COMMUNICATIONS ON
PURE AND APPLIED ANALYSIS
Volume **18**, Number **1**, January **2019**

doi:10.3934/cpaa.2019009

pp. **159–180**

EXISTENCE AND A GENERAL DECAY RESULTS FOR A VISCOELASTIC PLATE EQUATION WITH A LOGARITHMIC NONLINEARITY

Mohammad M. Al-Gharabli

King Fahd University of Petroleum and Minerals
The Preparatory Year Program, Department of Mathematics
Dhahran 31261, Saudi Arabia

Aissa Guesmia

Institut Elie Cartan de Lorraine, UMR 7502
Université de Lorraine, 3 Rue Augustin Fresnel
BP 45112, 57073 Metz Cedex 03, France
and
King Fahd University of Petroleum and Minerals
The Preparatory Year Program, Department of Mathematics
Dhahran 31261, Saudi Arabia

Salim A. Messaoudi

King Fahd University of Petroleum and Minerals
Department of Mathematics and Statistics
Dhahran 31261, Saudi Arabia

(Communicated by Daoyuan Fang)

Abstract. In this paper, we consider a viscoelastic plate equation with a logarithmic nonlinearity. Using the Galaerkin method and the multiplier method, we establish the existence of solutions and prove an explicit and general decay rate result. This result extends and improves many results in the literature such as Gorka [19], Hiramatsu et al. [27] and Han and Wang [26].

1. **Introduction.** In this paper, we deal with the existence and decay of solutions of the following plate problem:

$$\begin{cases} u_{tt} + \Delta^2 u + u - \int_0^t g(t-s)\Delta^2 u(s)ds = ku\ln|u|, & \text{in } \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{in } \partial\Omega \times (0,\infty), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain of $\mathbb{R}^2$ with a smooth boundary $\partial\Omega$, ν is the unit outer normal to $\partial\Omega$ and k is a small postive real number. The kernel g satisfies some conditions to be specified later.

2000 *Mathematics Subject Classification.* 35B35, 35L55, 75D05, 74D10, 93D20.
Key words and phrases. General decay, logarithmic nonlinearity, viscoelasticity, plate equation.

1.1. **Plate problems.** Concerning the study of plates, Lagnese [31] studied a viscoelastic plate equation and showed that the energy decays to zero as time goes to infinity by intorducing a dissipative mechanism on the boundary of the system. Rivera et al. [43] proved that the first and second order energy, associated with the solutions of the viscoelastic plate equation, decay exponentially provided that the kernel of the memory also decays exponentially. Komornik [29] investigated the energy decay of a plate model under weak growth assumptions on the feedback function. Messaoudi [36] studied the following problem:

$$\begin{cases} u_{tt} + \Delta^2 u + |u_t|^{m-2}u_t = |u|^{p-2}u, & \text{in } Q_T = \Omega \times (0,T), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_T = \partial\Omega \times [0,T), \\ u(x,0) = u_0(x), \ u_t(x,0) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.2}$$

established an existence result and showed that the solution continues to exist globally if $m \geq p$, and blows up in finite time if $m < p$ and the initial energy is negative. This result was later improved by Chen and Zhou [13].

For boundary damping, Santos and Junior [44] studied the stability of the following system:

$$\begin{cases} u_{tt} + \Delta^2 u = 0, & \text{in } \Omega \times (0,\infty), \\ u = \frac{\partial u}{\partial \nu} = 0, & \text{on } \Gamma_0 \times (0,\infty), \\ -u + \int_0^t g_1(t-s)\beta_1 u(s)ds = 0, & \text{on } \Gamma_1 \times (0,\infty), \\ \frac{\partial u}{\partial \nu} + \int_0^t g_2(t-s)\beta_2 u(s)ds = 0, & \text{on } \Gamma_2 \times (0,\infty), \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & \text{in } \Omega, \end{cases} \tag{1.3}$$

where

$$\beta_1 u = \Delta u + (1-\mu)B_1 u \quad \text{and} \quad \beta_2 u = \frac{\partial \Delta u}{\partial \mu} + (1-\mu)\frac{\partial B_2 u}{\partial \eta}$$

with

$$B_1 u = 2\nu_1\nu_2 u_{xy} - \nu_1^2 u_{yy} - \nu_2^2 u_{xx} \quad \text{and} \quad B_2 u = (\nu_1 - \nu_2)\, u_{xy} + \nu_1\nu_2\,(u_{yy} - u_{xx})\,.$$

For more results in this direction, see [3, 22, 26, 30, 32].

1.2. **Viscoelastic problems.** Since the pioneer works of Dafermos [15, 16] in 1970, where the general decay was discussed, problems related to viscoelasticity have attracted a great deal of attention and many results of existence and long-time behavior have been established. The importance of the viscoelastic properties of materials has been realized because of the rapid developments in rubber and plastic industry. Many advances in the studies of constitutive relations, failure theories and life prediction of viscoelastic materials and structures were reported and reviewed in the last two decades [14]. Hrusa [28] considered a one-dimensional nonlinear viscoelastic equation of the form

$$u_{tt} - cu_{xx} + \int_0^t m(t-s)(\psi(u_x(s)))_x ds = f(x,t)$$

and proved several global existence results for large data and an exponential decay result for strong solutions when $m(s) = e^{-s}$ and ψ satisfies certain conditions. In [17], Dassios and Zafiropoulos considered a viscoelastic problem in $\mathbb{R}^3$ and proved a polynomial decay result for exponentially decaying kernels. After that, Rivera [42] considered equations for linear isotropic homogeneous viscoelastic solids of integral type which occupy bounded domains or the whole space $\mathbb{R}^n$. In the bounded-domain case and for exponentially decaying memory kernels and regular solutions, he showed that the sum of the first and the second energy decays exponentially. Whereas, the decay is polynomial when the body occupies the whole space $\mathbb{R}^n$, even if the relaxation function is of an exponential decay.

For quasilinear problems, Cavalcanti et al. [7] studied, in a bounded domain, the following equation:

$$|u_t|^\rho u_{tt} - \Delta u - \Delta u_{tt} + \int_0^t g(t-\tau)\Delta u(\tau)d\tau - \gamma \Delta u_t = 0, \quad \text{in } \Omega \times (0,\infty), \quad (1.4)$$

for $\rho > 0$. A global existence result for $\gamma \geq 0$, as well as an exponential decay result for $\gamma > 0$, have been established. This latter result was then extended to a situation, where $\gamma = 0$, by Messaoudi and Tatar [39, 40], and exponential and polynomial decay results have been established in the absence, as well as in the presence, of a source term. In all the above mentioned works, the rates of decay in relaxation functions were either of exponential or polynomial type. In [8], Cavalcanti et al. considered

$$u_{tt} - \Delta u + \int_0^t g(t-s)\Delta u(s)ds + a(x)u_t + |u|^{p-1}u = 0, \quad \text{in } \Omega \times (0,\infty),$$

where $a : \Omega \to \mathbb{R}^+$ is a function which may vanish on a part of the domain Ω but satisfies $a(x) \geq a_0$ on $\omega \subset \Omega$ and g satisfies, for two positive constants ξ_1 and ξ_2,

$$-\xi_1 g(t) \leq g'(t) \leq -\xi_2 g(t), \quad \forall t \in \mathbb{R}^+$$

and established an exponential decay result under some restrictions on ω. Berrimi and Messaoudi [4] established the result of [8], under weaker conditions on both a and g, to a problem where a source term is competing with the damping term. Cavalcanti and Oquendo [10] considered the following problem:

$$u_{tt} - k_0 \Delta u + \int_0^t \operatorname{div}[a(x)g(t-s)\Delta u(s)]ds + b(x)h(u_t) + f(u) = 0, \text{ in } \Omega \times (0,\infty) \quad (1.5)$$

and established, for $a(x) + b(x) \geq \rho > 0$, an exponential stability result for g decaying exponentially and h linear, and a polynomial stability result for g decaying polynomially and h nonlinear. Li et al. [34] treated (1.5) with $b(x) = 0$ and $f(u) = -|u|^\gamma u,\ \gamma > 0$. They showed the global existence and uniqueness of global solution of problem (1.5) and established uniform decay rate of the energy under suitable conditions on the initial data and the relaxation function g. For more general decaying relaxation functions, Messaoudi [37, 38] considered

$$u_{tt} - \Delta u + \int_0^t g(t-\tau)\Delta u(\tau)d\tau = b|u|^{q-2}u, \text{ in } \Omega \times (0,\infty) \quad (1.6)$$

for $q \geq 2$, $b \in \{0,1\}$ and g satisfying $(A1)$ and $(A2)$ below with $p = 1$, and established a more general decay result, from which the usual exponential and polynomial decay rates are only special cases. Guesmia et al. [24] studied the well-posedness and stability of the following coupled two wave equations:

$$\begin{cases} u_{tt} - \Delta u + \int_0^t g_1(t-\tau)\Delta u(\tau)d\tau + |u_t|^{m-1}u_t = f_1(u,v), & \text{in } \Omega \times (0,\infty), \\ v_{tt} - \Delta v + \int_0^t g_2(t-\tau)\Delta v(\tau)d\tau + |v_t|^{r-1}v_t = f_2(u,v), & \text{in } \Omega \times (0,\infty), \end{cases} \quad (1.7)$$

where $m,\ r \geq 1$, f_1 and f_2 are given functions satisfying some hypotheses, and g_1 and g_2 are like g in $(A1)$ and $(A2)$ below (with $p = 1$, ξ_1 and ξ_2 instead of ξ). They established the same stability estimate as in [37, 38] with $\xi = \min\{\xi_1, \xi_2\}$. Very recently, Messaoudi and Al-Khulaifi [41] considered (1.4) with $\gamma = 0$, where the relaxation function satisfies (2.2) below, and established a more general decay result. For the case of memories acting on the boundary of domain, we refer the readers to [9, 23] and the references therein.

1.3. **Problems with logarithmic nonlinearity.** The logarithmic nonlinearity is of much interest in physics, since it appears naturally in inflation cosmology and supersymmetric filed theories, quantum mechanics and nuclear physics [1, 18]. This type of problems has applications in many branches of physics such as nuclear physics, optics and geophysics [2, 5, 19]. Birula and Mycielski [5, 6] studied the following problem:

$$\begin{cases} u_{tt} - u_{xx} + u - \varepsilon u \ln |u|^2 = 0, & \text{in } [a,b] \times (0,T), \\ u(a,t) = u(b,t) = 0, & \text{in } (0,T), \\ u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x), & \text{in } [a,b], \end{cases} \tag{1.8}$$

which is a relativistic version of logarithmic quantum mechanics and can also be obtained by taking the limit $p \to 1$ for the p-adic string equation [20, 45]. In [11], Cazenave and Haraux considered

$$u_{tt} - \Delta u = u \ln |u|^k, \text{ in } \mathbb{R}^3 \tag{1.9}$$

and established the existence and uniqueness of the solution for the Cauchy problem. Gorka [19] used some compactness arguments and obtained the global existence of weak solutions for all $(u_0, u_1) \in H_0^1 \times L^2$ to the initial-boundary value problem (1.9) in the one-dimensional case. Bartkowski and Gorka [2] proved the existence of classical solutions and investigated the weak solutions for the corresponding one-dimensional Cauchy problem for equation (1.9). Hiramatsu et al. [27] introduced the following equation:

$$u_{tt} - \Delta u + u + u_t + |u|^2 u = u \ln |u| \tag{1.10}$$

to study the dynamics of Q-ball in theoretical physics and presented a numerical study. However, there was no theoretical analysis for the problem. In [25], Han proved the global existence of weak solutions, for all $(u_0, u_1) \in H_0^1(\Omega) \times L^2(\Omega)$, to the initial boundary value problem (1.10) in $\Omega \subset \mathbb{R}^3$.

In this paper, we are concerned with the well-posedness and stability of the plate problem (1.1) with a kernel g having an arbitrary growth at infinity (condition (2.2) below). The obtained stability results improve and generalize many results in the literature.

This paper is organized as follows. In section 2, we present some notations and material needed for our work. In section 3, we establish the local existence of the solutions of the problem. The global existence and the decay results are presented in section 4 and section 5, respectively.

2. **Preliminaries.** In this section, we present some notations and material needed in the proof of our results. We use the standard Lebesgue space $L^2(\Omega)$ and Sobolev space $H_0^2(\Omega)$ with their usual scalar products and norms. Throughout this paper, c is used to denote a generic positive constant.

We consider the following hypotheses:

$(A1)$ $g : \mathbb{R}^+ \to \mathbb{R}^+$ is a C^1- nonincreasing function satisfying

$$g(0) > 0 \quad \text{and} \quad 1 - \int_0^\infty g(s)ds = \ell > 0. \tag{2.1}$$

$(A2)$ There exist a nonincreasing differentiable function $\xi : \mathbb{R}^+ \to \mathbb{R}^+$, with $\xi(0) > 0$, and a constant $1 \le p < \frac{3}{2}$ such that

$$g'(t) \le -\xi(t) g^p(t), \ \forall t \in \mathbb{R}^+. \tag{2.2}$$

$(A3)$ The constant k in (1.1) satisfies $0 < k < k_0$, where k_0 is the positive real number satisfying

$$\sqrt{\frac{2\pi\ell}{k_0 c_p}} = e^{-\frac{3}{2}-\frac{1}{k_0}} \tag{2.3}$$

and c_p is the smallest positive number satisfying

$$\|\nabla u\|_2^2 \le c_p \|\Delta u\|_2^2, \quad \forall u \in H_0^2(\Omega),$$

where $\|.\|_2 = \|.\|_{L^2(\Omega)}$.

Remark 2.1. The function $f(s) = \sqrt{\frac{2\pi\ell}{c_p s}} - e^{-\frac{3}{2}-\frac{1}{s}}$ is a continuous and decreasing function on $(0, \infty)$, with

$$\lim_{s\to 0^+} f(s) = \infty \text{ and } \lim_{x\to\infty} f(x) = -e^{-\frac{3}{2}}.$$

Then, there exists a unique $k_0 > 0$ such that $f(k_0) = 0$. Moreover,

$$e^{-\frac{3}{2}-\frac{1}{s}} < \sqrt{\frac{2\pi\ell}{c_p s}}, \ \forall s \in (0, k_0). \tag{2.4}$$

The energy functional associated with problem (1.1) is

$$\begin{aligned} E(t) = \frac{1}{2}\left(\|u_t\|_2^2 + \left(1 - \int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \frac{k+2}{2}\|u\|_2^2\right) \\ - \frac{1}{2}\int_\Omega u^2 \ln |u|^k dx + \frac{1}{2}(go\Delta u)(t), \end{aligned} \tag{2.5}$$

where

$$(go\Delta u)(t) = \int_0^t g(t-s)\|\Delta u(s) - \Delta u(t)\|_2^2 ds.$$

Direct differentiation of (2.5), using (1.1), leads to

$$E'(t) = \frac{1}{2}(g'o\Delta u)(t) - \frac{1}{2}g(t)\|\Delta u\|_2^2 \le \frac{1}{2}(g'o\Delta u)(t) \le 0. \tag{2.6}$$

Lemma 2.2 ([21, 12], Logarithmic Sobolev inequality)**.** *Let u be any function in $H_0^1(\Omega)$ and a be any positive real number. Then*

$$\int_\Omega u^2 \ln |u| dx \le \frac{1}{2}\|u\|_2^2 \ln \|u\|_2^2 + \frac{a^2}{2\pi}\|\nabla u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{2.7}$$

Corollary 2.3. Let u be any function in $H_0^2(\Omega)$ and a be any positive real number. Then

$$\int_\Omega u^2 \ln |u| dx \le \frac{1}{2}\|u\|_2^2 \ln \|u\|_2^2 + \frac{c_p a^2}{2\pi}\|\Delta u\|_2^2 - (1 + \ln a)\|u\|_2^2. \tag{2.8}$$

Lemma 2.4 ([11], Logarithmic Gronwall inequality)**.** *Let $c > 0$, $\gamma \in L^1(0, T; \mathbb{R}^+)$ and assume that the function $w : [0, T] \to [1, \infty)$ satisfies*

$$w(t) \le c\left(1 + \int_0^t \gamma(s) w(s) \ln w(s) ds\right), \quad 0 \le t \le T. \tag{2.9}$$

Then

$$w(t) \le c \exp\left(c\int_0^t \gamma(s)ds\right), \quad 0 \le t \le T. \tag{2.10}$$

3. **Local existence.** In this section, we state and prove the local existence result for problem (1.1).

Definition 3.1. Let $T > 0$. A function

$$u \in C([0,T], H_0^2(\Omega)) \cap C^1([0,T], L^2(\Omega)) \cap C^2([0,T], H^{-2}(\Omega))$$

is called a weak solution of (1.1) on $[0,T]$ if, for any $w \in H_0^2(\Omega)$ and $t \in [0,T]$,

$$\begin{cases} \int_\Omega u_{tt}(x,t)w(x)dx + \int_\Omega \Delta u(x,t)\Delta w(x)dx + \int_\Omega u(x,t)w(x)dx \\ \quad - \int_\Omega \Delta w(x) \int_0^t g(t-s)\Delta u(s)dsdx = \int_\Omega u(x,t)w(x)\ln|u(x,t)|^k dx, \\ \quad u(x,0) = u_0(x), \qquad u_t(x,0) = u_1(x). \end{cases} \tag{3.1}$$

Theorem 3.2. *Assume that (A1) and (A3) hold and let $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$. Then problem (1.1) has a weak solution*

$$u \in C([0,T], H_0^2(\Omega)) \cap C^1([0,T], L^2(\Omega)) \cap C^2([0,T], H^{-2}(\Omega)). \tag{3.2}$$

Proof. To establish the existence of a solution to problem (1.1), we use the Faedo-Galerkin method. Let $\{w_j\}_{j=1}^{\infty}$ be an orthogonal basis of the "separable" space $H_0^2(\Omega)$ which is orthonormal in $L^2(\Omega)$. Let $V_m = span\{w_1, w_2, ..., w_m\}$ and let the projections of the initial data on the finite dimensional subspace V_m be given by

$$u_0^m(x) = \sum_{j=1}^m a_j w_j(x) \quad \text{and} \quad u_1^m(x) = \sum_{j=1}^m b_j w_j(x),$$

where

$$u_0^m \to u_0 \text{ in } H_0^2(\Omega) \quad \text{and} \quad u_1^m \to u \text{ in } L^2(\Omega), \text{ as } m \to \infty. \tag{3.3}$$

We search for an approximate solution

$$u^m(x,t) = \sum_{j=1}^m h_j^m(t) w_j(x)$$

of the approximate problem in V_m:

$$\begin{cases} \int_\Omega \left(u_{tt}^m w + \Delta u^m \Delta w + u^m w - \int_0^t g(t-s)\Delta u^m(s)ds\Delta w\right)dx \\ \qquad = \int_\Omega w u^m \ln|u^m|^k dx, \ \forall w \in V_m, \\ \quad u^m(0) = u_0^m = \sum_{j=1}^m (u_0, w_j)w_j, \\ \quad u_t^m(0) = u_1^m = \sum_{j=1}^m (u_1, w_j)w_j. \end{cases} \tag{3.4}$$

This leads to a system of ODEs for unknown functions $h_j^m(t)$. Based on standard existence theory for ODE, one can obtain functions

$$h_j : [0, t_m) \to \mathbb{R}, \ j = 1, 2, ..., m,$$

which satisfy (3.4) in a maximal interval $[0, t_m), t_m \in (0, T]$. Next, we show that $t_m = T$ and that the local solution is uniformly bounded independent of m and t. For this purpose, let us replace w by u_t^m in (3.4) and integrate by parts to obtain

$$\frac{d}{dt}E^m(t) \le \frac{1}{2}(g' o \Delta u^m)(t) \le 0, \tag{3.5}$$

where

$$\begin{aligned} E^m(t) = \frac{1}{2}\left(\|u_t^m\|_2^2 + \left(1 - \int_0^t g(s)\right)\|\Delta u^m\|_2^2 + \frac{k+2}{2}\|u^m\|_2^2\right) \\ - \frac{1}{2}\int_\Omega |u^m|^2 \ln |u^m|^k dx + \frac{1}{2}(go\Delta u^m)(t). \end{aligned} \tag{3.6}$$

From (3.5), we have

$$E^m(t) \leq E^m(0).$$

The last inequality together with (2.1) and the Logarithmic Sobolev inequality (2.8) lead to

$$\begin{aligned} \|u_t^m\|_2^2 + (go\Delta u^m)(t) + \left(\ell - \frac{ka^2c_p}{2\pi}\right)\|\Delta u^m\|_2^2 + \left[\frac{k+2}{2} + k\,(1+\ln a)\right]\|u^m\|_2^2 \\ \leq C + \frac{k}{2}\|u^m\|_2^2 \ln \|u^m\|_2^2, \end{aligned} \tag{3.7}$$

where $C = 2E^m(0)$. Choosing

$$e^{-\frac{3}{2}-\frac{1}{k}} < a < \sqrt{\frac{2\pi\ell}{kc_p}} \tag{3.8}$$

will make

$$\ell - \frac{ka^2c_p}{2\pi} > 0 \quad \text{and} \quad \frac{k+2}{2} + k\,(1+\ln a) > 0.$$

This selection is possible thanks to $(A3)$. So, we get

$$(go\Delta u^m)(t) + \|u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 \leq c\left(1 + \|u^m\|_2^2 \ln \|u^m\|_2^2\right). \tag{3.9}$$

Let us note that

$$u^m(.,t) = u^m(.,0) + \int_0^t \frac{\partial u^m}{\partial s}(.,s)ds.$$

Then, using Cauchy-Schwarz' inequality, we get

$$\begin{aligned} \|u^m(t)\|_2^2 &\leq 2\|u^m(0)\|_2^2 + 2\left\|\int_0^t \frac{\partial u^m}{\partial s}(s)ds\right\|_2^2 \\ &\leq 2\|u^m(0)\|_2^2 + 2T\int_0^t \|u_t^m(s)\|_2^2 ds, \end{aligned} \tag{3.10}$$

hence, inequality (3.9) gives

$$\|u^m\|_2^2 \leq 2\|u^m(0)\|_2^2 + 2cT\left(1 + \int_0^t \|u^m\|_2^2 \ln \|u^m\|_2^2 ds\right). \tag{3.11}$$

If we put $C_1 = \max\{2cT, 2\|u^m(0)\|_2^2\}$, (3.11) leads to

$$\|u^m\|_2^2 \leq 2C_1\left(1 + \int_0^t \|u^m\|_2^2 \ln \|u^m\|_2^2 ds\right).$$

Without loss of generality, we take $C_1 \geq 1$, which gives

$$\|u^m\|_2^2 \leq 2C_1\left(1 + \int_0^t \left(C_1 + \|u^m\|_2^2\right)\ln\left(C_1 + \|u^m\|_2^2\right)ds\right).$$

Applying the Logarithmic Gronwall inequality to the last inequality, we obtain the following estimate:

$$\|u^m\|_2^2 \leq 2C_1 e^{2C_1T} := C_2.$$

Hence, from inequality (3.9) it follows that:

$$(go\Delta u^m)(t) + \|u_t^m\|_2^2 + \|\Delta u^m\|_2^2 + \|u^m\|_2^2 \le c\,(1 + C_2 \ln C_2) := C_3,$$

where C_3 is a positive constant independent of m and t. This implies

$$\sup_{t\in(0,t_m)} \|u_t^m\|_2^2 + \sup_{t\in(0,t_m)} \|\Delta u^m\|_2^2 + \sup_{t\in(0,t_m)} \|u^m\|_2^2 \le 3C_3. \tag{3.12}$$

So, the approximate solution is uniformly bounded independent of m and t. Therefore, we can extend t_m to T. Moreover, we obtain, from (3.12),

$$\begin{cases} u^m \text{ is uniformly bounded in } L^\infty(0,T;H_0^2(\Omega)), \\ u_t^m \text{ is uniformly bounded in } L^\infty(0,T;L^2(\Omega)), \end{cases} \tag{3.13}$$

which implies that there exists a subsequence of (u^m) (still denoted by (u^m)), such that

$$\begin{cases} u^m \rightharpoonup u \ \text{ weakly * in } L^\infty(0,T;H_0^2(\Omega)), \\ u_t^m \rightharpoonup u_t \ \text{ weakly * in } L^\infty(0,T;L^2(\Omega)), \\ u^m \rightharpoonup u \ \text{ weakly in } L^2(0,T;H_0^2(\Omega)), \\ u_t^m \rightharpoonup u_t \ \text{ weakly in } L^2(0,T;L^2(\Omega)). \end{cases} \tag{3.14}$$

Making use of Aubin-Lions' theorem, we find, up to a subsequence, that

$$u^m \to u \text{ strongly in } L^2(0,T;L^2(\Omega))$$

and

$$u^m \to u \text{ a.e. in } \Omega \times (0,T).$$

Since the map $s \to s \ln|s|^k$ is continuous on $\mathbb{R}$, we have the convergence

$$u^m \ln|u^m|^k \to u \ln|u|^k \text{ a.e. in } \Omega \times (0,T).$$

Using the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$ (since $\Omega \subset \mathbb{R}^2$), it is clear that $u^m \ln|u^m|^k$ is bounded in $L^\infty(\Omega \times (0,T))$. Next, taking into account the Lebesgue bounded convergence theorem (Ω is bounded), we get

$$u^m \ln|u^m|^k \to u \ln|u|^k \text{ strongly in } L^2(0,T;L^2(\Omega)). \tag{3.15}$$

Now, we integrate (3.4) over $(0,t)$ to obtain, for every $w \in V_m$,

$$\begin{aligned} &\int_\Omega u_t^m w dx - \int_\Omega u_1^m w dx + \int_0^t \int_\Omega \Delta u^m(s) \Delta w dx ds + \int_0^t \int_\Omega u^m(s) w dx ds \\ &\quad - \int_0^t \int_\Omega \int_0^\tau g(\tau - s) \Delta u^m(s) ds \Delta w dx ds d\tau = \int_\Omega \int_0^t w u^m(s) \ln|u^m(s)|^k dx ds. \end{aligned} \tag{3.16}$$

Convergences (3.3), (3.14) and (3.15) are sufficient to pass to the limit in (3.16), as $m \to +\infty$, and get, for any $w \in V_m$ and $m \ge 1$,

$$\begin{aligned} \int_\Omega u_t w dx &= \int_\Omega u_1 w dx - \int_0^t \int_\Omega \Delta u(s) \Delta w dx ds - \int_0^t \int_\Omega u(s) w dx ds \\ &\quad + \int_0^\tau \int_\Omega \Delta w(x) \int_0^t g(t-s) \Delta u(s) ds dx dt + \int_\Omega \int_0^t u(s) w \ln|u(s)|^k ds dx, \end{aligned} \tag{3.17}$$

which implies that (3.17) is valid for any $w \in H_0^2(\Omega)$. Using the fact that the terms in the right-hand side of (3.17) are absolutely continuous since they are functions of t defined by integrals over $(0,t)$, hence it is differentiable for a.e. $t \in \mathbb{R}^+$. Thus, differentiating (3.17), we obtain, for a.e. $t \in (0,T)$ and any $w \in H_0^2(\Omega)$,

$$\begin{aligned}\int_\Omega u_{tt}(x,t)w(x)dx + \int_\Omega \Delta u(x,t)\Delta w(x)dx + \int_\Omega u(x,t)w(x)dx \\ - \int_\Omega \Delta w(x)\int_0^t g(t-s)\Delta u(s)dsdx = \int_\Omega w(x)u(x,t)\ln|u(x,t)|^k dx.\end{aligned} \tag{3.18}$$

To handle the initial conditions, we note that

$$u^m \rightharpoonup u \quad \text{weakly in } L^2(0,T;H_0^2(\Omega)) \quad \text{and} \quad u_t^m \rightharpoonup u_t \quad \text{weakly in } L^2(0,T;L^2(\Omega)). \tag{3.19}$$

Thus, using Lion's Lemma [33], we obtain

$$u^m \to u \text{ in } C([0,T],L^2(\Omega)). \tag{3.20}$$

Therefore, $u^m(x,0)$ makes sense and

$$u^m(x,0) \to u(x,0) \text{ in } L^2(\Omega).$$

Also, we have

$$u^m(x,0) = u_0^m(x) \to u_0(x) \text{ in } H_0^2(\Omega).$$

Hence

$$u(x,0) = u_0(x).$$

Now, multiply (3.4) by $\phi \in C_0^\infty(0,T)$ and integrate over $(0,T)$, we obtain, for any $w \in V_m$,

$$\begin{aligned}-\int_0^T\int_\Omega u_t^m(t)w\phi'(t)dxdt &= -\int_0^T\int_\Omega \Delta u^m(t)\Delta w\phi(t)dxdt \\ &- \int_0^T\int_\Omega u^m w\phi(t)dxdt + \int_0^T\int_\Omega\int_0^t g(t-s)\Delta u^m(s)ds\Delta w\phi(t)dxdt \\ &+ \int_0^T\int_\Omega u^m w\phi(t)\ln|u^m|^k dxdt.\end{aligned} \tag{3.21}$$

As $m \to \infty$, we have, for any $w \in H_0^2(\Omega)$ and any $\phi \in C_0^\infty((0,T))$,

$$\begin{aligned}-\int_0^T\int_\Omega u_t(t)w\phi'(t)dxdt &= -\int_0^T\int_\Omega \Delta u(t)\Delta w\phi(t)dxdt - \int_0^T\int_\Omega uw\phi(t)dxdt \\ &+ \int_0^T\int_\Omega\int_0^t g(t-s)\Delta u(s)ds\Delta w\phi(t)dxdt + \int_0^T\int_\Omega w\phi(t)u\ln|u|^k dxdt.\end{aligned} \tag{3.22}$$

This means (see [35]),

$$u_{tt} \in L^2([0,T),H^{-2}(\Omega)).$$

Recalling that $u_t \in L^2((0,T),L^2(\Omega))$, we obtain

$$u_t \in C([0,T),H^{-2}(\Omega)).$$

So, $u_t^m(x,0)$ makes sense and

$$u_t^m(x,0) \to u_t(x,0) \text{ in } H^{-2}(\Omega).$$

But

$$u_t^m(x,0) = u_1^m(x) \to u_1(x) \text{ in } L^2(\Omega).$$

Hence

$$u_t(x,0) = u_1(x).$$

□

4. **Global existence.** In this section, we state and prove a global existence result under smallness conditions on the initial data (u_0, u_1). For this purpose, we introduce the following functionals:

$$J(t) = \frac{1}{2}\left(\left(1 - \int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (go\Delta u)(t) - \int_\Omega u^2 \ln|u|^k dx\right) + \frac{k}{4}\|u\|_2^2 \tag{4.1}$$

and

$$I(t) = \left(1 - \int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (go\Delta u)(t) - 3\int_\Omega u^2 \ln|u|^k dx. \tag{4.2}$$

Lemma 4.1. *The following inequalities hold:*

$$-kd_0\sqrt{|\Omega|c_*^3}\|\Delta u\|_2^{\frac{3}{2}} \le \int_\Omega u^2 \ln|u|^k dx \le kc_*^3\|\Delta u\|_2^3, \qquad \forall u \in H_0^2(\Omega), \tag{4.3}$$

where $d_0 = \sup_{0<s<1}\sqrt{s}|\ln s|$, $|\Omega|$ *is the Lebegue measure of* Ω *and* c_* *is the smallest embedding constant*

$$\left(\int_\Omega |u|^3 dx\right)^{\frac{1}{3}} \le c_*\|\Delta u\|_2, \qquad \forall u \in H_0^2(\Omega) \tag{4.4}$$

(c_* exists thanks to the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$).

Proof. Let

$$\Omega_1 = \{x \in \Omega : |u(x)| > 1\} \text{ and } \Omega_2 = \{x \in \Omega : |u(x)| \le 1\}.$$

So, using (4.4), we have

$$\begin{aligned}\int_\Omega u^2 \ln|u|^k dx &= \int_{\Omega_2} u^2 \ln|u|^k dx + \int_{\Omega_1} u^2 \ln|u|^k dx \\ &\le k\int_{\Omega_1} u^2 \ln|u| dx \le k\int_{\Omega_1}|u|^3 dx \le k\int_\Omega |u|^3 dx \le kc_*^3\|\Delta u\|_2^3,\end{aligned}$$

this gives the right inequality in (4.3).

On the other hand, using Hölder's inequality and (4.4), we find

$$\begin{aligned}-\int_\Omega u^2 \ln|u|^k dx &= -\int_{\Omega_2} u^2 \ln|u|^k dx - \int_{\Omega_1} u^2 \ln|u|^k dx \\ &\le -k\int_{\Omega_2} u^2 \ln|u| dx = k\int_{\Omega_2} u^2|\ln|u||dx \\ &\le kd_0\int_\Omega |u|^{\frac{3}{2}} dx \le kd_0\sqrt{|\Omega|}\left(\int_\Omega |u|^3 dx\right)^{\frac{1}{2}} \le kd_0\sqrt{|\Omega|c_*^3}\|\Delta u\|_2^{\frac{3}{2}},\end{aligned}$$

which implies the left inequality in (4.3). ◻

Lemma 4.2. *Assume that* $(A1) - (A3)$. *Let* $(u_0, u_1) \in H_0^2(\Omega) \times L^2(\Omega)$ *such that*

$$I(0) > 0 \text{ and } \sqrt{54}kc_*^3\left(\frac{E(0)}{\ell}\right)^{\frac{1}{2}} < \ell. \tag{4.5}$$

Then

$$I(t) > 0, \ \forall t \in [0, T). \tag{4.6}$$

Proof. From (4.2), we have

$$\int_\Omega u^2 \ln |u|^k dx = \frac{1}{3}\left(1-\int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \frac{1}{3}\|u\|_2^2 + \frac{1}{3}(go\Delta u)(t) - \frac{1}{3}I(t). \quad (4.7)$$

Substitute (4.7) in (4.1), we find

$$J(t) = \frac{1}{3}\left[\left(1-\int_0^t g(s)ds\right)\|\Delta u\|_2^2 + \|u\|_2^2 + (go\Delta u)(t)\right] + \frac{k}{4}\|u\|_2^2 + \frac{1}{6}I(t). \quad (4.8)$$

Since $I(0) > 0$ and I is continuous on $[0,T]$, there exists $t_0 \in (0,T]$ such that $I(t) > 0$, for all $t \in [0,t_0)$. Let us denote by t_0 the biggest real number in $(0,T]$ such that $I > 0$ on $[0,t_0)$. If $t_0 = T$, then (4.6) is satisfied.

We assume by contradiction that $t_0 \in (0,T)$. Thus $I(t_0) = 0$ and

$$\|\Delta u(t)\|_2^2 \le \frac{3}{\ell}J(t) \le \frac{3}{\ell}E(t) \le \frac{3}{\ell}E(0), \ \forall t \in [0,t_0). \quad (4.9)$$

If $\|\Delta u(t_0)\|_2^2 = 0$, then (4.3) and (4.4) give

$$0 = I(t_0) = (go\Delta u)(t_0) = \int_0^{t_0} g(s)\|\Delta u(s)\|_2^2 ds. \quad (4.10)$$

Consequently, if $g > 0$ on $[0,t_0)$, we get

$$\|\Delta u(s)\|_2 = 0, \quad \forall s \in [0,t_0).$$

Then

$$I(t) = 0, \quad \forall t \in [0,t_0),$$

which is not true since $I > 0$ on $[0,t_0)$. If g is not positive on $[0,t_0)$, then let $t_1 \in [0,t_0)$ the smallest real number such that $g(t_1) = 0$. Because $g(0) > 0$ and g is positive, nonincreasing and continuous on $\mathbb{R}^+$ (condition $(A1)$), then $t_1 > 0$ and $g = 0$ on $[t_1,\infty)$. Therefore, from (4.10), we deduce that

$$0 = \int_0^{t_0} g(s)\|\Delta u(s)\|_2^2 ds = \int_0^{t_1} g(s)\|\Delta u(s)\|_2^2 ds,$$

then $\|\Delta u(s)\|_2 = 0$, for any $s \in [0,t_1)$, which implies that $I(t) = 0$, for any $t \in [0,t_1)$. As before, this is a contraduction with the fact that $I > 0$ on $[0,t_0)$. Then we conclude that $\|\Delta u(t_0)\|_2^2 > 0$. On the other hand, we have

$$I(t_0) \ge \ell\|\Delta u(t_0)\|_2^2 - 3\int_\Omega u(t_0)^2 \ln |u(t_0)|^k dx.$$

By using (4.9) and Lemma 4.1, we have

$$I(t_0) \ge \left[\ell - 3kc_*^3\left(\frac{6E(0)}{\ell}\right)^{\frac{1}{2}}\right]\|\Delta u(t_0)\|_2^2.$$

By recalling (4.5), we arrive at $I(t_0) > 0$, which contradicts the assumption $I(t_0) = 0$. Hence, $t_0 = T$ and then

$$I(t) > 0, \ \forall t \in [0,T).$$

□

The global existence can be easily established by repeating the steps of the proof of Theorem 3.1 [40].

5. **Stability.** In this section, we state and prove our stability result. We start by establishing several lemmas needed for the proof of our main result.

Lemma 5.1. *Assume that* g *satisfies* $(A1)$. *Then, for* $u \in H_0^2(\Omega)$, *we have*

$$\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 dx \leq c(go\Delta u)(t)$$

and

$$\int_\Omega \left(\int_0^t g'(t-s)(u(t)-u(s))ds\right)^2 dx \leq -c(g'o\Delta u)(t).$$

Proof.

$$\begin{aligned}&\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 dx\\ &= \int_\Omega \left(\int_0^t \sqrt{g(t-s)}\sqrt{g(t-s)}(u(t)-u(s))ds\right)^2 dx.\end{aligned}$$

By applying Cauchy-Schwarz' and Poincaré's inequalities, we can show that

$$\begin{aligned}&\int_\Omega \left(\int_0^t g(t-s)(u(t)-u(s))ds\right)^2 dx\\ &\leq \int_\Omega \left(\int_0^t g(t-s)ds\right)\left(\int_0^t g(t-s)(u(t)-u(s))^2 ds\right) dx \qquad (5.1)\\ &\leq (1-\ell)c(go\Delta u)(t)\\ &\leq c(go\Delta u)(t).\end{aligned}$$

Similarly, the second inequality in Lemma 5.1 can be proved. □

Lemma 5.2. *Assume that* g *satisfies* $(A1)$ *and* $(A2)$. *Then*

$$\int_0^\infty \xi(t)g^{1-\sigma}(t)dt < \infty, \quad \forall \sigma < 2-p. \tag{5.2}$$

Proof. Using $(A1)$ and $(A2)$, we easily see that, for any $\sigma < 2-p$,

$$\xi(t)g^{1-\sigma}(t) = \xi(t)g^{1-\sigma}(t)g^p(t)g^{-p}(t) \leq -g'(t)g^{1-\sigma-p}(t).$$

Integrate the last inequality over $(0,\infty)$, we obtain

$$\int_0^\infty \xi(t)g^{1-\sigma}(t)dt \leq -\int_0^\infty g'(t)g^{1-\sigma-p}(t)dt = \left[-\frac{g^{2-p-\sigma}(t)}{2-p-\sigma}\right]_0^\infty < \infty.$$

□

Similar to Cavalcanti and Oquendo [10], we can easily have the following lemma:

Lemma 5.3. *Assume that* $(A1)-(A3)$ *and* (4.5) *hold and* u *is a solution of* (1.1). *Then, for any* $0<\sigma<1$, *we have*

$$(go\Delta u)(t) \leq c\left[\left(\int_0^\infty g^{1-\sigma}(t)dt\right)E(0)\right]^{\frac{p-1}{p-1+\sigma}} (g^p o\Delta u)^{\frac{\sigma}{p-1+\sigma}}(t).$$

By taking $\sigma = \frac{1}{2}$, *we get*

$$(go\Delta u)(t) \leq c\left(\int_0^t g^{\frac{1}{2}}(s)ds\right)^{\frac{2p-2}{2p-1}} (g^p o\Delta u)^{\frac{1}{2p-1}}(t) \tag{5.3}$$

and, for any $\epsilon_0 \in (0,1)$,

$$(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t) \leq c^{\frac{1}{1+\epsilon_0}} \left(\int_0^t g^{\frac{1}{2}}(s)ds \right)^{\frac{2p-2}{(2p-1)(1+\epsilon_0)}} (g^p o\Delta u)^{\frac{1}{(2p-1)(1+\epsilon_0)}}(t). \tag{5.4}$$

Corollary 5.4. Assume that $(A1)-(A3)$ and (4.5) hold and u is a solution of (1.1). Then

$$\xi(t)(go\Delta u)(t) \leq c\left(-E'(t)\right)^{\frac{1}{2p-1}} \tag{5.5}$$

and, for any $\epsilon_0 \in (0,1)$,

$$\xi(t)(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t) \leq c_{\epsilon_0}\left(-E'(t)\right)^{\frac{1}{(2p-1)(1+\epsilon_0)}}. \tag{5.6}$$

Proof. Multiply both sides of (5.3) by $\xi(t)$ and use (5.2) and (2.6) to obtain

$$\begin{aligned}
\xi(t)(go\Delta u)(t) &\leq c\xi^{\frac{2p-2}{2p-1}}(t)\left(\int_0^t g^{\frac{1}{2}}(s)ds\right)^{\frac{2p-2}{2p-1}} \xi^{\frac{1}{2p-1}}(t)(g^p o\Delta u)^{\frac{1}{2p-1}}(t) \\
&\leq c\left(\int_0^t \xi(s)g^{\frac{1}{2}}(s)ds\right)^{\frac{2p-2}{2p-1}} (\xi g^p o\Delta u)^{\frac{1}{2p-1}}(t) \\
&\leq c\left(\int_0^\infty \xi(s)g^{\frac{1}{2}}(s)ds\right)^{\frac{2p-2}{2p-1}} (-g' o\Delta u)^{\frac{1}{2p-1}}(t) \\
&\leq c\left(-E'(t)\right)^{\frac{1}{2p-1}}.
\end{aligned} \tag{5.7}$$

For the proof of (5.6), using (5.5) and because ξ is nonincreasing, we obtain

$$\xi(t)(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t) = \xi^{\frac{\epsilon_0}{1+\epsilon_0}}(t)\left(\xi(t)(go\Delta u)(t)\right)^{\frac{1}{1+\epsilon_0}} \leq c_{\epsilon_0}\left(-E'(t)\right)^{\frac{1}{(2p-1)(1+\epsilon_0)}}.$$

□

Lemma 5.5. *Assume that* $(A1)-(A3)$ *and* (4.5) *hold. Then the functional*

$$\psi(t) = \int_\Omega u u_t dx$$

satisfies, along the solutions of (1.1),

$$\psi'(t) \leq ||u_t||_2^2 - \frac{\ell}{2}||\Delta u||_2^2 - ||u||_2^2 + \int_\Omega u^2 \ln|u|^k dx + c(go\Delta u)(t). \tag{5.8}$$

Proof. By using Eq. (1.1), we easily see that

$$\begin{aligned}
\psi'(t) = ||u_t||_2^2 - ||\Delta u||_2^2 - ||u||_2^2 + \int_\Omega \Delta u \int_0^t g(t-s)\Delta u(s)dsdx \\
+ \int_\Omega u^2 \ln|u|^k dx.
\end{aligned} \tag{5.9}$$

We now use Lemma 5.1 and Young's inequality, to obtain, for any $\mu > 0$,

$$\begin{aligned}
&\int_\Omega \Delta u(t)\left(\int_0^t g(t-s)\Delta u(s)ds\right)dx \\
&\leq \left(1-\ell+\frac{\mu}{2}\right)||\Delta u||_2^2 + \frac{1}{2\mu}(1-\ell)(go\Delta u)(t).
\end{aligned} \tag{5.10}$$

By choosing $\mu = \ell$ and combining (5.9) and (5.10), we obtain (5.8). □

Lemma 5.6. *Assume that* $(A1)-(A3)$ *and* (4.5) *hold. Then the functional*

$$\chi(t)=-\int_\Omega u_t\int_0^t g(t-s)(u(t)-u(s))dsdx$$

satisfies, along the solutions of (1.1) *and for any* $\epsilon_0\in(0,1)$ *and* $\delta>0$,

$$\begin{aligned}\chi'(t)\le\delta||\Delta u||_2^2+\frac{c}{\delta}(go\Delta u)(t)+\frac{c}{\delta}(-g'o\Delta u)(t)+\left(\delta-\int_0^t g(s)ds\right)\|u_t\|_2^2\\+c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t).\end{aligned}\tag{5.11}$$

Proof. Direct computations, using (1.1), yield

$$\begin{aligned}\chi'(t)=&\int_\Omega\Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx+\int_\Omega u\int_0^t g(t-s)(u(t)-u(s))dsdx\\&+\int_\Omega\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\int_0^t g(t-s)\Delta u(s)dsdx\\&-\int_\Omega u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\\&-\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx\\&-\left(\int_0^t g(s)ds\right)\int_\Omega u_t^2dx.\end{aligned}\tag{5.12}$$

Similarly to (5.9), we estimate the right-hand side terms of (5.12). So, by using Young's inequality, the first term gives, for any $\delta>0$,

$$\int_\Omega\Delta u\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))dsdx\le\frac{\delta}{4}||\Delta u||_2^2+\frac{c}{\delta}(go\Delta u)(t).\tag{5.13}$$

Using Lemma 5.1, Young's and Poincaré's inequalities, the second and fifth terms lead to

$$\int_\Omega u\int_0^t g(t-s)(u(t)-u(s))dsdx\le\frac{\delta}{4}||\Delta u||_2^2+\frac{c}{\delta}(go\Delta u)(t)\tag{5.14}$$

and

$$-\int_\Omega u_t\int_0^t g'(t-s)(u(t)-u(s))dsdx\le\delta||u_t||_2^2-\frac{c}{\delta}(g'o\Delta u)(t).\tag{5.15}$$

Similarly, the third term can be estimated as follows

$$\begin{aligned}\int_\Omega\int_0^t g(t-s)(\Delta u(t)-\Delta u(s))ds\int_0^t g(t-s)\Delta u(s)dsdx\\\le\frac{\delta}{4}||\Delta u||_2^2+c\left(1+\frac{1}{\delta}\right)(go\Delta u)(t).\end{aligned}\tag{5.16}$$

Let $\epsilon_0\in(0,1)$ and $f(s)=s^{\epsilon_0}(|\ln s|-s)$. Notice that f is continuous on $(0,\infty)$ and its limit at 0 is 0, and its limit at ∞ is $-\infty$. Then f has a maximum d_{ϵ_0} on $[0,\infty)$, so the following inequality holds:

$$s|\ln s|\le s^2+d_{\epsilon_0}s^{1-\epsilon_0},\ \forall s>0.\tag{5.17}$$

Applying this inequality to $u\ln|u|$, using the embedding of $H_0^2(\Omega)$ in $L^\infty(\Omega)$ and performing the same calulactions as before, we get, for any $\delta_1>0$,

$$\begin{aligned}
&\int_\Omega u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\\
\leq &k\int_\Omega \left(u^2+d_{\epsilon_0}|u|^{1-\epsilon_0}\right)\left|\int_0^t g(t-s)(u(t)-u(s))dsdx\right|\\
\leq &c\int_\Omega |u|\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|dx\\
&+\delta_1\int_\Omega u^2dx+c_{\epsilon_0,\delta_1}\int_\Omega\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^{\frac{2}{1+\epsilon_0}}dx\\
\leq &c\delta_1||\Delta u||_2^2+\frac{c}{\delta_1}\int_\Omega\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^2dx\\
&+c_{\epsilon_0,\delta_1}\int_\Omega\left|\int_0^t g(t-s)(u(t)-u(s))ds\right|^{\frac{2}{1+\epsilon_0}}dx,
\end{aligned}$$

then, puting $\frac{\delta}{4}=c\delta_1$ and using Hölder's inequality and Lemma 5.1, we find

$$\begin{aligned}\int_\Omega u\ln|u|^k\int_0^t g(t-s)(u(t)-u(s))dsdx\leq &\frac{\delta}{4}||\Delta u||_2^2+\frac{c}{\delta}(go\Delta u)(t)\\
&+c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t).\end{aligned}\tag{5.18}$$

The above inequalities imply (5.11). □

Lemma 5.7. *Assume that* $(A1)-(A3)$ *and* (4.5) *hold and let* $\epsilon_0\in(0,1)$. *Then, for* k *small enough, there exist two positive constants* ε_1 *and* ε_2 *such that the functional*

$$L(t)=E(t)+\varepsilon_1\psi(t)+\varepsilon_2\chi(t)$$

satisfies

$$L\sim E\tag{5.19}$$

and, for any $t_0>0$, *there exists a positive constant* m *such that*

$$L'(t)\leq -mE(t)+c(go\Delta u)(t)+c_{\epsilon_0}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t),\quad \forall t\geq t_0.\tag{5.20}$$

Proof. For the proof of (5.19), we see that, using similar calculations as before,

$$\begin{aligned}|L(t)-E(t)|&=|\varepsilon_1\psi(t)+\varepsilon_2\chi(t)|\\
&\leq c\left(\varepsilon_1+\varepsilon_2\right)\left(||u_t||_2^2+||\Delta u||_2^2+(go\Delta u)(t)\right),\end{aligned}$$

therefore, from (4.6) and (4.8), we obtain

$$|L(t)-E(t)|\leq c\left(\varepsilon_1+\varepsilon_2\right)\left(\frac{1}{2}||u_t||_2^2+J(t)\right)=c\left(\varepsilon_1+\varepsilon_2\right)E(t),$$

then

$$\left(1-c\left(\varepsilon_1+\varepsilon_2\right)\right)E(t)\leq L(t)\leq\left(1+c\left(\varepsilon_1+\varepsilon_2\right)\right)E(t).$$

Hence, for ε_1, $\varepsilon_2>0$ satisfying

$$1-c\left(\varepsilon_1+\varepsilon_2\right)>0,\tag{5.21}$$

the equivalence (5.19) holds.

Now, we prove inequality (5.20). Since g is positive and $g(0) > 0$ then, for any $t_0 > 0$, we have

$$\int_0^t g(s)ds \geq \int_0^{t_0} g(s)ds = g_0 > 0, \ \forall t \geq t_0.$$

By using (2.6), (5.8), (5.11) and the definition of $E(t)$, then, for $t \geq t_0$ and any $m > 0$, we have

$$\begin{aligned}
L'(t) \leq & -mE(t) - \left(\varepsilon_2(g_0 - \delta) - \varepsilon_1 - \frac{m}{2}\right)\|u_t\|_2^2 \\
& - \left(\frac{\ell}{2}\varepsilon_1 - \varepsilon_2\delta - \frac{m}{2}\right)\|\Delta u\|_2^2 - \left(\varepsilon_1 - \frac{(k+2)m}{4}\right)\|u\|_2^2 \\
& + \left(k\varepsilon_1 - k\frac{m}{2}\right)\int_\Omega u^2 \ln|u|dx + \left(c\varepsilon_1 + \varepsilon_2\frac{c}{\delta} + \frac{m}{2}\right)(go\Delta u)(t) \\
& + \left(\frac{1}{2} - \frac{c\varepsilon_2}{\delta}\right)(g'o\Delta u)(t) + \varepsilon_2 c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t).
\end{aligned} \tag{5.22}$$

Using the Logarithmic Sobolev inequality, for $0 < m < 2\varepsilon_1$, we get

$$\begin{aligned}
L'(t) \leq & -mE(t) - \left(\varepsilon_2(g_0 - \delta) - \varepsilon_1 - \frac{m}{2}\right)\|u_t\|_2^2 \\
& - \left(\frac{\ell}{2}\varepsilon_1 - \varepsilon_2\delta - \frac{m}{2} - k\left(\varepsilon_1 - \frac{m}{2}\right)\frac{c_p a^2}{2\pi}\right)\|\Delta u\|_2^2 \\
& - \left(\varepsilon_1 - \frac{m(k+2)}{4} + k\left(\varepsilon_1 - \frac{m}{2}\right)(1 + \ln a) + k\left(\frac{m}{4} - \frac{\varepsilon_1}{2}\right)\ln\|u\|_2^2\right)\|u\|_2^2 \\
& + \left(c\varepsilon_1 + \varepsilon_2\frac{c}{\delta} + \frac{m}{2}\right)(go\Delta u)(t) \\
& + \left(\frac{1}{2} - \frac{c\varepsilon_2}{\delta}\right)(g'o\Delta u)(t) + \varepsilon_2 c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t).
\end{aligned} \tag{5.23}$$

At this point we choose δ so small that

$$g_0 - \delta > \frac{1}{2}g_0 \quad \text{and} \quad \delta < \frac{\ell g_0}{16}.$$

Whence δ is fixed, the choice of any two positive constants ε_1 and ε_2 satisfying

$$\frac{g_0}{4}\varepsilon_2 < \varepsilon_1 < \frac{g_0}{2}\varepsilon_2 \tag{5.24}$$

will make

$$k_1 := \varepsilon_2(g_0 - \delta) - \varepsilon_1 > 0 \quad \text{and} \quad k_2 := \frac{\ell}{2}\varepsilon_1 - \varepsilon_2\delta > 0.$$

Then, we choose ε_1 and ε_2 so small so that (5.21) and (5.24) remain valid and, further,

$$\frac{1}{2} - \frac{c\varepsilon_2}{\delta} > 0.$$

Consequently, we get (5.19) and

$$
\begin{aligned}
L'(t) \le & -mE(t) - \left(k_1 - \frac{m}{2}\right) \|u_t\|_2^2 \\
& - \left(k_2 - \frac{m}{2} - k\left(\varepsilon_1 - \frac{m}{2}\right)\frac{c_p a^2}{2\pi}\right) \|\Delta u\|_2^2 \\
& - \left(\varepsilon_1 - \frac{m(k+2)}{4} + k\left(\varepsilon_1 - \frac{m}{2}\right)(1+\ln a) + k\left(\frac{m}{4} - \frac{\varepsilon_1}{2}\right)\ln \|u\|_2^2\right)\|u\|_2^2 \\
& + c(go\Delta u)(t) + c_{\epsilon_0,\delta}(go\Delta u)^{\frac{1}{1+\epsilon_0}}(t).
\end{aligned}
\tag{5.25}
$$

Then, using (3.8) and selecting m and k so small that

$$
\alpha_1 = k_1 - \frac{m}{2} > 0,\ \ \alpha_2 = k_2 - \frac{m}{2} - k\left(\varepsilon_1 - \frac{m}{2}\right)\frac{c_p a^2}{2\pi} > 0
$$

and

$$
\alpha_3 = \varepsilon_1 - \frac{m(k+2)}{4} + k\left(\varepsilon_1 - \frac{m}{2}\right)(1+\ln a) + k\left(\frac{m}{4} - \frac{\varepsilon_1}{2}\right)\ln \|u\|_2^2 > 0.
$$

Therefore, we arrive at the desired result (5.20). □

Remark 5.8. Using (2.1), (2.5), (4.1), (4.6) and (4.8), we have

$$
E(t) = J(t) + \frac{1}{2}\|u_t(t)\|_2^2 \ge J(t) \ge \frac{l}{6}\|\Delta u(t)\|_2^2,
$$

then, using (2.6),

$$
\|\Delta u(t)\|_2^2 \le \frac{6}{l}E(t) \le \frac{6}{l}E(0). \tag{5.26}
$$

So, from (2.6) and using Young's inequality, we get

$$
\begin{aligned}
|E'(t)| &= \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - \frac{1}{2}(g'o\Delta u)(t) \\
&\le \frac{1}{2}g(t)\|\Delta u(t)\|_2^2 - \int_0^t g'(t-s)\left(\|\Delta u(t)\|_2^2 + \|\Delta u(s)\|_2^2\right)ds \\
&\le \frac{6}{l}\left(\frac{1}{2}g(t) + 2g(0) - 2g(t)\right)E(0) \\
&\le cE(0).
\end{aligned}
\tag{5.27}
$$

Theorem 5.9. *Let $(u_0,u_1) \in H_0^2(\Omega)\times L^2(\Omega)$, $\epsilon \in (0,2p-1)$ and $t_0 > 0$. Assume that $(A1)-(A3)$ and (4.5) hold. Then, for k small enough, there exists a positive constant K such that the solution of (1.1) satisfies*

$$
E(t) \le K\left(1 + \int_{t_0}^t \xi^{2p-1+\epsilon}(s)ds\right)^{\frac{-1}{2p-2+\epsilon}}, \quad \forall t \ge t_0. \tag{5.28}
$$

Moreover, if there exist $\epsilon_1 \in (0,2p-1)$ and $t_0 > 0$ such that

$$
\int_{t_0}^{\infty}\left(1 + \int_{t_0}^t \xi^{2p-1+\epsilon_1}(s)ds\right)^{\frac{-1}{2p-2+\epsilon_1}} dt < \infty, \tag{5.29}
$$

then, for any $\epsilon \in (0,p)$ and $t_0 > 0$, there exists a positive constant $\widetilde{K}$ such that the solution of (1.1) satisfies

$$
E(t) \le \widetilde{K}\left(1 + \int_{t_0}^t \xi^{p+\epsilon}(s)ds\right)^{\frac{-1}{p-1+\epsilon}}, \quad \forall t \ge t_0. \tag{5.30}
$$

Proof. We multiply (5.20) by $\xi(t)$ and use Corollary 5.4 and (5.27) to get, for any $t \geq t_0$,

$$\begin{aligned} \xi(t)L'(t) &\leq -m\xi(t)E(t) + c\left(-E'(t)\right)^{\frac{1}{2p-1}} + c\left(-E'(t)\right)^{\frac{1}{(2p-1)(1+\epsilon_0)}} \\ &\leq -m\xi(t)E(t) + c\left(-E'(t)\right)^{\frac{\epsilon_0}{(2p-1)(1+\epsilon_0)}}\left(-E'(t)\right)^{\frac{1}{(2p-1)(1+\epsilon_0)}} \\ &+ c\left(-E'(t)\right)^{\frac{1}{(2p-1)(1+\epsilon_0)}} \\ &\leq -m\xi(t)E(t) + c\left(-E'(t)\right)^{\frac{1}{(2p-1)(1+\epsilon_0)}}, \quad \forall t \geq t_0. \end{aligned} \tag{5.31}$$

Multiply the last inequality by $\xi^{\gamma}(t)E^{\gamma}(t)$, where $\gamma = (2p-1)(1+\epsilon_0)-1$, and notice that $\xi' \leq 0$ to obtain

$$\xi^{\gamma+1}(t)E^{\gamma}(t)L'(t) \leq -m\xi^{\gamma+1}(t)E^{\gamma+1}(t) + c\left(\xi E\right)^{\gamma}(t)\left(-E'(t)\right)^{\frac{1}{\gamma+1}}, \quad \forall t \geq t_0.$$

Use of Young's inequality, with $q = \gamma+1$ and $q^* = \frac{\gamma+1}{\gamma}$, gives, for any $\varepsilon' > 0$,

$$\begin{aligned} \xi^{\gamma+1}(t)E^{\gamma}(t)L'(t) &\leq -m\xi^{\gamma+1}(t)E^{\gamma+1}(t) + c\left(\varepsilon'\xi^{\gamma+1}(t)E^{\gamma+1} - c_{\varepsilon'}E'(t)\right) \\ &= -(m-\varepsilon' c)\xi^{\gamma+1}(t)E^{\gamma+1} - cE'(t), \quad \forall t \geq t_0. \end{aligned}$$

We then choose $0 < \varepsilon' < \frac{m}{c}$ and recall that $\xi' \leq 0$ and $E' \leq 0$, to get, for $c_1 = m - \varepsilon' c$,

$$\left(\xi^{\gamma+1}E^{\gamma}L\right)'(t) \leq \xi^{\gamma+1}(t)E^{\gamma}(t)L'(t) \leq -c_1\xi^{\gamma+1}(t)E^{\gamma+1}(t) - cE'(t), \quad \forall t \geq t_0,$$

which implies

$$\left(\xi^{\gamma+1}E^{\gamma}L + cE\right)'(t) \leq -c_1\xi^{\gamma+1}(t)E^{\gamma+1}(t), \quad \forall t \geq t_0.$$

Let $F = \xi^{\gamma+1}E^{\gamma}L + cE$. Then $F \sim E$ (thanks to (5.19)) and

$$F'(t) \leq -c\xi^{\gamma+1}(t)F^{\gamma+1}(t) = -c\xi^{(2p-1)(1+\epsilon_0)}(t)F^{(2p-1)(1+\epsilon_0)}(t), \ \forall t \geq t_0.$$

Integrating over (t_0, t) and using the fact that $F \sim E$, we obtain (5.28) with $\epsilon = (2p-1)\epsilon_0$.

To establish (5.30), we use the idea of Messaoudi and Al-Khulaifi [41]. Let

$$\eta(t) = \int_0^t \|\Delta u(t) - \Delta u(t-s)\|_2^2 ds.$$

Using (5.28), (5.29) and (5.26), we have

$$\begin{aligned} \eta(t) &\leq 2\int_0^t \left(\|\Delta u(t)\|_2^2 + \|\Delta u(t-s)\|_2^2\right) ds \\ &\leq \frac{12}{l}\int_0^t \left(E(t) + E(t-s)\right) ds \\ &\leq \frac{24}{l}\int_0^t E(s)ds < \frac{24}{l}\int_0^\infty E(s)ds < \infty. \end{aligned}$$

This implies that

$$\sup_{t>0} \eta^{1-\frac{1}{p}}(t) < \infty. \tag{5.32}$$

Assume that $\eta(t) > 0$. Then, because ξ is nonincreasing, we find

$$\xi(t)(g \circ \Delta u)(t) \leq \frac{\eta(t)}{\eta(t)}\int_0^t \left(\xi^p(s)g^p(s)\right)^{\frac{1}{p}} \|\Delta u(t) - \Delta u(t-s)\|_2^2 ds.$$

Applying Jensen's inequality to get

$$\xi(t)(g\circ\Delta u)(t)\le \eta(t)\left(\frac{1}{\eta(t)}\int_0^t \xi^p(s)g^p(s)\|\Delta u(t)-\Delta u(t-s)\|_2^2\,ds\right)^{\frac{1}{p}}.$$

Therefore, using $(A2)$ and (5.32) we obtain

$$\begin{aligned}\xi(t)(g\circ\Delta u)(t)&\le \eta^{1-\frac{1}{p}}(t)\left(\xi^{p-1}(0)\int_0^t \xi(s)g^p(s)\|\Delta u(t)-\Delta u(t-s)\|_2^2 ds\right)^{\frac{1}{p}}\\&\le c(-g'\circ\Delta u)^{\frac{1}{p}}(t),\end{aligned}$$

and then, according to (2.6),

$$\xi(t)(g\circ\Delta u)(t)\le c(-E'(t))^{\frac{1}{p}}. \tag{5.33}$$

So, since ξ is nonincreasing,

$$\begin{aligned}\xi(t)(g\circ\Delta u)^{\frac{1}{1+\epsilon_0}}(t)&=(\xi^{\epsilon_0}(t)\xi(t)(g\circ\Delta u)(t))^{\frac{1}{1+\epsilon_0}}\\&\le(\xi^{\epsilon_0}(0)\xi(t)(g\circ\Delta u)(t))^{\frac{1}{1+\epsilon_0}}\\&\le c\,(\xi(t)(g\circ\Delta u)(t))^{\frac{1}{1+\epsilon_0}}\\&\le c(-E'(t))^{\frac{1}{p(1+\epsilon_0)}}.\end{aligned} \tag{5.34}$$

If $\eta(t)=0$, then $s\to\Delta u(s)$ is a constant function on $[0,t]$. Therefore

$$(g\circ\Delta u)(t)=0,$$

and hence (5.33) and (5.34) hold.

Now, multiplying (5.20) by $\xi(t)$ and using (5.27), (5.33) and (5.34) to find, for any $t\ge t_0$ (as for (5.31)),

$$\begin{aligned}\xi(t)L'(t)&\le -m\xi(t)E(t)+c\,(-E'(t))^{\frac{1}{p}}+c\,(-E'(t))^{\frac{1}{p(1+\epsilon_0)}}\\&\le -m\xi(t)E(t)+c\,(-E'(t))^{\frac{\epsilon_0}{p(1+\epsilon_0)}}\,(-E'(t))^{\frac{1}{p(1+\epsilon_0)}}+c\,(-E'(t))^{\frac{1}{p(1+\epsilon_0)}}\\&\le -m\xi(t)E(t)+c\,(-E'(t))^{\frac{1}{p(1+\epsilon_0)}},\quad \forall t\ge t_0.\end{aligned} \tag{5.35}$$

Inequality (5.31) with $2p-1$ replaced by p is exactely (5.35). Then, the proof of (5.30) can be completed as for the one of (5.28) (by taking $\gamma=p(1+\epsilon_0)-1$ and $\epsilon=p\epsilon_0$). This completes the proof of our main result. □

Remark 5.10. We note here that $2p-2+\epsilon$ and $p-1+\epsilon$ can be arbitrary close to $2p-2$ and $p-1$, respectively, since ϵ can be arbitrary close to zero. On the other hand, in the absence of the logarithmic "forcing" term ($k=0$), the estimates (5.17) and (5.18) drop out and, consequently, (5.20) takes the form

$$L'(t)\le -mE(t)+c(go\Delta u)(t),\quad \forall t\ge t_0. \tag{5.36}$$

In this case, we obtain the following result:

Theorem 5.11. *Let $(u_0,u_1)\in H_0^2(\Omega)\times L^2(\Omega)$ and $t_0>0$. Assume that $(A1)-(A2)$ hold. Then, there exists a positive constant K such that the solution of (1.1) satisfies, for all $t\ge t_0$,*

$$E(t)\le Ke^{-\lambda\int_{t_0}^t \xi(s)ds}\qquad \textit{if}\ \ p=1 \tag{5.37}$$

and

$$E(t) \leq K \left(1 + \int_{t_0}^{t} \xi^{2p-1}(s)ds\right)^{\frac{-1}{2p-2}} \qquad \text{if} \ \ 1 < p < \frac{3}{2}. \tag{5.38}$$

Moreover, if $1 < p < \frac{3}{2}$ *and*

$$\int_{0}^{\infty} \left(1 + \int_{t_0}^{t} \xi^{2p-1}(s)ds\right)^{\frac{-1}{2p-2}} dt < \infty, \tag{5.39}$$

then

$$E(t) \leq K \left(1 + \int_{t_0}^{t} \xi^{p}(s)ds\right)^{\frac{-1}{p-1}}, \quad \forall t \geq t_0. \tag{5.40}$$

Remark 5.12. This result ($k = 0$) improves and generalizes many results in the literature such as Han and Wang [26].

Acknowledgment. The authors thank KFUPM for its continuous support. This work was funded by KFUPM under Project #IN151023. This work was finished during the visit of the second author to KFUPM in December 2016.

REFERENCES

[1] J. Barrow and P. Parsons, Inflationary models with logarithmic potentials, *Phys. Rev. D*, **52** (1995), 5576–5587.

[2] K. Bartkowski and P. Gorka, One-dimensional Klein-Gordon equation with logarithmic nonlinearities, *J. Phys. A*, **41** (2008), 355201, 11 pp.

[3] A. Benaissa and A. Guesmia, Energy decay of solutions of a wave equation of ϕ-Laplacian type with a general weakly nolinear dissipation, *Elec. J. Diff. Equa.*, **109** (2008), 1–22.

[4] S. Berrimi and S. Messaoudi, Exponential decay of solutions to a viscoelastic equation with nonlinear localized damping, *Electron. J. Differential Equations*, **88** (2004), 1–10.

[5] I. Bialynicki-Birula and J. Mycielski, Wave equations with logarithmic nonlinearities, *Bull. Acad. Polon. Sci. Ser. Sci. Math. Astronom. Phys.*, **23** (1975), 461–466.

[6] I. Bialynicki-Birula and J. Mycielski, Nonlinear wave mechanics, *Ann. Physics*, **100** (1976), 62–93.

[7] M. Cavalcanti, V. Domingos Cavalcanti and J. Ferreira, Existence and uniform decay for nonlinear viscoelastic equation with strong damping, *Math. Methods Appl. Sci.*, **24** (2001), 1043–1053.

[8] M. Cavalcanti, V. Domingos Cavalcanti and J. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, *E. J. Differ. Eq.*, **44** (2002), 1–14.

[9] M. Cavalcanti and A. Guesmia, General decay rates of solutions to a nonlinear wave equation with boundary condition of memory type, *Diff. Integ. Equa.*, **18** (2005), 583–600.

[10] M. Cavalcanti and H. Oquendo, Frictional versus viscoelastic damping in a semilinear wave equation, *SIAM J. Control Optim.*, **42** (2003), 1310–1324.

[11] T. Cazenave and A. Haraux, Equations d'evolution avec non-linearite logarithmique, *Ann. Fac. Sci. Toulouse Math.*, **2** (1980), 21–51.

[12] H. Chen, P. Luo and G. W. Liu, Global solution and blow-up of a semilinear heat equation with logarithmic nonlinearity, *J. Math. Anal. Appl.*, **422** (2015), 84–98.

[13] W. Chen and Y. Zhou, Global nonexistence for a semilinear Petrovsky equation, *Nonlinear Analysis A*, **70** (2009), 3203–3208.

[14] R. Christensen, *Theory of Viscoelasticity, An Introduction*, Academic Press: New York, 1982.

[15] C. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Ration. Mech. Anal.*, **37** (1970), 297–308.

[16] C. Dafermos, On abstract volterra equations with applications to linear viscoelasticity, *J. Differ. Equ.*, **7** (1970), 554–569.

[17] G. Dasios and F. Zafiropoulos, Equipartition of energy in linearized 3-D viscoelasticity, *Quart. Appl. Math.*, **48** (1990), 715–730.

[18] K. Enqvist and J. McDonald, Q-balls and baryogenesis in the MSSM, *Phys. Lett. B*, **425** (1998), 309–321.

[19] P.Gorka, Logarithmic Klein-Gordon equation, *Acta Phys. Polon. B*, **40** (2009), 59–66.

[20] P. Gorka, H. Prado and E. G. Reyes, Nonlinear equations with infinitely many derivatives, *Complex Anal. Oper. Theory*, **5** (2011), 313–323.

[21] L. Gross, Logarithmic Sobolev inequalities, *Amer. J. Math.*, **97** (1975), 1061–1083.

[22] A. Guesmia, Existence globale et stabilisation interne non linéaire d'un système de Petrovsky, *Bull. Belg. Math. Soc.*, **5** (1998), 583–594.

[23] A. Guesmia, Stabilisation de l'équation des ondes avec conditions aux limites de type mémoire, *Afrika Matematika*, **10** (1999), 14–25.

[24] A. Guesmia, S. Messaoudi and B. Said-Houari, General decay of solutions of a nonlinear system of viscoelastic wave equations, *NoDEA*, **18** (2011), 659–684.

[25] X. Han, Global existence of weak solutions for a logarithmic wave equation arising from Q-ball dynamics, *Bull. Korean Math. Soc.*, **50** (2013), 275–283.

[26] X. Han and M. Wang, General decay estimate of energy for the second order evolution equations with memory, *Act Appl. Math.*, **110** (2010), 194–207.

[27] T. Hiramatsu, M. Kawasaki and F. Takahashi, Numerical study of Q-ball formation in gravity mediation, *Journal of Cosmology and Astroparticle Physics*, **6** (2010), 008.

[28] W. Hrusa, Global existence and asymptotic stability for a semilinear Volterra equation with large initial data, *SIAM J. Math. Anal.*, **16** (1985), 110–134.

[29] V. Komornik, On the nonlinear boundary stabilization of Kirchoff plates, *NoDEA Nonlinear Differential Equations Appl.*, **1** (1994), 323–337.

[30] J. Lagnese, *Boundary Stabilization of Thin Plates*, SIAM, Philadelphia, 1989.

[31] J. Lagnese, Asymptotic energy estimates for Kirchhoff plates subject to weak viscoelastic damping, *International Series of Numerical Mathematics*, vol. 91. Birhauser: Verlag, Bassel, 1989.

[32] I. Lasiecka, Exponential decay rates for the solutions of Euler-Bernoulli moments only, *J. Differential Equations*, **95** (1992), 169–182.

[33] J. Lions, *Quelques Methodes de Resolution des Problemes aux Limites Non Lineaires*, second Edition, Dunod, Paris, 2002.

[34] F. Li, Z. Zhao and Y. Chen, Global existence uniqueness and decay estimates for nonlinear viscoelastic wave equation with boundary dissipation, *Nonlinear Anal.: RealWorld Applications*, **12** (2011), 1759–1773.

[35] M-T. Lacroix-Sonrier, *Distrubutions Espace de Sobolev Application*, Ellipses Edition Marketing S.A, 1998.

[36] S. Messaoudi, Global existence and nonexistence in a system of Petrovsky, *Journal of Mathematical Analysis and Applications*, **265** (2002), 296–308.

[37] S. Messaoudi, General decay of solution energy in a viscoelastic equation with a nonlinear source, *Nonlinear Anal.*, **69** (2008), 2589–2598.

[38] S. Messaoudi, General decay of solutions of a viscoelastic equation, *J. Math. Anal. App.*, **341** (2008), 1457–1467.

[39] S. Messaoudi and N.-E Tatar, Global existence asymptotic behavior for a non-linear viscoelastic problem, *Math. Methods Sci. Res.*, **7** (2003), 136–149.

[40] S. Messaoudi and N.-E Tatar, Global existence and uniform stability of solutions for a quasilinear viscoelastic problem, *Math. Methods Appl. Sci.*, **30** (2007), 665–680.

[41] S. Messaoudi and W. Al-Khulaifi, General and optimal decay for a quasilinear viscoelastic equation, *Applied Mathematics Letters*, **66** (2017), 16–22.

[42] Rivera J. Muñoz, Asymptotic behavior in linear viscoelasticity, *Quart. Appl. Math.*, **52** (1994), 628–648.

[43] Rivera J. Muñoz, E. C. Lapa and R. Barreto, Decay rates for viscoelastic paltes with memory, *Journal of Elasticity*, **44** (1996), 61–87.

[44] M. Santos and F. junior, A boundary condition with memory for Kirchoff plates equations, *Appl. Math. Comput.*, **148** (2004), 475–496.

[45] V. S. Vladimirov, The equation of the p-adic open string for the scalar tachyon field, *Izv. Math.*, **69** (2005), 487–512.

Received September 2017; revised January 2018.

E-mail address: mahfouz@kfupm.edu.sa
E-mail address: aissa.guesmia@univ-lorraine.fr
E-mail address: messaoud@kfupm.edu.sa