STABILITY OF A LINEAR THERMOELASTIC BRESSE SYSTEM WITH SECOND SOUND UNDER NEW CONDITIONS ON THE COEFFICIENTS

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Abstract. In this paper, we discuss the stability **of the mathematical model** of a linear one-dimensional thermoelastic Bresse system, where the coupling is given through the first component of the Bresse model with the heat conduction of second sound type. We state the well-posedness and show the polynomial stability of the system, where the decay rate depends on the smoothness of initial data. Moreover, we prove the non exponential and the exponential decay depending on a new conditions on the parameters of the system. The proof is based on a combination of the energy method and the frequency domain approach.

Keywords and phrases: Bresse system, heat conduction, Cattaneo law, asymptotic behavior, energy method, frequency domain approach.

AMS classification: 35B40, 35L45, 74H40, 93D20, 93D15.

1. Introduction

In this paper, we consider the following **mathematical model consisting of a linear Bresse system coupled with heat equation via the first equation**:

$$
\begin{cases}
\rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) + \delta\theta_x = 0 & \text{in } (0,1)\times(0,\infty)\,, \\
\rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty)\,, \\
\rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty)\,, \\
\rho_3\theta_t + q_x + \delta\varphi_{xt} = 0 & \text{in } (0,1)\times(0,\infty)\,, \\
\tau q_t + \beta q + \theta_x = 0 & \text{in } (0,1)\times(0,\infty)
\end{cases}
\tag{1.1}
$$

along with the initial and boundary conditions of the form

$$
\begin{cases}
\varphi(x,0) = \varphi_0(x)\,, \quad \varphi_t(x,0) = \varphi_1(x) & \text{in } (0,1)\,, \\
\psi(x,0) = \psi_0(x)\,, \quad \psi_t(x,0) = \psi_1(x) & \text{in } (0,1)\,, \\
w(x,0) = w_0(x)\,, \quad w_t(x,0) = w_1(x) & \text{in } (0,1)\,, \\
\theta(x,0) = \theta_0(x)\,, \quad q(x,0) = q_0(x) & \text{in } (0,1)\,, \\
\varphi(0,t) = \psi_x(0,t) = w_x(0,t) = q(0,t) = 0 & \text{in } (0,\infty)\,, \\
\varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta(1,t) = 0 & \text{in } (0,\infty)\,,
\end{cases}
\tag{1.2}
$$

where ρ_1, ρ_2, ρ_3, b, k, k_0, τ, β, δ and l are positive constants, the initial data φ_0, φ_1, ψ_0, ψ_1, w_0, w_1, θ_0 and q_0 belong to a suitable Hilbert space, and the unknowns of (1.1)-(1.2) are the following variables:

$$(\varphi,\psi,w,\theta,q) : (0,1)\times(0,\infty) \to \mathbb{R}^5.$$

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Many researchers studied the well-posedness and stability of Bresse systems as well as the thermoelastic Bresse systems. Under different types of feedbacks, many stability results in the literature have been obtained depending on the following wave speeds parameters:

$$s_1 = \frac{k}{\rho_1}, \quad s_2 = \frac{b}{\rho_2} \quad \text{and} \quad s_3 = \frac{k_0}{\rho_1},$$

for this purpose, we refer the reader to [1, 3, 4, 5, 7, 9] and the references therein.

In [7], the authors considered the following coupled system:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta\psi_{xt} = 0 & \text{in } (0,1)\times(0,\infty), \\ \tau q_t + \beta q + \theta_x = 0 & \text{in } (0,1)\times(0,\infty). \end{cases} \tag{1.3}$$

They proved that (1.3) is exponentially stable if

$$s_1 = s_3, \quad \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k\rho_3}{\rho_1}\right) = \frac{\tau\delta^2}{b} \quad \text{and} \quad l \ \text{ is small},$$

and (1.3) is not exponentially stable if

$$s_1 \neq s_3 \quad \text{or} \quad \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k\rho_3}{\rho_1}\right) \neq \frac{\tau\delta^2}{b}.$$

Moreover, when

$$s_1 = s_3, \quad \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k\rho_3}{\rho_1}\right) \neq \frac{\tau\delta^2}{b} \quad \text{and} \quad l \ \text{ is small},$$

the polynomial stability for (1.3) was proved in [7] with the decay rate $t^{-\frac{1}{2}}$.

Recently, in [1], the authors considered the following system:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0 & \text{in } (0,1)\times(0,\infty), \\ \rho_3\theta_t + q_x + \delta w_{xt} = 0 & \text{in } (0,1)\times(0,\infty), \\ \tau q_t + \beta q + \theta_x = 0 & \text{in } (0,1)\times(0,\infty), \end{cases} \tag{1.4}$$

under the restriction

$$l \neq \frac{\pi}{2} + p\pi, \quad \forall p \in \mathbb{N}. \tag{1.5}$$

They proved that the solution is not exponentially stable if (1.6) or (1.7) does not hold, where

$$(k - k_0)\left(\rho_3 - \frac{\rho_1}{\tau k}\right) - \delta^2 = b\rho_1 - k\rho_2 = 0, \tag{1.6}$$

and

$$l^2 \neq \frac{b\rho_1 + k_0\rho_2}{k_0\rho_2}\left(\frac{\pi}{2} + p\pi\right)^2 + \frac{\rho_1 k}{\rho_2(k + k_0)}, \quad \forall p \in \mathbb{Z}. \tag{1.7}$$

Also, they proved that the solution is exponentially stable if (1.6) and (1.7) hold. Moreover, the polynomial stability for (1.4) with the decay rate $t^{-\frac{1}{8}}$ was proved in [1] when (1.7) holds and (1.6) does not hold.

The heat conduction in (1.1), (1.3) and (1.4) is of second sound type; known also as Cattaneo's law (for more details, see [7]). On the other hand, in (1.3) and (1.4), the Bresse system is indirectly stabilized via only its second or third equation, while in our case, the first hyperbolic equation in (1.1) is indirectly damped through the coupling with the last two ones in (1.1) (which describe the heat conduction of Cattaneo's law).

The stability of Bresse system via only its first equation was treated in [3, 4, 5] by the second author of the present paper using a linear frictional damping or an infinite memory or a heat conduction of type

I (known as Fourier's law) or type III. More precisely, it was proved in [3, 4, 5] that, independently of the values of the coefficients, the Bresse system

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) + F = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) = 0, & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0,1)\times(0,\infty)\,, \end{cases}$$

is not exponentially stable but it is at least polynomially stable with a decay rate depending on the smoothness of the initial data, where $F = \gamma\varphi_t$ and γ is a positive constant (a linear frictional damping; see [5]), or

$$F = \int_0^\infty g(s)\varphi_{xx}(x, t-s)\,ds,$$

and $g : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a differentiable function converging exponentially to zero at infinity (an infinite memory; see [3]) or $F = \delta\theta_x$ or $F = \delta\eta_{xt}$ (heat conduction of type I or type III, respectively; see [4]), where

$$\rho_3\theta_t - \beta\theta_{xx} + \delta\varphi_{xt} = 0 \quad \text{in } (0,1)\times(0,\infty)$$

and

$$\rho_3\eta_{tt} - \beta\eta_{xx} - \gamma\eta_{xxt} + \delta\varphi_{xt} = 0 \quad \text{in } (0,1)\times(0,\infty)\,.$$

Our objective **in this paper is to check from mathematical viewpoint whether the indirect damping via the coupling with the heat equation is enough to stabilize the full system**, we establish some stability results for the solutions: non exponential stability, polynomial stability and exponential stability. Contrary to the cases considered in [3, 4, 5], we prove that, under new relationships between the coefficients of (1.1), the heat conduction of Cattaneo's law is strong enough to stabilize (1.1)-(1.2) exponentially. When these relationships are not satisfied, we show that (1.1)-(1.2) is not exponentially stable and it is polynomially stable with a decay rate depending on the smoothness of the initial data. The stability results are proved using the energy method combining with the frequency domain approach.

Our paper is organized as follows. In section 2, we state the well-posedness of (1.1)-(1.2). In sections 3 and 4, we prove the lack of exponential stability as well as the polynomial decay of solutions for (1.1)-(1.2), respectively. Section 5 is devoted to the proof of the exponential decay of the solutions for (1.1)-(1.2). We give some concluding remarks in the last section.

2. Well-posedness

In this section, we state an existence, uniqueness and smoothness result for problem (1.1)-(1.2) using the semigroup theory and following the same procedure as in [1]. Introducing the vector functions

$$\Phi = \left(\varphi, u, \psi, v, w, z, \theta, q\right)^T \quad \text{and} \quad \Phi_0 = \left(\varphi_0, \varphi_1, \psi_0, \psi_1, w_0, w_1, \theta_0, q_0\right)^T,$$

where $u = \varphi_t$, $v = \psi_t$ and $z = w_t$, system (1.1)-(1.2) can be written as

$$\begin{cases} \Phi_t = \mathcal{A}\Phi, \quad \forall t > 0, \\ \Phi(0) = \Phi_0, \end{cases} \tag{2.1}$$

where the operator $\mathcal{A}$ is linear and defined by

$$\mathcal{A}\Phi = \begin{bmatrix} u \\ \dfrac{k}{\rho_1}\left(\varphi_x + \psi + l\,w\right)_x + \dfrac{lk_0}{\rho_1}\left(w_x - l\varphi\right) - \dfrac{\delta}{\rho_1}\theta_x \\ v \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}\left(\varphi_x + \psi + l\,w\right) \\ z \\ \dfrac{k_0}{\rho_1}\left(w_x - l\varphi\right)_x - \dfrac{lk}{\rho_1}\left(\varphi_x + \psi + l\,w\right) \\ -\dfrac{1}{\rho_3}q_x - \dfrac{\delta}{\rho_3}u_x \\ -\dfrac{\beta}{\tau}q - \dfrac{1}{\tau}\theta_x \end{bmatrix}. \tag{2.2}$$

We consider the following spaces:

$$H_*^1(0,1) = \left\{ f \in H^1(0,1) : f(0) = 0 \right\}, \ \widetilde{H}_*^1(0,1) = \left\{ f \in H^1(0,1) : f(1) = 0 \right\},$$

$$H_*^2(0,1) = H^2(0,1) \cap H_*^1(0,1), \ \widetilde{H}_*^2(0,1) = H^2(0,1) \cap \widetilde{H}_*^1(0,1)$$

and

$$\mathcal{H} = H_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \times L^2(0,1) \times \widetilde{H}_*^1(0,1) \times \left(L^2(0,1)\right)^3,$$

equipped with the inner product

$$\begin{aligned} \langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} &= k \langle \varphi_{1,x} + \psi_1 + l\, w_1, \varphi_{2,x} + \psi_2 + l\, w_2 \rangle + k_0 \langle w_{1,x} - l\varphi_1, w_{2,x} - l\varphi_2 \rangle \\ &\quad + b \langle \psi_{1,x}, \psi_{2,x} \rangle + \rho_1 \langle u_1, u_2 \rangle + \rho_2 \langle v_1, v_2 \rangle + \rho_1 \langle z_1, z_2 \rangle + \rho_3 \langle \theta_1, \theta_2 \rangle + \tau \langle q_1, q_2 \rangle, \end{aligned}$$

where $\langle \cdot, \cdot \rangle$ denotes the classical inner product of $L^2(0,1)$. The corresponding energy norm will be defined as follow:

$$\|\Phi\|_{\mathcal{H}}^2 = k \|\varphi_x + \psi + l\, w\|^2 + k_0 \|w_x - l\varphi\|^2 + b \|\psi_x\|^2 + \rho_1 \|u\|^2 + \rho_2 \|v\|^2 + \rho_1 \|z\|^2 + \rho_3 \|\theta\|^2 + \tau \|q\|^2,$$

where $\|\cdot\|$ is the standard norm of $L^2(0,1)$. Then $\mathcal{A}$, formally given in (2.2), has the domain

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \Phi \in \mathcal{H} : \ \varphi \in H_*^2(0,1), \ \psi, w \in \widetilde{H}_*^2(0,1), \ u, q \in H_*^1(0,1), \\ v, z, \theta \in \widetilde{H}_*^1(0,1), \ \varphi_x(1) = w_x(0) = \psi_x(0) = 0 \end{array} \right\}.$$

Using the same arguments and steps as in [1], we prove that, under the condition (1.5), the space $(\mathcal{H}, \langle \cdot, \cdot \rangle_{\mathcal{H}})$ **is a Hilbert space, the norm** $\|\cdot\|_{\mathcal{H}}$ **is equivalent to the one of**

$$H^1(0,1) \times L^2(0,1) \times H^1(0,1) \times L^2(0,1) \times H^1(0,1) \times \left(L^2(0,1)\right)^3,$$

$0 \in \rho(\mathcal{A})$ and the operator $\mathcal{A}$ is a maximal monotone operator and its domain is dense in $\mathcal{H}$. Therefore, from Lummer-Phillip's theorem, we have that $\mathcal{A}$ is the infinitesimal generator of a linear contraction C_0-semigroup in $\mathcal{H}$. So, the following well-posedness result holds (see [10]):

Theorem 2.1. *Assume that* (1.5) *holds. Then, for any* $m \in \mathbb{N}$ *and* $\Phi_0 \in D(\mathcal{A}^m)$, *system* (2.1) *admits a unique solution*

$$\Phi \in \cap_{j=0}^m C^{m-j}(\mathbb{R}^+, \ D(\mathcal{A}^j)),$$

where $D(\mathcal{A}^j)$ *is endowed by the graph norm* $\|\cdot\|_{D(\mathcal{A}^j)} = \sum_{r=0}^j \|\mathcal{A}^r \cdot\|_{\mathcal{H}}$.

Remark 1. *1.* **In the particular case** $m = 0$**; that is,** $\Phi_0 \in D(\mathcal{A}^0) = \mathcal{H}$**,** Φ **is a weak solution. For** $m \in \mathbb{N}^*$**,** Φ **is at least a classical solution**.

2. ***The operator*** A^{-1} ***is bounded and it is a bijection between*** H ***and the domain*** $D(A)$. *So* A ***has a nonempty resolvant and its spectrum is consisting entirely of eigenvalues***.

3. Lack of exponential stability

In this section, we state and prove a result regarding the lack of exponential stability of the solutions of (2.1) depending on the following constants:

$$\left\{ \begin{array}{l} \xi_0 = b\rho_1 - k_0\rho_2, \\ \xi_1 = \delta^2 - \left(\rho_1 - \dfrac{k\rho_2}{b} \right) \left(\dfrac{b\rho_3}{\rho_2} - \dfrac{1}{\tau} \right), \\ \xi_2 = \delta^2 - \left(1 - \dfrac{k}{k_0} \right) \left(\rho_3 k_0 - \dfrac{\rho_1}{\tau} \right), \end{array} \right.$$

and the following additional restriction on l:

$$l^2 \neq \frac{k_0\rho_2 - b\rho_1}{k_0\rho_2} \left(\frac{\pi}{2} + p\pi \right)^2 - \frac{k\rho_1}{\rho_2(k + k_0)}, \quad \forall p \in \mathbb{Z}. \tag{3.1}$$

First, we will state and prove the following crucial lemma needed for the proofs of our main results.

Lemma 3.1. *Assume that* (1.5) *holds. Then* (3.1) *and* $i\,\mathbb{R} \subset \rho(\mathcal{A})$ *are equivalent.*

Proof. Let $a \in \mathbb{R}^*$ and let $\Phi \in D(\mathcal{A})$ with

$$\mathcal{A}\Phi = i\,a\,\Phi. \tag{3.2}$$

It is sufficient to prove the equivalence between $\Phi = 0$ (that is, ia is not an eigenvalue of $\mathcal{A}$) and (3.1). We see that (3.2) is equivalent to

$$\begin{cases} u = ia\varphi, \quad v = ia\psi, \ z = iaw, \\ \dfrac{k}{\rho_1}\left(\varphi_x + \psi + l\,w\right)_x + \dfrac{lk_0}{\rho_1}\left(w_x - l\varphi\right) - \dfrac{\delta}{\rho_1}\theta_x = iau, \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}\left(\varphi_x + \psi + l\,w\right) = iav, \\ \dfrac{k_0}{\rho_1}\left(w_x - l\varphi\right)_x - \dfrac{lk}{\rho_1}\left(\varphi_x + \psi + l\,w\right) = iaz, \\ -\dfrac{1}{\rho_3}q_x - \dfrac{\delta}{\rho_3}u_x = ia\theta, \\ -\dfrac{\beta}{\tau}q - \dfrac{1}{\tau}\theta_x = iaq. \end{cases} \tag{3.3}$$

As in [1], computing $\langle \mathcal{A}\Phi, \Phi\rangle$, we get

$$\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = -\beta \left\|q\right\|^2. \tag{3.4}$$

Therefore, using (3.2),

$$-\beta \left\|q\right\|^2 = Re\,\langle \mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = Re\,\langle ia\Phi, \Phi\rangle_{\mathcal{H}} = Re\,ia\left\|\Phi\right\|^2_{\mathcal{H}} = 0.$$

So we deduce that

$$q = 0. \tag{3.5}$$

Taking into account that $\theta \in \tilde{H}{}^1_*(0,1)$ and using (3.5) and the eighth equation of (3.3), we deduce that

$$\theta = 0. \tag{3.6}$$

By using the seventh equation of (3.3), (3.5) and (3.6), we find

$$u_x = 0,$$

and with the first equation of (3.3), we obtain that

$$\varphi_x = 0.$$

As $\varphi \in H^1_*(0,1)$ and thanks again to the first equation of (3.3), we have

$$\varphi = u = 0. \tag{3.7}$$

Using (3.5), (3.6) and (3.7), we remark that (3.3) is reduced to

$$\begin{cases} v = ia\psi, \ z = iaw, \\ \psi_x + l\left(1 + \dfrac{k_0}{k}\right)w_x = 0, \\ \dfrac{b}{k}\psi_{xx} - \left(\psi + l\,w\right) = -\dfrac{\rho_2 a^2}{k}\psi, \\ \dfrac{k_0}{lk}w_{xx} - \left(\psi + l\,w\right) = -\dfrac{\rho_1 a^2}{lk}w. \end{cases} \tag{3.8}$$

Taking into account that $\psi(1) = w(1) = 0$, we remark that the third equation of (3.8) is equivalent to

$$\psi = -l\left(1 + \frac{k_0}{k}\right)w. \tag{3.9}$$

Using the last two equations of (3.8), we obtain

$$\frac{b}{k}\psi_{xx} - \frac{k_0}{lk}w_{xx} = -\frac{\rho_2 a^2}{k}\psi + \frac{\rho_1 a^2}{lk}w,$$

and by (3.9), we have

$$-\left(\frac{b}{k}l\left(1 + \frac{k_0}{k}\right) + \frac{k_0}{lk}\right)w_{xx} = \frac{a^2}{lk}\left(\rho_2 l^2\left(1 + \frac{k_0}{k}\right) + \rho_1\right)w \tag{3.10}$$

with the boundary conditions

$$w\,(1) = w_x\,(0) = 0.$$

Equation (3.10) is equivalent to, for some constants C_1 and C_2,

$$w(x) = C_1 \cos(Ax) + C_2 \sin(Ax) \quad \text{with } A = \sqrt{\frac{a^2\left(\rho_2 l^2\left(1+\frac{k_0}{k}\right)+\rho_1\right)}{bl^2\left(1+\frac{k_0}{k}\right)+k_0}}.$$

Then, the boundary condition $w_x\,(0) = 0$ implies that $C_2 = 0$, and so, according to (3.9),

$$w(x) = C_1 \cos(Ax) \quad \text{and} \quad \psi(x) = -C_1 l\left(1+\frac{k_0}{k}\right)\cos(Ax). \tag{3.11}$$

Assume that (3.1) holds. We have to prove that $C_1 = 0$. Assume by contradiction that $C_1 \neq 0$. Using (3.11) and the definition of A, we observe that the last two equations of (3.8) are equivalent to

$$a^2\left(\rho_1 b - k_0\rho_2\right) + \frac{kk_0}{k+k_0}\left(bl^2\left(1+\frac{k_0}{k}\right)+k_0\right) = 0. \tag{3.12}$$

On the other hand, (3.11) and the boundary condition $w(1) = 0$ lead to

$$\exists p \in \mathbb{Z} :\ A = \frac{\pi}{2} + p\pi. \tag{3.13}$$

By combining (3.12), (3.13) and the definition of A, we arrive at

$$\exists p \in \mathbb{Z} :\ l^2 = \frac{k_0\rho_2 - b\rho_1}{k_0\rho_2}\left(\frac{\pi}{2}+p\pi\right)^2 - \frac{k\rho_1}{\rho_2(k+k_0)}, \tag{3.14}$$

which is a contradiction with (3.1). Hence $C_1 = 0$, and consequently, $\psi = w = v = z = 0$ according to (3.11) and the first two equations of (3.8). Then, with (3.5), (3.6) and (3.7), it is clear that $\Phi = 0$. This shows that (3.15) is satisfied.

Now, assume that (3.1) is not satisfied; that is (3.14) holds. We notice that, for

$$a = \left(\frac{\pi}{2}+p\pi\right)\sqrt{\frac{bl^2\left(1+\frac{k_0}{k}\right)+k_0}{\rho_2 l^2\left(1+\frac{k_0}{k}\right)+\rho_1}},$$

and for any $C_1 \in \mathbb{C}$, the function

$$\Phi(x) = \left(0, 0, -l\left(1+\frac{k_0}{k}\right)C_1\cos(Ax), -il\left(1+\frac{k_0}{k}\right)C_1 a\cos(Ax), C_1\cos(Ax), iC_1 a\cos(Ax), 0, 0\right)^T,$$

is in $D(\mathcal{A})$ and satisfies (3.2). Hence $ia \notin \rho(\mathcal{A})$, which implies that (3.15) does not hold. Conclusion, (3.1) and (3.15) are equivalent. □

Theorem 3.2. *Assume that* (1.5) *holds. Then the semigroup associated to problem* (2.1) *is not exponentially stable if* (3.1) *does not hold or* $\xi_0 = 0$ *or* $\xi_1 \neq 0$ *or* $\xi_2 \neq 0$.

Proof. It is known that the exponential stability holds if and only if (see [6, 11])

$$i\,\mathbb{R} \subset \rho(\mathcal{A}) \tag{3.15}$$

and

$$\sup_{\lambda\in\mathbb{R}}\left\|(i\lambda I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{3.16}$$

We know, from Lemma 3.1 that (3.15) is not satisfied if (3.1) does not hold. Now, we need to prove that (3.16) does not hold if $\xi_0 = 0$ or $\xi_1 \neq 0$ or $\xi_2 \neq 0$.

Assume that $\xi_0 = 0$ or $\xi_1 \neq 0$ or $\xi_2 \neq 0$. We follow the same procedures as in [1], where we prove that there exists a sequence $(\lambda_n)_{n\in\mathbb{N}} \subset \mathbb{R}$ such that

$$\lim_{n\longrightarrow\infty} \left\| (i\lambda_n I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} = \infty,$$

which is equivalent to prove that there exists $(F_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ with

$$\|F_n\|_{\mathcal{H}} \leq 1, \quad \forall n \in \mathbb{N}, \tag{3.17}$$

and, for $\Phi_n = (i\lambda_n I - \mathcal{A})^{-1} F_n$,

$$\lim_{n\longrightarrow\infty} \|\Phi_n\|_{\mathcal{H}} = \infty, \tag{3.18}$$

therefore, we have

$$i\lambda_n \Phi_n - \mathcal{A}\Phi_n = F_n. \tag{3.19}$$

So to say, we have to look at the solution of spectral equation (3.19) and show that the corresponding solution Φ_n is not bounded when F_n is bounded in $\mathcal{H}$. Rewriting the spectral equation in term of its components, we have

$$\begin{cases} i\lambda_n\varphi_n - u_n = f_{n,1}, \\ i\lambda_n\rho_1 u_n - k\left(\varphi_{n,x} + \psi_n + l\, w_n\right)_x - lk_0\left(w_{n,x} - l\varphi_n\right) + \delta\theta_{n,x} = \rho_1 f_{n,2}, \\ i\lambda_n\psi_n - v_n = f_{n,3}, \\ i\lambda_n\rho_2 v_n - b\psi_{n,xx} + k\left(\varphi_{n,x} + \psi_n + l\, w_n\right) = \rho_2 f_{n,4}, \\ i\lambda_n w_n - z_n = f_{n,5}, \\ i\lambda_n\rho_1 z_n - k_0\left(w_{n,x} - l\varphi_n\right)_x + lk\left(\varphi_{n,x} + \psi_n + l\, w_n\right) = \rho_1 f_{n,6}, \\ i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x} = \rho_3 f_{n,7}, \\ i\lambda_n\tau q_n + \beta q_n + \theta_{n,x} = \tau f_{n,8}, \end{cases} \tag{3.20}$$

where $F_n = (f_{n,1}, \cdots, f_{n,8}) \in \mathcal{H}$ and

$$\Phi_n = (\varphi_n, u_n, \psi_n, v_n, w_n, z_n, \theta_n, q_n) \in D(\mathcal{A}). \tag{3.21}$$

We will prove that there exists a sequence of real numbers $(\lambda_n)_{n\in\mathbb{N}}$ and functions $(F_n)_{n\in\mathbb{N}} \subset \mathcal{H}$ verifying (3.17), (3.18) and (3.20). To do this, we take

$$f_{n,1} = f_{n,3} = f_{n,5} = 0. \tag{3.22}$$

So, $(3.20)_1$, $(3.20)_3$ and $(3.20)_5$ are equivalent to

$$u_n = i\lambda_n\varphi_n, \quad v_n = i\lambda_n\psi_n, \quad \text{and} \quad z_n = i\lambda_n w_n. \tag{3.23}$$

Then solving (3.20) is reduced to solving

$$\begin{cases} -\lambda_n^2\rho_1\varphi_n - k\left(\varphi_{n,x} + \psi_n + l\, w_n\right)_x - lk_0\left(w_{n,x} - l\varphi_n\right) + \delta\theta_{n,x} = \rho_1 f_{n,2}, \\ -\lambda_n^2\rho_2\psi_n - b\psi_{n,xx} + k\left(\varphi_{n,x} + \psi_n + l\, w_n\right) = \rho_2 f_{n,4}, \\ -\lambda_n^2\rho_1 w_n - k_0\left(w_{n,x} - l\varphi_n\right)_x + lk\left(\varphi_{n,x} + \psi_n + l\, w_n\right) = \rho_1 f_{n,6}, \\ i\lambda_n\rho_3\theta_n + q_{n,x} + i\delta\lambda_n\varphi_{n,x} = \rho_3 f_{n,7}, \\ i\lambda_n\tau q_n + \beta q_n + \theta_{n,x} = \tau f_{n,8}. \end{cases} \tag{3.24}$$

To simplify the calculations, let $N = \dfrac{(2n+1)\pi}{2}$. Now, according to our hypotheses in Theorem 3.2, we consider the three cases $\xi_0 = 0$, $[\xi_0 \neq 0$ and $\xi_1 \neq 0]$ and $[\xi_0 \neq 0$ and $\xi_2 \neq 0]$.

Case 1: $\xi_0 = 0$. We have $\dfrac{b}{\rho_2} = \dfrac{k_0}{\rho_1}$, then we choose

$$f_{n,4}(x) = -\frac{lk_0}{\rho_2} D\cos\left(Nx\right), \;\; f_{n,6}(x) = -\frac{l^2 k_0}{\rho_1} D\cos\left(Nx\right) \tag{3.25}$$

and

$$f_{n,2} = f_{n,7} = f_{n,8} = 0, \tag{3.26}$$

where $D \in \mathbb{R}$, which will be fixed. We will look for a particular solution $\Phi_n \in D(\mathcal{A})$ of (3.19) as follow:

$$\Phi_n = \left(0, 0, B\cos\left(Nx\right), iB\lambda_n\cos\left(Nx\right), D\cos\left(Nx\right), iD\lambda_n\cos\left(Nx\right), 0, 0\right)^T,$$

where $B \in \mathbb{R}$ that will be chosen. So (3.23) is satisfied and $\Phi_n \in D(\mathcal{A})$. On the other hand, Φ_n satisfies (3.24) if and only if the coefficients B and D satisfy the following system:

$$\begin{cases} kB + l(k+k_0)D = 0, \\ \left(-\lambda_n^2 + \dfrac{b}{\rho_2}N^2 + \dfrac{k}{\rho_2}\right)B + \dfrac{lk}{\rho_2}D = -\dfrac{lk_0}{\rho_2}D, \\ \dfrac{lk}{\rho_1}B + \left(-\lambda_n^2 + \dfrac{k_0}{\rho_1}N^2 + \dfrac{l^2 k}{\rho_1}\right)D = -\dfrac{l^2 k_0}{\rho_1}D. \end{cases} \tag{3.27}$$

Now, we will take

$$\lambda_n = N\sqrt{\frac{k_0}{\rho_1}}.$$

Because $\dfrac{b}{\rho_2} = \dfrac{k_0}{\rho_1}$, we have

$$-\lambda_n^2 + \frac{b}{\rho_2}N^2 = -\lambda_n^2 + \frac{k_0}{\rho_1}N^2 = 0,$$

and therefore, the system (3.27) will be reduced to

$$kB + l\,(k+k_0)\,D = 0,$$

which is equivalent to

$$B = -l\left(1 + \frac{k_0}{k}\right)D.$$

Choosing

$$B = -l\left(1 + \frac{k_0}{k}\right)\frac{\rho_1\rho_2}{lk_0\sqrt{\rho_1^2 + l^2\rho_2^2}} \quad \text{and} \quad D = \frac{\rho_1\rho_2}{lk_0\sqrt{\rho_1^2 + l^2\rho_2^2}},$$

and using (3.22), (3.25) and (3.26), we obtain

$$\begin{aligned} \|F_n\|_{\mathcal{H}}^2 = \|f_{n,4}\|^2 + \|f_{n,6}\|^2 &= \left(\frac{lk_0}{\rho_2}\right)^2\left[1 + \left(\frac{l\rho_2}{\rho_1}\right)^2\right]D^2\int_0^1 \cos^2(Nx)\,dx \\ &\le \left(\frac{lk_0}{\rho_2}\right)^2\left[1 + \left(\frac{l\rho_2}{\rho_1}\right)^2\right]D^2 = 1, \end{aligned}$$

which implies (3.17). On the other hand, we have

$$\begin{aligned} \|\Phi_n\|_{\mathcal{H}}^2 \ge k_0\,\|w_{n,x} - l\varphi_n\|^2 &= k_0\,\|w_{n,x}\|^2 = k_0 D^2 N^2 \int_0^1 \sin^2(Nx)\,dx \\ &= \frac{k_0}{2}D^2N^2\int_0^1 [1 - \cos(2Nx)]\,dx = \frac{k_0}{2}D^2N^2, \end{aligned}$$

hence (3.18) is satisfied.

Case 2: $\xi_0 \neq 0$ **and** $\xi_1 \neq 0$. We have $\dfrac{b}{\rho_2} \neq \dfrac{k_0}{\rho_1}$, and $\delta^2 - \left(\rho_1 - \dfrac{k\rho_2}{b}\right)\left(\dfrac{b\rho_3}{\rho_2} - \dfrac{1}{\tau}\right) \neq 0$, then we choose

$$f_{n,2} = f_{n,6} = f_{n,7} = f_{n,8} = 0 \quad \text{and} \quad f_{n,4} = \cos(Nx), \tag{3.28}$$

we consider (3.23) and we take

$$\begin{cases} \varphi_n = \alpha_1 \sin(Nx), \quad \psi_n = \alpha_2\cos(Nx), \quad w_n = \alpha_3\cos(Nx), \\ \theta_n = \alpha_4\cos(Nx), \quad q_n = \alpha_5 \sin(Nx), \\ \lambda_n^2 = \dfrac{b}{\rho_2}N^2 - \dfrac{k_0}{\rho_2}, \quad \text{for } n \text{ large}, \end{cases} \tag{3.29}$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are constants that will be fixed. It is clear that (3.17) is satisfied and $\Phi_n \in D(\mathcal{A})$. Using (3.28) and (3.29), we observe that (3.24) is equivalent to

$$
\left\{
\begin{array}{l}
\left(kN^2 - \lambda_n^2\rho_1 + l^2k_0\right)\alpha_1 + kN\alpha_2 + l\left(k+k_0\right)N\alpha_3 - \delta N\alpha_4 = 0,\\
kN\alpha_1 + \left(bN^2 - \lambda_n^2\rho_2 + k\right)\alpha_2 + kl\alpha_3 = \rho_2,\\
l\left(k+k_0\right)N\alpha_1 + lk\alpha_2 + \left(k_0N^2 - \lambda_n^2\rho_1 + l^2k\right)\alpha_3 = 0,\\
i\lambda_n\rho_3\alpha_4 + N\alpha_5 + \delta i\lambda_n N\alpha_1 = 0,\\
\left(i\lambda_n\tau + \beta\right)\alpha_5 - N\alpha_4 = 0.
\end{array}
\right. \tag{3.30}
$$

Using the definition of λ_n given in (3.29), we see that (3.30) is reduced to

$$
\left\{
\begin{array}{l}
\left(\left(k - \dfrac{\rho_1 b}{\rho_2}\right)N^2 + \dfrac{k_0\rho_1}{\rho_2} + l^2k_0\right)\alpha_1 + kN\alpha_2 + l\left(k+k_0\right)N\alpha_3 - \delta N\alpha_4 = 0,\\
\alpha_2 = \dfrac{\rho_2}{k_0+k} - \dfrac{kN}{k_0+k}\alpha_1 - \dfrac{kl}{k_0+k}\alpha_3,\\
\left(\left(k_0 - \dfrac{\rho_1 b}{\rho_2}\right)N^2 + \dfrac{\rho_1 k_0}{\rho_2} + l^2k\right)\alpha_3 + l\left(k+k_0\right)N\alpha_1 + lk\alpha_2 = 0,\\
\alpha_5 = \dfrac{iN^2\delta\lambda_n}{\left(\dfrac{\tau\rho_3 b}{\rho_2} - 1\right)N^2 - \dfrac{\tau\rho_3k_0}{\rho_2} - i\lambda_n\rho_3\beta}\alpha_1,\\
\alpha_4 = \dfrac{-\dfrac{b}{\rho_2}N^3\tau\delta + \dfrac{\tau\delta k_0 N}{\rho_2} + i\delta\lambda_n N\beta}{\left(\dfrac{\tau\rho_3 b}{\rho_2} - 1\right)N^2 - \dfrac{\tau\rho_3k_0}{\rho_2} - i\lambda_n\rho_3\beta}\alpha_1,
\end{array}
\right. \tag{3.31}
$$

inserting $(3.31)_2$ **into** $(3.31)_3$, we deduce that

$$
\alpha_3 = -\frac{\left(k_0^2 + 2kk_0\right)lN\alpha_1 + lk\rho_2}{\left(k_0+k\right)\left(\left(k_0 - \dfrac{\rho_1 b}{\rho_2}\right)N^2 + \dfrac{k_0\rho_1}{\rho_2} + \dfrac{kk_0l^2}{k_0+k}\right)}, \tag{3.32}
$$

and then $(3.31)_1 \times \dfrac{1}{N^3}$ is equivalent to

$$
\begin{aligned}
&\frac{b\tau}{\rho_2}\left[\left(\frac{kk_0\rho_2}{\left(k_0+k\right)b} - \rho_1\right)\left(\frac{\rho_3 b}{\rho_2} - \frac{1}{\tau}\right) + \delta^2\right]N\alpha_1 - i\beta\left[\delta^2 + \left(\frac{kk_0}{k_0+k} - \frac{\rho_1 b}{\rho_2}\right)\rho_3\right]\frac{\lambda_n}{N}\alpha_1 \\
&\quad + \left(\frac{\rho_1}{\rho_2} + l^2\right)\left(\frac{\tau\rho_3 b}{\rho_2} - 1\right)\frac{k_0}{N}\alpha_1 - \left[\delta^2 + \left(\frac{kk_0}{k_0+k} - \frac{\rho_1 b}{\rho_2}\right)\rho_3\right]\frac{\tau k_0}{N\rho_2}\alpha_1 \\
&\quad - \frac{\left[\left(\dfrac{\tau\rho_3 b}{\rho_2} - 1\right) - \dfrac{\tau\rho_3k_0}{N^2\rho_2} - i\dfrac{\lambda_n}{N^2}\rho_3\beta\right]\left(k_0^2 + 2kk_0\right)^2 l^2}{\left(k_0+k\right)^2\left[\left(k_0 - \dfrac{\rho_1 b}{\rho_2}\right) + \dfrac{k_0\rho_1}{N^2\rho_2} + \dfrac{kk_0l^2}{\left(k_0+k\right)N^2}\right]N}\alpha_1 - k_0\left(\frac{\rho_1}{\rho_2} + l^2\right)\left(\frac{\tau\rho_3k_0}{\rho_2N^3} + i\frac{\lambda_n}{N^3}\rho_3\beta\right)\alpha_1 \\
&= \frac{\left[\left(\dfrac{\tau\rho_3 b}{\rho_2} - 1\right) - \dfrac{\tau\rho_3k_0}{N^2\rho_2} - i\dfrac{\lambda_n}{N^2}\rho_3\beta\right]\left(k_0^2 + 2kk_0\right)l^2k\rho_2}{\left(k_0+k\right)^2\left[\left(k_0 - \dfrac{\rho_1 b}{\rho_2}\right)N^2 + \dfrac{k_0\rho_1}{\rho_2} + \dfrac{kk_0l^2}{k_0+k}\right]} - \frac{k\rho_2\left[\left(\dfrac{\tau\rho_3 b}{\rho_2} - 1\right) - \dfrac{\tau\rho_3k_0}{N^2\rho_2} - i\rho_3\beta\dfrac{\lambda_n}{N^2}\right]}{k_0+k}.
\end{aligned} \tag{3.33}
$$

then, we have

$$
x_n\,\alpha_1 = y_n,
$$

where

$$\begin{cases} x_n = \frac{b\tau}{\rho_2}\left[\left(\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right)\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2\right]N - i\beta\left[\delta^2+\left(\frac{kk_0}{k_0+k}-\frac{\rho_1 b}{\rho_2}\right)\rho_3\right]\frac{\lambda_n}{N} \\ +\left(\frac{\rho_1}{\rho_2}+l^2\right)\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)\frac{k_0}{N}-\left[\delta^2+\left(\frac{kk_0}{k_0+k}-\frac{\rho_1 b}{\rho_2}\right)\rho_3\right]\frac{\tau k_0}{N\rho_2} \\ -\frac{\left[\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)-\frac{\tau\rho_3 k_0}{N^2\rho_2}-i\frac{\lambda_n}{N^2}\rho_3\beta\right]\left(k_0^2+2kk_0\right)^2 l^2}{(k_0+k)^2\left[\left(k_0-\frac{\rho_1 b}{\rho_2}\right)+\frac{k_0\rho_1}{N^2\rho_2}+\frac{kk_0 l^2}{(k_0+k)\,N^2}\right]N} \\ -k_0\left(\frac{\rho_1}{\rho_2}+l^2\right)\left(\frac{\tau\rho_3 k_0}{\rho_2 N^3}+i\frac{\lambda_n}{N^3}\rho_3\beta\right), \\ \\ y_n = \frac{\left[\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)-\frac{\tau\rho_3 k_0}{N^2\rho_2}-i\frac{\lambda_n}{N^2}\rho_3\beta\right]\left(k_0^2+2kk_0\right)l^2 k\rho_2}{(k_0+k)^2\left[\left(k_0-\frac{\rho_1 b}{\rho_2}\right)N^2+\frac{k_0\rho_1}{\rho_2}+\frac{kk_0 l^2}{k_0+k}\right]} \\ -\frac{k\rho_2\left[\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)-\frac{\tau\rho_3 k_0}{N^2\rho_2}-i\rho_3\beta\frac{\lambda_n}{N^2}\right]}{k_0+k}, \end{cases}$$

using the fact that $\lambda_n^2 = \frac{b}{\rho_2}N^2 - \frac{k_0}{\rho_2}$, we deduce that

$$\begin{aligned} \lim_{n\to\infty}|x_n| &= \lim_{n\to\infty}\left|\frac{b\tau}{\rho_2}\left[\left(\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right)\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2\right]N - i\beta\left[\delta^2+\left(\frac{kk_0}{k_0+k}-\frac{\rho_1 b}{\rho_2}\right)\rho_3\right]\frac{\lambda_n}{N}\right| \\ &= \lim_{n\to\infty}\left|\frac{b\tau}{\rho_2}\left[\left(\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right)\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2\right]\right|N \\ &= \infty \end{aligned}$$

Therefore, $x_n \neq 0$ for every n sufficiently large, This shows that (3.33) has indeed a solution α_1 (for all n large enough), which is given by

$$\alpha_1 = \frac{y_n}{x_n}.$$

Now, we distinguish three subcases.

Subcase 2.1: $\left[\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right]\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2 \neq 0$ **and** $\frac{\tau\rho_3 b}{\rho_2}-1 \neq 0$. **Throughout this section, the notation $\simeq$ means that "asymptotically equal"**, then we deduce from (3.32) and (3.33), as $n \to \infty$,

$$\alpha_1 \simeq -\frac{k\rho_2^2\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)}{b\tau\,(k_0+k)\left[\left(\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right)\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2\right]N},$$

$$\alpha_3 \simeq -\frac{\left(k_0^2+2kk_0\right)l}{(k_0+k)\left(k_0-\frac{\rho_1 b}{\rho_2}\right)N}\alpha_1 - \frac{lk\rho_2}{(k_0+k)\left(k_0-\frac{\rho_1 b}{\rho_2}\right)N^2} \tag{3.34}$$

and

$$\alpha_2 \simeq \frac{\rho_2\left[\left(\frac{k\rho_2}{b}-\rho_1\right)\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2\right]}{(k_0+k)\left[\left(\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right)\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2\right]}.$$

As $\xi_1 \neq 0$; that is, $\left(\frac{k\rho_2}{b}-\rho_1\right)\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2 \neq 0$, then

$$\lim_{n\to\infty}|\alpha_2| > 0. \tag{3.35}$$

Subcase 2.2: $\left[\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right]\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2\neq 0$ **and** $\frac{\tau\rho_3 b}{\rho_2}-1=0$. We deduce from (3.34), (3.32), (3.33) and the choice of λ_n in (3.29) that, when $n\to\infty$,

$$\alpha_1\simeq\frac{i\rho_2^2\rho_3\beta k\sqrt{\frac{b}{\rho_2}}}{(k_0+k)\,b\tau\delta^2N^2}\quad\text{and}\quad\alpha_2\simeq\frac{\rho_2}{k_0+k},$$

which implies (3.35).

Subcase 2.3: $\left[\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right]\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2=0$. We see that (3.33) becomes

$$-i\left[\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right]\frac{\beta}{\tau}\lambda_nN^2\alpha_1+\left(\frac{\rho_1}{\rho_2}+l^2\right)\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)k_0N^2\alpha_1-\left(\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right)\frac{k_0}{\rho_2}N^2\alpha_1$$
$$-\frac{\left[\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)N^2-\frac{\tau\rho_3k_0}{\rho_2}-i\lambda_n\rho_3\beta\right]\left(k_0^2+2kk_0\right)^2l^2}{(k_0+k)^2\left[\left(k_0-\frac{\rho_1 b}{\rho_2}\right)N^2+\frac{k_0\rho_1}{\rho_2}+\frac{kk_0l^2}{k_0+k}\right]}N^2\alpha_1-k_0\left(\frac{\rho_1}{\rho_2}+l^2\right)\left(\frac{\tau\rho_3k_0}{\rho_2}+i\lambda_n\rho_3\beta\right)\alpha_1$$
$$=\frac{\left[\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)N^2-\frac{\tau\rho_3k_0}{\rho_2}-i\lambda_n\rho_3\beta\right]\left(k_0^2+2kk_0\right)l^2k\rho_2N}{(k_0+k)^2\left[\left(k_0-\frac{\rho_1 b}{\rho_2}\right)N^2+\frac{k_0\rho_1}{\rho_2}+\frac{kk_0l^2}{k_0+k}\right]}-\frac{k\rho_2\left[\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)N^3-\frac{\tau\rho_3k_0N}{\rho_2}-i\rho_3\beta\lambda_nN\right]}{k_0+k}. \tag{3.36}$$

As $\left[\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right]\left(\frac{\rho_3 b}{\rho_2}-\frac{1}{\tau}\right)+\delta^2=0$ and $\delta^2>0$, then we have $\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\neq 0$, and $\frac{\tau\rho_3 b}{\rho_2}-1\neq 0$. From (3.34), (3.32), (3.36) and the definition of λ_n in (3.29), we have, when $n\to\infty$,

$$\alpha_1\simeq-\frac{ik\tau\rho_2\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)\sqrt{\frac{\rho_2}{b}}}{(k_0+k)\left[\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right]\beta}\quad\text{and}\quad\alpha_2\simeq\frac{ik^2\tau\rho_2\left(\frac{\tau\rho_3 b}{\rho_2}-1\right)\sqrt{\frac{\rho_2}{b}}}{(k_0+k)^2\left[\frac{kk_0\rho_2}{(k_0+k)\,b}-\rho_1\right]\beta}N,$$

this leads to

$$\lim_{n\to\infty}|\alpha_2|=\infty. \tag{3.37}$$

Moreover, as

$$\|\Phi_n\|_{\mathcal{H}}^2\geq b\,\|\psi_{n,x}\|^2=b\,|N\alpha_2|^2\int_0^1\sin^2\,(Nx)\,dx=\frac{b}{2}\,|N\alpha_2|^2\int_0^1(1-\cos\,(2Nx))\,dx=\frac{b}{2}\,|N\alpha_2|^2\,,$$

then, for all these three subcases, (3.35) and (3.37) lead to (3.18).

Case 3. $\xi_0\neq 0$ **and** $\xi_2\neq 0$. We have $\frac{b}{\rho_2}\neq\frac{k_0}{\rho_1}$ and $\delta^2-\left(1-\frac{k}{k_0}\right)\left(\rho_3k_0-\frac{\rho_1}{\tau}\right)\neq 0$, then we choose

$$f_{n,2}=f_{n,4}=f_{n,7}=f_{n,8}=0\quad\text{and}\quad f_{n,6}=\cos\left(Nx\right), \tag{3.38}$$

we consider (3.23) and we take (3.29) by replacing the third equation by

$$\lambda_n^2=\frac{k_0}{\rho_1}N^2-\frac{l^2k_0}{\rho_1},\quad\text{for } n \text{ large}, \tag{3.39}$$

where the constants $\alpha_1,\alpha_2,\alpha_3,\alpha_4$ and α_5 will be determined. Then (3.17) is satisfied and $\Phi_n\in D(\mathcal{A})$. On the other hand, by using (3.24), (3.38) and (3.39), we obtain

$$\begin{cases}\left(kN^2-\lambda_n^2\rho_1+l^2k_0\right)\alpha_1+kN\alpha_2+l\left(k+k_0\right)N\alpha_3-\delta N\alpha_4=0,\\ \left(bN^2-\lambda_n^2\rho_2+k\right)\alpha_2+kN\alpha_1+kl\alpha_3=0,\\ \left(k_0N^2-\lambda_n^2\rho_1+l^2k\right)\alpha_3+l\left(k+k_0\right)N\alpha_1+lk\alpha_2=\rho_1,\\ i\lambda_n\rho_3\alpha_4+N\alpha_5+\delta i\lambda_nN\alpha_1=0,\\ \left(i\lambda_n\tau+\beta\right)\alpha_5-N\alpha_4=0.\end{cases} \tag{3.40}$$

From $(3.40)_4$, $(3.40)_5$ and the definition of λ_n in (3.39), we have

$$\alpha_4=\frac{-i\delta\lambda_n N\left(i\lambda_n\tau+\beta\right)\alpha_1}{\left(1-\frac{\tau\rho_3k_0}{\rho_1}\right)N^2+\frac{\tau\rho_3k_0l^2}{\rho_1}+i\lambda_n\rho_3\beta}\text{ and }\alpha_5=\frac{-i\delta\lambda_n N^2\alpha_1}{\left(1-\frac{\tau\rho_3k_0}{\rho_1}\right)N^2+\frac{\tau\rho_3k_0l^2}{\rho_1}+i\lambda_n\rho_3\beta}. \tag{3.41}$$

Using (3.41), we deduce from $(3.40)_1$-$(3.40)_3$ and the definition of λ_n in (3.39) that

$$\begin{cases}\left((k-k_0)N^2+2l^2k_0+\frac{i\delta^2N^2\lambda_n(i\lambda_n\tau+\beta)}{\left(1-\frac{\tau\rho_3k_0}{\rho_1}\right)N^2+\frac{\tau\rho_3k_0l^2}{\rho_1}+i\lambda_n\rho_3\beta}\right)\alpha_1\\ \quad+kN\alpha_2+l\left(k+k_0\right)N\alpha_3=0,\\ kN\alpha_1+\left(\left(b-\frac{\rho_2k_0}{\rho_1}\right)N^2+\frac{l^2k_0\rho_2}{\rho_1}+k\right)\alpha_2+kl\alpha_3=0,\\ \alpha_3=\frac{\rho_1}{l^2\left(k+k_0\right)}-\frac{N}{l}\alpha_1-\frac{k}{l\left(k+k_0\right)}\alpha_2,\end{cases} \tag{3.42}$$

which implies that

$$\begin{cases}\left(-2k_0N^2+2l^2k_0+\frac{i\delta^2N^2\lambda_n(i\lambda_n\tau+\beta)}{\left(1-\frac{\tau\rho_3k_0}{\rho_1}\right)N^2+\frac{l^2\tau\rho_3k_0}{\rho_1}+i\lambda_n\rho_3\beta}\right)\alpha_1=-\frac{\rho_1}{l}N,\\ \left[\left(b-\frac{\rho_2k_0}{\rho_1}\right)N^2+\frac{l^2k_0\rho_2}{\rho_1}+\frac{kk_0}{k+k_0}\right]\alpha_2=-\frac{k\rho_1}{l\left(k+k_0\right)}.\end{cases}$$

Therefore, using the definition of λ_n given in (3.39),

$$\begin{cases}\left(\left(2+\frac{\tau\delta^2}{\rho_1}-4\frac{\tau\rho_3k_0}{\rho_1}\right)l^2k_0N^2+2ik_0\rho_3\beta l^2\sqrt{\frac{k_0}{\rho_1}-\frac{l^2k_0}{\rho_1N^2}}N+\frac{2\tau\rho_3l^4k_0^2}{\rho_1}\right)\alpha_1\\ +\left(-\left(\delta^2+2\left(\frac{\rho_1}{\tau}-\rho_3k_0\right)\right)\frac{\tau k_0}{\rho_1}N^4+i(\delta^2-2k_0\rho_3)\beta\sqrt{\frac{k_0}{\rho_1}-\frac{l^2k_0}{\rho_1N^2}}N^3\right)\alpha_1\\ =-\frac{\rho_1}{l}\left(1-\frac{\tau\rho_3k_0}{\rho_1}\right)N^3-l\tau\rho_3k_0N-i\frac{\rho_1\rho_3\beta}{l}\sqrt{\frac{k_0}{\rho_1}-\frac{l^2k_0}{\rho_1N^2}}N^2,\\ \alpha_2=-\frac{k\rho_1}{l\left(k+k_0\right)\left[\left(b-\frac{\rho_2k_0}{\rho_1}\right)N^2+\frac{l^2k_0\rho_2}{\rho_1}+\frac{kk_0}{k+k_0}\right]}.\end{cases} \tag{3.43}$$

Now, we distinguish three subcases.

Subcase 3.1: $\delta^2+2\left(\frac{\rho_1}{\tau}-\rho_3k_0\right)\neq 0$ **and** $1-\frac{\tau\rho_3k_0}{\rho_1}\neq 0$. We deduce from $(3.42)_3$ and (3.43) that, as $n\to\infty$,

$$\begin{cases}\alpha_1\simeq\frac{\rho_1^2\left(1-\frac{\tau\rho_3k_0}{\rho_1}\right)}{\tau lk_0\left[\delta^2+2\left(\frac{\rho_1}{\tau}-\rho_3k_0\right)\right]N},\\ \alpha_2\simeq-\frac{k\rho_1}{l\left(k+k_0\right)\left(b-\frac{\rho_2k_0}{\rho_1}\right)N^2},\\ \alpha_3\simeq\frac{\tau\rho_1k_0}{\tau k_0l^2\left(k+k_0\right)\left[\delta^2+2\left(\frac{\rho_1}{\tau}-\rho_3k_0\right)\right]}\left[\delta^2-\left(1-\frac{k}{k_0}\right)\left(\rho_3k_0-\frac{\rho_1}{\tau}\right)\right].\end{cases}$$

As $\xi_2\neq 0$; that is, $\delta^2-\left(1-\frac{k}{k_0}\right)\left(\rho_3k_0-\frac{\rho_1}{\tau}\right)\neq 0$, then we get

$$\lim_{n\to\infty}\left|N\alpha_3+l\alpha_1\right|=\infty. \tag{3.44}$$

Subcase 3.2: $\delta^2+2\left(\frac{\rho_1}{\tau}-\rho_3 k_0\right)\neq 0$ **and** $1-\frac{\tau\rho_3 k_0}{\rho_1}=0$. We deduce from (3.43) that, when $n\to\infty$,

$$\alpha_1 \simeq \frac{i\rho_1^2\rho_3\beta\sqrt{\frac{k_0}{\rho_1}}}{\tau l k_0\left[\delta^2+2\left(\frac{\rho_1}{\tau}-\rho_3 k_0\right)\right]N^2},\ \alpha_2\simeq -\frac{k\rho_1}{l\left(k+k_0\right)\left(b-\frac{\rho_2 k_0}{\rho_1}\right)N^2} \text{ and } \alpha_3\simeq\frac{\rho_1}{l^2\left(k+k_0\right)}.$$

Hence (3.44) holds.

Subcase 3.3: $\delta^2+2\left(\frac{\rho_1}{\tau}-\rho_3 k_0\right)=0$. Then (3.43) becomes

$$\begin{cases} \left[-2i\frac{\rho_1}{\tau}\beta\sqrt{\frac{k_0}{\rho_1}-\frac{l^2k_0}{\rho_1 N^2}}N^3-\frac{2\tau\rho_3 l^2 k_0^2}{\rho_1}N^2+2ik_0\rho_3\beta l^2\sqrt{\frac{k_0}{\rho_1}-\frac{l^2k_0}{\rho_1 N^2}}N+\frac{2\tau\rho_3 l^4 k_0^2}{\rho_1}\right]\alpha_1 \\ =-\frac{\rho_1}{l}\left(1-\frac{\tau\rho_3 k_0}{\rho_1}\right)N^3-l\tau\rho_3 k_0 N-i\frac{\rho_1\rho_3\beta}{l}\sqrt{\frac{k_0}{\rho_1}-\frac{l^2k_0}{\rho_1 N^2}}N^2, \\ \alpha_2=-\frac{k\rho_1}{l\left(k+k_0\right)\left[\left(b-\frac{\rho_2 k_0}{\rho_1}\right)N^2+\frac{l^2k_0\rho_2}{\rho_1}+\frac{kk_0}{k+k_0}\right]}. \end{cases}$$

As $\delta^2+2\left(\frac{\rho_1}{\tau}-\rho_3 k_0\right)=0$ and $\delta^2>0$, then $1-\frac{\tau\rho_3 k_0}{\rho_1}\neq 0$, using the previous system and (3.42), we have as $n\to\infty$,

$$\alpha_1\simeq -\frac{i\tau\left(1-\frac{\tau\rho_3 k_0}{\rho_1}\right)}{2l\beta\sqrt{\frac{k_0}{\rho_1}}},\ \alpha_2\simeq -\frac{k\rho_1}{l\left(k+k_0\right)\left(b-\frac{\rho_2 k_0}{\rho_1}\right)N^2} \text{ and } \alpha_3\simeq\frac{i\tau\left(1-\frac{\tau\rho_3 k_0}{\rho_1}\right)}{2l^2\beta\sqrt{\frac{k_0}{\rho_1}}}N,$$

hence (3.44) holds. Finally, because

$$\|\Phi_n\|_{\mathcal{H}}^2\geq k_0\|w_{n,x}-l\varphi_n\|^2=\frac{k_0}{2}\left|N\alpha_3+l\alpha_1\right|^2\int_0^1\left[1-\cos\left(2Nx\right)\right]dx=\frac{k_0}{2}\left|N\alpha_3+l\alpha_1\right|^2,$$

then, by (3.44), we obtain (3.18). This concludes the proof of our Theorem 3.2. ☐

4. Polynomial stability

In this section, we prove the polynomial decay of the solutions of (2.1). **Here and after we will use the notation** $[\langle f(x),g(x)\rangle]_0^1$ **to refer to the usual scalar product in** $\mathbb{C}$ **and given by**

$$[\langle f(x),g(x)\rangle]_0^1:=\left[f(x)\overline{g(x)}\right]_0^1.$$

Our main result is stated as follow:

Theorem 4.1. *We assume that* (1.5) *and* (3.1) *hold. Then, for any* $m\in\mathbb{N}$*, there exists a constant* $C_m>0$ *such that*

$$\forall\Phi_0\in D\left(\mathcal{A}^m\right),\ \forall t>2,\ \left\|e^{t\mathcal{A}}\Phi_0\right\|_{\mathcal{H}}\leq C_m\|\Phi_0\|_{D(\mathcal{A}^m)}\left(\frac{\ln t}{t}\right)^{\frac{m}{4}}\ln t. \tag{4.1}$$

Proof. It is known (see [8]) that (4.1) holds if (3.15) is satisfied and

$$\sup_{|\lambda|\geq 1}\lambda^{-4}\left\|\left(i\lambda I-\mathcal{A}\right)^{-1}\right\|_{\mathcal{L}(\mathcal{H})}<\infty. \tag{4.2}$$

First, the condition (3.15) is satisfied thanks to (3.1) as shown in Lemma 3.1.

Next, we establish condition (4.2) by contradiction. So, assume that (4.2) is false, then there exist a sequence $(\Phi_n)_{n\in\mathbb{N}}\subset D\left(\mathcal{A}\right)$ and a sequence $(\lambda_n)_{n\in\mathbb{N}}\subset\mathbb{R}$ satisfying

$$\|\Phi_n\|_{\mathcal{H}}=1,\quad\forall n\in\mathbb{N}, \tag{4.3}$$

$$\lim_{n\to\infty}|\lambda_n|=\infty \tag{4.4}$$

and

$$\lim_{n\to\infty} \lambda_n^4 \left\|(i\lambda_n I - \mathcal{A})\Phi_n\right\|_{\mathcal{H}} = 0. \tag{4.5}$$

Let Φ_n be define by (3.21). Then (4.5) is equivalent to

$$\begin{cases} \lambda_n^4 \left(i\lambda_n\varphi_n - u_n\right) \to 0 & \text{in } H_*^1(0,1), \\ \lambda_n^4 \left(i\lambda_n\rho_1 u_n - k\left(\varphi_{n,x} + \psi_n + lw_n\right)_x - lk_0\left(w_{n,x} - l\varphi_n\right) + \delta\theta_{n,x}\right) \to 0 & \text{in } L^2(0,1), \\ \lambda_n^4 \left(i\lambda_n\psi_n - v_n\right) \to 0 & \text{in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4 \left(i\lambda_n\rho_2 v_n - b\psi_{n,xx} + k\left(\varphi_{n,x} + \psi_n + lw_n\right)\right) \to 0 & \text{in } L^2(0,1), \\ \lambda_n^4 \left(i\lambda_n w_n - z_n\right) \to 0 & \text{in } \widetilde{H}_*^1(0,1), \\ \lambda_n^4 \left(i\lambda_n\rho_1 z_n - k_0\left(w_{n,x} - l\varphi_n\right)_x + lk\left(\varphi_{n,x} + \psi_n + lw_n\right)\right) \to 0 & \text{in } L^2(0,1), \\ \lambda_n^4 \left(i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x}\right) \to 0 & \text{in } L^2(0,1), \\ \lambda_n^4 \left(i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}\right) \to 0 & \text{in } L^2(0,1). \end{cases} \tag{4.6}$$

Our goal is to derive

$$\lim_{n\to\infty} \left\|\Phi_n\right\|_{\mathcal{H}} = 0 \tag{4.7}$$

as a contradiction to (4.3). This will be established through several steps.

Step 1. Taking the inner product of $\lambda_n^4 \left(i\,\lambda_n\, I\, -\, \mathcal{A}\right)\ \Phi_n$ with Φ_n in $\mathcal{H}$ and using (3.4), we get

$$Re\left\langle \lambda_n^4 \left(i\,\lambda_n\, I\, -\, \mathcal{A}\right)\ \Phi_n, \Phi_n\right\rangle = \beta\lambda_n^4 \left\|q_n\right\|^2.$$

So we have, according to (4.3) and (4.5),

$$\lambda_n^2 q_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.8}$$

Step 2. Applying triangular inequality, we obtain

$$\left\|\lambda_n\theta_{n,x}\right\| \leq \left\|\frac{\lambda_n^4 \left(i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}\right)}{\lambda_n^3}\right\| + \left\|i\lambda_n^2\tau q_n + \beta\lambda_n q_n\right\|,$$

and by using (4.4), $(4.6)_8$ and (4.8), we have

$$\lambda_n\theta_{n,x} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.9}$$

As $\theta_n \in \widetilde{H}_*^1(0,1)$, then we get

$$\lambda_n\theta_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.10}$$

Step 3. By multiplying $(4.6)_1$, $(4.6)_3$ and $(4.6)_5$ by $\dfrac{1}{\lambda_n^5}$ and using (4.4), we obtain

$$\varphi_n \longrightarrow 0, \quad \psi_n \longrightarrow 0 \quad \text{and} \quad w_n \longrightarrow 0 \quad \text{in } L^2(0,1). \tag{4.11}$$

Step 4. Taking the inner product of $(4.6)_7$ with $\dfrac{i\varphi_{n,x}}{\lambda_n^4}$ in $L^2(0,1)$ and using (4.3) and (4.4), we get

$$\left\langle i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x}, i\varphi_{n,x}\right\rangle \longrightarrow 0,$$

that is,

$$\rho_3\left\langle \lambda_n\theta_n, \varphi_{n,x}\right\rangle + \left\langle q_{n,x}, i\varphi_{n,x}\right\rangle - \delta\left\langle i\lambda_n\varphi_{n,x} - u_{n,x}, i\varphi_{n,x}\right\rangle + \delta\lambda_n\left\|\varphi_{n,x}\right\|^2 \longrightarrow 0,$$

integrating by parts and taking into account the boundary conditions, we have

$$\rho_3\left\langle \lambda_n\theta_n, \varphi_{n,x}\right\rangle - \left\langle \lambda_n q_n, i\frac{\varphi_{n,xx}}{\lambda_n}\right\rangle - \delta\left\langle i\lambda_n\varphi_{n,x} - u_{n,x}, i\varphi_{n,x}\right\rangle + \delta\lambda_n\left\|\varphi_{n,x}\right\|^2 \longrightarrow 0. \tag{4.12}$$

Multiplying $(4.6)_2$ with $\dfrac{1}{\lambda_n^5}$ and using (4.4), we obtain

$$i\rho_1 u_n - \frac{k}{\lambda_n}\left(\varphi_{n,x} + \psi_n + lw_n\right)_x - \frac{lk_0}{\lambda_n}\left(w_{n,x} - l\varphi_n\right) + \delta\frac{\theta_{n,x}}{\lambda_n} \longrightarrow 0 \text{ in } L^2(0,1),$$

then, using (4.3), (4.4) and (4.9), we deduce that

$$\left(\left\|\frac{\varphi_{n,xx}}{\lambda_n}\right\|\right)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{4.13}$$

So, by (4.3), (4.4), $(4.6)_1$, (4.8), (4.10), (4.12) and (4.13), we have

$$\lambda_n \left\|\varphi_{n,x}\right\|^2 \longrightarrow 0. \tag{4.14}$$

Step 5. Taking the inner product of $(4.6)_2$ with $\frac{\varphi_n}{\lambda_n^4}$ in $L^2(0,1)$ and using (4.3) and (4.4), we get

$$\rho_1 \langle i\lambda_n u_n, \varphi_n\rangle - k\left\langle (\varphi_{n,x}+\psi_n+lw_n)_x, \varphi_n\right\rangle - lk_0 \langle w_{n,x} - l\varphi_n, \varphi_n\rangle + \delta\langle\theta_{n,x},\varphi_n\rangle \longrightarrow 0,$$

then, by integrating by parts, we find

$$\begin{aligned}&-\rho_1\langle i\lambda_n(i\lambda_n\varphi_n - u_n), \varphi_n\rangle - \rho_1\|\lambda_n\varphi_n\|^2 - k\left[\langle\varphi_{n,x}+\psi_n+lw_n,\varphi_n\rangle\right]_0^1\\ &+k\langle\varphi_{n,x}+\psi_n+lw_n,\varphi_{n,x}\rangle - lk_0\left[\langle w_n,\varphi_n\rangle\right]_0^1 + lk_0\langle w_n,\varphi_{n,x}\rangle + l^2k_0\|\varphi_n\|^2 + \delta\langle\theta_{n,x},\varphi_n\rangle \longrightarrow 0,\end{aligned}$$

by using the boundary conditions, (4.3), (4.4), $(4.6)_1$, (4.9), (4.11) and (4.14), we obtain

$$\lambda_n\varphi_n \longrightarrow 0 \text{ in } L^2(0,1), \tag{4.15}$$

using (4.4) and $(4.6)_1$, we deduce that

$$u_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.16}$$

Step 6. We have, by integrating by parts and using the boundary conditions,

$$\begin{aligned}\langle q_{n,x}, u_{n,x}\rangle &= -\langle q_{n,x}, i\lambda_n\varphi_{n,x} - u_{n,x}\rangle + \langle q_{n,x}, i\lambda_n\varphi_{n,x}\rangle\\ &= -\left\langle \frac{q_{n,x}}{\lambda_n}, \lambda_n(i\lambda_n\varphi_{n,x}-u_{n,x})\right\rangle + \left[\langle q_n, i\lambda_n\varphi_{n,x}\rangle\right]_0^1 - \left\langle \lambda_n^2 q_n, i\frac{\varphi_{n,xx}}{\lambda_n}\right\rangle\\ &= -\left\langle \frac{q_{n,x}}{\lambda_n}, \lambda_n(i\lambda_n\varphi_{n,x}-u_{n,x})\right\rangle - \left\langle \lambda_n^2 q_n, i\frac{\varphi_{n,xx}}{\lambda_n}\right\rangle.\end{aligned} \tag{4.17}$$

Multiplying $(4.6)_1$ and $(4.6)_7$ by $\frac{1}{\lambda_n^5}$ and using (4.4), (4.10) and (4.14), we have

$$\frac{u_{n,x}}{\lambda_n} \longrightarrow 0 \quad \text{and} \quad \frac{q_{n,x}}{\lambda_n} + \delta\frac{u_{n,x}}{\lambda_n} \longrightarrow 0 \text{ in } L^2(0,1), \tag{4.18}$$

so

$$\frac{q_{n,x}}{\lambda_n} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.19}$$

By $(4.6)_1$, (4.8), (4.13), (4.17) and (4.19), we deduce that

$$\langle q_{n,x}, u_{n,x}\rangle \longrightarrow 0. \tag{4.20}$$

Taking the inner product of $(4.6)_7$ with $\frac{u_{n,x}}{\lambda_n^4}$ in $L^2(0,1)$ and using the first convergence of (4.18) and (4.4), we get

$$\langle i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x}, u_{n,x}\rangle \to 0,$$

therefore

$$\langle i\lambda_n\rho_3\theta_n, u_{n,x}\rangle + \langle q_{n,x}, u_{n,x}\rangle + \delta\|u_{n,x}\|^2 \to 0,$$

so, integrating by parts, we obtain

$$\left[\langle i\lambda_n\rho_3\theta_n, u_n\rangle\right]_0^1 - \langle i\lambda_n\rho_3\theta_{n,x}, u_n\rangle + \langle q_{n,x}, u_{n,x}\rangle + \delta\|u_{n,x}\|^2 \to 0,$$

by using the boundary conditions, (4.3), (4.9), (4.16) and (4.20), we deduce that

$$u_{n,x} \to 0 \text{ in } L^2(0,1). \tag{4.21}$$

Also with (4.4) and $\frac{1}{\lambda_n^4}\times(4.6)_1$, we have

$$i\lambda_n\varphi_{n,x} - u_{n,x} \to 0 \quad \text{in } H_*^1(0,1),$$

then, by (4.21), we obtain

$$\lambda_n \varphi_{n,x} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.22}$$

Step 7. By multiplying $(4.6)_3$ and $(4.6)_5$ by $\frac{1}{\lambda_n^4}$ and using (4.3) and (4.4), we have

$$\left(\|\lambda_n \psi_n\|\right)_{n\in\mathbb{N}} \text{ and } \left(\|\lambda_n w_n\|\right)_{n\in\mathbb{N}} \text{ are uniformly bounded.} \tag{4.23}$$

Taking the inner product of $(4.6)_2$ with $\frac{iu_n}{\lambda_n^3}$ in $L^2(0,1)$ and using (4.3) and (4.4), we get

$$\left\langle i\lambda_n^2 \rho_1 u_n - \lambda_n k\left(\varphi_{n,x} + \psi_n + lw_n\right)_x - lk_0\lambda_n\left(w_{n,x} - l\varphi_n\right) + \delta\lambda_n\theta_{n,x}, iu_n\right\rangle \to 0,$$

integrating by parts, we obtain

$$\begin{aligned} &\rho_1\|\lambda_n u_n\|^2 - k\lambda_n\left[\langle\varphi_{n,x}+\psi_n+lw_n, iu_n\rangle\right]_0^1 + k\langle\lambda_n\varphi_{n,x}+\lambda_n\psi_n+l\lambda_n w_n, iu_{n,x}\rangle \\ &-lk_0\lambda_n\left[\langle w_n, iu_n\rangle\right]_0^1 + lk_0\langle\lambda_n w_n, iu_{n,x}\rangle + l^2k_0\langle\lambda_n\varphi_n, iu_n\rangle + \delta\langle\lambda_n\theta_{n,x}, iu_n\rangle \to 0, \end{aligned} \tag{4.24}$$

so, using the boundary conditions, (4.3), (4.9), (4.15), (4.21), (4.22), (4.23) and (4.24), we deduce that

$$\lambda_n u_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.25}$$

Step 8. Taking the inner product of $(4.6)_2$ with $\frac{1}{\lambda_n^4}\left(k\psi_{n,x} + l\left(k+k_0\right)w_{n,x}\right)$ in $L^2(0,1)$ and using (4.3) and (4.4), we get

$$\left\langle i\lambda_n\rho_1 u_n - k\left(\varphi_{n,x}+\psi_n+lw_n\right)_x - lk_0\left(w_{n,x}-l\varphi_n\right) + \delta\theta_{n,x}, k\psi_{n,x} + l\left(k+k_0\right)w_{n,x}\right\rangle \to 0,$$

that is,

$$\begin{aligned} &\rho_1\langle i\lambda_n u_n, k\psi_{n,x}+l(k+k_0)w_{n,x}\rangle - k\langle\varphi_{n,xx}, k\psi_{n,x}+l(k+k_0)w_{n,x}\rangle \\ &-\|k\psi_{n,x}+l(k+k_0)w_{n,x}\|^2 + l^2k_0\langle\varphi_n, k\psi_{n,x}+l(k+k_0)w_{n,x}\rangle + \delta\langle\theta_{n,x}, k\psi_{n,x}+l(k+k_0)w_{n,x}\rangle \to 0. \end{aligned} \tag{4.26}$$

Also, by integrating by parts and using the boundary conditions, we have

$$\begin{aligned} \langle\varphi_{n,xx}, k\psi_{n,x}+l(k+k_0)w_{n,x}\rangle &= \left[\langle\varphi_{n,x}, k\psi_{n,x}+l(k+k_0)w_{n,x}\rangle\right]_0^1 - \langle\varphi_{n,x}, k\psi_{n,xx}+l(k+k_0)w_{n,xx}\rangle \\ &= -\left\langle\lambda_n\varphi_{n,x}, k\frac{\psi_{n,xx}}{\lambda_n} + l(k+k_0)\frac{w_{n,xx}}{\lambda_n}\right\rangle. \end{aligned} \tag{4.27}$$

On the other hand, by multiplying $(4.6)_4$ and $(4.6)_6$ by $\frac{1}{\lambda_n^5}$ and using (4.4), we arrive at

$$i\rho_2 v_n - b\frac{\psi_{n,xx}}{\lambda_n} + \frac{k}{\lambda_n}\left(\varphi_{n,x}+\psi_n+lw_n\right) \to 0 \text{ in } L^2(0,1)$$

and

$$i\rho_1 z_n - k_0\frac{w_{n,xx}}{\lambda_n} + lk_0\frac{\varphi_{n,x}}{\lambda_n} + \frac{lk}{\lambda_n}\left(\varphi_{n,x}+\psi_n+lw_n\right) \to 0 \text{ in } L^2(0,1).$$

So, by (4.3) and (4.4), we deduce that

$$\left(\left\|\frac{\psi_{n,xx}}{\lambda_n}\right\|\right)_{n\in\mathbb{N}} \text{ and } \left(\left\|\frac{w_{n,xx}}{\lambda_n}\right\|\right)_{n\in\mathbb{N}} \text{ are uniformly bounded.} \tag{4.28}$$

Using (4.28), we deduce from (4.22) and (4.27) that

$$\langle\varphi_{n,xx}, k\psi_{n,x}+l(k+k_0)w_{n,x}\rangle \to 0,$$

and by (4.3) and (4.4), (4.9), (4.11), (4.25) and (4.26), we see that

$$k\psi_{n,x}+l(k+k_0)w_{n,x} \to 0 \text{ in } L^2(0,1). \tag{4.29}$$

Step 9. Taking the inner product of $(4.6)_4$ with $\frac{\psi_n}{\lambda_n^4}$ in $L^2(0,1)$ and using (4.3) and (4.4), we get

$$\langle i\lambda_n\rho_2 v_n - b\psi_{n,xx} + k\left(\varphi_{n,x}+\psi_n+lw_n\right), \psi_n\rangle \to 0,$$

that is,

$$-\rho_2\langle v_n, i\lambda_n\psi_n - v_n\rangle - \rho_2\|v_n\|^2 - b\left[\langle\psi_{n,x},\psi_n\rangle\right]_0^1 + b\|\psi_{n,x}\|^2 + k\langle\varphi_{n,x}+\psi_n+lw_n,\psi_n\rangle \to 0,$$

then by the boundary conditions, (4.3), $(4.6)_3$ and (4.11), we deduce that

$$b\left\|\psi_{n,x}\right\|^2-\rho_2\left\|v_n\right\|^2\to 0. \tag{4.30}$$

Taking the inner product of $(4.6)_6$ with $\frac{w_n}{\lambda_n^4}$ in $L^2(0,1)$ and using (4.3) and (4.4), we get

$$\left\langle i\lambda_n\rho_1 z_n-k_0\left(w_{n,x}-l\varphi_n\right)_x+lk\left(\varphi_{n,x}+\psi_n+lw_n\right),w_n\right\rangle\to 0,$$

by integrating by parts, we have

$$\begin{aligned}&-\rho_1\left\langle z_n,i\lambda_n w_n-z_n\right\rangle-\rho_1\left\|z_n\right\|^2-k_0\left[\left\langle w_{n,x}-l\varphi_n,w_n\right\rangle\right]_0^1\\&+k_0\left\|w_{n,x}\right\|^2-lk_0\left\langle\varphi_n,w_{n,x}\right\rangle+lk\left\langle\varphi_{n,x}+\psi_n+lw_n,w_n\right\rangle\to 0,\end{aligned}$$

using the boundary conditions, (4.3), (4.4), $(4.6)_5$ and (4.11), we see that

$$k_0\left\|w_{n,x}\right\|^2-\rho_1\left\|z_n\right\|^2\to 0. \tag{4.31}$$

Step 10. Taking the inner product of $(4.6)_4$ with $\frac{w_n}{\lambda_n^4}$ and $(4.6)_6$ with $\frac{\psi_n}{\lambda_n^4}$ and using (4.3) and (4.4), we get

$$\begin{cases}\left\langle i\lambda_n\rho_2 v_n-b\psi_{n,xx}+k\left(\varphi_{n,x}+\psi_n+lw_n\right),w_n\right\rangle\to 0\,,\\\left\langle i\lambda_n\rho_1 z_n-k_0\left(w_{n,x}-l\varphi_n\right)_x+lk\left(\varphi_{n,x}+\psi_n+lw_n\right),\psi_n\right\rangle\to 0,\end{cases}$$

then, by integrating by parts and using the boundary conditions, we observe that

$$\begin{cases}-\rho_2\left\langle v_n,i\lambda_n w_n-z_n\right\rangle-\rho_2\left\langle v_n,z_n\right\rangle+b\left\langle\psi_{n,x},w_{n,x}\right\rangle+k\left\langle\varphi_{n,x}+\psi_n+lw_n,w_n\right\rangle\to 0,\\-\rho_1\left\langle z_n,i\lambda_n\psi_n-v_n\right\rangle-\rho_1\left\langle z_n,v_n\right\rangle+k_0\left\langle w_{n,x}-l\varphi_n,\psi_{n,x}\right\rangle+lk\left\langle\varphi_{n,x}+\psi_n+lw_n,\psi_n\right\rangle\to 0,\end{cases}$$

by using (4.4), (4.11), $(4.6)_3$ and $(4.6)_5$, we obtain

$$-\rho_2\left\langle v_n,z_n\right\rangle+b\left\langle\psi_{n,x},w_{n,x}\right\rangle\to 0,\ \text{ and }\ -\rho_1\left\langle v_n,z_n\right\rangle+k_0\left\langle\psi_{n,x},w_{n,x}\right\rangle\to 0,$$

hence

$$\left(\frac{\rho_2}{b}-\frac{\rho_1}{k_0}\right)\left\langle v_n,z_n\right\rangle\to 0\quad\text{and}\quad\left(\frac{b}{\rho_2}-\frac{k_0}{\rho_1}\right)\left\langle\psi_{n,x},w_{n,x}\right\rangle\to 0. \tag{4.32}$$

Step 11. Now, we distinguish two cases.

Case 1: $\xi_0\neq 0$. We have $\frac{b}{\rho_2}-\frac{k_0}{\rho_1}\neq 0$, then (4.32) implies that

$$\left\langle v_n,z_n\right\rangle\to 0,\quad\text{and}\quad\left\langle\psi_{n,x},w_{n,x}\right\rangle\to 0. \tag{4.33}$$

Therefore, taking the inner product in $L^2(0,1)$ of $k\psi_{n,x}+l(k+k_0)w_{n,x}$ with $\psi_{n,x}$ and $w_{n,x}$, and using (4.29) and (4.33), we find

$$\psi_{n,x}\to 0\quad\text{and}\quad w_{n,x}\to 0\text{ in }L^2(0,1)\,, \tag{4.34}$$

and by (4.30), (4.31) and (4.34), we deduce that

$$v_n\to 0\quad\text{and}\quad z_n\to 0\text{ in }L^2(0,1)\,. \tag{4.35}$$

Finally, (4.4), (4.8), (4.10), (4.11), (4.16), (4.22), (4.34) and (4.35) imply (4.7).

Case 2: $\xi_0=0$. We have $\frac{b}{\rho_2}-\frac{k_0}{\rho_1}=0$, then, using $(4.6)_3$-$(4.6)_6$, we obtain

$$\begin{cases}\lambda_n^4\left(-\lambda_n^2\frac{\rho_2}{b}\psi_n-\psi_{n,xx}+\frac{k}{b}\left(\varphi_{n,x}+\psi_n+lw_n\right)\right)\to 0\text{ in }L^2(0,1)\,,\\\lambda_n^4\left(-\lambda_n^2\frac{\rho_2}{b}w_n-\left(w_{n,x}-l\varphi_n\right)_x+\frac{lk}{k_0}\left(\varphi_{n,x}+\psi_n+lw_n\right)\right)\to 0\text{ in }L^2(0,1)\,.\end{cases} \tag{4.36}$$

Multiplying $(4.36)_1$ and $(4.36)_2$ with $\frac{1}{\lambda_n^4}$, and using (4.4), (4.11) and (4.22), we get

$$\lambda_n^2\frac{\rho_2}{b}\psi_n+\psi_{n,xx}\to 0\quad\text{and}\quad\lambda_n^2\frac{\rho_2}{b}w_n+w_{n,xx}\to 0\text{ in }L^2(0,1)\,. \tag{4.37}$$

Adding $k\times(4.37)_1$ with $l(k+k_0)\times$ $(4.37)_2$, and $k\times(4.37)_1$ with $-l(k+k_0)\times$ $(4.37)_2$, we obtain

$$\begin{cases} \lambda_n^2 \frac{\rho_2}{b}\left[k\psi_n + l(k+k_0)w_n\right] + k\psi_{n,xx} + l(k+k_0)w_{n,xx} \to 0 \text{ in } L^2(0,1), \\ \lambda_n^2 \frac{\rho_2}{b}\left[k\psi_n - l(k+k_0)w_n\right] + k\psi_{n,xx} - l(k+k_0)w_{n,xx} \to 0 \text{ in } L^2(0,1). \end{cases} \tag{4.38}$$

Taking the inner product in $L^2(0,1)$ of $(4.38)_1$ and $(4.38)_2$ with $k\psi_n + l(k+k_0)w_n$, integrating by parts and using (4.3) and the boundary conditions, we get

$$\begin{cases} \frac{\rho_2}{b}\left\|k\lambda_n\psi_n + l(k+k_0)\lambda_n w_n\right\|^2 - \left\|k\psi_{n,x} + l(k+k_0)w_{n,x}\right\|^2 \to 0, \\ \left\langle \lambda_n^2 \frac{\rho_2}{b}\left[k\psi_n - l(k+k_0)w_n\right], k\psi_n + l(k+k_0)w_n \right\rangle - \left\langle k\psi_{n,x} - l(k+k_0)w_{n,x}, k\psi_{n,x} + l(k+k_0)w_{n,x}\right\rangle \to 0, \end{cases}$$

then, by using (4.3) and (4.29), we obtain

$$k\lambda_n\psi_n + l(k+k_0)\lambda_n w_n \to 0 \text{ in } L^2(0,1) \text{ and } k^2\left\|\lambda_n\psi_n\right\|^2 - l^2(k+k_0)^2\left\|\lambda_n w_n\right\|^2 \to 0. \tag{4.39}$$

Taking the inner product in $L^2(0,1)$ of $(4.36)_1$ with $\frac{w_n}{\lambda_n^2}$, and $(4.36)_2$ with $\frac{\psi_n}{\lambda_n^2}$, and using (4.3) and (4.4), we get

(4.40)

$$\begin{cases} -\lambda_n^4 \frac{\rho_2}{b}\langle \psi_n, w_n\rangle + \lambda_n^2\langle \psi_{n,x}, w_{n,x}\rangle + \frac{k}{b}\langle \lambda_n\varphi_{n,x}, \lambda_n w_n\rangle + \frac{k}{b}\langle \lambda_n\psi_n, \lambda_n w_n\rangle + \frac{lk}{b}\left\|\lambda_n w_n\right\|^2 \to 0, \\ -\lambda_n^4 \frac{\rho_2}{b}\langle \psi_n, w_n\rangle + \lambda_n^2\langle \psi_{n,x}, w_{n,x}\rangle + l\left(1+\frac{k}{k_0}\right)\langle \lambda_n\psi_n, \lambda_n\varphi_{n,x}\rangle + \frac{lk}{k_0}\left\|\lambda_n\psi_n\right\|^2 + \frac{l^2k}{k_0}\langle \lambda_n\psi_n, \lambda_n w_n\rangle \to 0, \end{cases}$$

then, by using (4.22) and (4.23), and adding $\frac{bk_0}{k}\times(4.40)_1$ and $-\frac{bk_0}{k}\times(4.40)_2$, we obtain

$$lk_0\left\|\lambda_n w_n\right\|^2 - lb\left\|\lambda_n\psi_n\right\|^2 + \left(k_0 - l^2 b\right)\langle \lambda_n\psi_n, \lambda_n w_n\rangle \to 0. \tag{4.41}$$

By taking the inner product in $L^2(0,1)$ of $(4.39)_1$ with $\lambda_n\psi_n$ and using (4.23), we arrive at

$$k\left\|\lambda_n\psi_n\right\|^2 + l(k+k_0)\langle \lambda_n\psi_n, \lambda_n w_n\rangle \to 0. \tag{4.42}$$

On the other hand, combining $k_0\times(4.39)_2$ and using $l(k+k_0)^2\times(4.41)$, it follows that

$$\left[k_0k^2 - bl^2(k+k_0)^2\right]\left\|\lambda_n\psi_n\right\|^2 + l(k+k_0)^2\left(k_0 - l^2 b\right)\langle \lambda_n\psi_n, \lambda_n w_n\rangle \to 0. \tag{4.43}$$

Adding $(k+k_0)(k_0-bl^2)\times(4.42)$ and $-(4.43)$, we find

$$k_0\left(kk_0 + bl^2(k+k_0)\right)\left\|\lambda_n\psi_n\right\|^2 \to 0,$$

then, we have

$$\lambda_n\psi_n \to 0 \text{ in } L^2(0,1), \tag{4.44}$$

and by using $(4.39)_1$, we obtain

$$\lambda_n w_n \to 0 \text{ in } L^2(0,1). \tag{4.45}$$

Using (4.4), $(4.6)_3$, $(4.6)_5$, (4.44) and (4.45), we deduce that

$$v_n \to 0 \quad \text{and} \quad z_n \to 0 \text{ in } L^2(0,1).$$

Taking the inner product in $L^2(0,1)$ of $(4.37)_1$ with ψ_n, and $(4.37)_2$ with w_n, integrating by parts and using the boundary conditions, we get

$$\frac{\rho_2}{b}\left\|\lambda_n\psi_n\right\|^2 - \left\|\psi_{n,x}\right\|^2 \to 0 \quad \text{and} \quad \frac{\rho_2}{b}\left\|\lambda_n w_n\right\|^2 - \left\|w_{n,x}\right\|^2 \to 0,$$

then by (4.44) and (4.45), we deduce that

$$\psi_{n,x} \to 0 \quad \text{and} \quad w_{n,x} \to 0 \text{ in } L^2(0,1).$$

Consequently, as in case 1, we see that (4.7) holds. Finally, the proof of our Theorem 4.1 is completed. □

5. Exponential stability

In this section, we prove that the semigroup associated to (2.1) is exponentially stable provided (1.5), (3.1) and the following new conditions hold:

$$\xi_0 \neq 0 \quad \text{and} \quad \xi_1 = \xi_2 = 0. \tag{5.1}$$

Theorem 5.1. *We assume that* (1.5), (3.1) *and* (5.1) *hold. Then the semigroup associated with* (2.1) *is exponentially stable.*

Proof. We will use the method introduced in [6, 11] by proving (3.15) and (3.16). We have proved in Lemma 3.1 that (3.1) and (3.15) are equivalent. So the semigroup associated with (2.1) is exponentially stable if (3.16) holds. We assume by contradiction that the condition (3.16) is false. Then there is a real sequence $(\lambda_n)_{n\in\mathbb{N}}$ and a sequence $(\Phi_n)_{n\in\mathbb{N}} \in D(\mathcal{A})$ such that (4.3) and (4.4) are satisfied and

$$\lim_{n\to\infty} \|(i\lambda_n I - \mathcal{A})\,\Phi_n\|_{\mathcal{H}} = 0, \tag{5.2}$$

i.e., defining Φ_n by (3.21), we have the following convergence:

$$\begin{cases} i\lambda_n\varphi_n - u_n \to 0 & \text{in } H_*^1(0,1), \\ i\lambda_n\rho_1 u_n - k\left(\varphi_{n,x}+\psi_n+l\,w_n\right)_x - lk_0\left(w_{n,x}-l\varphi_n\right) + \delta\theta_{n,x} \to 0 & \text{in } L^2(0,1), \\ i\lambda_n\psi_n - v_n \to 0 & \text{in } \widetilde{H}_*^1(0,1), \\ i\lambda_n\rho_2 v_n - b\psi_{n,xx} + k\left(\varphi_{n,x}+\psi_n+l\,w_n\right) \to 0 & \text{in } L^2(0,1), \\ i\lambda_n w_n - z_n \to 0 & \text{in } \widetilde{H}_*^1(0,1), \\ i\lambda_n\rho_1 z_n - k_0\left(w_{n,x}-l\varphi_n\right)_x + lk\left(\varphi_{n,x}+\psi_n+l\,w_n\right) \to 0 & \text{in } L^2(0,1), \\ i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x} \to 0 & \text{in } L^2(0,1), \\ i\lambda_n\tau q_n + \beta q_n + \theta_{n,x} \to 0 & \text{in } L^2(0,1). \end{cases} \tag{5.3}$$

In the following, we will check the condition (3.16) by finding the contradiction (4.7) with (4.3). Our proof is divided into several steps.

Step 1. Taking the inner product of $(i\lambda_n I - \mathcal{A})\,\Phi_n$ with Φ_n in $\mathcal{H}$ and using (3.4), we get

$$Re\,\langle (i\lambda_n I - \mathcal{A})\,\Phi_n, \Phi_n\rangle_{\mathcal{H}} = \beta\,\|q_n\|^2,$$

using (4.3) and (5.2), we deduce that

$$q_n \to 0 \text{ in } L^2(0,1). \tag{5.4}$$

By the triangular inequality, we get

$$\left\|\frac{\theta_{n,x}}{\lambda_n}\right\| \leq \frac{1}{|\lambda_n|}\,\|i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}\| + \left\|i\tau q_n + \frac{\beta}{\lambda_n}q_n\right\|.$$

From (4.4), $(5.3)_8$ and (5.4), we deduce that

$$\frac{\theta_{n,x}}{\lambda_n} \to 0 \text{ in } L^2(0,1). \tag{5.5}$$

Step 2. Multiplying $(5.3)_1$ by $\overline{\dfrac{i\varphi_n}{\lambda_n}}$, we obtain

$$\|\varphi_n\|^2 - \frac{1}{\lambda_n}\,\langle u_n, i\varphi_n\rangle \to 0.$$

Multiplying $(5.3)_3$ by $\overline{\dfrac{i\psi_n}{\lambda_n}}$, we find

$$\|\psi_n\|^2 - \frac{1}{\lambda_n}\,\langle v_n, i\psi_n\rangle \to 0.$$

Multiplying $(5.3)_5$ by $\overline{\dfrac{iw_n}{\lambda_n}}$, we arrive at

$$\|w_n\|^2 - \frac{1}{\lambda_n}\,\langle z_n, iw_n\rangle \to 0.$$

Hence, using (4.3) and (4.4), we observe that

$$\varphi_n \to 0 \text{ in } L^2(0,1), \tag{5.6}$$

$$\psi_n \to 0 \text{ in } L^2(0,1) \tag{5.7}$$

and

$$w_n \to 0 \text{ in } L^2(0,1). \tag{5.8}$$

Step 3. Multiplying $(5.3)_7$ by $\dfrac{\overline{\theta_n}}{\lambda_n}$ and integration by parts, we get

$$i\rho_3 \|\theta_n\|^2 + \left[\left\langle q_n, \frac{\theta_n}{\lambda_n}\right\rangle\right]_0^1 - \left\langle q_n, \frac{\theta_{n,x}}{\lambda_n}\right\rangle + \delta\left[\left\langle u_n, \frac{\theta_n}{\lambda_n}\right\rangle\right]_0^1 - \delta\left\langle u_n, \frac{\theta_{n,x}}{\lambda_n}\right\rangle \to 0,$$

and by using the boundary conditions, (4.3), (5.4) and (5.5), we find

$$\theta_n \to 0 \text{ in } L^2(0,1). \tag{5.9}$$

Using the triangular inequality, we have

$$\begin{aligned}\left\|\frac{\varphi_{n,xx}}{\lambda_n}\right\| \;\leq\; & \left\|\frac{1}{k\lambda_n}\left(i\lambda_n\rho_1 u_n - k\left(\varphi_{n,x}+\psi_n+l\,w_n\right)_x - lk_0\left(w_{n,x}-l\varphi_n\right)+\delta\theta_{n,x}\right)\right\| \\ & + \left\|\frac{i\rho_1}{k}u_n - \frac{1}{\lambda_n}\left(\psi_{n,x}+l\,w_{n,x}\right) - \frac{lk_0}{k\lambda_n}\left(w_{n,x}-l\varphi_n\right) + \frac{\delta}{k}\frac{\theta_{n,x}}{\lambda_n}\right\|,\end{aligned}$$

and by (4.3), (4.4), $(5.3)_2$ and (5.5), we obtain

$$\left(\left\|\frac{\varphi_{n,xx}}{\lambda_n}\right\|\right)_{n\in\mathbb{N}} \text{ is uniformly bounded.} \tag{5.10}$$

Multiplying $(5.3)_7$ by $\dfrac{\overline{i\varphi_{n,x}}}{\lambda_n}$, we obtain

$$\rho_3\langle\theta_n,\varphi_{n,x}\rangle + \frac{1}{\lambda_n}\langle q_{n,x}, i\varphi_{n,x}\rangle - \delta\left\langle i\lambda_n\varphi_{n,x}-u_{n,x}, \frac{i\varphi_{n,x}}{\lambda_n}\right\rangle + \delta\|\varphi_{n,x}\|^2 \to 0,$$

using (4.3), (4.4) and $(5.3)_1$ and integration by parts, we get

$$\rho_3\langle\theta_n,\varphi_{n,x}\rangle + \frac{1}{\lambda_n}\left[\langle q_n, i\varphi_{n,x}\rangle\right]_0^1 - \left\langle q_n, \frac{i\varphi_{n,xx}}{\lambda_n}\right\rangle + \delta\|\varphi_{n,x}\|^2 \to 0, \tag{5.11}$$

by using the boundary conditions, (4.3), (5.4), (5.9) and (5.10), we deduce from (5.11) that

$$\varphi_{n,x} \to 0 \text{ in } L^2(0,1), \tag{5.12}$$

and by (4.4) and $(5.3)_1$, we deduce that

$$\frac{u_{n,x}}{\lambda_n} \to 0 \text{ in } L^2(0,1). \tag{5.13}$$

As $u_n \in H^1_*(0,1)$, then, by (5.13), we get

$$\frac{u_n}{\lambda_n} \to 0 \text{ in } L^2(0,1).$$

Step 4. Multiplying $(5.3)_2$ by $\dfrac{\overline{iu_n}}{\lambda_n}$ and integration by parts, we obtain

$$\begin{aligned}&\rho_1\|u_n\|^2 + k\left\langle\frac{\varphi_{n,xx}}{\lambda_n}, i\left(i\lambda_n\varphi_n - u_n\right)\right\rangle + k\langle\varphi_{n,xx},\varphi_n\rangle - \frac{k}{\lambda_n}\langle\psi_{n,x}, iu_n\rangle \\ &- \frac{l\left(k+k_0\right)}{\lambda_n}\langle w_{n,x}, iu_n\rangle + \frac{l^2k_0}{\lambda_n}\langle\varphi_n, iu_n\rangle + \delta\left\langle\frac{\theta_{n,x}}{\lambda_n}, iu_n\right\rangle \to 0,\end{aligned}$$

then, by integration by parts and using (4.3), (4.4), $(5.3)_1$, (5.5) and (5.10), we have

$$\rho_1\|u_n\|^2 + k\left[\langle\varphi_{n,x},\varphi_n\rangle\right]_0^1 - k\|\varphi_{n,x}\|^2 \to 0,$$

using the boundary conditions and (5.12), we get

$$u_n \to 0 \text{ in } L^2(0,1), \tag{5.14}$$

and by $(5.3)_1$, we deduce that

$$\lambda_n \varphi_n \to 0 \text{ in } L^2(0,1). \tag{5.15}$$

Step 5. Multiplying $(5.3)_4$ by $\overline{w}_n$, we obtain

$$\langle i\lambda_n \rho_2 v_n, w_n\rangle - b\langle \psi_{n,xx}, w_n\rangle + k\langle \varphi_{n,x}+\psi_n + l\,w_n, w_n\rangle \to 0,$$

and with integration by parts, we get

$$-\rho_2\langle v_n, i\lambda_n w_n - z_n\rangle - \rho_2\langle v_n, z_n\rangle - b\left[\langle \psi_{n,x}, w_n\rangle\right]_0^1 + b\langle \psi_{n,x}, w_{n,x}\rangle + k\langle \varphi_{n,x}+\psi_n+l\,w_n, w_n\rangle \to 0,$$

then, using the boundary conditions, (4.3), $(5.3)_5$, (5.7), (5.8) and (5.12), we deduce that

$$b\langle \psi_{n,x}, w_{n,x}\rangle - \rho_2\langle v_n, z_n\rangle \to 0,$$

then, by using (4.3) and (5.6), we have

$$b\langle \psi_{n,x}, w_{n,x} - l\varphi_n\rangle - \rho_2\langle v_n, z_n\rangle \to 0. \tag{5.16}$$

Step 6. Multiplying $(5.3)_2$ by $\overline{w_{n,x} - l\varphi_n}$, we obtain

$$\rho_1\langle i\lambda_n u_n w_{n,x} - l\varphi_n\rangle - k\left\langle (\varphi_{n,x}+\psi_n+l\,w_n)_x, w_{n,x}-l\varphi_n\right\rangle - lk_0\|w_{n,x}-l\varphi_n\|^2 + \delta\langle \theta_{n,x}, w_{n,x}-l\varphi_n\rangle \to 0,$$

then, we have

$$\begin{aligned} &-\frac{\rho_1}{k}\langle u_n, i\lambda_n w_{n,x} - z_{n,x}\rangle - \frac{\rho_1}{k}\langle u_n, z_{n,x}\rangle + \frac{l\rho_1}{k}\langle u_n, i\lambda_n\varphi_n\rangle \\ &-\left\langle (\varphi_{n,x}+\psi_n+l\,w_n)_x, w_{n,x}-l\varphi_n\right\rangle - \frac{lk_0}{k}\|w_{n,x}-l\varphi_n\|^2 + \frac{\delta}{k}\langle \theta_{n,x}, w_{n,x}-l\varphi_n\rangle \to 0. \end{aligned}$$

By using (4.3), $(5.3)_5$ and (5.15), we get

$$\begin{aligned} &-\frac{\rho_1}{k}\langle u_n, z_{n,x}\rangle - \left\langle (\varphi_{n,x}+\psi_n+l\,w_n)_x, w_{n,x}-l\varphi_n\right\rangle \\ &-\frac{lk_0}{k}\|w_{n,x}-l\varphi_n\|^2 + \frac{\delta}{k}\langle \theta_{n,x}, w_{n,x}-l\varphi_n\rangle \to 0. \end{aligned} \tag{5.17}$$

Multiplying $(5.3)_6$ by $\overline{\varphi_{n,x}+\psi_n+l\,w_n}$, we obtain

$$\langle i\lambda_n\rho_1 z_n, \varphi_{n,x}+\psi_n+l\,w_n\rangle - k_0\left\langle (w_{n,x}-l\varphi_n)_x, \varphi_{n,x}+\psi_n+l\,w_n\right\rangle + lk\|\varphi_{n,x}+\psi_n+l\,w_n\|^2 \to 0,$$

then, with integration by parts and using the boundary conditions, we get

$$\begin{aligned} &-\rho_1\langle z_n, i\lambda_n\varphi_{n,x}\rangle - \rho_1\langle z_n, i\lambda_n\psi_n - v_n\rangle - \rho_1\langle z_n, v_n\rangle - l\rho_1\langle z_n, i\lambda_n\, w_n - z_n\rangle \\ &-l\rho_1\|z_n\|^2 + k_0\left\langle w_{n,x}-l\varphi_n, (\varphi_{n,x}+\psi_n+l\,w_n)_x\right\rangle + lk\|\varphi_{n,x}+\psi_n+l\,w_n\|^2 \to 0, \end{aligned}$$

therefore, using (4.3), $(5.3)_3$, $(5.3)_5$, (5.7), (5.8), (5.12) and (5.16), we deduce that

$$\begin{aligned} &-\frac{\rho_1}{k_0}\langle z_n, i\lambda_n\varphi_{n,x}\rangle - \frac{b\rho_1}{k_0\rho_2}\langle w_{n,x}-l\varphi_n, \psi_{n,x}\rangle - \frac{l\rho_1}{k_0}\|z_n\|^2 \\ &+\left\langle w_{n,x}-l\varphi_n, (\varphi_{n,x}+\psi_n+l\,w_n)_x\right\rangle \to 0, \end{aligned} \tag{5.18}$$

combining (5.17) and (5.18), we find

$$\begin{aligned} &-\frac{\rho_1}{k_0}\langle z_n, i\lambda_n\varphi_{n,x}\rangle - \frac{b\rho_1}{k_0\rho_2}\langle w_{n,x}-l\varphi_n, \psi_{n,x}\rangle - \frac{l\rho_1}{k_0}\|z_n\|^2 \\ &-\frac{\rho_1}{k}\langle z_{n,x}, u_n\rangle - \frac{lk_0}{k}\|w_{n,x}-l\varphi_n\|^2 + \frac{\delta}{k}\langle w_{n,x}-l\varphi_n, \theta_{n,x}\rangle \to 0, \end{aligned}$$

then, with integration by parts and using the boundary conditions, we obtain

$$\begin{aligned} &-\frac{\rho_1}{k_0}\langle z_n, i\lambda_n\varphi_{n,x}\rangle - \frac{b\rho_1}{k_0\rho_2}\langle w_{n,x}-l\varphi_n, \psi_{n,x}\rangle - \frac{l\rho_1}{k_0}\|z_n\|^2 - \frac{\rho_1}{k}\langle z_n, i\lambda_n\varphi_{n,x} - u_{n,x}\rangle \\ &+\frac{\rho_1}{k}\langle z_n, i\lambda_n\varphi_{n,x}\rangle - \frac{lk_0}{k}\|w_{n,x}-l\varphi_n\|^2 + \frac{\delta}{k}\langle w_{n,x}-l\varphi_n, \theta_{n,x}\rangle \to 0, \end{aligned}$$

using (4.3) and $(5.3)_1$, we arrive at

$$\begin{aligned} &\frac{\rho_1}{k_0}\left(\frac{k_0}{k}-1\right)\langle z_n, i\lambda_n\varphi_{n,x}\rangle - \frac{b\rho_1}{k_0\rho_2}\langle w_{n,x}-l\varphi_n, \psi_{n,x}\rangle - \frac{l\rho_1}{k_0}\|z_n\|^2 \\ &-\frac{lk_0}{k}\|w_{n,x}-l\varphi_n\|^2 + \frac{\delta}{k}\langle w_{n,x}-l\varphi_n, \theta_{n,x}\rangle \to 0. \end{aligned} \tag{5.19}$$

Step 7. From (4.3), $(5.3)_3$ and $(5.3)_5$, we observe that

$$(\|\lambda_n\psi_n\|)_{n\in\mathbb{N}} \text{ and } (\|\lambda_n w_n\|)_{n\in\mathbb{N}} \text{ are uniformly bounded.} \tag{5.20}$$

We have, by integrating by parts,

$$\begin{aligned} \left\langle \lambda_n^2\rho_2\psi_n + i\lambda_n\rho_2 v_n, i\theta_n\right\rangle &= -i\rho_2\langle i\lambda_n\psi_n - v_n, i\lambda_n\theta_n\rangle \\ &= -\frac{i\rho_2}{\rho_3}\langle i\lambda_n\psi_n - v_n, i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x}\rangle \\ &\quad +\frac{i\rho_2}{\rho_3}\langle i\lambda_n\psi_n - v_n, q_{n,x}\rangle + \frac{i\rho_2}{\rho_3}\langle i\lambda_n\psi_n - v_n, \delta u_{n,x}\rangle \\ &= -\frac{i\rho_2}{\rho_3}\langle i\lambda_n\psi_n - v_n, i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x}\rangle - \frac{i\rho_2\delta}{\rho_3}\langle (i\lambda_n\psi_n - v_n)_x, u_n\rangle \\ &\quad +\frac{i\rho_2}{\rho_3}\left[\langle i\lambda_n\psi_n - v_n, q_n\rangle\right]_0^1 + \frac{i\rho_2}{\rho_3}\left[\langle i\lambda_n\psi_n - v_n, \delta u_n\rangle\right]_0^1 - \frac{i\rho_2}{\rho_3}\langle (i\lambda_n\psi_n - v_n)_x, q_n\rangle, \end{aligned}$$

by using the boundary conditions, (4.3), $(5.3)_3$ and $(5.3)_7$, we deduce that

$$\left\langle \lambda_n^2\rho_2\psi_n + i\lambda_n\rho_2 v_n, i\theta_n\right\rangle \to 0. \tag{5.21}$$

Also, we have

$$\begin{aligned} \langle \lambda_n\psi_n, u_{n,x}\rangle &= \left[\langle \lambda_n\psi_n, u_n\rangle\right]_0^1 - \langle \lambda_n\psi_{n,x}, u_n\rangle \\ &= -\langle i\lambda_n\psi_{n,x} - v_{n,x}, iu_n\rangle - \langle v_{n,x}, iu_n\rangle \\ &= -\langle i\lambda_n\psi_{n,x} - v_{n,x}, iu_n\rangle + \langle v_n, iu_{n,x}\rangle. \end{aligned} \tag{5.22}$$

Using again integration by parts and the boundary conditions, we have

$$\begin{aligned} \lambda_n\langle \psi_{n,x}, q_n\rangle &= \lambda_n\left[\langle \psi_n, q_n\rangle\right]_0^1 - \lambda_n\langle \psi_n, q_{n,x}\rangle \\ &= -\lambda_n\langle \psi_n, i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x}\rangle + \lambda_n\langle \psi_n, i\lambda_n\rho_3\theta_n\rangle + \lambda_n\langle \psi_n, \delta u_{n,x}\rangle \\ &= -\lambda_n\langle \psi_n, i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x}\rangle + \frac{\rho_3}{\rho_2}\left\langle \lambda_n^2\rho_2\psi_n + i\lambda_n\rho_2 v_n, i\theta_n\right\rangle \\ &\quad -\frac{\rho_3}{\rho_2}\langle i\lambda_n\rho_2 v_n - b\psi_{n,xx} + k(\varphi_{n,x} + \psi_n + l\,w_n), i\theta_n\rangle \\ &\quad -\frac{b\rho_3}{\rho_2}\langle \psi_{n,xx}, i\theta_n\rangle + \frac{k\rho_3}{\rho_2}\langle \varphi_{n,x} + \psi_n + l\,w_n, i\theta_n\rangle + \delta\langle \lambda_n\psi_n, u_{n,x}\rangle, \end{aligned}$$

then, by (5.22) and integration by parts, we obtain

$$\begin{aligned} \lambda_n\langle \psi_{n,x}, q_n\rangle &= -\langle \lambda_n\psi_n, i\lambda_n\rho_3\theta_n + q_{n,x} + \delta u_{n,x}\rangle + \frac{\rho_3}{\rho_2}\left\langle \lambda_n^2\rho_2\psi_n + i\lambda_n\rho_2 v_n, i\theta_n\right\rangle \\ &\quad -\frac{\rho_3}{\rho_2}\langle i\lambda_n\rho_2 v_n - b\psi_{n,xx} + k(\varphi_{n,x} + \psi_n + l\,w_n), i\theta_n\rangle \\ &\quad -\frac{b\rho_3}{\rho_2}\left[\langle \psi_{n,x}, i\theta_n\rangle\right]_0^1 + \frac{b\rho_3}{\rho_2}\langle \psi_{n,x}, i\theta_{n,x}\rangle + \frac{k\rho_3}{\rho_2}\langle \varphi_{n,x} + \psi_n + l\,w_n, i\theta_n\rangle \\ &\quad -\delta\langle i\lambda_n\psi_{n,x} - v_{n,x}, iu_n\rangle + \delta\langle v_n, iu_{n,x}\rangle, \end{aligned} \tag{5.23}$$

using the boundary conditions, (4.3), $(5.3)_3$, $(5.3)_4$, $(5.3)_7$, (5.7), (5.8), (5.12), (5.20) and (5.21), we deduce from (5.23) that

$$\lambda_n\langle \psi_{n,x}, q_n\rangle - \frac{b\rho_3}{\rho_2}\langle \psi_{n,x}, i\theta_{n,x}\rangle - \delta\langle v_n, iu_{n,x}\rangle \to 0. \tag{5.24}$$

Also, we have

$$\begin{aligned}\lambda_n \langle \psi_{n,x}, q_n \rangle &= \frac{i}{\tau} \langle \psi_{n,x}, i\tau\lambda_n q_n \rangle \\ &= \frac{i}{\tau} \langle \psi_{n,x}, i\lambda_n \tau q_n + \beta q_n + \theta_{n,x} \rangle - \frac{i\beta}{\tau} \langle \psi_{n,x}, q_n \rangle + \frac{1}{\tau} \langle \psi_{n,x}, i\theta_{n,x} \rangle ,\end{aligned} \tag{5.25}$$

therefore, by using (4.3), $(5.3)_8$, (5.4), (5.24) and (5.25), we obtain

$$\left(\frac{b\rho_3}{\rho_2} - \frac{1}{\tau}\right) \langle \psi_{n,x}, \theta_{n,x} \rangle + \delta \langle v_n, u_{n,x} \rangle \to 0,$$

and so

$$\left(\frac{b\rho_3}{\rho_2} - \frac{1}{\tau}\right) \langle \psi_{n,x}, \theta_{n,x} \rangle - \delta \langle v_n, (i\lambda_n \varphi_n - u_n)_x \rangle + \delta \langle v_n, i\lambda_n \varphi_{n,x} \rangle \to 0,$$

and moreover, by (4.3) and $(5.3)_1$, we find

$$\left(\frac{b\rho_3}{\rho_2} - \frac{1}{\tau}\right) \langle \psi_{n,x}, \theta_{n,x} \rangle + \delta \langle v_n, i\lambda_n \varphi_{n,x} \rangle \to 0. \tag{5.26}$$

Step 8. Multiplying $(5.3)_4$ by $\overline{\varphi_{n,x} + \psi_n + l\, w_n}$, we obtain

$$\langle i\lambda_n \rho_2 v_n, \varphi_{n,x} + \psi_n + l\, w_n \rangle - b \langle \psi_{n,xx}, \varphi_{n,x} + \psi_n + l\, w_n \rangle + k \left\| \varphi_{n,x} + \psi_n + l\, w_n \right\|^2 \to 0,$$

with integration by parts and using the boundary conditions, (5.7), (5.8) and (5.12), we get

$$\begin{aligned}&-\rho_2 \langle v_n, i\lambda_n \varphi_{n,x} \rangle - \rho_2 \langle v_n, i\lambda_n \psi_n - v_n \rangle - \rho_2 \left\| v_n \right\|^2 \\ &-l\rho_2 \langle v_n, i\lambda_n\, w_n - z_n \rangle - l\rho_2 \langle v_n, z_n \rangle + b \left\langle \psi_{n,x}, (\varphi_{n,x} + \psi_n + l\, w_n)_x \right\rangle \to 0,\end{aligned}$$

using (4.3), $(5.3)_3$ and $(5.3)_5$, we deduce that

$$\begin{aligned}&-\rho_2 \langle v_n, i\lambda_n \varphi_{n,x} \rangle - \rho_2 \left\| v_n \right\|^2 - l\rho_2 \langle v_n, z_n \rangle + \frac{b}{k} \langle \psi_{n,x}, i\lambda_n \rho_1 u_n - lk_0 (w_{n,x} - l\varphi_n) + \delta\theta_{n,x} \rangle \\ &-\frac{b}{k} \left\langle \psi_{n,x}, i\lambda_n \rho_1 u_n - k (\varphi_{n,x} + \psi_n + l\, w_n)_x - lk_0 (w_{n,x} - l\varphi_n) + \delta\theta_{n,x} \right\rangle \to 0,\end{aligned}$$

using (4.3) and $(5.3)_2$, we have

$$\begin{aligned}&-\rho_2 \langle v_n, i\lambda_n \varphi_{n,x} \rangle - \rho_2 \left\| v_n \right\|^2 - l\rho_2 \langle v_n, z_n \rangle - \frac{b\rho_1}{k} \langle i\lambda_n \psi_{n,x} - v_{n,x}, u_n \rangle \\ &-\frac{b\rho_1}{k} \langle v_{n,x}, u_n \rangle - \frac{blk_0}{k} \langle \psi_{n,x}, w_{n,x} - l\varphi_n \rangle + \frac{b\delta}{k} \langle \psi_{n,x}, \theta_{n,x} \rangle \to 0.\end{aligned} \tag{5.27}$$

As, by integrating by parts and using the boundary conditions,

$$\langle v_{n,x}, u_n \rangle = - \langle v_n, u_{n,x} \rangle = \langle v_n, i\lambda_n \varphi_{n,x} - u_{n,x} \rangle - \langle v_n, i\lambda_n \varphi_{n,x} \rangle ,$$

and with (4.3), $(5.3)_1$, $(5.3)_3$ and (5.27), we see that

$$\left(\frac{b\rho_1}{k} - \rho_2\right) \langle v_n, i\lambda_n \varphi_{n,x} \rangle - \rho_2 \left\| v_n \right\|^2 - l\rho_2 \langle v_n, z_n \rangle - \frac{blk_0}{k} \langle \psi_{n,x}, w_{n,x} - l\varphi_n \rangle + \frac{b\delta}{k} \langle \psi_{n,x}, \theta_{n,x} \rangle \to 0,$$

combining with (5.16) and (5.26), we obtain

$$-\frac{1}{\delta} \left(\frac{b\rho_1}{k} - \rho_2\right) \left(\frac{b\rho_3}{\rho_2} - \frac{1}{\tau}\right) \langle \psi_{n,x}, \theta_{n,x} \rangle + \frac{b}{k} \delta \langle \psi_{n,x}, \theta_{n,x} \rangle - \rho_2 \left\| v_n \right\|^2 - lb \left(1 + \frac{k_0}{k}\right) \langle \psi_{n,x}, w_{n,x} - l\varphi_n \rangle \to 0,$$

then, we get

$$\frac{b}{\delta k} \left[\delta^2 - \left(\rho_1 - \frac{k\rho_2}{b}\right) \left(\frac{b\rho_3}{\rho_2} - \frac{1}{\tau}\right)\right] \langle \psi_{n,x}, \theta_{n,x} \rangle - \rho_2 \left\| v_n \right\|^2 - lb \left(1 + \frac{k_0}{k}\right) \langle \psi_{n,x}, w_{n,x} - l\varphi_n \rangle \to 0. \tag{5.28}$$

Step 9. We have

$$\begin{aligned} \langle z_n,\, q_{n,x}\rangle &= [\langle z_n,\, q_n\rangle]_0^1 - \langle z_{n,x},\, q_n\rangle \\ &= \langle i\lambda_n w_{n,x} - z_{n,x},\, q_n\rangle - \langle i\lambda_n w_{n,x},\, q_n\rangle \\ &= \langle i\lambda_n w_{n,x} - z_{n,x},\, q_n\rangle + \frac{1}{\tau}\langle w_{n,x}, i\lambda_n \tau q_n + \beta q_n + \theta_{n,x}\rangle \\ &\quad -\frac{\beta}{\tau}\langle w_{n,x},\, q_n\rangle - \frac{1}{\tau}\langle w_{n,x},\, \theta_{n,x}\rangle . \end{aligned} \tag{5.29}$$

Also, we see that

$$\begin{aligned} \langle i\lambda_n \rho_1 z_n,\, \theta_n\rangle &= -\rho_1 \langle z_n, i\lambda_n\, \theta_n\rangle \\ &= -\frac{\rho_1}{\rho_3}\langle z_n, i\lambda_n\, \rho_3\theta_n + q_{n,x} + \delta u_{n,x}\rangle + \frac{\rho_1}{\rho_3}\langle z_n, q_{n,x}\rangle + \frac{\delta\rho_1}{\rho_3}\langle z_n, u_{n,x}\rangle \\ &= -\frac{\rho_1}{\rho_3}\langle z_n, i\lambda_n\, \rho_3\theta_n + q_{n,x} + \delta u_{n,x}\rangle + \frac{\rho_1}{\rho_3}\langle z_n, q_{n,x}\rangle \\ &\quad -\frac{\delta\rho_1}{\rho_3}\langle z_n, i\lambda_n\varphi_{n,x} - u_{n,x}\rangle + \frac{\delta\rho_1}{\rho_3}\langle z_n, i\lambda_n\varphi_{n,x}\rangle , \end{aligned}$$

by using (5.29), we obtain

$$\begin{aligned} \langle i\lambda_n \rho_1 z_n,\, \theta_n\rangle &= -\frac{\rho_1}{\rho_3}\langle z_n, i\lambda_n\, \rho_3\theta_n + q_{n,x} + \delta u_{n,x}\rangle - \frac{\delta\rho_1}{\rho_3}\langle z_n, i\lambda_n\varphi_{n,x} - u_{n,x}\rangle \\ &\quad +\frac{\rho_1}{\rho_3}\langle i\lambda_n w_{n,x} - z_{n,x},\, q_n\rangle + \frac{\rho_1}{\tau\rho_3}\langle w_{n,x}, i\lambda_n\tau q_n + \beta q_n + \theta_{n,x}\rangle \\ &\quad +\frac{\delta\rho_1}{\rho_3}\langle z_n, i\lambda_n\varphi_{n,x}\rangle - \frac{\beta\rho_1}{\tau\rho_3}\langle w_{n,x},\, q_n\rangle - \frac{\rho_1}{\tau\rho_3}\langle w_{n,x},\, \theta_{n,x}\rangle . \end{aligned} \tag{5.30}$$

Multiplying $(5.3)_6$ by $\overline{\theta_n}$, we find

$$\langle i\lambda_n\rho_1 z_n,\, \theta_n\rangle - k_0\left\langle (w_{n,x} - l\varphi_n)_x\, ,\, \theta_n\right\rangle + kl\,\langle \varphi_{n,x} + \psi_n + l w_n, \theta_n\rangle \to 0,$$

then by integration by parts and using the boundary conditions, (4.3), $(5.3)_1$, $(5.3)_5$, $(5.3)_7$, $(5.3)_8$, (5.4), (5.7), (5.8), (5.12) and (5.30), we obtain

$$-\frac{\rho_1}{\tau\rho_3}\langle w_{n,x},\, \theta_{n,x}\rangle + \frac{\delta\rho_1}{\rho_3}\langle z_n, i\lambda_n\varphi_{n,x}\rangle + k_0\,\langle w_{n,x} - l\varphi_n, \theta_{n,x}\rangle \to 0.$$

As (thanks to (5.5) and (5.15))

$$\langle \varphi_n,\, \theta_{n,x}\rangle = \left\langle \lambda_n\varphi_n, \frac{\theta_{n,x}}{\lambda_n}\right\rangle \to 0,$$

we get

$$\left(k_0 - \frac{\rho_1}{\tau\rho_3}\right)\langle w_{n,x} - l\varphi_n, \theta_{n,x}\rangle + \frac{\delta\rho_1}{\rho_3}\langle z_n, i\lambda_n\varphi_{n,x}\rangle \to 0. \tag{5.31}$$

Step 10. By using (5.19) and (5.31), we observe that

$$\begin{aligned} &\frac{1}{k\delta}\left[\delta^2 - \left(1 - \frac{k}{k_0}\right)\left(\rho_3 k_0 - \frac{\rho_1}{\tau}\right)\right]\langle w_{n,x} - l\varphi_n, \theta_{n,x}\rangle \\ &-\frac{b\rho_1}{k_0\rho_2}\langle w_{n,x} - l\varphi_n, \psi_{n,x}\rangle - \frac{l\rho_1}{k_0}\|z_n\|^2 - \frac{lk_0}{k}\|w_{n,x} - l\varphi_n\|^2 \to 0. \end{aligned} \tag{5.32}$$

Multiplying $(5.3)_4$ by $\overline{w_n}$, and $(5.3)_6$ by $\overline{\psi_n}$, we get

$$\begin{cases} \langle i\lambda_n v_n, w_n\rangle - \dfrac{b}{\rho_2}\langle \psi_{n,xx}, w_n\rangle + \dfrac{k}{\rho_2}\langle \varphi_{n,x} + \psi_n + l\, w_n, w_n\rangle \to 0, \\ \langle i\lambda_n z_n, \psi_n\rangle - \dfrac{k_0}{\rho_1}\left\langle (w_{n,x} - l\varphi_n)_x\, , \psi_n\right\rangle + \dfrac{lk}{\rho_1}\langle \varphi_{n,x} + \psi_n + l\, w_n, \psi_n\rangle \to 0, \end{cases}$$

then

$$\begin{cases} -\langle v_n, i\lambda_n w_n - z_n\rangle - \langle v_n, z_n\rangle - \dfrac{b}{\rho_2}\langle \psi_{n,xx}, w_n\rangle + \dfrac{k}{\rho_2}\langle \varphi_{n,x} + \psi_n + l\, w_n, w_n\rangle \to 0, \\ -\langle z_n, i\lambda_n \psi_n - v_n\rangle - \langle z_n, v_n\rangle - \dfrac{k_0}{\rho_1}\left\langle (w_{n,x} - l\varphi_n)_x, \psi_n\right\rangle + \dfrac{lk}{\rho_1}\langle \varphi_{n,x} + \psi_n + l\, w_n, \psi_n\rangle \to 0, \end{cases}$$

by integration by parts and using (4.3), $(5.3)_3$, $(5.3)_5$, (5.7), (5.8) and (5.12), we obtain

$$\begin{cases} -\langle v_n, z_n\rangle - \dfrac{b}{\rho_2}\left[\langle \psi_{n,x}, w_n\rangle\right]_0^1 + \dfrac{b}{\rho_2}\langle \psi_{n,x}, w_{n,x}\rangle \to 0, \\ -\langle v_n, z_n\rangle - \dfrac{k_0}{\rho_1}\left[\langle \psi_n, w_{n,x} - l\varphi_n\rangle\right]_0^1 + \dfrac{k_0}{\rho_1}\langle \psi_{n,x}, w_{n,x} - l\varphi_n\rangle \to 0, \end{cases}$$

by using the boundary conditions, we find

$$\frac{b}{\rho_2}\langle \psi_{n,x}, w_{n,x}\rangle - \frac{k_0}{\rho_1}\langle \psi_{n,x}, w_{n,x} - l\varphi_n\rangle \to 0.$$

As $\langle \psi_{n,x}, \varphi_n\rangle \to 0$ (according to (4.3) and (5.6)), then

$$\left(\frac{b}{\rho_2} - \frac{k_0}{\rho_1}\right)\langle \psi_{n,x}, w_{n,x} - l\varphi_n\rangle \to 0.$$

As $\xi_0 \neq 0$; that is, $\dfrac{b}{\rho_2} \neq \dfrac{k_0}{\rho_1}$, then we obtain

$$\langle \psi_{n,x}, w_{n,x} - l\varphi_n\rangle \to 0. \tag{5.33}$$

As $\xi_1 = 0$; that is, $\delta^2 - \left(\rho_1 - \frac{k\rho_2}{b}\right)\left(\dfrac{b\rho_3}{\rho_2} - \dfrac{1}{\tau}\right) = 0$, then, using (5.28) and (5.33), we find

$$v_n \to 0 \text{ in } L^2(0,1). \tag{5.34}$$

By $(5.3)_3$ and (5.34), we have

$$\lambda_n \psi_n \to 0 \text{ in } L^2(0,1). \tag{5.35}$$

Multiplying $(5.3)_4$ by $\overline{\psi_n}$, we get

$$\langle i\lambda_n \rho_2 v_n, \psi_n\rangle - b\langle \psi_{n,xx}, \psi_n\rangle + k\langle \varphi_{n,x} + \psi_n + l\, w_n, \psi_n\rangle \to 0,$$

then, by integrating by parts, we remark that

$$\langle i\rho_2 v_n, \lambda_n \psi_n\rangle - b\left[\langle \psi_{n,x}, \psi_n\rangle\right]_0^1 + \frac{b}{2}\|\psi_{n,x}\|^2 + k\langle \varphi_{n,x} + \psi_n + l\, w_n, \psi_n\rangle \to 0. \tag{5.36}$$

By using the boundary conditions, (4.3), (5.7), (5.8), (5.12), (5.34), (5.35) and (5.36), we arrive at

$$\psi_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.37}$$

As $\xi_2 = 0$; that is, $\delta^2 - \left(1 - \dfrac{k}{k_0}\right)\left(\rho_3 k_0 - \dfrac{\rho_1}{\tau}\right) = 0$, then, using (5.32) and (5.33), we deduce that

$$z_n \to 0 \text{ in } L^2(0,1) \tag{5.38}$$

and

$$w_{n,x} - l\varphi_n \to 0 \text{ in } L^2(0,1),$$

and so, using (5.6),

$$w_{n,x} \to 0 \text{ in } L^2(0,1). \tag{5.39}$$

Finally, (5.4), (5.6), (5.7), (5.8), (5.9), (5.12), (5.14), (5.34), (5.37), (5.38) and (5.39) lead to (4.7), which is a contradiction with (4.3). Hence, the proof of Theorem 5.1 is completed. □

Remark 2. Our stability results hold for some other boundary conditions such as

$$\begin{cases} \varphi_x(0,t) = \psi(0,t) = w(0,t) = \theta(0,t) = 0 & in\ (0,\infty), \\ \varphi_x(1,t) = \psi(1,t) = w(1,t) = \theta(1,t) = 0 & in\ (0,\infty), \end{cases}$$

$$\begin{cases} \varphi(0,t) = \psi_x(0,t) = w_x(0,t) = q(0,t) = 0 & in\ (0,\infty), \\ \varphi(1,t) = \psi_x(1,t) = w_x(1,t) = q(1,t) = 0 & in\ (0,\infty) \end{cases}$$

and

$$\begin{cases} \varphi_x(0,t) = \psi(0,t) = w(0,t) = \theta(0,t) = 0 & in\ (0,\infty), \\ \varphi(1,t) = \psi_x(1,t) = w_x(1,t) = q(1,t) = 0 & in\ (0,\infty). \end{cases}$$

The question is posed when [φ and ψ] or [φ and w] or [φ and θ] has the same boundary condition at 0 or at 1, and when [φ and q] or [ψ and w] or [ψ and θ] or [w and θ] do not have the same boundary condition at 0 or at 1.

6. Concluding Remarks

In this work, we proved that, under new relationships between the coefficients of the considered model, the coupling of the first component in Bresse system with the heat conduction of Cattaneo's law is strong enough to stabilize exponentially the solutions of the considered model. When these relationships are not satisfied, we showed that the total energy of the system is not decaying exponentially and it is decaying at least polynomially with a decay rate depending on the smoothness of the initial data. It will interesting to study the optimality of the decay rate for the polynomial stability case and to extend our results to other kind of heat conduction models.

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