LOCAL DECAY IN NON-RELATIVISTIC QED

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Abstract. We prove the limiting absorption principle for a dressed electron at a fixed total momentum in the standard model of non-relativistic quantum electrodynamics. Our proof is based on an application of the smooth Feshbach-Schur map in conjunction with Mourre's theory.

1. Introduction

In this paper, we study the dynamics of a single charged non-relativistic quantum-mechanical particle - an electron - coupled to the quantized electromagnetic field. Its quantum Hamiltonian is given by (in what follows, we will employ units such that the bare electron mass and the speed of light are $m = 1$ and $c = 1$)

$$H := \frac{1}{2}\big(p_{\mathrm{el}} + \alpha^{\frac{1}{2}} A(x_{\mathrm{el}})\big)^2 + H_f, \tag{1.1}$$

acting on $\mathcal{H} = \mathcal{H}_{\mathrm{el}} \otimes \mathcal{F}$, where $\mathcal{H}_{\mathrm{el}} = \mathrm{L}^2(\mathbb{R}^3)$ is the Hilbert space for an electron (for the sake of simplicity, the spin of the electron is neglected), and $\mathcal{F}$ is the symmetric Fock space for the photons defined as

$$\mathcal{F} := \Gamma_s(\mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)) \equiv \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} S_n \left[\mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)^{\otimes n}\right], \tag{1.2}$$

where S_n denotes the symmetrization operator on $\mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)^{\otimes n}$. In Eq. (1.1), x_{el} denotes the position of the electron, $p_{\mathrm{el}} := -\mathrm{i}\nabla_{x_{\mathrm{el}}}$ is the electron momentum operator, α is the fine structure constant (in our units the electron charge is $e = -\alpha^{1/2}$), $A(x_{\mathrm{el}})$ is the quantized electromagnetic vector potential,

$$A(x_{\mathrm{el}}) := \frac{1}{\sqrt{2}} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\kappa^{\Lambda}(k)}{|k|^{\frac{1}{2}}} \varepsilon_\lambda(k) (a_\lambda^*(k) e^{-\mathrm{i}k\cdot x_{\mathrm{el}}} + a_\lambda(k) e^{\mathrm{i}k\cdot x_{\mathrm{el}}}) \mathrm{d}k, \tag{1.3}$$

and H_f is the Hamiltonian for the free quantized electromagnetic field given by

$$H_f := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| a_\lambda^*(k) a_\lambda(k) \mathrm{d}k. \tag{1.4}$$

The photon creation- and annihilation operators, $a_\lambda^*(k)$, $a_\lambda(k)$, are operator-valued distributions on $\mathcal{F}$ obeying the canonical commutation relations

$$[a_\lambda^\#(k), a_{\lambda'}^\#(k')] = 0, \quad [a_\lambda(k), a_{\lambda'}^*(k')] = \delta_{\lambda\lambda'}\delta(k-k'), \tag{1.5}$$

where $a^\#$ stands for a^* or a; $\varepsilon_\lambda(k)$, $\lambda = 1, 2$, are normalized polarization vectors, i.e., vector fields orthogonal to one another and to k (we assume, in addition, that $\varepsilon_\lambda(k) = \varepsilon_\lambda(k/|k|)$, so that $(k \cdot \nabla_k \varepsilon_\lambda)(k) = 0$), and κ^Λ is an ultraviolet cutoff function, chosen such that

$$\kappa^\Lambda \in \mathrm{C}_0^\infty(\{k, |k| \leq \Lambda\}; [0,1]) \text{ and } \kappa^\Lambda = 1 \text{ on } \{k, |k| \leq 3\Lambda/4\}. \tag{1.6}$$

There is no external potential acting on the electron. It can, however, absorb and emit photons, (i.e., field quanta of the electromagnetic field), which dramatically affects its dynamical properties. This is the simplest system of quantum electrodynamics. In the present paper, we take an important step towards understanding the dynamics of this system: We exhibit a local decay property saying, roughly speaking, that the probability of finding all photons within a ball of an arbitrary radius $R < \infty$ centered at the position, x_{el}, of the electron tends to 0, as time t tends to ∞. In other words, asymptotically, as time t tends to ∞, the distance between some photons and the electron tends to ∞, and the electron relaxes into a "lowest-energy state".

The above result is proven for an arbitrary initial state of the system, assuming only that its maximal total momentum has a magnitude smaller than $p_{\mathrm{c}} < mc = 1$; (recall that $m = 1$ and $c = 1$). In the following, we set $p_{\mathrm{c}} = 1/40$, but we expect our result to hold for any value of p_{c} smaller than 1. The physical origin of the restriction on the total momentum will be described below.

It has long been expected and has recently been proven that an electron coupled to the quantized electromagnetic field is an *"infra-particle"*: The infimum, $E(P)$, of the spectrum of the Hamiltonian at total momentum P is *not* an eigenvalue, except when $P = 0$. (This result is sometimes referred to as "infrared catastrophe". Precise notions will be given later in this introduction.) However, there is an "infrared representation" of the canonical commutation relations of the electromagnetic field that is disjoint from the Fock representation and such that the corresponding representation space contains an eigenvector associated to $\inf \sigma(H|_P)$; see [Fr2, Pi, CF, CFP2]. This suggests that if we prepare the system, at some initial time $t(=0)$, in an arbitrary state described by a vector in the tensor product of the one-electron Hilbert space and the photon Fock space, whose maximal total momentum has a magnitude strictly smaller than $mc = 1$, and then study the time evolution of this vector we will find that the probability of finding photons within a ball of an arbitrary radius $R < \infty$ centered at the position, x_{el}, of the electron tends to 0, as time t tends to ∞.

This intuitive picture is expressed in precise language in terms of the *local decay property*, which is formulated as

$$\left\| \, (\mathrm{d}\Gamma(\langle x_{\mathrm{ph}} - x_{\mathrm{el}} \rangle) + 1)^{-s} \, e^{-\mathrm{i}tH} \, g(H, P_{\mathrm{tot}})\Phi \, \right\| \, \leq \, \mathrm{C} \, t^{-(s-\frac{1}{2})} \, , \tag{1.7}$$

with $\langle a \rangle := \sqrt{a^2+1}$. Here $\mathrm{d}\Gamma(b)$ denotes the usual (Lie-algebra) second quantization of an operator b acting on $\mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, x_{ph} denotes the photon "position" operator, $x_{\mathrm{ph}} = \mathrm{i}\nabla_k$, acting on $L^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, $P_{\mathrm{tot}} := p_{\mathrm{el}} + P_f$ is the total momentum operator, where the field momentum, P_f, is given by

$$P_f := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} k a_\lambda^*(k) a(k) \mathrm{d}k, \tag{1.8}$$

g is an arbitrary smooth function compactly supported on the set

$$\mathcal{M}_{a.c.} := \left\{ (\lambda, P) \in \mathbb{R} \times \mathcal{S} \,|\, \lambda > E(P) \right\}, \tag{1.9}$$

where $\mathcal{S} := \{P \in \mathbb{R}^3 | \, |P| < p_{\mathrm{c}}\}$, and Φ ranges over a certain dense set in $\mathcal{H}$. (Inequality (1.7) states that photons move out of any bounded domain around the electron with probability one, as time tends to infinity.) This is one of the key results of this paper. Another related consequence of our analysis is that the spectrum of the Hamiltonian of the system at total momentum P different from 0, with $|P| < p_c$, is purely absolutely continuous.

One expects, in fact, that, asymptotically, as time t tends to ∞, the system approaches a scattering state describing an electron and an outgoing cloud of infinitely many freely moving photons of finite total energy, with the spatial separation between the electron and the photon cloud diverging linearly in t; (Compton scattering, see [CFP1]).

The system studied in this paper is translation invariant, in the sense that H commutes with the total momentum operator $P_{\mathrm{tot}} = p_{\mathrm{el}} + P_f$. This implies that H admits a "fiber decomposition"

$$UHU^{-1} = \int_{\mathbb{R}^3}^{\oplus} H(P) \mathrm{d}P, \tag{1.10}$$

over the spectrum of P_{tot}. The r.h.s. of (1.10) acts on the direct integral $\tilde{\mathcal{H}} := \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}_P \mathrm{d}P$, with fibers $\mathcal{H}_P \cong \mathcal{F}$, (i.e $\tilde{\mathcal{H}} = L^2(\mathbb{R}^3, \mathrm{d}P; \mathcal{F})$), the fiber operators $H(P)$, $P \in \mathbb{R}^3$, are self-adjoint operators on the spaces $\mathcal{H}_P$, and U is the unitary operator

$$(U\Psi)(P) := \int_{\mathbb{R}^3} e^{\mathrm{i}(P-P_f)\cdot y} \Psi(y) \mathrm{d}y. \tag{1.11}$$

It maps the state space $\mathcal{H} = \mathcal{H}_{\mathrm{el}} \otimes \mathcal{F}$ onto the direct integral $\tilde{\mathcal{H}} = \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}_P \mathrm{d}P$. (The inverse is given by $(U^{-1}\Phi)(x_{\mathrm{el}}) = \int_{\mathbb{R}^3} e^{-\mathrm{i}x_{\mathrm{el}}\cdot(P-P_f)} \Phi(P) \mathrm{d}P$.)

The quantity $E(P)$ mentioned above is defined as $E(P) := \inf \sigma(H(P))$. It is the energy of a dressed one-particle state of momentum P, provided $|P|$ is small enough. Its regularity, which turns out to be essential in our analysis, has been investigated in [Ch, BCFS2, CFP2, FP]. In [AFGG], related results for a model of a dressed non-relativistic electron in a magnetic field are established.

For the uncoupled system, $\alpha = 0$, at total momentum P, $E(P) = P^2/2$ is an eigenvalue of the Hamiltonian $H(P)$. For $|P|$ smaller than or equal to $mc = 1$, it is at the bottom of the spectrum of $H(P)$. But if $|P| > 1$ the bottom of the spectrum of the Hamiltonian of the uncoupled system at total momentum P reaches down to

$$|P| - 1/2,$$

which is strictly smaller than $P^2/2$, and hence the eigenvalue $P^2/2$ is embedded in the continuous spectrum; see Figure 1, below. In this range of momenta, the charged particle may propagate faster than the speed of light and, hence, it emits Cerenkov radiation. Thus, one expects the dynamics of the system to be quite different depending on whether $|P| < 1$ or $|P| > 1$. This is the physical origin of our restriction on the total momentum ($|P| \leq p_c < 1$) which appeared above.

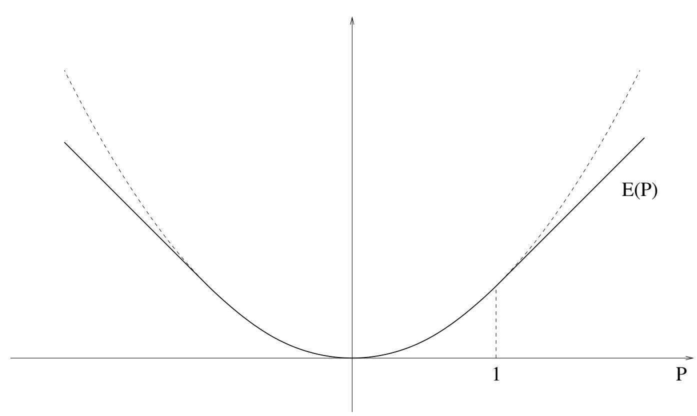

FIGURE 1. **The map** $E(P) = \inf \sigma(H(P))$**, for** $\alpha = 0$:
If $|P| \leq 1$, $E(P) = P^2/2 \in \sigma_{\rm pp}(H(P))$,
If $|P| > 1$, $E(P) = |P| - 1/2 \notin \sigma_{\rm pp}(H(P))$.

We will analyze the spectra of the fiber Hamiltonians $H(P)$ at a fixed total momentum $P \in \mathbb{R}^3$, with $|P| \leq p_c$. We prove the limiting absorption principle (LAP) for $H(P)$, for $\alpha^{1/2}$ small enough and $|P| \leq p_c$. As a consequence, we obtain local decay estimates and absolute continuity of the spectrum of $H(P)$ in the interval $(E(P), +\infty)$. (In an appendix, we explain how to modify the proof given in this paper to arrive at a LAP for electrons bound to static nuclei and linearly coupled to the radiation field.) Our method can be also easily adapted to the P-fibers of particle systems, like atoms and molecules (see, e.g., [LMS]). If such a system, in the center-of-mass frame, has a ground state at the bottom of its spectrum, then of course the approach simplifies considerably and becomes similar to the one outlined in Appendix C.

Our proof of the LAP is based on an application of the isospectral *smooth Feshbach-Schur map* introduced in [BCFS1]; see also [GH, FGS3]. This map depends on the choice of an unperturbed Hamiltonian. An important and *new* point in our analysis is to choose an unperturbed Hamiltonian obtained by decoupling the low-energy photons from the electron; (a similar idea was suggested independently by M. Griesemer [Gr]; such Hamiltonians were used previously, but in a different context, in, e.g., [BFP, FGS1, FP].) We combine the Feshbach-Schur map with *Mourre's theory* (see [Mo, PSS, ABG, HS]). Our proofs incorporate many important earlier ideas, methods and results; (especially from [BCFS1, GH, FGS1, FP]). To compare our approach with that of [FGS1, FGS2], we apply it in Appendix C to the Nelson model involving bound particles linearly coupled to the quantized radiation field. We emphasize that our methods are well adapted to coping with the infrared singularity of the form factor in the interaction Hamiltonian.

If one attempted to establish local decay for the Hamiltonian in (1.1) directly, i.e., without using the fiber decomposition (1.10), one would face a major difficulty: One would have to deal with a continuum of thresholds, $E(P)$, potentially leading to extremely slow decay.

For the standard model of charged non-relativistic particles bound to a static nucleus and interacting with the quantized electromagnetic field, a LAP just above the ground state energy has been recently proven in [FGS1] and [FGS3]. The proof in [FGS1] is based on an infrared decomposition of the photon Fock space: In order to establish a LAP in an interval located at a distance σ from the bottom of the spectrum, the initial Hamiltonian is approximated by an infrared-cutoff Hamiltonian (which is obtained by turning off the interaction between the charged particles and photons of energies smaller than σ). The Mourre estimate is then established in a perturbative way. A feature of the infrared-cutoff Hamiltonian, which the method of [FGS1] is based upon, is that only the free-field energy operator affects the low-energy photons.

The proof in [FGS3] is based on a spectral renormalization group analysis; (see [BFS, BCFS1, FGS2]) and could possibly be adapted to our context. However, the proof we present in the following is significantly simpler, in that we require only one application of the smooth Feshbach-Schur map, whereas renormalization group methods are based on an iteration of this map.

While progress on understanding the standard model of charged non-relativistic particles bound to static nuclei and interacting with the quantized electromagnetic field has been fairly robust, our understanding of free electrons coupled to the quantized electromagnetic field has emerged rather slowly and has always come at the price of very involved and lengthy arguments. Many techniques that work beautifully for the former (see, e.g., the extensive literature on existence of ground states) are hitting upon walls in the latter case. To begin with, an important ingredient in various proofs, including the one in [FGS1], namely the use of a unitary Pauli-Fierz transformation (combined with exponential decay of states bound to nuclei in the position variables of the electrons), is not available in the free-electron model. Furthermore, the important feature that, after an infrared cutoff has been imposed, only the free-field energy operator determines the dynamics of the low energy photons, is no longer true in our model. More precisely, a term coupling the low- and high-energy photons appears in the infrared cutoff Hamiltonian (see (1.30) and the discussion after it), so that the methods in [FGS1] do not apply directly.

Main results

We now state our main results and outline the strategy of our proof. Whenever the readers meet an unfamiliar notation they are encouraged to consult Appendix D.

We prove a limiting absorption principle for $H(P)$ in an energy interval just above $E(P) = \inf \sigma(H(P))$, for $|P| \leq p_{\mathrm{c}}$, where $0 < p_{\mathrm{c}} < 1$. In this paper we choose $p_{\mathrm{c}} = 1/40$, and we do not attempt to find an optimal estimate on p_{c}.

The main result of this paper can be formulated as follows: For an interval $J \subseteq \mathbb{R}$, we set $J_\pm := \big\{ z \in \mathbb{C}, \mathrm{Re} z \in J, 0 < \pm \mathrm{Im} z \leq 1 \big\}$. Since the operator $\mathrm{d}\Gamma(\langle x_{\mathrm{ph}} - x_{\mathrm{el}} \rangle)$ is translationally invariant (it commutes with P_{tot}), it is represented as the fiber integral,

$$U \mathrm{d}\Gamma(\langle x_{\mathrm{ph}} - x_{\mathrm{el}} \rangle) U^{-1} = \int_{\mathbb{R}^3}^{\oplus} \mathrm{d}\Gamma(\langle y \rangle) \mathrm{d}P, \tag{1.12}$$

where $y := \mathrm{i}\nabla_k$ is the "position" operator of the photon, but now relative to the electron position. (To distinguish it from the original photon "position" operator $x_{\mathrm{ph}} = \mathrm{i}\nabla_k$, we use the symbol y.) We have

Theorem 1.1. *There exists an $\alpha_0 > 0$ such that, for any $|P| \leq p_{\mathrm{c}} \, (= 1/40)$, $0 \leq \alpha \leq \alpha_0$, $1/2 < s \leq 1$, and any compact interval $J \subset (E(P), \infty)$, we have that*

$$\sup_{z \in J_\pm} \big\| (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \big[H(P) - z \big]^{-1} (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \big\| \leq \mathrm{C}, \tag{1.13}$$

where C *is a constant depending on J and s. Moreover, the map*

$$J \ni \lambda \mapsto (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \big[H(P) - \lambda \pm \mathrm{i}0^+ \big]^{-1} (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \in B(\mathcal{H}) \tag{1.14}$$

is uniformly Hölder continuous in λ of order $s - 1/2$.

This theorem follows from Corollaries 2.3 and 5.4 below. Our proof will show that, if $\mathrm{dist}(E(P), J) = \sigma$ then the constant C in (1.13) is bounded by $O(\sigma^{-1})$. Finding an optimal upper bound on C with respect to σ is beyond the scope of this paper.

As a consequence of Theorem 1.1, we have the following

Corollary 1.2. *There exists $\alpha_0 > 0$ such that for any $|P| \le p_{\mathrm{c}}$ and $0 \le \alpha \le \alpha_0$, the spectrum of $H(P)$ is purely absolutely continuous in the interval $(E(P), +\infty)$.*

Physical interpretation of our results

We describe a consequence of Theorem 1.1 pointing to a key physical property of the system. We consider an initial state consisting of a dressed electron together with a cloud of photons located in a finite ball centered at the position of the electron.

Corollary 1.3. *Recall that $\mathcal{S} = \{P \in \mathbb{R}^3 | \, |P| < p_{\mathrm{c}}\}$, and let $\Phi \in \mathcal{H} = \mathcal{H}_{\mathrm{el}} \otimes \mathcal{F}$ denote an arbitrary state such that $U\Phi = \int_{\mathcal{S}}^{\oplus} \Phi(P) \mathrm{d}P$ and*

$$\| \, (\mathrm{d}\Gamma(\langle y \rangle) + 1)^s \, \Phi(P) \, \| \, < \, \infty \, , \tag{1.15}$$

for some $1/2 < s \le 1$ and for all $P \in \mathcal{S}$. Then our system has the local decay property (1.7)*.*

Proof. Let $\Phi_g \, := \, g(H \, , P_{\mathrm{tot}}) \, \Phi$. The state $U\Phi_g \in \int_{\mathbb{R}^3}^{\oplus} \mathcal{H}_P \mathrm{d}P$ can be written as

$$U\Phi_g \, = \, Ug(H \, , P_{\mathrm{tot}})\Phi \, = \, \int_{\mathcal{S}}^{\oplus} \, g(H(P), P) \, \Phi(P) \, \mathrm{d}P. \tag{1.16}$$

We note that

$$Ue^{-\mathrm{i}tH} \Phi_g \, = \, \lim_{\varepsilon \to 0} \frac{1}{2\mathrm{i}\pi} \int_{\mathcal{S}} \mathrm{d}P \int \mathrm{d}\lambda \, f(\lambda, P) \, e^{-\mathrm{i}t\lambda} \mathrm{Im} \frac{1}{H(P) - \lambda - \mathrm{i}\varepsilon} \Phi(P) \, , \tag{1.17}$$

so that

$$\begin{aligned} &\Big\| \, (\mathrm{d}\Gamma(\langle x_{\mathrm{ph}} - x_{\mathrm{el}} \rangle) + 1)^{-s} \, e^{-\mathrm{i}tH} \, \Phi_g \, \Big\| \\ &= \sup_{\|\Phi'\|=1} \Big| \, \lim_{\varepsilon \to 0} \int_{\mathcal{S}} \mathrm{d}P \int \mathrm{d}\lambda \, e^{-\mathrm{i}t\lambda} \, f(\lambda, P) \\ &\qquad \Big\langle \, \Phi' \, , \, (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \mathrm{Im} \frac{1}{H(P) - \lambda - \mathrm{i}\varepsilon} \Phi(P) \, \Big\rangle \, \Big| \, . \end{aligned} \tag{1.18}$$

Since $g(\lambda, P)$ is supported on the set $\{\lambda > E(P)\}$, Theorem 1.1 implies that the scalar product $\langle \cdots \rangle$ in (1.18) is $(s - \frac{1}{2})$-Hölder continuous in λ, for any choice of Φ', and for a Hölder constant independent of Φ', because $\widetilde{\Phi}(P) := (\mathrm{d}\Gamma(\langle y \rangle) + 1)^s \Phi(P) \in \mathcal{F}$. The Fourier transform $\widehat{h}(t) = \int e^{it\lambda} h(\lambda) \mathrm{d}\lambda$ of an $(s - \frac{1}{2})$-Hölder continuous function $h(\lambda)$ satisfies $|\widehat{h}(t)| \le \mathrm{C}t^{-(s-1/2)}$. Thus, (1.7) follows. $\square$

This corollary implies that photons that are not permanently bound to the dressed electron move out of any bounded domain around the dressed electron with probability one, as time tends to ∞.

We consider an observable A, given by a selfadjoint operator on $\mathcal{H}$ which we assume to satisfy

$$\|(\mathrm{d}\Gamma(\langle x_{\mathrm{ph}} - x_{\mathrm{el}}\rangle) + 1)^s A(\mathrm{d}\Gamma(\langle x_{\mathrm{ph}} - x_{\mathrm{el}}\rangle) + 1)^s\| \,<\, \infty\,, \tag{1.19}$$

Then,

$$\lim_{t\to 0} \Big\langle \, \Phi_g \, , \, e^{\mathrm{i}tH} \, A \, e^{-\mathrm{i}tH} \, \Phi_g \, \Big\rangle \,=\, 0\,. \tag{1.20}$$

Indeed, we have

$$\begin{aligned} \Big| \Big\langle \, e^{-\mathrm{i}tH} \Phi_g \, , \, A \, e^{-\mathrm{i}tH} \Phi_g \Big\rangle \Big| &\leq \|(\mathrm{d}\Gamma(\langle x_{\mathrm{ph}} - x_{\mathrm{el}}\rangle) + 1)^s A(\mathrm{d}\Gamma(\langle x_{\mathrm{ph}} - x_{\mathrm{el}}\rangle) + 1)^s\| \\ &\quad \times \| \, (\mathrm{d}\Gamma(\langle x_{\mathrm{ph}} - x_{\mathrm{el}}\rangle) + 1)^{-s} \, e^{-\mathrm{i}tH} \, \Phi_g \, \|^2 \\ &\leq \mathrm{C} \, t^{-2(s-\frac{1}{2})}. \end{aligned} \tag{1.21}$$

More generally, we expect the following picture to hold true. We assume that $h \in C^\infty((-\infty, E_{\mathrm{c}}) \times \mathcal{S})$, where $E_{\mathrm{c}} = E(P)$ with $|P| = p_{\mathrm{c}}$, and consider the state

$$\Phi_h \,:=\, h(H, P_{\mathrm{tot}}) \, \Phi$$

where $\Phi \in \mathcal{H}$ is as in Corollary 1.3. Let $A = U^{-1} \int^{\oplus} A_P \mathrm{d}P U$ denote a bounded translation invariant observable. Then, we expect that

$$\lim_{t\to\infty} \Big\langle \, e^{-\mathrm{i}tH} \Phi_h \, , \, A \, e^{-\mathrm{i}tH} \Phi_h \, \Big\rangle \,=\, \int_{\mathcal{S}} \mathrm{d}\mu_{\Phi_h}(P) \Big\langle \, \Psi_P \, , \, A_P \, \Psi_P \, \Big\rangle \,, \tag{1.22}$$

where $\mathrm{supp}\{\mathrm{d}\mu_{\Phi_h}\} \subseteq \mathcal{S}$. Here, $\langle \, \Psi_P \, , (\, \cdot \,) \, \Psi_P \, \rangle$ denotes an expectation in the generalized ground state of the fiber Hamiltonian $H(P)$. This describes the relaxation of the state Φ_h to the mass shell, asymptotically as $t \to \infty$, under emission of photons that disperse to spatial infinity. (Note that, for $P \neq 0$, Ψ_P does not belong to the Fock space, but to a Hilbert space carrying an infrared representation of the canonical commutation relations.)

We end this discussion by presenting the explicit expression for the fiber Hamiltonians $H(P)$. Using (1.10) and (1.11) and using that $A(x_{\mathrm{el}}) e^{\mathrm{i}x_{\mathrm{el}} \cdot (P - P_f)} = e^{\mathrm{i}x_{\mathrm{el}} \cdot (P - P_f)} A(0)$, we compute $H(U^{-1}\Phi)(x_{\mathrm{el}}) = \int_{\mathbb{R}^3} e^{\mathrm{i}x_{\mathrm{el}} \cdot (P - P_f)} H(P) \Phi(P) \mathrm{d}P$, where $H(P)$ is given explicitly by

$$H(P) = \frac{1}{2} \big(P - P_f + \alpha^{\frac{1}{2}} A \big)^2 + H_f, \tag{1.23}$$

with

$$A := A(0) = \frac{1}{\sqrt{2}} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\kappa^\Lambda(k)}{|k|^{\frac{1}{2}}} \varepsilon_\lambda(k) (a_\lambda^*(k) + a_\lambda(k)) \mathrm{d}k. \tag{1.24}$$

Strategy of the proof of Theorem 1.1
First, we prove an easy part - a LAP in any compact interval $J \subset (E(P), \infty)$ with the property that $\inf J \geq E(P) + \mathrm{C}_0 \alpha^{1/2}$, where C_0 is a sufficiently large, positive constant (Section 2). This follows from a Mourre estimate of the form

$$\mathbb{1}_J(H(P))[H(P), \mathrm{i}B]\mathbb{1}_J(H(P)) \geq \mathrm{c}\mathbb{1}_J(H(P)), \tag{1.25}$$

where B is the generator of dilatations on Fock space (see Equation (2.1)) and c is positive. Using the assumption that $\inf J \geq E(P) + \mathrm{C}_0\alpha^{1/2}$ and standard estimates, Equation (1.25) can be proven in a straightforward way.

A considerably more difficult task is to prove a limiting absorption principle near $E(P)$. We use a theorem due to [FGS3] (see Theorem B.2 in Appendix B of the present paper), which essentially says that one can derive a LAP for $H(P)$ from a LAP for an operator resulting from applying a smooth Feshbach-Schur map to $H(P)$. We explain these points in detail.

Our construction of the smooth Feshbach-Schur map is based on a *low-energy decomposition* of the Hamiltonian $H(P)$:

$$H(P) = H_\sigma(P) + U_\sigma(P), \tag{1.26}$$

where $\sigma \geq 0$, $U_\sigma(P)$ is defined by this equation and the infrared cutoff Hamiltonian $H_\sigma(P)$, $\sigma \geq 0$, is given by

$$H_\sigma(P) := \frac{1}{2}(P - P_f + \alpha^{\frac{1}{2}} A_\sigma)^2 + H_f, \tag{1.27}$$

for every $P \in \mathbb{R}^3$, with

$$A_\sigma := \frac{1}{\sqrt{2}} \sum_{\lambda=1,2} \int_{\{|k| \geq \sigma\}} \frac{\kappa^\Lambda(k)}{|k|^{\frac{1}{2}}} \varepsilon_\lambda(k)(a^*_\lambda(k) + a_\lambda(k))\mathrm{d}k, \tag{1.28}$$

(see Section 3). Note that $H_\sigma(P)$ is defined by decoupling photons of energy less than σ from the electron. Such a decomposition was used previously in the analysis of non-relativistic QED; (see, e.g., [BFP, FGS1]).

Next we use the fact that the Hilbert space $\mathcal{F}$ is isometrically isomorphic to $\mathcal{F}_\sigma \otimes \mathcal{F}^\sigma$ where $\mathcal{F}_\sigma := \Gamma_s(\mathrm{L}^2(\{(k,\lambda), |k| \geq \sigma\}))$ and $\mathcal{F}^\sigma := \Gamma_s(\mathrm{L}^2(\{(k,\lambda), |k| \leq \sigma\}))$. Below we will use this representation without always mentioning it. The operator $H_\sigma(P)$ leaves invariant the Fock space $\mathcal{F}_\sigma$ of photons of energies larger than σ, and its restriction to $\mathcal{F}_\sigma$,

$$K_\sigma(P) := H_\sigma(P)|_{\mathcal{F}_\sigma}, \tag{1.29}$$

has a gap of order $O(\sigma)$ in its spectrum above the ground state energy. Moreover, in $\mathcal{F}_\sigma \otimes \mathcal{F}^\sigma$, $H_\sigma(P)$ decomposes as

$$H_\sigma(P) = K_\sigma(P) \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + H_f\big) - \nabla K_\sigma(P) \otimes P_f, \tag{1.30}$$

where $\nabla K_\sigma(P) := P - P_f + \alpha^{1/2} A_\sigma$. The central difficulty in our analysis comes from the presence of the last term in (1.30), which couples the low- and high-energy photons. This is the main reason why we are not able to prove a Mourre estimate for $H(P)$ near $E(P)$ by using a suitable σ-dependent conjugate operator (as is done in [FGS1]). To circumvent this difficulty, we apply the Feshbach-Schur map.

We use the projection, $P_\sigma(P)$, onto the ground state of $K_\sigma(P)$ in order to construct a smooth Feshbach-Schur map F_χ, where $\chi = P_\sigma(P) \otimes \chi_f^\sigma(H_f)$, with $\chi_f^\sigma(H_f)$ a smoothed "projection" onto the spectral subspace $H_f \leq \sigma$; (see Section 4). This map projects out the degrees of freedom corresponding to photons of energies larger than σ. The resulting operator $F(\lambda) := F_\chi(H(P) - \lambda)$, where λ is the spectral parameter, is of the form

$$F(\lambda) = K_\sigma(P) \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2} P_f^2 + H_f\big) - \nabla E_\sigma(P) \otimes P_f - \lambda + W, \tag{1.31}$$

where $E_\sigma(P) := \inf \sigma(H_\sigma(P))$ and W is defined by this relation. We notice that, due to the last term in (1.30), the unperturbed operator chosen to construct $F(\lambda)$ cannot be $H_\sigma(P)$. Instead we choose the following operator:

$$T_\sigma(P) = K_\sigma(P) \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2} P_f^2 + H_f\big) - \nabla E_\sigma(P) \otimes P_f. \tag{1.32}$$

Thanks to the uniform regularity of $E_\sigma(P)$ with respect to P (see Proposition 3.1) and using the Feynman-Hellman formula (see Lemma 5.6), we see that the difference $H_\sigma(P) - T_\sigma(P)$ is small in an appropriate sense. In particular, the operator W in (1.31) can be estimated to be $O(\alpha^{1/2}\sigma)$.

Next, in order to obtain a LAP for $F(\lambda)$, we use again Mourre's theory, choosing a conjugate operator B^σ defined as the generator of dilatations with a cutoff in the photon momentum variable,

$$B^\sigma := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} a_\lambda^*(k) \kappa^\sigma(k) b \kappa^\sigma(k) a_\lambda(k) \mathrm{d}k, \tag{1.33}$$

with $\kappa^\sigma(k)$ a cutoff in the photon momentum variable, see (1.6), and $b := \frac{\mathrm{i}}{2}(k \cdot \nabla_k + \nabla_k \cdot k)$ the generator of dilatations; (see Section 5). Let λ be in the interval

$$J_\sigma^< := [E(P) + 11\rho\sigma/128, E(P) + 13\rho\sigma/128], \tag{1.34}$$

where σ satisfies $\sigma \leq \mathrm{C}_0' \alpha^{1/2}$ for some fixed, sufficiently large positive constant $\mathrm{C}_0' \geq \mathrm{C}_0$, and $\rho\sigma$ is the size of the gap above $E_\sigma(P)$ in the spectrum of $K_\sigma(P)$. The Mourre estimate for $F(\lambda)$, on the spectral interval $\Delta_\sigma = [-\rho\sigma/128, \rho\sigma/128]$, is established as follows. By energy localization and the facts that the operator $K_\sigma(P)$ commutes with B^σ and that $|\nabla E_\sigma(P)| \leq |P| + \mathrm{C}\alpha \leq 1/4$, for $|P| \leq 1/40$ and α sufficiently small, the commutator of the "unperturbed" part in $F(\lambda)$ with B^σ yields a positive term of order $O(\sigma)$. This and the fact that the commutator with the "perturbation" W is of order $O(\alpha^{1/2}\sigma)$ lead to the Mourre estimate and, therefore, to the LAP for $F(\lambda)$. Once the LAP is established for $F(\lambda)$, it is transferred by the theorem of [FGS3] mentioned above (see Theorem B.2 in the present paper), to the original Hamiltonian $H(P)$ on the interval $J_\sigma^<$. Finally, we use that the intervals $J_\sigma^>$ in (1.34) with $\sigma \le \mathrm{C}_0'\alpha^{1/2}$ cover the interval $(E(P), \mathrm{C}_0\alpha^{1/2}]$.

Organization of the paper

Our paper is organized as follows. In the next section, we prove the LAP for $H(P)$ outside a certain neighborhood of $E(P) = \inf \sigma(H(P))$. Section 3 is concerned with the approximation of $H(P)$ by the infrared cutoff Hamiltonian $H_\sigma(P)$. In Section 4, we prove the existence of the Feshbach-Schur operator $F(\lambda)$ mentioned above. We establish the Mourre estimate for $F(\lambda)$ in Section 5, from which we deduce the LAP for $H(P)$ near $E(P)$. In Appendix A, we collect some technical estimates used in Sections 4 and 5. Appendix B recalls the definition of the smooth Feshbach-Schur map and some of its main properties. In Appendix C, we briefly explain how to adapt the methods used in this paper to a model of bound non-relativistic electrons coupled to the radiation field. Finally, for the convenience of the reader, a list of notations used in this paper is contained in Appendix D.

Throughout the paper, $\mathrm{C}, \mathrm{C}', \mathrm{C}''$ denote positive constants that may vary from one line to another.

Acknowledgements

J.Fr. and I.M.S. are grateful to Marcel Griesemer for all he has taught them in the course of joint work on [FGS1]. The ideas of this paper are fundamental for the present paper. We also thank Alessandro Pizzo for sharing his important insights with us; (see [CFP2, FP]). J.Fa., I.M.S., and T.C. are grateful to J.Fr. for hospitality at ETH Zürich. T.C. thanks I.M.S. for hospitality at the University of Toronto. The authors acknowledge the support of the Oberwolfach Institute. Part of this work was done during I.M.S.'s stay at the IAS, Princeton. The research of I.M.S. has been supported by NSERC under Grant NA 7901. T.C. has been supported by the NSF under grant DMS-070403/DMS-0940145.

2. Limiting absorption principle outside a neighborhood of the ground state energy

In this section we shall prove Theorem 1.1 for any interval J of the form

$$J = J_\sigma^> := E(P) + [\sigma, 2\sigma],$$

where the parameter σ is chosen to satisfy $\sigma \ge \mathrm{C}_0\alpha^{\frac{1}{2}}$, for some fixed positive constant C_0. Our proof is based on the standard Mourre theory ([Mo]), the conjugate operator

B being chosen as the generator of dilatations on $\mathcal{F}$, i.e.,

$$B := \mathrm{d}\Gamma(b), \quad \text{with} \quad b := \frac{\mathrm{i}}{2}(k \cdot \nabla_k + \nabla_k \cdot k). \tag{2.1}$$

One can verify that

$$[H_f, \mathrm{i}B] = H_f, \tag{2.2}$$

in the sense of quadratic forms on $D(H_f) \cap D(B)$, and that, for $j \in \{1, 2, 3\}$,

$$[\mathrm{d}\Gamma(k_j), \mathrm{i}B] = \mathrm{d}\Gamma(k_j), \tag{2.3}$$

in the sense of quadratic forms on $D(\mathrm{d}\Gamma(k_j)) \cap D(B)$. Likewise, for any $f \in D(b)$,

$$[\Phi(f), \mathrm{i}B] = -\Phi(\mathrm{i}bf) \tag{2.4}$$

in the sense of quadratic forms on $D(\Phi(f)) \cap D(B)$. Here

$$\Phi(h) := \frac{1}{\sqrt{2}}(a^*(h) + a(h)), \tag{2.5}$$

where, as usual, for any $h \in \mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, we set

$$a^*(h) := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} h(k, \lambda) a^*_\lambda(k) \mathrm{d}k, \quad a(h) := \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \bar{h}(k, \lambda) a_\lambda(k) \mathrm{d}k, \tag{2.6}$$

so that

$$A = \Phi(h), \quad h(k, \lambda) := \frac{\kappa^\Lambda(k)}{|k|^{\frac{1}{2}}} \varepsilon_\lambda(k). \tag{2.7}$$

We recall that our choice of the polarization vectors $\varepsilon_\lambda(k)$ implies that $k \cdot \nabla_k \varepsilon_\lambda(k) = 0$.

Theorem 2.1. *There exist constants $\alpha_0 > 0$ and $\mathrm{C}_0 > 0$ such that, for all $|P| \le p_{\mathrm{c}}$, $0 \le \alpha \le \alpha_0$ and $\sigma \ge \mathrm{C}_0 \alpha^{1/2}$,*

$$\mathbb{1}_{J^>_\sigma}(H(P)) [H(P), \mathrm{i}B] \mathbb{1}_{J^>_\sigma}(H(P)) \ge \frac{\sigma}{2} \mathbb{1}_{J^>_\sigma}(H(P)). \tag{2.8}$$

Proof. Note that $H(P)$ can be written as

$$\begin{aligned} H(P) = &\frac{1}{2} P^2 + \frac{1}{2} P_f^2 + H_f - P \cdot P_f + \alpha^{\frac{1}{2}} P \cdot \Phi(h) \\ &- \frac{\alpha^{\frac{1}{2}}}{2} \big(\Phi(h) \cdot P_f + P_f \cdot \Phi(h) \big) + \frac{\alpha}{2} \Phi(h)^2. \end{aligned} \tag{2.9}$$

It follows from (2.2), (2.3) and (2.4) that

$$\begin{aligned} [H(P), \mathrm{i}B] = &-\frac{1}{2} \big(P - P_f + \alpha^{\frac{1}{2}} \Phi(h) \big) \cdot \big(P_f + \alpha^{\frac{1}{2}} \Phi(\mathrm{i}bh) \big) \\ &- \frac{1}{2} \big(P_f + \alpha^{\frac{1}{2}} \Phi(\mathrm{i}bh) \big) \cdot \big(P - P_f + \alpha^{\frac{1}{2}} \Phi(h) \big) + H_f, \end{aligned} \tag{2.10}$$

in the sense of quadratic forms on $D(H(P))\cap D(B)$. Since $D(H(P)) = D(P_f^2/2+H_f)$, one can check, in the same way as in [FGS1, Proposition 9], that for all $t \in \mathbb{R}$,

$$e^{\mathrm{i}tB}D(H(P)) \subset D(H(P)). \tag{2.11}$$

Hence $D(H(P)) \cap D(B)$ is a core for $H(P)$ and (2.10) extends by continuity to an identity between quadratic forms on $D(H(P))$. Now, by (2.9), we get

$$\begin{aligned}[H(P), \mathrm{i}B] \geq & H(P) - \frac{1}{2}P^2 - \alpha^{\frac{1}{2}} P \cdot \big(\Phi(h) + \Phi(\mathrm{i}bh)\big) - \frac{\alpha}{2}\Phi(h)^2 \\ &+ \frac{\alpha^{\frac{1}{2}}}{2}\big(\Phi(\mathrm{i}bh)\cdot(P_f - \alpha^{\frac{1}{2}}\Phi(h)) + (P_f - \alpha^{\frac{1}{2}}\Phi(h))\cdot\Phi(\mathrm{i}bh)\big).\end{aligned} \tag{2.12}$$

Multiplying both sides of Inequality (2.12) by $\mathbb{1}_{J_\sigma^>}(H(P))$, using in particular that P_f, $\Phi(h)$ and $\Phi(\mathrm{i}bh)$ are $H(P)$-bounded, this yields

$$\mathbb{1}_{J_\sigma^>}(H(P))[H(P), \mathrm{i}B]\mathbb{1}_{J_\sigma^>}(H(P)) \geq \big(E(P) - \frac{1}{2}P^2 + \sigma - \mathrm{C}\alpha^{\frac{1}{2}}\big)\mathbb{1}_{J_\sigma^>}(H(P)). \tag{2.13}$$

Since $|E(P) - P^2/2| \leq \mathrm{C}'\alpha$ (see Proposition 3.1), we obtain

$$\begin{aligned}\mathbb{1}_{J_\sigma^>}(H(P))[H(P), \mathrm{i}B]\mathbb{1}_{J_\sigma^>}(H(P)) &\geq \big(\sigma - \mathrm{C}''\alpha^{\frac{1}{2}}\big)\mathbb{1}_{J_\sigma^>}(H(P)) \\ &\geq \frac{\sigma}{2}\mathbb{1}_{J_\sigma^>}(H(P)),\end{aligned} \tag{2.14}$$

provided that $\sigma \geq \mathrm{C}_0\alpha^{1/2}$, the constant C_0 being chosen sufficiently large. □

Corollary 2.2. *There exists $\alpha_0 > 0$ such that, for any $|P| \leq p_c$, $0 \leq \alpha \leq \alpha_0$ and $1/2 < s \leq 1$, and for any compact interval $J \subset [E(P) + \mathrm{C}_0\alpha^{1/2}, \infty)$,*

$$\sup_{z \in J_\pm} \big\| \langle B \rangle^{-s} \big[H(P) - z\big]^{-1} \langle B \rangle^{-s} \big\| < \infty. \tag{2.15}$$

Here $\mathrm{C}_0 > 0$ is given by Theorem 2.1. Moreover, the map

$$J \ni \lambda \mapsto \langle B \rangle^{-s}\big[H(P) - \lambda \pm \mathrm{i}0^+\big]^{-1}\langle B \rangle^{-s} \in B(\mathcal{H}) \tag{2.16}$$

is uniformly Hölder continuous in λ of order $s - 1/2$.

Proof. Using the well-known conjugate operator method (see [Mo], [ABG]), it suffices to show that $H(P) \in \mathrm{C}^2(B)$. Since (2.11) holds, in order to obtain the C^2-property of $H(P)$ with respect to B, it is sufficient to verify that $[H(P), \mathrm{i}B]$ and $[[H(P), \mathrm{i}B], \mathrm{i}B]$ extend to $H(P)$-bounded operators. This follows easily from the expression of the commutator of $H(P)$ with $\mathrm{i}B$, Equation (2.10), and by computing similarly the double commutator $[[H(P), \mathrm{i}B], \mathrm{i}B]$. □

Corollary 2.3. *Under the conditions of Corollary 2.2,*

$$\sup_{z \in J_\pm} \big\| (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s}\big[H(P) - z\big]^{-1}(\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s}\big\| < \infty, \tag{2.17}$$

and the map

$$J \ni \lambda \mapsto (\mathrm{d}\Gamma(\langle y\rangle)+1)^{-s}\big[H(P)-\lambda \pm \mathrm{i}0\big]^{-1}(\mathrm{d}\Gamma(\langle y\rangle)+1)^{-s} \in B(\mathcal{H}) \tag{2.18}$$

is uniformly Hölder continuous in λ of order $s-1/2$.

Proof. Let $\phi \in \mathrm{C}_0^\infty(\mathbb{R};[0,1])$ be such that $\phi = 1$ on a neighborhood of J. Let $\bar\phi = 1-\phi$. It follows from the spectral theorem that

$$\sup_{z\in J_\pm} \big\|\bar\phi(H(P))\big[H(P)-z\big]^{-1}\big\| < \infty. \tag{2.19}$$

Therefore, to establish (2.17), it suffices to prove that

$$\sup_{z\in J_\pm} \big\|(\mathrm{d}\Gamma(\langle y\rangle)+1)^{-s}\phi(H(P))\big[H(P)-z\big]^{-1}(\mathrm{d}\Gamma(\langle y\rangle)+1)^{-s}\big\| < \infty. \tag{2.20}$$

Let us show that

$$\big\|(\mathrm{d}\Gamma(\langle y\rangle)+1)^{-1}\phi(H(P))B\big\| < \infty. \tag{2.21}$$

Since $[H(P),\mathrm{i}B]$ extends to an $H(P)$-bounded operator (see (2.10)), an easy application of the Helffer-Sjöstrand functional calculus shows that $[\phi(H(P)),\mathrm{i}B]$ extends to a bounded operator on $\mathcal{F}$. Moreover, considering the restriction of the operator below to all n-particles subspaces of the Fock space, one verifies that

$$\big\|(\mathrm{d}\Gamma(\langle y\rangle)+1)^{-1}B(H_f+1)^{-1}\big\| < \infty. \tag{2.22}$$

Since H_f is relatively bounded with respect to $H(P)$, it follows that $(\mathrm{d}\Gamma(\langle y\rangle)+1)^{-1}B\phi(H(P))$ extends to a bounded operator on $\mathcal{F}$. Hence, writing

$$\begin{aligned}(\mathrm{d}\Gamma(\langle y\rangle)+1)^{-1}\phi(H(P))B &= (\mathrm{d}\Gamma(\langle y\rangle)+1)^{-1}[\phi(H(P)),B] \\ &\quad + (\mathrm{d}\Gamma(\langle y\rangle)+1)^{-1}B\phi(H(P)),\end{aligned} \tag{2.23}$$

this proves (2.21). Now, using an interpolation argument, (2.21) implies that

$$\big\|(\mathrm{d}\Gamma(\langle y\rangle)+1)^{-s}\phi(H(P))\langle B\rangle^s\big\| < \infty, \tag{2.24}$$

for any $0 \le s \le 1$. Likewise, if $\tilde\phi \in \mathrm{C}_0^\infty(\mathbb{R};[0,1])$ is such that $\tilde\phi\phi = \phi$, we have that

$$\big\|\langle B\rangle^s\tilde\phi(H(P))(\mathrm{d}\Gamma(\langle y\rangle)+1)^{-s}\big\| < \infty. \tag{2.25}$$

Combining Corollary 2.2 with (2.24) and (2.25), we obtain (2.20), which concludes the proof of (2.17). The Hölder continuity stated in (2.18) follows similarly. □

Henceforth and throughout the remainder of this paper, we assume that

$$\sigma \le \mathrm{C}_0'\alpha^{\frac{1}{2}}, \tag{2.26}$$

where C_0' is a positive constant such that $\mathrm{C}_0' \ge \mathrm{C}_0$ (here C_0 is given by Theorem 2.1).

3. Low energy decomposition

With this section we begin our proof of the LAP in a neighborhood of $E(P)$. Recall the infrared cutoff Hamiltonian $H_\sigma(P)$ we defined for $\sigma \geq 0$,

$$H_\sigma(P) := \frac{1}{2}(P - P_f + \alpha^{\frac{1}{2}} A_\sigma)^2 + H_f, \tag{3.1}$$

where

$$A_\sigma := \Phi(h_\sigma), \quad h_\sigma(k,\lambda) := \frac{\kappa_\sigma^\Lambda(k)}{|k|^{\frac{1}{2}}} \varepsilon_\lambda(k), \tag{3.2}$$

and

$$\kappa_\sigma^\Lambda(k) := \mathbb{1}_{\{|k| \geq \sigma\}}(k) \kappa^\Lambda(k). \tag{3.3}$$

Note that $H_0(P) = H(P)$. Let

$$E_\sigma(P) := \inf \sigma(H_\sigma(P)). \tag{3.4}$$

For $\sigma = 0$ we set $E(P) := E_0(P)$. Let $\mathcal{F}_\sigma := \Gamma_s(\mathrm{L}^2(\{(k,\lambda), |k| \geq \sigma\}))$ and

$$K_\sigma(P) := H_\sigma(P)|_{\mathcal{F}_\sigma}. \tag{3.5}$$

Let $\mathrm{Gap}(H)$ be defined by $\mathrm{Gap}(H) := \inf\{\sigma(H) \backslash \{E(H)\}\} - E(H)$, where $E(H) := \inf\{\sigma(H)\}$, for any self-adjoint and semi-bounded operator H. The following proposition is proven in [Ch, BCFS2, CFP2, FP].

Proposition 3.1. *There exists $\alpha_0 > 0$ such that, for all $0 \leq \alpha \leq \alpha_0$, the following properties hold:*

1) *For all $\sigma > 0$ and $|P| \leq p_\mathrm{c}$,*

$$\mathrm{Gap}(K_\sigma(P)) \geq \rho\sigma, \ \textit{for some } 0 < \rho < 1. \tag{3.6}$$

Moreover $\inf \sigma(K_\sigma(P)) = E_\sigma(P)$ is a non-degenerate (isolated) eigenvalue of $K_\sigma(P)$.

2) *For all $\sigma \geq 0$ and $|P| \leq p_\mathrm{c}$,*

$$\big| E_\sigma(P) - E(P) \big| \leq \mathrm{C}\alpha\sigma, \tag{3.7}$$

where C is a positive constant independent of σ.

3) *For all $\sigma > 0$, the map $P \mapsto E_\sigma(P)$ is twice continuously differentiable on $\{P \in \mathbb{R}^3, |P| \leq p_\mathrm{c}\}$ and satisfies*

$$\Big| E_\sigma(P) - \frac{P^2}{2} \Big| \leq \mathrm{C}\alpha, \quad \big| \nabla E_\sigma(P) - P \big| \leq \mathrm{C}\alpha, \tag{3.8}$$

$$\big| \nabla E_\sigma(P) - \nabla E_\sigma(P') \big| \leq \mathrm{C}|P - P'|, \quad \textit{for all } |P|, |P'| \leq p_\mathrm{c}, \tag{3.9}$$

where C is a positive constant independent of σ.

4) *For all* $\sigma \geq 0$, $|P| \leq p_{\rm c}$ *and* $k \in \mathbb{R}^3$,

$$E_\sigma(P-k) \geq E_\sigma(P) - \frac{1}{3}|k|. \tag{3.10}$$

We fix $P \in \mathbb{R}^3$ and, to simplify notations, we drop, from now on, the dependence on P everywhere unless some confusion may arise. Note that the Hilbert space $\mathcal{F}$ is isometric to $\mathcal{F}_\sigma \otimes \mathcal{F}^\sigma$ where $\mathcal{F}^\sigma := \Gamma_s(\mathrm{L}^2(\{(k,\lambda), |k| \leq \sigma\}))$. In this representation, we have that

$$H_\sigma = K_\sigma \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + H_f\big) - \nabla K_\sigma \otimes P_f, \tag{3.11}$$

where we use (with obvious abuse of notation) that $P_f = P_f \otimes \mathbb{1} + \mathbb{1} \otimes P_f$, $H_f = H_f \otimes \mathbb{1} + \mathbb{1} \otimes H_f$ and $A_\sigma = A_\sigma \otimes \mathbb{1}$, and where we use the notation

$$\nabla K_\sigma := \nabla H_\sigma|_{\mathcal{F}_\sigma}, \quad \text{with} \quad \nabla H_\sigma := P - P_f + \alpha^{\frac{1}{2}} A_\sigma. \tag{3.12}$$

In conclusion of this section we mention the decomposition

$$H = H_\sigma + U_\sigma, \tag{3.13}$$

where

$$U_\sigma := \alpha^{\frac{1}{2}} \nabla K_\sigma \otimes A^\sigma - \frac{\alpha^{\frac{1}{2}}}{2} \mathbb{1} \otimes \Big(A^\sigma \cdot P_f + P_f \cdot A^\sigma\Big) + \frac{\alpha}{2} \mathbb{1} \otimes (A^\sigma)^2, \tag{3.14}$$

and

$$A^\sigma := \Phi(h^\sigma), \quad h^\sigma(k,\lambda) := h(k,\lambda) - h_\sigma(k,\lambda) = \frac{\mathbb{1}_{\{|k| \leq \sigma\}}(k)}{|k|^{\frac{1}{2}}} \varepsilon_\lambda(k). \tag{3.15}$$

4. Feshbach-Schur operator

In this section we use the "smooth Feshbach-Schur map", F_χ, introduced in [BCFS1] to map the operators $H - \lambda$ onto more tractable operators. Define

$$\chi_f^\sigma := \chi_f^\sigma(H_f) \equiv \kappa^{\rho\sigma}(H_f), \quad \bar{\chi}_f^\sigma := \sqrt{\mathbb{1} - (\chi_f^\sigma)^2}, \tag{4.1}$$

with $\kappa^{\rho\sigma}$ as defined in (1.6), ρ the same as in (3.6), and

$$\chi := P_\sigma \otimes \chi_f^\sigma, \quad \bar{\chi} := P_\sigma \otimes \bar{\chi}_f^\sigma + \bar{P}_\sigma \otimes \mathbb{1}, \tag{4.2}$$

where

$$P_\sigma := \mathbb{1}_{\{E_\sigma\}}(K_\sigma) \text{ and } \bar{P}_\sigma := \mathbb{1} - P_\sigma. \tag{4.3}$$

Note that $\chi^2 + \bar{\chi}^2 = \mathbb{1}$ and $[\chi, \bar{\chi}] = 0$.

To define the smooth Feshbach-Schur map F_χ for $H - \lambda$, we have to choose an "unperturbed" operator - we call it T - around which we construct our perturbation theory (see Appendix B). It is tempting to choose it as $T = H_\sigma - \lambda$. However this choice is not suitable, since, due to the term $-\nabla K_\sigma \otimes P_f$ in H_σ (see Equation (3.11)), the commutator $[H_\sigma, \chi]$ does not vanish; (hence Hypothesis (1) of Appendix B is not satisfied). Another choice would be $T = H_\sigma + \nabla K_\sigma \otimes P_f - \lambda$. However, as far as the Mourre estimate of Section 5 is concerned, this choice does not work either, since it gives rise to “perturbation” terms of order $O(\sigma)$ in $F_\chi(H-\lambda)$, that is the same order as the leading order terms in $F_\chi(H-\lambda)$.

To circumvent this difficulty, we set $T_\sigma := H_\sigma + (\nabla K_\sigma - \nabla E_\sigma) \otimes P_f$, that is

$$T_\sigma = K_\sigma \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2} P_f^2 + H_f\big) - \nabla E_\sigma \otimes P_f. \tag{4.4}$$

Notice that $[\chi, T_\sigma] = 0$, and that

$$H = T_\sigma + W_\sigma, \quad \text{where} \quad W_\sigma := U_\sigma - (\nabla K_\sigma - \nabla E_\sigma) \otimes P_f. \tag{4.5}$$

Using the Feynman-Hellman formula, we shall see in the following that the term $(\nabla K_\sigma - \nabla E_\sigma) \otimes P_f$ can indeed be treated as a perturbation, and leads to terms of order $O(\alpha^{1/2}\sigma)$ in $F_\chi(H-\lambda)$; (see Lemmata 5.6, 5.7 and 5.8).

On operators of the form $H - \lambda$ we introduce the Feshbach-Schur map (see Appendix B):

$$F_\chi(H-\lambda) = T_\sigma - \lambda + \chi W_\sigma \chi - \chi W_\sigma \bar{\chi} \big[H_{\bar{\chi}} - \lambda \big]^{-1} \bar{\chi} W_\sigma \chi, \tag{4.6}$$

where (cf. Appendix B)

$$H_{\bar{\chi}} := T_\sigma + \bar{\chi} W_\sigma \bar{\chi}. \tag{4.7}$$

This family is well-defined as follows from the fact that the operators χW_σ and $W_\sigma \chi$ are bounded and from Proposition 4.1 below. The Feynman-Hellman formula says that $P_\sigma \nabla K_\sigma P_\sigma = \nabla E_\sigma P_\sigma$ and hence $\chi W_\sigma \chi = \chi U_\sigma \chi$. Thus Equations (4.4) and (4.6) imply

$$\begin{aligned} F_\chi(H-\lambda) =& K_\sigma \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2} P_f^2 + H_f\big) - \nabla E_\sigma \otimes P_f - \lambda \\ &+ \chi U_\sigma \chi - \chi W_\sigma \bar{\chi} \big[H_{\bar{\chi}} - \lambda \big]^{-1} \bar{\chi} W_\sigma \chi. \end{aligned} \tag{4.8}$$

Proposition 4.1. *For any* $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for all* $|P| \le p_{\mathrm{c}}$*,* $0 \le \alpha \le \alpha_0$ *and* $0 < \sigma \le \mathrm{C}_0 \alpha^{1/2}$*, for all* $\lambda \le E_\sigma + \rho\sigma/4$*,* $H_{\bar{\chi}} - \lambda$ *is bounded invertible on* $\mathrm{Ran}(\bar{\chi})$ *and*

$$\big\| \bar{\chi} \big[H_{\bar{\chi}} - \lambda \big]^{-1} \bar{\chi} \big\| \le \mathrm{C} \sigma^{-1}, \tag{4.9}$$

$$\big\| \bar{\chi} \big[H_{\bar{\chi}} - \lambda \big]^{-1} \bar{\chi} W_\sigma \chi \big\| \le \mathrm{C}. \tag{4.10}$$

Proof. By (4.5), the perturbation W_σ consists of two terms. As a first step in the proof of Proposition 4.1, we focus on the term $(\nabla K_\sigma - \nabla E_\sigma) \otimes P_f$, which is analyzed in the following lemma.

Lemma 4.2. *Let*

$$H^1_{\bar\chi} := T_\sigma - \bar\chi(\nabla K_\sigma - \nabla E_\sigma) \otimes P_f \bar\chi \,. \tag{4.11}$$

For any $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for all* $|P| \leq p_{\mathrm{c}}$*,* $0 \leq \alpha \leq \alpha_0$ *and* $0 < \sigma \leq \mathrm{C}_0 \alpha^{1/2}$*, for all* $\lambda \leq E_\sigma + \rho\sigma/4$*,* $H^1_{\bar\chi} - \lambda$ *is bounded invertible on* $\mathrm{Ran}(\bar\chi)$ *and*

$$\big\| \bar\chi \big[H^1_{\bar\chi} - \lambda \big]^{-1} \bar\chi \big\| \leq \mathrm{C} \sigma^{-1}, \tag{4.12}$$

$$\big\| \bar\chi \big[H^1_{\bar\chi} - \lambda \big]^{-1} \bar\chi (\nabla K_\sigma - \nabla E_\sigma) \otimes P_f \chi \big\| \leq \mathrm{C}. \tag{4.13}$$

Proof. Let $\Phi = \bar\chi \Psi \in D(H_\sigma) \cap \mathrm{Ran}(\bar\chi)$, $\|\Phi\| = 1$. Let us first prove that

$$(\Phi, H_\sigma \Phi) \geq E_\sigma + \frac{3}{8} \rho\sigma. \tag{4.14}$$

We decompose

$$(\Phi, H_\sigma \Phi) = (\Phi, H_\sigma (\mathbb{1} \otimes \mathbb{1}_{H_f \leq 3\rho\sigma/4}) \Phi) + (\Phi, H_\sigma (\mathbb{1} \otimes \mathbb{1}_{H_f \geq 3\rho\sigma/4}) \Phi), \tag{4.15}$$

and use that $\Phi = \bar\chi \Psi = (\bar P_\sigma \otimes \mathbb{1}) \Psi + (P_\sigma \otimes \bar\chi^\sigma_f) \Psi$. Using Lemma A.4 and the fact that $\mathbb{1}_{H_f \leq 3\rho\sigma/4} \, \bar\chi^\sigma_f = 0$, we obtain that

$$\begin{aligned}(\Phi, H_\sigma (\mathbb{1} \otimes \mathbb{1}_{H_f \leq 3\rho\sigma/4}) \Phi) &\geq (1 - \frac{3}{4}\rho\sigma)(\Phi, K_\sigma \otimes \mathbb{1} (\mathbb{1} \otimes \mathbb{1}_{H_f \leq 3\rho\sigma/4}) \Phi) \\ &= (1 - \frac{3}{4}\rho\sigma)((\bar P_\sigma \otimes \mathbb{1})\Psi, K_\sigma \otimes \mathbb{1} (\bar P_\sigma \otimes \mathbb{1}_{H_f \leq 3\rho\sigma/4}) \Psi).\end{aligned} \tag{4.16}$$

Since, by Proposition 3.1, $\bar P_\sigma K_\sigma \bar P_\sigma \geq E_\sigma + \rho\sigma$, this implies that

$$\begin{aligned}(\Phi, H_\sigma (\mathbb{1} \otimes \mathbb{1}_{H_f \leq 3\rho\sigma/4}) \Phi) &\geq (1 - \frac{3}{4}\rho\sigma)(E_\sigma + \rho\sigma)(\Phi, (\mathbb{1} \otimes \mathbb{1}_{H_f \leq 3\rho\sigma/4}) \Phi) \\ &\geq (E_\sigma + \frac{3}{8}\rho\sigma)(\Phi, (\mathbb{1} \otimes \mathbb{1}_{H_f \leq 3\rho\sigma/4}) \Phi).\end{aligned} \tag{4.17}$$

Note that in the last inequality we used that, by Proposition 3.1, $E_\sigma \leq 1/100$ for $|P| \leq 1/40$ and α sufficiently small. The second term on the right-hand side of (4.15) is estimated with the help of Lemma A.3, which gives:

$$\begin{aligned}(\Phi, H_\sigma (\mathbb{1} \otimes \mathbb{1}_{H_f \geq 3\rho\sigma/4}) \Phi) &\geq E_\sigma + \frac{1}{2}(\Phi, (\mathbb{1} \otimes H_f)(\mathbb{1} \otimes \mathbb{1}_{H_f \geq 3\rho\sigma/4}) \Phi) \\ &\geq (E_\sigma + \frac{3}{8}\rho\sigma)(\Phi, (\mathbb{1} \otimes \mathbb{1}_{H_f \geq 3\rho\sigma/4}) \Phi).\end{aligned} \tag{4.18}$$

Hence (4.14) is proven.

From the definition of $H^1_{\bar\chi}$, we infer that

$$\begin{aligned} H^1_{\bar\chi} =& H_\sigma + (\nabla K_\sigma - \nabla E_\sigma)\otimes P_f - \bar\chi(\nabla K_\sigma - \nabla E_\sigma)\otimes P_f \bar\chi \\ =& H_\sigma + \big(P_\sigma \otimes (\bar\chi^\sigma_f - \mathbb{1})\big)\nabla K_\sigma \otimes P_f \big(\bar P_\sigma \otimes \mathbb{1}\big) \\ &+ \big(\bar P_\sigma \otimes \mathbb{1}\big)\nabla K_\sigma \otimes P_f \big(P_\sigma \otimes (\bar\chi^\sigma_f - \mathbb{1})\big) \end{aligned} \tag{4.19}$$

where we used that $\bar\chi = P_\sigma \otimes (\bar\chi^\sigma_f - \mathbb{1}) + \mathbb{1}\otimes\mathbb{1}$, and

$$\begin{aligned} &\big(\mathbb{1}\otimes\mathbb{1}\big)(\nabla K_\sigma - \nabla E_\sigma)\otimes P_f\big(P_\sigma\otimes(\bar\chi^\sigma_f - \mathbb{1})\big) \\ &= \big(\bar P_\sigma \otimes \mathbb{1}\big)\nabla K_\sigma \otimes P_f\big(P_\sigma \otimes (\bar\chi^\sigma_f - \mathbb{1})\big). \end{aligned} \tag{4.20}$$

Equation (4.20) follows from the Feynman-Hellman formula, $P_\sigma \nabla K_\sigma P_\sigma = \nabla E_\sigma P_\sigma$, and orthogonality, $P_\sigma \bar P_\sigma = 0$. By Proposition 3.1, for $|P| \le p_{\rm c} = 1/40$ and α sufficiently small,

$$\big\| \nabla K_\sigma P_\sigma \big\|^2 \le 2E_\sigma \le P^2 + \mathrm{C}\alpha \le \frac{1}{36^2}. \tag{4.21}$$

Thus, when combined with

$$\|P_f(\bar\chi^\sigma_f - \mathbb{1})\| \le 2\|H_f(\bar\chi^\sigma_f - \mathbb{1})\| \le 2\rho\sigma \tag{4.22}$$

and (4.14), Equations (4.19)–(4.21) imply that

$$(\Phi, H^1_{\bar\chi}\Phi) \ge E_\sigma + (\frac{3}{8} - \frac{1}{9})\rho\sigma = E_\sigma + \frac{19}{72}\rho\sigma, \tag{4.23}$$

provided that α is sufficiently small. This implies that $H^1_{\bar\chi} - \lambda$ is bounded invertible for any $\lambda \le E_\sigma + \rho\sigma/4$, and leads to (4.12). To obtain (4.13), it suffices to combine (4.12) with (4.21) and the fact that $\|P_f \chi^\sigma_f\| \le \mathrm{C}\sigma$. □

We now return to the proof of Proposition 4.1. Using the operator $H^1_{\bar\chi}$ introduced in the statement of Lemma 4.2, we have that

$$H_{\bar\chi} = H^1_{\bar\chi} + \bar\chi U_\sigma \bar\chi. \tag{4.24}$$

Consider the Neumann series

$$\bar\chi\big[H_{\bar\chi} - \lambda\big]^{-1}\bar\chi = \bar\chi\big[H^1_{\bar\chi} - \lambda\big]^{-1}\sum_{n\ge0}\Big(- \bar\chi U_\sigma \bar\chi\big[H^1_{\bar\chi} - \lambda\big]^{-1}\Big)^n \bar\chi. \tag{4.25}$$

We claim that

$$\big\| \big[H^1_{\bar\chi} - \lambda\big]^{-\frac12}\bar\chi U_\sigma \bar\chi \big[H^1_{\bar\chi} - \lambda\big]^{-\frac12}\bar\chi \big\| \le \mathrm{C}\alpha^{\frac12}. \tag{4.26}$$

Indeed, inserting the expression (3.14) of U_σ into the left-hand side of (4.26), we obtain three terms: The first one is given by

$$\big\| \alpha^{\frac12}\big[H^1_{\bar\chi} - \lambda\big]^{-\frac12}\bar\chi \nabla K_\sigma \otimes A^\sigma \bar\chi \big[H^1_{\bar\chi} - \lambda\big]^{-\frac12}\bar\chi \big\|. \tag{4.27}$$

It follows from Lemmata A.1, A.3 and 4.2 that

$$\big\|(\mathbb{1} \otimes a(h^\sigma))\bar{\chi}\big[H^1_{\bar{\chi}} - \lambda\big]^{-\frac{1}{2}}\bar{\chi}\big\| \leq \mathrm{C}\sigma^{\frac{1}{2}}. \tag{4.28}$$

Using in addition that, by Lemma 4.2,

$$\big\|(\nabla K_\sigma \otimes \mathbb{1})\bar{\chi}\big[H^1_{\bar{\chi}} - \lambda\big]^{-\frac{1}{2}}\bar{\chi}\big\| \leq \mathrm{C}\sigma^{-\frac{1}{2}}, \tag{4.29}$$

we get $\|(4.27)\| \leq \mathrm{C}\alpha^{1/2}$. The second and third terms from (3.14) are estimated similarly, which leads to (4.26). Together with (4.12) from Lemma 4.2, this implies that, for any $n \in \mathbb{N}$,

$$\Big\|\bar{\chi}\big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\Big(- \bar{\chi}U_\sigma\bar{\chi}\big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\Big)^n\bar{\chi}\Big\| \leq \mathrm{C}\sigma^{-1}(\mathrm{C}'\alpha^{\frac{1}{2}})^n. \tag{4.30}$$

Hence, for α sufficiently small, the right-hand-side of (4.25) is convergent and (4.9) holds. Estimate (4.10) follows similarly. □

5. Mourre estimate for the Feshbach-Schur operator

In this section we shall prove Theorem 1.1 in the case where

$$J = J_\sigma^< := [E(P) + 11\rho\sigma/128, E(P) + 13\rho\sigma/128],$$

and σ is such that $\sigma \leq \mathrm{C}_0\alpha^{1/2}$. We shall begin with proving a limiting absorption principle for the Feshbach-Schur operator

$$F(\lambda) := F_\chi(H - \lambda)|_{\mathrm{Ran}(P_\sigma \otimes \mathbb{1})}, \tag{5.1}$$

defined in (4.6), Section 4. Note that the operator $F(\lambda)$ is self-adjoint $\forall \lambda \in J_\sigma^<$. Here the parameter λ shall be fixed such that $\lambda \in J_\sigma^<$ and we shall prove a LAP for $F(\lambda)$ on the interval Δ_σ defined in this section by

$$\Delta_\sigma = [-\rho\sigma/128, \rho\sigma/128]. \tag{5.2}$$

Then we shall deduce a limiting absorption principle for H near the ground state energy E by applying Theorem B.2.

We begin with showing the Mourre estimate for $F(\lambda)$, $\lambda \in J_\sigma^<$.

Recall that κ^σ denotes a function in $\mathrm{C}_0^\infty(\{k, |k| \leq \sigma\}; [0,1])$ such that $\kappa^\sigma = 1$ on $\{k, |k| \leq 3\sigma/4\}$. The conjugate operator we shall use in this section is the operator B^σ, defined by:

$$B^\sigma = \mathrm{d}\Gamma(b^\sigma), \quad \text{with} \quad b^\sigma = \kappa^\sigma b \kappa^\sigma. \tag{5.3}$$

Clearly, B^σ acts on the second component of the tensor product $\mathcal{F}_\sigma \otimes \mathcal{F}^\sigma$. The main theorem of this section is:

Theorem 5.1. *For any* $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for all* $|P| \leq p_{\mathrm{c}}$*,* $0 \leq \alpha \leq \alpha_0$*,* $0 < \sigma \leq \mathrm{C}_0 \alpha^{1/2}$*, and* $\lambda \in J_\sigma^<$*,*

$$\mathbb{1}_{\Delta_\sigma}(F(\lambda))[F(\lambda), \mathrm{i}B^\sigma]\mathbb{1}_{\Delta_\sigma}(F(\lambda)) \geq \frac{\rho\sigma}{128}\mathbb{1}_{\Delta_\sigma}(F(\lambda)). \tag{5.4}$$

Before proceeding to the proof of this theorem we draw the desired conclusions from it.

Proposition 5.2. *For any* $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for any* $|P| \leq p_{\mathrm{c}}$*,* $0 \leq \alpha \leq \alpha_0$*,* $0 < \sigma \leq \mathrm{C}_0\alpha^{1/2}$*,* $1/2 < s \leq 1$*, and* $\lambda \in J_\sigma^<$*,*

$$\sup_{z \in (\Delta_\sigma)_\pm} \left\| \langle B^\sigma \rangle^{-s} \big[F(\lambda) - z \big]^{-1} \langle B^\sigma \rangle^{-s} \right\| < \infty. \tag{5.5}$$

Here $(\Delta_\sigma)_\pm = \{z \in \mathbb{C}, \mathrm{Re} z \in [-\rho\sigma/128, \rho\sigma/128], 0 < \pm \mathrm{Im} z \leq 1\}$*. Moreover, the map*

$$J_\sigma^< \times \Delta_\sigma \ni (\lambda, \mu) \mapsto \langle B^\sigma \rangle^{-s} \big[F(\lambda) - \mu \pm \mathrm{i}0^+ \big]^{-1} \langle B^\sigma \rangle^{-s} \in B(\mathcal{H}) \tag{5.6}$$

is uniformly Hölder continuous in (λ, μ) *of order* $s - 1/2$*.*

Proof. It follows from Equations (4.4) and (4.6) that

$$\begin{aligned} F(\lambda) =& \mathbb{1} \otimes \big(\frac{1}{2} P_f^2 + H_f \big) - \nabla E_\sigma \otimes P_f + E_\sigma - \lambda, \\ &+ \chi W_\sigma \chi - \chi W_\sigma \bar{\chi} \big[H_{\bar{\chi}} - \lambda \big]^{-1} \bar{\chi} W_\sigma \chi. \end{aligned} \tag{5.7}$$

By standard Mourre theory (see for instance [ABG]) and in view of Theorem 5.1, the limiting absorption principle (5.5) and the Hölder continuity in μ follow from the fact that $F(\lambda) \in \mathrm{C}^2(B^\sigma)$. Since χW_σ and $W_\sigma \chi$ are bounded operators, it follows that $D(F(\lambda)) = D(\mathbb{1} \otimes (\frac{1}{2}P_f^2 + H_f))$, and, using the method of [FGS1, Proposition 9], one verifies that

$$e^{\mathrm{i}tB^\sigma} D(\mathbb{1} \otimes (\frac{1}{2}P_f^2 + H_f)) \subset D(\mathbb{1} \otimes (\frac{1}{2}P_f^2 + H_f)), \tag{5.8}$$

for all $t \in \mathbb{R}$. Hence it suffices to show that $[F(\lambda), \mathrm{i}B^\sigma]$ and $[[F(\lambda), \mathrm{i}B^\sigma], \mathrm{i}B^\sigma]$ are bounded with respect to $\mathbb{1} \otimes (\frac{1}{2}P_f^2 + H_f)$, which follows easily from the expressions of the commutators; (see, in particular, the proofs of Lemmata 5.5 and 5.8).

Now, for $\lambda, \lambda' \in J_\sigma^<$, we have

$$F(\lambda) - F(\lambda') = (\lambda' - \lambda)\big(P_\sigma \otimes \mathbb{1} + \chi W_\sigma \bar{\chi} \big[H_{\bar{\chi}} - \lambda \big]^{-1} \big[H_{\bar{\chi}} - \lambda' \big]^{-1} \bar{\chi} W_\sigma \chi \big). \tag{5.9}$$

Equation (4.10) in the statement of Proposition 4.1 implies that

$$\left\| \chi W_\sigma \bar{\chi} \big[H_{\bar{\chi}} - \lambda \big]^{-1} \big[H_{\bar{\chi}} - \lambda' \big]^{-1} \bar{\chi} W_\sigma \chi \right\| \leq \mathrm{C}, \tag{5.10}$$

where C is independent of λ and λ'. Thus, the Hölder continuity in (λ, μ) stated in (5.6) follows again by standard arguments of Mourre theory (see [PSS, AHS, HS]). $\square$

This proposition and Theorem B.2 imply the following

Corollary 5.3. *For any $\mathrm{C}_0 > 0$, there exists $\alpha_0 > 0$ such that, for any $|P| \leq p_c$, $0 \leq \alpha \leq \alpha_0$, $0 < \sigma \leq \mathrm{C}_0 \alpha^{1/2}$ and $1/2 < s \leq 1$,*

$$\sup_{z \in (J_\sigma^<)_\pm} \big\| \langle B^\sigma \rangle^{-s} \big[H(P) - z \big]^{-1} \langle B^\sigma \rangle^{-s} \big\| < \infty, \tag{5.11}$$

where $(J_\sigma^<)_\pm = \{ z \in \mathbb{C}, \mathrm{Re} z \in [E(P) + 11\rho\sigma/128, E(P) + 13\rho\sigma/128], 0 < \pm \mathrm{Im} z \leq 1 \}$. Moreover, the map

$$[E(P) + \frac{11\rho\sigma}{128}, E(P) + \frac{13\rho\sigma}{128}] \ni \lambda \mapsto \langle B^\sigma \rangle^{-s} \big[H(P) - \lambda \pm \mathrm{i}0^+ \big]^{-1} \langle B^\sigma \rangle^{-s} \in B(\mathcal{H}) \tag{5.12}$$

is uniformly Hölder continuous in λ of order $s - 1/2$.

By arguments similar to ones used in the proof of Corollary 2.3, Corollary 5.3 implies the following result.

Corollary 5.4. *Under the conditions of Corollary 5.3, for any compact interval $J \subset (E(P), \mathrm{C}_0 \alpha^{\frac{1}{2}}]$,*

$$\sup_{z \in J_\pm} \big\| (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \big[H(P) - z \big]^{-1} (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \big\| < \infty, \tag{5.13}$$

and the map

$$J \ni \lambda \mapsto (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \big[H(P) - \lambda \pm \mathrm{i}0^+ \big]^{-1} (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \in B(\mathcal{H}) \tag{5.14}$$

is uniformly Hölder continuous in λ of order $s - 1/2$.

Now we proceed to the proof of Theorem 5.1. It will be divided into a sequence of Lemmata. In what follows we often do not display the argument λ. First, let us write

$$F = F_0 + W_1 + W_2, \tag{5.15}$$

where

$$F_0 := \mathbb{1} \otimes \big(\frac{1}{2} P_f^2 + H_f \big) - \nabla E_\sigma \otimes P_f + E_\sigma - \lambda, \tag{5.16}$$

$$W_1 := \chi U_\sigma \chi, \ (= \chi W_\sigma \chi \text{ by Feynman-Hellman; see above}) \tag{5.17}$$

$$W_2 := -\chi W_\sigma \bar{\chi} \big[H_{\bar{\chi}} - \lambda \big]^{-1} \bar{\chi} W_\sigma \chi. \tag{5.18}$$

Let us begin by estimating $[F_0, \mathrm{i}B^\sigma]$ from below on the range of $\mathbb{1} \otimes \mathbb{1}_{H_f \leq \delta\rho\sigma}$, for some suitably chosen $\delta > 0$.

Lemma 5.5. *Let* $|P| \le p_c$ *and* $\delta > 0$ *be such that* $\delta\rho\sigma < 3\sigma/4$*. Then on* $\mathrm{Ran}(\mathbb{1} \otimes \mathbb{1}_{H_f \le \delta\rho\sigma})$*,*

$$\big[F_0, \mathrm{i}B^\sigma\big] \ge \frac{1}{2}(\mathbb{1} \otimes H_f) - \mathrm{C}\sigma^2, \tag{5.19}$$

where C *is a positive constant.*

Proof. We have that

$$[H_f, \mathrm{i}B^\sigma] = \mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|), \quad [P_f, \mathrm{i}B^\sigma] = \mathrm{d}\Gamma(\kappa^\sigma(k)^2 k). \tag{5.20}$$

Therefore,

$$\big[F_0, \mathrm{i}B^\sigma\big] = \mathbb{1} \otimes (P_f \cdot \mathrm{d}\Gamma(\kappa^\sigma(k)^2 k) + \mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|)) - \nabla E_\sigma \otimes \mathrm{d}\Gamma(\kappa^\sigma(k)^2 k). \tag{5.21}$$

For $j = 1, 2, 3$, we have

$$\pm\mathrm{d}\Gamma(\kappa^\sigma(k)^2 k_j) \le \mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|) \le \mathbb{1} \otimes H_f, \tag{5.22}$$

so that

$$\begin{aligned}\nabla E_\sigma \otimes \mathrm{d}\Gamma(\kappa^\sigma(k)^2 k) &\ge -(\sum_j |(\nabla E_\sigma)_j|)\mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|) \\ &\ge -2|\nabla E_\sigma|\mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|).\end{aligned} \tag{5.23}$$

Moreover, using again (5.22), it can easily be checked that

$$\mathbb{1} \otimes \big(P_f \cdot \mathrm{d}\Gamma(\kappa^\sigma(k)^2 k)\mathbb{1}_{H_f \le \delta\rho\sigma}\big) \ge -\mathrm{C}\sigma^2. \tag{5.24}$$

Hence Equations (5.21), (5.23) and (5.24) yield

$$\begin{aligned}&\big[F_0, \mathrm{i}B^\sigma\big](\mathbb{1} \otimes \mathbb{1}_{H_f \le \delta\rho\sigma}) \\ &\ge (1 - 2|\nabla E_\sigma|)(\mathbb{1} \otimes \mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|))(\mathbb{1} \otimes \mathbb{1}_{H_f \le \delta\rho\sigma}) - \mathrm{C}\sigma^2(\mathbb{1} \otimes \mathbb{1}_{H_f \le \delta\rho\sigma}) \\ &\ge \frac{1}{2}(\mathbb{1} \otimes \mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|))(\mathbb{1} \otimes \mathbb{1}_{H_f \le \delta\rho\sigma}) - \mathrm{C}\sigma^2(\mathbb{1} \otimes \mathbb{1}_{H_f \le \delta\rho\sigma}).\end{aligned} \tag{5.25}$$

In the second inequality we used that, by Proposition 3.1, $|\nabla E_\sigma| \le |P| + \mathrm{C}\alpha^{\frac{1}{2}} \le 1/4$ for $|P| \le 1/40$ and α sufficiently small. To conclude the proof of the lemma, it remains to justify that the operator $\mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|)$ in (5.25) can be replaced by H_f. To this end, we define

$$\begin{aligned}H^\sigma_{f,3\sigma/4} &= \sum_{\lambda=1,2} \int_{3\sigma/4 \le |k| \le \sigma} |k| a^*_\lambda(k) a_\lambda(k)\mathrm{d}k, \\ N^\sigma_{3\sigma/4} &= \sum_{\lambda=1,2} \int_{3\sigma/4 \le |k| \le \sigma} a^*_\lambda(k) a_\lambda(k)\mathrm{d}k,\end{aligned} \tag{5.26}$$

and $P^\sigma_{3\sigma/4} = \mathbb{1}_{\{0\}}(H^\sigma_{f,3\sigma/4})$, $\bar{P}^\sigma_{3\sigma/4} = \mathbb{1} - P^\sigma_{3\sigma/4}$. Then we have that

$$(\mathbb{1} \otimes H_f)\bar{P}^\sigma_{3\sigma/4} \geq H^\sigma_{f,3\sigma/4}\bar{P}^\sigma_{3\sigma/4} \geq \frac{3\sigma}{4} N^\sigma_{3\sigma/4}\bar{P}^\sigma_{3\sigma/4} \geq \frac{3\sigma}{4}\bar{P}^\sigma_{3\sigma/4}. \tag{5.27}$$

Therefore, since $\mathbb{1} \otimes H_f$ commutes with $P^\sigma_{3\sigma/4}$, we get

$$\begin{aligned}\delta\rho\sigma\bar{P}^\sigma_{3\sigma/4}(\mathbb{1} \otimes \mathbb{1}_{H_f \leq \delta\rho\sigma}) &\geq (\mathbb{1} \otimes H_f)\bar{P}^\sigma_{3\sigma/4}(\mathbb{1} \otimes \mathbb{1}_{H_f \leq \delta\rho\sigma}) \\ &\geq \frac{3\sigma}{4}\bar{P}^\sigma_{3\sigma/4}(\mathbb{1} \otimes \mathbb{1}_{H_f \leq \delta\rho\sigma})\end{aligned} \tag{5.28}$$

and since $\delta\rho\sigma < 3\sigma/4$ by assumption, this implies

$$(\mathbb{1} \otimes \mathbb{1}_{H_f \leq \delta\rho\sigma}) = P^\sigma_{3\sigma/4}(\mathbb{1} \otimes \mathbb{1}_{H_f \leq \delta\rho\sigma}). \tag{5.29}$$

Since $\kappa^\sigma(k) = 1$ for any $|k| \leq 3\sigma/4$, we obtain that

$$\big(\mathbb{1} \otimes \mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|)\big)P^\sigma_{3\sigma/4} = (\mathbb{1} \otimes H_f)P^\sigma_{3\sigma/4}. \tag{5.30}$$

We conclude the proof using (5.25), (5.29), (5.30), and the fact that

$$\mathbb{1} \otimes \mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|) \geq \big(\mathbb{1} \otimes \mathrm{d}\Gamma(\kappa^\sigma(k)^2|k|)\big)P^\sigma_{3\sigma/4}. \tag{5.31}$$

□

The following lemma is an important ingredient in showing Theorem 5.1. It justifies the fact that one can consider the term $(\nabla K_\sigma - \nabla E_\sigma) \otimes P_f$ in W_σ as a small perturbation. The idea of its proof is due to [AFGG], and is based on the C^2-regularity of the map $P \mapsto E_\sigma(P)$ uniformly in σ, established in [Ch] and [FP] (see more precisely inequality (3.9) in Proposition 3.1).

Let $(e_j,\ j = 1,2,3)$ be the canonical orthonormal basis of $\mathbb{R}^3$. For any $y \in \mathbb{R}^3$, we set $y_j = y \cdot e_j$.

Lemma 5.6. *For any $\mathrm{C}_0 > 0$, there exists $\alpha_0 > 0$ such that, for all $|P| \leq p_{\mathrm{c}}$, $0 \leq \alpha \leq \alpha_0$, $0 < \sigma \leq \mathrm{C}_0\alpha^{1/2}$, $\lambda \in J^<_\sigma$, $j \in \{1,2,3\}$, and $0 < \delta \ll 1$,*

$$\Big\| \big[H_{\bar\chi} - \lambda\big]^{-\frac12}\bar\chi\big((\nabla K_\sigma - \nabla E_\sigma)_j P_\sigma\big) \otimes \mathbb{1}_{H_f \leq \delta} \Big\| \leq \mathrm{C}\big(1 + \delta^{\frac12}\sigma^{-\frac12}\big). \tag{5.32}$$

Proof. For any $u > 0$, we can write

$$(\nabla K_\sigma)_j = \frac{1}{u}\big(K_\sigma(P + ue_j) - K_\sigma(P)\big) - \frac{u}{2}. \tag{5.33}$$

Using that $K_\sigma(P)P_\sigma = E_\sigma(P)P_\sigma$, this implies

$$\begin{aligned}(\nabla K_\sigma - \nabla E_\sigma)_j P_\sigma =& \frac{1}{u}(K_\sigma(P + ue_j) - E_\sigma(P + ue_j))P_\sigma \\ &+ \Big(\frac{1}{u}(E_\sigma(P + ue_j) - E_\sigma(P)) - (\nabla E_\sigma)_j - \frac{u}{2}\Big)P_\sigma.\end{aligned} \tag{5.34}$$

By Proposition 3.1,

$$\Big|\frac{1}{u}(E_\sigma(P+ue_j)-E_\sigma(P))-(\nabla E_\sigma)_j\Big|\leq \mathrm{C}u, \tag{5.35}$$

where C is independent of σ. Consequently, it follows from the Feynman-Hellman formula, $P_\sigma(\nabla K_\sigma)_j P_\sigma = (\nabla E_\sigma)_j P_\sigma$, together with Equation (5.33) that, for any $\Phi \in \mathrm{Ran}(P_\sigma)$, $\|\Phi\|=1$,

$$\begin{aligned}
&\big\|(K_\sigma(P+ue_j)-E_\sigma(P+ue_j))^{\frac12}\Phi\big\|^2\\
&=\big(\Phi,(K_\sigma(P+ue_j)-E_\sigma(P+ue_j))\Phi\big)\\
&=\big(\Phi,(K_\sigma(P)+u(\nabla K_\sigma)_j+\frac{u^2}{2}-E_\sigma(P+ue_j))\Phi\big)\\
&=E_\sigma(P)-E_\sigma(P+ue_j)+u(\nabla E_\sigma)_j+\frac{u^2}{2}\leq \mathrm{C}u^2.
\end{aligned}\tag{5.36}$$

From (5.34), we obtain that

$$(\nabla K_\sigma-\nabla E_\sigma)_j P_\sigma=(K_\sigma(P+ue_j)-E_\sigma(P+ue_j))^{\frac12}B_1+B_2, \tag{5.37}$$

where

$$B_1:=\frac{1}{u}(K_\sigma(P+ue_j)-E_\sigma(P+ue_j))^{\frac12}P_\sigma, \tag{5.38}$$

$$B_2:=\Big(\frac{1}{u}(E_\sigma(P+ue_j)-E_\sigma(P))-(\nabla E_\sigma)_j-\frac{u}{2}\Big)P_\sigma. \tag{5.39}$$

By (5.36) and (5.35), the operators B_1, B_2 are bounded and satisfy

$$\|B_1\|\leq \mathrm{C},\quad \|B_2\|\leq \mathrm{C}u. \tag{5.40}$$

Thus, choosing $u\leq\sigma$, the lemma will follow if we show that

$$\Big\|\bar\chi\big[H_{\bar\chi}-\lambda\big]^{-\frac12}\bar\chi(K_\sigma(P+ue_j)-E_\sigma(P+ue_j))^{\frac12}\otimes\mathbb{1}_{H_f\leq\delta}\Big\|^2\leq \mathrm{C}\delta\sigma^{-1}. \tag{5.41}$$

Let us prove (5.41). To simplify notations, we set

$$\bar\chi_{\leq\delta}:=(\mathbb{1}\otimes\mathbb{1}_{H_f\leq\delta})\bar\chi \tag{5.42}$$

Let $\Phi\in\mathrm{Ran}(\bar\chi)$, $\|\Phi\|=1$. Since

$$\big\|\big(H^1_{\bar\chi}-\lambda\big)\big[H_{\bar\chi}-\lambda\big]^{-1}\bar\chi\big\|\leq \mathrm{C}, \tag{5.43}$$

(see the proof of Proposition 4.1), it suffices to estimate

$$\Big(\Phi,\bar\chi\big[H^1_{\bar\chi}-\lambda\big]^{-\frac12}\bar\chi_{\leq\delta}\big((K_\sigma(P+ue_j)-E_\sigma(P+ue_j))\otimes\mathbb{1}\big)\bar\chi_{\leq\delta}\big[H^1_{\bar\chi}-\lambda\big]^{-\frac12}\bar\chi\Phi\Big). \tag{5.44}$$

Using that

$$\Big\| \bar\chi \big[H^1_{\bar\chi} - \lambda \big]^{-\frac12} \bar\chi \big((\nabla K_\sigma - \nabla E_\sigma) \otimes \mathbb{1} \big) \bar\chi \big[H^1_{\bar\chi} - \lambda \big]^{-\frac12} \bar\chi \Big\| \le \mathrm{C} \sigma^{-1}, \tag{5.45}$$

and since $0 < u \le \sigma$, we get

$$(5.44) \le \Big(\Phi, \bar\chi \big[H^1_{\bar\chi} - \lambda \big]^{-\frac12} \bar\chi_{\le\delta} \big((K_\sigma(P) - E_\sigma(P)) \otimes \mathbb{1} \big) \bar\chi_{\le\delta} \big[H^1_{\bar\chi} - \lambda \big]^{-\frac12} \bar\chi \Phi \Big) + \mathrm{C}. \tag{5.46}$$

Next, by Lemma A.4,

$$\begin{aligned} &\bar\chi_{\le\delta} \big((K_\sigma(P) - E_\sigma(P)) \otimes \mathbb{1} \big) \bar\chi_{\le\delta} \\ &\le \frac{1}{1-\delta} \bar\chi_{\le\delta} \big(\big(H_\sigma(P) - E_\sigma(P) \big) + 4\delta E_\sigma \big) \bar\chi_{\le\delta}. \end{aligned} \tag{5.47}$$

Using the expression (4.19) of $H^1_{\bar\chi}$, we conclude from (5.47) that

$$\begin{aligned} &\bar\chi_{\le\delta} \big((K_\sigma(P) - E_\sigma(P)) \otimes \mathbb{1} \big) \bar\chi_{\le\delta} \\ &\le \bar\chi_{\le\delta} \big(\big(H^1_{\bar\chi}(P) - E_\sigma(P) \big) + \mathrm{C}(\sigma + \delta) \big) \bar\chi_{\le\delta}. \end{aligned} \tag{5.48}$$

The statement of the lemma follows from (5.46), (5.48) and Lemma 4.2. □

In the following lemma, we prove that the "perturbation" operators W_1, W_2 in (5.17)–(5.18) are of order $O(\alpha^{1/2}\sigma)$.

Lemma 5.7. *For any* $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for all* $|P| \le p_{\mathrm{c}}$*,* $0 \le \alpha \le \alpha_0$*,* $0 < \sigma \le \mathrm{C}_0 \alpha^{1/2}$*, and* $\lambda \in J_\sigma^<$*,*

$$\big\| W_i \big\| \le \mathrm{C} \alpha^{\frac12} \sigma, \ \ i = 1, 2, \tag{5.49}$$

where W_1 *and* W_2 *are as in* (5.17), (5.18)*.*

Proof. Let us first prove (5.49) for $i = 1$. Equation (3.14) combined with the Feynman-Hellman formula gives

$$\begin{aligned} \chi U_\sigma \chi = &\alpha^{\frac12} \big(\nabla E_\sigma P_\sigma \big) \otimes \big(\chi_f^\sigma A^\sigma \chi_f^\sigma \big) - \frac{\alpha^{\frac12}}{2} P_\sigma \otimes \Big(\chi_f^\sigma \big(P_f \cdot A^\sigma + A^\sigma \cdot P_f \big) \chi_f^\sigma \Big) \\ &+ \frac{\alpha}{2} P_\sigma \otimes \Big(\chi_f^\sigma (A^\sigma)^2 \chi_f^\sigma \Big). \end{aligned} \tag{5.50}$$

It follows from Lemma A.1 that

$$\big\| A^\sigma \chi_f^\sigma \big\| \le \mathrm{C} \sigma^{\frac12} \big\| [H_f + \sigma]^{\frac12} \chi_f^\sigma \big\| \le \mathrm{C}' \sigma, \tag{5.51}$$

$$\big\| (A^\sigma \cdot P_f) \chi_f^\sigma \big\| \le \mathrm{C} \sigma^{\frac12} \big\| [H_f + \sigma]^{\frac12} |P_f| \chi_f^\sigma \big\| \le \mathrm{C}' \sigma^2. \tag{5.52}$$

Therefore (5.49) for $i = 1$ follows.

To prove (5.49) for $i = 2$ it suffices to show that for $\lambda \in J_\sigma^<$,

$$\Big\| \big[H_{\bar\chi} - \lambda \big]^{-\frac12} \bar\chi W_\sigma \chi \Big\| \le \mathrm{C} \alpha^{\frac14} \sigma^{\frac12}. \tag{5.53}$$

By Equations (3.14) and (4.5),

$$W_\sigma \chi = \alpha^{\frac{1}{2}} \big(\nabla K_\sigma P_\sigma \big) \otimes \big(A^\sigma \chi_f^\sigma \big) \tag{5.54}$$

$$- \frac{\alpha^{\frac{1}{2}}}{2} P_\sigma \otimes \Big(\big(P_f \cdot A^\sigma + A^\sigma \cdot P_f \big) \chi_f^\sigma \Big) \tag{5.55}$$

$$+ \frac{\alpha}{2} P_\sigma \otimes \big((A^\sigma)^2 \chi_f^\sigma \big) \tag{5.56}$$

$$- \Big(\big(\nabla K_\sigma - \nabla E_\sigma \big) P_\sigma \Big) \otimes \big(P_f \chi_f^\sigma \big). \tag{5.57}$$

We insert this expression into (5.53) and estimate each term separately. First, it follows from Proposition 4.1 and Estimate (5.51) that

$$\Big\| \big[H_{\bar\chi} - \lambda \big]^{-\frac{1}{2}} \bar\chi (5.54) \Big\| \leq \mathrm{C} \alpha^{\frac{1}{2}} \sigma^{\frac{1}{2}}. \tag{5.58}$$

Similarly, Lemma A.2 combined with Proposition 4.1 and (5.51)–(5.52) implies

$$\Big\| \big[H_{\bar\chi} - \lambda \big]^{-\frac{1}{2}} \bar\chi \big((5.55) + (5.56) \big) \Big\| \leq \mathrm{C} \alpha^{\frac{1}{2}} \sigma^{\frac{3}{2}}. \tag{5.59}$$

Finally the contribution from (5.57) is estimated thanks to Lemma 5.6: Using (5.32) with $\delta = \rho\sigma$, we get, for $j \in \{1, 2, 3\}$,

$$\Big\| \big[H_{\bar\chi} - \lambda \big]^{-\frac{1}{2}} \bar\chi \Big(\big(\nabla K_\sigma - \nabla E_\sigma \big)_j P_\sigma \Big) \otimes \mathbb{1}_{H_f \leq \rho\sigma} \Big\| \leq \mathrm{C}. \tag{5.60}$$

Together with $\|(P_f)_j \chi_f^\sigma\| \leq \mathrm{C}\sigma$, this yields

$$\Big\| \big[H_{\bar\chi} - \lambda \big]^{-\frac{1}{2}} \bar\chi (5.57) \Big\| \leq \mathrm{C}\sigma \leq \mathrm{C}' \alpha^{\frac{1}{4}} \sigma^{\frac{1}{2}}. \tag{5.61}$$

Estimates (5.58), (5.59) and (5.61) imply (5.53), so (5.49), $i = 2$, follows. □

In the next lemma, we estimate the commutators $[W_i, \mathrm{i}B^\sigma]$, $i = 1, 2$.

Lemma 5.8. *For any* $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for all* $|P| \leq p_{\mathrm{c}}$*,* $0 \leq \alpha \leq \alpha_0$*,* $0 < \sigma \leq \mathrm{C}_0 \alpha^{1/2}$*, and* $\lambda \in J_\sigma^<$

$$\| [W_i, \mathrm{i}B^\sigma] \| \leq \mathrm{C} \alpha^{\frac{1}{2}} \sigma, \ \ i = 1, 2, \tag{5.62}$$

where W_1 *and* W_2 *are as in* (5.17), (5.18).

Proof. Using for instance the Helffer-Sjöstrand functional calculus, the following identities follow straightforwardly from (5.20):

$$[\chi, \mathrm{i}B^\sigma] = P_\sigma \otimes \big(\mathrm{d}\Gamma(\kappa^\sigma(k)^2 |k|) (\chi_f^\sigma)'(H_f) \big), \tag{5.63}$$

$$[\bar\chi, \mathrm{i}B^\sigma] = P_\sigma \otimes \big(\mathrm{d}\Gamma(\kappa^\sigma(k)^2 |k|) (\bar\chi_f^\sigma)'(H_f) \big). \tag{5.64}$$

Furthermore,

$$[A^\sigma, \mathrm{i}B^\sigma] = -\Phi(\mathrm{i}b^\sigma h^\sigma). \tag{5.65}$$

We first prove (5.62) for $i = 1$. We have that

$$[W_1, \mathrm{i}B^\sigma] = [\chi, \mathrm{i}B^\sigma]U_\sigma \chi + \chi[U_\sigma, \mathrm{i}B^\sigma]\chi + \chi U_\sigma[\chi, \mathrm{i}B^\sigma]. \tag{5.66}$$

As in the proof of (5.49), $i = 1$, in Lemma 5.7, we obtain, using (5.63), that

$$\big\|[\chi, \mathrm{i}B^\sigma]U_\sigma \chi\big\| = \big\|\chi U_\sigma[\chi, \mathrm{i}B^\sigma]\big\| \leq \mathrm{C}\alpha^{\frac{1}{2}}\sigma. \tag{5.67}$$

It follows from (3.14), (5.20) and (5.65) that

$$\begin{aligned}
[U_\sigma, \mathrm{i}B^\sigma] = &- \alpha^{\frac{1}{2}} \nabla K_\sigma \otimes \Phi(\mathrm{i}b^\sigma h^\sigma) + \frac{\alpha^{\frac{1}{2}}}{2} \mathbb{1} \otimes \big(\Phi(\mathrm{i}b^\sigma h^\sigma) \cdot P_f + P_f \cdot \Phi(\mathrm{i}b^\sigma h^\sigma)\big) \\
&- \frac{\alpha^{\frac{1}{2}}}{2} \mathbb{1} \otimes \big(\Phi(h^\sigma) \cdot \mathrm{d}\Gamma(\kappa^\sigma(k)^2 k) + \mathrm{d}\Gamma(\kappa^\sigma(k)^2 k) \cdot \Phi(h^\sigma)\big) \\
&- \frac{\alpha}{2} \mathbb{1} \otimes \big(\Phi(h^\sigma) \cdot \Phi(\mathrm{i}b^\sigma h^\sigma) + \Phi(\mathrm{i}b^\sigma h^\sigma) \cdot \Phi(h^\sigma)\big).
\end{aligned} \tag{5.68}$$

Arguing as in the proof of (5.49), $i = 1$, in Lemma 5.7, we then obtain

$$\big\|\chi[U_\sigma, \mathrm{i}B^\sigma]\chi\big\| \leq \mathrm{C}\alpha^{\frac{1}{2}}\sigma. \tag{5.69}$$

Hence (5.62), $i = 1$, is proven. In order to prove (5.62), $i = 2$, let us decompose

$$[W_2, \mathrm{i}B^\sigma] = - [\chi, \mathrm{i}B^\sigma] W_\sigma \bar{\chi} \big[H_{\bar{\chi}} - \lambda\big]^{-1} \bar{\chi} W_\sigma \chi + \mathrm{h.c.} \tag{5.70}$$

$$- \chi[W_\sigma, \mathrm{i}B^\sigma] \bar{\chi} \big[H_{\bar{\chi}} - \lambda\big]^{-1} \bar{\chi} W_\sigma \chi + \mathrm{h.c.} \tag{5.71}$$

$$- \chi W_\sigma [\bar{\chi}, \mathrm{i}B^\sigma] \big[H_{\bar{\chi}} - \lambda\big]^{-1} \bar{\chi} W_\sigma \chi + \mathrm{h.c.} \tag{5.72}$$

$$- \chi W_\sigma \bar{\chi} \big[\big[H_{\bar{\chi}} - \lambda\big]^{-1}, \mathrm{i}B^\sigma\big] \bar{\chi} W_\sigma \chi. \tag{5.73}$$

Using Equations (5.20), (5.63), (5.64) and (5.65) for the different commutators entering the terms (5.70), (5.71) and (5.72), one can check in the same way as in the proof of (5.49), $i = 2$, in Lemma 5.7 that

$$\big\|(5.70) + (5.71) + (5.72)\big\| \leq \mathrm{C}\alpha^{\frac{1}{2}}\sigma. \tag{5.74}$$

To conclude we need to estimate (5.73). We expand $[H_{\bar{\chi}} - \lambda]^{-1}$ into the Neumann series (4.25), which leads to

$$
\begin{aligned}
&\big[\big[H_{\bar{\chi}} - \lambda\big]^{-1}, \mathrm{i}B^\sigma\big] \\
&= -\big[H_{\bar{\chi}} - \lambda\big]^{-1}\big[H_{\bar{\chi}}, \mathrm{i}B^\sigma\big]\big[H_{\bar{\chi}} - \lambda\big]^{-1} \\
&= -\big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\sum_{n\geq 0}\Big(- \bar{\chi}U_\sigma\bar{\chi}\big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\Big)^n\big[H_{\bar{\chi}}, \mathrm{i}B^\sigma\big] \\
&\qquad \times \big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\sum_{n'\geq 0}\Big(- \bar{\chi}U_\sigma\bar{\chi}\big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\Big)^{n'}.
\end{aligned} \tag{5.75}
$$

Inserting this series into (5.73) yields a sum of terms of the form

$$
\begin{aligned}
&\chi W_\sigma\bar{\chi}\big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\Big(\bar{\chi}U_\sigma\bar{\chi}\big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\Big)^n\big[H_{\bar{\chi}}, \mathrm{i}B^\sigma\big] \\
&\times \big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\Big(\bar{\chi}U_\sigma\bar{\chi}\big[H^1_{\bar{\chi}} - \lambda\big]^{-1}\Big)^{n'}\bar{\chi}W_\sigma\chi,
\end{aligned} \tag{5.76}
$$

where $n, n' \in \mathbb{N}$. To estimate (5.76), we notice that, by Lemma A.2, $W_\sigma\chi^\sigma_f = (\mathbb{1}\otimes\mathbb{1}_{H_f\leq 3\sigma})W_\sigma\chi^\sigma_f$, and likewise with U_σ replacing W_σ. Thus, since $\mathbb{1}\otimes H_f$ commutes with $H^1_{\bar{\chi}}$, we conclude from (5.53) and (4.26) that

$$
\begin{aligned}
\big\|(5.76)\big\| \leq \mathrm{C}\alpha^{\frac{1}{2}}\sigma\big(\mathrm{C}'\alpha^{\frac{1}{2}}\big)^{n+n'}\Big\|&\big[H^1_{\bar{\chi}} - \lambda\big]^{-\frac{1}{2}}(\mathbb{1}\otimes\mathbb{1}_{H_f\leq(2n+1)\sigma}) \\
&\big[H_{\bar{\chi}}, \mathrm{i}B^\sigma\big](\mathbb{1}\otimes\mathbb{1}_{H_f\leq(2n'+1)\sigma})\big[H^1_{\bar{\chi}} - \lambda\big]^{-\frac{1}{2}}\Big\|.
\end{aligned} \tag{5.77}
$$

Using identities (5.20) and (5.63)–(5.65), one can check that, for any $\gamma \geq 1$,

$$
\big\|\big[H_{\bar{\chi}}, \mathrm{i}B^\sigma\big](\mathbb{1}\otimes\mathbb{1}_{H_f\leq\gamma\sigma})\big\| \leq \mathrm{C}\gamma^2\sigma. \tag{5.78}
$$

This implies

$$
\big\|(5.76)\big\| \leq \mathrm{C}\alpha^{\frac{1}{2}}\sigma(n+n'+1)^2\big(\mathrm{C}'\alpha^{\frac{1}{2}}\big)^{n+n'}. \tag{5.79}
$$

Summing over n, n', we get that

$$
\big\|(5.73)\big\| \leq \mathrm{C}\alpha^{\frac{1}{2}}\sigma, \tag{5.80}
$$

for α small enough, which concludes the proof of (5.62), $i = 2$. □

In the proof of Theorem 5.1, it will be convenient to replace F by an operator $\tilde{F}$, translated from F in such a way that the unperturbed part in $\tilde{F}$ do not depend on the spectral parameter λ anymore. More precisely, let

$$
\tilde{F} := F + \lambda - E_\sigma. \tag{5.81}
$$

Then we have that $\tilde{F} = \tilde{F}_0 + W_1 + W_2$, where

$$\tilde{F}_0 := F_0 + \lambda - E_\sigma = \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + H_f\big) - \nabla E_\sigma \otimes P_f, \tag{5.82}$$

and W_1, W_2 are defined as in (5.17), (5.18).

Lemma 5.9. *For any* $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for all* $|P| \leq p_{\mathrm{c}}$*,* $0 \leq \alpha \leq \alpha_0$*,* $0 < \sigma \leq \mathrm{C}_0\alpha^{1/2}$*, and* $\lambda \in J_\sigma^<$*,*

$$\mathbb{1}_{\Delta_\sigma}(F) = \mathbb{1}_{\Delta_\sigma}(F)\mathbb{1}_{\Delta'_\sigma}(\tilde{F}), \tag{5.83}$$

where $\Delta'_\sigma := [\rho\sigma/16, \rho\sigma/8]$ *and* Δ_σ *is given in* (5.2).

Proof. Since $\tilde{F}$ is a translate of F, it is only necessary to check that $\Delta_\sigma \subseteq \Delta'_\sigma - \lambda + E_\sigma$ for all $\lambda \in J_\sigma^<$, or equivalently, that $\Delta_\sigma \subseteq \Delta'_\sigma - J_\sigma^< + E_\sigma$ in the sense of "sumsets". Using the definitions of Δ_σ, Δ'_σ, $J_\sigma^<$, and the fact that $|E - E_\sigma| \leq \mathrm{C}\alpha\sigma$ by Proposition 3.1, one can verify that this is the case for α sufficiently small. □

Let $f_\sigma \in \mathrm{C}_0^\infty(\mathbb{R}; [0,1])$ be such that $f_\sigma = 1$ on $\Delta'_\sigma = [\rho\sigma/16, \rho\sigma/8]$ and

$$\mathrm{supp}(f_\sigma) \subset [\frac{3}{64}\rho\sigma, \frac{9}{64}\rho\sigma]. \tag{5.84}$$

Lemma 5.10. *For any* $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for all* $|P| \leq p_{\mathrm{c}}$*,* $0 \leq \alpha \leq \alpha_0$*,* $0 < \sigma \leq \mathrm{C}_0\alpha^{1/2}$*, and* $\lambda \in J_\sigma^<$*,*

$$\big\| f_\sigma(\tilde{F}) - f_\sigma(\tilde{F}_0) \big\| \leq \mathrm{C}\alpha^{\frac{1}{2}}. \tag{5.85}$$

Proof. Let $\tilde{f}_\sigma$ be an almost analytic extension of f_σ obeying

$$\mathrm{supp}(\tilde{f}_\sigma) \subset \big\{ z \in \mathbb{C}, \mathrm{Re}(z) \in \mathrm{supp}(f_\sigma), |\mathrm{Im}(z)| \leq \sigma \big\}, \tag{5.86}$$

$\partial_{\bar{z}}\tilde{f}_\sigma(z) = 0$ if $\mathrm{Im}(z) = 0$, and

$$\big| \frac{\partial \tilde{f}_\sigma}{\partial \bar{z}}(z) \big| \leq \frac{\mathrm{C}_n}{\sigma}\big(\frac{|y|}{\sigma}\big)^n, \tag{5.87}$$

for any $n \in \mathbb{N}$ (see for instance [HS]). Here we used the notations

$$z = x + \mathrm{i}y, \quad \frac{\partial}{\partial \bar{z}} = \frac{\partial}{\partial x} + \mathrm{i}\frac{\partial}{\partial y}. \tag{5.88}$$

By the Helffer-Sjöstrand functional calculus and the second resolvent equation,

$$f_\sigma(\tilde{F}) - f_\sigma(\tilde{F}_0) = \frac{\mathrm{i}}{2\pi}\int \frac{\partial \tilde{f}_\sigma}{\partial \bar{z}}(z)\big[\tilde{F} - z\big]^{-1}\big(\tilde{F} - \tilde{F}_0\big)\big[\tilde{F}_0 - z\big]^{-1}\mathrm{d}z \wedge \mathrm{d}\bar{z}. \tag{5.89}$$

Lemma 5.7 implies

$$\big\|\tilde{F} - \tilde{F}_0\big\| = \big\|F - F_0\big\| = \big\|W_1 + W_2\big\| \leq \mathrm{C}\alpha^{\frac{1}{2}}\sigma. \tag{5.90}$$

The statement of the lemma then follows from (5.86)–(5.90). □

Lemma 5.10 will allow us to replace $f_\sigma(\tilde{F})$ by $f_\sigma(\tilde{F}_0)$ in our proof of Theorem 5.1. In view of Lemma 5.5, we shall also need to replace $f_\sigma(\tilde{F}_0)$ by some function of H_f. This is the purpose of the following lemma.

Lemma 5.11. *For any* $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for all* $|P| \le p_{\mathrm{c}}$*,* $0 \le \alpha \le \alpha_0$*,* $0 < \sigma \le \mathrm{C}_0\alpha^{1/2}$*, and* $\lambda \in J_\sigma^<$*,*

$$f_\sigma(\tilde{F}_0)(\mathbb{1} \otimes \mathbb{1}_{\frac{1}{32}\rho\sigma \le H_f \le \frac{1}{4}\rho\sigma}) = f_\sigma(\tilde{F}_0). \tag{5.91}$$

Proof. We recall that

$$\tilde{F}_0 = \tilde{F}_0(H_f, P_f) = \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + H_f\big) - \nabla E_\sigma \otimes P_f\,. \tag{5.92}$$

The claim of the lemma is equivalent to the statement that whenever $\tilde{F}_0(X_0, X) \in \mathrm{supp}(f_\sigma)$ with $|X| \le X_0$, then $X_0 \in [\frac{1}{32}\rho\sigma, \frac{1}{4}\rho\sigma]$.

Let $[a, b] \equiv [\frac{3}{64}\rho\sigma, \frac{9}{64}\rho\sigma] \supset \mathrm{supp}(f_\sigma)$. We assume that

$$a \le \tilde{F}(X_0, X) = X_0 + \frac{1}{2}X^2 - \nabla E_\sigma \cdot X \le b \tag{5.93}$$

with $|X| \le X_0$. Clearly, this implies, on the one hand, that

$$X_0 - |\nabla E_\sigma| X_0 \le \tilde{F}(X_0, X) \le b \tag{5.94}$$

so that $X_0 \le (1 - |\nabla E_\sigma|)^{-1} b$, and, on the other hand,

$$X_0 + \frac{1}{2}X_0^2 + |\nabla E_\sigma| X_0 \ge \tilde{F}(X_0, X) \ge a \tag{5.95}$$

so that $X_0 \ge (1 + |\nabla E_\sigma|)^{-1}(a - \frac{1}{2}(1 - |\nabla E_\sigma|)^{-2} b^2)$.

By Proposition 3.1, $|\nabla E_\sigma| \le |P| + \mathrm{C}\alpha \le 1/10$ for $|P| \le 1/40$ and α sufficiently small. Thus, one concludes that $X_0 \in [\frac{1}{32}\rho\sigma, \frac{1}{4}\rho\sigma]$, as claimed. □

We will also make use of the following easy lemma.

Lemma 5.12. *For any* $\mathrm{C}_0 > 0$*, there exists* $\alpha_0 > 0$ *such that, for all* $|P| \le p_{\mathrm{c}}$*,* $0 \le \alpha \le \alpha_0$*,* $0 < \sigma \le \mathrm{C}_0\alpha^{1/2}$*, and* $\lambda \in J_\sigma^<$*, the operators* $[F, \mathrm{i}B^\sigma] f_\sigma(\tilde{F}_0)$ *and* $[F, \mathrm{i}B^\sigma] f_\sigma(\tilde{F})$ *are bounded on* $\mathrm{Ran}(P_\sigma \otimes \mathbb{1})$ *and satisfy*

$$\big\|[F, \mathrm{i}B^\sigma] f_\sigma(\tilde{F}_0)\big\| \le \mathrm{C}\sigma, \quad \big\|[F, \mathrm{i}B^\sigma] f_\sigma(\tilde{F})\big\| \le \mathrm{C}\sigma. \tag{5.96}$$

Proof. The first bound in (5.96) is a consequence of Lemmata 5.8 and 5.11. Indeed, using expression (5.21) for $[F_0, \mathrm{i}B^\sigma]$, we get

$$\begin{aligned} \big\|[F, \mathrm{i}B^\sigma] f_\sigma(\tilde{F}_0)\big\| &\le \big\|[F_0, \mathrm{i}B^\sigma](\mathbb{1} \otimes \mathbb{1}_{H_f \le \frac{1}{4}\rho\sigma})\big\| + \big\|[W_1, \mathrm{i}B^\sigma]\big\| + \big\|[W_2, \mathrm{i}B^\sigma]\big\| \\ &\le \mathrm{C}\sigma. \end{aligned} \tag{5.97}$$

Likewise, to prove the second bound in (5.96), it suffices to show that

$$f_\sigma(\tilde{F}) = (\mathbb{1} \otimes \mathbb{1}_{H_f \leq \rho\sigma}) f_\sigma(\tilde{F}). \tag{5.98}$$

Since $\chi_f^\sigma \mathbb{1}_{H_f \leq \rho\sigma} = \chi_f^\sigma$, and since $\tilde{F}_0$ commutes with $\mathbb{1} \otimes \mathbb{1}_{H_f \leq \rho\sigma}$, it follows that $\tilde{F}$ commutes with $\mathbb{1} \otimes \mathbb{1}_{H_f \leq \rho\sigma}$. By Lemma 5.7,

$$\tilde{F}(\mathbb{1} \otimes \mathbb{1}_{H_f \geq \rho\sigma}) \geq \tilde{F}_0(\mathbb{1} \otimes \mathbb{1}_{H_f \geq \rho\sigma}) - \mathrm{C}\alpha^{\frac{1}{2}}\sigma(\mathbb{1} \otimes \mathbb{1}_{H_f \geq \rho\sigma}). \tag{5.99}$$

Using the fact that $|\nabla E_\sigma| \leq 1/8$ for $|P| \leq 1/40$ and α sufficiently small (see Proposition 3.1), we obtain

$$\begin{aligned} \tilde{F}_0(\mathbb{1} \otimes \mathbb{1}_{H_f \geq \rho\sigma}) &= \big(\mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + H_f\big) - \nabla E_\sigma \otimes P_f\big)(\mathbb{1} \otimes \mathbb{1}_{H_f \geq \rho\sigma}) \\ &\geq (1 - 2|\nabla E_\sigma|)(\mathbb{1} \otimes H_f)(\mathbb{1} \otimes \mathbb{1}_{H_f \geq \rho\sigma}) \\ &\geq \frac{3}{4}\rho\sigma(\mathbb{1} \otimes \mathbb{1}_{H_f \geq 2\rho\sigma}). \end{aligned} \tag{5.100}$$

Hence, for α sufficiently small,

$$\tilde{F}(\mathbb{1} \otimes \mathbb{1}_{H_f \geq \rho\sigma}) \geq \frac{1}{2}\rho\sigma(\mathbb{1} \otimes \mathbb{1}_{H_f \geq \rho\sigma}). \tag{5.101}$$

Since $\mathrm{supp}(f_\sigma) \subset [3\rho\sigma/64, 9\rho\sigma/64]$, it follows that $(\mathbb{1} \otimes \mathbb{1}_{H_f \geq \rho\sigma}) f_\sigma(\tilde{F}) = 0$, which establishes (5.98) and concludes the proof. □

Next, we turn to the proof of Theorem 5.1. Recall that the intervals Δ_σ, Δ'_σ are given by $\Delta_\sigma = [-\rho\sigma/128, \rho\sigma/128]$, $\Delta'_\sigma = [\rho\sigma/16, \rho\sigma/8]$, and that the function $f_\sigma \in \mathrm{C}_0^\infty(\mathbb{R}; [0,1])$ is such that $f_\sigma = 1$ on Δ'_σ and $\mathrm{supp}(f_\sigma) \subset [3\rho\sigma/64, 9\rho\sigma/64]$. Let us also recall the notations $\tilde{F} = F + \lambda - E_\sigma$, $\tilde{F}_0 = F_0 + \lambda - E_\sigma$. By Lemma 5.9, we have that

$$\begin{aligned} &\mathbb{1}_{\Delta_\sigma}(F)[F, \mathrm{i}B^\sigma]\mathbb{1}_{\Delta_\sigma}(F) \\ &= \mathbb{1}_{\Delta_\sigma}(F)\mathbb{1}_{\Delta'_\sigma}(\tilde{F})[F, \mathrm{i}B^\sigma]\mathbb{1}_{\Delta'_\sigma}(\tilde{F})\mathbb{1}_{\Delta_\sigma}(F) && (5.102) \\ &= \mathbb{1}_{\Delta_\sigma}(F)\mathbb{1}_{\Delta'_\sigma}(\tilde{F})f_\sigma(\tilde{F})[F, \mathrm{i}B^\sigma]f_\sigma(\tilde{F})\mathbb{1}_{\Delta'_\sigma}(\tilde{F})\mathbb{1}_{\Delta_\sigma}(F). && (5.103) \end{aligned}$$

Next, we write

$$\begin{aligned} &f_\sigma(\tilde{F})[F, \mathrm{i}B^\sigma]f_\sigma(\tilde{F}) \\ &= f_\sigma(\tilde{F}_0)[F, \mathrm{i}B^\sigma]f_\sigma(\tilde{F}_0) && (5.104) \\ &\quad + (f_\sigma(\tilde{F}) - f_\sigma(\tilde{F}_0))[F, \mathrm{i}B^\sigma]f_\sigma(\tilde{F}) + f_\sigma(\tilde{F}_0)[F, \mathrm{i}B^\sigma](f_\sigma(\tilde{F}) - f_\sigma(\tilde{F}_0)). && (5.105) \end{aligned}$$

Lemmata 5.10 and 5.12 imply

$$\|(5.105)\| \leq \mathrm{C}\alpha^{\frac{1}{2}}\sigma. \tag{5.106}$$

Using Lemmata 5.5, 5.8, 5.10 and 5.11, we estimate (5.104) from below as follows:

$$\begin{aligned}
&f_\sigma(\tilde{F}_0)[F, \mathrm{i}B^\sigma]f_\sigma(\tilde{F}_0)\\
&\geq f_\sigma(\tilde{F}_0)[F_0, \mathrm{i}B^\sigma]f_\sigma(\tilde{F}_0) - \mathrm{C}\alpha^{\frac{1}{2}}\sigma f_\sigma(\tilde{F}_0)^2\\
&\geq f_\sigma(\tilde{F}_0)[F_0, \mathrm{i}B^\sigma](\mathbb{1}\otimes \mathbb{1}_{\frac{1}{32}\rho\sigma\leq H_f\leq\frac{1}{4}\rho\sigma})f_\sigma(\tilde{F}_0) - \mathrm{C}\alpha^{\frac{1}{2}}\sigma f_\sigma(\tilde{F}_0)^2\\
&\geq \frac{1}{2}f_\sigma(\tilde{F}_0)(\mathbb{1}\otimes H_f)(\mathbb{1}\otimes \mathbb{1}_{\frac{1}{32}\rho\sigma\leq H_f\leq\frac{1}{4}\rho\sigma})f_\sigma(\tilde{F}_0) - \mathrm{C}'\alpha^{\frac{1}{2}}\sigma f_\sigma(\tilde{F}_0)^2\\
&\geq \frac{\rho\sigma}{64}f_\sigma(\tilde{F}_0)^2 - \mathrm{C}'\alpha^{\frac{1}{2}}\sigma f_\sigma(\tilde{F}_0)^2\\
&\geq \frac{\rho\sigma}{64}f_\sigma(\tilde{F})^2 - \mathrm{C}''\alpha^{\frac{1}{2}}\sigma.
\end{aligned} \tag{5.107}$$

Inequality (5.107) combined with (5.106) yield

$$\begin{aligned}
f_\sigma(\tilde{F})[F, \mathrm{i}B^\sigma]f_\sigma(\tilde{F}) &\geq \frac{\rho\sigma}{64}f_\sigma(\tilde{F})^2 - \mathrm{C}\alpha^{\frac{1}{2}}\sigma\\
&\geq \frac{\rho\sigma}{128}f_\sigma(\tilde{F})^2 - \mathrm{C}\alpha^{\frac{1}{2}}\sigma\big(\mathbb{1} - f_\sigma(\tilde{F})^2\big),
\end{aligned} \tag{5.108}$$

provided that α is sufficiently small. Multiplying both sides of (5.108) by $\mathbb{1}_{\Delta'_\sigma}(\tilde{F})$ gives

$$\mathbb{1}_{\Delta'_\sigma}(\tilde{F})[F, \mathrm{i}B^\sigma]\mathbb{1}_{\Delta'_\sigma}(\tilde{F}) \geq \frac{\rho\sigma}{128}\mathbb{1}_{\Delta'_\sigma}(\tilde{F}). \tag{5.109}$$

Inserting this into (5.102) and using Lemma 5.9 conclude the proof of the theorem. □

Appendix A. Technical estimates

In this appendix we collect some estimates that were used in Sections 4 and 5. For $f : \mathbb{R}^3 \times \mathbb{Z}_2 \mapsto \mathbb{C}$ and $\gamma > 0$, we define

$$f^\gamma(k, \lambda) = f(k, \lambda)\mathbb{1}_{|k|\leq\gamma}. \tag{A.1}$$

Similarly we set

$$H_f^\gamma = \sum_{\lambda=1,2}\int_{|k|\leq\gamma}|k|a^*_\lambda(k)a_\lambda(k)\mathrm{d}k. \tag{A.2}$$

We begin with two well-known lemmata; (see for instance [BFS] for a proof).

Lemma A.1. *For any $f \in \mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$ such that $|k|^{-1/2}f \in \mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, and any $\gamma > 0$,*

$$\|a(f^\gamma)[H_f^\gamma + \gamma]^{-1/2}\| \leq \||k|^{-\frac{1}{2}}f^\gamma\|, \tag{A.3}$$

$$\|a^*(f^\gamma)[H_f^\gamma + \gamma]^{-1/2}\| \leq \||k|^{-\frac{1}{2}}f^\gamma\| + \gamma^{-\frac{1}{2}}\|f^\gamma\|. \tag{A.4}$$

Lemma A.2. *For any $f \in \mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)$, and any $\gamma > 0$, $\delta > 0$,*

$$a(f^\gamma)\mathbb{1}_{H_f^\gamma \le \delta} = \mathbb{1}_{H_f^\gamma \le \delta} a(f^\gamma)\mathbb{1}_{H_f^\gamma \le \delta} \tag{A.5}$$

$$a^*(f^\gamma)\mathbb{1}_{H_f^\gamma \le \delta} = \mathbb{1}_{H_f^\gamma \le \gamma+\delta} a^*(f^\gamma)\mathbb{1}_{H_f^\gamma \le \delta} \tag{A.6}$$

Proof. The statement of the lemma follows directly from the "pull-through formula"

$$a(k)g(H_f^\gamma) = g(H_f^\gamma + |k|)a(k), \tag{A.7}$$

which holds for any bounded measurable function $g : [0,\infty) \to \mathbb{C}$, and any $k \in \mathbb{R}^3$, $|k| \le \gamma$. □

In the following, the parameters α, σ and P are fixed with $0 \le \alpha \le \alpha_0$, where α_0 is sufficiently small, $0 < \sigma \le \mathrm{C}_0\alpha^{1/2}$, where C_0 is a positive constant, and $|P| \le p_{\mathrm{c}} = 1/40$. We use the notations introduced in Section 3.

Lemma A.3. *For any $\mathrm{c} \ge 1/2$, we have that*

$$K_\sigma \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + \mathrm{c}H_f\big) - \nabla K_\sigma \otimes P_f \ge E_\sigma. \tag{A.8}$$

In particular,

$$\mathbb{1} \otimes H_f \le 2(H_\sigma - E_\sigma). \tag{A.9}$$

Proof. To simplify notations, we set

$$H_{\sigma,\mathrm{c}} = H_{\sigma,\mathrm{c}}(P) = K_\sigma \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + \mathrm{c}H_f\big) - \nabla K_\sigma \otimes P_f. \tag{A.10}$$

Note that

$$\begin{aligned} H_{\sigma,\mathrm{c}} &= \frac{1}{2}\big(P - P_f - \alpha^{\frac{1}{2}}A_\sigma\big)^2 + H_f \otimes \mathbb{1} + \mathrm{c}\,\mathbb{1} \otimes H_f \\ &= \frac{1}{2}\big(\nabla H_\sigma\big)^2 + H_f \otimes \mathbb{1} + \mathrm{c}\,\mathbb{1} \otimes H_f. \end{aligned} \tag{A.11}$$

Let $\Phi \in D(H_{\sigma,\mathrm{c}})$, $\|\Phi\| = 1$. We propose to show that

$$(\Phi, H_{\sigma,\mathrm{c}}\Phi) \ge E_\sigma. \tag{A.12}$$

Since the number operator $N^\sigma = \sum_{\lambda=1,2}\int_{|k|\le\sigma} a_\lambda^*(k)a_\lambda(k)\mathrm{d}k$ commutes with $H_{\sigma,\mathrm{c}}$, in order to prove (A.12), it suffices to consider $\Phi \in D(H_{\sigma,\mathrm{c}})$ of the form $\Phi = \Phi_1 \otimes \Phi_2$ where $\Phi_1 \in \mathcal{F}_\sigma$ and Φ_2 is an eigenstate of $N^\sigma|_{\mathcal{F}^\sigma}$. Let us prove the following assertion by induction:

($\mathbf{h_n}$) For all $\Phi = \Phi_1 \otimes \Phi_2 \in D(H_{\sigma,\mathrm{c}})$ such that $\|\Phi_1\| = \|\Phi_2\| = 1$ and $N^\sigma\Phi_2 = n\Phi_2$, (A.12) holds.

Since $H_{\sigma,\mathrm{c}}(\Phi_1 \otimes \Omega) = (K_\sigma \Phi_1) \otimes \Omega$ and since $E_\sigma = \inf \sigma(K_\sigma)$ (see Proposition 3.1), ($\mathbf{h_0}$) is obviously satisfied. Assume that ($\mathbf{h_n}$) holds and let $\Phi = \Phi_1 \otimes \Phi_2 \in D(H_{\sigma,c})$ with $\|\Phi_1\| = \|\Phi_2\| = 1$ and $N^\sigma \Phi_2 = (n+1)\Phi_2$. Let us write

$$\Phi_2\big((k,\lambda),(k_1,\lambda_1),\dots,(k_n,\lambda_n)\big) = \Phi_2(k,\lambda)\big((k_1,\lambda_1),\dots,(k_n,\lambda_n)\big). \tag{A.13}$$

One can compute

$$(\Phi, H_{\sigma,\mathrm{c}}\Phi) = \sum_{\lambda=1,2} \int_{|k|\le\sigma} \big(\Phi_1 \otimes \Phi_2(k,\lambda), \\ (H_{\sigma,\mathrm{c}}(P-k) + \mathrm{c}|k|)\Phi_1 \otimes \Phi_2(k,\lambda)\big)\mathrm{d}k. \tag{A.14}$$

Next, it follows from (A.11) that

$$\begin{aligned}&\big(\Phi_1 \otimes \Phi_2(k,\lambda), (H_{\sigma,\mathrm{c}}(P-k) + \mathrm{c}|k|)\Phi_1 \otimes \Phi_2(k,\lambda)\big) \\ &= \Big(\Phi_1 \otimes \Phi_2(k,\lambda), \Big(H_{\sigma,\mathrm{c}} - k\cdot\nabla H_\sigma + \frac{k^2}{2} + \mathrm{c}|k|\Big)\Phi_1 \otimes \Phi_2(k,\lambda)\Big).\end{aligned} \tag{A.15}$$

Using that $k \cdot \nabla H_\sigma \le |k|/4 + |k|(\nabla H_\sigma)^2$ and that $(\nabla H_\sigma)^2 \le 2H_{\sigma,\mathrm{c}}$, we obtain that

$$\begin{aligned}&\big(\Phi_1 \otimes \Phi_2(k,\lambda), (H_{\sigma,\mathrm{c}}(P-k) + \mathrm{c}|k|)\Phi_1 \otimes \Phi_2(k,\lambda)\big) \\ &\ge \Big(\Phi_1 \otimes \Phi_2(k,\lambda), \Big(H_{\sigma,\mathrm{c}} - |k|(\nabla H_\sigma)^2 + \frac{k^2}{2} + (\mathrm{c} - \frac{1}{4})|k|\Big)\Phi_1 \otimes \Phi_2(k,\lambda)\Big) \\ &\ge \Big(\Phi_1 \otimes \Phi_2(k,\lambda), \Big((1-2|k|)H_{\sigma,\mathrm{c}} + (\mathrm{c} - \frac{1}{4})|k|\Big)\Phi_1 \otimes \Phi_2(k,\lambda)\Big).\end{aligned} \tag{A.16}$$

Since by the induction hypothesis $(\Phi_1 \otimes \Phi_2(k,\lambda), H_{\sigma,\mathrm{c}}\Phi_1 \otimes \Phi_2(k,\lambda)) \ge E_\sigma \|\Phi_2(k,\lambda)\|^2$, this implies

$$\begin{aligned}&\big(\Phi_1 \otimes \Phi_2(k,\lambda), (H_{\sigma,\mathrm{c}}(P-k) + |k|)\Phi_1 \otimes \Phi_2(k,\lambda)\big) \\ &\ge \Big((1-2|k|)E_\sigma + (\mathrm{c} - \frac{1}{4})|k|\Big)\|\Phi_2(k,\lambda)\|^2 \\ &\ge \Big(E_\sigma + |k|(\mathrm{c} - \frac{1}{4} - 2E_\sigma)\Big)\|\Phi_2(k,\lambda)\|^2.\end{aligned} \tag{A.17}$$

By Rayleigh-Ritz (see Proposition 3.1),

$$E_\sigma \le \frac{1}{2}P^2 + \mathrm{C}\alpha \le \frac{1}{100} \tag{A.18}$$

for α sufficiently small and $|P| \le 1/40$, so that, in particular, $\mathrm{c} - 1/4 - 2E_\sigma \ge 0$; (recall that $\mathrm{c} \ge 1/2$). Therefore ($\mathbf{h_{n+1}}$) holds, and hence (A.12) is proven.

To prove (A.9), it suffices to write, using (A.8) with c = 1/2,

$$\begin{aligned} H_\sigma &= K_\sigma \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + \frac{1}{2}H_f\big) - \nabla K_\sigma \otimes P_f + \frac{1}{2}\big(\mathbb{1} \otimes H_f\big) \\ &\geq E_\sigma + \frac{1}{2}\big(\mathbb{1} \otimes H_f\big). \end{aligned} \tag{A.19}$$

□

Lemma A.4. *Let* $0 < \delta < 1$*. Then*

$$H_\sigma(\mathbb{1} \otimes \mathbb{1}_{H_f \leq \delta}) \geq (1-\delta)\big(K_\sigma \otimes \mathbb{1}\big)(\mathbb{1} \otimes \mathbb{1}_{H_f \leq \delta}). \tag{A.20}$$

Proof. Note that $\mathbb{1} \otimes \mathbb{1}_{H_f \leq \delta}$ commutes both with H_σ and $K_\sigma \otimes \mathbb{1}$. In addition, since the number operator N^σ also commutes with H_σ and $K_\sigma \otimes \mathbb{1}$, it suffices to prove (A.20) on states $\Phi \in D(H_\sigma)$ of the form $\Phi = \Phi_1 \otimes \Phi_2$ with $\|\Phi_1\| = \|\Phi_2\| = 1$, $\Phi_1 \in D(K_\sigma)$, and $\Phi_2 \in \mathrm{Ran}(\mathbb{1}_{H_f \leq \delta})$ is an eigenstate of $N^\sigma|_{\mathcal{F}^\sigma}$. For such a vector Φ, we have

$$\begin{aligned} \big(\Phi, H_\sigma \Phi\big) = \big(\Phi_1, K_\sigma \Phi_1\big) + \big(\Phi_2, (\frac{1}{2}P_f^2 + H_f)\Phi_2\big) \\ - \big(\Phi_1, \nabla K_\sigma \Phi_1\big)\big(\Phi_2, P_f \Phi_2\big). \end{aligned} \tag{A.21}$$

One can check that

$$\big|\big(\Phi_1, \nabla K_\sigma \Phi_1\big)\big| \leq \big(\Phi_1, (\nabla K_\sigma)^2 \Phi_1\big)^{1/2}, \tag{A.22}$$

$$\big|\big(\Phi_2, P_f \Phi_2\big)\big| \leq \big(\Phi_2, H_f \Phi_2\big), \tag{A.23}$$

and hence

$$\big(\Phi_1, \nabla K_\sigma \Phi_1\big)\big(\Phi_2, P_f \Phi_2\big) \leq \frac{1}{2}\big(\Phi_1, (\nabla K_\sigma)^2 \Phi_1\big)\big(\Phi_2, H_f \Phi_2\big) + \frac{1}{2}\big(\Phi_2, H_f \Phi_2\big). \tag{A.24}$$

Inserting this into (A.21) and using that $(\nabla K_\sigma)^2 \leq 2K_\sigma$, we obtain

$$\begin{aligned} \big(\Phi, H_\sigma \Phi\big) &\geq \big(\Phi_1, K_\sigma \Phi_1\big) + \frac{1}{2}\big(\Phi_2, H_f \Phi_2\big) - \frac{1}{2}\big(\Phi_1, (\nabla K_\sigma)^2 \Phi_1\big)\big(\Phi_2, H_f \Phi_2\big) \\ &\geq \big(\Phi_1, K_\sigma \Phi_1\big) - \delta\big(\Phi_1, \frac{1}{2}(\nabla K_\sigma)^2 \Phi_1\big) \\ &\geq (1-\delta)\big(\Phi_1, K_\sigma \Phi_1\big), \end{aligned} \tag{A.25}$$

which concludes the proof. □

Appendix B. The smooth Feshbach-Schur map

In this appendix we recall the definition and some of the main properties of the smooth Feshbach-Schur map introduced in [BCFS1]. The version we present uses aspects developed in [GH] and [FGS3].

Let $\mathcal{H}$ be a separable Hilbert space. Let χ, $\bar\chi$ be nonzero bounded operators on $\mathcal{H}$, such that $[\chi, \bar\chi] = 0$ and $\chi^2 + \bar\chi^2 = 1$. Let H and T be two closed operators on $\mathcal{H}$ such that $D(H) = D(T)$. Define $W = H - T$ on $D(T)$ and

$$H_\chi = T + \chi W \chi, \quad H_{\bar\chi} = T + \bar\chi W \bar\chi \tag{B.1}$$

We make the following hypotheses:

(1) $\chi T \subset T\chi$ and $\bar\chi T \subset T\bar\chi$.
(2) $T, H_{\bar\chi} : D(T) \cap \mathrm{Ran}(\bar\chi) \to \mathrm{Ran}(\bar\chi)$ are bijections with bounded inverses.
(3) $W\chi$ and χW extend to bounded operators on $\mathcal{H}$.

Given the above assumptions, the (smooth) Feshbach-Schur map $F_\chi(H)$ is defined by

$$F_\chi(H) = H_\chi - \chi W \bar\chi H_{\bar\chi}^{-1} \bar\chi W \chi. \tag{B.2}$$

Note that $F_\chi(H)$ is well-defined on $D(T)$. If Hypotheses (1),(2),(3) above are satisfied, we say that H is in the domain of F_χ. In addition, we consider the two auxiliary bounded operators $Q_\chi(H)$ and $Q_\chi^\#(H)$ defined by

$$Q_\chi(H) = \chi - \bar\chi H_{\bar\chi}^{-1} \bar\chi W \chi, \quad Q_\chi^\#(H) = \chi - \chi W \bar\chi H_{\bar\chi}^{-1} \bar\chi. \tag{B.3}$$

It follows from [BCFS1, GH, FGS3] that the smooth Feshbach-Schur map F_χ is isospectral in the following sense:

Theorem B.1. *Let $H, T, \chi, \bar\chi$ be as above. Then the following holds:*

(i) *Let V be a subspace such that $\mathrm{Ran}\chi \subset V \subset \mathcal{H}$, $T : D(T) \cap V \to V$ and $\bar\chi T^{-1} \bar\chi V \subset V$. Then $H : D(T) \to \mathcal{H}$ is bounded invertible if and only if $F_\chi(H) : D(T) \cap V \to V$ is bounded invertible, and we have*

$$H^{-1} = Q_\chi(H) F_\chi(H)^{-1} Q_\chi^\#(H) + \bar\chi H_{\bar\chi}^{-1} \bar\chi, \tag{B.4}$$

$$F_\chi(H)^{-1} = \chi H^{-1} \chi + \bar\chi T^{-1} \bar\chi. \tag{B.5}$$

(ii) *If $\phi \in \mathcal{H} \setminus \{0\}$ solves $H\phi = 0$ then $\psi := \chi\phi \in \mathrm{Ran}\chi \setminus \{0\}$ solves $F_\chi(H)\,\psi = 0$.*
(iii) *If $\psi \in \mathrm{Ran}\,\chi \setminus \{0\}$ solves $F_\chi(H)\,\psi = 0$ then $\phi := Q_\chi(H)\psi \in \mathcal{H} \setminus \{0\}$ solves $H\phi = 0$.*
(iv) *The multiplicity of the spectral value $\{0\}$ is conserved in the sense that*

$$\dim \mathrm{Ker} H = \dim \mathrm{Ker} F_\chi(H). \tag{B.6}$$

Next, we recall a result given in [FGS3] showing that a LAP for H can be deduced from a corresponding LAP for $F_\chi(H-\lambda)$, for suitably chosen λ's. Notice that, in [FGS3], $F_\chi(H-\lambda)$ is considered as an operator on $\mathcal{H}$, whereas its restriction to some closed subspace V is considered here. However, the the following theorem can be proven is the same way. For the convenience of the reader, we recall the proof.

Theorem B.2. *Let $H, T, \chi, \bar{\chi}$ be as above. Let Δ be an open interval in $\mathbb{R}$. Let V be a closed subspace of $\mathcal{H}$ satisfying the assumptions of Theorem B.1(i). Let B a self-adjoint operator on $\mathcal{H}$ such that $B : D(B)\cap V \to V$ and $[B \pm \mathrm{i}]^{-1}V \subset V$. Assume that $\forall \lambda \in \Delta$,*

$$[A_\lambda, B] \text{ extends to a bounded operator,} \tag{B.7}$$

where A_λ stands for one of the operators $A_\lambda = \chi,\ \overline{\chi},\ \chi W,\ W\chi,\ \bar{\chi}[H_{\bar{\chi}}-\lambda]^{-1}\bar{\chi}$. If $H-\lambda$ is in the domain of F_χ, then for any $\nu \geq 0$ and $0 < s \leq 1$,

$$\begin{aligned} &\lambda \mapsto \langle B\rangle^{-s}(F_\chi(H-\lambda)-\mathrm{i}0)^{-1}\langle B\rangle^{-s} \in C^\nu(\Delta;\mathcal{B}(V)) \\ &\textit{implies that} \quad \lambda \mapsto \langle B\rangle^{-s}(H-\lambda-\mathrm{i}0)^{-1}\langle B\rangle^{-s} \in C^\nu(\Delta;\mathcal{B}(\mathcal{H})). \end{aligned} \tag{B.8}$$

Proof. It follows form Equation (B.4) with H replaced by $H-\lambda-\mathrm{i}\varepsilon$ that

$$\begin{aligned} [H-\lambda-\mathrm{i}\varepsilon]^{-1} =& Q_\chi(H-\lambda-\mathrm{i}\varepsilon)F_\chi(H-\lambda-\mathrm{i}\varepsilon)^{-1}Q_\chi^\#(H-\lambda-\mathrm{i}\varepsilon) \\ &+ \bar{\chi}[H_{\bar{\chi}}-\lambda-\mathrm{i}\varepsilon]^{-1}\bar{\chi}. \end{aligned} \tag{B.9}$$

The map $\varepsilon \mapsto [H_{\bar{\chi}}-\lambda-\mathrm{i}\varepsilon]^{-1} \in \mathcal{B}(\mathrm{Ran}(\bar{\chi}))$ is analytic in a neighborhood of 0, and can be expanded as

$$[H_{\bar{\chi}}-\lambda-\mathrm{i}\varepsilon]^{-1} = [H_{\bar{\chi}}-\lambda]^{-1} + \mathrm{i}\varepsilon[H_{\bar{\chi}}-\lambda]^{-1}\bar{\chi}^2[H_{\bar{\chi}}-\lambda]^{-1} + O(\varepsilon^2). \tag{B.10}$$

This yields

$$\lim_{\varepsilon\to 0}\langle B\rangle^{-s}F_\chi(H-\lambda-\mathrm{i}\varepsilon)^{-1}\langle B\rangle^{-s} = \langle B\rangle^{-s}[F_\chi(H-\lambda)-\mathrm{i}0]^{-1}\langle B\rangle^{-s}. \tag{B.11}$$

Note that

$$\langle B\rangle^{-s} = \mathrm{C}_s\int_0^\infty \frac{\mathrm{d}\omega}{\omega^{s/2}}(\omega+1+B^2)^{-1}, \tag{B.12}$$

where $\mathrm{C}_s := \big[\int_0^\infty \frac{\mathrm{d}\omega}{\omega^{s/2}}(\omega+1)^{-1}\big]^{-1}$. Hence, Conditions (B.7) imply that the operators

$$\langle B\rangle^{-s}\chi\langle B\rangle^{s}, \quad \langle B\rangle^{-s}\overline{\chi}\langle B\rangle^{s}, \quad \langle B\rangle^{s}\chi\langle B\rangle^{-s}, \quad \langle B\rangle^{s}\bar{\chi}\langle B\rangle^{-s} \tag{B.13}$$

are bounded. Similarly, the maps

$$\lambda \mapsto \langle B\rangle^{-s}\bar{\chi}[H_{\bar{\chi}}-\lambda]^{-1}\bar{\chi}\langle B\rangle^{s} \quad \text{and} \quad \lambda \mapsto \langle B\rangle^{s}\bar{\chi}[H_{\bar{\chi}}-\lambda]^{-1}\bar{\chi}\langle B\rangle^{-s} \tag{B.14}$$

are in $C^\infty(\Delta;\mathcal{B}(\mathcal{H}))$. This property shows that

$$\langle B\rangle^{-s}Q_\chi(H-\lambda)\langle B\rangle^{s} \quad \text{and} \quad \langle B\rangle^{s}Q_\chi^\#(H-\lambda)\langle B\rangle^{-s} \tag{B.15}$$

are bounded and smooth in $\lambda \in \Delta$. The theorem then follows from (B.11), the fact that $H - \lambda$ is in the domain of F_χ, and (B.4). $\square$

Appendix C. Bound particles coupled to a quantized radiation field

In this appendix, we explain how to adapt the proof of Theorem 1.1 to the case of non-relativistic particles interacting with an infinitely heavy nucleus and coupled to a massless radiation field. To simplify matters, we assume that the non-relativistic particles are spinless, and that the bosons are scalar (Nelson's model). The Hamiltonian H^{N} associated to this system acts on $\mathcal{H} = \mathcal{H}_{\mathrm{el}} \otimes \mathcal{F}$, where $\mathcal{H}_{\mathrm{el}} = \mathrm{L}^2(\mathbb{R}^{3N})$, and $\mathcal{F} = \Gamma_s(\mathrm{L}^2(\mathbb{R}^3))$ is the symmetric Fock space over $\mathrm{L}^2(\mathbb{R}^3)$. It is given by

$$H^{\mathrm{N}} := H_{\mathrm{el}} \otimes \mathbb{1} + \mathbb{1} \otimes H_f + W. \tag{C.1}$$

Here, $H_{\mathrm{el}} = \sum_{j=1}^N p_j^2/2m_j + V$ denotes an N-particle Schrödinger operator on $\mathcal{H}_{\mathrm{el}}$. For k in $\mathbb{R}^3$, we denote by $a^*(k)$ and $a(k)$ the usual phonon creation and annihilation operators on $\mathcal{F}$ obeying the canonical commutation relations

$$[a^*(k), a^*(k')] = [a(k), a(k')] = 0 \quad , \quad [a(k), a^*(k')] = \delta(k - k'). \tag{C.2}$$

The operator associated with the energy of the free boson field, H_f, is given by the expression (1.4), except that the operators $a^*(k)$ and $a(k)$ now are scalar creation and annihilation operators as given above. The interaction W in (C.1) is assumed to be of the form $W = g\phi(G_x)$ where g is a small coupling constant, $x = (x_1, x_2, \dots, x_n)$ and

$$\phi(G_x) := \frac{1}{\sqrt{2}} \sum_{j=1}^N \int_{\mathbb{R}^3} \frac{\kappa^\Lambda(k)}{|k|^{1/2-\mu}} \left[e^{-ik\cdot x_j} a^*(k) + e^{ik\cdot x_j} a(k) \right] \mathrm{d}k. \tag{C.3}$$

As above, the function κ^Λ denotes an ultraviolet cutoff, and the parameter μ is assumed to be non-negative.

We assume that V is infinitely small with respect to $\sum_j p_j^2$, and that the spectrum of H_{el} consists of a sequence of discrete eigenvalues, $e_0, e_1, \dots$, below some semi-axis $[\Sigma, \infty)$. Let $E^{\mathrm{N}} := \inf(\sigma(H^{\mathrm{N}}))$ and $y := \mathrm{i}\nabla_k$. Adapting the proof of Theorem 1.1, one can show the following

Theorem C.1. *Let H^{N} be given as above. For any $\mu \geq 0$, there exists $g_0 > 0$ such that, for any $0 \leq g \leq g_0$, $1/2 < s \leq 1$, and any compact interval $J \subset (E^{\mathrm{N}}, (e_0 + e_1)/2)$,*

$$\sup_{z \in J_\pm} \left\| (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \left[H^{\mathrm{N}} - z \right]^{-1} (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \right\| \leq \mathrm{C}, \tag{C.4}$$

where C is a positive constant depending on J and s. In particular, the spectrum of H^{N} in $(E^{\mathrm{N}}, (e_0 + e_1)/2)$ is absolutely continuous. Moreover, the map

$$J \ni \lambda \mapsto (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \left[H^{\mathrm{N}} - \lambda \pm \mathrm{i}0^+ \right]^{-1} (\mathrm{d}\Gamma(\langle y \rangle) + 1)^{-s} \in B(\mathcal{H}) \tag{C.5}$$

is uniformly Hölder continuous in λ of order $s-1/2$.

Let us emphasize that Theorem C.1 does not require any infrared regularization in the form factor. In comparison, the proof of [FGS1] would give Theorem C.1 for any $\mu \geq 1$, and the one in [FGS3] for any $\mu > 0$. In [FGS1], this restriction comes from the estimate $\|f(H^{\mathrm{N}} - E^{\mathrm{N}}) - f(H^{\mathrm{N}}_\sigma - E^{\mathrm{N}}_\sigma)\| \leq \mathrm{C} g\sigma$ which holds for $\mu \geq 1$ (where f is a smooth function compactly supported in $[\sigma/3, 2\sigma/3]$, H^{N}_σ is the infrared cutoff Hamiltonian, see (C.6) below, and $E^{\mathrm{N}}_\sigma = \inf \sigma(H^{\mathrm{N}}_\sigma)$). In [FGS3], the assumption that $\mu > 0$ is needed to apply the renormalization group. However, for the standard model of non-relativistic QED (which is considered in [FGS1] and [FGS3]), thanks to a Pauli-Fierz transformation, the methods given in [FGS1] and [FGS3] work without any infrared regularization.

Proof. We briefly explain how to adapt the proof of Theorem 1.1. First, using the generator of dilatations on Fock space, B, as a conjugate operator, it follows from standard estimates that a Mourre estimate holds outside a neighborhood of E^{N}; see [BFS].

To obtain the LAP near E^{N}, we modify Sections 4 and 5 as follows: We take $T_\sigma = H^{\mathrm{N}}_\sigma$, where H^{N}_σ is the infrared cutoff Hamiltonian

$$H^{\mathrm{N}}_\sigma := H_{\mathrm{el}} \otimes \mathbb{1} + \mathbb{1} \otimes H_f + W_\sigma. \tag{C.6}$$

Here $W_\sigma = g\phi(G_{x,\sigma})$, and $\phi(G_{x,\sigma})$ is given by (C.3) except that the integral over $\mathbb{R}^3$ is replaced by the integral over $\{k \in \mathbb{R}^3, |k| \geq \sigma\}$. We define similarly $W^\sigma = H^{\mathrm{N}} - H^{\mathrm{N}}_\sigma = g\phi(G^\sigma_x)$ with the obvious notation. The Hilbert space $\mathcal{H}$ is unitarily equivalent to $\mathcal{H}_\sigma \otimes \mathcal{F}^\sigma$, where $\mathcal{H}_\sigma = \mathcal{H}_{\mathrm{el}} \otimes \mathcal{F}_\sigma$ and $\mathcal{F}_\sigma = \Gamma_s(\mathrm{L}^2(\{k \in \mathbb{R}^3, |k| \geq \sigma\}))$, respectively $\mathcal{F}^\sigma = \Gamma_s(\mathrm{L}^2(\{k \in \mathbb{R}^3, |k| \leq \sigma\}))$. In this representation, we can write

$$H^{\mathrm{N}} = K^{\mathrm{N}}_\sigma \otimes \mathbb{1} + \mathbb{1} \otimes H_f + W^\sigma, \tag{C.7}$$

where K^{N}_σ denotes the restriction of H^{N}_σ to $\mathcal{H}_\sigma$. It is known that the ground state energy E^{N}_σ of K^{N}_σ is separated from the rest of the spectrum by a gap of order $O(\sigma)$. Thus, letting $P_\sigma = \mathbb{1}_{\{E^{\mathrm{N}}_\sigma\}}(K^{\mathrm{N}}_\sigma)$ and $\chi = P_\sigma \otimes \chi^\sigma_f$, one can define the smooth Feshbach-Schur operator in the same way as in Section 4, that is

$$\begin{aligned} F(\lambda) &= F_\chi(H^{\mathrm{N}} - \lambda)|_{\mathrm{Ran}(P_\sigma \otimes \mathbb{1})} \\ &= E^{\mathrm{N}}_\sigma - \lambda + \mathbb{1} \otimes H_f + \chi W^\sigma \chi - \chi W^\sigma \bar{\chi}\big[H_{\bar{\chi}} - \lambda\big]^{-1} \bar{\chi} W^\sigma \chi, \end{aligned} \tag{C.8}$$

for λ in a neighborhood of E^{N}_σ. The proof of the Mourre estimate for $F(\lambda)$ follows then in the same way as in Section 5, using B^σ as a conjugate operator. Note that the "perturbation" W^σ is simpler here than the one considered in Section 4, in that it only consists of the sum of a creation and an annihilation operator. However, some exponential decay in the electronic position variables x_j has to be used in order to control the commutator of W^σ with B^σ. (We do not present details.) □

Appendix D. List of notations

Hilbert spaces

$$\mathcal{H} = \mathrm{L}^2(\mathbb{R}^3) \otimes \mathcal{F}, \tag{D.1}$$

$$\mathcal{F} = \Gamma_s(\mathrm{L}^2(\mathbb{R}^3 \times \mathbb{Z}_2)), \tag{D.2}$$

$$\mathcal{F}_\sigma = \Gamma_s(\mathrm{L}^2(\{(k,\lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2, |k| \geq \sigma\})), \tag{D.3}$$

$$\mathcal{F}^\sigma = \Gamma_s(\mathrm{L}^2(\{(k,\lambda) \in \mathbb{R}^3 \times \mathbb{Z}_2, |k| \leq \sigma\})). \tag{D.4}$$

Hamiltonians

$$H = \frac{1}{2}(P - P_f + \alpha^{\frac{1}{2}} A)^2 + H_f, \tag{D.5}$$

$$H_\sigma = \frac{1}{2}(P - P_f + \alpha^{\frac{1}{2}} A_\sigma)^2 + H_f \text{ (as an operator on } \mathcal{F}), \tag{D.6}$$

$$= K_\sigma \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + H_f\big) - \nabla K_\sigma \otimes P_f \text{ (as an operator on } \mathcal{F}_\sigma \otimes \mathcal{F}^\sigma), \tag{D.7}$$

$$\nabla H_\sigma = P - P_f + \alpha^{\frac{1}{2}} A_\sigma, \tag{D.8}$$

$$K_\sigma = H_\sigma|_{\mathcal{F}_\sigma}, \quad \nabla K_\sigma = \nabla H_\sigma|_{\mathcal{F}_\sigma}, \tag{D.9}$$

$$U_\sigma = H - H_\sigma, \tag{D.10}$$

$$T_\sigma = K_\sigma \otimes \mathbb{1} + \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + H_f\big) - \nabla E_\sigma \otimes P_f, \tag{D.11}$$

$$W_\sigma = H - T_\sigma = U_\sigma - (\nabla K_\sigma - \nabla E_\sigma) \otimes P_f, \tag{D.12}$$

$$H_\chi = T_\sigma + \chi W_\sigma \chi, \quad H_{\bar\chi} = T_\sigma + \bar\chi W_\sigma \bar\chi, \tag{D.13}$$

$$H^1_{\bar\chi} = T_\sigma - \bar\chi(\nabla K_\sigma - \nabla E_\sigma) \otimes P_f \bar\chi, \tag{D.14}$$

$$F = F_\chi(H - \lambda)|_{\mathrm{Ran}(P_\sigma \otimes \mathbb{1})} \tag{D.15}$$

$$= E_\sigma - \lambda + \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + H_f\big) - \nabla E_\sigma \otimes P_f + \chi U_\sigma \chi - \chi W_\sigma \bar\chi \big[H_{\bar\chi} - \lambda\big]^{-1} \bar\chi W_\sigma \chi, \tag{D.16}$$

$$F_0 = E_\sigma - \lambda + \mathbb{1} \otimes \big(\frac{1}{2}P_f^2 + H_f\big) - \nabla E_\sigma \otimes P_f, \tag{D.17}$$

$$W_1 = \chi U_\sigma \chi, \tag{D.18}$$

$$W_2 = -\chi W_\sigma \bar\chi \big[H_{\bar\chi} - \lambda\big]^{-1} \bar\chi W_\sigma \chi, \tag{D.19}$$

$$\tilde F = F + \lambda - E_\sigma, \quad \tilde F_0 = F_0 + \lambda - E_\sigma. \tag{D.20}$$

Conjugate operators

$$B = \mathrm{d}\Gamma(b), \quad b = \frac{\mathrm{i}}{2}(k \cdot \nabla_k + \nabla_k \cdot k), \tag{D.21}$$

$$B^\sigma = \mathrm{d}\Gamma(b^\sigma), \quad b^\sigma = \kappa^\sigma b \kappa^\sigma. \tag{D.22}$$

Intervals

$$E = \inf \sigma(H), \quad E_\sigma = \inf \sigma(H_\sigma), \tag{D.23}$$

$$J_\sigma^> = E + [\sigma, 2\sigma] \text{ (for } \sigma \geq \mathrm{C}_0 \alpha^{\frac{1}{2}}), \tag{D.24}$$

$$J_\sigma^< = E + [11\rho\sigma/128, 13\rho\sigma/128] \text{ (for } \sigma \leq \mathrm{C}_0' \alpha^{\frac{1}{2}}), \tag{D.25}$$

$$\rho : \text{ fixed parameter such that } 0 < \rho < 1 \text{ and } \mathrm{Gap}(K_\sigma) \geq \rho\sigma, \tag{D.26}$$

$$\Delta_\sigma = [-\rho\sigma/128, \rho\sigma/128], \tag{D.27}$$

$$\Delta_\sigma' = [\rho\sigma/16, \rho\sigma/8], \tag{D.28}$$

FIGURE 2. **The intervals $J_\sigma^<$ and $J_\sigma^>$**

Functions

$$\kappa^\Lambda \in \mathrm{C}_0^\infty(\{k, |k| \leq \Lambda\}; [0,1]) \text{ and } \kappa^\Lambda = 1 \text{ on } \{k, |k| \leq 3\Lambda/4\}, \tag{D.29}$$

$$f_\sigma \in \mathrm{C}_0^\infty([3\rho\sigma/64; 9\rho\sigma/64]; [0,1]) \text{ and } f_\sigma = 1 \text{ on } \Delta_\sigma', \tag{D.30}$$

$$\tilde{f}_\sigma : \text{ almost analytic extension of } f_\sigma. \tag{D.31}$$

(Almost) projections

$$P_\sigma = \mathbb{1}_{\{E_\sigma\}}(K_\sigma), \quad \bar{P}_\sigma = \mathbb{1} - P_\sigma, \tag{D.32}$$

$$\chi_f^\sigma = \kappa^{\rho\sigma}(H_f), \quad \bar{\chi}_f^\sigma = \sqrt{\mathbb{1} - (\chi_f^\sigma)^2}, \tag{D.33}$$

$$\chi = P_\sigma \otimes \chi_f^\sigma, \quad \bar{\chi} = P_\sigma \otimes \bar{\chi}_f^\sigma + \bar{P}_\sigma \otimes \mathbb{1}. \tag{D.34}$$

References

[AHS] S. Agmon, I. Herbst and E. Skibsted, *Perturbation of embedded eigenvalues in the generalized N-body problem*, Comm. Math. Phys. **122**, (1989), 411–438.

[AFGG] L. Amour, J. Faupin, B. Grébert and J-C. Guillot, *On the infrared problem for the dressed non-relativistic electron in a magnetic field*, In *Spectral and Scattering Theory for Quantum Magnetic Systems*, vol. **500** of *Contemp. Math.*, Amer. Math. Soc., Providence, RI, (2009), 1–24.

[ABG] W. Amrein, A. Boutet de Monvel, V. Georgescu, *C_0-groups, commutators methods and spectral theory for N-body Hamiltonians*, Vol. **135** of Progress in Mathematics. Basel-Boston: Birkhäuser, 1996.

[BCFS1] V. Bach, T. Chen, J. Fröhlich and I.M. Sigal, *Smooth Feshbach map and operator-theoretic renormalization group methods* , J. Funct. Anal., **203**, (2003), 44–92.

[BCFS2] V. Bach, T. Chen, J. Fröhlich and I.M. Sigal, *The renormalized electron mass in non-relativistic quantum electrodynamics*, J. Funct. Anal., **243**, (2007), 426–535.

[BFP] V. Bach, J. Fröhlich and A. Pizzo, *Infrared-Finite Algorithms in QED: The Groundstate of an Atom Interacting with the Quantized Radiation Field*, Comm. Math. Phys., **264**, (2006), 145–165.

[BFS] V. Bach, J. Fröhlich and I.M. Sigal, *Quantum electrodynamics of confined nonrelativistic particles*, Adv. Math., **137**, (1998), 299–395.

[Ch] T. Chen, *Infrared renormalization in non-relativistic qed and scaling criticality*, J. Funct. Anal., **254**, (2008), 2555–2647.

[CF] T. Chen and J. Fröhlich, *Coherent infrared representations in non-relativistic QED*, In *Spectral theory and mathematical physics: a Festschrift in honor of Barry Simon's 60th birthday*, vol. **76** of *Proc. Sympos. Pure Math.*, Amer. Math. Soc., Providence, RI, (2007), 25–45.

[CFP1] T. Chen, J. Fröhlich and A. Pizzo, *Infraparticle scattering states in non-relativistic QED. I. The Bloch-Nordsieck paradigm*, Comm. Math. Phys., to appear.

[CFP2] T. Chen, J. Fröhlich and A. Pizzo, *Infraparticle scattering states in non-relativistic QED. II. Mass shell properties*, J. Math. Phys., **50**, 012103, (2009).

[Fr1] J. Fröhlich, *On the infrared problem in a model of scalar electrons and massless, scalar bosons*, Ann. Inst. H. Poincaré Sect. A, **19**, (1973), 1–103.

[Fr2] J. Fröhlich, *Existence of dressed one electron states in a class of persistent models*, Fortschr. Phys., **22**, (1974), 159–198.

[FGSch] J. Fröhlich, M. Griesemer, and B. Schlein, *Asymptotic completeness for Compton scattering*, Comm. Math. Phys., **252**, (2004), 415–476.

[FGS1] J. Fröhlich, M. Griesemer and I.M. Sigal, *Spectral theory for the standard model of non-relativistic QED*, Comm. Math. Phys., **283**, (2008), 613–646.

[FGS2] J. Fröhlich, M. Griesemer and I.M. Sigal, *On Spectral Renormalization Group*, Rev. Math. Phys., **21**, (2009), 511-548.

[FGS3] J. Fröhlich, M. Griesemer and I.M. Sigal, *Spectral Renormalization Group and Local Decay in the Standard Model of the Non-relativistic Quantum Electrodynamics*, Preprint, arXiv:0904.1014, (2009).

[FP] J. Fröhlich and A. Pizzo, *Renormalized Electron Mass in Nonrelativistic QED*, Preprint, arXiv:0908.1858, (2009).

[GH] M. Griesemer and D. Hasler, *On the smooth Feshbach-Schur map*, J. Funct. Anal., **254**, (2008), 2329–2335.

[Gr] M. Griesemer, Private communication.

[HH] D. Hasler and I. Herbst, *Absence of Ground States for a Class of Translation Invariant Models of Non-relativistic QED*, Comm. Math. Phys., **279**, (2008), 769–787.

[HS] W. Hunziker and I.M. Sigal, *The quantum N-body problem*, J. Math. Phys., **41**, (2000), 3448–3510.

[LMS] M. Loss, T. Miyao and H. Spohn, *Lowest energy states in nonrelativisic QED: Atoms and ions in motion*, J. Funct. anal., **243**, (2007), 353–393.

[Mø] J.S. Møller, *On the essential spectrum of the translation invariant Nelson model*, Mathematical physics of quantum mechanics, 179–195, Lecture Notes in Phys., **690**, Springer, Berlin, (2006).

[Mo] E. Mourre, *Absence of singular continuous spectrum for certain selfadjoint operators*, Comm. Math. Phys., **78**, (1981), 391–408.

[PSS] P. Perry, I.M. Sigal and B. Simon, *Spectral analysis of N-body Schrödinger operators*, Ann. Math., **114**, (1981), 519–567.

[Pi] A. Pizzo, *One-particle (improper) states in Nelson's massless model*, Ann. Henri Poincaré, **4**, (2003), 439–486.

[Sp] H. Spohn, *Dynamics of charged particles and their radiation field*, Cambridge University Press, Cambridge, (2004).

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