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SOME WELL-POSEDNESS AND STABILITY RESULTS FOR ABSTRACT HYPERBOLIC EQUATIONS WITH INFINITE MEMORY AND DISTRIBUTED TIME DELAY

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Abstract. In this paper, we consider a class of second order abstract linear hyperbolic equations with infinite memory and distributed time delay. Under appropriate assumptions on the infinite memory and distributed time delay convolution kernels, we prove well-posedness and stability of the system. Our estimation shows that the dissipation resulting from the infinite memory alone guarantees the asymptotic stability of the system in spite of the presence of distributed time delay. The decay rate of solutions is found explicitly in terms of the growth at infinity of the infinite memory and the distributed time delay convolution kernels. An application of our approach to the discrete time delay case is also given.

1. **Introduction.** Let H be a real Hilbert space with inner product and related norm denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, respectively. Let $A: D(A) \to H$ and $B: D(B) \to H$ be self-adjoint linear positive definite operators with domains $D(A) \subset D(B) \subset H$ such that the embeddings are dense and compact. Let $g: \mathbb{R}_+ \to \mathbb{R}_+$ and $f: \mathbb{R}_+ \to \mathbb{R}$ be given functions. We consider the following class of second-order linear hyperbolic equations:

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s) Bu(t-s)ds + \int_0^{+\infty} f(s) u_t(t-s)ds = 0, \quad \forall t > 0 \tag{1}$$

with initial conditions

$$\begin{cases} u(-t) = u_0(t), & \forall t \in \mathbb{R}_+, \\ u_t(-t) = u_1(t), & \forall t \in \mathbb{R}_+, \end{cases} \tag{2}$$

where (u_0, u_1) are given initial data belonging to a suitable space and $u: \mathbb{R}_+ \to H$ is the unknown of the system (1)-(2). The infinite integrals represent, respectively,

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the infinite memory and the distributed time delay terms. The subscript t denotes the derivative with respect to t.

Our objective here is to prove the well-posedness and investigate the asymptotic behavior as time goes to infinity of solutions of (1)-(2) under appropriate assumptions on the operators A and B, and the convolution kernels g and f.

The questions related to well-posedness and stability/instability of evolution equations with delay and/or memory have attracted considerable attention in recent years and many researchers have shown that the memory plays the role of a damper, whereas the time delay can destabilize a system that was asymptotically stable in the absence of time delay. Let us recall some works related to the problem we address here. For more details, we refer the reader to the list of references in this paper.

In the absence of time delay term (i.e. $f \equiv 0$):

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Bu(t-s)ds = 0, \quad \forall t > 0, \tag{3}$$

a large amount of literature is available on this model (3), addressing the issues of the existence, uniqueness and asymptotic behavior in time; see [10, 11, 14, 16, 17, 23, 26, 31, 40] and the references cited therein. Different decay estimates (exponential or polynomial or others) depending on the growth of g at infinity have been obtained.

Also, in case $f \equiv 0$ and the infinite memory is replaced by a finite one:

$$u_{tt}(t) + Au(t) - \int_0^{t} g(s)Bu(t-s)ds = 0, \quad \forall t > 0, \tag{4}$$

a large number of papers on this subject are available in the literature, where various decay estimates for the solutions of (4) were obtained; see in this regard [6, 8, 9, 22, 27, 28, 29, 30] and the references cited therein. For the particular case of the wave equation with (internal or boundary) finite memory; see [2, 1, 7, 43] and [44]-[52]. See also [21] for the wave equation with complementary finite and infinite memories, and [20] for the Timoshenko systems with complementary finite memory and nonlinear frictional damping.

When the memory term is replaced by a frictional damping $Bu_t(t)$, and the distributed time delay is replaced by the discrete one $\mu u_t(t-\tau)$:

$$u_{tt}(t) + Au(t) + Bu_t(t) + \mu u_t(t-\tau) = 0, \quad \forall t > 0, \tag{5}$$

where μ and τ are fixed constants, there exist in the literature different stability/instability results for (5) depending, in particular, on the connection between B and μ. We refer the reader to [13, 12] and [38] for the one-dimensional wave equation with internal and/or boundary feedbak and constant discrete time delay, [3, 33, 34, 35] and [36] for the N-dimensional case, and [15] and [37] for an abstract system with constant or variable discrete time delay. These results show that the damping $Bu_t(t)$ is strong enough to stabilize the system (5) in presence of a constant discrete time delay provided that $|\mu|$ is small enough.

The wave equation in an N-dimensional bounded domain with linear frictional damping $\mu_1 u_t(t)$, constant discrete time delay $\mu_2 u_t(t-\tau)$, and finite memory:

$$u_{tt}(x,t) - \Delta u(x,t) + \int_0^{t} g(s)\Delta u(x,t-s)ds + \mu_1 u_t(x,t) + \mu_2 u_t(x,t-\tau) = 0, \ \forall t > 0, \tag{6}$$

where μ_1 and μ_2 are fixed nonnegative constants, was considered in [24]. The exponential stability was proved under the assumptions that $0 \leq \mu_2 \leq \mu_1$ and g

converges exponentially to zero at infinity. A similar result was obtained in [42] for Timoshenko systems, in [4] for thermoelasticity type III systems, and in [5] for the wave equation with nonlinear discrete time delay and frictional damping terms. We recall that, when $0 \leq \mu_1 \leq \mu_2$ and $g = 0$, the system (6) is instable for some values of τ; see [33]. For the case of boundary time delay, see [32] and the references therein.

Recently, the stability of the following second order abstract linear equation with infinite memory and discrete time delay terms:

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s)Au(t-s)ds + \mu u_t(t-\tau) = 0, \quad \forall t > 0 \tag{7}$$

was considered in [18]. It was proved that the dissipation generated by the infinite memory alone guarantees the exponential stability of the system provided that the weight $|\mu|$ of the discrete time delay is small enough and the memory kernel g converges exponentially to zero at infinity.

Unlike the discrete time delay models which ignore the inherent memory effects, the distributed time delay models (we will consider in this paper) do take into account the whole (infinite) past history of the solution. More precisely, we are in presence of an "indefinite" frictional damping which depends on all previous states (the past information is stored and used later). This is what makes the present model more realistic. In fact the discrete case will be a special case which corresponds to the Dirac delta distribution kernel (at some time τ).

In this paper, we shall prove, under some appropriate assumptions on A, B, g and f, that (1)-(2) is well-posed in an appropriate underlying space and that the only dissipation generated by the infinite memory guarantees the asymptotic stability of (1)-(2) in spite of the presence of a distributed time delay. Moreover, the decay rate of solutions is explicitly found in terms of, in particular, the growth of g and f at infinity. The proof is based on the semigroup theory for the well-posedness, and the energy method for the stability. We introduce new functionals to get crucial estimates on the distributed time delay and the infinite memory, and overcome subsequently the difficulties generated by the nondissipativity character of our system (1)-(2). Moreover, we will appeal to some ideas and functionals introduced in [44]-[51]. These ideas will, in particular, allow us to deal with some arbitrary decaying kernels without assuming explicit conditions on their derivative. The approach presented in this paper can be applied to the case of finite memory and/or discrete time delay. On the other hand, our model includes various practical applications like the wave equation, the Petrovsky system (as well as coupled wave-wave, wave-Petrovsky and Petrovsky-Petrovsky systems) and some elastic systems in N-dimensional open bounded domain.

The case of Timoshenko-type systems (two coupled wave equations in one-dimensional open bounded domain) with infinite memory and (distributed or discrete) time delay was treated in [19] where similar results to the ones of the present paper were obtained. But the coupling in the Timoshenko-type systems is of order one (the coupling terms depend on the derivatives with respect to the space variable), and so, they are not included by our general model (1)-(2) which generates some new difficulties (related to the treatment of the operators A and B) surmounted in the abstract framework introduced in the present paper.

The plan of the paper is as follows. In Section 2, we present our assumptions on A, B, g and f, and state and prove the well-posedness of (1)-(2). Section 3 is devoted to the statement and proof of the asymptotic stability results of (1)-(2) under some additional assumptions on A, B, g and f. Section 4 will be devoted to an application to the discrete time delay case. Finally, in Section 5, we discuss some general comments and issues.

2. **Well-posedness.** In this section, we state our assumptions on A, B, g and f, and prove the global existence, uniqueness and smoothness of the solution of (1)-(2). We assume that

(A1): There exist positive constants a and b satisfying

$$b\|w\|^2 \leq \|B^{\frac{1}{2}}w\|^2 \leq a\|A^{\frac{1}{2}}w\|^2, \quad \forall w \in D(A^{\frac{1}{2}}). \tag{8}$$

(A2): The function g is of class $C^1(\mathbb{R}_+, \mathbb{R}_+)$, nonincreasing and satisfies

$$g_0 := \int_0^{+\infty} g(s)ds < \frac{1}{a}. \tag{9}$$

Moreover, for some positive constant θ,

$$-g'(s) \leq \theta g(s), \quad \forall s \in \mathbb{R}_+. \tag{10}$$

(A3): The function f is of class $C^1(\mathbb{R}_+, \mathbb{R})$ and satisfies, for some positive constant α,

$$|f(s)| \leq \alpha g(s) \quad \text{and} \quad |f'(s)| \leq \alpha g(s), \quad \forall s \in \mathbb{R}_+. \tag{11}$$

Following a method devised in [11], we consider a new auxiliary variable η to treat the infinite memory and distributed time delay terms, and formulate the system (1)-(2) in the following abstract linear first-order system:

$$\begin{cases} \mathscr{U}_t(t) = (\mathscr{A} + \mathscr{B})\mathscr{U}(t), & \forall t > 0, \\ \mathscr{U}(0) = \mathscr{U}_0, \end{cases} \tag{12}$$

where $\mathscr{U} = (u, u_t, \eta)^T$, $\mathscr{U}_0 = (u_0(0), u_1(0), \eta_0)^T \in \mathscr{H}$,

$$\mathscr{H} = D(A^{\frac{1}{2}}) \times H \times L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))$$

and

$$\begin{cases} \eta(t, s) = u(t) - u(t - s), & \forall t,\, s \in \mathbb{R}_+, \\ \eta_0(s) = \eta(0, s) = u_0(0) - u_0(s), & \forall s \in \mathbb{R}_+. \end{cases} \tag{13}$$

The set $L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))$ is the weighted space with respect to the measure $g(s)ds$ defined by

$$L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}})) = \left\{ w : \mathbb{R}_+ \to D(B^{\frac{1}{2}}), \quad \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}w(s)\|^2 ds < +\infty \right\}$$

and endowed with the classical inner product

$$\langle w_1, w_2 \rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))} = \int_0^{+\infty} g(s) \left\langle B^{\frac{1}{2}}w_1(s), B^{\frac{1}{2}}w_2(s) \right\rangle ds.$$

The operators $\mathscr{A}$ and $\mathscr{B}$ are linear and given by

$$\mathscr{A}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_2 \\ (-A + g_0 B)w_1 - \epsilon_0 c_0 w_2 - \int_0^{+\infty} g(s)Bw_3(s)ds - \int_0^{+\infty} f(s)\frac{\partial w_3(s)}{\partial s}ds \\ -\frac{\partial w_3}{\partial s} - \frac{c_0}{\epsilon_0}w_3 + w_2 \end{pmatrix}, \tag{14}$$

where

$$\begin{cases} c_0 = \dfrac{\alpha}{2}\sqrt{\dfrac{g_0}{b}}, \\ \epsilon_0 = \dfrac{\|f\|_\infty}{c_0} \end{cases} \tag{15}$$

(ϵ_0 is a positive constant, since $\|f\|_\infty > 0$, otherwise, $f \equiv 0$ and then no distributed time delay is considered in (1)) and

$$\mathscr{B}(w_1, w_2, w_3)^T = c_0 \left(0, \epsilon_0 w_2, \frac{1}{\epsilon_0} w_3\right)^T. \tag{16}$$

The domains $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}(\mathscr{B})$ of $\mathscr{A}$ and $\mathscr{B}$, respectively, are given by

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{l} (w_1, w_2, w_3)^T \in \mathscr{H}, \, \dfrac{\partial w_3}{\partial s} \in L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}})), \, w_2 \in D(A^{\frac{1}{2}}), \\ (A - g_0 B) w_1 + \displaystyle\int_0^{+\infty} g(s) B w_3(s) ds \in H, \, w_3(0) = 0 \end{array} \right\}, \tag{17}$$

since, thanks to the Cauchy-Schwarz inequality and the first inequalities in (8) and (11),

$$\frac{\partial w_3}{\partial s} \in L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}})) \implies \int_0^{+\infty} f(s) \frac{\partial w_3(s)}{\partial s} ds \in H, \tag{18}$$

and $\mathscr{D}(\mathscr{B}) = \mathscr{H}$. On the other hand, keeping in mind the definition of η in (13), we have

$$\begin{cases} \eta_t(t,s) + \eta_s(t,s) = u_t(t), & \forall t, \, s \in \mathbb{R}_+, \\ \eta_s(t,s) = u_t(t-s), & \forall t, \, s \in \mathbb{R}_+, \\ \eta(t,0) = 0, & \forall t \in \mathbb{R}_+. \end{cases} \tag{19}$$

Therefore, we conclude from (14), (16) and (19) that the systems (1)-(2) and (12) are equivalent.

Thanks to (9) and the second inequality in (8), $\mathscr{H}$ endowed with the inner product

$$\left\langle (w_1, w_2, w_3)^T, (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3)^T \right\rangle_{\mathscr{H}} = \left\langle A^{\frac{1}{2}} w_1, A^{\frac{1}{2}} \tilde{w}_1 \right\rangle - g_0 \left\langle B^{\frac{1}{2}} w_1, B^{\frac{1}{2}} \tilde{w}_1 \right\rangle + \langle w_2, \tilde{w}_2 \rangle + \langle w_3, \tilde{w}_3 \rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))}$$

is a Hilbert space and $\mathscr{D}(\mathscr{A}) \subset \mathscr{H}$ with dense embedding (see, for example, [31]). The well-posedness of problem (12) is ensured by the following theorem:

Theorem 2.1. *Assume that (A1)-(A3) hold. Then, for any $\mathscr{U}_0 \in \mathscr{H}$, the system (12) has a unique weak solution*

$$\mathscr{U} \in C(\mathbb{R}_+, \mathscr{H}). \tag{20}$$

Moreover, if $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, then the solution of (12) satisfies (classical solution)

$$\mathscr{U} \in C^1(\mathbb{R}_+, \mathscr{H}) \cap C(\mathbb{R}_+, \mathscr{D}(\mathscr{A})). \tag{21}$$

Proof. To prove Theorem 2.1, we use the semigroup approach. So, first, we show that the linear operator $\mathscr{A}$ is dissipative. Indeed, let $W = (w_1, w_2, w_3)^T \in \mathscr{D}(\mathscr{A})$, then

$$\begin{aligned}\langle \mathscr{A}W, W\rangle_{\mathscr{H}} =& \left\langle A^{\frac{1}{2}}w_2, A^{\frac{1}{2}}w_1\right\rangle - g_0\left\langle B^{\frac{1}{2}}w_2, B^{\frac{1}{2}}w_1\right\rangle - \left\langle \int_0^{+\infty} f(s)\frac{\partial w_3(s)}{\partial s}ds, w_2\right\rangle \\ &+ \left\langle (-A+g_0B)w_1 - c_0\epsilon_0 w_2 - \int_0^{+\infty} g(s)Bw_3(s)ds, w_2\right\rangle \\ &+ \left\langle -\frac{\partial w_3}{\partial s} - \frac{c_0}{\epsilon_0}w_3 + w_2, w_3\right\rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))}. \end{aligned} \tag{22}$$

It is clear that, by the definitions of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, and the fact that H is a real Hilbert space,

$$\langle (-A+g_0B)w_1, w_2\rangle = -\left\langle A^{\frac{1}{2}}w_2, A^{\frac{1}{2}}w_1\right\rangle + g_0\left\langle B^{\frac{1}{2}}w_2, B^{\frac{1}{2}}w_1\right\rangle$$

and

$$\left\langle -\int_0^{+\infty} g(s)Bw_3(s)ds, w_2\right\rangle = -\langle w_2, w_3\rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))}.$$

On the other hand, integrating by parts and using the fact that

$$\lim_{s\to+\infty} g(s)B^{\frac{1}{2}}w_3(s) = \lim_{s\to+\infty} f(s)w_3(s) = 0$$

(due to **(A2)** and (11)) and $w_3(0) = 0$ (definition of $\mathscr{D}(\mathscr{A})$), we find

$$\left\langle -\frac{\partial w_3}{\partial s}, w_3\right\rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))} = \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}w_3(s)\|^2 ds.$$

Moreover

$$-\left\langle \int_0^{+\infty} f(s)\frac{\partial w_3(s)}{\partial s}ds, w_2\right\rangle = \int_0^{+\infty} f'(s)\langle w_3(s), w_2\rangle ds.$$

Consequently, inserting these four formulas in the previous identity (22), we get

$$\begin{aligned}\langle \mathscr{A}W, W\rangle_{\mathscr{H}} =& \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}w_3(s)\|^2 ds + \int_0^{+\infty} f'(s)\langle w_3(s), w_2\rangle ds \\ &- c_0\left(\epsilon_0\|w_2\|^2 + \frac{1}{\epsilon_0}\|w_3\|^2_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))}\right). \end{aligned} \tag{23}$$

Note that, thanks to (10) and the fact that g is nonincreasing and $w_3 \in L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))$,

$$\begin{aligned}\left|\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}w_3(s)\|^2 ds\right| =& -\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}w_3(s)\|^2 ds \\ \leq& \theta\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}w_3(s)\|^2 ds \\ <& +\infty, \end{aligned}$$

so the first integral in the right hand side of (23) is well defined.

Now, using Cauchy-Schwarz and Young's inequalities, the first inequality in (8) and the second one in (11) imply that

$$\begin{aligned}\int_0^{+\infty} f'(s)\langle w_3(s), w_2\rangle\, ds \leq & \frac{\alpha}{2}\int_0^{+\infty} g(s)\left(\frac{\epsilon_0}{\sqrt{g_0 b}}\|w_2\|^2 + \frac{\sqrt{g_0 b}}{\epsilon_0 b}\|B^{\frac{1}{2}} w_3(s)\|^2\right) ds \\ \leq & c_0\left(\epsilon_0\|w_2\|^2 + \frac{1}{\epsilon_0}\|w_3\|^2_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))}\right). \end{aligned} \tag{24}$$

Finally, combining (23) and (24), and using the fact that g is nonincreasing, we obtain

$$\langle \mathscr{A} W, W\rangle_{\mathscr{H}} \leq \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}} w_3(s)\|^2 ds \leq 0, \tag{25}$$

which means that $\mathscr{A}$ is dissipative.

Next, we shall prove that $Id - \mathscr{A}$ is surjective. Indeed, let $F = (f_1, f_2, f_3)^T \in \mathscr{H}$, we show that there exists $W = (w_1, w_2, w_3)^T \in \mathscr{D}(\mathscr{A})$ satisfying

$$(Id - \mathscr{A})W = F, \tag{26}$$

which is equivalent to

$$\begin{aligned} & w_2 = w_1 - f_1, \\ & (A - g_0 B + (1+\epsilon_0 c_0) Id)\, w_1 + \int_0^{+\infty} g(s) B w_3(s) ds + \int_0^{+\infty} f(s)\frac{\partial w_3(s)}{\partial s} ds \\ & 2cm = (1+\epsilon_0 c_0) f_1 + f_2, \\ & (1 + \frac{c_0}{\epsilon_0}) w_3 + \frac{\partial w_3}{\partial s} = w_1 + f_3 - f_1. \end{aligned} \tag{27}$$

We note that the third equation in (27) with $w_3(0) = 0$ has the unique solution

$$\begin{aligned} w_3(s) = & e^{-(1+\frac{c_0}{\epsilon_0})s}\int_0^s e^{(1+\frac{c_0}{\epsilon_0})y}(w_1 + f_3(y) - f_1) dy \\ = & (1+\frac{c_0}{\epsilon_0})^{-1}\left(1 - e^{-(1+\frac{c_0}{\epsilon_0})s}\right) w_1 + e^{-(1+\frac{c_0}{\epsilon_0})s}\int_0^s e^{(1+\frac{c_0}{\epsilon_0})y}(f_3(y) - f_1) dy. \end{aligned} \tag{28}$$

Next, plugging (28) into the second equation in (27), we get

$$(A - l_1 B + l_2 Id) w_1 = \tilde{f}, \tag{29}$$

where

$$\begin{aligned} & l_1 = g_0 - (1+\frac{c_0}{\epsilon_0})^{-1} g_0 + (1+\frac{c_0}{\epsilon_0})^{-1}\int_0^{+\infty} g(s) e^{-(1+\frac{c_0}{\epsilon_0})s} ds, \\ & l_2 = 1 + c_0\epsilon_0 + \int_0^{+\infty} f(s) e^{-(1+\frac{c_0}{\epsilon_0})s} ds \end{aligned}$$

and

$$\begin{aligned}\tilde{f} =&(1+c_0\epsilon_0)f_1 + f_2 + \int_0^{+\infty} f(s)(f_1 - f_3(s))ds \\ &+ (1+\frac{c_0}{\epsilon_0})\int_0^{+\infty} f(s)e^{-(1+\frac{c_0}{\epsilon_0})s}\left(\int_0^s e^{(1+\frac{c_0}{\epsilon_0})y}(f_3(y)-f_1)dy\right)ds \\ &- \int_0^{+\infty} g(s)e^{-(1+\frac{c_0}{\epsilon_0})s}\left(\int_0^s e^{(1+\frac{c_0}{\epsilon_0})y}B(f_3(y)-f_1)dy\right)ds.\end{aligned}$$

It remains only to prove that (29) has a solution $w_1 \in D(A^{\frac{1}{2}})$. Then, substituting in (28) and the first equation in (27), we obtain $W \in \mathscr{D}(\mathscr{A})$ satisfying (26). Since $l_1 < g_0$, then $A - l_1 B$ is a positive definite operator thanks to (9) and the second inequality in (8). On the other hand, using (15), we see that

$$l_2 \geq 1 + c_0\epsilon_0 - \|f\|_\infty \frac{\epsilon_0}{\epsilon_0 + c_0} \geq 1 + c_0\epsilon_0 - \|f\|_\infty = 1.$$

Therefore, $A - l_1 B + l_2 Id$ is a self-adjoint linear positive definite operator. Applying the Lax-Milgram theorem and classical regularity arguments, we conclude that (29) has a unique solution $w_1 \in D(A^{\frac{1}{2}})$ satisfying, using (18) and (28),

$$(A - g_0 B)w_1 + \int_0^{+\infty} g(s)Bw_3(s)ds \in H.$$

This proves that $Id - \mathscr{A}$ is surjective. Finally, we note that (25) and (26) mean that $-\mathscr{A}$ is a maximal monotone operator. Hence, using Lummer-Phillips theorem (see [41]), we deduce that $\mathscr{A}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$.

On the other hand, as the linear operator $\mathscr{B}$ is Lipschitz continuous, it follows that $\mathscr{A} + \mathscr{B}$ also is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$ (see [41]: Chapter 3 - Theorem 1.1). Consequently, (12) is well-posed in the sense of Theorem 2.1 (see [25] and [41]). □

3. **Asymptotic stability.**

In this section, we investigate the asymptotic behavior of the solution of (12) by the use of the energy method. We produce a suitable Lyapunov functional and prove some decay estimates depending on the asymptotic behavior of g, and the connection between g and f. Our asymptotic stability results hold under the following additional assumptions:

(A4): There exists a positive constant d such that

$$\|A^{\frac{1}{2}}w\|^2 \leq d\|B^{\frac{1}{2}}w\|^2, \quad \forall w \in D(A^{\frac{1}{2}}). \tag{30}$$

(A5): The function g satisfies

$$g_0 := \int_0^{+\infty} g(s)ds > 0 \tag{31}$$

and there exist a positive constant δ and a positive function $\xi : \mathbb{R}_+ \to \mathbb{R}_+^*$ of class $C(\mathbb{R}_+, \mathbb{R}_+^*)$ satisfying $\lim_{s\to+\infty} \xi(s)$ exists such that

$$g'(s) \leq -\delta g(s), \quad \forall s \in \mathbb{R}_+ \tag{32}$$

or

$$\begin{cases} g(t-s) \geq \xi(t) \displaystyle\int_t^{+\infty} g(\tau - s)d\tau, & \forall t \in \mathbb{R}_+,\ \forall s \in [0,t], \\ g'(s) < 0, & \forall s \in \mathbb{R}_+. \end{cases} \tag{33}$$

(A6): There exist a positive even function $\gamma : \mathbb{R} \to \mathbb{R}_+^*$ of class $C(\mathbb{R}, \mathbb{R}_+^*)$ and nonincreasing on $\mathbb{R}_+$, and a positive function $\beta : \mathbb{R}_+ \to \mathbb{R}_+^*$ of class $C(\mathbb{R}_+, \mathbb{R}_+^*)$ such that

$$\beta_0 := \int_0^{+\infty} \beta(s)ds < +\infty \tag{34}$$

and

$$|f(s)| \leq e^{-\tilde{\gamma}(s)}\beta(s)g(s), \quad \forall s \in \mathbb{R}_+, \tag{35}$$

where

$$\tilde{\gamma}(s) = 2\int_0^{\frac{s}{2}} \gamma(\tau)d\tau, \quad \forall s \in \mathbb{R}_+. \tag{36}$$

Remark 3.1. 1. The conditions (32) and (33) include, respectively, the class of functions g which converge to zero at least exponentially or less than exponentially. When

$$\lim_{t \to +\infty} \xi(t) > 0,$$

the first inequality in (33), introduced in [47] and [50], implies that g converges to zero at least exponentially but it does not involve the derivative of g. We distinguish the cases (32) and (33) because they lead to different kinds of decay.

2. Assumptions **(A1)** and **(A4)** imply that A and B are equivalent.

Theorem 3.2. *Assume that* **(A1)-(A6)** *hold. Then there exists a positive constant δ_0 independent of f such that, if*

$$\int_0^{+\infty} |f(s)|ds < \delta_0, \tag{37}$$

then, for any $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, there exist positive constants δ_1 and δ_2 (depending on $\|\mathscr{U}_0\|_{\mathscr{D}(\mathscr{A})}$, a, b, d, g_0, $g(0)$, δ, δ_0, ξ, γ and β) such that the classical solution of (12) satisfies

1. Exponential decay of g. *If (32) holds, then*

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \delta_2 e^{-\delta_1 t}, \quad \forall t \in \mathbb{R}_+ \tag{38}$$

if $\lim_{t\to+\infty} \gamma(t) > 0$, *and*

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \delta_2 e^{-\delta_1 \hat{\gamma}(t)}, \quad \forall t \in \mathbb{R}_+ \tag{39}$$

if $\lim_{t\to+\infty} \gamma(t) = 0$, *where*

$$\hat{\gamma}(s) = \int_0^s \gamma(\tau)d\tau, \quad \forall s \in \mathbb{R}. \tag{40}$$

2. Arbitrary decay of g. *If (33) holds and (32) does not hold, then*

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \delta_2 e^{-\delta_1 t}\left(1+\int_0^t e^{\delta_1 s}\int_s^{+\infty} g(\tau)\|B^{\frac{1}{2}}u_0(\tau-s)\|^2 d\tau ds\right), \quad \forall t \in \mathbb{R}_+ \tag{41}$$

if $(\lim_{t\to+\infty} \gamma(t))(\lim_{t\to+\infty} \xi(t)) > 0$,

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \delta_2 e^{-\delta_1 \hat{\gamma}(t)}\left(1+\int_0^t e^{\delta_1 \hat{\gamma}(s)}\int_s^{+\infty} g(\tau)\|B^{\frac{1}{2}}u_0(\tau-s)\|^2 d\tau ds\right), \ \forall t \in \mathbb{R}_+ \tag{42}$$

if $\lim_{t\to+\infty}\gamma(t)=0$ *and* $\lim_{t\to+\infty}\xi(t)>0$,

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \le \delta_2 e^{-\delta_1\hat{\xi}(t)}\left(1+\int_0^t e^{\delta_1\hat{\xi}(s)}\int_s^{+\infty} g(\tau)\|B^{\frac{1}{2}}u_0(\tau-s)\|^2 d\tau ds\right),\ \forall t\in\mathbb{R}_+ \tag{43}$$

if $\lim_{t\to+\infty}\gamma(t)>0$ *and* $\lim_{t\to+\infty}\xi(t)=0$, *where*

$$\hat{\xi}(s)=\int_0^s \xi(\tau)d\tau, \quad \forall s\in\mathbb{R}_+, \tag{44}$$

and

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \le \delta_2 e^{-\delta_1\hat{\phi}(t)}\left(1+\int_0^t e^{\delta_1\hat{\phi}(s)}\int_s^{+\infty} g(\tau)\|B^{\frac{1}{2}}u_0(\tau-s)\|^2 d\tau ds\right),\ \forall t\in\mathbb{R}_+ \tag{45}$$

if $\lim_{t\to+\infty}\gamma(t)=\lim_{t\to+\infty}\xi(t)=0$, *where*

$$\hat{\phi}(s)=\int_0^s \phi(\tau)d\tau \quad \textit{and} \quad \phi(s)=\min\{\gamma(s),\xi(s)\}, \quad \forall s\in\mathbb{R}_+. \tag{46}$$

Remark 3.3. Let us illustrate our decay estimates by the following simple examples:

Case 1. Exponential decay of g: (32) holds. Let us consider the class $g(s)=\alpha_2 e^{-\alpha_1 s}$, with α_1, $\alpha_2>0$ and α_2 small enough so that (9) holds. This class satisfies (32) with $\delta=\alpha_1$.

1. If

$$|f(s)|\le \beta_2 e^{-\beta_1(s+1)^p}g(s), \quad \forall s\in\mathbb{R}_+, \tag{47}$$

for some positive constants β_1, β_2 and p, then (35) is satisfied with $\beta(s)=\beta_2 e^{-\beta_0(s+1)^p}$ and

$$\gamma(s)=q(\beta_1-\beta_0)(2|s|+1)^{q-1}, \quad \forall s\in\mathbb{R}_+, \tag{48}$$

for any $\beta_0\in]0,\beta_1[$ and $q=\min\{p,1\}$ (so γ is positive on $\mathbb{R}$ and nonincreasing on $\mathbb{R}_+$), and therefore, for some positive constants c' and c'',

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \le c'' e^{-c'(t+1)^q}, \quad \forall t\in\mathbb{R}_+.$$

2. If

$$|f(s)|\le \beta_2 e^{-\beta_1(\ln(s+1))^p}g(s), \quad \forall s\in\mathbb{R}_+, \tag{49}$$

for some constants β_1, $\beta_2>0$ and $p>1$, then (35) holds with $\beta(s)=\beta_3 e^{-\beta_0(\ln(s+1))^p}$ and

$$\gamma(s)=\begin{cases} p(\beta_1-\beta_0)\dfrac{(\ln(2|s|+1))^{p-1}}{2|s|+1} & \text{if } |s|\ge \dfrac{1}{2}(e^{p-1}-1):=s_0, \\ p(\beta_1-\beta_0)(p-1)^{p-1}e^{1-p}:=\tilde{c}_0 & \text{if } |s|\in[0,s_0], \end{cases} \tag{50}$$

for $\beta_3=\beta_2 e^{2\tilde{c}_0 s_0}$ and any $\beta_0\in]0,\beta_1[$ (so γ is positive and continuous on $\mathbb{R}$, and nonincreasing on $\mathbb{R}_+$), and therefore, for some positive constants c' and c'',

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \le c'' e^{-c'(\ln(t+1))^p}, \quad \forall t\in\mathbb{R}_+.$$

3. If

$$|f(s)|\le \frac{\beta_1}{(s+1)^p}g(s), \quad \forall s\in\mathbb{R}_+, \tag{51}$$

for some constants $\beta_1>0$ and $p>1$, then (35) holds with $\beta(s)=\frac{\beta_1}{(s+1)^{p-\beta_0}}$ and

$$\gamma(s)=\frac{\beta_0}{2|s|+1}, \quad \forall s\in\mathbb{R}_+, \tag{52}$$

for any $\beta_0 \in]0, p-1[$ (so β is integrable on $\mathbb{R}_+$), and therefore, for some positive constants c' and c'',

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \frac{c''}{(t+1)^{c'}}, \quad \forall t \in \mathbb{R}_+.$$

Case 2. Arbitrary decay of g: (33) holds and (32) does not hold. Let us consider the classes (47), (49) and (51) of f, and the following two classes of g which satisfy (33) and do not converge exponentially to zero at infinity:

1. If

$$g(s) = \frac{\alpha_2}{s+e^{r-1}} (\ln(s+e^{r-1}))^{r-1} e^{-\alpha_1 (\ln(s+e^{r-1}))^r}, \quad \forall s \in \mathbb{R}_+, \tag{53}$$

for some constants $\alpha_2 > 0$ small enough so that (9) holds, $\alpha_1 > 0$ and $r > 1$, then (33) holds with $\xi(s) = r\alpha_1 (s+e^{r-1})^{-1} (\ln(s+e^{r-1}))^{r-1}$, and therefore, (45) holds with

$$\phi = \begin{cases} \xi & \text{in case (47) and in case (49) with } r \leq p, \\ \gamma & \text{in case (51) and in case (49) with } r > p. \end{cases}$$

If, for example, for some positive constants λ and M_0,

$$\|B^{\frac{1}{2}} u_0(s)\|^2 \leq M_0 (s+1)^{\lambda}, \quad \forall s \in \mathbb{R}_+, \tag{54}$$

then we get, for some positive constants c' and c'',

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \begin{cases} c'' e^{-c'(\ln(t+1))^r} & \text{in case (47)}, \\ c'' e^{-c'(\ln(t+1))^{\min\{p,r\}}} & \text{in case (49)}, \\ c''(t+1)^{-c'} & \text{in case (51)}. \end{cases} \tag{55}$$

2. If

$$g(s) = \alpha_1 (s+1)^{-r}, \quad \forall s \in \mathbb{R}_+, \tag{56}$$

for some constants $\alpha_1 > 0$ small enough so that (9) holds and $r > 1$, then (33) is satisfied with $\xi(s) = (r-1)(s+1)^{-1}$, and therefore, (45) holds with $\phi = \xi$.

If, for example, for some positive constants λ and M_0,

$$\|B^{\frac{1}{2}} u_0(s)\|^2 \leq M_0 (\ln(s+2))^{\lambda}, \quad \forall s \in \mathbb{R}_+, \tag{57}$$

then we get, for some positive constants c' and c'',

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \begin{cases} c''(t+1)^{-c'} & \text{if } r > 2, \\ c''(t+1)^{c'} & \text{if } r \leq 2. \end{cases} \tag{58}$$

Proof of Theorem 3.2. Assume that **(A1)-(A6)** are satisfied and let $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, so that all the calculations below are justified. We start our proof by giving the modified energy functional E associated with the solution of (12) corresponding to $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$

$$\begin{aligned} E(t) =& \frac{1}{2} \|\mathscr{U}(t)\|_{\mathscr{H}}^2 \\ =& \frac{1}{2} \left(\|u_t(t)\|^2 + \|A^{\frac{1}{2}} u(t)\|^2 - g_0 \|B^{\frac{1}{2}} u(t)\|^2 \right) \\ &+ \frac{1}{2} \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \end{aligned} \tag{59}$$

Multiplying (1) by $u_t(t)$ and integrating by parts, we easily get (similarly to the proof of (25)) (see, for example, [31] in case $f \equiv 0$)

$$E'(t) = \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds - \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, u_t(t)\right\rangle, \quad \forall t \in \mathbb{R}_+. \tag{60}$$

Note that, in contrast to the situation where there is a frictional damping in (5) and (6) and no time delay in (3) and (4), we are unable to determine the sign of E' from (60), and therefore the system (12) is not necessarily dissipative with respect to E at this stage. □

1. Exponential decay of g: (32) holds. In order to prove (38) and (39), we first prove the following three lemmas. The first two are classical (see, for example, [17, 27, 28] and [31]), whilst the third one is introduced to cope with the new situation due to the distributed time delay in the present problem.

Lemma 3.4. *Let us consider the functional*

$$I_1(t) = -\left\langle u_t(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle, \quad \forall t \in \mathbb{R}_+. \tag{61}$$

Then

$$\begin{aligned} I_1'(t) \leq & -(g_0-\epsilon)\|u_t(t)\|^2 + \epsilon\|A^{\frac{1}{2}}u(t)\|^2 + c_1\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ & - c_2\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ & + \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle, \quad \forall t \in \mathbb{R}_+, \end{aligned} \tag{62}$$

where (the constants a, b, d and g_0 are defined in (8), (9) and (30))

$$\begin{cases} \epsilon = \dfrac{g_0(1-ag_0)}{3+g_0-ag_0}, \\ c_1 = g_0 + \dfrac{g_0}{2\epsilon}(d+ag_0^2), \\ c_2 = \dfrac{g(0)}{4b\epsilon}. \end{cases} \tag{63}$$

Proof. Multiplying (1) by $\int_0^{+\infty} g(s)\eta(t,s)ds$, we get

$$\begin{aligned} 0 = & \left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle + \left\langle Au(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle \\ & - g_0\left\langle Bu(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle + \left\langle \int_0^{+\infty} g(s)B\eta(t,s)ds, \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle \\ & + \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle. \end{aligned}$$

Then, the definitions of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$ yield

$$\begin{aligned}0 =& \left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle + \left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds \right\rangle \\ &- g_0 \left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds \right\rangle \\ &+ \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds \right\rangle \\ &+ \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle. \end{aligned} \tag{64}$$

On the other hand, by using the fact that $\eta_t(t,s) = -\eta_s(t,s) + u_t(t)$, we find

$$\begin{aligned}&\left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle \\ =&\frac{\partial}{\partial t} \left\langle u_t(t), \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle - \left\langle u_t(t), \int_0^{+\infty} g(s)\eta_t(t,s)ds \right\rangle \\ =& - I_1'(t) - g_0\|u_t(t)\|^2 + \left\langle u_t(t), \int_0^{+\infty} g(s)\eta_s(t,s)ds \right\rangle. \end{aligned}$$

Integrating by parts with respect to s in the infinite memory integral, and using the fact that $\lim_{s\to+\infty} g(s)\eta(t,s) = 0$ and $\eta(t,0) = 0$, we get

$$\left\langle u_{tt}(t), \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle = -I_1'(t) - g_0\|u_t(t)\|^2 - \left\langle u_t(t), \int_0^{+\infty} g'(s)\eta(t,s)ds \right\rangle. \tag{65}$$

We deduce from (64)-(65) that

$$\begin{aligned}I_1'(t) =& - g_0\|u_t(t)\|^2 + \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle \\ &- \left\langle u_t(t), \int_0^{+\infty} g'(s)\eta(t,s)ds \right\rangle + \left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds \right\rangle \\ &- g_0 \left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds \right\rangle + \left\| \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds \right\|^2. \end{aligned} \tag{66}$$

Using Cauchy-Schwarz and Young's inequalities for the last four terms of (66), and (8) and (30) to estimate $\|\eta(t,s)\|^2$, $\|A^{\frac{1}{2}}\eta(t,s)\|^2$ and $\|B^{\frac{1}{2}}u(t)\|^2$ by $\frac{1}{b}\|B^{\frac{1}{2}}\eta(t,s)\|^2$, $d\|B^{\frac{1}{2}}\eta(t,s)\|^2$ and $a\|A^{\frac{1}{2}}u(t)\|^2$, respectively, we get, for ϵ defined in (63) (ϵ is a positive constant according to (9) and (31), and does not depend on f),

$$\begin{aligned}&- \left\langle u_t(t), \int_0^{+\infty} g'(s)\eta(t,s)ds \right\rangle \\ \le&\epsilon\|u_t(t)\|^2 + \frac{1}{4\epsilon} \left(\int_0^{+\infty} \sqrt{-g'(s)}\sqrt{-g'(s)}\|\eta(t,s)\|ds \right)^2 \\ \le&\epsilon\|u_t(t)\|^2 - \frac{g(0)}{4b\epsilon} \int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds, \end{aligned} \tag{67}$$

$$\begin{aligned}
&\left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds\right\rangle \\
\leq & \frac{\epsilon}{2}\|A^{\frac{1}{2}}u(t)\|^2 + \frac{1}{2\epsilon}\left(\int_0^{+\infty}\sqrt{g(s)}\sqrt{g(s)}\|A^{\frac{1}{2}}\eta(t,s)\|ds\right)^2 \\
\leq & \frac{\epsilon}{2}\|A^{\frac{1}{2}}u(t)\|^2 + \frac{g_0 d}{2\epsilon}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds, \qquad (68)
\end{aligned}$$

$$\begin{aligned}
&-g_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\rangle \\
\leq & \frac{\epsilon}{2a}\|B^{\frac{1}{2}}u(t)\|^2 + \frac{ag_0^2}{2\epsilon}\left(\int_0^{+\infty}\sqrt{g(s)}\sqrt{g(s)}\|B^{\frac{1}{2}}\eta(t,s)\|ds\right)^2 \\
\leq & \frac{\epsilon}{2}\|A^{\frac{1}{2}}u(t)\|^2 + \frac{ag_0^3}{2\epsilon}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \qquad (69)
\end{aligned}$$

and

$$\begin{aligned}
\left\|\int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\|^2 \leq & \left(\int_0^{+\infty}\sqrt{g(s)}\sqrt{g(s)}\|B^{\frac{1}{2}}\eta(t,s)\|ds\right)^2 \\
\leq & g_0\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds. \qquad (70)
\end{aligned}$$

Inserting these four inequalities (67)-(70) into (66), we get (62) with c_1 and c_2 defined in (63) (note that c_1 and c_2 are positive and do not depend on f according to (9) and (31)). □

Lemma 3.5. *Consider the functional*

$$I_2(t) = \langle u_t(t), u(t)\rangle, \quad \forall t \in \mathbb{R}_+. \qquad (71)$$

Then

$$\begin{aligned}
I_2'(t) \leq & \|u_t(t)\|^2 - (1 - ag_0 - \epsilon)\|A^{\frac{1}{2}}u(t)\|^2 + c_3\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \\
& - \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, u(t)\right\rangle, \quad \forall t \in \mathbb{R}_+, \qquad (72)
\end{aligned}$$

where ϵ is defined in (63) and

$$c_3 = \frac{ag_0}{4\epsilon}. \qquad (73)$$

Proof. Multiplying (1) by $u(t)$, we find

$$\begin{aligned}
0 = & \langle u_{tt}(t), u(t)\rangle + \langle Au(t), u(t)\rangle - g_0\langle Bu(t), u(t)\rangle \\
& + \left\langle \int_0^{+\infty} g(s)B\eta(t,s)ds, u(t)\right\rangle + \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, u(t)\right\rangle.
\end{aligned}$$

Consequently, using the definition of $A^{\frac{1}{2}}$ and $B^{\frac{1}{2}}$, we obtain

$$\begin{aligned}
0 = & \frac{\partial}{\partial t}\langle u_t(t), u(t)\rangle - \|u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2 \\
& + \left\langle \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds, B^{\frac{1}{2}}u(t)\right\rangle + \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, u(t)\right\rangle,
\end{aligned}$$

which implies that

$$I_2'(t) = \|u_t(t)\|^2 - \|A^{\frac{1}{2}}u(t)\|^2 + g_0\|B^{\frac{1}{2}}u(t)\|^2 \\ - \left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds \right\rangle - \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, u(t) \right\rangle. \quad (74)$$

By using Cauchy-Schwarz and Young's inequalities and the second inequality in (8), we get (as in the proof of (69))

$$-\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds \right\rangle \leq \epsilon\|A^{\frac{1}{2}}u(t)\|^2 + \frac{ag_0}{4\epsilon}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds.$$

Substituting this inequality in (74) and using (8) again, (72) holds with c_3 defined in (73), which is positive and does not depend on f according to (9), (31) and (63). □

Now, we introduce a new functional I_3 which plays a crucial role in dealing with the distributed time delay.

Lemma 3.6. *Let*

$$I_3(t) = e^{-\hat{\gamma}(t)}\int_0^{+\infty} e^{\tilde{\gamma}(s)}|f(s)|\left(\int_{t-s}^{t} e^{\hat{\gamma}(\tau)}\|u_t(\tau)\|^2 d\tau\right)ds, \quad \forall t \in \mathbb{R}_+, \quad (75)$$

where $\tilde{\gamma}$ and γ are defined in assumption **(A6)**, *and $\hat{\gamma}$ is defined in (40). The functional I_3 satisfies*

$$I_3'(t) \leq g(0)\beta_0\|u_t(t)\|^2 - \gamma(t)I_3(t) - \int_0^{+\infty}|f(s)|\|u_t(t-s)\|^2 ds, \quad \forall t \in \mathbb{R}_+, \quad (76)$$

where β_0 is defined in (34).

Proof. First, thanks to the first inequality in (8), (13), (35) and the fact that $\eta_s \in L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))$ (in virtue of (21)), we have (note also that g in nonincreasing and $\hat{\gamma}$ is increasing)

$$\begin{aligned}
I_3(t) \leq& \int_0^{+\infty}\beta(s)g(s)\left(\int_{t-s}^{t}\|u_t(\tau)\|^2 d\tau\right)ds \\
\leq& \int_0^{+\infty}\beta(s)\left(\int_{t-s}^{t} g(t-\tau)\|u_t(\tau)\|^2 d\tau\right)ds \\
\leq& \frac{1}{b}\int_0^{+\infty}\beta(s)\left(\int_0^{s} g(\tau)\|B^{\frac{1}{2}}u_t(t-\tau)\|^2 d\tau\right)ds \\
\leq& \frac{1}{b}\int_0^{+\infty}\beta(s))\left(\int_0^{+\infty} g(\tau)\|B^{\frac{1}{2}}\eta_s(t,\tau)\|^2 d\tau\right)ds \\
\leq& \frac{1}{b}\beta_0\|\eta_s(t,s)\|^2_{L^2_g(\mathbb{R}_+, D(B^{\frac{1}{2}}))} \\
<& +\infty,
\end{aligned}$$

because of (34), and therefore the functional I_3 is well-defined. Moreover, using (35), a simple and direct differentiation gives

$$\begin{aligned} I_3'(t) =& -\gamma(t)I_3(t) + \left(\int_0^{+\infty} e^{\tilde{\gamma}(s)}|f(s)|ds\right)\|u_t(t)\|^2 \\ & - \int_0^{+\infty} e^{\tilde{\gamma}(s)+\hat{\gamma}(t-s)-\hat{\gamma}(t)}|f(s)|\|u_t(t-s)\|^2ds \\ \leq & -\gamma(t)I_3(t) + \left(\int_0^{+\infty} \beta(s)g(s)ds\right)\|u_t(t)\|^2 \\ & - \int_0^{+\infty} e^{\tilde{\gamma}(s)+\hat{\gamma}(t-s)-\hat{\gamma}(t)}|f(s)|\|u_t(t-s)\|^2ds \\ \leq & -\gamma(t)I_3(t) + g(0)\beta_0\|u_t(t)\|^2 - \int_0^{+\infty} e^{\tilde{\gamma}(s)+\hat{\gamma}(t-s)-\hat{\gamma}(t)}|f(s)|\|u_t(t-s)\|^2ds. \end{aligned} \tag{77}$$

On the other hand, the function $h(t) := \tilde{\gamma}(s) + \hat{\gamma}(t-s) - \hat{\gamma}(t)$, for $s \geq 0$ fixed and $t \geq 0$, satisfies $h'(t) = \gamma(t-s) - \gamma(t)$. Then h is nondecreasing for $t \geq \frac{s}{2}$, and nonincreasing for $t \leq \frac{s}{2}$ because γ is even and nonincreasing on $\mathbb{R}_+$. Therefore, $h(t) \geq h(\frac{s}{2}) = 0$, and (76) follows at once. □

We denote

$$\begin{cases} c_4 = \dfrac{1}{2}\left(\dfrac{c_5 + g(0)\beta_0}{g_0 - \epsilon} + \dfrac{c_5(1 - ag_0 - \epsilon)}{\epsilon}\right), \\ c_5 = \dfrac{\epsilon g(0)\beta_0 + 1}{(g_0 - \epsilon)(1 - ag_0 - \epsilon) - \epsilon}. \end{cases} \tag{78}$$

Note that c_4 and c_5 are positive and do not depend on f according to (9), (31) and (63). We define

$$I_4 = c_4 I_1 + c_5 I_2 + I_3. \tag{79}$$

Then, combining (62), (72) and (76), we entail

$$\begin{aligned} I_4'(t) \leq & -((g_0 - \epsilon)c_4 - c_5 - g(0)\beta_0)\|u_t(t)\|^2 \\ & - ((1 - ag_0 - \epsilon)c_5 - \epsilon c_4)\|A^{\frac{1}{2}}u(t)\|^2 \\ & - \gamma(t)I_3(t) + (c_1c_4 + c_3c_5)\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2ds \\ & - \int_0^{+\infty} |f(s)|\|u_t(t-s)\|^2ds - c_2c_4\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2ds \\ & + \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, c_4\int_0^{+\infty} g(s)\eta(t,s)ds - c_5u(t)\right\rangle. \end{aligned} \tag{80}$$

Let

$$\begin{cases} c_6 = 2c_2c_4 + \dfrac{2}{\delta}(c_1c_4 + c_3c_5 + c_7) + c_4\max\left\{1, \frac{g_0}{b}\right\} + c_5\max\left\{1, \dfrac{a}{b(1 - ag_0)}\right\}, \\ c_7 = \min\left\{(g_0 - \epsilon)c_4 - c_5 - g(0)\beta_0, (1 - ag_0 - \epsilon)c_5 - \epsilon c_4\right\} \end{cases} \tag{81}$$

(note that c_6 and c_7 are positive and do not depend on f according to (9), (31), (63) and (78)) and

$$F = c_6 E + I_4. \tag{82}$$

In virtue of (32), (60) and (80), we infer that

$$\begin{aligned}F'(t) \leq & - c_7 \left(\|u_t(t)\|^2 + \|A^{\frac{1}{2}} u(t)\|^2 + \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2 ds \right) - \gamma(t) I_3(t) \\ & - \int_0^{+\infty} |f(s)| \|u_t(t-s)\|^2 ds \\ & + \left\langle \int_0^{+\infty} f(s) u_t(t-s) ds, c_4 \int_0^{+\infty} g(s) \eta(t,s) ds - c_5 u(t) - c_6 u_t(t) \right\rangle . \quad (83)\end{aligned}$$

Applying Cauchy-Schwarz and Young's inequalities for the last term of (83), and (8) to estimate $\|\eta(t,s)\|^2$ and $\|u(t)\|^2$ by $\frac{1}{b} \|B^{\frac{1}{2}} \eta(t,s)\|^2$ and $\frac{a}{b} \|A^{\frac{1}{2}} u(t)\|^2$, respectively, we get, for

$$\epsilon' = \frac{2}{\int_0^{+\infty} |f(s)| ds} \quad (84)$$

(if $\int_0^{+\infty} |f(s)| ds = 0$, then $f \equiv 0$, and therefore, the last two terms of (83) vanishes),

$$\begin{aligned}& \left\langle \int_0^{+\infty} f(s) u_t(t-s) ds, c_4 \int_0^{+\infty} g(s) \eta(t,s) ds - c_5 u(t) - c_6 u_t(t) \right\rangle \\ \leq & \left(\int_0^{+\infty} |f(s)| \|u_t(t-s)\| ds \right) \left(c_4 \int_0^{+\infty} g(s) \|\eta(t,s)\| ds + c_5 \|u(t)\| + c_6 \|u_t(t)\| \right) \\ \leq & \frac{\epsilon'}{2} \left(\int_0^{+\infty} |f(s)| ds \right) \int_0^{+\infty} |f(s)| \|u_t(t-s)\|^2 ds \\ & + \frac{3}{2\epsilon'} \left(\frac{g_0 c_4^2}{b} \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2 ds + \frac{a c_5^2}{b} \|A^{\frac{1}{2}} u(t)\|^2 + c_6^2 \|u_t(t)\|^2 \right) \\ \leq & \int_0^{+\infty} |f(s)| \|u_t(t-s)\|^2 ds \\ & + c_8 \left(\int_0^{+\infty} |f(s)| ds \right) \left(\int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2 ds + \|A^{\frac{1}{2}} u(t)\|^2 + \|u_t(t)\|^2 \right), \end{aligned} \quad (85)$$

where

$$c_8 = \frac{3}{4} \max \left\{ \frac{g_0 c_4^2}{b}, \frac{a c_5^2}{b}, c_6^2 \right\} \quad (86)$$

(note that c_8 is positive and does not depend on f). Then, under condition (37) with

$$\delta_0 = \frac{c_7}{c_8} \quad (87)$$

(noting that δ_0 is positive and does not depend on f), we find, by combining (59), (83) and (85),

$$F'(t) \leq -c_9 E(t) - \gamma(t) I_3(t), \quad \forall t \in \mathbb{R}_+, \quad (88)$$

where c_9 is the positive constant given by

$$c_9 = 2 \left(c_7 - c_8 \int_0^{+\infty} |f(s)| ds \right). \quad (89)$$

Now, by definition of E and using the second inequality in (8), we get

$$E(t) \geq \frac{1 - a g_0}{2} \left(\|u_t(t)\|^2 + \|A^{\frac{1}{2}} u(t)\|^2 + \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} \eta(t,s)\|^2 ds \right). \quad (90)$$

Therefore, by definition of I_1 and I_2, we have

$$\begin{aligned}|I_1(t)| \leq & \left|\left\langle u_t(t), \int_0^{+\infty} g(s)\eta(t,s)ds\right\rangle\right| \\ \leq & \frac{1}{2}\left(\|u_t(t)\|^2 + \frac{g_0}{b}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds\right) \\ \leq & \max\left\{1, \frac{g_0}{b}\right\} E(t) \end{aligned} \tag{91}$$

and

$$|I_2(t)| \leq |\langle u_t(t), u(t)\rangle| \leq \frac{1}{2}\left(\|u_t(t)\|^2 + \frac{a}{b}\|A^{\frac{1}{2}}u(t)\|^2\right) \leq \max\left\{1, \frac{a}{b(1-ag_0)}\right\} E(t). \tag{92}$$

Hence, (79), (82), (91) and (92) imply that

$$F(t) \leq c_{10}(E(t) + I_3(t)), \quad \forall t \in \mathbb{R}_+, \tag{93}$$

where

$$c_{10} = c_6 + c_4 \max\left\{1, \frac{g_0}{b}\right\} + c_5 \max\left\{1, \frac{a}{b(1-ag_0)}\right\} + 1. \tag{94}$$

Now, we distinguish two cases depending on $\lim_{t\to+\infty}\gamma(t)$.

(a) Case $\lim_{t\to+\infty}\gamma(t) := \gamma_0 > 0$. By combining (88) and (93) (note that γ is nonincreasing on $\mathbb{R}_+$), we obtain

$$F'(t) \leq -\delta_1 F(t), \quad \forall t \in \mathbb{R}_+, \tag{95}$$

where $\delta_1 = \frac{\min\{c_9,\gamma_0\}}{c_{10}}$. Then, an integration of the differential inequality (95) over $[0,t]$ gives

$$F(t) \leq F(0)e^{-\delta_1 t}, \quad \forall t \in \mathbb{R}_+. \tag{96}$$

On the other hand, (79), (82), (91) and (92) imply that

$$F(t) \geq c_{11}E(t), \quad \forall t \in \mathbb{R}_+, \tag{97}$$

where

$$c_{11} = c_6 - c_4 \max\left\{1, \frac{g_0}{b}\right\} - c_5 \max\left\{1, \frac{a}{b(1-ag_0)}\right\} \tag{98}$$

(note that $c_{11} > 0$ thanks to (81)). Consequently, the relations (59), (96) and (97) lead to

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} = 2E(t) \leq \frac{2}{c_{11}}F(t) \leq \frac{2F(0)}{c_{11}}e^{-\delta_1 t}, \quad \forall t \in \mathbb{R}_+,$$

which is the uniform decay estimate (38) with $\delta_2 = \frac{2F(0)}{c_{11}}$.

(b) Case $\lim_{t\to+\infty}\gamma(t) = 0$. Let $t_0 \geq 0$ be such that $\gamma(t) \leq c_9$, for all $t \geq t_0$. Then, taking into account (88) and (93), we infer that

$$F'(t) \leq -\delta_1\gamma(t)F(t), \quad \forall t \geq t_0, \tag{99}$$

where $\delta_1 = \frac{1}{c_{10}}$. An integration of (99) over $[t_0, t]$ yields ($\hat{\gamma}$ is defined in (40))

$$F(t) \leq F(t_0)e^{\delta_1\hat{\gamma}(t_0)}e^{-\delta_1\hat{\gamma}(t)}, \quad \forall t \geq t_0. \tag{100}$$

For $t \in [0, t_0]$, using the fact that F is nonincreasing (thanks to (99)) and $\hat{\gamma}$ is increasing, we get

$$F(t) = F(t)e^{\delta_1\hat{\gamma}(t)}e^{-\delta_1\hat{\gamma}(t)} \leq F(0)e^{\delta_1\hat{\gamma}(t_0)}e^{-\delta_1\hat{\gamma}(t)}, \quad \forall t \in [0, t_0]. \tag{101}$$

Inequalities (100) and (101) imply

$$F(t) \leq F(0)e^{\delta_1\hat{\gamma}(t_0)}e^{-\delta_1\hat{\gamma}(t)}, \quad \forall t \in \mathbb{R}_+. \tag{102}$$

Consequently, using (59), (97) and (102), we find

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 = 2E(t) \le \frac{2}{c_{11}} F(t) \le \frac{2F(0)e^{\delta_1 \hat{\gamma}(t_0)}}{c_{11}} e^{-\delta_1 \hat{\gamma}(t)}, \quad \forall t \in \mathbb{R}_+,$$

which gives the weak decay estimate (39) with $\delta_2 = \frac{2F(0)e^{\delta_1 \hat{\gamma}(t_0)}}{c_{11}}$.

2. Arbitrary decay of g: (33) holds and (32) does not hold. We prove here the decay estimates (41), (42), (43) and (45) under condition (33), which allows g to have an arbitrary decay at infinity that can vary from the exponential one e^{-qt} to $\frac{1}{t^p}$, for $q > 0$ and $p > 1$. The proof is based on new manipulations of the derivatives of I_1 and I_2 defined in (61) and (71), respectively, and on the use of new functionals J_1 and J_2 that are able to absorb some memory terms without passing by E'. We start our proof by giving a simple identity whose proof is straight forward. Let $\mathscr{U}_0 \in \mathscr{H}$, then, for any $v \in \left\{A^{\frac{1}{2}}u, B^{\frac{1}{2}}u\right\}$,

$$\begin{aligned} 2\left\langle v(t), \int_0^{+\infty} g(s)v(t-s)ds \right\rangle =& g_0\|v(t)\|^2 + \int_0^{+\infty} g(s)\|v(t-s)\|^2 ds \\ & - \int_0^{+\infty} g(s)\|v(t)-v(t-s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \end{aligned} \tag{103}$$

On the other hand, for $n \in \mathbb{N}^*$, we consider, as in [39], the set

$$A_n = \{s \in \mathbb{R}_+, \, g(s) \le -ng'(s)\}, \tag{104}$$

and we put $g_n = \displaystyle\int_{A_n^c} g(s)ds$. Note that $g_n > 0$, otherwise, $A_n^c = \emptyset$ and then (32) is satisfied for $\delta = \frac{1}{n}$, which is the case of exponential decay of g treated previously. On the other hand, thanks to the second inequality in (33), we have $\lim_{n\to+\infty} A_n^c = \cap_{n\in\mathbb{N}^*} A_n^c = \emptyset$, and then

$$\lim_{n\to+\infty} g_n = 0. \tag{105}$$

Next, we go back to (66) and (74). Clearly we have

$$\begin{aligned} &\left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds \right\rangle \\ =& \left\langle A^{\frac{1}{2}}u(t), \int_{A_n} g(s)A^{\frac{1}{2}}\eta(t,s)ds \right\rangle + \left\langle A^{\frac{1}{2}}u(t), \int_{A_n^c} g(s)A^{\frac{1}{2}}\eta(t,s)ds \right\rangle. \end{aligned}$$

Then, similarly to (68) with $\epsilon = 2\epsilon_2$ and $\epsilon = \sqrt{dg_n}$ ($\epsilon_2 > 0$) for, respectively, the two terms of the left hand side of the above equality,

$$\begin{aligned} &\left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds \right\rangle \\ \le& \epsilon_2 \|A^{\frac{1}{2}}u(t)\|^2 + \frac{dg_0}{4\epsilon_2}\int_{A_n} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \\ &+ \frac{\sqrt{dg_n}}{2}\left(\|A^{\frac{1}{2}}u(t)\|^2 + \int_{A_n^c} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds\right). \end{aligned}$$

Using the definition (104) of A_n, we obtain

$$\begin{aligned}&\left\langle A^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)A^{\frac{1}{2}}\eta(t,s)ds\right\rangle\\ \le&\epsilon_2\|A^{\frac{1}{2}}u(t)\|^2 - \frac{dng_0}{4\epsilon_2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds\\ &+\frac{\sqrt{dg_n}}{2}\left(\|A^{\frac{1}{2}}u(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds\right). \qquad (106)\end{aligned}$$

As for (70), we find

$$\begin{aligned}&\left\|\int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\|^2\\ =&\left\|\int_{A_n} g(s)B^{\frac{1}{2}}\eta(t,s)ds + \int_{A_n^c} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\|^2\\ \le&2\left\|\int_{A_n} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\|^2 + 2\left\|\int_{A_n^c} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\|^2\\ \le&2g_0\int_{A_n} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds + 2g_n\int_{A_n^c} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds.\end{aligned}$$

Therefore, using again the definition (104) of A_n, we deduce that

$$\begin{aligned}&\left\|\int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\|^2\\ \le&-2ng_0\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds + 2g_n\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds. \qquad (107)\end{aligned}$$

The last term we discuss, using (103) (with $v = B^{\frac{1}{2}}u$), is

$$\begin{aligned}&-g_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)B^{\frac{1}{2}}\eta(t,s)ds\right\rangle\\ =&-g_0^2\|B^{\frac{1}{2}}u(t)\|^2 + g_0\left\langle B^{\frac{1}{2}}u(t), \int_0^{+\infty} g(s)B^{\frac{1}{2}}u(t-s)ds\right\rangle\\ =&-\frac{g_0^2}{2}\|B^{\frac{1}{2}}u(t)\|^2 + \frac{g_0}{2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds\\ &-\frac{g_0}{2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds. \qquad (108)\end{aligned}$$

Next, we insert (67) (with $\epsilon_1 > 0$ instead of ϵ), (106), (107) and (108) into (66) to obtain

$$\begin{aligned}
& I_1'(t) \\
\leq & -(g_0 - \epsilon_1)\|u_t(t)\|^2 + (\frac{\sqrt{dg_n}}{2} + \epsilon_2)\|A^{\frac{1}{2}}u(t)\|^2 - \frac{g_0^2}{2}\|B^{\frac{1}{2}}u(t)\|^2 \\
& + \left(-\frac{g_0}{2} + 2g_n + \frac{\sqrt{dg_n}}{2}\right)\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \\
& + \frac{g_0}{2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds - \left(\frac{g(0)}{4b\epsilon_1} + \frac{dng_0}{4\epsilon_2} + 2ng_0\right)\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \\
& + \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, \int_0^{+\infty} g(s)\eta(t,s)ds \right\rangle, \quad \forall t \in \mathbb{R}_+. \qquad (109)
\end{aligned}$$

On the other hand, by combining (74) and (108), we find

$$\begin{aligned}
I_2'(t) = & \|u_t(t)\|^2 - \|A^{\frac{1}{2}}u(t)\|^2 + \frac{g_0}{2}\|B^{\frac{1}{2}}u(t)\|^2 + \frac{1}{2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds \\
& - \frac{1}{2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds - \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, u(t)\right\rangle. \qquad (110)
\end{aligned}$$

Now, we introduce two functionals J_1 and J_2 and prove two crucial identities on their derivatives.

Lemma 3.7. *Let*

$$J_1(t) = \int_0^t \left(\int_t^{+\infty} g(\tau - s)d\tau\right)\|B^{\frac{1}{2}}u(s)\|^2 ds, \quad \forall t \in \mathbb{R}_+ \qquad (111)$$

and

$$J_2(t) = \int_0^t \left(\int_t^{+\infty} g(\tau - s)d\tau\right)\|A^{\frac{1}{2}}u(s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \qquad (112)$$

Then, for any $\lambda_1,\ \lambda_2 \in]0,1[$,

$$\begin{aligned}
J_1'(t) \leq & g_0\|B^{\frac{1}{2}}u(t)\|^2 - (1-\lambda_1)\xi(t)J_1(t) - \lambda_1\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds \\
& + \lambda_1\int_t^{+\infty} g(s)\|B^{\frac{1}{2}}u_0(s-t)\|^2 ds \quad \forall t \in \mathbb{R}_+ \qquad (113)
\end{aligned}$$

and

$$\begin{aligned}
J_2'(t) \leq & g_0\|A^{\frac{1}{2}}u(t)\|^2 - (1-\lambda_2)\xi(t)J_2(t) - \frac{\lambda_2}{a}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds \\
& + d\lambda_2\int_t^{+\infty} g(s)\|B^{\frac{1}{2}}u_0(s-t)\|^2 ds \quad \forall t \in \mathbb{R}_+. \qquad (114)
\end{aligned}$$

Proof. First, thanks to (33) and the fact that $\eta \in L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))$, we have

$$J_1(t) \leq \frac{1}{\xi(t)}\int_0^t g(t-s)\|B^{\frac{1}{2}}u(s)\|^2 ds \leq \frac{1}{\xi(t)}\int_0^t g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds < +\infty.$$

Consequently, the functional J_1 is well-defined. With the help of (30), we conclude that J_2 also is well defined. Moreover, by (33), a simple and direct differentiation gives

$$
\begin{aligned}
J_1'(t) =& \left(\int_t^{+\infty} g(\tau - t)d\tau\right) \|B^{\frac{1}{2}}u(t)\|^2 - \int_0^t g(t-s)\|B^{\frac{1}{2}}u(s)\|^2 ds \\
=& g_0\|B^{\frac{1}{2}}u(t)\|^2 - (1-\lambda_1)\int_0^t g(t-s)\|B^{\frac{1}{2}}u(s)\|^2 ds \\
& - \lambda_1 \int_{-\infty}^t g(t-s)\|B^{\frac{1}{2}}u(s)\|^2 ds + \lambda_1 \int_{-\infty}^0 g(t-s)\|B^{\frac{1}{2}}u(s)\|^2 ds \\
\leq& g_0\|B^{\frac{1}{2}}u(t)\|^2 - (1-\lambda_1)\xi(t)J_1(t) - \lambda_1 \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds \\
& + \lambda_1 \int_t^{+\infty} g(s)\|B^{\frac{1}{2}}u_0(s-t)\|^2 ds,
\end{aligned}
$$

which is exactly (113). A similar argument yields the relation (114). □

Let N, N_1, N_2, M_1, $M_2 > 0$ and

$$
F = NE + N_1 I_1 + N_2 I_2 + M_1 J_1 + M_2 J_2 + I_3. \tag{115}
$$

Taking into account the relations (60), (109), (110), (113), (114) and (76), we get

$$
\begin{aligned}
F'(t) \leq& C_1\|u_t(t)\|^2 + C_2\|A^{\frac{1}{2}}u(t)\|^2 + C_3 g_0\|B^{\frac{1}{2}}u(t)\|^2 \\
& + \int_0^{+\infty} g(s)\left(C_4\|B^{\frac{1}{2}}\eta(t,s)\|^2 + C_5\|B^{\frac{1}{2}}u(t-s)\|^2\right) ds \\
& + C_6 \int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds - \int_0^{+\infty} |f(s)|\|u_t(t-s)\|^2 ds \\
& + \frac{\sqrt{dg_n}N_1}{2}\|A^{\frac{1}{2}}u(t)\|^2 + \left(\frac{\sqrt{dg_n}}{2} + 2g_n\right) N_1 \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \\
& - \gamma(t)I_3(t) - M_1(1-\lambda_1)\xi(t)J_1(t) - M_2(1-\lambda_2)\xi(t)J_2(t) \\
& + (M_1\lambda_1 + M_2\lambda_2 d)\int_t^{+\infty} g(s)\|B^{\frac{1}{2}}u_0(s-t)\|^2 ds \\
& + \left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, N_1\int_0^{+\infty} g(s)\eta(t,s)ds - N_2 u(t) - N u_t(t) \right\rangle,
\end{aligned} \tag{116}
$$

where

$$
\begin{cases}
C_1 = N_1(\epsilon_1 - g_0) + N_2 + g(0)\beta_0, \\
C_2 = N_1\epsilon_2 - N_2 + g_0 M_2, \\
C_3 = \dfrac{-1}{2}N_1 g_0 + \frac{1}{2}N_2 + M_1, \\
C_4 = \dfrac{-1}{2}N_1 g_0 - \frac{1}{2}N_2, \\
C_5 = \dfrac{1}{2}N_1 g_0 + \frac{1}{2}N_2 - \lambda_1 M_1 - \dfrac{\lambda_2 M_2}{a}, \\
C_6 = \dfrac{1}{2}N - \left(\dfrac{g(0)}{4b\epsilon_1} + \dfrac{dng_0}{4\epsilon_2} + 2ng_0\right) N_1.
\end{cases} \tag{117}
$$

Now, we choose the different constants carefully so as to obtain some desired signs of the coefficients. First, we select

$$
\lambda_2 = \lambda_1, \quad M_2 = aM_1 \quad \text{and} \quad M_1 = \frac{g(0)\beta_0}{1 - ag_0}
$$

(note that M_1 exists according to (9)). Next, we pick N_2 such that

$$ag_0 M_1 < N_2 < M_1 - \frac{g(0)\beta_0}{2}$$

(the choice of M_1 guarantees the existence of N_2). Then, we choose N_1 such that

$$\max\left\{\frac{1}{g_0}(N_2 + g(0)\beta_0), \frac{1}{g_0}(2(1+ag_0)M_1 - N_2)\right\} < N_1 < \frac{1}{g_0}(N_2 + 2M_1)$$

(note that N_1 exists in vertue of (31) and our selection of M_1 and N_2). Next, we select ϵ_2, λ_1 and ϵ_1 such that

$$\epsilon_2 = \frac{1}{2N_1}(g_0 N_1 + N_2 - 2(1+ag_0)M_1)$$

(clearly $\epsilon_2 > 0$),

$$\frac{1}{4M_1}(N_2 + g_0 N_1) \le \lambda_1 < 1$$

and

$$0 < \epsilon_1 < g_0 - \frac{N_2 + g(0)\beta_0}{N_1}$$

(λ_1 and ϵ_1 exist by our choices of N_1 and N_2). These choices imply that $C_1 < 0$, $C_2 = -C_3$, $C_3 > 0$ and $C_5 \le 0$. Therefore, using the fact that $\|A^{\frac{1}{2}}u(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2 \ge 0$ (thanks to (9) and the second inequality in (8)), $C_4 < 0$ and the definition (59) of E, we see that

$$\begin{aligned}
&C_1\|u_t(t)\|^2 + C_2\|A^{\frac{1}{2}}u(t)\|^2 + C_3 g_0\|B^{\frac{1}{2}}u(t)\|^2 \\
&+ \int_0^{+\infty} g(s)\left(C_4\|B^{\frac{1}{2}}\eta(t,s)\|^2 + C_5\|B^{\frac{1}{2}}u(t-s)\|^2\right) ds \\
\le &\max\{C_1, C_2, C_4\}\left(\|u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2\right. \\
&\left. + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds\right) \\
\le &2\max\{C_1, C_2, C_4\}E(t), \quad \forall t \in \mathbb{R}_+.
\end{aligned} \tag{118}$$

Similarly, using (90), we see that

$$\begin{aligned}
&\frac{\sqrt{dg_n}N_1}{2}\|A^{\frac{1}{2}}u(t)\|^2 + \left(\frac{\sqrt{dg_n}}{2} + 2g_n\right)N_1\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds \\
\le &\left(\frac{\sqrt{dg_n}}{2} + 2g_n\right)N_1\left(\|A^{\frac{1}{2}}u(t)\|^2 + \int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds\right) \\
\le &\frac{\sqrt{dg_n} + 4g_n}{1 - ag_0}N_1 E(t), \quad \forall t \in \mathbb{R}_+.
\end{aligned} \tag{119}$$

Then, in vertue of (105), we can choose n big enough such that

$$2\max\{C_1, C_2, C_4\} + \frac{\sqrt{dg_n} + 4g_n}{1 - ag_0}N_1 < 0$$

(noting that $\max\{C_1, C_2, C_4\}$ is negative and does not depend on n). Finally, we choose N such that

$$N > \left(\frac{g(0)}{2b\epsilon_1} + \frac{dng_0}{2\epsilon_2} + 4ng_0\right)N_1$$

and

$$N > N_0 := N_1 \max\left\{1, \frac{g_0}{b}\right\} + N_2 \max\left\{1, \frac{a}{b(1-ag_0)}\right\},$$

which implies that $C_6 \geq 0$. Moreover, $E \sim NE + N_1 I_1 + N_2 I_2$ because (91) and (92) imply

$$(N - N_0)E \leq NE + N_1 I_1 + N_2 I_2 \leq (N + N_0)E. \tag{120}$$

On the other hand, for N_1, N_2 and N instead of c_4, c_5 and c_6, respectively, and using (90), we get (exactly as for (85))

$$\left\langle \int_0^{+\infty} f(s)u_t(t-s)ds, N_1 \int_0^{+\infty} g(s)\eta(t,s)ds - N_2 u(t) - N u_t(t) \right\rangle \tag{121}$$
$$\leq C_8 \left(\int_0^{+\infty} |f(s)|ds\right) E(t) + \int_0^{+\infty} |f(s)| \|u_t(t-s)\|^2 ds,$$

where

$$C_8 = \frac{3}{2(1-ag_0)} \max\left\{\frac{g_0 N_1^2}{b}, \frac{a N_2^2}{b}, N^2\right\}. \tag{122}$$

Then, under condition (37) with

$$\delta_0 = \frac{-1}{C_8}\left(2\max\{C_1, C_2, C_4\} + \frac{\sqrt{dg_n} + 4g_n}{1-ag_0} N_1\right),$$

and for

$$\begin{aligned} C_9 &= -2\max\{C_1, C_2, C_4\} - \frac{\sqrt{dg_n} + 4g_n}{1-ag_0} N_1 - C_8 \int_0^{+\infty} |f(s)|ds), \\ C_{10} &= (1-\lambda_1) M_1 \min\{1, a\}, \\ C_{11} &= M_1 \lambda_1 (1 + ad), \end{aligned}$$

we deduce from (116), (118), (119) and (121) that (notice that $\lambda_2 = \lambda_1$ and $M_2 = aM_1$)

$$\begin{aligned} F'(t) \leq &- C_9 E(t) - \gamma(t) I_3(t) - C_{10}\xi(t)(J_1(t) + J_2(t)) \\ &+ C_{11} \int_t^{+\infty} g(s) \|B^{\frac{1}{2}} u_0(s-t)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \end{aligned} \tag{123}$$

We distinguish four cases corresponding to the limits of γ and ξ at infinity.

(a) Case $(\lim_{t\to+\infty} \gamma(t))(\lim_{t\to+\infty} \xi(t)) > 0$. There exist $t_0 \geq 0$ and $\epsilon_0 > 0$ such that $\gamma(t) \geq \epsilon_0$ and $\xi(t) \geq \epsilon_0$, for all $t \geq t_0$. Therefore, using (115) and (120), we find, for

$$\delta_1 = \min\left\{\frac{C_9}{N + N_0}, \epsilon_0, \frac{C_{10}\epsilon_0}{M_1}, \frac{C_{10}\epsilon_0}{M_2}\right\},$$

$$F'(t) \leq -\delta_1 F(t) + C_{11} \int_t^{+\infty} g(s) \|B^{\frac{1}{2}} u_0(s-t)\|^2 ds, \quad \forall t \geq t_0. \tag{124}$$

By integrating the differential inequality (124) over $[t_0, t]$, we get

$$F(t) \leq e^{-\delta_1 t}\left(e^{\delta_1 t_0} F(t_0) + C_{11} \int_0^t e^{\delta_1 s} \int_s^{+\infty} g(\tau) \|B^{\frac{1}{2}} u_0(\tau - s)\|^2 d\tau ds\right), \ \forall t \geq t_0. \tag{125}$$

On the other hand, from (115) and (120), we have

$$F \geq (N - N_0)E, \tag{126}$$

consequently, using (59) and (125), we find, for all $t \geq t_0$,

$$\begin{aligned} \|\mathscr{U}(t)\|_{\mathscr{H}}^2 =& 2E(t) \\ \leq& \frac{2}{N-N_0} F(t) \\ \leq& C_{12} e^{-\delta_1 t} \left(1 + \int_0^t e^{\delta_1 s} \int_s^{+\infty} g(\tau) \|B^{\frac{1}{2}} u_0(\tau - s)\|^2 d\tau ds \right), \end{aligned} \tag{127}$$

where $C_{12} = \frac{2}{N-N_0} \max \left\{ C_{11}, e^{\delta_1 t_0} F(t_0) \right\}$. For $t \in [0, t_0]$, we have

$$\begin{aligned} \|\mathscr{U}(t)\|_{\mathscr{H}}^2 =& 2E(t) \\ \leq& \frac{2}{N-N_0} F(t) \\ \leq& \frac{2}{N-N_0} F(t) e^{\delta_1 t} e^{-\delta_1 t} \\ \leq& \frac{2}{N-N_0} \max_{s \in [0,t_0]} F(s) e^{\delta_1 t_0} e^{-\delta_1 t}, \quad \forall t \in [0, t_0]. \end{aligned} \tag{128}$$

Inequalities (127) and (128) gives (41) with

$$\delta_2 = \frac{2}{N-N_0} \max \left\{ C_{11}, e^{\delta_1 t_0} \max_{s \in [0,t_0]} F(s) \right\}.$$

(b) Case $\lim_{t \to +\infty} \gamma(t) = 0$ and $\lim_{t \to +\infty} \xi(t) > 0$. There exist $t_0 \geq 0$ and $\epsilon_0 > 0$ such that $\xi(t) \geq \epsilon_0$ and $\gamma(t) \leq \min\{\epsilon_0, C_9\}$, for all $t \geq t_0$. Therefore, using (115), (120) and (123), we find, for

$$\delta_1 = \min \left\{ \frac{1}{N+N_0}, 1, \frac{C_{10}}{M_1}, \frac{C_{10}}{M_2} \right\}, \tag{129}$$

$$F'(t) \leq -\delta_1 \gamma(t) F(t) + C_{11} \int_t^{+\infty} g(s) \|B^{\frac{1}{2}} u_0(s-t)\|^2 ds, \quad \forall t \geq t_0. \tag{130}$$

An integrating of the differential inequality (130) over $[t_0, t]$, yields ($\hat{\gamma}$ is defined in (40))

$$F(t) \leq e^{-\delta_1 \hat{\gamma}(t)} \left(e^{\delta_1 \hat{\gamma}(t_0)} F(t_0) + C_{11} \int_0^t e^{\delta_1 \hat{\gamma}(s)} \int_s^{+\infty} g(\tau) \|B^{\frac{1}{2}} u_0(\tau - \|!s)\|^2 d\tau ds \right) \tag{131}$$

for all $t \geq t_0$. On the other hand, from (59), (126) and (131), we obtain, for all $t \geq t_0$,

$$\begin{aligned} \|\mathscr{U}(t)\|_{\mathscr{H}}^2 =& 2E(t) \\ \leq& \frac{2}{N-N_0} F(t) \\ \leq& C_{12} e^{-\delta_1 \hat{\gamma}(t)} \left(1 + \int_0^t e^{\delta_1 \hat{\gamma}(s)} \int_s^{+\infty} g(\tau) \|B^{\frac{1}{2}} u_0(\tau - s)\|^2 d\tau ds \right), \end{aligned} \tag{132}$$

where $C_{12} = \frac{2}{N-N_0} \max\left\{C_{11}, e^{\delta_1 \hat{\gamma}(t_0)} F(t_0)\right\}$. For $t \in [0, t_0]$, we have (note that $\hat{\gamma}$ is increasing)

$$\begin{aligned} \|\mathscr{U}(t)\|_{\mathscr{H}}^2 =& 2E(t) \\ \leq& \frac{2}{N-N_0} F(t) \\ \leq& \frac{2}{N-N_0} F(t) e^{\delta_1 \hat{\gamma}(t)} e^{-\delta_1 \hat{\gamma}(t)} \\ \leq& \frac{2}{N-N_0} \max_{s \in [0,t_0]} F(s) e^{\delta_1 \hat{\gamma}(t_0)} e^{-\delta_1 \hat{\gamma}(t)}, \quad \forall t \in [0, t_0]. \end{aligned} \tag{133}$$

Inequalities (132) and (133) lead to (42) with

$$\delta_2 = \frac{2}{N-N_0} \max\left\{C_{11}, e^{\delta_1 \hat{\gamma}(t_0)} \max_{s \in [0,t_0]} F(s)\right\}.$$

(c) Case $\lim_{t \to +\infty} \gamma(t) > 0$ and $\lim_{t \to +\infty} \xi(t) = 0$. This case is similar to the previous one. There exist $t_0 \geq 0$ and $\epsilon_0 > 0$ such that $\gamma(t) \geq \epsilon_0$ and $\xi(t) \leq \min\{\epsilon_0, C_9\}$, for all $t \geq t_0$. Therefore, (130) holds with ξ instead of γ, and δ_1 is defined in (129). So the proof can be ended as in the case **(b)**.

(d) Case $\lim_{t \to +\infty} \gamma(t) = \lim_{t \to +\infty} \xi(t) = 0$. There exist $t_0 \geq 0$ such that $\min\{\gamma(t), \xi(t)\} \leq C_9$, for all $t \geq t_0$. Therefore, (130) holds with $\phi = \min\{\gamma, \xi\}$ instead of γ, and δ_1 is defined in (129). The rest of the proof is analogue to the one of case **(b)**.

4. **The discrete time delay case.** In this section, we discuss the well-posedness and stability of (1)-(2) with discrete time delay

$$u_{tt}(t) + Au(t) - \int_0^{+\infty} g(s) Bu(t-s) ds + \mu u_t(t-\tau) = 0, \quad \forall t > 0 \tag{134}$$

and initial conditions

$$\begin{cases} u(-t) = u_0(t), & \forall t \in \mathbb{R}_+, \\ u_t(t-\tau) = f_0(t-\tau), & \forall t \in]0, \tau[, \\ u_t(0) = u_1, \end{cases} \tag{135}$$

where $\tau \in]0, +\infty[$, $\mu \in \mathbb{R}^*$ and (u_0, u_1, f_0) are given initial data.

The problem of well-posedness and stability of (134)-(135) in the case $A = B$ was treated by the first author in [18]. He proved that (134)-(135) is well posed in a suitable space and is exponentially stable provided that (32) holds and $|\mu|$ is small enough. Applying the approach introduced in Section 3, we prove here that (134)-(135) is still stable for the much larger class of kernels g satisfying (33), and we establish decay estimates similar to the ones obtained for (1)-(2).

Well-posedness. We give here a brief idea of the proof of the well-posedness of (134)-(135) under assumptions **(A1)** and **(A2)**.

Following the idea in [33] (see also [34]-[38]) to deal with the delay term by considering a new auxiliary variable z, we can formulate the system (134)-(135) in the abstract form (12), where $\mathscr{U} = (u, u_t, \eta, z)^T$, $\mathscr{U}_0 = (u_0(0), u_1, \eta_0, z_0)^T \in \mathscr{H}$,

$$\mathscr{H} = D(A^{\frac{1}{2}}) \times H \times L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}})) \times L^2(]0, 1[, H),$$

$$L^2(]0, 1[, H) = \left\{w :]0, 1[\to H, \quad \int_0^1 \|w(p)\|^2 dp < +\infty\right\}$$

endowed with the inner product

$$\langle w_1, w_2\rangle_{L^2(]0,1[,H)} = \int_0^1 \langle w_1(p), w_2(p)\rangle \, dp,$$

and

$$\begin{cases} z(t,p) = u_t(t-\tau p), & \forall t \in \mathbb{R}_+, \forall p \in]0,1[, \\ z_0(p) = z(0,p) = f_0(-\tau p), & \forall p \in]0,1[. \end{cases} \tag{136}$$

The operators $\mathscr{A}$ and $\mathscr{B}$ are linear and are given by

$$\mathscr{A}\begin{pmatrix} w_1 \\ w_2 \\ w_3 \\ w_4 \end{pmatrix} = \begin{pmatrix} w_2 \\ -(A-g_0B)w_1 - \displaystyle\int_0^{+\infty} g(s)Bw_3(s)ds - |\mu|w_2 - \mu w_4(1) \\ -\dfrac{\partial w_3}{\partial s} + w_2 \\ -\dfrac{1}{\tau}\dfrac{\partial w_4}{\partial p} \end{pmatrix} \tag{137}$$

and

$$\mathscr{B}(w_1, w_2, w_3, w_4)^T = |\mu|\Big(0, w_2, 0, 0\Big)^T. \tag{138}$$

The domains $\mathscr{D}(\mathscr{A})$ and $\mathscr{D}(\mathscr{B})$ of $\mathscr{A}$ and $\mathscr{B}$, respectively, are given by

$$\mathscr{D}(\mathscr{A}) = \left\{ \begin{array}{c} (w_1, w_2, w_3, w_4)^T \in \mathscr{H}, \ \frac{\partial w_4}{\partial p} \in L^2(]0,1[,H), \ \frac{\partial w_3}{\partial s} \in L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}})), \\ w_2 \in D(A^{\frac{1}{2}}), \ (A-g_0B)w_1 + \displaystyle\int_0^{+\infty} g(s)Bw_3(s)ds \in H, \ w_3(0) = 0, \ w_4(0) = w_2 \end{array} \right\} \tag{139}$$

and $\mathscr{D}(\mathscr{B}) = \mathscr{H}$. Keeping in mind the definition (136) of z, we have

$$\begin{cases} \tau z_t(t,p) + z_p(t,p) = 0, & \forall t \in \mathbb{R}_+, \ \forall p \in]0,1[, \\ z(t,0) = u_t(t), & \forall t \in \mathbb{R}_+. \end{cases} \tag{140}$$

Therefore, we conclude from (19) and (140) that the systems (134)-(135) and (12) are equivalent.

Clearly, thanks to (9), $\mathscr{H}$ endowed with the inner product

$$\begin{aligned} &\left\langle (w_1, w_2, w_3, w_4)^T, (\tilde{w}_1, \tilde{w}_2, \tilde{w}_3, \tilde{w}_4)^T \right\rangle_{\mathscr{H}} \\ &= \left\langle A^{\frac{1}{2}} w_1, A^{\frac{1}{2}}\tilde{w}_1 \right\rangle - g_0 \left\langle B^{\frac{1}{2}} w_1, B^{\frac{1}{2}}\tilde{w}_1 \right\rangle + \langle w_2, \tilde{w}_2\rangle \\ &\quad + \langle w_3, \tilde{w}_3\rangle_{L_g^2(\mathbb{R}_+, D(B^{\frac{1}{2}}))} + \tau|\mu| \langle w_4, \tilde{w}_4\rangle_{L^2(]0,1[,H)} \end{aligned}$$

is a Hilbert space and $\mathscr{D}(\mathscr{A}) \subset \mathscr{H}$ with dense embedding. Similarly to the proof given by the first author in [18] for the case $A = B$, we can prove that the linear operator $-\mathscr{A}$ is a maximal monotone operator, and $\mathscr{B}$ is Lipschitz continuous. Then $\mathscr{A} + \mathscr{B}$ is an infinitesimal generator of a linear C_0-semigroup on $\mathscr{H}$ (see [41]: Chapter 3 - Theorem 1.1). Consequently, (12) associated with (134)-(135) is well-posed in the sense of Theorem 2.1 (see [25] and [41]).

Stability. We prove here that the system (12) associated with (134)-(135) is still stable under **(A1)**, **(A2)**, **(A4)**, (31) and (33) provided that $|\mu|$ is small enough, and provide two decay estimates depending on the limit of ξ at infinity. These results extend the one in [18] (obtained under condition (32)) to kernels g which do not necessarily converge exponentially to zero at infinity.

Theorem 4.1. *Assume that* **(A1), (A2), (A4)**, *(31) and (33) hold. Then there exists a positive constant δ_0 independent of μ such that, if*

$$|\mu| < \delta_0, \tag{141}$$

then, for any $\mathscr{U}_0 \in \mathscr{H}$, there exist positive constants δ_1 and δ_2 (depending on $\|\mathscr{U}_0\|_{\mathscr{H}}$, a, b, d, g_0, $g(0)$, ξ, μ and δ_0) such that the weak solution of (12) associated with (134)-(135) satisfies (41) if $\lim_{t\to+\infty}\xi(t) > 0$, and it satisfies (43) if $\lim_{t\to+\infty}\xi(t) = 0$.

Proof. Let $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$, so that all the calculations below are justified. By a simple density argument, the decay estimates of Theorem 4.1 remain valid for any weak solution ($\mathscr{U}_0 \in \mathscr{H}$). First, as in [18], we provide a bound on the derivative of the energy functional E associated with the solution of (12)

$$\begin{aligned} E(t) =& \frac{1}{2}\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \\ =& \frac{1}{2}\left(\|u_t(t)\|^2 + \|A^{\frac{1}{2}}u(t)\|^2 - g_0\|B^{\frac{1}{2}}u(t)\|^2\right) + \frac{\tau|\mu|}{2}\int_0^1 \|z(t,p)\|^2 dp \\ &+ \frac{1}{2}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \end{aligned} \tag{142}$$

We find

$$E'(t) \le \frac{1}{2}\int_0^{+\infty} g'(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds + |\mu|\|u_t(t)\|^2, \quad \forall t \in \mathbb{R}_+. \tag{143}$$

Note that, as in the distributed delay case, the sign of E' cannot be determined directly from (143). □

The only modification of the proof given in Section 3 for (41) and (43) is the use of the following functional J_3, introduced in [33], instead of I_3 to get a crucial estimate on the discrete delay term.

Lemma 4.2. *The functional*

$$J_3(t) = \tau e^{2\tau}\int_0^1 e^{-2\tau p}\|z(t,p)\|^2 dp, \quad \forall t \in \mathbb{R}_+,$$

satisfies

$$J_3'(t) = -2J_3(t) + e^{2\tau}\|u_t(t)\|^2 - \|z(t,1)\|^2, \quad \forall t \in \mathbb{R}_+. \tag{144}$$

Proof. Using (140), the derivative of J_3 has the form

$$\begin{aligned} J_3'(t) =& 2\tau e^{2\tau}\int_0^1 e^{-2\tau p}\langle z_t(t,p), z(t,p)\rangle dp \\ =& -2e^{2\tau}\int_0^1 e^{-2\tau p}\langle z_p(t,p), z(t,p)\rangle dp \\ =& -e^{2\tau}\int_0^1 e^{-2\tau p}\frac{\partial}{\partial p}(\|z(t,p)\|^2) dp. \end{aligned}$$

Integrating by parts and using the fact that $z(t,0) = u_t(t)$, we end up with

$$J_3'(t) = -2J_3(t) + e^{2\tau}\|u_t(t)\|^2 - \|z(t,1)\|^2,$$

which is exactly (144). □

Now, defining F by (115) with J_3 instead of I_3, we get (116) with J_3, 2, $e^{2\tau}$, $\|z(t,1)\|^2$ and $\mu z(t,1)$ instead of I_3, $\gamma(t)$, $g(0)\beta_0$, $\int_0^{+\infty}|f(s)|\|u_t(t-s)\|^2 ds$ and $\int_0^{+\infty} f(s)u_t(t-s)ds$, respectively. As in the case of distributed time delay, using $\epsilon' = \frac{2}{\mu^2}$ instead of (84), we arrive at, for C_8 defined by (122),

$$\left\langle \mu z(t,1), N_1 \int_0^{+\infty} g(s)\eta(t,s)ds - N_2 u(t) - N u_t(t) \right\rangle$$
$$\leq \|z(t,1)\|^2 + C_8\mu^2 \left(\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}\eta(t,s)\|^2 ds + \|A^{\frac{1}{2}}u(t)\|^2 + \|u_t(t)\|^2 \right).$$

Then, under condition (141) with

$$\delta_0 = \frac{1}{\sqrt{C_8}}\sqrt{-2\max\{C_1,C_2,C_4\} - \frac{\sqrt{dg_n}+4g_n}{1-ag_0}N_1},$$

we get, for $C_9 = \min\left\{-2\max\{C_1,C_2,C_4\} - \frac{\sqrt{dg_n}+4g_n}{1-ag_0}N_1 - C_8\mu^2, \frac{4}{|\mu|}\right\}$,

$$F'(t) \leq -C_9 E(t) - C_{10}\xi(t)(J_1(t)+J_2(t)) + C_{11}\int_t^{+\infty} g(s)\|B^{\frac{1}{2}}u_0(s-t)\|^2 ds, \quad \forall t \in \mathbb{R}_+,$$

which is identical to (123) with $\gamma = 0$. The rest of the proof carries out as in the case of distributed delay by distinguishing two cases corresponding to the limit of ξ at infinity.

Remark 4.3. Theorem 4.1 gives a positive answer to a question posed in [18], where only the exponential stability of (134)-(135) with $A = B$ was proved in [18] provided that $|\mu|$ is small enough and g converges exponentially to zero at infinity.

5. **General comments and issues.** 1. In the case of the problem with distributed time delay (1)-(2), the decay estimates are proved only for classical solutions (21) (that is, for $\mathscr{U}_0 \in \mathscr{D}(\mathscr{A})$), since the functional I_3 defined in (75) is not well-defined for weak solutions (20) (that is, when $\mathscr{U}_0 \in \mathscr{H}$). Probably, this technical difficulty can be overcome by using some density arguments. It would be interesting to study this question and get the decay estimates for weak solutions.

2. By taking $u_0(t) = 0$, for all $t \in \mathbb{R}_+$, our stability results for both the distributed and discrete time delay (1) and (134) hold for a finite memory.

3. The regularity of $g \in C^1(\mathbb{R}_+,\mathbb{R}_+)$ considered in **(A2)** can be weakened by assuming that g is differentiable almost everywhere on $\mathbb{R}_+$. On the other hand, in [10], it was proved that the weaker condition

$$\exists\alpha_1 \geq 1,\, \exists\alpha_2 > 0: \quad g(t+s) \leq \alpha_1 e^{-\alpha_2 t} g(s), \quad \forall t \in \mathbb{R}_+, \text{ for a.e. } s \in \mathbb{R}_+ \tag{145}$$

is necessary for (3) to be exponentially stable. Condition (145) implies that g converges exponentially to zero at infinity but it does not involve the derivative of g, which allows g to have horizontal inflection points or even flat zones; that is, the set

$$\{s \in \mathbb{R}_+ : g(s) > 0 \text{ and } g'(s) = 0\} \tag{146}$$

is not empty. In the particular case of the wave equation, it was proved in [40] and [50] that the exponential stability (38) holds if and only if g satisfies (145) and the set

$$\{s \in \mathbb{R}_+ : g'(s) < 0\}$$

has a positive Lebesgue measure. Using the first inequality of assumption (33), [47] and [50] proved that the stability of (3) holds with convolution kernels g having flat zones up to a certain extent; that is, the set in (146) is not negligeable but small enough in some sense. We believe that the exponential decay (38) of (1)-(2) can be extended to more general kernels which satisfy (145) and do not satisfy (33) necessarily. Similarily, we think that the other decay estimates can be extended to kernels satisfying the first inequality of assumption (33) and having flat zones up to a certain extent.

We have been constrained to the kernels in (32) and (33) only because of some technical difficulties. Of course, it would be nice to overcome these difficulties and get some more general stability results.

4. Our models (1)-(2) and (134)-(135) include various practical applications like the wave equation, the Petrovsky system (as well as coupled wave-wave, wave-Petrovsky and Petrovsky-Petrovsky systems) and some elastic systems.

5. The estimates (41), (42), (43) and (45) provide bounds for $\|\mathscr{U}(t)\|^2_{\mathscr{H}}$. These bounds do not always converge to zero at infinity because their limits depend on the connection between the growths of g and u_0 at infinity. If the first initial data u_0 satisfies

$$\int_0^{+\infty} \|B^{\frac{1}{2}} u_0(s)\|^2 ds < +\infty, \tag{147}$$

then the following decay estimates (instead of (41), (42), (43) and (45)) hold:

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} \leq \delta_2 e^{-\delta_1 t}, \quad \forall t \in \mathbb{R}_+ \tag{148}$$

if $(\lim_{t\to+\infty} \gamma(t))(\lim_{t\to+\infty} \xi(t)) > 0$,

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} \leq \delta_2 e^{-\delta_1 \hat{\gamma}(t)}, \quad \forall t \in \mathbb{R}_+ \tag{149}$$

if $\lim_{t\to+\infty} \gamma(t) = 0$ and $\lim_{t\to+\infty} \xi(t) > 0$,

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} \leq \delta_2 e^{-\delta_1 \hat{\xi}(t)}, \quad \forall t \in \mathbb{R}_+ \tag{150}$$

if $\lim_{t\to+\infty} \gamma(t) > 0$ and $\lim_{t\to+\infty} \xi(t) = 0$, and

$$\|\mathscr{U}(t)\|^2_{\mathscr{H}} \leq \delta_2 e^{-\delta_1 \hat{\phi}(t)}, \quad \forall t \in \mathbb{R}_+ \tag{151}$$

if $\lim_{t\to+\infty} \gamma(t) = \lim_{t\to+\infty} \xi(t) = 0$.

The idea of proof of (148)-(151) relies on the following functionals R_1 and R_2 instead of J_1 and J_2, respectively, defined by

$$R_1(t) = \int_0^{+\infty} g(s) \int_{t-s}^{t} \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau ds, \quad \forall t \in \mathbb{R}_+ \tag{152}$$

and

$$R_2(t) = \int_0^{+\infty} g(s) \int_{t-s}^{t} \|A^{\frac{1}{2}} u(\tau)\|^2 d\tau ds, \quad \forall t \in \mathbb{R}_+. \tag{153}$$

Indeed, first, we note that, thanks to (30) and (147),

$$\begin{aligned} R_1(t) \leq & \int_0^{+\infty} g(s) \int_{-\infty}^{t} \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau ds \\ \leq & g_0 \left(\int_0^{+\infty} \|B^{\frac{1}{2}} u_0(s)\|^2 ds + \int_0^{t} \|B^{\frac{1}{2}} u(s)\|^2 ds \right) < +\infty \end{aligned}$$

and

$$\begin{aligned} R_2(t) \leq & \int_0^{+\infty} g(s) \int_{-\infty}^{t} \|A^{\frac{1}{2}} u(\tau)\|^2 d\tau ds \\ \leq & d g_0 \left(\int_0^{+\infty} \|B^{\frac{1}{2}} u_0(s)\|^2 ds + \int_0^t \|B^{\frac{1}{2}} u(s)\|^2 ds \right) < +\infty, \end{aligned}$$

so that R_1 and R_2 are well defined. On the other hand, by a direct differentiation, we obtain, for $\lambda_1 \in]0,1[$,

$$\begin{aligned} R_1'(t) = & \int_0^{+\infty} g(s) \left(\|B^{\frac{1}{2}} u(t)\|^2 - \|B^{\frac{1}{2}} u(t-s)\|^2 \right) ds \\ = & g_0 \|B^{\frac{1}{2}} u(t)\|^2 - \lambda_1 \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} u(t-s)\|^2 ds \\ & - (1-\lambda_1) \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} u(t-s)\|^2 ds. \end{aligned} \tag{154}$$

Using (33) and integrating by parts, we get

$$\begin{aligned} & - \int_0^{+\infty} g(s) \|B^{\frac{1}{2}} u(t-s)\|^2 ds \\ = & - \int_{-\infty}^{t} g(t-s) \|B^{\frac{1}{2}} u(s)\|^2 ds \\ \leq & - \xi(t) \int_{-\infty}^{t} \left(\int_t^{+\infty} g(\tau - s) d\tau \right) \|B^{\frac{1}{2}} u(s)\|^2 ds \\ \leq & \xi(t) \left[\left(\int_t^{+\infty} g(\tau - s) d\tau \right) \left(\int_s^t \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau \right) \right]_{s=-\infty}^{s=t} \\ & - \xi(t) \int_{-\infty}^{t} \left(\frac{\partial}{\partial s} \int_t^{+\infty} g(\tau - s) d\tau \right) \left(\int_s^t \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau \right) ds \\ \leq & - \xi(t) \lim_{s \to -\infty} \left(\int_{t-s}^{+\infty} g(\tau) d\tau \right) \left(\int_s^t \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau \right) \\ & - \xi(t) \int_{-\infty}^{t} \left(\frac{\partial}{\partial s} \int_{t-s}^{+\infty} g(\tau) d\tau \right) \left(\int_s^t \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau \right) ds. \end{aligned} \tag{155}$$

From (147), we deduce that

$$\lim_{s \to -\infty} \left(\int_{t-s}^{+\infty} g(\tau) d\tau \right) \left(\int_s^t \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau \right) = 0. \tag{156}$$

The last integral in the right hand side of (155) satisfies

$$\begin{aligned} & \int_{-\infty}^{t} \left(\frac{\partial}{\partial s} \int_{t-s}^{+\infty} g(\tau) d\tau \right) \left(\int_s^t \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau \right) ds \\ = & \int_{-\infty}^{t} g(t-s) \int_s^t \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau ds \\ = & \int_0^{+\infty} g(s) \int_{t-s}^{t} \|B^{\frac{1}{2}} u(\tau)\|^2 d\tau ds \\ = & R_1(t). \end{aligned} \tag{157}$$

By combining (155)-(157), we find, for $\lambda_1 \in]0,1[$,

$$-(1-\lambda_1)\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds \leq -(1-\lambda_1)\xi(t)R_1(t). \tag{158}$$

Taking into account (158) into (154), we arrive at

$$R_1'(t) \leq g_0\|B^{\frac{1}{2}}u(t)\|^2 - (1-\lambda_1)\xi(t)R_1(t) - \lambda_1\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds, \ \forall t \in \mathbb{R}_+. \tag{159}$$

Similarly, using the second inequality in (8), we find, for $\lambda_2 \in]0,1[$,

$$R_2'(t) \leq g_0\|A^{\frac{1}{2}}u(t)\|^2 - (1-\lambda_1)\xi(t)R_2(t) - \frac{\lambda_2}{a}\int_0^{+\infty} g(s)\|B^{\frac{1}{2}}u(t-s)\|^2 ds, \ \forall t \in \mathbb{R}_+. \tag{160}$$

The estimates (159) and (160) are, respectively, identical to (113) and (114) with R_1 and R_2 instead of J_1 and J_2, respectively, but without the term $\int_t^{+\infty} g(s)\|B^{\frac{1}{2}}u_0(s-t)\|^2 ds$. This leads to the estimates (148)-(151).

6. If the second initial data u_1 satisfies

$$\int_0^{+\infty} \|u_1(s)\|^2 ds < +\infty, \tag{161}$$

the condition (35) is not needed. In this case, the estimates (41), (42), (43) and (45) are satisfied for $\hat{\gamma}$ defined by (40) and $\gamma : \mathbb{R}_+ \to \mathbb{R}_+^*$ is of class $C(\mathbb{R}_+, \mathbb{R}_+^*)$ such that $\lim_{s\to+\infty} \gamma(s)$ exists and

$$|f(t-s)| \geq \gamma(t)\int_t^{+\infty} |f(\tau - s)| d\tau, \quad \forall t \in \mathbb{R}_+, \ \forall s \in [0,t]. \tag{162}$$

Replacing (35) by (162) allows f to have more general decay rate at infinity. The idea of proof relies on the functional R_3 instead of I_3, where

$$R_3(t) = 2\int_0^{+\infty} |f(s)| \int_{t-s}^{t} \|u_t(\tau)\|^2 d\tau ds, \quad \forall t \in \mathbb{R}_+. \tag{163}$$

Indeed, using (9), the first inequality in (11) and (161), we see that

$$R_3(t) \leq 2\alpha g_0 \left(\int_0^{+\infty} \|u_1(s)\|^2 ds + \int_0^t \|u_t(s)\|^2 ds\right) < +\infty,$$

and therefore, R_3 is well defined. As for the proof of (159) (with $\lambda_1 = \frac{1}{2}$), we find

$$R_3'(t) \leq 2\alpha g_0\|u_t(t)\|^2 - \gamma(t)R_3(t) - \int_0^{+\infty} |f(s)|\|u_t(t-s)\|^2 ds, \quad \forall t \in \mathbb{R}_+. \tag{164}$$

The relation (164) looks like (76) with $2\alpha g_0$ and R_3 instead of $g(0)\beta_0$ and I_3, respectively.

7. If both (147) and (161) are satisfied, then (148)-(151) hold under (162) instead of (35).

8. As in the distributed delay case, if the first initial data u_0 satisfies (147), then the following decay estimates hold for the discrete time delay case (134)-(135):

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \delta_2 e^{-\delta_1 t}, \quad \forall t \in \mathbb{R}_+ \tag{165}$$

if $\lim_{t\to+\infty} \xi(t) > 0$, and

$$\|\mathscr{U}(t)\|_{\mathscr{H}}^2 \leq \delta_2 e^{-\delta_1 \hat{\xi}(t)}, \quad \forall t \in \mathbb{R}_+ \tag{166}$$

if $\lim_{t\to+\infty}\xi(t)=0$. The idea of proof of (165)-(166) is to replace, respectively, J_1 and J_2 (defined in (111)-(112)) by R_1 and R_2 (defined in (152)-(153)).

9. In [17], arbitrary decay estimates for (1) without delay (i.e. $f\equiv 0$) under the boundedness condition on u_0

$$\|B^{\frac{1}{2}}u_0(s)\|^2 \le M_0, \quad \forall s\in\mathbb{R}_+, \tag{167}$$

for some positive constant M_0, were proved. Our present decay estimates (41), (42), (43) and (45) show that (167) is not needed to have $\lim_{t\to+\infty}\|\mathscr{U}(t)\|^2_{\mathscr{H}}=0$; see (54)-(55) and (57)-(58).

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REFERENCES

[1] M. Aassila, M. M. Cavalcanti and V. N. Domingos Cavalcanti, Existence and uniform decay of the wave equation with nonlinear boundary damping and boundary memory source term, *Calc. Var. Partial Differential Equations*, **15** (2002), 155–180.

[2] M. Aassila, M. M. Cavalcanti and J. A. Soriano, Asymptotic stability and energy decay rates for solutions of the wave equation with memory in a star-shaped domain, *SIAM J. Control Optim.*, **38** (2000), 1581–1602.

[3] K. Ammari, S. Nicaise and C. Pignotti, Feedback boundary stabilization of wave equations with interior delay, *Systems Cont. Letters*, **59** (2010), 623–628.

[4] T. A. Apalara, S. A. Messaoudi and M. I. Mustafa, Energy decay in thermoelasticity type III with viscoelastic damping and delay term, *Elect. J. Diff. Equa.*, **2012** (2012), 1–15.

[5] A. Benaissa, A. K. Benaissa and S. A. Messaoudi, Global existence and energy decay of solutions for the wave equation with a time varying delay term in the weakly nonlinear internal feedbacks, *J. Math. Phys.*, **53** (2012), 1–19.

[6] S. Berrimi and S. A. Messaoudi, Existence and decay of solutions of a viscoelastic equation with a nonlinear source, *Nonlinear Analysis T. M. A.*, **64** (2006), 2314–2331.

[7] M. M. Cavalcanti, V. N. Domingos Cavalcanti and P. Martinez, General decay rate estimates for viscoelastic dissipative systems, *Nonlinear Analysis T. M. A.*, **68** (2008), 177–193.

[8] M. M. Cavalcanti, V. N. Domingos and J. A. Soriano, Exponential decay for the solution of semilinear viscoelastic wave equations with localized damping, *Elect. J. Diff. Equa.*, **44** (2002), 1–14.

[9] M. M. Cavalcanti and H. P. Oquendo, Frictional versus viscoelastic damping in a semilinear wave equation, *SIAM J. Control Optim.*, **42** (2003), 1310–1324.

[10] V. V. Chepyzhov and V. Pata, Some remarks on stability of semigroups arising from linear viscoelasticity, *Asymptotic Anal.*, **46** (2006), 251–273.

[11] C. M. Dafermos, Asymptotic stability in viscoelasticity, *Arch. Rational Mech. Anal.*, **37** (1970), 297–308.

[12] R. Datko, Two questions concerning the boundary control of certain elastic systems, *J. Diff. Equa.*, **1** (1991), 27–44.

[13] R. Datko, J. Lagnese and M. P. Polis, An example on the effect of time delays in boundary feedback stabilization of wave equations, *SIAM J. Control Optim.*, **1** (1986), 152–156.

[14] M. Fabrizio and B. Lazzari, On the existence and asymptotic stability of solutions for linear viscoelastic solids, *Arch. Rational Mech. Anal.*, **116** (1991), 139–152.

[15] E. Fridman, S. Nicaise and J. Valein, Stabilization of second order evolution equations with unbounded feedback with time-dependent delay, *SIAM J. Control Optim.*, **8** (2010), 5028–5052.

[16] C. Giorgi, J. E. Muñoz Rivera and V. Pata, Global attractors for a semilinear hyperbolic equation in viscoelasticity, *J. Math. Anal. Appl.*, **260** (2001), 83–99.

[17] A. Guesmia, Asymptotic stability of abstract dissipative systems with infinite memory, *J. Math. Anal. Appl.*, **382** (2011), 748–760.

[18] A. Guesmia, Well-posedness and exponential stability of an abstract evolution equation with infinite memory and time delay, *IMA J. Mathematical Control and Information*, **30** (2013), 507–526.

[19] A. Guesmia, Some well-posedness and general stability results in Timoshenko systems with infinite memory and distributed time delay, *J. Math. Phys.*, doi: 10.1063/1.4891489.

[20] A. Guesmia and S. Messaoudi, General energy decay estimates of Timoshenko systems with frictional versus viscoelastic damping, *Math. Meth. Appl. Sci.*, **32** (2009), 2102–2122.

[21] A. Guesmia and S. Messaoudi, A general decay result for a viscoelastic equation in the presence of past and finite history memories, *Nonlinear Analysis T. M. A.*, **13** (2012), 476–485.

[22] A. Guesmia, S. Messaoudi and B. Said-Houari, General decay of solutions of a nonlinear system of viscoelastic wave equations, *Nonl. Diff. Equa. Appl.*, **18** (2011), 659–684.

[23] A. Guesmia, S. Messaoudi and A. Soufyane, On the stabilization for a linear Timoshenko system with infinite history and applications to the coupled Timoshenko-heat systems, *Elect. J. Diff. Equa.*, **2012** (2012), 1–45.

[24] M. Kirane and B. Said-Houari, Existence and asymptotic stability of a viscoelastic wave equation with a delay, *Z. Angew. Math. Phys.*, **62** (2011), 1065–1082.

[25] V. Komornik, *Exact Controllability and Stabilization. The Multiplier Method*, Masson-John Wiley, Paris, 1994.

[26] Z. Liu and S. Zheng, On the exponential stability of linear viscoelasticity and thermoviscoelasticity, *Quart. Appl. Math.*, **54** (1996), 21–31.

[27] S. A. Messaoudi, General decay of solutions of a viscoelastic equation, *J. Math. Anal. Appl.*, **341** (2008), 1457–1467.

[28] S. A. Messaoudi, General decay of the solution energy in a viscoelastic equation with a nonlinear source, *Nonlinear Analysis T. M. A.*, **69** (2008), 2589–2598.

[29] S. A. Messaoudi and N. E. Tatar, Global existence and asymptotic behavior for a nonlinear viscoelastic problem, *Math. Meth. Sci. Res. J.*, **4** (2003), 136–149.

[30] S. A. Messaoudi and N. E. Tatar, Global existence and uniform stability of solutions for a quasilinear viscoelastic problem, *Math. Meth. Appl. Sci.*, **30** (2007), 665–680.

[31] J. E. Muñoz Rivera and M. G. Naso, Asymptotic stability of semigroups associated with linear weak dissipative systems with memory, *J. Math. Anal. Appl.*, **326** (2007), 691–707.

[32] M. I. Mustafa, Exponential decay in thermoelastic systems with boundary delay, *J. Abst. Diff. Equa. Appl.*, **2** (2011), 1–13.

[33] S. Nicaise and C. Pignotti, Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks, *SIAM J. Control Optim.*, **5** (2006), 1561–1585.

[34] S. Nicaise and C. Pignotti, Stabilization of the wave equation with boundary or internal distributed delay, *Diff. Integral Equa.*, **9-10** (2008), 935–958.

[35] S. Nicaise and C. Pignotti, Interior feedback stabilization of wave equations with time dependent delay, *Elect. J. Diff. Equa.*, **41** (2011), 1–20.

[36] S. Nicaise, C. Pignotti and J. Valein, Exponential stability of the wave equation with boundary time-varying delay, *Disc. Cont. Dyna. Syst. Series S*, **3** (2011), 693–722.

[37] S. Nicaise and J. Valein, Stabilization of second order evolution equations with unbounded feedback with delay, *ESAIM Control Optim.*, **2** (2010), 420–456.

[38] S. Nicaise, J. Valein and E. Fridman, Stability of the heat and of the wave equations with boundary time-varying delays, *Disc. Cont. Dyna. Syst. Series S*, **2** (2009), 559–581.

[39] V. Pata, Exponential stability in linear viscoelasticity, *Quart. Appl. Math.*, **3** (2006), 499–513.

[40] V. Pata, Exponential stability in linear viscoelasticity with almost flat memory kernels, *Commun. Pure. Appl. Anal.*, **9** (2010), 721–730.

[41] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[42] B. Said-Houari, A stability result for a Timoshenko system with past history and a delay term in the internal feedback, *Dynamic Systems and Applications*, **20** (2011), 327–354.

[43] B. Said-Houari and F. Falcão Nascimento, Global existence and nonexistence for the viscoelastic wave equation with nonlinear boundary damping-source interaction, *Comm. Pure Appl. Anal.*, **12** (2013), 375–403.

[44] N. E. Tatar, Exponential decay for a viscoelastic problem with a singular kernel, *Z. angew. Math. Phys.*, **60** (2009), 640–650.

[45] N. E. Tatar, On a large class of kernels yielding exponential stability in viscoelasticity, *Appl. Math. Comp.*, **215** (2009), 2298–2306.

[46] N. E. Tatar, How far can relaxation functions be increasing in viscoelastic problems? *Appl. Math. Letters*, **22** (2009), 336–340.
[47] N. E. Tatar, A new class of kernels leading to an arbitrary decay in viscoelasticity, *Mediterr. J. Math.*, **6** (2010), 139–150.
[48] N. E. Tatar, On a perturbed kernel in viscoelasticity, *Appl. Math. Letters*, **24** (2011), 766–770.
[49] N. E. Tatar, Arbitrary decays in linear viscoelasticity, *J. Math. Phys.*, **52** (2011), 1–12.
[50] N. E. Tatar, Uniform decay in viscoelasticity for kernels with small non-decreasingness zones, *Appl. Math. Comp.*, **218** (2012), 7939–7946.
[51] N. E. Tatar, Oscillating kernels and arbitrary decays in viscoelasticity, *Math. Nachr.*, **285** (2012), 1130–1143.
[52] A. Vicente, Wave equation with acoustic/memory boundary conditions, *Bol. Soc. Parana. Mat.*, **27** (2009), 29–39.

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