Differential and Integral Equations Volume 18, Number 5 (2005), 583–600

GENERAL DECAY RATES OF SOLUTIONS TO A NONLINEAR WAVE EQUATION WITH BOUNDARY CONDITION OF MEMORY TYPE

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(Submitted by: Viorel Barbu)

Abstract. In this article we study the hyperbolic problem

$$\begin{aligned}
&u_{tt} - \Delta u + F(x,t,u,\nabla u) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+,\\
&u = 0 \text{ on } \Gamma_0, \quad u + \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds = 0 \text{ on } \Gamma_1 \times \mathbb{R}_+,\\
&u(x,0) = u^0(x), \quad u'(x,0) = u^1(x) \quad \text{in} \quad \Omega,
\end{aligned}$$

where Ω is a bounded region in $\mathbb{R}^n$ whose boundary is partitioned into disjoint sets Γ_0, Γ_1. We prove that the dissipation given by the memory term is strong enough to assure stability of our system. The general decay estimates we obtain depend on the relaxation function. In particular, if the relaxation function decays exponentially (or polynomially), then the solution also decays exponentially (or polynomially) and with the same decay rate. Indeed, the main result of this paper is to give general relations between the decay of the solution and the decay of the relaxation function, under weaker hypotheses on the resolvent kernel function (defined in Section 2),and the potential functions $\phi(x)$ and $\varphi(t)$, which represent (in some sense) the linear and the nonlinear part of F with respect to ∇u, respectively. We assume only that φ is bounded and small enough at ∞ which means that, in fact, φ has no real influence on the stability of our system. In the case where the relaxation function decays exponentially or polynomially, we obtain the same decay for the solution, and then the results of [3,4] become just a particular case of ours. We also distinguish the case where the first data u^0 vanishes on Γ_1

Accepted for publication: September 2004.
AMS Subject Classifications: 35L05, 35L70, 74Dxx, 93D20.

and prove that, in this case, we have exponential or polynomial decay of solution, even if the relaxation function does not converge to 0 at ∞.

1. Introduction

In this work we study the existence of global solutions and the asymptotic behavior of the energy related to the following nonlinear wave equation with a boundary condition of memory type

$$u_{tt} - \Delta u + F(x,t,u,\nabla u) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+, \tag{1.1}$$

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+, \tag{1.2}$$

$$u + \int_0^t g(t-s)\frac{\partial u}{\partial \nu}(s)ds = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+, \tag{1.3}$$

$$u(x,0) = u^0(x), \quad u_t(x,0) = u^1(x) \quad \text{in} \quad \Omega, \tag{1.4}$$

where Ω is a bounded domain in $\mathbb{R}^n$, $n \geq 1$, with smooth boundary $\Gamma = \Gamma_0 \cup \Gamma_1$. Here, Γ_0 and Γ_1 are closed and disjoint, $\Gamma_0 \neq \emptyset$, and ν is the unit normal vector pointing towards the exterior of Ω. Equation (1.3) is a nonlocal boundary condition responsible for the memory effect. Considering the history condition, we must add to conditions (1.2)-(1.3) the one given by

$$u = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_-.$$

We observe that in problem (1.1)-(1.4), u represents the transverse displacement, and the relaxation function g is a positive nonincreasing function belonging to $W^{2,1}(\mathbb{R}_+)$. Furthermore, let γ be a constant such that $\gamma \geq 0$ for $n = 1, 2$, and $0 \leq \gamma \leq 2/(n-2)$ for $n \geq 3$, and suppose that the function $F : \overline{\Omega} \times \mathbb{R}_+ \times \mathbb{R}^{n+1} \to \mathbb{R}$ is of class C^1 and satisfies

$$|F(x,t,\xi,\zeta)| \leq C_0(1 + |\xi|^{\gamma+1} + |\zeta|) \tag{1.5}$$

where C_0 is a positive constant, and $\zeta = (\zeta_1, ..., \zeta_n)$.

Assume that there is a nonnegative function $\varphi(t)$ in the space $L^\infty(\mathbb{R}_+)$ and a function $\phi(x)$ in the space $W^{1,\infty}(\Omega)$ and a nonnegative constant D such that

$$(F(x,t,\xi,\zeta) + \nabla\phi(x)\cdot\zeta)\eta \geq D|\xi|^\gamma \xi\eta - \varphi(t)(1 + |\eta||\zeta|), \quad \forall \eta \in \mathbb{R}, \tag{1.6}$$

and, particularly,

$$(F(x,t,\xi,\zeta) + \nabla\phi(x)\cdot\zeta)(m\cdot\zeta) \geq D|\xi|^\gamma \xi(m\cdot\zeta) - \varphi(t)(1 + |\zeta||m\cdot\zeta|). \tag{1.7}$$

Additionally, there exist positive constants $C_0, \ldots, C_n$ such that

$$|F_t(x,t,\xi,\zeta)| \leq C_0\left(1 + |\xi|^{\gamma+1} + |\zeta|\right), \tag{1.8}$$

$$|F_\xi(x,t,\xi,\zeta)| \leq C_0(1 + |\xi|^\gamma), \tag{1.9}$$

$$|F_{\zeta_i}(x,t,\xi,\zeta)| \le C_i \quad \text{for } i=1,2,\dots,n. \tag{1.10}$$

We also assume that there exist positive constants D_1, D_2, such that for all $\xi, \hat{\xi}, \eta, \hat{\eta} \in \mathbb{R}$ and for all $\zeta, \hat{\zeta} \in \mathbb{R}^n$,

$$\begin{aligned}&(F(x,t,\xi,\zeta) - F(x,t,\hat{\xi},\hat{\zeta}))(\eta - \hat{\eta})\\ &\ge -D_1(|\xi|^\gamma + |\hat{\xi}|^\gamma)|\xi - \hat{\xi}||\eta - \hat{\eta}| - D_2|\eta - \hat{\eta}||\zeta - \hat{\zeta}|.\end{aligned} \tag{1.11}$$

Defining

$$F(x,t,u,\nabla u) = |u|^\gamma u + \varphi(t)\sum_{i=1}^n \sin\Big(\frac{\partial u}{\partial x_i}\Big) - \nabla\phi(x)\cdot\nabla u,$$

where φ and ϕ are sufficiently regular functions, we obtain an example of a function F which verifies the above hypotheses.

Remark 1.1. In fact assumption (1.6) implies that

$$(F(x,t,\xi,\zeta) + \nabla\phi\cdot\zeta)\eta \ge D|\xi|^\gamma\xi\eta - \varphi(t)|\eta||\zeta|, \quad \forall \eta \in \mathbb{R}. \tag{1.12}$$

Indeed, (1.6) implies that

$$(F(x,t,\xi,\zeta) + \nabla\phi\cdot\zeta - D|\xi|^\gamma\xi + \varphi(t)|\zeta|)\eta \ge -\varphi(t), \quad \forall \eta \in \mathbb{R}_+$$

and

$$(F(x,t,\xi,\zeta) + \nabla\phi\cdot\zeta - D|\xi|^\gamma\xi - \varphi(t)|\zeta|)\eta \ge -\varphi(t), \quad \forall \eta \in \mathbb{R}_-,$$

hence

$$\begin{aligned}F(x,t,\xi,\zeta) + \nabla\phi\cdot\zeta - D|\xi|^\gamma\xi + \varphi(t)|\zeta| \ge 0,\\ F(x,t,\xi,\zeta) + \nabla\phi\cdot\zeta - D|\xi|^\gamma\xi - \varphi(t)|\zeta| \le 0.\end{aligned}$$

Then we conclude (1.12). So we will assume (1.12) instead of (1.6).

The integral equation (1.3) describes the memory effect which can be caused, for example, by the interaction with another viscoelastic element. Indeed, from the physical point of view, condition (1.3) means that Ω is composed of a material which is clamped in a rigid body in Γ_0 and is clamped in a body with viscoelastic properties in the complementary part of its boundary named Γ_1. So, it is expected that the decay of solutions depends on the decay of the kernel of the memory. In particular, if the kernel of the memory decays (exponentially or polynomially) the same occurs with the solutions of problem (1.1)-(1.4).

In what follows we are going to assume that there exists $x_0 \in \mathbb{R}^n$ such that $\Gamma_0 = \{x \in \Gamma : \nu(x)\cdot(x - x_0) \le 0\}$, $\Gamma_1 = \{x \in \Gamma : \nu(x)\cdot(x - x_0) > 0\}$.

Defining $m(x) = x - x_0$, the compactness of Γ_1 implies that there exists a positive constant δ_0 such that

$$0 < \delta_0 \le m(x)\cdot\nu(x), \quad \forall x \in \Gamma_1. \tag{1.13}$$

For examples of a set Ω satisfying those properties, see [9].

There is not much in the literature regarding the existence and asymptotic behavior of evolution equations subject to memory conditions acting on the boundary. It is worth mentioning some papers in connection with viscoelastic effects on the boundary. In this direction we can cite the work by Aassila, Cavalcanti, and Soriano [1] who considered the linear wave equation subject to nonlinear feedback and viscoelastic effects on the boundary, and proved uniform (exponential and algebraic) decay rates. Also, we can cite the article of Andrade and Munõz Rivera [2] where there was considered a one-dimensional nonlinear wave equation subject to a nonlocal and nonlinear boundary memory effect. In this work the authors showed that the dissipation occasioned by the memory term was strong enough to guarantee global estimates and, consequently, allowed them to prove existence of a global smooth solution for small data and to obtain exponential (or polynomial) decay provided the kernel decays exponentially (or polynomially). In the same context we can mention the work of Santos [15], where decay rates were proved concerning the wave equation with coefficients depending on time and subject to a memory condition on the boundary.

A natural question that arises in this context concerns the nonexistence results for the wave equation in the presence of viscoelastic effects acting on the boundary. Related to this subject we can mention the work of Kirane and Tartar [11] who obtained nonexistence results and Qin [17] who proved a blow up result for the nonlinear one-dimensional wave equation with memory boundary condition.

In connection with the above discussion, regarding viscoelastic problems, it is important to cite the works of Ciarletta [5], Fabrizio and Morro [8] and Qin [16].

The most recent results in this direction were obtained by Cavalcanti, Domingos, and Santos [4] where the authors considered the same system under the same hypotheses with ϕ = constant and both φ and the relaxation function converging exponentially or polynomially to 0 at ∞, and proved that the solution has the same decay. The fact that φ converges exponentially or polynomially to 0 at ∞ is a strong hypothesis which is not satisfied if, for example, the function F does not depend on time t.

The main goal of the present paper is to complement and improve the above mentioned works. The main result of this paper is to give general relations between the decay of solutions and the decay of the relaxation function, under weaker hypotheses on the resolvent kernel function (defined in Section 2), and the potential functions $\phi(x)$ and $\varphi(t)$ which represent (in some sense) the linear and the nonlinear part of F with respect of ∇u, respectively. We assume only that φ is bounded and small enough at ∞ which means that, in fact, φ has no real influence on the stability of our system. In the case where the relaxation function decays exponentially or polynomially, we obtain the same decay for the solution, and then the results of [4] become just a particular case of ours. We also distinguish the case where the initial data u^0 vanishes on Γ_1 and prove that, in this case, we have exponential or polynomial decay of solutions, even if the relaxation function does not converge to 0 at ∞. In order to prove these results, we use a direct approach introduced by the second author in [9] and [10]. This approach is based on generalized integral inequalities for positive nondecreasing functions and the introduction of an equivalent energy which depends on ϕ.

On the other hand, the majority of the existing results are obtained in a one-dimensional domain while our paper deals with an n-dimensional problem bringing up some additional difficulties, mainly relating to the geometric conditions. In addition, as we have a nonlinear problem whose nonlinearity $F = F(x, t, u, \nabla u)$ depends on the gradient, we do not have any information about the influence of the integral $\int_\Omega F(x, t, u, \nabla u)u_t\, dx$ on the equivalent energy $E(t)$ or about the sign of the derivative $E'(t)$. In other words, we cannot guarantee that $E'(t) \leq 0$, which plays an essential role in establishing the desired decay rates.

Note that condition (1.2) implies that the solution of system (1.1)-(1.4) must belong to the following space $V := \{v \in H^1(\Omega) : v = 0 \quad \text{on} \quad \Gamma_0\}$. The notations we use in this paper are standard and can be found in Lion's book [12]. In the sequel, C (sometimes $C_0, C_1, \dots$) is going to denote various positive constants which do not depend on t, and depend on $\|(u^0, u^1)\|_{H^1(\Omega)\times L^2(\Omega)}$ in a continuous way. This paper is organized as follows. In Section 2 we establish the existence and uniqueness for regular and weak solutions to the system(1.1)-(1.4). In Sections 3 and 4 we prove the general decay estimates. In Section 5 we discuss some applications.

2. Notation and main results

In this section we present some notation and we study the existence of regular and weak solutions to the system (1.1)-(1.4). First, we will use equation (1.3) to estimate the term $\frac{\partial u}{\partial \nu}$.

Defining the convolution product operator by

$$(g * \varphi)(t) = \int_0^t g(t-s)\varphi(s)ds$$

and differentiating equation (1.3), we get

$$\frac{\partial u}{\partial \nu} + \frac{1}{g(0)}\Big(g' * \frac{\partial u}{\partial \nu}\Big) = -\frac{1}{g(0)} u_t \quad \text{on} \quad \Gamma_1 \times (0, +\infty).$$

Applying Volterra's inverse operator, we get

$$\frac{\partial u}{\partial \nu} = -\frac{1}{g(0)} \left(u_t + k * u_t\right) \quad \text{on} \quad \Gamma_1 \times (0, +\infty)$$

where the resolvent kernel satisfies

$$k + \frac{1}{g(0)} g' * k = -\frac{1}{g(0)} g'.$$

Defining $\eta = \frac{1}{g(0)}$, we get

$$\frac{\partial u}{\partial \nu} = -\eta \left(u_t + k(0)u - k(t)u^0 + k' * u\right) \quad \text{on} \quad \Gamma_1 \times (0, +\infty). \tag{2.1}$$

Reciprocally, considering that the initial data satisfies $u^0 = 0$ on Γ_1, (2.1) implies (1.3). Since we are interested here in relaxation functions of exponential or polynomial type and identity (2.1) involves the resolvent kernel k, we want to investigate if k has the same properties. The following lemma answers this question.

Let h be

$$k(t) - k * h(t) = h(t). \tag{2.2}$$

Lemma 2.1. *If h is a positive continuous function, then k is also a positive continuous function. Moreover,*

1. *If there exist positive constants c_0 and γ with $c_0 < \gamma$ such that $h(t) \leq c_0 e^{-\gamma t}$, we conclude that the function k satisfies $k(t) \leq \frac{c_0(\gamma-\epsilon)}{\gamma-\epsilon-c_0} e^{-\epsilon t}$, for all $0 < \epsilon < \gamma - c_0$.*

2. *Let us consider $p > 1$ and define by*

$$c_p := \sup_{t \geq 0} \int_0^t (1+t)^p (1+t-s)^{-p} (1+s)^{-p} \, ds.$$

Provided there exists a positive constant c_0 with $c_0 c_p < 1$ such that $h(t) \leq c_0(1+t)^{-p}$, the function k satisfies $k(t) \leq \frac{c_0}{1-c_0 c_p}(1+t)^{-p}$.

Proof. See [4].

Remark 2.1. In Racke [13, Lemma 7.4], it is assured that c_p is a finite positive constant. Also, according to this lemma, in what follows, we are going to use (2.1) instead of (1.3).

In order to prove the following lemma, let us define

$$(g \Box \varphi)(t) := \int_0^t g(t-s) |\varphi(t) - \varphi(s)|^2 ds.$$

Lemma 2.2. *For real functions $g, \varphi \in C^1(\mathbb{R}_+)$ we have*

$$(g * \varphi)\varphi_t = -\frac{1}{2}g(t)|\varphi(t)|^2 + \frac{1}{2}g'\Box\varphi - \frac{1}{2}\frac{d}{dt}\Big[g\Box\varphi - (\int_0^t g(s)ds)|\varphi|^2\Big].$$

Proof. The proof of this lemma follows by differentiating the term $g\Box\varphi$.

The first order equivalent energy of system (1.1)-(1.4) is defined by

$$E(t) := \frac{1}{2}\int_\Omega e^{\phi(x)}\left(|u_t(x,t)|^2 + |\nabla u(x,t)|^2\right)dx + \frac{D}{\gamma+2}\int_\Omega e^{\phi(x)}|u(x,t)|^{\gamma+2}dx$$
$$-\frac{\eta}{2}\int_{\Gamma_1} e^{\phi(x)}(k'\Box u)(t)d\Gamma + \frac{\eta}{2}k(t)\int_{\Gamma_1} e^{\phi(x)}|u(x,t)|^2 d\Gamma.$$

The well-posedness of system (1.1)-(1.4) as well as the decay rates expected are presented in the following theorem.

Theorem 2.1. *Let $k \in W^{2,1}(\mathbb{R}_+)$, assume that assumptions (1.5)-(1.11) hold, and suppose that $(u^0, u^1) \in \left(V \cap H^2(\Omega)\right)^2$, satisfying the compatibility condition*

$$\frac{\partial u^0}{\partial \nu} + \eta u^1 = 0 \quad on \quad \Gamma_1. \tag{2.3}$$

Then, problem (1-1)-(1.4) possesses a unique solution u such that

$$u \in L^\infty(\mathbb{R}_+, V \cap H^2(\Omega)),\ u' \in L^\infty(\mathbb{R}_+, V),\ u'' \in L^\infty(\mathbb{R}_+, V). \tag{2.4}$$

In addition, assuming that there exist $0 \le q < \frac{1}{2}$, $b > 0$ and $t_0 \ge 0$ such that

$$k(t) \ge 0, \quad k'(t) \le 0, \quad k''(t) \ge b(-k'(t))^{1+q}, \tag{2.5}$$

$$\|\nabla\phi\|_\infty \|m\|_\infty < \min\{1, \frac{\gamma n}{\gamma+4}\}, \tag{2.6}$$

$$\sup_{t\in[t_0,\infty[} \varphi(t) \quad and \quad \sup_{t\in[t_0,\infty[} k(t) \quad are\ small\ enough \tag{2.7}$$

and, moreover, that hypothesis (1.13) holds, we obtain that the equivalent energy $E(t)$ associated to problem (1.1)-(1.4) decays with the following rates of decay:

Case 1. $q = 0$:

$$E(t) \le Ce^{-\alpha t}\Big(1 + \int_0^t k^2(s)e^{\alpha s}ds\Big), \tag{2.8}$$

$$E(t) \le Ce^{-\alpha t} \quad if \quad u^0 = 0 \quad on \quad \Gamma_1, \tag{2.9}$$

where C and α are positive constants.

Case 2. $0 < q < \frac{1}{2}$:

$$E(t) \le C(t+1)^{-\lambda}\Big(1 + \int_0^t k^2(s)(s+1)^\lambda ds\Big), \tag{2.10}$$

where $\lambda = \frac{1}{q}$ *if* $\int_0^t k^2(s)(s+1)^{\lambda_0} ds$ *is bounded for some* $1 < \lambda_0 < \frac{1}{q} - 1$, *and* $1 < \lambda < \frac{1}{q} - 1$; *if not,*

$$E(t) \leq C(t+1)^{-\frac{1}{q}} \quad \text{if} \quad u^0 = 0 \quad \text{on} \quad \Gamma_1. \tag{2.11}$$

Theorem 2.2. *Let* $k \in W^{2,1}(\mathbb{R}_+)$, *suppose that* $(u^0, u^1) \in V \times L^2(\Omega)$, *and the assumptions* (1.5)-(1.11) *and* (2.5)-(2.7) *hold. Then, problem* (1.1)-(1.4) *has a unique weak solution* u *in the space* $C^0(\mathbb{R}_+; V) \cap C^1(\mathbb{R}_+; L^2(\Omega))$. *Furthermore, the decay rates presented in* (2.8)-(2.11) *hold for the weak solution* u.

Remark 2.2. We can take $t_0 = 0$ in (2.7) without lost of generality.

Remark 2.3. Thanks to (2.5), the assumption on γ, and the fact that ϕ is bounded, the equivalent energy E satisfies, for a positive constant d,

$$E(t) \geq d\|(u, u_t)\|_{H^1(\Omega)\times L^2(\Omega)}. \tag{2.12}$$

Remark 2.4. Thanks to Lemma 2.1 we have, if the relaxation function decays exponentially or polynomially to 0 at ∞, then the resolvent kernel k has the same properties. Then, in these cases, estimates (2.8) and (2.10) give the same decay for the solutions of system (1.1)-(1.4), respectively. That is, if $k^2(t) \leq Ce^{-\beta t}$ or $k^2(t) \leq C(1+t)^{-2p}$ with constants $\beta > 0$ and $p > \frac{1}{2}$, then, from (2.8) and (2.10), we have $E(t) \leq Ce^{-\min\{\alpha,\beta\}t}$ ($E(t) \leq Ce^{-(\alpha-\epsilon)t}$ for any $\epsilon > 0$ if $\alpha = \beta$) or $E(t) \leq C(1+t)^{-\min\{\lambda, 2p-1\}}$, respectively. These particular cases give the results of [4].

Proof. The proof of existence and uniqueness for regular and weak solutions can be obtained following exactly identical procedure as in the work [3] of the authors Cavalcanti, Domingos Cavalcanti, and Soriano. On the other hand, by using the usual density arguments, we can prove estimates (2.8)-(2.11) for weak solutions. Consequently, these two points will be omitted.

3. General decay: the case $q = 0$

In this section we shall study the asymptotic behavior of the solutions of system (1.1)-(1.4) when the resolvent kernel satisfies, for $b > 0$, the conditions

$$k(t) \geq 0, \quad k'(t) \leq 0, \quad k''(t) \geq -bk'(t). \tag{3.1}$$

These assumptions imply that k' converges exponentially to 0; that is, $0 \leq -k'(t) \leq Ce^{-bt}$. In [4] it was assumed the same property also for k.

Our point of departure will be to establish some inequalities for the solution of system (1.1)-(1.4).

Lemma 3.1. *Any regular solution u of the system* (1.1)–(1.4) *satisfies*

$$\frac{d}{dt}E(t) \le -\frac{\eta}{2}\int_{\Gamma_1} e^{\phi(x)}|u_t|^2 d\Gamma + \frac{\eta}{2}k^2(t)\int_{\Gamma_1} e^{\phi(x)}|u^0|^2 d\Gamma$$
$$-\frac{\eta}{2}\int_{\Gamma_1} e^{\phi(x)}(k''\Box u)d\Gamma + \varphi(t)\int_{\Omega} e^{\phi(x)}|u_t||\nabla u|dx.$$

Proof. Multiplying the equation (1.1) by $e^{\phi(x)}u_t$ and integrating by parts over Ω we get

$$\frac{1}{2}\frac{d}{dt}\int_{\Omega} e^{\phi(x)}(|u_t|^2 + |\nabla u|^2)dx$$
$$= -\int_{\Omega} e^{\phi(x)}(F(x,t,u,\nabla u) + \nabla\phi\cdot\nabla u)u_t dx + \int_{\Gamma_1} e^{\phi(x)}\frac{\partial u}{\partial \nu}u_t d\Gamma.$$

Taking (1.12), (2.1), and (3.1) into account and using Lemma 2.2 our conclusion follows.

Let us consider the following binary operator

$$(k\diamond\varphi)(t) := \int_0^t k(t-s)(\varphi(t)-\varphi(s))ds.$$

Then employing Hölder's inequality for $0 \le \mu \le 1$ we have

$$|(k\diamond\varphi)(t)|^2 \le \Big[\int_0^t |k(s)|^{2(1-\mu)}ds\Big](|k|^{2\mu}\Box\varphi)(t). \tag{3.2}$$

Let us define the functionals

$$\mathcal{N}(t) := \int_{\Omega} e^{\phi(x)}(|u_t|^2 + |\nabla u|^2 + |u|^{\gamma+2})dx,$$
$$\psi(t) = 2\int_{\Omega} e^{\phi(x)}(m\cdot\nabla u)u_t + \theta\int_{\Omega} e^{\phi(x)}uu_t dx, \tag{3.3}$$

where

$$\max\{n + \|m\|_\infty\|\nabla\phi\|_\infty - 2, \frac{2(n+\|m\|_\infty\|\nabla\phi\|_\infty)}{\gamma+2}\} < \theta < n - \|m\|_\infty\|\nabla\phi\|_\infty$$

(thanks to (2.6) the constant θ exists). The following lemma plays an important role for the construction of the Lyapunov functional.

Lemma 3.2. *For any regular solution of the system* (1.1)-(1.4) *we get*

$$\frac{d}{dt}\psi(t) \le \int_{\Gamma_1} e^{\phi(x)}(m\cdot\nu)|u_t|^2 d\Gamma + \int_{\Omega}(\theta - n - m\cdot\nabla\phi)e^{\phi(x)}|u_t|^2 dx$$
$$+\int_{\Omega}(n + m\cdot\nabla\phi - 2 - \theta)e^{\phi(x)}|\nabla u|^2 dx + \int_{\Gamma_1} e^{\phi(x)}\frac{\partial u}{\partial\nu}(2m\cdot\nabla u + \theta u)d\Gamma$$

$$-\int_{\Gamma_1} e^{\phi(x)}(m\cdot\nu)|\nabla u|^2 d\Gamma + \int_\Omega (\frac{2(n+m\cdot\nabla\phi)}{\gamma+2} - \theta) D e^{\phi(x)}|u|^{\gamma+2}dx$$
$$+\theta\varphi(t)\int_\Omega e^{\phi(x)}|u||\nabla u|dx + 2\varphi(t)\int_\Omega e^{\phi(x)}|\nabla u||m\cdot\nabla u|dx.$$

Proof. Differentiating the equation (3.3) with respect to t and substituting the equation (1.1) in the expression obtained we deduce

$$\frac{d}{dt}\psi(t) = \int_{\Gamma_1} e^{\phi(x)}(m\cdot\nu)|u_t|^2 d\Gamma + \int_\Omega (\theta - n - m\cdot\nabla\phi)e^{\phi(x)}|u_t|^2 dx$$
$$+\int_{\Gamma_1} e^{\phi(x)}\frac{\partial u}{\partial\nu}(2m\cdot\nabla u + \theta u)d\Gamma + \int_{\Gamma_0} e^{\phi(x)}(m\cdot\nu)|\nabla u|^2 d\Gamma$$
$$-\int_{\Gamma_1} e^{\phi(x)}(m\cdot\nu)|\nabla u|^2 d\Gamma + \int_\Omega (n + m\cdot\nabla\phi - 2 - \theta)e^{\phi(x)}|\nabla u|^2 dx$$
$$-2\int_\Omega e^{\phi(x)}(F(x,t,u,\nabla u) + \nabla\phi(x)\cdot\nabla u)(m\cdot\nabla u)dx$$
$$-\theta\int_\Omega e^{\phi(x)}(F(x,t,u,\nabla u) + \nabla\phi(x)\cdot\nabla u)u dx. \tag{3.4}$$

From the inequality (1.12) we obtain

$$-\theta\int_\Omega e^{\phi(x)}(F(x,t,u,\nabla u) + \nabla\phi(x)\cdot\nabla u)u dx$$
$$\le -\theta D\int_\Omega e^{\phi(x)}|u|^{\gamma+2}dx + \theta\varphi(t)\int_\Omega e^{\phi(x)}|u||\nabla u|dx, \tag{3.5}$$
$$-2\int_\Omega e^{\phi(x)}(F(x,t,u,\nabla u) + \nabla\phi(x)\cdot\nabla u)(m\cdot\nabla u)dx$$
$$\le -2D\int_\Omega e^{\phi(x)}|u|^\gamma u(m\cdot\nabla u)dx + 2\varphi(t)\int_\Omega e^{\phi(x)}|\nabla u||m\cdot\nabla u|dx. \tag{3.6}$$

Substituting the inequalities (3.5)-(3.6) into (3.4) and noting that

$$-\frac{2D}{\gamma+2}\int_{\Gamma_1}(m\cdot\nu)|u|^{\gamma+2}d\Gamma \le 0 \quad \text{and} \quad \int_{\Gamma_0} e^{\phi(x)}(m\cdot\nu)|\nabla u|^2 d\Gamma \le 0,$$

our conclusion follows.

Finally, we shall show that the equivalent energy E satisfies some integral inequalities. Using hypothesis (3.1) and Young's inequality in Lemma 3.1 we get

$$\frac{d}{dt}E(t) \le -\frac{\eta}{2}\int_{\Gamma_1} e^{\phi(x)}(|u_t|^2 - bk'\Box u - k^2(t)|u^0|^2)d\Gamma$$
$$+\frac{1}{2}\varphi(t)\int_\Omega e^{\phi(x)}(|u_t|^2 + |\nabla u|^2)dx. \tag{3.7}$$

Applying Young and Poincaré's inequalities in Lemma 3.2 and using (1.13) we obtain

$$\begin{aligned}\frac{d}{dt}\psi(t) \leq \int_{\Gamma_1} e^{\phi(x)}(m\cdot\nu)|u_t|^2 d\Gamma + (\theta - n + \|m\|_\infty\|\nabla\phi\|_\infty)\int_\Omega e^{\phi(x)}|u_t|^2 dx\\ -(\theta-(n+\|m\|_\infty\|\nabla\phi\|_\infty-2))\int_\Omega e^{\phi(x)}|\nabla u|^2 dx\\ -(\theta - \frac{2(n+\|m\|_\infty\|\nabla\phi\|_\infty)}{\gamma+2})D\int_\Omega e^{\phi(x)}|u|^{\gamma+2}dx\\ +C\Big(\epsilon\int_{\Gamma_1} e^{\phi(x)}(|\nabla u|^2+u^2)d\Gamma + \varphi(t)\mathcal{N}(t)\Big) + C_\epsilon\int_{\Gamma_1} e^{\phi(x)}|\frac{\partial u}{\partial\nu}|^2 d\Gamma\\ -\delta_0\int_{\Gamma_1} e^{\phi(x)}|\nabla u|^2 d\Gamma\end{aligned}$$

where ϵ is an arbitrary positive constant.

Noting that the boundary condition (2.1) can be written as

$$\frac{\partial u}{\partial\nu} = -\eta(u_t + k(t)u - k'\diamond u - k(t)u^0),$$

we arrive at

$$\begin{aligned}\frac{d}{dt}\psi(t) \leq (\theta - n + \|m\|_\infty\|\nabla\phi\|_\infty)\int_\Omega e^{\phi(x)}|u_t|^2 dx\\ -(\theta-(n+\|m\|_\infty\|\nabla\phi\|_\infty-2))\int_\Omega e^{\phi(x)}|\nabla u|^2 dx\\ -(\theta - \frac{2(n+\|m\|_\infty\|\nabla\phi\|_\infty)}{\gamma+2})D\int_\Omega e^{\phi(x)}|u|^{\gamma+2}dx\\ -\delta_0\int_{\Gamma_1} e^{\phi(x)}|\nabla u|^2 d\Gamma + C\Big(\epsilon\int_{\Gamma_1} e^{\phi(x)}(|\nabla u|^2+u^2)d\Gamma + \varphi(t)\mathcal{N}(t)\Big)\\ +C_\epsilon\int_{\Gamma_1} e^{\phi(x)}(|u_t|^2 + k^2(t)|u|^2 + |k'\diamond u|^2 + k^2(t)|u^0|^2)d\Gamma. \qquad (3.8)\end{aligned}$$

On the other hand, applying the inequality (3.2) for k' with $\mu = \frac{1}{2}$ and the trace theorem in inequality (3.8) with ϵ small enough we obtain

$$\begin{aligned}\frac{d}{dt}\psi(t) \leq (\theta - n + \|m\|_\infty\|\nabla\phi\|_\infty)\int_\Omega e^{\phi(x)}|u_t|^2 dx\\ -(\theta-(n+\|m\|_\infty\|\nabla\phi\|_\infty-2))\int_\Omega e^{\phi(x)}|\nabla u|^2 dx\\ -(\theta - \frac{2(n+\|m\|_\infty\|\nabla\phi\|_\infty)}{\gamma+2})D\int_\Omega e^{\phi(x)}|u|^{\gamma+2}dx\end{aligned}$$

$$+C(\varphi(t)+\epsilon)\mathcal{N}(t)+C_\epsilon\int_{\Gamma_1} e^{\phi(x)}(|u_t|^2+k^2(t)|u|^2-k'\Box u+k^2(t)|u^0|^2)d\Gamma. \quad (3.9)$$

Let us introduce the Lyapunov functional

$$\mathcal{L}(t) := NE(t)+\psi(t), \quad (3.10)$$

with $N>0$. Taking N large and ϵ small enough, using (2.7) for φ and (2.12), the previous inequalities (3.7) and (3.9) imply that, for some positive constants C_1 and C_2,

$$\frac{d}{dt}\mathcal{L}(t) \le -C_2E(t)+C_1\|u^0\|^2_{L^2(\Gamma_1)}k^2(t)+Ck^2(t)\int_{\Gamma_1} e^{\phi(x)}|u|^2d\Gamma,$$

hence, using (2.7) for k, (2.12) and the trace theorem,

$$\frac{d}{dt}\mathcal{L}(t) \le -C_0E(t)+C_1\|u^0\|^2_{L^2(\Gamma_1)}k^2(t)$$

for some positive constant C_0. Moreover, using Young's inequality and taking N sufficiently large we find that

$$q_0E(t) \le \mathcal{L}(t) \le q_1E(t), \quad (3.11)$$

for some positive constants q_0 and q_1. From the last two inequalities we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \le -\alpha\mathcal{L}(t)+C_1\|u^0\|^2_{L^2(\Gamma_1)}k^2(t), \quad (3.12)$$

for some positive constants α. Now we distinguish two cases.

Case 1. If $u^0=0$ on Γ_1, then $\|u^0\|^2_{L^2(\Gamma_1)}=0$ and (3.12) implies that

$$\frac{d}{dt}\mathcal{L}(t) \le -\alpha\mathcal{L}(t),$$

hence $\mathcal{L}(t) \le Ce^{-\alpha t}$. Then, using (3.11), we deduce the second estimate (2.9) of Theorem 2.1.

Case 2. If $u^0 \neq 0$ on Γ_1, we rewrite (3.12) as follows

$$\frac{d}{dt}\mathcal{L}(t) \le -\alpha\mathcal{L}(t)+C_2k^2(t) \quad (3.13)$$

where $C_2=C_1\|u^0\|^2_{L^2(\Gamma_1)}$. To show (2.8), we introduce the function

$$f(t)=\mathcal{L}(t)-C_2e^{-\alpha t}\int_0^t k^2(s)e^{\alpha s}ds.$$

Using (3.13) we easily see that $f'(t) \le -\alpha f(t)$ which implies that $f(t) \le Ce^{-\alpha t}$. Hence, using the definition of f and (3.11), we obtain (2.8). This completes the proof.

Remark 3.5. In [4, Lemma 3.3], the authors considered the inequality (3.13) with $k^2(t) = Ce^{-\beta t}$, for a positive constant β, and proved that $\mathcal{L}$ decays exponentially to 0. Our proof extends this lemma to any function $k^2(t)$.

4. General Decay: the case $0 < q < \frac{1}{2}$

Here our attention will be focused on the decay estimates (2.10) and (2.11) when the resolvent kernel k satisfies

$$k(t) \geq 0, \quad k'(t) \leq 0, \quad k''(t) \geq b(-k'(t))^{1+q} \tag{4.1}$$

for some $0 < q < \frac{1}{2}$ and some positive constant b. This assumption implies that k' (and not necessarily k as was assumed in [4]) decays polynomially to 0; that is,

$$0 \leq -k'(t) \leq C(1+t)^{-\frac{1}{q}}.$$

Let $p = \frac{1}{q} - 1 > 1$. The lemma below will play an important role in the sequel.

Lemma 4.1. *Let u, be a solution of system* (1.1)–(1.4). *Then, for $p > 1$, $0 < r < 1$ and $t \geq 0$, we have*

$$\Big(\int_{\Gamma_1} e^{\phi(x)}|k'|\Box u d\Gamma\Big)^{1+\frac{1}{(1-r)(p+1)}}$$

$$\leq C\Big(\|u\|^2_{L^\infty(0,t;L^2(\Gamma_1))}\int_0^t |k'(s)|^r ds\Big)^{\frac{1}{(1-r)(p+1)}} \int_{\Gamma_1} e^{\phi(x)}|k'|^{1+\frac{1}{p+1}}\Box u d\Gamma$$

while for $r = 0$ we get

$$\Big(\int_{\Gamma_1} e^{\phi(x)}|k'|\Box u d\Gamma\Big)^{1+\frac{1}{p+1}}$$

$$\leq C\Big(\int_0^t \|u(s,.)\|^2_{L^2(\Gamma_1)}ds + t\|u(t,.)\|^2_{L^2(\Gamma_1)}\Big)^{p+1}\int_{\Gamma_1} e^{\phi(x)}|k'|^{1+\frac{1}{p+1}}\Box u d\Gamma.$$

Proof. Because ϕ is bounded, we conclude these two inequalities from [4, Lemma 4.1] (see also [14]).

Finally, we shall prove the inequalities (2.10) and (2.11). Using hypothesis (4.1) in Lemma 3.1 yields

$$\frac{d}{dt}E(t) \leq -\frac{\eta}{2}\int_{\Gamma_1}(|u_t|^2 + b(-k')^{1+\frac{1}{p+1}}\Box u - k^2(t)|u^0|^2)d\Gamma$$

$$+\frac{1}{2}\varphi(t)\int_\Omega e^{\phi(x)}(|u_t|^2 + |\nabla u|^2)dx.$$

Considering inequality (3.2) for k' with $\mu = \frac{p+2}{2(p+1)}$ and taking hypothesis (4.1) into account we obtain the estimate

$$|k' \diamond u|^2 \le C(-k')^{1+\frac{1}{p+1}} \Box u.$$

Using the above inequalities and (1.13) in Lemma 3.2 yields

$$\begin{aligned}\frac{d}{dt}\psi(t) &\le (\theta - n + \|m\|_\infty \|\nabla\phi\|_\infty) \int_\Omega e^{\phi(x)} |u_t|^2 dx \\ &-(\theta - (n + \|m\|_\infty \|\nabla\phi\|_\infty - 2)) \int_\Omega e^{\phi(x)} |\nabla u|^2 dx \\ &-(\theta - \frac{2(n + \|m\|_\infty \|\nabla\phi\|_\infty)}{\gamma+2}) D \int_\Omega e^{\phi(x)} |u|^{\gamma+2} dx \\ &-\delta_0 \int_{\Gamma_1} e^{\phi(x)} |\nabla u|^2 d\Gamma + C\Big(\epsilon \int_{\Gamma_1} e^{\phi(x)} |\nabla u|^2 d\Gamma + (\varphi(t)+\epsilon)\mathcal{N}(t)\Big) \\ &+C \int_{\Gamma_1} e^{\phi(x)} (|u_t|^2 + k^2(t)|u|^2 + (-k')^{1+\frac{1}{p+1}} \Box u + k^2(t)|u^0|^2) d\Gamma\end{aligned}$$

where ϵ is an arbitrary positive constant and $\mathcal{N}$ is defined in Section 3.

On the other hand, using hypothesis (2.5) for $0 < q < \frac{1}{2}$ and Young's inequality in Lemma 3.1 we get

$$\begin{aligned}\frac{d}{dt}E(t) &\le -\frac{\eta}{2} \int_{\Gamma_1} e^{\phi(x)} (|u_t|^2 - b(-k')^{1+\frac{1}{p+1}} \Box u - k^2(t)|u^0|^2) d\Gamma \\ &+\frac{1}{2}\varphi(t) \int_\Omega e^{\phi(x)} (|u_t|^2 + |\nabla u|^2) dx.\end{aligned}$$

In these conditions, taking N sufficiently large, ϵ small enough, and using (2.7) for φ, the Lyapunov functional defined in (3.10) satisfies, for some positive constants C_i, $i = 0, 1, \cdots$,

$$\begin{aligned}\frac{d}{dt}\mathcal{L}(t) &\le -C_2\mathcal{N}(t) + C_1\|u^0\|^2_{L^2(\Omega)} k^2(t) - C_3 \int_{\Gamma_1} (-k')^{1+\frac{1}{p+1}} \Box u d\Gamma \\ &+C_4 k^2(t) \int_{\Gamma_1} e^{\phi(x)} |u|^2 d\Gamma,\end{aligned}$$

hence, using (2.7) for k and the trace formula,

$$\frac{d}{dt}\mathcal{L}(t) \le -C_0\Big(\mathcal{N}(t) + \int_{\Gamma_1} (-k')^{1+\frac{1}{p+1}} \Box u d\Gamma\Big) + C_1\|u^0\|^2_{L^2(\Gamma_1)} k^2(t). \quad (4.2)$$

Now let us fix $0 < r < 1$ such that $\frac{1}{p+1} < r < \frac{p}{p+1}$. From (4.1) we have that

$$\int_0^\infty |k'(t)|^r dt \le C \int_0^\infty (1+t)^{-r(p+1)} dt < \infty.$$

Using this estimate in Lemma 4.1 we get

$$\int_{\Gamma_1} e^{\phi(x)}(-k')^{1+\frac{1}{p+1}}\Box u d\Gamma \geq C\Big(\int_{\Gamma_1} e^{\phi(x)}(-k')\Box u d\Gamma\Big)^{1+\frac{1}{(1-r)(p+1)}}. \qquad (4.3)$$

On the other hand, thanks to the regularity (2.4), $\mathcal{N}$ is bounded. Then we deduce that

$$\mathcal{N}(t) + \Big(\int_{\Gamma_1} e^{\phi(x)}(-k')\Box u d\Gamma\Big)^{1+\frac{1}{(1-r)(p+1)}}$$
$$\geq C\Big(\mathcal{N}(t) + \int_{\Gamma_1} e^{\phi(x)}(-k')\Box u d\Gamma\Big)^{1+\frac{1}{(1-r)(p+1)}} \geq CE(t)^{1+\frac{1}{(1-r)(p+1)}}. \qquad (4.4)$$

Substituting (4.3)-(4.4) into (4.2) we obtain

$$\frac{d}{dt}\mathcal{L}(t) \leq -CE(t)^{1+\frac{1}{(1-r)(p+1)}} + C_1\|u^0\|^2_{L^2(\Gamma_1)}k^2(t).$$

Taking into account the inequalities (3.11) we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \leq -C\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + C_1\|u^0\|^2_{L^2(\Gamma_1)}k^2(t). \qquad (4.5)$$

Now we distinguish two cases.

Case 1. If $u^0 = 0$ on Γ_1, then $\|u^0\|^2_{L^2(\Gamma_1)} = 0$ and (4.5) implies that

$$\frac{d}{dt}\mathcal{L}(t) \leq -C\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}},$$

which implies, by integration,

$$\mathcal{L}(t) \leq C(1+t)^{-(1-r)(p+1)}. \qquad (4.6)$$

Case 2. If $u^0 \neq 0$ on Γ_1, we rewrite (4.5) as follows

$$\frac{d}{dt}\mathcal{L}(t) \leq -C\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + C_2k^2(t) \qquad (4.7)$$

where $C_2 = C_1\|u^0\|^2_{L^2(\Gamma_1)}$. We introduce the functions

$$f(t) = \mathcal{L}(t) - g(t), \quad g(t) = C_2(t+1)^{-(1-r)(p+1)}\int_0^t k^2(s)(s+1)^{(1-r)(p+1)}ds.$$

Using (4.1) we easily see that

$$g'(t) + Cg(t)^{1+\frac{1}{(1-r)(p+1)}} \geq C_2k^2(t).$$

Then, using (4.7),

$$f'(t) \leq -C\mathcal{L}(t)^{1+\frac{1}{(1-r)(p+1)}} + C_2k^2(t) - g'(t)$$
$$\leq -C\Big((f(t)+g(t))^{1+\frac{1}{(1-r)(p+1)}} + \frac{1}{C}g'(t) - \frac{C_2}{C}k^2(t)\Big)$$

$$\le -C\Big(f(t)^{1+\frac{1}{(1-r)(p+1)}} + g(t)^{1+\frac{1}{(1-r)(p+1)}} + \frac{1}{C}g'(t) - \frac{C_2}{C}k^2(t)\Big)$$
$$\le -Cf(t)^{1+\frac{1}{(1-r)(p+1)}},$$

which implies that

$$f(t) \le C(1+t)^{-(1-r)(p+1)}.$$

Hence, using the definition of f and g, we obtain that

$$\mathcal{L}(t) \le C(1+t)^{-(1-r)(p+1)}\Big(1 + \int_0^t k^2(s)(s+1)^{(1-r)(p+1)}ds\Big). \tag{4.8}$$

Then, using (3.11), we deduce (2.10) with $\lambda = (1-r)(p+1)$.

If in addition $\int_0^t k^2(s)(s+1)^{(1-r)(p+1)}ds$ is bounded or $u^0 = 0$ on Γ_1, then, using (2.12) and (3.11), we get, from (4.6) and (4.8), the following bounds (note that $(1-r)(p+1) > 1$)

$$t\|u\|^2_{L^2(\Gamma_1)} \le Ct\mathcal{L}(t) < \infty, \qquad \int_0^t \|u\|^2_{L^2(\Gamma_1)}ds \le C\int_0^t \mathcal{L}(s)ds < \infty.$$

Considering the above estimates in Lemma 4.1 with $r = 0$ it holds that

$$\int_{\Gamma_1} e^{\phi(x)}(-k')^{1+\frac{1}{p+1}}\Box u d\Gamma \ge C\Big(\int_{\Gamma_1} e^{\phi(x)}(-k')\Box u d\Gamma\Big)^{1+\frac{1}{p+1}}.$$

Using the last inequality instead of (4.3) and reasoning in the same way as above we conclude that

$$\frac{d}{dt}\mathcal{L}(t) \le -C\mathcal{L}(t)^{1+\frac{1}{p+1}} + C_1\|u^0\|^2_{L^2(\Gamma_1)}k^2(t)$$

which is an inequality similar to (4.5). Then by the same arguments as above we deduce that $\mathcal{L}(t) \le C(1+t)^{-(p+1)}$ if $u^0 = 0$ on Γ_1, and

$$\mathcal{L}(t) \le C(1+t)^{-(p+1)}\Big(1 + \int_0^t k^2(s)(s+1)^{p+1}ds\Big)$$

if $u^0 \ne 0$ on Γ_1. Finally, from (3.11) we conclude the estimates (2.10) and (2.11), which completes the proof.

Remark 4.1. In [4, Lemma 4.2], the authors considered the inequality (4.7) with $k^2(t) = C(t+1)^{-\beta}$, for a positive constant β, and proved that $\mathcal{L}$ decays polynomially to 0. Our proof extends this lemma to any function $k^2(t)$.

5. FURTHER REMARKS

In this section we would like to present other models where our technique can be applied. For instance, one can consider the degenerate coupled system ($\rho_1, \rho_2 \geq 0$) subject to memory conditions on the boundary given by

$$\rho_1(x)u_{tt} - \Delta u + F(x,t,u,\nabla u) + \alpha(u-v) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+,$$

$$\rho_2(x)v_{tt} - \Delta v + G(x,t,v,\nabla v) - \alpha(u-v) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+,$$

$$u = 0 \quad \text{on} \quad \Gamma_0, \quad u + \int_0^t g_1(t-s)\frac{\partial u}{\partial \nu}(s)ds = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+,$$

$$v = 0 \quad \text{on} \quad \Gamma_0, \quad v + \int_0^t g_2(t-s)\frac{\partial v}{\partial \nu}(s)ds = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+,$$

$$(u(0), v(0)) = (u^0, v^0), \quad (\sqrt{\rho_1}u_t(0), \sqrt{\rho_2}v_t(0)) = (\sqrt{\rho_1}u^1, \sqrt{\rho_2}v^1) \quad \text{in} \quad \Omega,$$

where α is a positive constant,

$$\nabla \rho_i \cdot m \geq 0 \quad \text{in} \quad \Omega \quad \text{for} \quad i = 1, 2$$

and $m(x) = x - x^0$, $x^0 \in \mathbb{R}^n$.

According to the physical point of view, if $\rho \geq 0$ is the mass density of the material which is modelled in order to have the shape of Ω, the above hypothesis informs us that the mass distribution is concentrated in such a way that the mass density grows as far as the points of Ω are distant from x^0.

Another interesting situation arises when one considers the models in connection with the nonlinear plate equation with boundary conditions of memory type, namely

$$u_{tt} + \Delta^2 u + F(x,t,u,\Delta u) = 0 \quad \text{in} \quad \Omega \times \mathbb{R}_+,$$

$$u = \frac{\partial u}{\partial \nu} = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}_+,$$

$$\frac{\partial u}{\partial \nu} + \int_0^t g_1(t-s)\Big(\Delta u(s) + \rho_1 \frac{\partial u}{\partial \nu}(s)\Big)ds = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+,$$

$$u - \int_0^t g_2(t-s)\Big(\frac{\partial(\Delta u)}{\partial \nu}(s) - \rho_2 u(s)\Big)ds = 0, \quad \text{on} \quad \Gamma_1 \times \mathbb{R}_+,$$

$$u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x) \quad \text{in} \quad \Omega.$$

Here, ρ_1 and ρ_2 are positive constants which come from the physical model.

References

[1] M. Aassila, M.M. Cavalcanti, and J.A. Soriano, *Asymptotic stability and energy decay rates for solutions of the wave equation with memory in a star-shaped domain,* SIAM J. Control Optim., 38 (2000), 1581–1602.

[2] D. Andrade and J.E. Muñoz Rivera, *Exponential decay of non-linear wave equation with viscoelastic boundary condition,* Math. Meth. Appl. Sci., 23 (2000), 41–61.

[3] M.M. Cavalcanti, V.N. Domingos Cavalcanti, and J.A. Soriano, *Existence and boundary stabilization of a nonlinear hyperbolic equation with time-dependent coefficients,* Electron. J. Differential Equations, 08 (1998), 1–21.

[4] M.M. Cavalcanti, V.N. Domingos Cavalcanti, and M.L. Santos, *Uniform decay rates of solutions to a nonlinear wave equation with boundary condition of memory type,* Proceedings of 21st IFIP TC 7 Conference on System Modeling and Optimization, Sophia Antipolis, Vol. 1(2004), 239–256.

[5] M. Ciarletta, *A differential problem for the heat equation with a boundary condition with memory,* Appl. Math. Lett., 10 (1997), 95–101.

[6] G. Chen and H.K. Wang, *Asymptotic behavior of solutions of the one-dimensional wave equation with a nonlinear boundary stabilizier,* SIAM J. Control Optim., 27 (1989), 758–775.

[7] C.M. Dafermos and J.A. Nohel, *A nonlinear hyperbolic Volterra equation in viscoelasticity,* Amer. J. Math., Supplement (1981), 87–116.

[8] M. Fabrizio and A. Morro, *A boundary condition with memory in Electromagnetism,* Arch. Rational Mech. Anal., 136 (1996), 359–381.

[9] A. Guesmia, *A new approach of stabilization of nondissipative distributed systems,* SIAM J. Control. Optim., 42 (1) (2003), 24–52.

[10] A. Guesmia, *Nouvelles inégalités intégrales et applications à la stabilisation des systèmes distribués non dissipatifs, C. R. Acad. Sci. Paris, Ser. I., 336 (2003), 801–804.*

[11] M. Kirane and N. Tatar, *Non-existence results for a semilinear hyperbolic problem with boundary condition of memory type,* Journal of Analysis and Its Applications, 19 (2) (2000), 453–468.

[12] J. L. Lions, *Quelques Méthodes de résolution de problèmes aux limites non linéaires,* Dunod Gauthiers Villars, Paris (1969).

[13] R. Racke, *Lectures on nonlinear evolution equations. Initial value problems,* Aspect of Mathematics E19. Friedr. Vieweg & Sohn, Braunschweig/Wiesbaden (1992).

[14] M. L. Santos, *Asymptotic behavior of solutions to wave equations with a memory condition at the boundary,* E. J. Diff. Eqs., 73 (2001), 1–11.

[15] M. L. Santos, *Decay rates for solutions of a system of wave equations with memory,* E. J. Diff. Eqs., 38 (2002), 1–17.

[16] T. Qin, *Global solvability of nonlinear wave equation with a viscoelastic boundary condition,* Chin. Ann. Math., 14B (3) (1993), 335–346.

[17] T. Qin, *Breakdown of solutions to nonlinear wave equations with a viscoelastic boundary condition,* Arab. J. Sci. Engng., 19 (2A) (1994), 195–201.