vec{l}) - \Big(r - \frac{\vec{p}}{m} \cdot \vec{l}\Big) \\ &= \sum_{j=1}^{3} \int_0^{l_j} \Big((\partial_{l_j} t)(\vec{p}, z, r, \hat{l}_j(l')) + \frac{p_j}{m}\Big)\, dl' + \int_0^r \Big((\partial_{r'} t)(\vec{p}, z, r', \vec{0}) - 1\Big)\, dr', \end{aligned}$$

with $\hat{l}_1(l') = (l', 0, 0)$, $\hat{l}_2(l') = (l_1, l', 0)$, $\hat{l}_3(l') = (l_1, l_2, l')$. Since $H(\underline{w}(\cdot,\cdot))$ belongs to the polydisc $\mathcal{B}(\gamma, \delta, \varepsilon)$, we deduce that $\|t(\vec{p}, z, r, \vec{l}) - (r - m^{-1}\vec{p} \cdot \vec{l})\|^\sharp \leq \gamma$, and hence the previous relation implies

$$\Big|t(\vec{p}, z, r, \vec{l}) - \Big(r - \frac{\vec{p}}{m} \cdot \vec{l}\Big)\Big| \leq r\gamma.$$

Since $(\vec{p}, z) \in \mathcal{U}[w_{0,0}]$, $|w_{0,0}(\vec{p}, z, 0, \vec{0})| < \mu\rho/2$, and since $|\vec{p} - \vec{p}^*| < \mu m$, where $\mu = (m - |\vec{p}^*|)/2m$, we have that $|\vec{p}| < (1 - \mu)m$. Therefore (2.29) gives

$$|w_{0,0}(\vec{p}, z, r, \vec{l})| \geq r\,(\mu - \gamma) - \frac{\mu\rho}{2}. \tag{2.30}$$

Since $r \geq 3\rho/4$, we conclude that, for $\gamma \ll \mu$, $W_{0,0}(\underline{w}(\vec{p}, z))$ is bounded invertible on $\operatorname{Ran} \overline{\chi}_\rho(H_f)$ with an inverse bounded by $\mathcal{O}((\mu\rho)^{-1})$.

To prove that the operator $W_{0,0}(\underline{w}(\vec{p},z))+\overline{\chi}_\rho(H_f)W_{\geq 1}(\underline{w}(\vec{p},z))\overline{\chi}_\rho(H_f)$ is bounded invertible on $\mathrm{Ran}\,\overline{\chi}_\rho(H_f)$, we use Lemma 2.4 together with the fact that $H(\underline{w}) \in \mathcal{B}(\gamma,\delta,\varepsilon)$, which implies that $\|W_{\geq 1}(\underline{w}(\vec{p},z))\| \leq \varepsilon$. For $0<\varepsilon \ll \rho\mu$, it follows that

$$\begin{aligned}
&\left\| \big(W_{0,0}(\underline{w}(\vec{p},z))\big)^{-1}\overline{\chi}_\rho(H_f)W_{\geq 1}(\underline{w}(\vec{p},z))\right\| < 1,\\
&\left\| W_{\geq 1}(\underline{w}(\vec{p},z))\big(W_{0,0}(\underline{w}(\vec{p},z))\big)^{-1}\overline{\chi}_\rho(H_f)\right\| < 1,
\end{aligned} \tag{2.31}$$

and hence $W_{0,0}(\underline{w}(\vec{p},z)) + \overline{\chi}_\rho(H_f)W_{\geq 1}(\underline{w}(\vec{p},z))\overline{\chi}_\rho(H_f)$ is bounded invertible on $\mathrm{Ran}\,\overline{\chi}_\rho(H_f)$. Condition (c) is a direct consequence of (2.31). □

2.3.4. *Definition of the renormalization map.* The renormalization map is the composition of the scale transformation, S_ρ, the smooth Feshbach-Schur map, $F_{\chi_\rho(H_f)}$, and the inverse of the rescaling of the spectral parameter E_ρ.

Definition 2.11 (The renormalization map $\mathcal{R}_\rho$). *Let $0<\rho<1/2$, $0<\xi<1/\sqrt{8\pi}$, $0<\delta,\varepsilon \ll \rho\mu$ and $0<\gamma \ll \mu$. The renormalization transformation $\mathcal{R}_\rho : \mathcal{B}(\gamma,\delta,\varepsilon) \to \mathcal{W}_{\mathrm{op}}$ is defined by*

$$\mathcal{R}_\rho(H(\underline{w}))(\vec{p},\zeta) := S_\rho\left(F_{\chi_\rho(H_f)}\left(H\left(\underline{w}\left(E_\rho^{-1}(\vec{p},\zeta)\right)\right), W_{0,0}\left(\underline{w}\left(E_\rho^{-1}(\vec{p},\zeta)\right)\right)\right)\right), \tag{2.32}$$

for any $(\vec{p},\zeta) \in U[\vec{p}^*]\times D_{\mu/2}$ and $H(\underline{w}(\cdot,\cdot)) \in \mathcal{B}(\gamma,\delta,\varepsilon)$ such that the map $(\vec{p},z)\mapsto w_{0,0}(\vec{p},z,0,\vec{0})$ is analytic on $U[\vec{p}^*]\times D_{\mu/2}$. We set

$$H(\underline{\hat{w}}(\vec{p},\zeta)) := \mathcal{R}_\rho(H(\underline{w}))(\vec{p},\zeta). \tag{2.33}$$

Remarks.

- *The rescaling of the spectral parameter z, given by $E_\rho^{-1}(\vec{p},\zeta)$ in Definition 2.11, allows us to control the expansion of the relevant part along the iteration of the renormalization map. More precisely, if we set*

$$\tilde{\mathcal{R}}_\rho(H(\underline{w}))(\vec{p},z) := S_\rho\left(F_{\chi_\rho(H_f)}\left(H\left(\underline{w}(\vec{p},z)\right), W_{0,0}\left(\underline{w}(\vec{p},z)\right)\right)\right)$$

  and assume that z is sufficiently small to iterate the Feshbach-Schur map and the scale transformation, then the absolute value of the relevant part increases, and after a large number of iterations, it is not possible anymore to apply the Feshbach-Schur map. This phenomenon is illustrated in Figure 2.1; The marginal part coincides with the fixed point manifold $\mathcal{M}_{fp}$, the irrelevant part coincides with the stable manifold $\mathcal{M}_s$, and the relevant part with the unstable manifold $\mathcal{M}_u$.

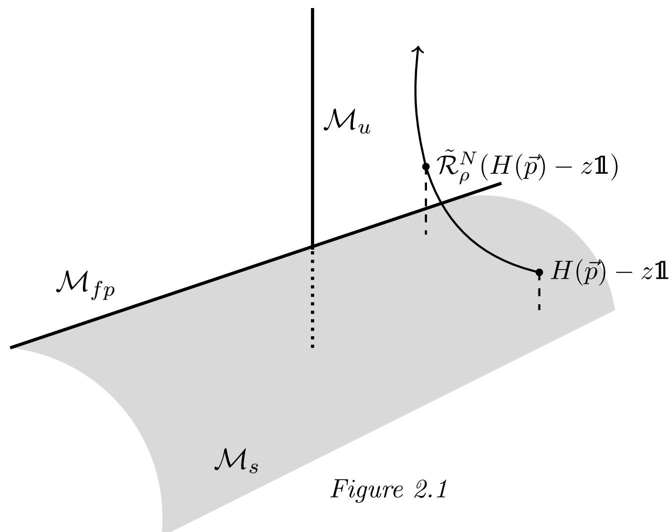


Figure 2.1

- *We anticipate that the operator-valued function $\mathcal{R}_\rho(H(\underline{w}))(\cdot,\cdot)$ belongs to $\mathcal{W}_{\rm op}$, i.e., can be written as a series of Wick monomials. The proof is sketched in the next subsection. In particular, the sequence of kernels $\underline{w}$ defined in (2.33) is uniquely determined by the procedure we use to write $\mathcal{R}_\rho(H(\underline{w}))(\cdot,\cdot)$ as a sequence of Wick monomials.*
- *The condition that $(\vec{p},z)\mapsto w_{0,0}(\vec{p},z,0,\vec{0})$ is analytic on $U[\vec{p}^*]\times D_{\mu/2}$ can be weakened. The proof of Lemma 2.9 (step 3) indeed shows that the condition that $z\mapsto w_{0,0}(\vec{p},z,0,\vec{0})$ is analytic on $D_{\mu/2}$ for all $\vec{p}\in U[\vec{p}^*]$ is sufficient to ensure existence of the inverse of the scale transformation E_ρ.*

2.4. **Wick ordering.** We sketch the proof of the claim that $\mathcal{R}_\rho:\mathcal{B}(\gamma,\delta,\varepsilon)\to\mathcal{W}_{\rm op}$. In fact, if $\gamma,\delta,\varepsilon$ are "small enough", a stronger results holds: $\mathcal{R}_\rho$ is codimension-4 contractive and maps $\mathcal{B}(\gamma,\delta,\varepsilon)$ to $\mathcal{B}(\gamma+\varepsilon/2,\varepsilon/2,\varepsilon/2)$. The discussion of this essential property is done in Subsection 4.2.2 and the exact expression of the new operator $H(\hat{\underline{w}})$ is given in Appendix B. In this paragraph, we focus on the reason why $\mathcal{R}_\rho(H(w))$ can be rewritten as a series of Wick monomials. We omit the arguments $(\vec{p},z)$ to shorten the length of the formulas. We have that

$$\begin{aligned}F_{\chi_\rho(H_f)}\left(H\left(\underline{w}\right),W_{0,0}\left(\underline{w}\right)\right)&=W_{0,0}\left(\underline{w}\right)+\chi_\rho(H_f)W_{\geq1}\left(\underline{w}\right)\chi_\rho(H_f)\\&-\chi_\rho(H_f)W_{\geq1}\left(\underline{w}\right)\overline{\chi}_\rho(H_f)\Big(W_{0,0}\left(\underline{w}\right)+\overline{\chi}_\rho(H_f)W_{\geq1}\left(\underline{w}\right)\overline{\chi}_\rho(H_f)\Big)^{-1}\\&\quad\overline{\chi}_\rho(H_f)W_{\geq1}\left(\underline{w}\right)\chi_\rho(H_f).\end{aligned}\tag{2.34}$$

The two terms on the first line of the right-hand side of (2.34) are already written with Wick monomials. To rewrite the second line of (2.34) with Wick monomials, we need to expand the inverse of $W_{0,0}\left(\underline{w}\right)+\overline{\chi}_\rho(H_f)W_{\geq1}\left(\underline{w}\right)\overline{\chi}_\rho(H_f)$ in a convergent Neumann series, and to normal order the product of annihilation and creation operators of each term of this series. The Neumann series reads

$$\sum_{L=2}^{\infty}(-1)^L\chi_\rho(H_f)W_{\geq1}\left(\underline{w}\right)\frac{\overline{\chi}_\rho(H_f)}{W_{0,0}\left(\underline{w}\right)}\left(\overline{\chi}_\rho(H_f)W_{\geq1}\left(\underline{w}\right)\frac{\overline{\chi}_\rho(H_f)}{W_{0,0}\left(\underline{w}\right)}\right)^{L-2}\overline{\chi}_\rho(H_f)W_{\geq1}\left(\underline{w}\right)\chi_\rho(H_f).\tag{2.35}$$

Let $L\in\mathbb{N}$, $L\geq2$, and consider the L^{th} term in the sum (2.35). Each operator $W_{\geq1}\left(\underline{w}\right)$ is a series of Wick monomials; We label the L operators $W_{\geq1}\left(\underline{w}\right)$ by an index i, $i=1,...,L$. For each $W_{\geq1,i}\left(\underline{w}\right)$, we choose one term $W_{M_i,N_i}(\underline{w})$ in the series defining $W_{\geq1,i}\left(\underline{w}\right)$ and we normal order the annihilation and creation operators appearing in the product

$$\chi_\rho(H_f)W_{M_1,N_1}(\underline{w})\,\frac{\overline{\chi}_\rho(H_f)}{W_{0,0}(\underline{w})}\prod_{i=2}^{L-1}\left(\overline{\chi}_\rho(H_f)W_{M_i,N_i}(\underline{w})\,\frac{\overline{\chi}_\rho(H_f)}{W_{0,0}(\underline{w})}\right)$$
$$\overline{\chi}_\rho(H_f)W_{M_L,N_L}(\underline{w})\,\chi_\rho(H_f). \tag{2.36}$$

Normal ordering is carried out with the help of Wick's Theorem and the pull-through formula (whose proofs are standards; see [7]).

Lemma 2.12 (Wick ordering). *Set* $\mathbb{N}_p := \{1,...,p\}$, $b^+(\underline{k}_j) := b^*(\underline{k}_j)$, $b^-(\underline{k}_j) := b(\underline{k}_j)$, *and let* $\sigma_i = \pm$, $i = 1,...,p$. *For any ordered finite subset* $\mathcal{P} = \{i_1,...,i_p\}$ *of* $\mathbb{N}$, $i_1 < ... < i_p$, *we set*

$$\prod_{i\in\mathcal{P}} b^{\sigma_i}(\underline{k}_i) = b^{\sigma_{i_1}}(\underline{k}_{i_1})...b^{\sigma_{i_p}}(\underline{k}_{i_p}).$$

Then,

$$\prod_{i=1}^{p} b^{\sigma_i}(\underline{k}_i) = \sum_{\mathcal{P}\subset\mathbb{N}_p}\langle\Omega|\prod_{i\in\mathbb{N}_p\setminus\mathcal{P}} b^{\sigma_i}(\underline{k}_i)\Omega\rangle\ :\prod_{j\in\mathcal{P}} b^{\sigma_j}(\underline{k}_j):, \tag{2.37}$$

where $:\cdot:$ *denotes the Wick-ordered product of creation and annihilation operators:*

$$:\prod_{j\in\mathcal{P}} b^{\sigma_j}(\underline{k}_j) := \prod_{j\in\mathcal{P},\sigma_j=+} b^{\sigma_j}(\underline{k}_j)\prod_{l\in\mathcal{P},\sigma_l=-} b^{\sigma_l}(\underline{k}_l).$$

Lemma 2.13 (Pull-through formula). *Let* $f : \mathbb{R}\times\mathbb{R}^3 \to \mathbb{C}$ *be a measurable function. Then, for a.e.* $\vec{k}\in\mathbb{R}^3$,

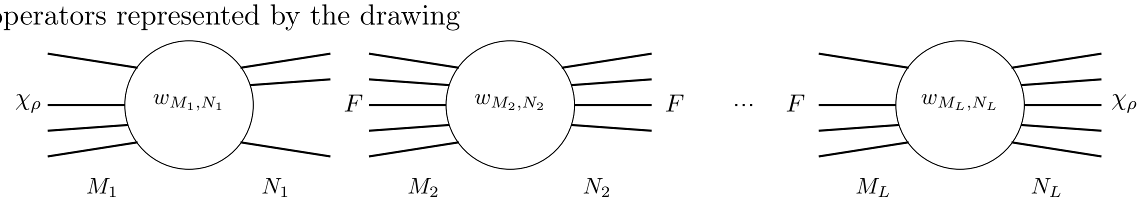
$$b(\vec{k})f(H_f,\vec{P}_f) = f(H_f+|\vec{k}|,\vec{P}_f+\vec{k})b(\vec{k}), \tag{2.38}$$

$$f(H_f,\vec{P}_f)b^*(\vec{k}) = b^*(\vec{k})f(H_f+|\vec{k}|,\vec{P}_f+\vec{k}). \tag{2.39}$$

To explain how to use Lemmas 2.12 and 2.13, we illustrate Formula (2.36) with some pictures. We represent the operators $w_{M_i,N_i}(H_f,\vec{P}_f,\underline{k}_1,...,\underline{k}_{M_i},\underline{\tilde{k}}_1,...,\underline{\tilde{k}}_{N_i})$ by circular domains and the creation/annihilation operators by plain lines. We set

$$F := \frac{\overline{\chi}_\rho^2(H_f)}{W_{0,0}(\underline{w})}. \tag{2.40}$$

The operator (2.36) is an integral over the momenta, $\underline{k}_1^{(M_1)},...,\underline{k}_L^{(M_L)},\underline{\tilde{k}}_1^{(N_1)},...,\underline{\tilde{k}}_L^{(N_L)}$, of the operators represented by the drawing



In the picture above, we have set

$$\chi_\rho := \chi_\rho(H_f)$$

and

$$w_{M_i,N_i} := w_{M_i,N_i}(H_f,\vec{P}_f,\underline{k}_1,...,\underline{k}_{M_i},\underline{\tilde{k}}_1,...,\underline{\tilde{k}}_{N_i})$$

to shorten notation. Plain lines starting from the left hand-side of an operator w_{M_i,N_i} represent creation operators, whereas plain lines starting from its right-hand side represent annihilation operators. We use Formula (2.37) to normal order the product of creation and annihilation operators that appears in (2.36). This amounts to picking out m_1/n_1 creation/annihilation operators over the M_1/N_1 creation/annihilation operators available from W_{M_1,N_1}, ... , m_L/n_L creation/annihilation operators over the M_L/N_L creation/annihilation operators available from W_{M_L,N_L}, and to contracting all the remaining creation/annihilation operators that appear in (2.36). As the kernels w_{M_i,N_i} are symmetric in $\underline{k}_1, ..., \underline{k}_{M_i}$ and $\tilde{\underline{k}}_1, ..., \tilde{\underline{k}}_{N_i}$, the contractions with the same numbers $m_1, ..., m_L/n_1, ..., n_L$ give rise to the same contribution. There are therefore

$$C_{\underline{m},\underline{n}}^{\underline{M},\underline{N}} := \prod_{i=1}^{L} \begin{pmatrix} M_i \\ m_i \end{pmatrix} \begin{pmatrix} N_i \\ n_i \end{pmatrix}$$

contraction schemes giving rise to the same operator, where we have set $\underline{M} := (M_1, ..., M_L)$. Finally, we pull-through the uncontracted $m_1 + ... + m_L$ creation operators to the left of the operator $\chi_\rho(H_f)$ located on the left-hand side of (2.36), and the $n_1 + ... + n_L$ uncontracted annihilation operators to the right of the operator $\chi_\rho(H_f)$ located on the right-hand side of (2.36). This causes a shift in the kernel arguments via Formula (2.13). Formula (2.37) tells us that the contracted part can be rewritten as the vacuum expectation value of an operator $\mathscr{V}_{\underline{m},\underline{n}}^{\underline{M},\underline{N}}(\vec{p}, z, r, \vec{l}, \underline{K}^{(\underline{m},\underline{n})})$; See Appendix B for more details. The Wick-ordered operator corresponding to (2.36) is therefore a sum over $m_1, ..., m_L, n_1, ..., n_L$ (with $0 \leq m_i \leq M_i$, $0 \leq n_i \leq N_i$), of operators of the form

$$C_{\underline{m},\underline{n}}^{\underline{M},\underline{N}} \int d\underline{K}^{(\underline{m},\underline{n})} b^*(\underline{k}^{(\underline{m})}) \, \langle \Omega | \mathscr{V}_{\underline{m},\underline{n}}^{\underline{M},\underline{N}}(\vec{p}, z, r, \vec{l}, \underline{K}^{(\underline{m},\underline{n})}) \Omega \rangle_{r=H_f, \vec{l}=\vec{P}_f} \; b(\tilde{\underline{k}}^{(\underline{n})}), \tag{2.41}$$

where the operator $\mathscr{V}_{\underline{m},\underline{n}}^{\underline{M},\underline{N}}(\vec{p}, z, r, \vec{l}, \underline{K}^{(\underline{m},\underline{n})})$ sitting within the expectation value depends on the operator-valued functions F and w_{M_i,N_i}, $i = 1, ..., L$. The operator in (2.41) is Wick-ordered. Applying this procedure for each $(M_1, N_1), ..., (M_L, N_L)$, and for each $L \in \mathbb{N}$, $L \geq 2$, (2.34) can be rewritten as a series of Wick monomials. The operator norm convergence of this series and the fact that the kernel $\underline{\hat{w}}$ belongs to $\mathcal{W}$ requires a little more effort to be established. We refer the reader to [4] or [28] for detailed proofs.

3. The first decimation step

Let $0 < \gamma, \delta, \varepsilon < 1$. Constraining the coupling constant λ_0, any polydisc $\mathcal{B}(\gamma, \delta, \varepsilon)$ (see Definition 2.7) can be reached from the initial Hamiltonian

$$H(\vec{p}, z) := \frac{\vec{P}_f^2}{2m} - \frac{\vec{p}}{m} \cdot \vec{P}_f + \omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \lambda_0 H_I + H_f - z\mathbb{1},$$

by applying to $H(\vec{p}, z)$ some isospectral transformations. The aim of this section is to highlight how this can be achieved.

3.1. **The first step: Applying the smooth Feshbach-Schur map to $H(\vec{p}, z)$.**

3.1.1. *Some notations.* Let $(|\uparrow\rangle, |\downarrow\rangle)$ be the orthonormal basis of $\mathbb{C}^2$ in which

$$\begin{pmatrix} \omega_0 & 0 \\ 0 & 0 \end{pmatrix} = \omega_0 P_\uparrow + 0 \cdot P_\downarrow,$$

where we introduced the orthogonal projections $P_\uparrow := |\uparrow\rangle\langle\uparrow|$ and $P_\downarrow := |\downarrow\rangle\langle\downarrow|$. Let χ and $\overline{\chi}$ be the smooth functions introduced in Paragraph 2.3.3. Let $\rho_0 > 0$. We define two new bounded operators on $\mathbb{C}^2 \otimes \mathcal{F}_+(L^2(\mathbb{R}^3))$ by

$$\boldsymbol{\chi} := P_\downarrow \otimes \chi_{\rho_0}(H_f), \qquad \overline{\boldsymbol{\chi}} := P_\downarrow \otimes \overline{\chi}_{\rho_0}(H_f) + P_\uparrow \otimes \mathbb{1}. \tag{3.1}$$

The reader can check that $\boldsymbol{\chi}^2 + \overline{\boldsymbol{\chi}}^2 = \mathbb{1}$. We denote by $H_0(\vec{p}, z)$ the operator

$$H_0(\vec{p}, z) := \frac{\vec{P}_f^2}{2m} - \frac{\vec{p}}{m} \cdot \vec{P}_f + \omega_0 \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + H_f - z\mathbb{1}.$$

3.1.2. *The Feshbach pair.*

Lemma 3.1. *Let $\rho_0 < \omega_0$. There exists $\lambda_c > 0$ such that, for all $0 \leq \lambda_0 < \lambda_c$, for all $|z| < \frac{\mu\rho_0}{2}$, $(H(\vec{p}, z), H_0(\vec{p}, z))$ is a Feshbach pair for $\boldsymbol{\chi}$.*

Proof. The proof is standard, but we show that Condition (b) of Definition 2.1 is satisfied, in order to illustrate how the functional calculus outlined in Paragraph 1.2.3 can be used to control the norm of bounded operators. The restriction of $H_0(\vec{p}, z)$ to $\mathrm{Ran}(\overline{\boldsymbol{\chi}})$ can be decomposed into a sum of two operators,

$$H_0(\vec{p}, z)_{|\mathrm{Ran}(\overline{\boldsymbol{\chi}})} = P_\downarrow \otimes b_1(\vec{p}, z, H_f, \vec{P}_f) + P_\uparrow \otimes b_2(\vec{p}, z, H_f, \vec{P}_f), \tag{3.2}$$

where

$$b_1(\vec{p}, z, r, \vec{l}) := \left(r + \frac{\vec{l}^2}{2m} - \frac{1}{m}\vec{p} \cdot \vec{l} - z \right) \mathbb{1}_{r \geq \frac{3\rho_0}{4}}(r), \tag{3.3}$$

$$b_2(\vec{p}, z, r, \vec{l}) := r + \frac{\vec{l}^2}{2m} - \frac{1}{m}\vec{p} \cdot \vec{l} + \omega_0 - z, \tag{3.4}$$

with $b_1(\vec{p}, z, H_f, \vec{P}_f)$ and $b_2(\vec{p}, z, H_f, \vec{P}_f)$ defined by the functional calculus of Paragraph 1.2.3. Using the bound (1.23), $b_1(\vec{p}, z, H_f, \vec{P}_f)$ is bounded invertible, if, for any $|z| < \frac{\mu\rho_0}{2}$,

$$\sup_{r \geq \frac{3\rho_0}{4}, |\vec{l}| \leq r} \left(\frac{1}{|r + \frac{\vec{l}^2}{2m} - \frac{1}{m}\vec{p} \cdot \vec{l} - z|} \right) < \infty.$$

This is indeed the case, as

$$\Re \left(r + \frac{\vec{l}^2}{2m} - \frac{1}{m}\vec{p} \cdot \vec{l} - z \right) \geq r \left(1 - \frac{1}{m}|\vec{p}| \right) - |\Re(z)| > \frac{\mu\rho_0}{4},$$

and we see that the inverse of $b_1(\vec{p}, z, H_f, \vec{P}_f)$ is bounded by a constant of order $(\mu\rho_0)^{-1}$. Similarly, $b_2(\vec{p}, z, H_f, \vec{P}_f)$ is bounded invertible and its inverse is bounded by a constant of order $(\omega_0 - \mu\rho_0/2)^{-1}$, which implies that the restriction of $H_0(\vec{p}, z)$ to $\mathrm{Ran}(\overline{\boldsymbol{\chi}})$ is bounded invertible. The bounded invertibility of the restriction of the operator

$$H_{\overline{\boldsymbol{\chi}}}(\vec{p}, z) := H_0(\vec{p}, z) + \lambda_0 \overline{\boldsymbol{\chi}} H_I \overline{\boldsymbol{\chi}}$$

to $\mathrm{Ran}(\overline{\boldsymbol{\chi}})$ is shown with a Neumann expansion and the standard estimate

$$\left\| (H_f + \rho_0)^{-\frac{1}{2}} H_I (H_f + \rho_0)^{-\frac{1}{2}} \right\| \leq C\rho_0^{-\frac{1}{2}}. \tag{3.5}$$

The Neumann expansion for $H_{\overline{\boldsymbol{\chi}}}^{-1}(\vec{p}, z)$ is formally equal to

$$(H_0(\vec{p}, z))^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})} \sum_{n=0}^{\infty} (-\lambda_0)^n \left[\overline{\boldsymbol{\chi}} H_I \overline{\boldsymbol{\chi}} \left(H_0(\vec{p}, z) \right)^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})} \right]^n. \tag{3.6}$$

Introducing the operator $(H_f+\rho_0)^{-\frac{1}{2}}$ on the left and on the right of each operator H_I, we obtain the bound

$$\| \, [H_{\overline{\boldsymbol{\chi}}}(\vec{p},z)]^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})} \, \| \leq \, \rho_0^{-1} \sum_{n=0}^{\infty} \Big(C\lambda_0 \rho_0^{-\frac{1}{2}} \Big)^n \, \Big\| \, \Big[\frac{H_f+\rho_0}{H_0(\vec{p},z)} \Big]_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})} \Big\|^{n+1}. \tag{3.7}$$

Again, the functional calculus of section 1.2 shows that $\|[(H_f+\rho_0)(H_0(\vec{p},z))^{-1}]_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})}\|$ is bounded by

$$\sup_{r\geq 3\rho_0/4, |\vec{l}|\leq r} \frac{r+\rho_0}{|r+\frac{\vec{l}^2}{2m}-\frac{1}{m}\vec{p}\cdot\vec{l}-z|} + \sup_{r\geq 0, |\vec{l}|\leq r} \frac{r+\rho_0}{|r+\frac{\vec{l}^2}{2m}-\frac{1}{m}\vec{p}\cdot\vec{l}+\omega_0-z|} = \mathcal{O}(\mu^{-1}).$$

Therefore, there exists a positive constant λ_c of order $\mu\rho_0^{\frac{1}{2}}$, such that, for any $0\leq\lambda_0<\lambda_c$, the Neumann series in (3.6) converges in norm. □

3.2. The second step: Reaching a polydisc $\mathcal{B}(\gamma,\delta,\varepsilon)$.

As the Feshbach-Schur map is isospectral, the study of the invertibility of the operator $F_{\boldsymbol{\chi}}(H(\vec{p},z),H_0(\vec{p},z))$ restricted on $\mathrm{Ran}(\boldsymbol{\chi})$ should provide important insight about the spectrum of $H(\vec{p})$ near the origin of the complex plane. The restriction of $F_{\boldsymbol{\chi}}(H(\vec{p},z),H_0(\vec{p},z))$ to $\mathrm{Ran}(\boldsymbol{\chi})$ is equal to

$$P_\downarrow \otimes \Big\langle \Big(H_f + \frac{\vec{P}_f^2}{2m} - \frac{\vec{p}}{m}\cdot\vec{P}_f + \lambda_0 \boldsymbol{\chi} H_I \boldsymbol{\chi} - \lambda_0^2 \boldsymbol{\chi} H_I \overline{\boldsymbol{\chi}} (H_{\overline{\boldsymbol{\chi}}}(\vec{p},z))^{-1} \overline{\boldsymbol{\chi}} H_I \boldsymbol{\chi} - z\mathbb{1} \Big) \mathbb{1}_{H_f\leq\rho_0} \Big\rangle_\downarrow, \tag{3.8}$$

where $\langle A\rangle_\downarrow \in \mathcal{B}(\mathcal{H}_f)$ is the bounded operator associated to the quadratic form

$$\langle \psi | \langle A \rangle_\downarrow \phi \rangle := \langle \downarrow \otimes \psi | A (\downarrow \otimes \phi) \rangle, \qquad \forall \psi, \phi \in \mathcal{H}_f,$$

for any operator A in $\mathcal{B}(\mathbb{C}^2\otimes\mathcal{H}_f)$. If we apply the scale transformation S_{ρ_0} defined in Subsection 2.3.1 to (3.8) and Wick-order the resulting operator following the procedure sketched in Section 2.4, we conclude that the next result holds.

Lemma 3.2. *Let $0<\rho_0<\omega_0$, $0<\gamma,\delta,\varepsilon<1$, and $\rho_0^{3/2}<\xi<1/\sqrt{8\pi}$. There exists $\lambda_c>0$ such that, for all $0\leq\lambda_0<\lambda_c$, for all $|z|<\frac{\mu}{2}$,*

$$\mathcal{S}_{\rho_0}\Big([F_{\boldsymbol{\chi}}(H(\vec{p},\rho_0 z), H_0(\vec{p},\rho_0 z))]_{\mathrm{Ran}(\boldsymbol{\chi})} \Big) =: P_\downarrow \otimes H(\underline{w}^{(0)}(\vec{p},z)),$$

$$\textit{with} \quad H(\underline{w}^{(0)}(\cdot,\cdot)) \in \mathcal{B}(\frac{\sqrt{3}\rho_0}{m}+\gamma,\delta,\varepsilon).$$

The proof is given in Appendix C. In particular, we show there that the critical value λ_c can be chosen such that $\lambda_c\leq C\min(\gamma,\delta,\varepsilon)\mu^2\rho_0^{1/2}$ for some positive constant C independent of the problem parameters.

3.3. Analyticity of the effective Hamiltonian $H(w^{(0)}(\vec{p},z))$ in $\vec{p}$ and z.

Lemma 3.3. *If the assumptions of Lemma 3.2 are satisfied, then $(\vec{p},z)\mapsto H(\underline{w}^{(0)}(\vec{p},z))$ defined in Lemma 3.2 is analytic on $U[\vec{p}^*]\times D_{\mu/2}$.*

Proof. The proof follows from (3.8) and the Neumann expansion for $[H_{\overline{\boldsymbol{\chi}}}(\vec{p},z)]^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})}$ in the proof of Lemma 3.1. Since $(\vec{p},z)\mapsto(H_f+\frac{\vec{P}_f^2}{2m}-\frac{\vec{p}}{m}\cdot\vec{P}_f-\rho_0 z\mathbb{1})\mathbb{1}_{H_f\leq 1}\in\mathcal{B}(\mathcal{H}_{\mathrm{red}})$ is analytic on $U[\vec{p}^*]\times D_{\mu/2}$, and since H_I is independent of $(\vec{p},z)$, it remains to show that

$$\boldsymbol{\chi} H_I \overline{\boldsymbol{\chi}} \, [H_{\overline{\boldsymbol{\chi}}}(\vec{p},\rho_0 z)]^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})} \, \overline{\boldsymbol{\chi}} H_I \boldsymbol{\chi} \tag{3.9}$$

is analytic in $\vec{p}$ and z. Introducing $(H_f+\rho_0)^{-1/2}$ on the left and on the right of H_I in (3.6), we have seen in the proof of Lemma 3.1 that the Neumann expansion for (3.9) converges uniformly in $(\vec{p},z)\in U[\vec{p}^*]\times D_{\mu/2}$; see (3.7). As $(H_f+\rho_0)^{-1/2}H_I(H_f+\rho_0)^{-1/2}$ is bounded and independent of $\vec{p}$ and z, we only need to show that the operator

$$\left[\frac{H_f+\rho_0}{H_0(\vec{p},\rho_0 z)}\right]_{\mathrm{Ran}(\overline{\chi})}$$

is analytic on $U[\vec{p}^*]\times D_{\mu/2}$. By the equivalence between weak and strong analyticity stated in Theorem A.2, and the polarization identity for bilinear forms, it is sufficient to check that

$$\langle\downarrow\otimes\psi|\left[(H_f+\rho_0)(H_0(\vec{p},\rho_0 z))^{-1}\right]_{\mathrm{Ran}(\overline{\chi})}\downarrow\otimes\psi\rangle$$

and

$$\langle\uparrow\otimes\psi|\left[(H_f+\rho_0)(H_0(\vec{p},\rho_0 z))^{-1}\right]_{\mathrm{Ran}(\overline{\chi})}\uparrow\otimes\psi\rangle$$

are analytic in $\vec{p}$ and z for any $|\psi\rangle\in\mathcal{H}_f$. The functions

$$(\vec{p},z,r,\vec{l})\mapsto\frac{r+\rho_0}{\frac{\vec{l}^2}{2m}-\frac{\vec{p}}{m}\cdot\vec{l}+r-\rho_0 z},\qquad(\vec{p},z,r,\vec{l})\mapsto\frac{r+\rho_0}{\frac{\vec{l}^2}{2m}-\frac{\vec{p}}{m}\cdot\vec{l}+r+\omega_0-\rho_0 z},$$

are analytic in $(\vec{p},z)$ for any $(r,\vec{l})\in\mathbb{R}_+\times\mathbb{R}^3$ with $r\geq 3\rho_0/4$ and $|\vec{l}|\leq r$, and for any $(r,\vec{l})\in\mathbb{R}_+\times\mathbb{R}^3$ with $|\vec{l}|\leq r$, respectively. Using the functional calculus of Paragraph 1.2.3, we deduce that

$$\langle\downarrow\otimes\psi|\left[(H_f+\rho_0)(H_0(\vec{p},\rho_0 z))^{-1}\right]_{\mathrm{Ran}(\overline{\chi})}\downarrow\otimes\psi\rangle$$

and

$$\langle\uparrow\otimes\psi|\left[(H_f+\rho_0)(H_0(\vec{p},\rho_0 z))^{-1}\right]_{\mathrm{Ran}(\overline{\chi})}\uparrow\otimes\psi\rangle$$

are analytic on $U[\vec{p}^*]\times D_{\mu/2}$. This is a direct consequence of Morera's theorem for several complex variables; See Theorem A.4 in Appendix A. For a similar and detailed argumentation, the reader can consult the proof of Lemma 4.2 in Section 4.1. □

4. Iteration of the renormalization map and Preservation of analyticity in $\vec{p}$ and z

The renormalization map $\mathcal{R}_\rho$ can be iterated arbitrarily many times if, initially, it is restricted to a sufficiently small polydisc $\mathcal{B}(\gamma,\delta,\varepsilon)$ of operator valued functions. The purpose of this section is to explain under which conditions this iteration can be done and to show that the analyticity of the operator $H(\underline{w}(\vec{p},z))$ in $(\vec{p},z)$ is preserved under iterations of the renormalization map.

4.1. Results on Analyticity.

We denote by $\mathcal{B}^{\mathrm{an}}(\gamma,\delta,\varepsilon)$ the subset of $\mathcal{B}(\gamma,\delta,\varepsilon)$ composed of the analytic maps $U[\vec{p}^*]\times D_{\mu/2}\ni(\vec{p},z)\mapsto H(\underline{w}(\vec{p},z))\in\mathcal{B}(\mathcal{H}_{\mathrm{red}})$. Lemma 4.1 states that the renormalization map preserves analyticity.

Lemma 4.1. *Let $0<\rho<1/2$, $0<\xi<1/(4\sqrt{8\pi})$, $0<\gamma\ll\mu$, and $0<\delta,\varepsilon\ll\rho\mu$. If $H(\underline{w}(\cdot,\cdot))\in\mathcal{B}^{an}(\gamma,\delta,\varepsilon)$, then the $\mathcal{B}(\mathcal{H}_{red})$-valued function $H(\underline{\hat{w}}(\vec{p},\zeta))=\mathcal{R}_\rho(H(\underline{w}))(\vec{p},\zeta)$ is analytic on $U[\vec{p}^*]\times D_{\mu/2}$.*

Remarks.

- *Different strategies lead to Lemma 4.1. The preservation of analyticity can be proven for kernels: If* $(\vec{p}, z) \mapsto \underline{w}(\vec{p}, z) \in \mathcal{W}^\sharp$ *is analytic on* $U[\vec{p}^*] \times D_{\mu/2}$*, one can show that* $(\vec{p}, z) \mapsto \hat{\underline{w}}(\vec{p}, z) \in \mathcal{W}^\sharp$ *is analytic. To do so, it suffices to consider the explicit expression of the kernel* $\hat{\underline{w}}$*, as given in Appendix B. One shows that all the terms appearing in the series* (B.6) *are analytic. The uniform convergence in* $\vec{p}$ *and* z *of the series in* (B.6) *is then sufficient to establish the result. Such a procedure has been used in* [28] *to show the analyticity of the ground state energy with respect to the coupling constant in the spin boson model. Here, we prefer to work on the operator level and we adopt an approach similar to* [25].
- *We recall that, to pass from the operator* $\mathcal{R}_\rho(H(\underline{w}))(\vec{p}, \zeta)$ *given by the expression* (2.32) *to its Wick-ordered expression* $H(\hat{\underline{w}}(\vec{p}, \zeta))$*, it suffices to follow the procedure described in Paragraph 2.4 (and detailed in Appendix B), based on the combination of the pull-through formula and Wick-ordering. This uniquely determines the sequence of kernels* $\hat{\underline{w}}(\vec{p}, \zeta)$*. Naturally, since the two operators coincide, proving analyticity of* $\mathcal{R}_\rho(H(\underline{w}))(\vec{p}, \zeta)$ *is equivalent to proving analyticity of* $H(\hat{\underline{w}}(\vec{p}, \zeta))$*.*

To prove Lemma 4.1, we first state Lemma 4.2 which shows that the analyticity of $(\vec{p}, z) \mapsto H(\underline{w}(\cdot, \cdot)) \in \mathcal{W}_{\rm op}$ implies the analyticity of its "components" in the following sense.

Lemma 4.2. *Assume that the* $\mathcal{B}(\mathcal{H}_{red})$*-valued function* $H(\underline{w}(\cdot, \cdot)) \in \mathcal{B}(\gamma, \delta, \varepsilon)$ *is analytic on* $U[\vec{p}^*] \times D_{\mu/2}$*. Then* $(\vec{p}, z) \mapsto w_{0,0}(\vec{p}, z, 0, \vec{0}) \in \mathbb{C}$*,* $(\vec{p}, z) \mapsto W_{0,0}(\underline{w}(\vec{p}, z)) \in \mathcal{B}(\mathcal{H}_{red})$ *and* $(\vec{p}, z) \mapsto W_{\geq 1}(\underline{w}(\vec{p}, z)) \in \mathcal{B}(\mathcal{H}_{red})$ *are analytic on* $U[\vec{p}^*] \times D_{\mu/2}$*.*

Proof of Lemma 4.1. The proof follows from Lemmas 2.9 and 4.2. We remind the reader that

$$\mathcal{R}_\rho(H(\underline{w}))(\vec{p}, \zeta) = S_\rho \left(F_{\chi_\rho(H_f)} \left(H \left(\underline{w} \left(E_\rho^{-1}(\vec{p}, \zeta) \right) \right), W_{0,0} \left(\underline{w} \left(E_\rho^{-1}(\vec{p}, \zeta) \right) \right) \right) \right).$$

We have seen in Lemma 2.9 that $E_\rho^{-1}(\vec{p}, z)$ is holomorphic on $U[\vec{p}^*] \times D_{\mu/2}$. The composition of two analytic maps being analytic, $H \left(\underline{w} \left(E_\rho^{-1}(\vec{p}, \zeta) \right) \right)$ and $W_{0,0} \left(\underline{w} \left(E_\rho^{-1}(\vec{p}, \zeta) \right) \right)$ are analytic on $U[\vec{p}^*] \times D_{\mu/2}$. We now remark that the smooth Feshbach-Schur map preserves analyticity. Indeed, if $H(\vec{p}, z)$ and $T(\vec{p}, z)$ are analytic $\mathcal{B}(\mathcal{H}_{\rm red})$-valued functions, then, with $W \equiv H - T$,

$$F_\chi(H(\vec{p}, z), T(\vec{p}, z)) = H_\chi(\vec{p}, z) - \chi W(\vec{p}, z) \overline{\chi} [H_{\overline{\chi}}(\vec{p}, z)]^{-1}_{\mathrm{Ran}(\overline{\chi})} \overline{\chi} W(\vec{p}, z) \chi$$

is analytic as a product of bounded analytic operators, provided that the Feshbach-Schur map is well-defined. As the scale transformation S_ρ preserves analyticity, we deduce that $\mathcal{R}_\rho(H(\underline{w}))(\vec{p}, \zeta)$ is analytic on $U[\vec{p}^*] \times D_{\mu/2}$. □

Proof of Lemma 4.2. We first prove that $(\vec{p}, z) \mapsto w_{0,0}(\vec{p}, z, r, \vec{l}) \in \mathbb{C}$ is analytic for any fixed $(r, \vec{l}) \in \mathcal{B}$. In order to "extract" $w_{0,0}$ from $H(\underline{w}(\cdot, \cdot))$, we consider a sequence of smooth functions whose modulus squared converges weakly to a Dirac distribution. Let η be a smooth function with compact support, such that $\int_{\mathbb{R}^3} |\eta(\vec{x})|^2 d^3x = 1$. For any $\vec{l}$ in the unit ball of $\mathbb{R}^3$, for any $n \in \mathbb{N}$, we set

$$\eta_{n,\vec{l}}(\vec{k}) := n^{3/2} \eta \left(n(\vec{k} - \vec{l}) \right). \tag{4.1}$$

Let $\vec{l}, \vec{l}'$ be in the unit ball of $\mathbb{R}^3$. We introduce the two-particle state $\Psi_n := b^*(\eta_{n,\vec{l}}) b^*(\eta_{n,\vec{l}'}) \Omega$ and consider the sequence of holomorphic functions $(g_n)_{n \in \mathbb{N}}$ defined by

$$g_n(\vec{p}, z) := \langle \Psi_n | \mathbb{1}_{H_f \leq 1} H(\underline{w}(\vec{p}, z)) \mathbb{1}_{H_f \leq 1} \Psi_n \rangle, \tag{4.2}$$

for all $n \in \mathbb{N}$ and all $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$. We show in Appendix D that $(g_n)_n$ converges uniformly on $U[\vec{p}^*] \times D_{\mu/2}$ to a multiple of $w_{0,0}(\vec{p}, z, |\vec{l}| + |\vec{l}'|, \vec{l} + \vec{l}')$. This condition is sufficient to ensure that $(\vec{p}, z) \mapsto w_{0,0}(\vec{p}, z, r, \vec{l})$ is analytic for any fixed $(r, \vec{l}) \in \mathcal{B}$; see Lemma D.1.

We now show that $(\vec{p}, z) \mapsto W_{0,0}(\underline{w}(\vec{p}, z)) \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ is analytic on $U[\vec{p}^*] \times D_{\mu/2}$. To do so, we make use of the equivalence between weak and strong analyticity stated in Theorem A.2, and we show that $(\vec{p}, z) \mapsto W_{0,0}(\underline{w}(\vec{p}, z)) \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ is weakly analytic. Thanks to the polarization identity for bilinear forms, it is sufficient to prove that $\langle \Psi | W_{0,0}(\underline{w}(\vec{p}, z)) \Psi \rangle$ is analytic for any $\Psi \in \mathcal{H}_{\mathrm{red}}$. Let $\Psi \in \mathcal{H}_{\mathrm{red}}$. We write Ψ as a sequence of n-photon state functions $(\psi^{(n)})_{n \in \mathbb{N}}$; By the functional calculus outlined in Paragraph 1.2.3, we have that

$$\begin{aligned} \langle \Psi | W_{0,0}(\underline{w}(\vec{p}, z)) \Psi \rangle = \sum_{n=1}^{\infty} \int_{\mathcal{D}(n)} \prod_{i=1}^{n} d\underline{k}_i \ |\psi^{(n)}(\underline{k}_1, ..., \underline{k}_n)|^2 w_{0,0}(\vec{p}, z, \Sigma(\underline{k}^{(n)}), \vec{\Sigma}(\underline{k}^{(n)})) \\ + w_{0,0}(\vec{p}, z, 0, \vec{0}) |\psi^{(0)}|^2, \end{aligned} \tag{4.3}$$

where

$$\mathcal{D}(n) := \{(\underline{k}_1, ..., \underline{k}_n) \in \underline{B}_1^n \mid |\vec{k}_1| + ... + |\vec{k}_n| \le 1\}.$$

We use the theorem of Morera for several complex variables (see Theorem A.4 in Appendix A) to show the analyticity of the map $(\vec{p}, z) \mapsto \langle \Psi | W_{0,0}(\underline{w}(\vec{p}, z)) \Psi \rangle \in \mathbb{C}$. Let $V_i := l_1 \times \Delta_i \times l_3 \times l_4$, $(i = 1, ..., 4)$, be a surface included in the domain $U[\vec{p}^*] \times D_{\mu/2}$ that satisfies the criteria of Theorem A.4. $H(\underline{w}) \in \mathcal{B}(\gamma, \delta, \epsilon)$, and there exists a constant $C > 0$ such that $|w_{0,0}(\vec{p}, z, r, \vec{l})| \le C$, for all $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$ and $(r, \vec{l}) \in \mathcal{B}$. As

$$\sum_{n=0}^{\infty} \int_{\mathcal{D}(n)} \prod_{i=1}^{n} d\underline{k}_i \ |\psi(\underline{k}_1, ..., \underline{k}_n)|^2 = \|\Psi\|^2 < \infty,$$

we deduce that the function $(\vec{p}, z) \mapsto \langle \Psi | W_{0,0}(\underline{w}(\vec{p}, z)) \Psi \rangle$ is continuous; Furthermore, by integrating (4.3) over ∂V_i, the sum over n and the integral over ∂V_i can be exchanged. Fubini's theorem implies that

$$\begin{aligned} \int_{\partial V_i} \langle \Psi | W_{0,0}(\underline{w}) \Psi \rangle = \int_{\partial V_i} \ w_{0,0}(\cdot, \cdot, 0, \vec{0}) |\psi^{(0)}|^2 \\ + \sum_{n=1}^{\infty} \int_{\underline{\mathbb{R}}^n} \prod_{i=1}^{n} d\underline{k}_i |\psi^{(n)}(\underline{k}_1, ..., \underline{k}_n)|^2 \int_{\partial V_i} \ w_{0,0}(\cdot, \cdot, \Sigma(\underline{k}^{(n)}), \vec{\Sigma}(\underline{k}^{(n)})). \end{aligned} \tag{4.4}$$

The analyticity of $U[\vec{p}^*] \times D_{\mu/2} \ni (\vec{p}, z) \mapsto w_{0,0}(\vec{p}, z, \Sigma(\underline{k}^{(n)}), \vec{\Sigma}(\underline{k}^{(n)}))$ for any $(\Sigma(\underline{k}^{(n)}), \vec{\Sigma}(\underline{k}^{(n)})) \in \mathcal{B}$ shows that the right-hand side of (4.4) is equal to zero. Hence, Morera's theorem implies that $(\vec{p}, z) \mapsto \langle \Psi | W_{0,0}(\underline{w}(\vec{p}, z)) | \Psi \rangle$ is analytic for any Ψ in $\mathcal{H}_f$, and we conclude that $(\vec{p}, z) \mapsto W_{0,0}(\underline{w}(\vec{p}, z)) \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ is analytic on $U[\vec{p}^*] \times D_{\mu/2}$. Obviously, since $W_{\ge 1}(\underline{w}(\vec{p}, z)) = H(\underline{w}(\vec{p}, z)) - W_{0,0}(\underline{w}(\vec{p}, z))$, $U[\vec{p}^*] \times D_{\mu/2} \ni (\vec{p}, z) \mapsto W_{\ge 1}(\underline{w}(\vec{p}, z)) \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ is also analytic. □

4.2. **Iterating the renormalization map.** In Subsection 4.1 it has been shown that the renormalization map preserves analyticity. We now investigate under which condition the renormalization map can be iterated.

4.2.1. *Codimension-4 contractivity.* The renormalization map is codimension-4 contractive, in the sense of the following lemma.

Lemma 4.3. *Let* $0 < \rho \ll \mu$, $0 < \xi < 1/(4\sqrt{8\pi})$, $0 < \gamma \ll \mu$, $0 < \delta \ll \rho\mu$, *and* $0 < \varepsilon \ll \rho^2\mu^2$. *Then,*

$$\mathcal{R}_\rho : \mathcal{B}^{an}(\gamma, \delta, \varepsilon) \to \mathcal{B}^{an}(\gamma + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}). \tag{4.5}$$

The proof of Lemma 4.3 is similar to the proof of [4, Theorem 3.8]. We sketch its main idea in the rest of this paragraph. For the sake of completeness, a detailed exposition is given in Appendix E. If ρ and ε are sufficiently small, we expect from the scaling properties of the kernels in (2.24) that the irrelevant part $\hat{W}_{\geq 1}$ of the effective operator $H(\underline{\hat{w}}) := \mathcal{R}_\rho(H(w))$ will have a smaller norm than $W_{\geq 1}$. To check it, we need to bound the Wick-ordered series $\hat{W}_{\geq 1}$. $\hat{W}_{\geq 1}$ is a sum over $L \geq 1$, followed by a sum over $M_1, \dots, M_L/N_1, \dots, N_L$ and $m_1, \dots, m_L/n_1, \dots, n_L$ ($\sum_i m_i + n_i \geq 1$, $0 \leq m_i \leq M_i$, $0 \leq n_i \leq N_i$), of operators of the form

$$C^{\underline{M},\underline{N}}_{\underline{m},\underline{n}} \int d\underline{K}^{(\underline{m},\underline{n})} b^*(\underline{k}^{(\underline{m})}) \; \langle \Omega | \mathscr{V}^{\underline{M},\underline{N}}_{\underline{m},\underline{n}}(\vec{p}, z, r, \vec{l}, \underline{K}^{(\underline{m},\underline{n})}) \Omega \rangle_{r=H_f, \vec{l}=\vec{P}_f} \; b(\underline{\tilde{k}}^{(\underline{n})}). \tag{4.6}$$

The norm of the operator $\mathscr{V}^{\underline{M},\underline{N}}_{\underline{m},\underline{n}}(\vec{p}, z, r, \vec{l}, \underline{K}^{(\underline{m},\underline{n})})$ in (4.6) can be bounded by the product of the norms of the kernels w_{M_i,N_i} with the norms of the operator valued functions F. After rescaling, (4.6) is bounded in norm by

$$C^{\underline{M},\underline{N}}_{\underline{m},\underline{n}} \rho^{-1} \left(\prod_{i=1}^{L} \varepsilon (\sqrt{8\pi}\xi)^{M_i+N_i} \rho^{2(m_i+n_i)} \right) \left(\frac{C}{\rho\mu} \right)^{L-1}. \tag{4.7}$$

In (4.7), the term $(C/\rho\mu)^{L-1}$ comes from the norm of the rescaled operators F, and the factors $\varepsilon\xi^{M_i+N_i}$ arise because $||\underline{w}_{\geq 1}||^\sharp \leq \varepsilon$. It is not difficult to realize that the sum over L, $M_1, \dots, M_L/N_1, \dots, N_L$ and $m_1, \dots, m_L/n_1, \dots, n_L$, of (4.7), is bounded if ρ and ε are small enough. In particular, it becomes smaller than $\varepsilon/2$ for sufficiently small ρ, ε.

The relevant part does not contract under renormalization. Indeed,

$$\hat{W}_{0,0}(\vec{p}, \zeta) = S_\rho W_{0,0}(E_\rho^{-1}(\vec{p}, \zeta)) + \tilde{W}(E_\rho^{-1}(\vec{p}, \zeta)), \tag{4.8}$$

where $\tilde{W}(E_\rho^{-1}(\vec{p}, \zeta))$ is the sum over all Wick-ordered operators of the form (4.7) with $m_1 = \dots = m_L = n_1 = \cdots = n_L = 0$. This term can be bounded by $\varepsilon/2$ if ρ and ε are sufficiently small. However, the norm of $S_\rho W_{0,0}(E_\rho^{-1}(\vec{p}, \zeta))$ does not change much. The reader can consult Appendix E for more detailed calculations.

4.2.2. *Iteration of the renormalization map.* Let $\rho, \xi, \gamma, \delta, \varepsilon$ satisfying the constraints of Lemma 4.3. Let $0 < \rho_0 < \min(\xi^{2/3}, \omega_0)$. Lemmas 3.2 and 3.3 show that there exists a constant $\lambda_c > 0$, such that, for all $0 \leq \lambda_0 < \lambda_c$, the operator-valued function $U[\vec{p}^*] \times D_{\mu/2} \ni (\vec{p}, z) \mapsto H(\underline{w}^{(0)}(\vec{p}, z))$ obtained from $(\vec{p}, z) \mapsto H(\vec{p}, z)$ after a rescaling of z by ρ_0, a smooth Feshbach-Schur transformation, and a scale transformation, S_{ρ_0}, lies in $\mathcal{B}^{\text{an}}(\gamma, \varepsilon, \varepsilon)$. The codimension-4 contractivity of Lemma 4.3 implies that

$$\mathcal{R}_\rho : \mathcal{B}^{\text{an}}(\gamma, \varepsilon, \varepsilon) \to \mathcal{B}^{\text{an}}\left(\gamma + \frac{\varepsilon}{2}, \frac{\varepsilon}{2}, \frac{\varepsilon}{2}\right). \tag{4.9}$$

As $\varepsilon \ll \rho^2\mu^2 < \mu$, $\gamma + \varepsilon \ll \mu$ and the renormalization map can be iterated indefinitely. For any $n \geq 0$, we set

$$H(\underline{w}^{(n)}) := \mathcal{R}_\rho^n(H(\underline{w}^{(0)})) \in \mathcal{B}^{\text{an}}\left(\gamma + \varepsilon r_n, \frac{\varepsilon}{2^n}, \frac{\varepsilon}{2^n}\right), \tag{4.10}$$

where $r_n = 1 - 2^{-n}$ for $n \geq 0$. Thanks to the isospectrality of the renormalization map, we keep track of the eigenvalue 0 of the effective Hamiltonians $H(\underline{w}^{(n)})$ in order to find an eigenvalue $z_\infty(\vec{p})$ of the initial operator $H(\vec{p})$. We set, for any $n \in \mathbb{N} \cup \{0\}$,

$$\mathcal{U}^{(n)} := \{(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2} \mid \ |w_{0,0}^{(n)}(\vec{p}, z, 0, \vec{0})| < \frac{\mu\rho}{2}\}, \tag{4.11}$$

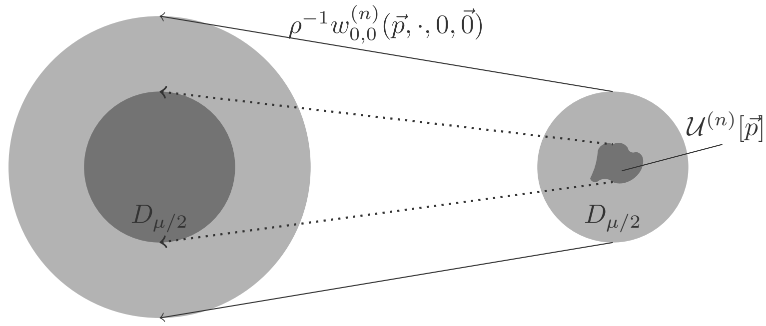
and we introduce the maps $E_{\rho,n} : \mathcal{U}^{(n)} \to U[\vec{p}^*] \times D_{\mu/2}$ defined by

$$E_{\rho,n}(\vec{p}, z) := (\vec{p}, -\rho^{-1} w_{0,0}^{(n)}(\vec{p}, z, 0, \vec{0})),$$

for all $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$. We explain with some pictures why the sets $\mathcal{U}^{(n)}$ and the maps $E_{\rho,n}$ are relevant to find the ground state eigenvalue of the initial Hamiltonian $H(\vec{p})$. It is more convenient to work with $\vec{p} \in U[\vec{p}^*]$ fixed and to look at the sets

$$\mathcal{U}^{(n)}[\vec{p}] := \{z \in D_{\mu/2} \mid \ |w_{0,0}^{(n)}(\vec{p}, z, 0, \vec{0})| < \frac{\mu\rho}{2}\}. \tag{4.12}$$

$\mathcal{U}^{(n)}$ is the union over $\vec{p} \in U[\vec{p}^*]$ of all sets $\{\vec{p}\} \times \mathcal{U}^{(n)}[\vec{p}]$. For any $n \geq 0$, $\mathcal{U}^{(n)}[\vec{p}] \subset D_{\mu/2}$ and $\rho^{-1} w_{0,0}^{(n)}(\vec{p}, \mathcal{U}^{(n)}[\vec{p}], 0, \vec{0}) \subset D_{\mu/2}$.



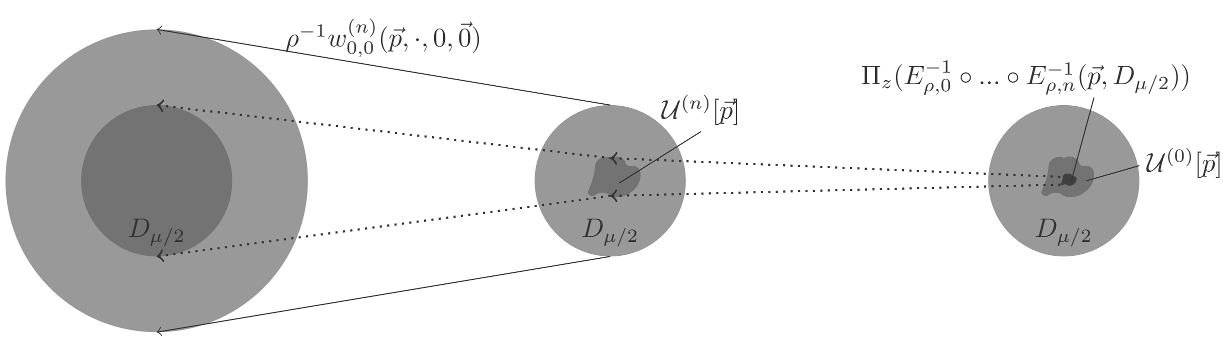
The isospectrality of the renormalization map shows that $\dim[\ker H(\underline{w}^{(n+1)}(\vec{p}, z))] \neq 0$ iff

$$\dim[\ker H(\underline{w}^{(0)}(E_{\rho,0}^{-1} \circ ... \circ E_{\rho,n}^{-1}(\vec{p}, z)))] \neq 0.$$

If we want z to be located inside the disc $D_{\mu/2}$ for all $n \geq 0$, the set of initial spectral values $\Pi_z(E_{\rho,0}^{-1} \circ ... \circ E_{\rho,n}^{-1}(\vec{p}, D_{\mu/2})) \subset D_{\mu/2}$ has to shrink with n. Π_z is the projection along the z-component in $\mathbb{C}^3 \times \mathbb{C}$, that is

$$\Pi_z(\vec{p}', z') := z',$$

for any $(\vec{p}', z') \in \mathbb{C}^3 \times \mathbb{C}$.



In the limit where n tends to infinity, we expect the set $\Pi_z(E_{\rho,0}^{-1} \circ ... \circ E_{\rho,n}^{-1}(\vec{p}, D_{\mu/2}))$ to shrink to a point located near 0, which will turn out to be the eigenvalue of $H(\vec{p})$ rescaled by a factor ρ_0^{-1}. We define the sequence $(e_{0,n})_n$ of complex-valued functions on $U[\vec{p}^*] \times D_{\mu/2}$ by

$$e_{0,n}(\vec{p}, z) := \Pi_z(E_{\rho,0}^{-1} \circ ... \circ E_{\rho,n}^{-1}(\vec{p}, z)), \tag{4.13}$$

for all $n \geq 0$ and for all $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$.

Lemma 4.4. *Let $0 < \rho \ll \mu$, $0 < \xi < 1/(4\sqrt{8\pi})$, $0 < \gamma \ll \mu$, $0 < \delta \ll \rho\mu$, and $0 < \varepsilon \ll \rho^2\mu^2$. The sequence of functions $(e_{0,n})_n$ converges uniformly on $U[\vec{p}^*] \times D_{\mu/2}$ to a function $e_{0,\infty}$. $e_{0,\infty}(\vec{p}, z)$ is independent of z.*

Proof. We define the sequence $(e_{(m,n)})_{m,n}$ of functions on $U[\vec{p}^*] \times D_{\mu/2}$, $n \geq m$, by

$$e_{(m,n)}(\vec{p}, z) := \Pi_z(E_{\rho,m}^{-1} \circ ... \circ E_{\rho,n}^{-1}(\vec{p}, z)). \tag{4.14}$$

Let $m \geq 0$. We show that the sequence $(e_{(m,n)}(\vec{p}, z))_n$ converges uniformly in $(\vec{p}, z)$ to some function $e_{m,\infty}(\vec{p}, z)$. We have seen in the proof of Lemma 2.9 that $|\partial_z w_{0,0}^{(m)}(\vec{p}, z, 0, \vec{0}) + 1| < C < 1$, for all $(\vec{p}, z) \in \mathcal{U}^{(m)}$. The constant C does not depend on m, because $H(\underline{w}^{(m)}) \in \mathcal{B}^{\mathrm{an}}(\gamma + \epsilon, \epsilon, \epsilon)$ for all $m \geq 0$. We denote by $h^{(m)}$ the complex-valued analytic function defined by $(\vec{p}, h^{(m)}(\vec{p}, z)) := E_{\rho,m}^{-1}(\vec{p}, z)$, for any $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$. $|\partial_z h^{(m)}(\vec{p}, z)| < \rho/(1 - C)$ for all $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$. If ρ is sufficiently small, $\rho/(1 - C) < 1$. Let $n, k \geq 0$, $n \geq m$, and $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$. Then, $e_{(m,n)} = h^{(m)} \circ ... \circ h^{(n)}$, and

$$\begin{aligned} |e_{m,n}(\vec{p}, z) - e_{m,n+k}(\vec{p}, z)| &< \frac{\rho}{1 - C} |e_{m+1,n}(\vec{p}, z) - e_{m+1,n+k}(\vec{p}, z)| < ... \\ &< \left(\frac{\rho}{1 - C}\right)^{n-m} |e_{n,n}(\vec{p}, z) - e_{n,n+k}(\vec{p}, z)| \leq \left(\frac{\rho}{1 - C}\right)^{n-m} \mu. \end{aligned}$$

The sequence $(e_{m,n}(\vec{p}, z))_n$ is Cauchy and converges to $e_{m,\infty}(\vec{p}, z)$. The convergence is uniform in $(\vec{p}, z)$, as $|e_{m,n}(\vec{p}, z) - e_{m,\infty}(\vec{p}, z)| \leq (\rho/(1 - C))^{n-m}\mu$ for any $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$. Let $z, z' \in D_{\mu/2}$ and $\vec{p} \in U[\vec{p}^*]$. Then

$$|e_{m,n}(\vec{p}, z) - e_{m,n}(\vec{p}, z')| < \left(\frac{\rho}{1 - C}\right)^{n-m} |z - z'|,$$

and we deduce that $e_{m,\infty}(\vec{p}, z)$ does not depend on z. As $H(\underline{w}^{(m)}) \in \mathcal{B}^{\mathrm{an}}\left(\gamma + \varepsilon r_m, \frac{\varepsilon}{2^m}, \frac{\varepsilon}{2^m}\right)$,

$$|e_{m,\infty}(\vec{p}, z) - \rho e_{m+1,\infty}(\vec{p}, z)| \leq 2^{-m}\varepsilon.$$

We deduce that

$$|e_{0,\infty}(\vec{p}, 0)| \leq \sum_{j=0}^{\infty} \rho^j |e_{j,\infty}(\vec{p}, 0) - \rho e_{j+1,\infty}(\vec{p}, 0)| \leq \frac{2\varepsilon}{2 - \rho},$$

for any $\vec{p} \in U[\vec{p}^*]$. Therefore, $e_{0,\infty}(\vec{p}, z) = e_{0,\infty}(\vec{p})$ belongs to a small disk of radius 2ε centered at the origin of the complex plane. □

When n goes to infinity, the perturbation $W_{\geq 1}^{(n)}$ tends to zero, and we expect the operator $H^{(n)}(\underline{w}(\vec{p}, e_{n,\infty}(\vec{p}, 0)))$ to tend to $\alpha(\vec{p})H_f + \vec{\beta}(\vec{p}) \cdot \vec{P}_f$. $\alpha(\vec{p})H_f + \vec{\beta}(\vec{p}) \cdot \vec{P}_f$ has an eigenvalue 0 with associated eigenvector $|\Omega\rangle$, and we expect the complex sequence $(\rho_0 e_{0,n}(\vec{p}, 0))_n$ to converge to an eigenvalue of the initial "Hamiltonian" $H(\vec{p})$.

Lemma 4.5. *Let $0 < \rho \ll \mu$, $0 < \xi < 1/(4\sqrt{8\pi})$, $0 < \gamma \ll \mu$, $0 < \delta \ll \rho\mu$, $0 < \varepsilon \ll \rho^2\mu^2$, and $\vec{p} \in U[\vec{p}^*]$. Then $H(\underline{w}^{(n)}(\vec{p}, e_{n,\infty}(\vec{p}, 0)))$ converges in norm to an operator $H_{Fix}(\vec{p}) := \alpha(\vec{p})H_f + \vec{\beta}(\vec{p}) \cdot \vec{P}_f$, where $\alpha(\vec{p}) \in \mathbb{C}$ and $\vec{\beta}(\vec{p}) \in \mathbb{C}^3$. If $\vec{p} \in \mathbb{R}^3 \cap U[\vec{p}^*]$, $\alpha(\vec{p}) \in \mathbb{R}$ and $\vec{\beta}(\vec{p}) \in \mathbb{R}^3$.*

Proof. Let $\vec{p} \in U[\vec{p}^*]$. We show that the sequence of kernels $(w_{0,0}^{(n)}(\vec{p}, e_{n,\infty}(\vec{p}, 0), r, \vec{l}))_n$ converges uniformly on $\mathcal{B}$ to $\alpha(\vec{p})r + \vec{\beta}(\vec{p}) \cdot \vec{l}$. We introduce the function

$$T_n(\vec{p}, r, \vec{l}) := w_{0,0}^{(n)}(\vec{p}, e_{n,\infty}(\vec{p}, 0), r, \vec{l}) - w_{0,0}^{(n)}(\vec{p}, e_{n,\infty}(\vec{p}, 0), 0, \vec{0}),$$

defined for any $(r, \vec{l}) \in \mathcal{B}$. We show that the continuous functions $(r, \vec{l}) \mapsto (\partial_r T_n)(\vec{p}, r, \vec{l}) \in \mathbb{C}$ and $(r, \vec{l}) \mapsto (\partial_{l_j} T_n)(\vec{p}, r, \vec{l}) \in \mathbb{C}$ converge to constants $\alpha(\vec{p})$ and $\beta_j(\vec{p})$ uniformly on $\mathcal{B}$, respectively. We set

$$\Delta T_n(\vec{p}, r, \vec{l}) := T_n(\vec{p}, r, \vec{l}) - \rho^{-1} T_{n-1}(\vec{p}, \rho r, \rho \vec{l}).$$

For any $m, n \in \mathbb{N}$ with $n > m$,

$$T_n(\vec{p}, r, \vec{l}) - \rho^{m-n} T_m(\vec{p}, \rho^{n-m} r, \rho^{n-m} \vec{l}) = \sum_{i=m+1}^{n} \rho^{i-n} \Delta T_i(\vec{p}, \rho^{n-i} r, \rho^{n-i} \vec{l}). \tag{4.15}$$

From the exact formula (B.8) for the kernels $w_{0,0}^{(m)}$, we deduce that

$$\begin{aligned} &\Delta T_n(\vec{p}, r, \vec{l}) \\ &= \rho^{-1} \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{\substack{\underline{p}, \underline{q} \\ p_i + q_i \geq 1}} \left(V_{\underline{0,p,0,q}}^{(n)}(\vec{p}, e_{n-1,\infty}(\vec{p}, 0), r, \vec{l}) - V_{\underline{0,p,0,q}}^{(n)}(\vec{p}, e_{n-1,\infty}(\vec{p}, 0), 0, \vec{0}) \right). \end{aligned}$$

The proof of Lemma 4.3 (see Appendix E) shows that $\Delta T_n(\vec{p}, r, \vec{l})$ is differentiable with respect to the variables r and l_j, for $j = 1, \ldots, 3$, as the series of partial derivatives converge uniformly on $\mathcal{B}$. Furthermore, $\|\partial_r \Delta T_n(\vec{p})\|_\infty \leq \varepsilon/2^{n-1}$ and $\|\partial_{l_j} \Delta T_n(\vec{p})\|_\infty \leq \varepsilon/2^{n-1}$. Differentiating (4.15) with respect to r, we have that

$$|(\partial_r T_n)(\vec{p}, r, \vec{l}) - (\partial_r T_m)(\vec{p}, \rho^{n-m} r, \rho^{n-m} \vec{l})| \leq \frac{\varepsilon}{2^{m-1}},$$

for all $(r, \vec{l}) \in \mathcal{B}$ and $n > m$. In particular, for $(r, \vec{l}) = (0, \vec{0})$,

$$|(\partial_r T_n)(\vec{p}, 0, \vec{0}) - (\partial_r T_m)(\vec{p}, 0, \vec{0})| \leq \frac{\varepsilon}{2^{m-1}}.$$

$((\partial_r T_n)(\vec{p}, 0, \vec{0}))_n$ is a complex Cauchy sequence and converges to a complex number that we denote by $\alpha(\vec{p})$. The same result holds for the partial derivatives with respect to l_j, and we denote their limits by $\beta_j(\vec{p})$. $\alpha(\vec{p})$ and $\beta_j(\vec{p}) \in \mathbb{R}$ if $\vec{p} \in \mathbb{R}^3 \cap U[\vec{p}^*]$. We show that the sequence of functions $(T_n)_n$ converges to $\alpha(\vec{p})r + \vec{\beta}(\vec{p}) \cdot \vec{l}$ uniformly on $\mathcal{B}$. Let $\eta > 0$. There exists $N(\eta) \in \mathbb{N}$ such that, for any $n \geq N(\eta)$, $|(\partial_r T_n)(\vec{p}, 0, \vec{0}) - \alpha(\vec{p})| < \eta$ and $\frac{\varepsilon}{2^{n-1}} < \eta$. Then, for any $n > N(\eta)$, for any $(r, \vec{l}) \in \mathcal{B}$,

$$\begin{aligned}|(\partial_r T_n)(\vec{p}, r, \vec{l}) - \alpha(\vec{p})| &\leq |(\partial_r T_n)(\vec{p}, r, \vec{l}) - (\partial_r T_{N(\eta)})(\vec{p}, \rho^{n-N(\eta)} r, \rho^{n-N(\eta)} \vec{l})| \\ &+ |(\partial_r T_{N(\eta)})(\vec{p}, 0, \vec{0}) - \alpha(\vec{p})| + |(\partial_r T_{N(\eta)})(\vec{p}, \rho^{n-N(\eta)} r, \rho^{n-N(\eta)} \vec{l}) - (\partial_r T_{N(\eta)})(\vec{p}, 0, \vec{0})| \\ &< 2\eta + |(\partial_r T_{N(\eta)})(\vec{p}, \rho^{n-N(\eta)} r, \rho^{n-N(\eta)} \vec{l}) - (\partial_r T_{N(\eta)})(\vec{p}, 0, \vec{0})|.\end{aligned}$$

The function $\partial_r T_{N(\eta)}$ is continuous in $(0, \vec{0})$ and there exists $M(\eta) > N(\eta)$, such that , for any $n > M(\eta)$, for any $(r, \vec{l}) \in \mathcal{B}$,

$$|(\partial_r T_n)(\vec{p}, r, \vec{l}) - \alpha(\vec{p})| < 3\eta.$$

We deduce that $(\partial_r T_n)_n$ converges uniformly on $\mathcal{B}$ to the constant $\alpha(\vec{p})$. A similar result holds for the partial derivatives with respect to l_j. As $T_n(\vec{p}, 0, \vec{0}) = 0$, $T_n(\vec{p}, r, \vec{l})$ converges uniformly to $\alpha(\vec{p})r + \vec{\beta}(\vec{p}) \cdot \vec{l}$ on $\mathcal{B}$. $w_{0,0}^{(n)}(\vec{p}, e_{n,\infty}(\vec{p}, 0), 0, \vec{0}) = -\rho e_{n+1,\infty}(\vec{p}, 0)$ converges to zero, and therefore, $w_{0,0}^{(n)}(\vec{p}, e_{n,\infty}(\vec{p}, 0), r, \vec{l})$ converges uniformly on $\mathcal{B}$ to $\alpha(\vec{p})r + \vec{\beta}(\vec{p}) \cdot \vec{l}$. □

Remarks. *$|\alpha(\vec{p}) - 1| \leq \gamma + \varepsilon \ll \mu$, and $|\vec{\beta}(\vec{p}) + \vec{p}/m| \leq \gamma + \varepsilon \ll \mu$. Therefore, the "effective" momentum $-m\vec{\beta}(\vec{p})$ stays close to $\vec{p}$ and the renormalization map can be iterated indefinitely.*

We set

$$z_\infty(\vec{p}) := \rho_0 e_{0,\infty}(\vec{p}, 0). \tag{4.16}$$

The next lemma states that $z_\infty(\vec{p})$ is an eigenvalue of $H(\vec{p})$.

Lemma 4.6. *$z_\infty(\vec{p})$ is a non-degenerate eigenvalue of $H(\vec{p})$. If $\vec{p} \in \mathbb{R}^3 \cap U[\vec{p}^*]$, $z_\infty(\vec{p})$ lies at the bottom of the spectrum of $H(\vec{p})$, i.e.*

$$z_\infty(\vec{p}) = \inf \sigma(H(\vec{p})). \tag{4.17}$$

The vector $|\Omega\rangle$ is non-zero eigenvector of H_{Fix} with eigenvalue 0. The Feshbach-Schur theorem 2.2 asserts that we can find an eigenvector of $H(\vec{p})$ corresponding to $z_\infty(\vec{p})$ with the help of the auxiliary operators Q_χ introduced in (2.7). We consider the sequence of vectors

$$\Psi_n(\vec{p}) := Q_{-1}(\vec{p}) \Gamma_{\rho_0}^* Q_0(\vec{p}) \Gamma_\rho^* \cdots \Gamma_\rho^* Q_n(\vec{p}) (\downarrow \otimes \Omega), \tag{4.18}$$

where

$$Q_n(\vec{p}) := Q_{\chi_\rho(H_f)} \left(H(\underline{w}^{(n)}(\vec{p}, e_{(n,\infty)}(\vec{p}))), W_{0,0}(w_{0,0}^{(n)}(\vec{p}, e_{(n,\infty)}(\vec{p}))) \right), \tag{4.19}$$

and

$$Q_{(-1)}(\vec{p}) := Q_{\boldsymbol{\chi}} \left(H(\vec{p}, z_\infty(\vec{p})), H(\vec{p}, z_\infty(\vec{p})) - \lambda_0 H_I \right). \tag{4.20}$$

Lemma 4.7. *Under the conditions of Lemma 4.4, the limit*

$$\Psi_\infty(\vec{p}) := \lim_{n \to \infty} \Psi_n(\vec{p}) \tag{4.21}$$

exists, is non zero, and belongs to $\ker[H(\vec{p}, z_\infty(\vec{p}))]$. The convergence of $\Psi_n(\vec{p})$ to $\Psi_\infty(\vec{p})$ is uniform in $\vec{p} \in U[\vec{p}^*]$.

The detailed proof of the assertion – "$\rho_0 e_{0,\infty}(\vec{p})$ is an eigenvalue of $H(\vec{p})$" – in Lemma 4.6 is similar to [4, Theorem 3.12], and we refer the reader to this paper for details. The proof of the non-degeneracy of $z_\infty(\vec{p})$ is a direct consequence of [29, Theorem 2.1]. Furthermore, the fact that $z_\infty(\vec{p})$ lies at the bottom of the spectrum if $\vec{p} \in U[\vec{p}^*] \cap \mathbb{R}^3$, is, in all points, similar to the proof of [28, Theorem 2.1 (i)]. The proof of Lemma 4.7 is carried out in detail in [4].

5. Proof of Theorem 1.1

5.1. **Proof of the main theorem.** The proof of Theorem 1.1 follows by gathering the results of Lemmas 3.3, 4.1, 4.3, 4.6 and 4.7. Lemmas 3.3 and 4.3 show that the map $(\vec{p}, z) \mapsto H(\underline{w}^{(n)}(\vec{p}, z)) \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ is analytic on $U[\vec{p}^*] \times D_{\mu/2}$ for all $n \in \mathbb{N} \cup \{0\}$. We deduce from Lemmas 4.2 and 2.9 that $E_{\rho,n}^{-1}(\vec{p}, z)$ is analytic on $U[\vec{p}^*] \times D_{\mu/2}$, for any $n \geq 0$. As a composition of analytic maps, $e_{(0,n)}(\cdot, 0)$ defined in (4.13) is analytic on $U[\vec{p}^*]$ for any $n \geq 0$. The uniform convergence of the sequence of functions $(e_{(0,n)}(\cdot, 0))_n$ to $e_{(0,\infty)}$ on $U[\vec{p}^*]$ shows that $e_{(0,\infty)}$, and therefore z_∞, are analytic on $U[\vec{p}^*]$.

The analyticity of $\vec{p} \mapsto \Psi_\infty(\vec{p})$ is proven in a similar way. To shorten notations, we set $H^{(n)}(\vec{p}) := H(\underline{w}^{(n)}(\vec{p}, e_{(n,\infty)}(\vec{p})))$ and $W_{\geq 1}^{(n)}(\vec{p}) := W_{\geq 1}(\underline{w}^{(n)}(\vec{p}, e_{(n,\infty)}(\vec{p})))$, for any $n \geq 0$. Lemmas 4.2 and 4.3 imply that $\vec{p} \mapsto H^{(n)}(\vec{p}) \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ and $\vec{p} \mapsto W_{\geq 1}^{(n)}(\vec{p}) \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$ are analytic on $U[\vec{p}^*]$. From (4.19) and the definition of Q_χ in (2.7), we deduce that

$$Q_n(\vec{p}) = \chi_\rho(H_f) - \overline{\chi}_\rho(H_f)[H^{(n)}(\vec{p})]^{-1}_{\mathrm{Ran}(\overline{\chi}(H_f))} \overline{\chi}_\rho(H_f) W_{\geq 1}^{(n)}(\vec{p}) \chi_\rho(H_f) \in \mathcal{B}(\mathcal{H}_{\mathrm{red}})$$

is analytic on $U[\vec{p}^*]$, for any $n \geq 0$, being a product of bounded analytic operators. Furthermore,

$$Q_{-1}(\vec{p}) = \boldsymbol{\chi} - \lambda_0 \overline{\boldsymbol{\chi}} [H_{\overline{\boldsymbol{\chi}}}(\vec{p}, z_\infty(\vec{p}))]^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})} \overline{\boldsymbol{\chi}} H_I \boldsymbol{\chi} \in \mathcal{B}(\mathcal{H})$$

is analytic in $\vec{p}$. Indeed, the Neumann expansion of $\overline{\boldsymbol{\chi}}[H_{\overline{\boldsymbol{\chi}}}(\vec{p}, z_\infty(\vec{p}))]^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})}(H_f + \rho_0)^{1/2}\overline{\boldsymbol{\chi}}$ converges uniformly on $U[\vec{p}^*]$ (see the proof of Lemma 3.1), and the bounded operator $[(H_f + \rho_0)(H_0(\vec{p}, z_\infty(\vec{p})))]^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})}$ is analytic on $U[\vec{p}^*]$; See the proof of Lemma 3.3. Therefore, $\Psi_n(\vec{p})$ is analytic on $U[\vec{p}^*]$, for any $n \geq 0$. The uniform convergence of $(\Psi_n)_n$ to Ψ_∞ on $U[\vec{p}^*]$ completes the proof.

Appendix A. Lemmas about holomorphic functions

In this appendix, we gather some useful theorems for Banach space-valued and complex-valued analytic functions of several variables. The reader is referred to [36], [35], [30] and [8] for more detailed expositions. Let U be a connected open set in $\mathbb{C}^n$.

Definition A.1 (Analytic function)**.** *Let $n \geq 1$ and let X be a Banach space. A function $f : U \subset \mathbb{C}^n \to X$ is said to be analytic on U (or strongly analytic) if, for any $u = (u_1, \ldots, u_n) \in U$, there exists a neighborhood $\mathcal{V}(u) \subset U$ of u, such that*

$$f(z) = \sum_{\alpha_1, \ldots, \alpha_n \geq 0} x_{\alpha_1 \ldots \alpha_n} (z_1 - u_1)^{\alpha_1} \cdots (z_n - u_n)^{\alpha_n}, \tag{A.1}$$

for any $z = (z_1, \ldots, z_n) \in \mathcal{V}(u)$, and where $x_{\alpha_1 \ldots \alpha_n} \in X$.

It is possible to show that strong and weak analyticity are equivalent for Banach space-valued holomorphic functions (see e.g. [35] for functions of one variable, and [8] for a generalization to several complex variables).

Theorem A.2. *Let X be a Banach space, and X' its dual. Let $f : U \subset \mathbb{C}^n \to X$ be a function. Then the following assertions are equivalent:*

(i) f is analytic on U;
(ii) For any $\Phi \in X'$, $\Phi \circ f$ is analytic on U.

Theorem A.2 is useful, since we can use the standard results for complex-valued analytic functions to investigate the analyticity of Banach space-valued functions. In particular, for functions $f : U \subset \mathbb{C}^n \to B(\mathcal{H})$ - where $B(\mathcal{H})$ is the Banach space of bounded operators on the Hilbert space $\mathcal{H}$- it is sufficient to check that $\langle \phi | f(\cdot) \psi \rangle$ is analytic for any $\phi, \psi \in \mathcal{H}$. A theorem due to Hartogs says that a complex-valued function of several variables is analytic if and only if it is holomorphic with respect to each variable individually. This theorem establishes the equivalence between analyticity and complex differentiability (or holomorphy) for complex-valued functions of several variables. To find out whether a complex-valued function of several variables is holomorphic on U or not, we will use the generalization of Cauchy-Poincaré and Morera's theorems to several complex variables (see [36]):

Theorem A.3 (Cauchy-Poincaré)**.** *If $f : U \subset \mathbb{C}^n \to \mathbb{C}$ is holomorphic, then, for any $(n+1)-$dimensional bounded surface $V \subset U$ with class C^1 boundary such that $\partial V$is an n-dimensional piecewise-smooth surface,*

$$\int_{\partial V} f = 0. \tag{A.2}$$

Theorem A.4 (Morera)**.** *Let $f : U \subset \mathbb{C}^n \to \mathbb{C}$ be a continuous function. We assume that*

$$\int_{\partial V_k} f = 0 \tag{A.3}$$

for any arbitrary surface $V_k = l_1 \times ... \times l_{k-1} \times \Delta_k \times l_{k+1} \times ... \times l_n$ included in U, where the l_j's are class C^1 curves with ends z_j' and z_j'' in the z_j-plane, and Δ_k is a closed bounded simply connected domain in the z_k-plane with a piecewise-smooth boundary, $k = 1, ..., n$. Then f is holomorphic on U.

Finally, we state a well known convergence theorem for sequences of holomorphic functions.

Lemma A.5. *Let X a Banach space, and $(f_k)_{k \in \mathbb{N}}$ a sequence of holomorphic functions $f_k : U \to X$, which converges to a function f uniformly on any compact subset $K \subset U$. Then f is holomorphic on U.*

Appendix B. Wick ordered expression of the renormalized operator

In Subsection 2.4, we explained shortly why $\mathcal{R}_\rho(H(\underline{w}))$ can be rewritten as a series of Wick monomials. In this appendix, we give the exact expression of the new kernel $\underline{\hat{w}}$ and complete the intuitive picture of Subsection 2.4. We use the notations of Paragraph 2.2.1, and, to avoid a large increase of the length of our formulas, we introduce

$$\begin{aligned} r_i &:= \sum_{j=1}^{i-1} \Sigma[\underline{\tilde{k}}_j^{(n_j)}] + \sum_{j=i+1}^{L} \Sigma[\underline{k}_j^{(m_j)}], \quad \vec{l}_i := \sum_{j=1}^{i-1} \vec{\Sigma}[\underline{\tilde{k}}_j^{(n_j)}] + \sum_{j=i+1}^{L} \vec{\Sigma}[\underline{k}_j^{(m_j)}], \\ \tilde{r}_i &:= \sum_{j=1}^{i} \Sigma[\underline{\tilde{k}}_j^{(n_j)}] + \sum_{j=i+1}^{L} \Sigma[\underline{k}_j^{(m_j)}], \quad \vec{\tilde{l}}_i := \sum_{j=1}^{i} \vec{\Sigma}[\underline{\tilde{k}}_j^{(n_j)}] + \sum_{j=i+1}^{L} \vec{\Sigma}[\underline{k}_j^{(m_j)}]. \end{aligned} \tag{B.1}$$

We remind the reader that

$$F(\vec{p},z,r,\vec{l}) := \frac{\overline{\chi}_\rho^2(r)}{w_{0,0}(\vec{p},z,r,\vec{l})}. \tag{B.2}$$

In order to get the expression of the new kernels, we have to investigate how the arguments of the kernels w_{M_i,N_i} are modified by the pull-through of the annihilation and creation operators. We set $p_i := M_i - m_i$, $q_i := N_i - n_i$ to follow the notations already present in the literature. Once we have contracted $p_1 + ... + p_L$ creation operators with $q_1 + ... + q_L$ annihilation operators and pulled through the $m_1 + ... + m_L$ uncontracted creation operators to the left and the $n_1 + ... + n_L$ uncontracted annihilation operators to the right, the third and the fourth arguments of the operators $w_{m_i+p_i,n_i+q_i}(\vec{p},z,H_f,\vec{P}_f,\underline{K}_i^{(m_i+p_i,n_i+q_i)})$ and $F(\vec{p},z,H_f,\vec{P}_f)$ are shifted by a certain amount, which depends on their position i. Namely, they are modified by the uncontracted creation operators that were originally sitting on their right hand-side and have been pulled-through to the left, and the uncontracted annihilation operators that were originally sitting on their left hand-side, and have been pulled-through to the right. This leads to a shift

$$H_f \to H_f + \sum_{j=i+1}^{L} \Sigma[\underline{k}^{(m_j)}] + \sum_{j=1}^{i-1} \Sigma[\underline{\tilde{k}}^{(n_j)}] = H_f + r_i$$

for the third argument of $w_{m_i+p_i,n_i+q_i}$, and a shift

$$H_f \to H_f + \sum_{j=i+1}^{L} \Sigma[\underline{k}^{(m_j)}] + \sum_{j=1}^{i} \Sigma[\underline{\tilde{k}}^{(n_j)}] = H_f + \tilde{r}_i$$

for the third argument of the operator-valued function $F(\vec{p},z,H_f,\vec{P}_f)$ sitting directly on the right of $w_{m_i+p_i,n_i+q_i}(\vec{p},z,H_f,\vec{P}_f)$. The shifts for $\vec{P}_f$ are given by the same formulas, with Σ replaced by $\vec{\Sigma}$. Thanks to Formula (2.37), the contracted part can be rewritten as the vacum expectation value of an operator $\mathscr{V}_{\underline{m,n}}^{\underline{m+p,n+q}}(\vec{p},z,r,\vec{l},\underline{K}^{(\underline{m},\underline{n})})$, whose expression after rescaling is given inside the brackets of Equation (B.4).

We are ready to state the exact expression of the new kernel $\underline{\hat{w}} = (\hat{w}_{M,N})$ which satisfies $H(\underline{\hat{w}}) = \mathcal{R}_\rho(H(\underline{w}))$. We introduce

$$W_{p,q}^{m,n}(\vec{p},z,r,\vec{l},\underline{K}^{(m,n)}) := \mathbb{1}_{H_f\leq 1} \int_{\underline{B}_1^{p+q}} b^*(\underline{x}^p) w_{m+p,n+q}(\vec{p},z,H_f+r,\vec{P}_f+\vec{l},\underline{k}^{(m)},\underline{x}^{(p)},\underline{\tilde{k}}^{(n)},\underline{\tilde{x}}^{(q)}) \\ b(\underline{\tilde{x}}^q) d\underline{X}^{(p,q)} \mathbb{1}_{H_f\leq 1}. \tag{B.3}$$

We also define the function

$$V_{\underline{m,p,n,q}}(\vec{p},z,r,\vec{l},\underline{K}^{(M,N)}) := \chi_1(r+\tilde{r}_0)\chi_1(r+\tilde{r}_L) \\ \Big\langle \Omega \Big| \prod_{i=1}^{L-1} \Big(W_{p_i,q_i}^{m_i,n_i}(\vec{p},z,\rho(r+r_i),\rho(\vec{l}+\vec{l}_i),\rho\underline{K}_i^{(m_i,n_i)}) F(\vec{p},z,H_f+\rho(r+\tilde{r}_i),\vec{P}_f+\rho(\vec{l}+\vec{\tilde{l}}_i)) \Big) \\ W_{p_L,q_L}^{m_L,n_L}(\vec{p},z,\rho(r+r_L),\rho(\vec{l}+\vec{l}_L),\rho\underline{K}_L^{(m_L,n_L)}) \Omega \Big\rangle, \tag{B.4}$$

where we have set $\underline{m} := (m_1,\cdots,m_L)$, $\underline{n} := (n_1,\cdots,n_L)$, $\underline{p} := (p_1,\cdots,p_L)$, $\underline{q} := (q_1,\cdots,q_L)$, and $\underline{m,p,n,q} := (m_1,p_1,n_1,q_1,\cdots,m_L,p_L,n_L,q_L)$ to simplify notation. We introduce the constants

$$C_{\underline{m},\underline{n}}^{\underline{m+p},\underline{n+q}} := \prod_{i=1}^{L} \begin{pmatrix} m_i + p_i \\ p_i \end{pmatrix} \begin{pmatrix} n_i + q_i \\ q_i \end{pmatrix}. \tag{B.5}$$

Setting $(\vec{p}, z) := E_\rho^{-1}(\vec{p}, \zeta)$, the sequence $(\hat{w}_{M,N})$ is given by the following expressions: For $M + N \geq 1$,

$$\begin{aligned} &\hat{w}_{M,N}[\vec{p}, \zeta, r, \vec{l}, \underline{K}^{(M,N)}] \\ &= \sum_{L=1}^{\infty} (-1)^{L-1} \rho^{\frac{3}{2}(M+N)-1} \sum_{\substack{\underline{m}, \underline{p}, \underline{n}, \underline{q} \\ m_1 + \ldots + m_L = M, \\ n_1 + \ldots + n_L = N \\ m_i + n_i + p_i + q_i \geq 1}} C_{\underline{m},\underline{n}}^{\underline{m+p},\underline{n+q}} \, V^{\mathrm{sym}}_{\underline{m,p,n,q}}(\vec{p}, z, r, \vec{l}, \underline{K}^{(M,N)}). \end{aligned} \tag{B.6}$$

The notation $f^{\mathrm{sym}}\left(\underline{K}^{(M,N)}\right)$ appearing in (B.6) denotes the symmetrization of f w.r.t. the variables $\underline{k}^{(M)}$ and $\underline{\tilde{k}}^{(N)}$, that is

$$f^{\mathrm{sym}}\left(\underline{K}^{(M,N)}\right) := \frac{1}{M!N!} \sum_{\pi \in S_M} \sum_{\tilde{\pi} \in S_N} f\left(\underline{k}_{\pi(1)}, \ldots, \underline{k}_{\pi(M)}, \underline{\tilde{k}}_{\tilde{\pi}(1)}, \ldots, \underline{\tilde{k}}_{\tilde{\pi}(N)}\right). \tag{B.7}$$

Finally, for $(M, N) = (0, 0)$,

$$\hat{w}_{0,0}(\vec{p}, \zeta, r, \vec{l}) = \rho^{-1} w_{0,0}(\vec{p}, z, \rho r, \rho \vec{l}) + \rho^{-1} \sum_{L=2}^{\infty} (-1)^{L-1} \sum_{\substack{\underline{p}, \underline{q} \\ p_i + q_i \geq 1}} V_{\underline{0,p,0,q}}(\vec{p}, z, r, \vec{l}), \tag{B.8}$$

where

$$\begin{aligned} &V_{\underline{0,p,0,q}}(\vec{p}, z, r, \vec{l}) \\ &:= \chi_1^2(r) \Big\langle \Omega \Big| \prod_{i=1}^{L-1} \Big(W^{0,0}_{p_i,q_i}(\vec{p}, z, \rho r, \rho \vec{l}) F(\vec{p}, z, H_f + \rho r, \vec{P}_f + \rho \vec{l}) \Big) W^{0,0}_{p_L,q_L}(\vec{p}, z, \rho r, \rho \vec{l}) \; \Omega \Big\rangle. \end{aligned} \tag{B.9}$$

Appendix C. Proof of Lemma 3.2

To render the structure of the proof of Lemma 3.2 clearer, we subdivide it into three steps. All the positive constants of order one appearing in the estimates are denoted by the capital letter C if they are independent of the parameters ρ_0, λ_0 and μ.

C.1. **Step 1: Explicit expressions for the kernels.** We start from (3.8). We set

$$\begin{aligned} H^{(0)}(\vec{p}, z) := \mathcal{S}_{\rho_0} \Big(\Big\langle \Big(H_f + \frac{\vec{P}_f^2}{2m} - \frac{\vec{p}}{m} \cdot \vec{P}_f + \lambda_0 \boldsymbol{\chi} H_I \boldsymbol{\chi} \\ - \lambda_0^2 \boldsymbol{\chi} H_I \overline{\boldsymbol{\chi}} (H_{\overline{\boldsymbol{\chi}}}(\vec{p}, \rho_0 z))^{-1} \overline{\boldsymbol{\chi}} H_I \boldsymbol{\chi} - \rho_0 z \mathbb{1} \Big) \mathbb{1}_{H_f \leq \rho_0} \Big\rangle_{\downarrow} \Big). \end{aligned}$$

One has that

$$\begin{aligned} H^{(0)}(\vec{p}, z) = &\hat{w}_{0,0}(\vec{p}, z, H_f, \vec{P}_f) \mathbb{1}_{H_f \leq 1} \\ &+ \lambda_0 \mathcal{S}_{\rho_0} \left(\left\langle \boldsymbol{\chi} H_I \boldsymbol{\chi} \right\rangle_{\downarrow} \right) - \lambda_0^2 \mathcal{S}_{\rho_0} \left(\left\langle \boldsymbol{\chi} H_I \overline{\boldsymbol{\chi}} (H_{\overline{\boldsymbol{\chi}}}(\vec{p}, \rho_0 z))^{-1} \overline{\boldsymbol{\chi}} H_I \boldsymbol{\chi} \right\rangle_{\downarrow} \right), \end{aligned} \tag{C.1}$$

where $\hat{w}_{0,0}(\vec{p}, z, r, \vec{l}) := r + \rho_0 \frac{\vec{l}^2}{2m} - \frac{1}{m}\vec{p} \cdot \vec{l} - z$. An obvious calculation shows that

$$\sup_{(\vec{p},z)\in U[\vec{p}^*]\times D_{\mu/2}} \|\hat{w}_{0,0}(\vec{p}, z, r, \vec{l}) - \hat{w}_{0,0}(\vec{p}, z, 0, \vec{0}) - (r - m^{-1}\vec{p} \cdot \vec{l})\|^\sharp \leq \frac{\sqrt{3}\rho_0}{m},$$

where we used that $|\vec{l}| \leq 1$. We remind the reader that

$$\|\hat{w}_{0,0}\|^\sharp := |\hat{w}_{0,0}(0, \vec{0})| + \|\partial_r \hat{w}_{0,0}\|_\infty + \sum_{i=1}^{3} \|\partial_{l_i} \hat{w}_{0,0}\|_\infty.$$

The second term on the right hand side of (C.1) is already cast into a sum of Wick monomials. A straightforward calculation making use of the Pull-through formula to restrict the integration range implies that

$$\lambda_0 \mathcal{S}_{\rho_0}\left(\left\langle \boldsymbol{\chi} H_I \boldsymbol{\chi}\right\rangle_\downarrow\right) = -i\lambda_0\rho_0 \int_{\underline{B}_1} |\vec{k}|^{\frac{1}{2}} \vec{\epsilon}(\underline{k}) \cdot \vec{e}_z \; \chi(H_f)\,(b(\underline{k}) - h.c.)\,\chi(H_f)\, d\underline{k} \tag{C.2}$$

The associated kernels are independent of $\vec{p}, z$ and $\vec{l}$, and are given by

$$\hat{w}_{1,0}(r, \underline{k}) = -\hat{w}_{0,1}(r, \underline{k}) = i\lambda_0\rho_0 |\vec{k}|^{1/2} \; \vec{\epsilon}(\underline{k}) \cdot \vec{e}_z \; \chi(r + |k|)\chi(r). \tag{C.3}$$

Their norm $\|\cdot\|_{\frac{1}{2}}$ is of order $\lambda_0\rho_0$. The analysis of the third term on the right-hand side of (C.1) requires more work. The procedure is similar to the discussion of Paragraph 2.4. We write down the Neumann expansion for $[H_{\overline{\boldsymbol{\chi}}}(\vec{p}, \rho_0 z)]^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})}$ and Wick-order it to rewrite

$$\mathcal{S}_{\rho_0}\left(\left\langle \boldsymbol{\chi} H_I \overline{\boldsymbol{\chi}}\,[H_{\overline{\boldsymbol{\chi}}}(\vec{p}, \rho_0 z)]^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})}\, H_I \boldsymbol{\chi}\right\rangle_\downarrow\right)$$

as a series of Wick monomials. We use the formulas in Appendix B and get that

$$-\lambda_0^2 \mathcal{S}_{\rho_0}\left(\left\langle \boldsymbol{\chi} H_I \overline{\boldsymbol{\chi}}\,[H_{\overline{\boldsymbol{\chi}}}(\vec{p}, \rho_0 z)]^{-1}_{\mathrm{Ran}(\overline{\boldsymbol{\chi}})}\, H_I \boldsymbol{\chi}\right\rangle_\downarrow\right) = H(\underline{\tilde{w}}(\vec{p}, z)), \tag{C.4}$$

where, for $M, N \geq 0$,

$$\begin{aligned} &\tilde{w}_{M,N}(\vec{p}, z, r, \vec{l}, \underline{K}^{(M,N)}) \\ &= \sum_{L=2}^{\infty} (-1)^{L-1} \rho_0^{\frac{3}{2}(M+N)-1} \sum_{\substack{\underline{m}, \underline{p}, \underline{n}, \underline{q}, \\ m_1 + \cdots + m_L = M, \\ n_1 + \cdots + n_L = N \\ m_i + n_i + p_i + q_i = 1}} V^{\mathrm{sym}}_{\underline{m,p,n,q}}(\vec{p}, \rho_0 z, r, \vec{l}, \underline{K}^{(M,N)}), \end{aligned} \tag{C.5}$$

with

$$\begin{aligned} V_{\underline{m,p,n,q}}(\vec{p}, \rho_0 z, r, \vec{l}, \underline{K}^{(M,N)}) &:= \chi_1(r + \tilde{r}_0)\chi_1(r + \tilde{r}_L)\Big\langle \downarrow \otimes \Omega \Big| \prod_{i=1}^{L-1} \Big(W^{m_i,n_i}_{p_i,q_i}(\rho_0 \underline{K}_i^{(m_i,n_i)}) \\ &F(\vec{p}, \rho_0 z, H_f + \rho_0(r + \tilde{r}_i), \vec{P}_f + \rho_0(\vec{l} + \vec{\tilde{l}}_i))\Big) W^{m_L,n_L}_{p_L,q_L}(\rho_0 \underline{K}_L^{(m_L,n_L)}) \; \downarrow \otimes \Omega \Big\rangle. \end{aligned} \tag{C.6}$$

The function F is given by

$$F(\vec{p}, z, r, \vec{l}) = P_\downarrow \otimes b_1^{-1}(\vec{p}, z, r, \vec{l})\overline{\chi}^2_{\rho_0}(r) + P_\uparrow \otimes b_2^{-1}(\vec{p}, z, r, \vec{l}), \tag{C.7}$$

with

$$b_1(\vec{p},z,r,\vec{l}) = \Big(r + \frac{\vec{l}^2}{2m} - \frac{\vec{p}}{m}\cdot\vec{l} - z\Big)\mathbb{1}_{r\geq 3\rho_0/4} \tag{C.8}$$

$$b_2(\vec{p},z,r,\vec{l}) = r + \frac{\vec{l}^2}{2m} + \omega_0 - \frac{\vec{p}}{m}\cdot\vec{l} - z. \tag{C.9}$$

The sum in (C.5) is only carried out for $m_i + p_i + n_i + q_i = 1$ and the operators $W^{m_i,n_i}_{p_i,q_i}$ do not depend on $(\vec{p},z)$. In (C.6),

$$\begin{aligned} &W^{m_i,n_i}_{p_i,q_i}(\underline{K}^{(m_i,n_i)}) \\ &= \int_{\underline{B}_1} b^*(\underline{x}^{(p_i)})\ w_{m_i+p_i,n_i+q_i}(\underline{k}^{(m_i)},\underline{x}^{(p_i)},\underline{\tilde{k}}^{(n_i)},\underline{\tilde{x}}^{(q_i)})\ b(\underline{\tilde{x}}^{(q_i)})d\underline{X}^{(p_i,q_i)}. \end{aligned} \tag{C.10}$$

As H_I is linear in annihilation and creation operators, the only non-zero kernels are the $\mathbb{C}^2$-valued functions $w_{0,1}$ and $w_{1,0}$, given by

$$w_{0,1}(\underline{k}) = -w_{1,0}(\underline{k}) = i\lambda_0|\vec{k}|^{1/2}\vec{\epsilon}(\underline{k})\cdot\vec{\sigma}\ \mathbb{1}_{|\vec{k}|\leq 1}. \tag{C.11}$$

Therefore, $W^{m,n}_{p,q}$ is non-zero if, and only if,

$$m+p=0,\ n+q=1 \qquad \text{or} \qquad m+p=1,\ n+q=0.$$

If $m+n=1$,

$$W^{m,n}_{0,0}(\rho_0\underline{k}) = (-1)^m i\lambda_0\rho_0^{1/2}|\vec{k}|^{1/2}\vec{\epsilon}(\underline{k})\cdot\vec{\sigma}\ \mathbb{1}_{|\vec{k}|\leq 1}(\vec{k}). \tag{C.12}$$

If $m+n=0$,

$$W^{0,0}_{1,0} = -i\lambda_0\int_{\underline{B}_1}|\vec{k}|^{1/2}\ \vec{\epsilon}(\underline{k})\cdot\vec{\sigma}\ b^*(\underline{k})d\underline{k}, \tag{C.13}$$

and

$$W^{0,0}_{0,1} = i\lambda_0\int_{\underline{B}_1}|\vec{k}|^{1/2}\ \vec{\epsilon}(\underline{k})\cdot\vec{\sigma}\ b(\underline{k})d\underline{k}. \tag{C.14}$$

Note that $\mathbb{1}_{|\vec{k}|\leq 1}(\vec{k})$ appears in (C.12), because the term $\chi(r+\tilde{r}_0)$ in (C.6) forces $|\vec{k}|$ to be smaller than one.

C.2. **Step 2: Bounds on the norm of kernels.** We construct the sequence of kernels $\underline{\hat{w}} = (\hat{w}_{M,N})$ by setting $\hat{w}_{0,0}(\vec{p},z,r,\vec{l}) = r + \rho_0\frac{\vec{l}^2}{2m} - \frac{1}{m}\vec{p}\cdot\vec{l} - z$, $\hat{w}_{1,0}$ and $\hat{w}_{0,1}$ as given in (C.3), and $\hat{w}_{M,N} = 0$ for $M+N>1$. $H^{(0)}(\vec{p},z) = H((\underline{\hat{w}}+\underline{\tilde{w}})(\vec{p},z))$, with $\tilde{w}_{M,N}$ as in (C.5). In order to show that the Wick-ordered operator $H(\underline{\hat{w}}+\underline{\tilde{w}})$ belongs to a polydisc $\mathcal{B}(\gamma,\delta,\epsilon)$, we need to investigate the convergence of the series in (C.5). The assertion of the lemma follows if this series converges uniformly in norm $\|\cdot\|^\sharp$ on $U[\vec{p}^*]\times D_{\mu/2}$, and, can be bounded by an arbitrary small constant after an adequate tuning of the coupling λ_0. We first bound $\|V_{\underline{m,p,n,q}}\|^\sharp$. To avoid a long proof, we only explicitly bound $\|V_{\underline{m,p,n,q}}\|_{\frac{1}{2}}$. The reader can check that the bounds are similar for the partial derivatives. Our proof works if the ultraviolet cut-off function in the interacting Hamiltonian, $\mathbb{1}(|\vec{k}|\leq 1)$, is replaced by an arbitrary cut-off function in $L^2(\mathbb{R}^3)$ that equals one in a neighborhood of the origin. To shorten notations, we set $R_i := H_f + \rho_0(r + \tilde{r}_i)$, and $\vec{L}_i := \vec{P}_f + \rho_0(\vec{l} + \tilde{\vec{l}}_i)$. We introduce $(H_f + \rho_0)^{-1/2}$ to the right and to the left of $W^{0,0}_{0,1}$ and $W^{0,0}_{1,0}$ in (C.6). Then,

$$\begin{aligned} V_{\underline{m,p,n,q}}(\vec{p}, \rho_0 z, r, \vec{l}, \underline{K}^{(M,N)}) &= \rho_0 \chi_1(r + \tilde{r}_0)\chi_1(r + \tilde{r}_L)\Big\langle \downarrow \otimes \Omega \Big| \\ &\prod_{i=1}^{L-1} \Big[(H_f + \rho_0)^{-\frac{1}{2}} W^{m_i,n_i}_{p_i,q_i}(\rho_0 \underline{K}_i^{(m_i,n_i)})(H_f + \rho_0)^{-\frac{1}{2}}(H_f + \rho_0)F(\vec{p}, \rho_0 z, R_i, \vec{L}_i)\Big] \\ &(H_f + \rho_0)^{-\frac{1}{2}} W^{m_L,n_L}_{p_L,q_L}(\rho_0 \underline{K}_L^{(m_L,n_L)})(H_f + \rho_0)^{-\frac{1}{2}} \ \downarrow \otimes \Omega \Big\rangle. \end{aligned} \tag{C.15}$$

For $i \neq L$, we give an upper bound for

$$A^{m_i,n_i}_{p_i,q_i} := \|(H_f + \rho_0)^{-\frac{1}{2}} W^{m_i,n_i}_{p_i,q_i}(\rho_0 \underline{K}_i^{(m_i,n_i)})(H_f + \rho_0)^{-\frac{1}{2}}(H_f + \rho_0)F(\vec{p}, \rho_0 z, R_i, \vec{L}_i)\|, \tag{C.16}$$

$m_i + n_i + p_i + q_i = 1$. There are two different cases:

(i) $m_i + n_i = 1$. Then,

$$\begin{aligned} A^{m_i,n_i}_{0,0} &\leq \lambda_0 \|F(\vec{p}, \rho_0 z, R_i, \vec{L}_i)\| |\rho_0 \vec{k}_i|^{1/2} \|\vec{\epsilon}_{\lambda_i}(\vec{k}_i) \cdot \vec{\sigma}\| \mathbb{1}_{|\vec{k}_i| \leq 1}(\vec{k}_i) \\ &\leq C \frac{\lambda_0}{\mu} |\vec{k}_i|^{1/2} \mathbb{1}_{|\vec{k}_i| \leq 1}(\vec{k}_i) \rho_0^{-1/2}. \end{aligned} \tag{C.17}$$

(ii) $m_i + n_i = 0$. Then,

$$A^{0,0}_{p_i,q_i} \leq C \lambda_0 \rho_0^{-\frac{1}{2}} \|(H_f + \rho_0)F(\vec{p}, \rho_0 z, R_i, \vec{L}_i)\|. \tag{C.18}$$

We use the functional calculus outlined in Paragraph 1.2.3 to bound $\|(H_f + \rho_0)F(\vec{p}, \rho_0 z, R_i, \vec{L}_i)\|$. It is bounded by the sum of the constants C_1 and C_2, where

$$C_1 := \sup_{r' \geq \frac{3\rho_0}{4}, |\vec{l}'| \leq r'} \left| \frac{r' + \rho_0}{r' + \rho_0(r + \tilde{r}_i) + \frac{|\vec{l}' + \rho_0(\vec{l} + \tilde{\vec{l}}_i)|^2}{2} - \frac{\vec{p}}{m} \cdot (\vec{l}' + \rho_0(\vec{l} + \tilde{\vec{l}}_i)) - \rho_0 z} \right|,$$

$$C_2 := \sup_{r' \geq 0, |\vec{l}'| \leq r'} \left| \frac{r' + \rho_0}{r' + \rho_0(r + \tilde{r}_i) + \omega_0 + \frac{|\vec{l}' + \rho_0(\vec{l} + \tilde{\vec{l}}_i)|^2}{2} - \frac{\vec{p}}{m} \cdot (\vec{l}' + \rho_0(\vec{l} + \tilde{\vec{l}}_i)) - \rho_0 z} \right|.$$

Using the assumptions $|z| < \mu/2$ together with $|\vec{l}| \leq r$ and $|\tilde{\vec{l}}_i| \leq \tilde{r}_i$, we obtain that

$$C_1 \leq \sup_{r' \geq \frac{3\rho_0}{4}, |\vec{l}'| \leq r'} \frac{r' + \rho_0}{\mu r' - \frac{\mu \rho_0}{2}}, \qquad C_2 \leq \sup_{r' \geq 0, |\vec{l}'| \leq r'} \frac{r' + \rho_0}{\mu r' + \omega_0 - \mu \frac{\rho_0}{2}}.$$

It follows that there is a constant $C > 1$, such that $A^{0,0}_{p_i,q_i} \leq C \mu^{-1} \lambda_0 \rho_0^{-\frac{1}{2}}$, and the norm $\| \cdot \|_{\frac{1}{2}}$ of $V_{\underline{m,p,n,q}}(\vec{p}, \rho_0 z, r, \vec{l}, \underline{K}^{(M,N)})$ is bounded by

$$\|V_{\underline{m,p,n,q}}(\vec{p}, \rho_0 z)\|_{\frac{1}{2}} \leq C^{L+1} \left(\frac{\lambda_0}{\mu \rho_0^{1/2}} \right)^L \rho_0.$$

Similar calculations for the partial derivatives show that

$$\left\|\partial_r V_{\underline{m,p,n,q}}(\vec{p},\rho_0 z)\right\|_{\frac{1}{2}} \leq \mu^{-1}(L+1)C^{L+1}\left(\frac{\lambda_0}{\mu\rho_0^{1/2}}\right)^L \rho_0,$$

$$\left\|\partial_{l_k} V_{\underline{m,p,n,q}}(\vec{p},\rho_0 z)\right\|_{\frac{1}{2}} \leq \mu^{-1}(L-1)C^{L+1}\left(\frac{\lambda_0}{\mu\rho_0^{1/2}}\right)^L \rho_0,$$

which implies that

$$\rho_0^{\frac{3}{2}(M+N)-1}\left\|V_{\underline{m,p,n,q}}(\vec{p},\rho_0 z)\right\|^\sharp \leq 5\mu^{-1}(L+1)C^{L+1}\rho_0^{\frac{3}{2}(M+N)}\left(\mu^{-1}\rho_0^{-\frac{1}{2}}\lambda_0\right)^L. \tag{C.19}$$

C.3. **Step 3: Bound on the norms $\|\cdot\|_\xi^\sharp$.** By Plugging (C.19) into (C.5), we finally obtain that

$$\|\underline{\tilde{w}}(\vec{p},z)\|_{\xi,\geq 1}^\sharp \leq 5\mu^{-1}\sum_{L=2}^\infty (L+1)C^{L+1}\left(4\mu^{-1}\rho_0^{-\frac{1}{2}}\lambda_0\right)^L \left(\left(\sum_{m\geq 0}\left(\rho_0^{\frac{3}{2}}\xi^{-1}\right)^m\right)^2 - 1\right).$$

If we assume that $\xi > \rho_0^{\frac{3}{2}}$, we get the bound

$$\|\underline{\tilde{w}}(\vec{p},z)\|_{\xi,\geq 1}^\sharp \leq 10\mu^{-1}C\frac{\rho_0^{3/2}\xi^{-1}}{(1-\rho_0^{3/2}\xi^{-1})^2}\sum_{L=2}^\infty (L+1)\left(4\mu^{-1}C\lambda_0\rho_0^{-\frac{1}{2}}\right)^L. \tag{C.20}$$

If $0 \leq \lambda_0 < \lambda_c(\mu)$, with

$$\lambda_c(\mu) := \frac{\mu\rho_0^{\frac{1}{2}}}{4C},$$

the series in (C.20) converges. Setting $\alpha := \lambda_0/\lambda_c < 1$, we find that

$$\|\underline{\tilde{w}}(\vec{p},z)\|_{\xi,\geq 1}^\sharp \leq 10\mu^{-1}C\frac{\rho_0^{3/2}\xi^{-1}}{(1-\rho_0^{3/2}\xi^{-1})^2}\frac{3\alpha^2}{(1-\alpha)^2}. \tag{C.21}$$

The right-hand side of (C.21) can be made arbitrary small by tuning λ_0. To conclude the proof, it is sufficient to check that $\|\tilde{w}_{0,0}(\vec{p},z)\|^\#$ can also be made as small as we wish by an appropriate tuning of λ_0. To this end, we have to give an upper bound for the norm of

$$\rho_0^{-1}\sum_{L=2}^\infty (-1)^{L-1}\sum_{\substack{\underline{p},\underline{q}\\ p_i+q_i=1}} V_{\underline{0,p,0,q}}(\vec{p},\rho_0 z, r, \vec{l}).$$

Some easy calculations lead to

$$\left\|V_{\underline{0,p,0,q}}(\vec{p},\rho_0 z)\right\|^\sharp \leq 5\mu^{-1}(L+1)C^{L+1}\rho_0(\lambda_0\mu^{-1}\rho_0^{-\frac{1}{2}})^L. \tag{C.22}$$

We deduce from (C.22) that

$$\left\|\rho_0^{-1}\sum_{L=2}^\infty (-1)^{L-1}\sum_{\substack{\underline{p},\underline{q}\\ p_i+q_i=1}} V_{\underline{0,p,0,q}}(\vec{p},\rho_0 z)\right\|^\sharp \leq 5\mu^{-1}C\sum_{L=2}^\infty (L+1)(2\mu^{-1}\lambda_0\rho_0^{-\frac{1}{2}})^L.$$

For $\lambda_0 < \mu\rho_0^{1/2}/2$, the series on the right-hand side converges, and can be made as small as we wish by an appropriate tuning of λ_0. All the estimates are uniform in $(\vec{p}, z) \in U[\vec{p}^*]\times D_{\mu/2}$, and we deduce that $H(\underline{w}^{(0)})$ can belong to any polydisc $\mathcal{B}(\frac{\sqrt{3}\rho_0}{m}+\gamma,\delta,\epsilon)$ if the coupling constant λ_0 is small enough.

Appendix D. Completion of the proof of Lemma 4.2

D.1. **Proof of the uniform convergence in** $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$**.** We carry out the steps that were sketched in Lemma 4.2. As $|\Psi_n\rangle$ is a two-particle state, $g_n(\vec{p}, z)$ is the sum of three functions, $u_n(\vec{p}, z)$, $v_n(\vec{p}, z)$ and $w_n(\vec{p}, z)$, defined by

$$\begin{aligned} u_n(\vec{p}, z) &:= \langle \Psi_n | \mathbb{1}_{H_f\le 1} w_{0,0}(\vec{p}, z, H_f, \vec{P}_f)\mathbb{1}_{H_f\le 1}\Psi_n\rangle,\\ v_n(\vec{p}, z) &:= \langle \Psi_n | \mathbb{1}_{H_f\le 1} W_{1,1}(\underline{w}(\vec{p}, z))\mathbb{1}_{H_f\le 1}\Psi_n\rangle,\\ w_n(\vec{p}, z) &:= \langle \Psi_n | \mathbb{1}_{H_f\le 1} W_{2,2}(\underline{w}(\vec{p}, z))\mathbb{1}_{H_f\le 1}\Psi_n\rangle. \end{aligned}$$

We first look at the function u_n. We introduce the set

$$\mathcal{D}(n;p) := \{(\underline{k}_1, ..., \underline{k}_n, \underline{k}_{n+1}, ..., \underline{k}_{n+p}) \in \underline{B}_1^{n+p} \mid |\vec{k_1}| + ... + |\vec{k}_n| \le 1, |\vec{k}_{n+1}| + ... + |\vec{k}_{n+p}| \le 1\}$$

and use the notation $\mathcal{D}(n) := \mathcal{D}(n, 0)$. An easy application of the pull-through formula and the canonical commutation relations shows that

$$\begin{aligned} u_n(\vec{p}, z) = &\int_{\mathcal{D}(2)} d\underline{k}_1 d\underline{k}_2\ w_{0,0}(\vec{p}, z, |\vec{k}_1| + |\vec{k}_2|, \vec{k}_1 + \vec{k}_2)\ \overline{\eta}_{n,\vec{l}}(\underline{k}_1)\overline{\eta}_{n,\vec{l}'}(\underline{k}_2)\eta_{n,\vec{l}}(\underline{k}_2)\overline{\eta}_{n,\vec{l}'}(\underline{k}_1)\\ &+ \int_{\mathcal{D}(2)} d\underline{k}_1 d\underline{k}_2\ w_{0,0}(\vec{p}, z, |\vec{k}_1| + |\vec{k}_2|, \vec{k}_1 + \vec{k}_2)\ |\eta_{n,\vec{l}}(\underline{k}_1)|^2 |\eta_{n,\vec{l}'}(\underline{k}_2)|^2. \end{aligned} \tag{D.1}$$

We assume that $\vec{l} \ne \vec{l}'$. We show that the first term on the right-hand side of (D.1) tends to zero when n tends to infinity, uniformly in $(\vec{p}, z)$. We make the change of variables $\vec{k'}_1 := n(\vec{k}_1 - \vec{l})$, $\vec{k'}_2 := n(\vec{k}_2 - \vec{l})$. Let $R > 0$ such that $\mathrm{Supp}(\eta) \subset B_R$. As $H(\underline{w}) \in \mathcal{B}(\gamma, \delta, \epsilon)$, there exists a constant $C(\gamma) > 0$ so that the first term on the right-hand side of (D.1) is bounded by

$$C(\gamma)\int_{\underline{B}_R^2} d\underline{k}'_1 d\underline{k}'_2\ |\overline{\eta}(\vec{k'}_1)\overline{\eta}(\vec{k'}_2 + n(\vec{l} - \vec{l}'))\eta(\vec{k}_2)\eta(\vec{k'}_1 + n(\vec{l} - \vec{l}'))|, \tag{D.2}$$

for any $(\vec{p}, z) \in U[\vec{p}^*]\times D_{\mu/2}$. The function η has a compact support. Therefore, the integrand in equation (D.2) goes point-wise to zero for any $(\underline{k}'_1, \underline{k}'_2)$. Furthermore, the integrand in (D.2) is bounded because η is smooth. Lebesgue dominated convergence theorem implies that (D.2) goes to zero when n goes to infinity. In the case where $\vec{l} = \vec{l}'$, the two terms on the right-hand side of (D.1) are similar, and it is therefore sufficient to analyze the second term. This term converges to $4w_{0,0}(\vec{p}, z, |\vec{l}| + |\vec{l}'|, \vec{l} + \vec{l}')$,uniformly in $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$, as a direct consequence of Lebesgue convergence theorem. Indeed, the supremum of $\left|\vec{\nabla}_{r,\vec{l}}\, w_{0,0}(\vec{p}, z, r, \vec{l})\right|$ over $\mathcal{B}$ is uniformly bounded in $(\vec{p}, z)$ and $\mathcal{B}$ is convex. Therefore, there exists a constant $C > 0$ such that

$$|w_{0,0}(\vec{p}, z, r, \vec{l}) - w_{0,0}(\vec{p}, z, r', \vec{l}')| \le C|(r, \vec{l}) - (r', \vec{l}')|$$

for all $(\vec{p}, z) \in U[\vec{p}^*]\times D_{\mu/2}$ and all $(r, \vec{l}), (r', \vec{l}') \in \mathcal{B}$. The uniform convergence in $(\vec{p}, z)$ follows. We deduce that $u_n(\vec{p}, z)$ converges uniformly to $4w_{0,0}(\vec{p}, z, |\vec{l}|+|\vec{l}'|, \vec{l}+\vec{l}')+4w_{0,0}(\vec{p}, z, |2\vec{l}|, 2\vec{l})\delta_{\vec{l},\vec{l}'}$ on $U[\vec{p}^*] \times D_{\mu/2}$.

We now show that the sequence of functions $(v_n)_n$ converges to zero uniformly in $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$. We remind the reader that

$$v_n(\vec{p}, z) := \langle \Psi_n | \mathbb{1}_{H_f \le 1} W_{1,1}(\underline{w}(\vec{p}, z)) \mathbb{1}_{H_f \le 1} \Psi_n \rangle,$$

where $\Psi_n := b^*(\eta_{n,\vec{l}}) b^*(\eta_{n,\vec{l}'}) |\Omega\rangle$. One has that

$$v_n(\vec{p}, z) = \int_{\mathcal{D}(2;2) \times \underline{B}_1^2} d\underline{K}\ \overline{\eta}_{n,\vec{l}}(\underline{k}_1) \overline{\eta}_{n,\vec{l}'}(\underline{k}_2) \eta_{n,\vec{l}}(\underline{k}_3) \eta_{n,\vec{l}'}(\underline{k}_4)\ \langle \Omega | b(\underline{k}_1) b(\underline{k}_2) b^*(\underline{k}_5) \\ w_{1,1}(\vec{p}, z, H_f, \vec{P}_f, \vec{k}_5, \vec{k}_6) b(\underline{k}_6) b^*(\underline{k}_3) b^*(\underline{k}_4) \Omega \rangle, \tag{D.3}$$

where we have set $d\underline{K} := \prod_{i=1}^6 d\underline{k}_i$. The pull-through formula, together with the canonical commutation relations, imply that $v_n(\vec{p}, z) = \sum_{i=1}^4 v_{n;i}(\vec{p}, z)$, where

$$v_{n;1}(\vec{p}, z) = \int_{\mathcal{D}(2;1)} d\underline{k}_1 d\underline{k}_2 d\underline{k}_3\ \overline{\eta}_{n,\vec{l}}(\underline{k}_1) \overline{\eta}_{n,\vec{l}'}(\underline{k}_2) \eta_{n,\vec{l}}(\underline{k}_3) \eta_{n,\vec{l}'}(\underline{k}_2)\ w_{1,1}(\vec{p}, z, |\vec{k}_2|, \vec{k}_2, \vec{k}_1, \vec{k}_3) \mathbb{1}_{|\vec{k}_3| + |\vec{k}_2| \le 1},$$

$$v_{n;2}(\vec{p}, z) = \int_{\mathcal{D}(2;1)} d\underline{k}_1 d\underline{k}_2 d\underline{k}_4\ \overline{\eta}_{n,\vec{l}}(\underline{k}_1) \overline{\eta}_{n,\vec{l}'}(\underline{k}_2) \eta_{n,\vec{l}}(\underline{k}_2) \eta_{n,\vec{l}'}(\underline{k}_4)\ w_{1,1}(\vec{p}, z, |\vec{k}_2|, \vec{k}_2, \vec{k}_1, \vec{k}_4) \mathbb{1}_{|\vec{k}_4| + |\vec{k}_2| \le 1},$$

$$v_{n;3}(\vec{p}, z) = \int_{\mathcal{D}(2;1)} d\underline{k}_1 d\underline{k}_2 d\underline{k}_3\ \overline{\eta}_{n,\vec{l}}(\underline{k}_1) \overline{\eta}_{n,\vec{l}'}(\underline{k}_2) \eta_{n,\vec{l}}(\underline{k}_3) \eta_{n,\vec{l}'}(\underline{k}_1)\ w_{1,1}(\vec{p}, z, |\vec{k}_1|, \vec{k}_1, \vec{k}_2, \vec{k}_3) \mathbb{1}_{|\vec{k}_3| + |\vec{k}_1| \le 1},$$

$$v_{n;4}(\vec{p}, z) = \int_{\mathcal{D}(2;1)} d\underline{k}_1 d\underline{k}_2 d\underline{k}_4\ \overline{\eta}_{n,\vec{l}}(\underline{k}_1) \overline{\eta}_{n,\vec{l}'}(\underline{k}_2) \eta_{n,\vec{l}}(\underline{k}_1) \eta_{n,\vec{l}'}(\underline{k}_4)\ w_{1,1}(\vec{p}, z, |\vec{k}_1|, \vec{k}_1, \vec{k}_2, \vec{k}_4) \mathbb{1}_{|\vec{k}_4| + |\vec{k}_1| \le 1}.$$

We use that η has a compact support included in $B_R(0)$, $R > 0$. We only detail the upper bounds for $v_{n;1}$ and $v_{n;2}$. The reader can check that the bounds for $v_{n;3}$ and $v_{n;4}$ are similar. After a well-suited change of variables, we find that, for n large enough,

$$v_{n;1}(\vec{p}, z) = n^{-3} \int_{\underline{B}_R^3} d\underline{k}'_1 d\underline{k}'_2 d\underline{k}'_3\ \overline{\eta}(\underline{k}'_1) \eta(\underline{k}'_3) |\eta(\underline{k}'_2)|^2\ f_n(\vec{p}, z, \underline{k}'_2; \underline{k}'_1, \underline{k}'_3),$$
$$v_{n;2}(\vec{p}, z) = n^{-3} \int_{\underline{B}_R^3} d\underline{k}'_1 d\underline{k}'_2 d\underline{k}'_4\ \overline{\eta}(\underline{k}'_1) \overline{\eta}(\underline{k}'_2) \eta(\underline{k}'_2 + n(\vec{l}' - \vec{l})) \eta(\underline{k}'_4)\ g_n(\vec{p}, z, \underline{k}'_2; \underline{k}'_1, \underline{k}'_4),$$

where we have set

$$f_n(\vec{p}, z, \underline{k}; \underline{k}', \underline{k}'') = w_{1,1}(\vec{p}, z, |n^{-1}\vec{k} + \vec{l}'|, n^{-1}\vec{k} + \vec{l}', n^{-1}\vec{k}' + \vec{l}, n^{-1}\vec{k}'' + \vec{l}),$$
$$g_n(\vec{p}, z, \underline{k}; \underline{k}', \underline{k}'') = w_{1,1}(\vec{p}, z, |n^{-1}\vec{k} + \vec{l}'|, n^{-1}\vec{k} + \vec{l}', n^{-1}\vec{k}' + \vec{l}, n^{-1}\vec{k}'' + \vec{l}').$$

$H(\underline{w})$ belongs to $\mathcal{B}(\gamma, \delta, \epsilon)$. It follows that $\|w_{1,1}(\vec{p}, z)\|_{\frac{1}{2}} \le \epsilon$ for any $(\vec{p}, z) \in U[\vec{p}^*] \times D_{\mu/2}$, and

$$|v_{n;1}(\vec{p}, z)| \le C n^{-3} \int_{\underline{B}_R^3} d\underline{k}'_1 d\underline{k}'_2 d\underline{k}'_3\ |\overline{\eta}(\underline{k}'_1) \eta(\underline{k}'_3)|\ |\eta(\underline{k}'_2)|^2,$$
$$|v_{n;2}(\vec{p}, z)| \le C n^{-3} \int_{\underline{B}_R^3} d\underline{k}'_1 d\underline{k}'_2 d\underline{k}'_4\ |\overline{\eta}(\underline{k}'_1) \overline{\eta}(\underline{k}'_2) \eta(\underline{k}'_2 + n(\vec{l}' - \vec{l})) \eta(\underline{k}'_4)|.$$

The upper bounds go to zero when n goes to infinity, uniformly in $\vec{p}$ and z. A similar procedure would show that $(w_n)_n$ converges uniformly to zero.

D.2. **Knowing** $f(|\vec{x}|+|\vec{y}|,\vec{x}+\vec{y})$ **is sufficient to know** $f(r,\vec{l})$ **for any** $(r,\vec{l})\in\mathcal{B}$**.** We remind the reader that $\mathcal{B}$ has been defined in (2.11).

Lemma D.1. *Let* $f:\mathcal{B}\to\mathbb{C}$ *be a function. We suppose that for any* $\vec{x},\vec{y}\in B_1(0)$ *such that* $|\vec{x}|+|\vec{y}|\leq 1$*, the value of* $f(|\vec{x}|+|\vec{y}|,\vec{x}+\vec{y})$ *is known. Then the value of* f *is known for any point of* $\mathcal{B}$*.*

Proof. Let $(r_0,\vec{x}_0)\in\mathcal{B}$, with $\vec{x}_0\neq\vec{0}$ and $|\vec{x}_0|<r_0$. We can always choose a vector $\vec{x}\in\mathbb{R}^3$ orthogonal to $\vec{x}_0$, such that

$$|\vec{x}|=\frac{1}{2}\left(r_0-\frac{|\vec{x}_0|^2}{r_0}\right).$$

Then $0<|\vec{x}|<r_0/2\leq 1/2$. We consider the vector

$$\vec{y}:=\vec{x}_0-\vec{x}.$$

As $\vec{x}$ and $\vec{x}_0$ are orthogonal,

$$|\vec{y}|^2=|\vec{x}_0|^2+|\vec{x}|^2=\frac{1}{4}\left(r_0-\frac{|\vec{x}_0|^2}{r_0}\right)^2+|\vec{x}_0|^2=\frac{1}{4}\left(r_0+\frac{|\vec{x}_0|^2}{r_0}\right)^2.$$

Therefore, $|\vec{y}|<r_0\leq 1$, and

$$|\vec{x}|+|\vec{y}|=r_0.$$

We have found two vectors $(\vec{x},\vec{y})$ in the unit ball of $\mathbb{R}^3$ such that $\vec{x}+\vec{y}=\vec{x}_0$ and $|\vec{x}|+|\vec{y}|=r_0$. Consequently, $f(r_0,\vec{x}_0)=f(|\vec{x}|+|\vec{y}|,\vec{x}+\vec{y})$. The case $\vec{x}_0=\vec{0}$ is trivial, as we can always choose $\vec{x}\in B_1(0)$ of norm $|\vec{x}|=r_0/2$. Then, $|\vec{x}|+|-\vec{x}|=r_0$. □

Appendix E. Proof of Lemma 4.3

We prove Lemma 4.3 with the help of the explicit formulas given in Appendix B. This section is cut into three subsections. We start with a lemma that bounds the function F defined in (B.2) and its derivatives. We then show that the norm $\|\cdot\|^\sharp$ of the function $V_{\underline{m,p,n,q}}$ – see (C.6) – is finite and uniformly bounded in $(\vec{p},z)$. Finally, we prove Lemma 4.3.

E.1. A lemma to bound the kernels.

Lemma E.1. *There exists a constant* $C_\chi>0$ *such that, for any* $0<\rho<1/2$*,* $0<\gamma\ll\mu$*,* $\delta,\varepsilon>0$*,* $H(\underline{w})(\cdot,\cdot)\in\mathcal{B}(\gamma,\delta,\varepsilon)$*, and* $(\vec{p},z,r,\vec{l})\in\mathcal{U}[w_{0,0}]\times\mathcal{B}$*, the function* F *defined in* (B.2) *satisfies the upper bounds:*

$$|F(\vec{p},z,r,\vec{l})|\leq\frac{C_\chi}{\mu\rho},\quad|\partial_r F(\vec{p},z,r,\vec{l})|\leq\frac{C_\chi}{(\mu\rho)^2},\quad|\partial_{l_j}F(\vec{p},z,r,\vec{l})|\leq\frac{C_\chi}{(\mu\rho)^2}.\tag{E.1}$$

Proof. One has that

$$\partial_r F(\vec{p},z,r,\vec{l})=2\rho^{-1}\frac{\overline{\chi}'(\rho^{-1}r)\overline{\chi}_\rho(r)}{w_{0,0}(\vec{p},z,r,\vec{l})}-\frac{(\partial_r w_{0,0})(\vec{p},z,r,\vec{l})\ \overline{\chi}_\rho^2(r)}{w_{0,0}^2(\vec{p},z,r,\vec{l})},\tag{E.2}$$

and

$$\partial_{l_j}F(\vec{p},z,r,\vec{l})=-\frac{(\partial_{l_j}w_{0,0})(\vec{p},z,r,\vec{l})\ \overline{\chi}_\rho^2(r)}{w_{0,0}^2(\vec{p},z,r,\vec{l})}.\tag{E.3}$$

For all $r\in[\frac{3}{4}\rho,1]$, the bound (2.30) holds for the function $w_{0,0}(\vec{p},z,r,\vec{l})$, and hence $|F(\vec{p},z,r,\vec{l})|\leq C_\chi(\rho\mu)^{-1}$. The other bounds also follow directly from (2.30). □

E.2. **A bound for the norm of V.**

Lemma E.2. *Let $L \in \mathbb{N}$, and $\underline{m,p,n,q} \in \mathbb{N}^{4L}$ with $\sum_{i=1}^{L} m_i = M$, $\sum_{i=1}^{L} n_i = N$. $V_{\underline{m,p,n,q}}$ defined in* (B.4) *belongs to $\mathcal{W}_{M,N}$ and we have that*

$$\rho^{\frac{3}{2}(M+N)-1}\Big\|V_{\underline{m,p,n,q}}(\vec{p},z)\Big\|^{\sharp} \leq 5(L+1)\mu^{-L}C_{\chi}^{L+1}\rho^{2(M+N)-L}\prod_{i=1}^{L}(\sqrt{8\pi})^{p_i+q_i}\Big\|w_{m_i+p_i,n_i+q_i}(\vec{p},z)\Big\|^{\sharp}, \tag{E.4}$$

for all $(\vec{p},z) \in U[\vec{p}^*] \times D_{\mu/2}$.

Proof. To shorten notations, we set

$$\tilde{W}_i := W_{p_i,q_i}^{m_i,n_i}(\vec{p},z,\rho(r+r_i),\rho(\vec{l}+\vec{l}_i),\rho\underline{K}_i^{(m_i,n_i)}),$$

$$\tilde{W}'_{i,r} := (\partial_r W_{p_i,q_i}^{m_i,n_i})(\vec{p},z,\rho(r+r_i),\rho(\vec{l}+\vec{l}_i),\rho\underline{K}_i^{(m_i,n_i)}),$$

$$\tilde{W}'_{i,l_j} := (\partial_{l_j} W_{p_i,q_i}^{m_i,n_i})(\vec{p},z,\rho(r+r_i),\rho(\vec{l}+\vec{l}_i),\rho\underline{K}_i^{(m_i,n_i)}).$$

We remind the reader that the operators $W_{p_i,q_i}^{m_i,n_i}$ have been defined in (B.3). $\partial_r W_{p_i,q_i}^{m_i,n_i}$ and $\partial_{l_j} W_{p_i,q_i}^{m_i,n_i}$ have the same form as $W_{p_i,q_i}^{m_i,n_i}$, with the kernel $w_{m_i+p_i,n_i+q_i}$ replaced by $\partial_r w_{m_i+p_i,n_i+q_i}$ and $\partial_{l_j} w_{m_i+p_i,n_i+q_i}$, respectively. The formulas (B.4) and (E.1) imply that

$$\Big|V_{\underline{m,p,n,q}}(\vec{p},z,r,\vec{l},\underline{K}^{(M,N)})\Big| \leq C_{\chi}^{L+1}\,(\rho\mu)^{-L+1}\prod_{i=1}^{L}\|\tilde{W}_i\|,$$

$$\Big|\partial_r V_{\underline{m,p,n,q}}(\vec{p},z,r,\vec{l},\underline{K}^{(M,N)})\Big| \leq (L+1)\mu^{-1}C_{\chi}^{L+1}\,(\rho\mu)^{-L+1}\prod_{i=1}^{L}\Big(\|\tilde{W}_i\|+\rho\Big\|\tilde{W}'_{i,r}\Big\|\Big),$$

$$\Big|\partial_{l_j} V_{\underline{m,p,n,q}}(\vec{p},z,r,\vec{l},\underline{K}^{(M,N)})\Big| \leq (L+1)\mu^{-1}C_{\chi}^{L+1}\,(\rho\mu)^{-L+1}\prod_{i=1}^{L}\Big(\|\tilde{W}_i\|+\rho\Big\|\tilde{W}'_{i,l_j}\Big\|\Big).$$

Together with Lemma 2.6, the definition of the norm $\|\cdot\|^{\sharp}$ (see (2.13)–(2.14)) implies that

$$\Big\|V_{\underline{m,p,n,q}}(\vec{p},z)\Big\|_{\frac{1}{2}} \leq C_{\chi}^{L+1}\,(\rho\mu)^{-L+1}\prod_{i=1}^{L}(\sqrt{8\pi})^{p_i+q_i}\Big\|w_{m_i+p_i,n_i+q_i}(\vec{p},z)\Big\|^{\sharp}\rho^{\frac{M+N}{2}},$$

$$\Big\|\partial_r V_{\underline{m,p,n,q}}(\vec{p},z)\Big\|_{\frac{1}{2}} \leq (L+1)\mu^{-1}C_{\chi}^{L+1}\,(\rho\mu)^{-L+1}\prod_{i=1}^{L}(\sqrt{8\pi})^{p_i+q_i}\Big\|w_{m_i+p_i,n_i+q_i}(\vec{p},z)\Big\|^{\sharp}\rho^{\frac{M+N}{2}},$$

$$\Big\|\partial_{l_j} V_{\underline{m,p,n,q}}(\vec{p},z)\Big\|_{\frac{1}{2}} \leq (L+1)\mu^{-1}C_{\chi}^{L+1}\,(\rho\mu)^{-L+1}\prod_{i=1}^{L}(\sqrt{8\pi})^{p_i+q_i}\Big\|w_{m_i+p_i,n_i+q_i}(\vec{p},z)\Big\|^{\sharp}\rho^{\frac{M+N}{2}}.$$

□

E.3. **Proof of Lemma 4.3.**

Proof. The rest of the proof of Theorem 4.3 is similar to [4, Theorem 3.8]. Let $H(\underline{w}(\cdot,\cdot)) \in \mathcal{B}(\gamma,\delta,\varepsilon)$ and let $\underline{\hat{w}}$ be defined by (B.6). We have to show that $H(\underline{\hat{w}}(\cdot,\cdot)) \in \mathcal{B}(\gamma+\varepsilon/2,\varepsilon/2,\varepsilon/2)$.

It follows from (B.6) that

$$\left\|\hat{w}_{M,N}(\vec{p},\zeta)\right\|^{\sharp} \leq \sum_{L=1}^{\infty} \rho^{\frac{3}{2}(M+N)-1} \sum_{\substack{\underline{m,p,n,q},\\ m_1+\ldots+m_L=M,\\ n_1+\ldots+n_L=N,\\ m_i+n_i+p_i+q_i\geq 1}} C_{\underline{m},\underline{n}}^{\underline{m+p},\underline{n+q}} \left\|V_{\underline{m,p,n,q}}(\vec{p},z)\right\|^{\sharp},$$

where, recall, $(\vec{p},z) = E_\rho^{-1}(\vec{p},\zeta)$. Since $H(\underline{w}) \in \mathcal{B}(\gamma,\delta,\varepsilon)$, we have that

$$\left\|w_{m_i+p_i,n_i+q_i}(\vec{p},z)\right\|^{\sharp} \leq \varepsilon \xi^{m_i+p_i+n_i+q_i}$$

for any $m_i+p_i+n_i+q_i \geq 1$. Plugging this bound into (E.4), we get that

$$\xi^{-(M+N)}\left\|\hat{w}_{M,N}(\vec{p},\zeta)\right\|^{\sharp} \leq 5C_\chi (2\rho)^{2(M+N)} \sum_{L=1}^{\infty}(L+1)\left(\frac{\varepsilon C_\chi}{\mu\rho}\right)^L \sum_{\substack{\underline{m,p,n,q},\\ m_1+\ldots+m_L=M,\\ n_1+\ldots+n_L=N,\\ m_i+n_i+p_i+q_i\geq 1}} \prod_{i=1}^{L} f(m_i,p_i,n_i,q_i),$$

where

$$f(m_i,p_i,n_i,q_i) = (2\sqrt{8\pi}\xi)^{p_i+q_i}\left(\frac{1}{2}\right)^{m_i+n_i}.$$

It holds that

$$\sum_{M+N\leq K} \sum_{\substack{\underline{m,p,n,q},\\ m_1+\ldots+m_L=M,\\ n_1+\ldots+n_L=N,\\ m_i+n_i+p_i+q_i\geq 1}} \prod_{i=1}^{L} f(m_i,p_i,n_i,q_i) \leq \sum_{M,N\leq K} \sum_{\substack{\underline{m,p,n,q},\\ m_1+\ldots+m_L=M,\\ n_1+\ldots+n_L=N,\\ m_i+n_i+p_i+q_i\geq 1}} \prod_{i=1}^{L} f(m_i,p_i,n_i,q_i)$$

$$\leq \prod_{i=1}^{L}\Big(\sum_{\substack{p_i,q_i\\ m_i\leq K,\\ n_i\leq K,\\ m_i+n_i+p_i+q_i\geq 1}} f(m_i,p_i,n_i,q_i)\Big)$$

for any $K\in\mathbb{N}$. Since $2\sqrt{8\pi}\xi < 1/2$ by assumption, $f(m_i,p_i,n_i,q_i) < \left(\frac{1}{2}\right)^{m_i+n_i+p_i+q_i}$, and taking the limit $K\to\infty$, we finally get that

$$\left\|\hat{w}(\vec{p},\zeta)\right\|^{\sharp}_{\xi,\geq 1} \leq 20C_\chi\rho^2 \sum_{L=1}^{\infty}(L+1)\left(\frac{16\ \varepsilon C_\chi}{\mu\rho}\right)^L. \tag{E.5}$$

Choosing $\varepsilon \ll \rho\mu$, we obtain

$$\left\|\hat{w}(\vec{p},\zeta)\right\|^{\sharp}_{\xi,\geq 1} \leq C\frac{\rho\varepsilon}{\mu}. \tag{E.6}$$

(E.6) is smaller than $\varepsilon/2$ if $\rho \ll \mu$.

Next, writing $\hat{w}(\vec{p},\zeta,r,\vec{l}) =: \hat{t}(\vec{p},\zeta,r,\vec{l}) + \hat{w}(\vec{p},\zeta,0,\vec{0})$, we have to show that for any $(\vec{p},\zeta)\in U[\vec{p}^*]\times D_{\mu/2}$,

$$\left\|\hat{t}(\vec{p},\zeta,r,\vec{l}) - (r - m^{-1}\vec{p}\cdot\vec{l})\right\|^{\sharp} \leq \gamma + \varepsilon/2,$$

and that $|\hat{w}(\vec{p},\zeta,0,\vec{0})+\zeta| \le \varepsilon/2$. We have that

$$\hat{w}(\vec{p},\zeta,r,\vec{l}) = \rho^{-1} w_{0,0}(\vec{p},z,\rho r,\rho\vec{l}) + \rho^{-1}\sum_{L=2}^{\infty}(-1)^{L-1}\sum_{\substack{\underline{p},\underline{q}\\ p_i+q_i\ge 1}} V_{\underline{0,p,0,q}}(\vec{p},z,r,\vec{l}), \tag{E.7}$$

with $(\vec{p},\zeta) = E_\rho(\vec{p},z)$. Similar estimates as in Lemma E.2 lead us to

$$\rho^{-1}\Big\| V_{\underline{0,p,0,q}}(\vec{p},z)\Big\|^{\sharp} \le 5(L+1)C_\chi^{L+1}(\rho\mu)^{-L}\prod_{i=1}^{L}(\sqrt{8\pi})^{p_i+q_i}\|w_{p_i,q_i}(\vec{p},z)\|^{\sharp}. \tag{E.8}$$

We set

$$\begin{aligned} A(\zeta) :=& \Big\|\hat{t}(\vec{p},\zeta,r,\vec{l}) - (r - m^{-1}\vec{p}\cdot\vec{l})\Big\|^{\sharp} \\ &= \sup_{(r,\vec{l})\in\mathcal{B}} |\partial_r \hat{w}_{0,0}(\vec{p},\zeta,r,\vec{l}) - 1| + \sum_{j=1}^{3}\sup_{(r,\vec{l})\in\mathcal{B}}\Big|\partial_{l_j}\hat{w}_{0,0}(\vec{p},\zeta,r,\vec{l}) + \frac{p_j}{m}\Big|. \end{aligned} \tag{E.9}$$

From (E.7), we deduce that

$$\begin{aligned} A(\zeta) \le& \sup_{(r,\vec{l})\in\mathcal{B}} |\partial_r w_{0,0}(\vec{p},\zeta,r,\vec{l}) - 1| + \sum_{j=1}^{3}\sup_{(r,\vec{l})\in\mathcal{B}}\Big|\partial_{l_j} w_{0,0}(\vec{p},\zeta,r,\vec{l}) + \frac{p_j}{m}\Big| \\ &+ \rho^{-1}\sum_{L=2}^{\infty}\sum_{\substack{\underline{p},\underline{q}\\ p_i+q_i\ge 1}}\left(\sup_{(r,\vec{l})\in\mathcal{B}}|\partial_r V_{\underline{0,p,0,q}}(\vec{p},z,r,\vec{l})| + \sum_{j=1}^{3}\sup_{(r,\vec{l})\in\mathcal{B}}|\partial_{l_j} V_{\underline{0,p,0,q}}(\vec{p},z,r,\vec{l})|\right) \\ &\le \gamma + \rho^{-1}\sum_{L=2}^{\infty}\sum_{\substack{\underline{p},\underline{q}\\ p_i+q_i\ge 1}}\Big\|V_{\underline{0,p,0,q}}(\vec{p},z)\Big\|^{\sharp} \\ &\le \gamma + 5\sum_{L=2}^{\infty}(L+1)C_\chi^{L+1}(\rho\mu)^{-L}\left(\sum_{p+q\ge 1}(\sqrt{8\pi})^{p+q}\|w_{p,q}(\vec{p},z)\|^{\sharp}\right)^{L} \\ &\le \gamma + 5\sum_{L=2}^{\infty}(L+1)C_\chi^{L+1}(\rho\mu)^{-L}\left(\sqrt{8\pi}\xi\|\underline{w}(\vec{p},z)\|^{\sharp}_{\xi,\ge 1}\right)^{L} \\ &\le \gamma + 5C_\chi\sum_{L=2}^{\infty}(L+1)\left(\frac{C_\chi\xi\varepsilon}{\rho\mu}\right)^{L} \le \gamma + 60C_\chi\left(\frac{C_\chi\xi\varepsilon}{\rho\mu}\right)^{2} \le \gamma + \frac{\varepsilon}{2}, \end{aligned}$$

where we used that $\sum_{L=2}^{\infty}(L+1)a^L \le 12a^2$ for $a \le 1/2$, $\varepsilon \ll \rho^2\mu^2$, and absorbed $\sqrt{8\pi}$ into C_χ. To finish the proof of Theorem 4.3, we need to check that $|w_{0,0}(\vec{p},\zeta,0,\vec{0})+\zeta| \le \varepsilon/2$. By definition,

$$\zeta = -w_{0,0}(\vec{p},z,0,\vec{0})\rho^{-1}, \tag{E.10}$$

which together with equation (E.7) and the above calculation, implies that

$$|w_{0,0}(\vec{p},\zeta,0,\vec{0})+\zeta| \le \frac{\varepsilon}{2}. \tag{E.11}$$

□

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