*To Elliott Lieb in recognition of his groundbreaking work in mathematical and theoretical physics*

# MAXIMAL SPEED OF PROPAGATION IN OPEN QUANTUM SYSTEMS

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ABSTRACT. We prove a maximal velocity bound for the dynamics of Markovian open quantum systems. The dynamics are described by one-parameter semigroups of quantum channels satisfying the von Neumann-Lindblad equation. Our result says that dynamically evolving states are contained inside a suitable light cone up to polynomial errors. We also give a bound on the slope of the light cone, i.e., the maximal propagation speed. The result implies an upper bound on the speed of propagation of local perturbations of stationary states in open quantum systems.

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## 1. Introduction

In this paper, we prove a maximal velocity bound for the dynamics of Markovian open quantum systems.

Our work is inspired by the celebrated Lieb-Robinson bounds [75] on propagation of quantum correlations in quantum spin systems. The Lieb-Robinson bounds established a fundamental physical principle in Statistical Mechanics by showing rigorously that quantum correlations (specifically, commutators of local observables) are restricted to an effective light cone in space-time. They also provided an effective tool in many areas of quantum physics.

For examples of breakthroughs in quantum many-body theory that leveraged Lieb-Robinson bounds in essential ways, we mention (i) Hastings' proof of the area law for the entanglement entropy for ground states of 1D gapped Hamiltonians [55], (ii) the proofs by Hastings-Koma and Nachtergaele-Sims of the folklore assertion that a gap leads to exponential decay of correlations [58, 80], and (iii) the modern classification of topological quantum phases [9, 54, 59].

Many other applications of Lieb-Robinson bounds have been found since then in areas as diverse as condensed-matter physics, quantum information science and high-energy physics to name a few [8, 14, 15, 71, 73, 81, 88]. Effective light

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cones of Lieb-Robinson type have been observed in the laboratory [18, 95] and in sophisticated numerical experiments [74, 16], typically starting from a quench of the system.

The supreme usefulness of Lieb-Robinson bounds has also led to many extensions and variants of the original result. Indeed, exploring the scope of Lieb-Robinson bounds has developed into its own branch of research; see, e.g., [1, 20, 21, 22, 30, 31, 36, 37, 39, 46, 47, 48, 57, 53, 68, 72, 78, 79, 83, 84, 87, 90, 97, 98]. Recent efforts in this highly active area have been especially focused on extensions to fermionic [48, 84] or bosonic [36, 37, 72, 90, 97, 98] Hamiltonians and to long-range interactions [32, 40, 71, 77, 96]. See also [47, 57] for very novel directions. We particularly emphasize the works [30, 85] and [87] since they concern open quantum systems, which are also the topic of this paper. For more information on Lieb-Robinson bounds, we recommend the reviews [56, 69, 82].

In an independent later development in $n$-body quantum mechanics, it has been demonstrated in [91] that, up to small probability tails vanishing with time, the supports of wave function solutions of the Schrödinger equation spread with a finite speed. This result was further improved in [6, 61, 93], with [6] proving an energy dependent bound on the maximal speed of propagation. It was extended in [13] to photons interacting with an atomic or molecular system (see also [29, 42, 43, 44, 49]) The above bounds were used in a fundamental way in the scattering theory (see [26, 27, 28, 29, 38, 42, 43, 44, 49, 62, 92]). Furthermore, [36, 37] and [5] developed related techniques in condensed matter physics to prove the maximum velocity bounds for transport of particles in the Bose-Hubbard model in the thermodynamic regime and in the nonlinear Hartree many-body mean-field dynamics, respectively. In this paper, we extend this approach to Markov open quantum systems.

The link between this approach and Lieb-Robinson bounds was made in [37] when a Lieb-Robinson bound was proved for the Bose-Hubbard model by similar techniques. For more on the relation of the velocity bounds that we obtain here to Lieb-Robinson bounds see the end of Subsection 1.2.

1.1. **The von Neumann-Lindblad equation.** The dynamics of open quantum systems originate from the unitary dynamics of systems interacting with an environment by tracing out the latter. States of such systems are described by density operators $\rho$, i.e. positive trace class operators, $\rho = \rho^* \geq 0$, $\mathrm{Tr}(\rho) < \infty$, on some Hilbert space $\mathcal{H}$.

We are interested in open quantum dynamics under the usual Markov (semigroup) assumption. It has been proven in [51, 76] that, for finite-dimensional

Hilbert spaces, Markovian open quantum dynamics satisfy the von Neumann-Lindblad (vNL) equation [1]

$$\frac{\partial \rho_t}{\partial t} = -i[H, \rho_t] + \frac{1}{2}\sum_{j\geq 1}\big([W_j, \rho_t W_j^*] + [W_j\rho_t, W_j^*]\big). \tag{1.1}$$

Here $H$ is a self-adjoint operator on $\mathcal{H}$, the quantum Hamiltonian of a proper quantum system, and $W_j$ are bounded operators. We assume that $\sum_{j\geq 1} W_j^* W_j$ converges in the space of bounded operators $\mathcal{B}(\mathcal{H})$. In what follows, we always assume the above properties of $H$ and $W_j$.

We deal with (1.1) for infinite-dimensional Hilbert spaces, where it is conjectured that the statement above is true as well. In any case, it was shown in [23, 65, 70] that the converse statement, i.e. that the dynamics generated by (1.1) is a quantum dynamical semigroup, holds for both finite-dimensional and infinite-dimensional Hilbert spaces. Hence, if we wish to avoid the conjecture, we may confine ourselves to considering Markovian open quantum dynamics generated by the vNL equations.

For a discussion of existence results and, in particular, for a definition of the weak solution used in the main theorem below, see Subsection 1.3. For a discussion of open quantum systems and irreversibility, see [41, 52].

1.2. **Main result.** Now we suppose that $\mathcal{H} = L^2(\mathbb{R}^d)$. For a Borel set $A$, we let $\chi_A$ denote the characteristic function of $A$. For an operator $B$, the symbol $\mathcal{D}(B)$ denotes the domain of $B$.

We are interested in proving that the solution $\rho_t$ to (1.1) obeys a *maximal propagation speed bound (MSB)*. By this we mean that, under an appropriate localization assumption on the initial state, there is a constant $c < \infty$ s.t. the probability, $\mathrm{Tr}(\chi_{|x|\geq ct}\rho_t)$, that the system is localized in the domain $\{|x| \geq ct\}$ vanishes, as $t \to \infty$:

$$\mathrm{Tr}(\chi_{|x|\geq ct}\rho_t) \to 0. \tag{1.2}$$

In fact, the main result will allow for a non-localized stationary part and is thus more general. Moreover, the scalar $c_{\max} := \inf\{c : (1.2) \text{ holds}\}$ will be called the *maximal propagation speed.*

In this article we make no distinction in our notation between functions and the operators of multiplication defined by those functions.

**Assumptions.** Denote $\langle x\rangle = \sqrt{1+|x|^2}$. We assume that

$$\langle x\rangle^{-1}\mathcal{D}(H) \subset \mathcal{D}(H), \tag{1.3}$$

[1] Here and in what follows we use the units in which the Planck constant is set to $2\pi$, and the speed of light to 1: $\hbar = 1$ and $c = 1$.

and that for some positive integer $n$, the following estimates on $H$ and $W_j$ hold

$$\|\mathrm{ad}^k_{\langle x\rangle}(H)\| \lesssim 1, \quad \text{for} \quad 1 \le k \le n, \tag{1.4}$$

$$\sum_{j\ge1}\|\mathrm{ad}^k_{\langle x\rangle}W_j\|^2 \lesssim 1, \quad \text{for} \quad 1 \le k \le n. \tag{1.5}$$

Here the commutators $\mathrm{ad}^k_{\langle x\rangle}(H)$ are defined recursively by $\mathrm{ad}^0_{\langle x\rangle}(H) = H$ and, for all integer $k$, $\mathrm{ad}^{k+1}_{\langle x\rangle}(H) = [\mathrm{ad}^k_{\langle x\rangle}(H), \langle x\rangle]$.[2]

We describe examples of $H$ and $W_j$ of interest in Subsection 1.4 below. By Assumptions (1.4)-(1.5), with $k = 1$, the operator

$$\gamma := i[H, \langle x\rangle] + \frac{1}{2}\sum_{j\ge1}\left(W_j^*[\langle x\rangle, W_j] + [W_j^*, \langle x\rangle]W_j\right) \tag{1.6}$$

extends to a bounded operator. Its physical meaning is discussed in Subsection 1.4 below. Its norm

$$\kappa := \|\gamma\|\,, \tag{1.7}$$

will give a bound on the maximal propagation speed. We introduce the regions and the corresponding characteristic functions

$$A_\eta := \{x \in \mathbb{R}^d : \langle x\rangle \ge \eta\}, \qquad \text{and} \qquad \chi_b := \chi_{A_b}.$$

We say that a state $\rho_{\mathrm{st}}$ is a static solution to (1.1) if it is a time-independent bounded operator that solves (1.1).

Our main result is the following theorem.

**Theorem 1.1** (Maximal propagation speed bound)**.** *Suppose that Assumptions* (1.3)–(1.5) *hold for some positive integer $n$. Let $\rho_0 := \rho_{\mathrm{st}} + \lambda$, where $\rho_{\mathrm{st}} \ge 0$ is a static solution to* (1.1) *and $\lambda$ is a trace-class operator s.t. $\lambda \ge 0$ (or $-\rho_{\mathrm{st}} \le \lambda \le 0$) and $\chi_b\lambda = 0$ for some $b > 0$. Then, for all $a > b$, $c > \kappa$, there exists $C_n > 0$ such that the unique weak solution $\rho_t$ to* (1.1) *with the initial condition $\rho_0$ satisfies the estimate*

$$\mathrm{Tr}(\chi_\eta\,\rho_t) \le\, C_n\eta^{1-n} + \mathrm{Tr}(\chi_\eta\,\rho_{\mathrm{st}}), \qquad \text{for all } \eta \ge a + ct,\ t > 0. \tag{1.8}$$

In a nutshell, Theorem 1.1 says that under the vNL dynamics, the leakage of the particle probability outside of the light cone $\eta \sim a + ct$ is polynomially suppressed for any $c > \kappa$. In other words, $\kappa$ bounds the maximal propagation speed of particles. We remark that the initial condition $\rho_0 = \rho_{\mathrm{st}} + \lambda$ appearing in Theorem 1.1 is not localized around the origin, unless $\rho_{\mathrm{st}} = 0$.

To interpret the result, we recall that the dynamics generated by the vNL equation, are given by linear, strongly continuous, one-parameter semigroups of trace-preserving and completely positive contractions, called quantum dynamical

[2] As usual, the commutator between two operators $A$, $B$ is defined as a quadratic form on $\mathcal{D}(A) \cap \mathcal{D}(B)$. Assumptions (1.4)–(1.5) postulate that the commutators extend to elements of $\mathcal{B}(\mathcal{H})$.

semi-groups. (A converse statement was proven, for finite-dimensional Hilbert spaces, in [76].) As linear, completely positive maps define quantum channels with quantum information encoded in density operators, the vNL equation could be interpreted as describing transmission of quantum information along a quantum channel defined by the vNL equation (1.1). Then estimate (1.8) establishes that the quantum information is transmitted with a finite speed and gives an explicit bound on the maximal speed of the transmission.

This result may be compared to the MSB for the Schrödinger equation ([6]), on the one hand, and the LR bounds with the Lindblad term ([87]), on the other.

To compare our bounds with the Lieb-Robinson ones, the latter deal with the propagation of correlations in quantum statistical mechanics of macroscopic (or bulk) systems, while ours deal with the propagation of localization of probabilities in quantum mechanical systems at the zero density, i.e. with a finite number of particles propagating in an infinite physical space.

On a technical level, our approach works in both continuous and discrete cases and for unbounded interactions, while with exception of [48] and [79], the Lieb-Robinson bounds are obtained for discrete Hamiltonians and bounded interactions.

Moreover, we allow rather general interactions which could be of $N$-body type. While dependence of constants on the dimension, i.e. on the number of particles $N$, is not controlled here, our techniques are adaptable to the quantum statistical mechanics setting as shown in [36, 37].

1.3. **Existence of solutions to the von Neumann-Lindblad equation.** Denote by $S_1$ the Schatten space of trace class operators. Let $L$ be the operator on $S_1$ defined by the r.h.s. of (1.1), i.e.,

$$L\rho = -i[H,\rho] + \frac{1}{2}\sum_j \big([W_j, \rho W_j^*] + [W_j\rho, W_j^*]\big), \tag{1.9}$$

with the domain $\mathcal{D}(L) = \mathcal{D}(L_0)$, where $L_0\rho := -i[H,\rho]$, or explicitly

$$\mathcal{D}(L) := \big\{\rho \in S_1 \,|\, \rho\mathcal{D}(H) \subset \mathcal{D}(H) \text{ and } H\rho - \rho H \in S_1\big\} \subset S_1. \tag{1.10}$$

Let $L'$ be the operator on the space of observables $\mathcal{B}(\mathcal{H})$ dual of $L$ with respect to the coupling $(A,\rho) := \mathrm{Tr}(A\rho)$, i.e.

$$\mathrm{Tr}(AL\rho) = \mathrm{Tr}((L'A)\rho),$$

for $\rho \in \mathcal{D}(L)$ and $A \in \mathcal{D}(L') \subset \mathcal{B}(\mathcal{H})$ (see (3.2) for an explicit expression).[3] We say that (1.1) has a weak solution $\rho_t$ in $S_1$, if for any observable $A$, i.e. $A \in \mathcal{B}(\mathcal{H})$, in the domain of the operator $L'$, we have

$$\mathrm{Tr}\left(A\frac{\partial\rho_t}{\partial t}\right) = \mathrm{Tr}((L'A)\rho)$$

[3] $L'$ generates the dual Heisenberg-Lindblad evolution $\partial_t A_t = L'A_t$ of quantum observables.

(see e.g. [86]). By a standard argument, for any initial condition $\rho_0 \in S_1$, (1.1) has a unique weak solution in $S_1$ (see e.g. [23, Section 5.5], [34, Appendix A] or [86] for a detailed discussion). One can show further (see [4, 23, 34, 65, 70]) that $L$ defines a completely positive, trace preserving, strongly continuous semigroup of contractions so that, in particular:

$$\rho_t \geq 0 \quad \text{if} \quad \rho_0 \geq 0, \quad \text{and} \quad \mathrm{Tr}\rho_t = \mathrm{Tr}\rho_0.$$

1.4. **Discussion of Theorem 1.1.** The operator $\gamma$ given in (1.6) can be formally written as

$$\gamma = L'\langle x\rangle.$$

It essentially represents the component of the velocity operator $L'x$ along $x$. We expect that in many circumstances the environment will produce such quantum "friction" and even lead to equilibration (see [7, 45, 67, 89, 94] for analysis for quantum systems of finite degrees of freedom). Thus, we formulate the following conjecture:

*Conjecture*. For generic $W_j \neq 0$ and $H$, it holds that $\kappa < \|\mathrm{ad}_{\langle x\rangle}(H)\|$.

This conjecture would imply that the maximal propagation speed is smaller than $\|\mathrm{ad}_{\langle x\rangle}(H)\|$. A weaker version of the conjecture would be that the maximal propagation speed of any open quantum system with $W_j \neq 0$ is bounded by $\|\mathrm{ad}_{\langle x\rangle}(H)\|$. (This weaker version would be implied by the conjecture stated before since, by the result presented here, $\kappa$ bounds the maximal propagation speed of the open quantum system.)

Let us now discuss specific choices for the operators $H$ and $W_j$.

(A) The key example of the operator $H$ is the Schrödinger-type operator

$$H = \omega(p) + V(x), \tag{1.11}$$

with momentum operator $p := -i\nabla$. To satisfy our assumptions, we require that the for the kinetic energy symbol $\omega$ that $|\partial^\alpha \omega(\xi)| \lesssim 1$ for $1 \leq |\alpha| \leq n$ and for the potential $V(x)$ that it is $\omega(p)$-bounded with the relative bound $< 1$. We recall that relative boundedness means that

$$\exists\, 0 \leq a_1 < 1,\ a_2 > 0 : \quad \|Vu\| \leq a_1\|\omega(p)u\| + a_2\|u\|, \tag{1.12}$$

with $\|\cdot\|$ being the norm in $L^2(\mathbb{R}^d)$. By the Kato-Rellich theorem (see e.g. [19]), these assumptions ensure that $H$ is self-adjoint on the domain of $\omega(p)$.

(B) Another example in which the operators $H$ satisfy our assumptions arises if we consider the vNL equation on $\mathbb{Z}^d$, where the derivatives are automatically bounded.

(C) Examples of the Kraus-Lindblad operators $W_j$ such that $\mathrm{ad}^k_{\langle x\rangle} W_j$ are bounded are provided by pseudodifferential operators, $W_j = w_j(x,p)$ with symbols $w_j(x,\xi)$ satisfying the estimates

$$|\partial_x^\beta \partial_\xi^\alpha w_j(x,\xi)| \lesssim \langle \xi\rangle^{-\delta'|\alpha|+\delta|\beta|},$$

for $0 \leq \delta < \delta' \leq 1$ and the multi-indices $\alpha$ and $\beta$, with $|\alpha|$ and $|\beta|$ sufficiently large.

(D) For specific physical examples of the Kraus-Lindblad operators $W_j$, see [34, Section 4].

We note that in its current form, the assumption $|\partial^\alpha \omega(\xi)| \lesssim 1$ for $1 \leq |\alpha| \leq n$ excludes the Laplacian and thus the standard Schrödinger operator. The underlying reason is Assumption (1.4), with $k = 1$, which requires good ultraviolet behaviour.

*Open problem.* Relax Assumption (1.4), with $k = 1$, to $H$-boundedness of $\mathrm{ad}_{\langle x \rangle}(H)$. (In this case, we can allow $\omega(p)$ in (1.11) satisfying $|\partial\omega(\xi)| \lesssim \omega(\xi)$ instead of $|\partial\omega(\xi)| \lesssim 1$ and therefore the standard Schrödinger operators, with $\omega(p) = |p|^2 = -\Delta$.)

*Remark* 1.2. The scattering theory for von Neumann-Lindbald equations generated by unbounded operators has been studied in [2, 3, 24, 34, 35]. Assuming that $H$ has purely absolutely continuous spectrum and that the operators $W_j$ satisfy a suitable smallness condition, it is proven in these references that the dynamics given by (1.1) asymptotically converge, as $t \to \infty$, to the free, Hamiltonian dynamics given by the von Neumann equation

$$\frac{\partial \rho_t}{\partial t} = -i[H_0, \rho_t] \tag{1.13}$$

where $H_0$ is the Hamiltonian of the closed system. Note that under these special conditions, a weak version of the maximal velocity bound can be deduced from this scattering result. In the general case, the scattering theory must be modified to allow for a description of the phenomenon of absorption, or capture [3, 24, 34].

For earlier results on the problem of return to equilibrium, see [10, 11, 12, 33, 89, 94].

At the core of our proof lies a construction of *propagation observables* satisfying the *recursive monotonicity estimate* (RME). Section 2 presents the general method we use without using specificities of the dynamics. In this section, we derive Theorem 1.1 from the RME. The following Section 3 contains the proof of the RME and this is where the specific dynamics we consider enter. In Appendix A, we present, for convenience of the reader, known results on operator functional calculus, namely, expansions of commutators of operator functions with estimates of the remainders.

*Notation.* We write $\|\cdot\|$ for the operator norm.

## 2. Recursive monotonicity estimate and proof of Theorem 1.1

2.1. **Propagation observables.** By the linearity of the vNL equation (1.1), it suffices to consider the evolution $\rho_t := e^{Lt}\lambda$, with $\lambda$ satisfying $\chi_{A_b}\lambda = 0$.

Our goal is to estimate $\mathrm{Tr}(\chi_\eta e^{Lt}\lambda)$. To this end, we use the method of propagation observables.

Let $\rho_t = e^{Lt}\lambda$ be the solution to the vNL equation (1.1) and denote the average of $A$ in the state $\rho_t$ by

$$\langle A\rangle_t := \mathrm{Tr}(A\rho_t).$$

We consider a time-dependent, non-negative operator-family (*propagation observable*) $\Phi_t$ and try to obtain *propagation estimates* of the form $0 \le \mathrm{Tr}(\Phi_t\rho_t) \lesssim t^{-n}$.

By the definition $\mathrm{Tr}(AL\rho) = \mathrm{Tr}((L'A)\rho)$, we have the relation

$$\frac{d}{dt}\left\langle \Phi_t\right\rangle_t = \langle D\Phi_t\rangle_t; \qquad D\Phi_t = L'\Phi_t + \partial_t\Phi_t, \tag{2.1}$$

for all $t$, provided $\Phi_t \in \mathcal{D}(L')$ and $\partial_t\Phi_t \in \mathcal{B}(\mathcal{H})$. We call $D$ the *Heisenberg derivative*. We would like to show that $D\Phi_t \le 0$, modulo fast time-decaying and recursive terms (see (2.4) below), in which case the relation

$$\langle \Phi_t\rangle_t - \int_0^t \langle D\Phi_r\rangle_r dr = \langle \Phi_0\rangle_0, \tag{2.2}$$

which follows from the equation $\langle \Phi_t\rangle_t = \langle \Phi_0\rangle_0 + \int_0^t \frac{d}{dr}\left\langle \Phi_r\right\rangle_r dr$ and (2.1), gives estimates on the positive terms $\langle \Phi_t\rangle_t$ and $-\int_0^t\langle D\Phi_r\rangle_r dr$. We call (2.2) the basic equality.

2.2. **Function spaces.** We fix $c'$ such that $c > c' > \kappa$ and let $\mathcal{F}$ be the set of functions $0 \le f \in C^\infty(\mathbb{R})$, supported in $\mathbb{R}^+$ and satisfying $f(\mu) = 1$ for $\mu \ge c - c'$, and $f' \ge 0$, with $\mathrm{supp}(f') \subset (0, c - c')$ and $\sqrt{f'} \in C^\infty$.

Moreover, besides the notation of Theorem 1.1, we will use the following notation

$$f_{ts} := f(x_{ts}), \quad f'_{ts} = (f')_{ts}, \quad \text{ where } \; x_{ts} := s^{-1}(\langle x\rangle - a - c't) \tag{2.3}$$

The factor $s^{-1}$ is introduced to control multiple commutators and commutator products. It can be thought of as an adiabatic or semi-classical parameter.

2.3. **Recursive monotonicity estimate and proof of the main result.** The notation $O(s^{-m})$ denotes an operator $R \in \mathcal{B}(\mathcal{H})$ such that $\|R\| \le Cs^{-m}$ uniformly in $0 \le t \le s$. The following is the key estimate underlying the proof of Theorem 1.1:

**Proposition 2.1** (Recursive monotonicity estimate)**.** *Assume the hypotheses of Theorem 1.1. Then, for any $f \in \mathcal{F}$, there is $C > 0$ and $\tilde f \in \mathcal{F}$ s.t.*

$$Df_{ts} \le (\kappa - c')s^{-1}f'_{ts} + Cs^{-2}\tilde f'_{ts} + O(s^{-n}). \tag{2.4}$$

Proposition 2.1 is proved in Section 3 below. We call (2.4) the *recursive monotonicity* estimate. In the next proposition, we integrate this estimate.

**Proposition 2.2** (Propagation estimate)**.** *Under the hypotheses of Theorem 1.1, for any $f \in \mathcal{F}$, there is $\tilde{f} \in \mathcal{F}$ and $C > 0$ such that for all fixed $t > 0$ and all $s \geq t$,*

$$\langle f_{ts} \rangle_t + (c' - \kappa) s^{-1} \int_0^t \langle f'_{ts} \rangle_r dr \leq C s^{-2} \int_0^t \langle \tilde{f}'_{rs} \rangle_r dr + O(s^{1-n}). \tag{2.5}$$

*Proof of Proposition 2.2.* Taking the trace of (2.4) with respect to the density operator $\rho_t = e^{tL}\lambda$, we obtain

$$\langle D f_{ts} \rangle_t \leq (\kappa - c') s^{-1} \langle f'_{ts} \rangle_t + C s^{-2} \langle \tilde{f}'_{ts} \rangle_t + O(s^{-n}).$$

Integrating this over time and using Eqs. (2.1) and (2.2) we find

$$\langle f_{ts} \rangle_t + (c' - \kappa) s^{-1} \int_0^t \langle f'_{ts} \rangle_r dr \leq \langle f_{0s} \rangle_0 + C s^{-2} \int_0^t \langle \tilde{f}'_{rs} \rangle_r dr + O(ts^{-n}).$$

Finally, we claim that, for any $f \in \mathcal{F}$,

$$\langle f_{0s} \rangle_0 = \mathrm{Tr}(f_{0s}\lambda) = 0. \tag{2.6}$$

To see this, recall that for any $f \in \mathcal{F}$, we have $\operatorname{supp} f \subset \mathbb{R}^+$ and therefore $\operatorname{supp} f_{0s} \subset \{\langle x \rangle \geq a + \delta s\}$. Since $\chi_{A_b}\lambda = 0$ and $b < a$, we have that

$$f_{0s}\lambda = 0. \tag{2.7}$$

This proves (2.6) and therefore (2.5). □

We now show that Proposition 2.2 yields the main result via an iteration argument.

*Proof of Theorem 1.1 (assuming Proposition 2.1).* We consider an arbitrary $f \in \mathcal{F}$. Dropping the second term in (2.5) yields

$$\langle f_{ts} \rangle_t \leq C s^{-2} \int_0^t \langle \tilde{f}'_{rs} \rangle_r dr + O(s^{1-n}). \tag{2.8}$$

Now we apply Proposition 2.2 to the function $\tilde{f}$. It yields another function $\hat{f} \in \mathcal{F}$ such that, after we drop the first term,

$$(c' - \kappa) s^{-1} \int_0^t \langle \tilde{f}'_{rs} \rangle_r dr \leq C s^{-2} \int_0^t \langle \hat{f}'_{rs} \rangle_r dr + O(s^{1-n}).$$

Since $c' - \kappa > 0$ by assumption, it can be absorbed into the constant $C$. We iterate this procedure $n-1$ times and bound the final integral by the a priori bound $t \leq s$ using that the derivative of any function in $\mathcal{F}$ is uniformly bounded. This gives

$$s^{-1} \int_0^t \langle \tilde{f}'_{rs} \rangle_r dr \leq C s^{-n} \int_0^t \langle \hat{f}'_{rs} \rangle_r dr + O(s^{1-n}) \leq C s^{1-n}.$$

We apply this estimate to (2.8) and find

$$\langle f_{ts} \rangle_t \leq C s^{-2} \int_0^t \langle \tilde{f}'_{rs} \rangle_r dr + O(s^{1-n}) \leq C s^{1-n}. \tag{2.9}$$

It remains to choose $s$ appropriately. For any $f \in \mathcal{F}$, we have $f(\mu) = 1$ for $\mu \geq c - c'$, and therefore $f(x_{ts}) = 1$ on $\{\langle x \rangle \geq a + c't + (c - c')s\}$. Recall the notation $A_\eta := \{x \in \mathbb{R}^d : \langle x \rangle \geq \eta\}$. By our assumption, $\eta \geq a + ct$. We set

$$s = (\eta - a)/c \geq t.$$

This gives $\eta = a + cs \geq a + ct$ and therefore

$$A_\eta \subset \{\langle x \rangle \geq a + c't + (c - c')s\} \subset \{f(x_{ts}) = 1\}.$$

Using this together with estimate (2.9) and the definitions $s = (\eta - a)/c$, we obtain that $\langle \chi_\eta \rangle_t \leq C\eta^{1-n}$, which implies (1.8). □

## 3. Proof of the recursive monotonicity estimate

*Proof of Proposition 2.1.* In what follows, we often denote $\sum_{j \geq 1} \equiv \sum_j$. We use the time-dependent observable

$$\Phi_{ts} := f_{ts} \equiv f(x_{ts}), \quad f \in \mathcal{F}, \tag{3.1}$$

with $0 \leq t \leq s$. First, we observe that the operator $L'$ is given explicitly by

$$L' = L'_0 + G', \qquad L'_0 A = i[H, A], \tag{3.2}$$

$$G'A := \frac{1}{2} \sum_{j \geq 1} (W_j^*[A, W_j] + [W_j^*, A]W_j), \tag{3.3}$$

with domain

$$\mathcal{D}(L') \equiv \mathcal{D}(L'_0) \equiv \big\{A \in \mathcal{B}(\mathcal{H}) \,|\, A\mathcal{D}(H) \subset \mathcal{D}(H) \text{ and } HA - AH \text{ defined on } \mathcal{D}(H) \text{ extends to an element of } \mathcal{B}(\mathcal{H})\big\}.$$

It follows from Assumption (1.4) and Lemma A.1 that for all $0 \leq t \leq s$, $\Phi_{ts}\mathcal{D}(H) \subset \mathcal{D}(H)$ and that $[\Phi_{ts}, H]$ defined on $\mathcal{D}(H)$ extends to a bounded operator. Therefore $\Phi_{ts} \in \mathcal{D}(L')$. Hence, in order to estimate $\langle f_{ts} \rangle_t = \mathrm{Tr}\,(f_{ts}\rho_t)$, we can apply (2.1) and the basic equality (2.2). We start by computing $D\Phi_{ts}$. First, we have

$$\frac{\partial}{\partial t} f_{ts} = -s^{-1} c' f'_{ts}. \tag{3.4}$$

The more interesting term is the first term in the definition of $D$ in (2.1):

$$L' f_{ts} = i[H, f_{ts}] + \frac{1}{2} \sum_{j \geq 1} (W_j^*[f_{ts}, W_j] + [W_j^*, f_{ts}]W_j).$$

The terms on the r.h.s. are controlled via the following two key lemmas. First, it convenient to introduce the following definition. We say a function $h$ is *admissible* if it is smooth, non-negative with $\mathrm{supp}\, h \subset (0, c - c')$ and $\sqrt{h} \in C^\infty$. Note that if $h$ is admissible, then

$$h = f', \quad \text{with} \quad f/f(\infty) \in \mathcal{F}, \quad \text{where} \quad f(\mu) = \int_{-\infty}^{\mu} h(s)ds.$$

**Lemma 3.1** (Estimate of Hamiltonian contribution)**.** *Under the Hypotheses of Proposition 2.2, let* $f'_{ts} = (f')_{ts}$ *and* $u_{ts} = (f'_{ts})^{1/2}$*. Then, we have*

$$i[H, f_{ts}] = s^{-1} u_{ts} i[H, \langle x\rangle] u_{ts} + \mathrm{Rem}_H \tag{3.5}$$

*where the remainder satisfies the operator inequality*

$$\mathrm{Rem}_H \leq C s^{-2} \tilde{u}_{ts}^2 + O(s^{-n}), \tag{3.6}$$

*for a suitable admissible function* $\tilde{u}^2$*.*

The main novelty for the von Neumann-Lindblad equation is the following estimate on the interaction with the environment.

**Lemma 3.2** (Estimate on the environment contribution)**.** *Under the Hypotheses of Proposition 2.2 and with the definition* (3.3)*,*

$$G' f_{ts} = s^{-1} u_{ts} (G'\langle x\rangle) u_{ts} + \mathrm{Rem}_W \tag{3.7}$$

*where the remainder satisfies the operator inequality*

$$\mathrm{Rem}_W \leq C s^{-2} v_{ts}^2 + O(s^{-n}), \tag{3.8}$$

*for a suitable admissible function* $v^2$*.*

These lemmas will be proved in Subsections 3.1 and 3.2 below by using the commutator expansion in Lemma A.1 several times. This lemma is applicable due to Assumptions (1.3) and (1.4).

Combining (3.4), (3.5), (3.6), (3.7), and (3.8) and recalling the definitions $u_{ts}^2 = f'_{ts}$ and of $\kappa$ in (1.7), we obtain (2.4). □

3.1. **Proof of Lemma 3.1.** Thanks to Assumptions (1.3) and (1.4), we can use Lemma A.1, more precisely (A.3) and its adjoint, to obtain the commutator expansion

$$[H, f(x_{ts})] = \sum_{1\leq k<n} \frac{s^{-k}}{k!} f^{(k)}(x_{ts}) B_k + O(s^{-n}\|B_n\|) \tag{3.9}$$

where $B_k = \mathrm{ad}^k_{\langle x\rangle} H$.

In order to further use estimates on $B_k$ from Assumption (1.4), we need to symmetrize the appearance of the derivative. We set $u_1 = \sqrt{f'} \geq 0$ which satisfies $u_1 \in C^\infty(\mathbb{R}_+)$ since $f \in \mathcal{F}$. Furthermore, for $k \geqq 2$, we let $u_k \in C_c^\infty(\mathbb{R}_+)$ be s.t. $u_k = 1$ on $\operatorname{supp} f^{(k)}$.

We factor $f' = u_1^2$ and write $f^{(k)} = u_k^2 g_k$ with $g_k = f^{(k)}$ for $k \geq 2$ and $g_1 = 1$. Then we write

$$f^{(k)}(x_{ts}) B_k = u_k(x_{ts}) g_k(x_{ts}) B_k u_k(x_{ts}) + u_k(x_{ts}) g_k(x_{ts}) [u_k(x_{ts}), B_k], \quad 1 \leq k \leq n.$$

We can again expand the commutator via Lemma A.1,

$$[u_k(x_{ts}), B_k] = -\sum_{m=1}^{n-k-1} (-1)^m \frac{s^{-m}}{m!} u_k^{(m)}(x_{ts}) B_{k+m} + O(\|B_n\| s^{-n+k}). \tag{3.10}$$

Iterating this symmetrization procedure, we find

$$i[H,\Phi_{ts}] \ = s^{-1}\, u_1(x_{ts})\, [iH,\langle x\rangle]\, u_1(x_{ts}) + \mathrm{Rem}_H \tag{3.11}$$

where

$$\mathrm{Rem}_H = \sum_{k=2}^{n-1} s^{-k} v_k(x_{ts}) D_k v_k(x_{ts}) + O(s^{-n}\|B_n\|), \tag{3.12}$$

with $v_k \in C_c^\infty((0,c-c'))$, $v_k = 1$ on $\mathrm{supp}(f')$ and $D_k$ bounded operators satisfying

$$\|D_k\| \le C_k \|B_k\|, \qquad 2\le k\le n-1 \tag{3.13}$$

To obtain an operator bound from this norm bound, we rewrite (3.11) with a manifestly self-adjoint remainder term,

$$\begin{aligned} i[H,\Phi_{ts}] =& \frac{1}{2}\big(i[H,\Phi_{ts}] + (i[H,\Phi_{ts}])^*\big) \\ &= s^{-1}\, u_1(x_{ts})\, [iH,\langle x\rangle]\, u_1(x_{ts}) + \frac{1}{2}\big(\mathrm{Rem}_H + \mathrm{Rem}_H^*\big) \end{aligned} \tag{3.14}$$

where

$$\frac{\mathrm{Rem}_H + \mathrm{Rem}_H^*}{2} = \sum_{k=2}^{n-1} s^{-k} v_k(x_{ts}) \frac{D_k + D_k^*}{2} v_k(x_{ts}) + O(s^{-n}\|B_n\|). \tag{3.15}$$

Thanks to self-adjointness and (3.13), we have the operator inequality

$$\frac{D_k + D_k^*}{2} \le \|D_k + D_k^*\| \le 2\|D_k\| \le 2C_k\|B_k\|, \qquad 2\le k\le n-1,$$

and each $\|B_k\|$ is finite by Assumption (1.4). This implies

$$\frac{\mathrm{Rem}_H + \mathrm{Rem}_H^*}{2} \le C_n s^{-2} \max_{0\le k\le n} \|B_k\| \sum_{2\le k\le n} s^{-k+2} v_k(x_{ts})^2. \tag{3.16}$$

Since $\sum_{2\le k\le n} s^{-k+2} v_k^2$ is bounded by an $s$-independent admissible function, this proves Lemma 3.1. $\square$

## 3.2. **Proof of Lemma 3.2.**

*Proof of Lemma 3.2.* Fix $j\ge 1$ and recall the definition (3.3). We can restrict our attention to a single term $W_j^*[f(x_{ts}),W_j]$ and take adjoints and a sum over $j$ at the end to derive the lemma. Using Lemma A.1, we obtain the commutator expansion

$$W_j^*[f(x_{ts}),W_j] = W_j^* \sum_{k=1}^{n-1} \frac{s^{-k}}{k!} f^{(k)}(x_{ts}) A_j^k + W_j^* O(s^{-n}\|A_j^n\|). \tag{3.17}$$

where $A_j^k = \mathrm{ad}^k_{\langle x\rangle} W_j$. Notice that the last error term is summable in $j$ by Assumption (1.5) and the Cauchy-Schwarz inequality and yields $O(s^{-n})$, so it can be ignored in the following.

We consider the first term on the right-hand side of (3.17) and symmetrize the expression to the right of $W_j^*$. To this end, we write $f^{(k)} = u_k^2 g_k$ with $u_k$ and $g_k$ defined as in the proof of Lemma 3.1. Then we write, for $k \geq 1$,

$$W_j^* f^{(k)}(x_{ts}) A_j^k = u_k(x_{ts}) W_j^* g_k(x_{ts}) A_j^k u_k(x_{ts}) \\ + u_k(x_{ts}) W_j^* g_k(x_{ts}) [u_k(x_{ts}), A_j^k] + [W_j^*, u_k(x_{ts})] g_k(x_{ts}) u_k(x_{ts}) A_j^k. \quad (3.18)$$

We can again expand the first commutator via Lemma A.1,

$$[u_k(x_{ts}), A_j^k] = -\sum_{m=1}^{n-k-1} \frac{(-1)^m s^{-m}}{m!} u_k^{(m)}(x_{ts}) A_j^{k+m} + O(s^{-n+k} \|A_j^n\|). \quad (3.19)$$

For the second commutator in (3.18), we note that $W_j^* = (A_j^0)^*$ and use the adjoint version of Lemma A.1,

$$\begin{aligned}[(A_j^k)^*, u_k(x_{ts})] &= [u_k(x_{ts}), A_j^k]^* \\ &= \sum_{m=1}^{n-k-1} \frac{s^{-m}}{m!} (A_j^{k+m})^* u_k^{(m)}(x_{ts}) + O(s^{-n+k} \|A_j^n\|), \end{aligned} \quad (3.20)$$

Iterating this symmetrization procedure, we find

$$W_j^* [f(x_{ts}), W_j] = s^{-1} u(x_{ts}) W_j^* [\langle x \rangle, W_j] u(x_{ts}) + \mathrm{Rem}_{W,j} \quad (3.21)$$

with

$$\mathrm{Rem}_{W,j} = \sum_{k=2}^{n-1} s^{-k} v_k(x_{ts}) D_j^k v_k(x_{ts}) + O(s^{-n} \|D_j^n\|).$$

Here $v_k \in C_c^\infty((0, c - c'))$ and $v_k = 1$ on $\operatorname{supp} f'$, $v_k$ are independent of $j$, and $D_j^k$ are bounded operators satisfying the norm bound

$$\|D_j^k\| \leq C_k a_j^k, \quad (3.22)$$

where we introduced the shorthand

$$a_j^k := \max_{\substack{0 \leq \ell, m \leq k: \\ \ell + m = k}} \|A_j^\ell\| \|A_j^m\|.$$

We take the adjoint relation to find

$$W_j^* [f(x_{ts}), W_j] + [W_j^*, f(x_{ts})] W_j = s^{-1} u(x_{ts}) \big( W_j^* [\langle x \rangle, W_j] \\ + [W_j^*, \langle x \rangle] W_j \big) u(x_{ts}) + \mathrm{Rem}_{W,j} + (\mathrm{Rem}_{W,j})^*. \quad (3.23)$$

Now we take the sum over $j \geq 1$ (whose convergence is justified a posteriori) and recall the notation $u_{ts} := u(x_{ts})$ (see (2.3)) to obtain (3.7) with the remainder

$$\mathrm{Rem}_W = \sum_{j \geq 1} \left( \sum_{k=2}^{n-1} s^{-k} v_k(x_{ts}) (D_j^k + (D_j^k)^*) v_k(x_{ts}) + O(s^{-n} \|D_j^n\|) \right).$$

From self-adjointness and the norm bound (3.22), we conclude the operator inequality $D_j^k + (D_j^k)^* \le 2C_k a_j^k$ and hence

$$\mathrm{Rem}_W \le C_n s^{-2} \sum_{k=2}^{n-1} \left( \sum_{j\ge1} a_j^k \right) s^{-k+2} v_k(x_{ts}) v_k(x_{ts}) + O(s^{-n}) \sum_{j\ge1} a_j^n.$$

Assumption (1.5) implies that $\sum_{j\ge1} a_j^k < \infty$ for each $k \ge 2$. To complete the proof, it remains to note that $\sum_{k=2}^{n-1} s^{-k+2} v_k(x_{ts}) v_k(x_{ts})$ is bounded from above by an $s$-independent admissible function. □

## Appendix A. Commutator expansions

In this appendix, we present commutator expansions and estimates, first derived in [91] and then improved in [50, 63, 64, 93]. We follow [63] and refer to this paper for details and references. Here, we mention only that, by the Helffer-Sjöstrand formula, a function $f$ of a self-adjoint operator $A$ can be written as

$$f(A) = \int d\widetilde{f}(z)(z-A)^{-1}, \tag{A.1}$$

where $\widetilde{f}(z)$ is an almost analytic extension of $f$ to $\mathbb{C}$ supported in a complex neighbourhood of $\operatorname{supp} f$ [60]. For $f \in C^{n+2}(\mathbb{R})$, we can choose $\widetilde{f}$ satisfying the estimates (see (B.8) of [63], see also [28, 25, 66]):

$$\int |d\widetilde{f}(z)| |\mathrm{Im}(z)|^{-p-1} \lesssim \sum_{k=0}^{n+2} \|f^{(k)}\|_{k-p-1}, \tag{A.2}$$

where $\|f\|_m := \int \langle x\rangle^m |f(x)| dx$ and any integer $0 \le p \le n$.

The essential commutator expansions and remainder estimates are incorporated in the following lemma:

**Lemma A.1.** *Let $f \in C^\infty(\mathbb{R})$ be bounded, with $\sum_{k=0}^{n+2} \|f^{(k)}\|_{k-2} < \infty$, for some $n \ge 1$. Let $x_s = s^{-1}(\langle x\rangle - a)$ for $a > 0$ and $1 \le s < \infty$. Let $A$ be an operator such that $\langle x\rangle^{-1}\mathcal{D}(A) \subset \mathcal{D}(A)$. Define*

$$B_k = \mathrm{ad}^k_{\langle x\rangle} A$$

*and assume that $\|B_k\| < \infty$ for all $1 \le k \le n$. Then, for any $n \ge 1$,*

$$[A, f(x_s)] = \sum_{1\le k\le n-1} (-1)^{k-1} \frac{s^{-k}}{k!} B_k f^{(k)}(x_s) + O(s^{-n}\|B_n\|), \tag{A.3}$$

*uniformly in $a \in \mathbb{R}$.*

*Proof.* Using (A.1), we have

$$[A, f(x_s)] = \int d\widetilde{f}(z) \big[A, (z-x_s)^{-1}\big],$$

in the sense of quadratic forms on $\mathcal{D}(A)$. The hypothesis $\langle x\rangle^{-1}\mathcal{D}(A)\subset\mathcal{D}(A)$ shows that $(z-x_s)^{-1}=\langle x\rangle^{-1}(z\langle x\rangle^{-1}-x_s\langle x\rangle^{-1})^{-1}$ maps $\mathcal{D}(A)$ into itself for $z$ with large $|\mathrm{Im} z|$ and therefore for all $z$ with $\mathrm{Im} z\neq 0$. Hence, since $[A,\langle x\rangle]=\mathrm{ad}^1_{\langle x\rangle}(A)$ is bounded, the formula

$$s\big[A,(z-x_s)^{-1}\big]=(z-x_s)^{-1}[A,\langle x\rangle](z-x_s)^{-1}$$

holds in the sense of quadratic forms on $\mathcal{D}(A)$ ($\mathrm{Im}\langle Au,B^{-1}u\rangle=\mathrm{Im}\langle u,AB^{-1}u\rangle=\mathrm{Im}\langle BB^{-1}u,AB^{-1}u\rangle$). Since $[A,\langle x\rangle]=\mathrm{ad}^1_{\langle x\rangle}(A)$ is bounded, we can proceed as in (B.14)-(B.15) of [63], commuting successively the commutators $ad^1_{\langle x\rangle}(A)$ to the left. This yields

$$[A,f(x_s)]=\sum_{1\le k\le n-1}(-1)^{k-1}\frac{s^{-k}}{k!}B_k f^{(k)}(x_s)+s^{-n}\mathrm{Re}(s);$$
$$\mathrm{Re}(s)=\int d\widetilde{f}(z)(z-x_s)^{-1}B_n(z-x_s)^{-n}.$$

Since the operator $B_n$ is bounded, we have

$$\begin{aligned}\|\mathrm{Re}(s)\| &\le \|B_n\|\int|d\widetilde{f}(z)||z-x_s|^{-n-1}\\ &\le \|B_n\|\int|d\widetilde{f}(z)||\mathrm{Im} z|^{-n-1}\lesssim \|B_n\|\sum_{k=0}^{n+2}\|f^{(k)}\|_{k-n-1}.\end{aligned}$$

This concludes the proof. □

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