Spectral properties for Hamiltonians of weak interactions

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Abstract

We present recent results on the spectral theory for Hamiltonians of the weak decay. We discuss rigorous results on self-adjointness, location of the essential spectrum, existence of a ground state, purely absolutely continuous spectrum and limiting absorption principles. The last two properties heavily rely on the so-called Mourre Theory, which is used, depending on the Hamiltonian we study, either in its standard form, or in a more general framework using non self-adjoint conjugate operators.

1 Introduction

We study various mathematical models for the weak interactions that can be patterned according to the Standard Model of Quantum Field Theory. The reader may consult [30, (4.139)] and [50, (21.3.20)]) for a complete description of the physical Lagrangian of the lepton-gauge boson coupling. A full mathematical understanding of spectral properties for the associated Hamiltonians is not yet achieved, and a rigorous description of the dynamics of particles remains a tremendous task. It is however possible to obtain relevant results in certain cases, like for example a characterization of the absolutely continuous spectrum and limiting absorption principles. One of the main obstacles is to

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be able to establish rigorous results without denaturing the original (ill-defined) physical Hamiltonians, by imposing only mathematical mild and physically interpretable additional assumptions. Among other technical difficulties carried by each models, there are two common problems. A basic one is to prove that the interaction part of the Hamiltonian is relatively bounded with respect to the free Hamiltonian. Without this basic property, it is in general rather illusory to prove more than self-adjointness for the energy operator. This question can be reduced to the adaptation of the N_τ estimates of Glimm and Jaffe [21], as done e.g. in [8], with however serious difficulties for processes involving more than four particles or more than one massless particle. Another major difficulty is to prove a limiting absorption principle without imposing any infrared regularization. This problem can be partly overcome at the expense of a careful study of the Dirac and Boson fields, and thus a study of local properties for the generalized solutions to various partial differential equations, like e.g. the Dirac equations with or without external fields, or the Proca equation.

Derivation of spectral properties for weak interactions – or very similar – models have been achieved in [7, 8, 2, 22, 11, 13, 26, 4, 9, 10, 32, 33]. In the present article, we present a review of the results of [2, 11, 13, 32, 4, 9], focusing on two different processes, one for the gauge bosons $W^\pm$ and one for the gauge boson Z^0. These models already catch some of the main mathematical difficulties encountered in the above mentioned works. The first model is the decay of the intermediate vector bosons $W^\pm$ into the full family of leptons. The second is the decay of the vector boson Z^0 into pairs of electrons and positrons. Both processes involve only three different kind of particles, two fermions and one boson. However, they have a fundamental difference. The first one involves *massless* particles whereas the second one has only *massive* particles. This forces us to use rather different strategies to attack the study of spectral properties.

First model: In the weak decay of the intermediate vector bosons $W^\pm$ into the full family of leptons, the involved particles are the electron e^- and its antiparticle, the positron e^+, together with the associated neutrino ν_e and antineutrino $\bar{\nu}_e$, the muons μ^- and μ^+ together with the associated neutrino ν_μ and antineutrino $\bar{\nu}_\mu$ and the tau leptons τ^- and τ^+ together with the associated neutrino ν_τ and antineutrino $\bar{\nu}_\tau$.

A representative and well-known example of this general process is the decay of the gauge boson W^- into an electron and an antineutrino of the electron that occurs in the β-decay that led Pauli to conjecture the existence of the neutrino [39]

$$W^- \to e^- + \bar{\nu}_e.$$

For the sake of clarity, we shall stick to this case in the first model. The general situation with all other leptons can be recovered in a straightforward way.

The interaction for this $W^\pm$ decay, described in the Schrödinger representa-

tion, is formally given by (see [30, (4.139)] and [50, (21.3.20)])

$$I_{W^\pm} = \int \overline{\Psi_e}(x)\gamma^\alpha(1-\gamma_5)\Psi_{\nu_e}(x)W_\alpha(x)\mathrm{d}x + \int \overline{\Psi_{\nu_e}}(x)\gamma^\alpha(1-\gamma_5)\Psi_e(x)W_\alpha(x)^*\mathrm{d}x\,,$$

where γ^α, $\alpha = 0, 1, 2, 3$, and γ_5 are the Dirac matrices, $\Psi_\cdot(x)$ and $\overline{\Psi_\cdot}(x)$ are the Dirac fields for $e_\pm$, ν_e, and $\bar{\nu}_e$, and W_α are the boson fields (see [49, §5.3] and Section 2).

If one formally expands this interaction with respect to products of creation and annihilation operators, we are left with a finite sum of terms associated with kernels of the form

$$\delta(p_1 + p_2 - k)g(p_1,\ p_2,\ k)\ ,$$

with $g \in L^1$. Our restriction here only consists in approximating these kernels by square integrable functions with respect to momenta (see (2.3) and (2.4)-(2.6)).

Under this assumption, the total Hamiltonian, which is the sum of the free energy of the particles (see (2.2)) and of the interaction, is a well-defined self-adjoint operator (Theorem 2.2).

In addition, we can show (Theorem 2.6) that for a sufficiently small coupling constant, the total Hamiltonian has a unique ground state corresponding to the dressed vacuum. This property is not obvious since usual Kato's perturbation theory does not work here due to the fact that according to the standard model, neutrinos are massless particles (see discussion in Section 2), thus the unperturbed hamiltonian, namely the full Hamiltonian where the interaction between the different particles has been turned off, has a ground state with energy located at the bottom of the essential spectrum. The strategy for proving existence of a unique ground state for similar models has its origin in the seminal works of Bach, Fröhlich, and Sigal [6] (see also [40], [5] and [31]), for the Pauli-Fierz model of non-relativistic QED. Our proofs follow these techniques as adapted in [7, 8, 17] to a model of quantum electrodynamics and in [2] to a model of the Fermi weak interactions.

Under natural regularity assumptions on the kernels, we next establish a Mourre estimate (Theorem 2.8) and a limiting absorption principle (Theorem 2.10) for any spectral interval down to the energy of the ground state and below the mass of the electron, for small enough coupling constants. As a consequence, the whole spectrum between the ground state and the first threshold is shown to be purely absolutely continuous (Theorem 2.7).

Our method to achieve the spectral analysis above the ground state energy, follows [5, 19, 14], and is based on the proof of a spectral gap property for Hamiltonians with a cutoff interaction for small neutrino momenta and acting on neutrinos of strictly positive energies.

Eventually, as in [19, 13, 14], we use this gap property in combination with the conjugate operator method developed in [3] and [44] in order to establish a limiting absorption principle near the ground state energy of H_W. In [13], the chosen conjugate operator was the generator of dilatations in the Fock space for neutrinos and antineutrinos. As a consequence, an infrared regularization

was assumed in [13] in order to be able to implement the strategy of [19]. To overcome this difficulty and avoid infrared regularization, we choose in [4] a conjugate operator which is the generator of dilatations in the Fock space for neutrinos and antineutrinos *with a cutoff in the momentum variable*. Our conjugate operator thus only affects the massless particles of low energies. A similar choice is made in [14] for a model of non-relativistic QED for a free electron at fixed total momentum. Compared with [19] and [14], our method involves further estimates, which allows us to avoid any infrared regularization. Under stronger assumptions, the model of $W^{\pm}$ decay has been studied in [7, 13]. We present in section 2 the results obtained in [4], where the main achievement is that no infrared regularization is assumed.

Second Model: The physical phenomenon in the decay of the gauge boson we consider here only involves *massive* particles, the massive boson Z^0, electrons and positrons,

$$Z^0 \to e^- + e^+ .$$

In some respects, e.g. as far as the existence of a ground state is concerned, this feature renders trivial the spectral analysis of the Hamiltonian. On the other hand, due to the positive masses of the particles, an infinite number of thresholds occur in the spectrum of the unperturbed Hamiltonian. Understanding the nature of the spectrum of the full Hamiltonian near the thresholds as the interaction is turned on then becomes a subtle question, as it is known that spectral analysis near thresholds, in particular by means of perturbation theory, is a delicate subject. This question is the main concern in the analysis of the second model.

The interaction between the electrons, positrons and the boson vectors Z^0, in the Schrödinger representation, is given, up to coupling constants, by (see [30, (4.139)] and [50, (21.3.20)])

$$I_{Z^0} = \int \overline{\Psi_e}(x)\gamma^{\alpha}(g'_V - \gamma_5)\Psi_e(x)Z_{\alpha}(x)\,\mathrm{d}x \; + h.c., \tag{1.1}$$

where, as above, γ^{α}, $\alpha = 0, 1, 2, 3$, and γ_5 are the Dirac matrices and $\Psi_e(x)$ and $\overline{\Psi_e}(x)$ are the Dirac fields for the electron e^- and the positron e^+ of mass m_e. The field Z_{α} is the massive boson field for Z^0. The constant g'_V is a real parameter such that $g'_V \simeq 0,074$ (see e.g [30]).

The main results provide a complete description of the spectrum of the Hamiltonian below the boson mass. We will show that the spectrum is composed of a unique isolated eigenvalue E, the ground state energy corresponding to the dressed vacuum, and the semi-axis of essential spectrum $[E + \mathrm{m}_e, \infty)$, m_e being the electron mass (Theorem 3.4).

Moreover, with mild regularity assumptions on the kernel, using a version of Mourre's theory allowing for a non self-adjoint conjugate operator and requiring only low regularity of the Hamiltonian with respect to this conjugate operator, we establish a limiting absorption principle and prove that the essential spectrum below the boson mass is purely absolutely continuous (Theorem 3.5).

In order to establish these results, we need to use a spectral representation of the self-adjoint Dirac operator generated by the sequence of spherical waves (see [29] and Section 3). If we have been using the plane waves as for the first model above, for example the four ones associated with the helicity (see [47]), the two kernels $G^{(\alpha)}(\cdot)$ of the interaction would have had to satisfy an infrared regularization with respect to the fermionic variables. By our choice of the sequence of the spherical waves, our analysis only requires that the kernels of the interaction satisfy an infrared regularization for two values of the discrete parameters characterizing the sequence of spherical waves. For any other value of the discrete parameters, we do not need to introduce an infrared regularization.

The article is organized as follows. Section 2 is devoted to the study of the first model, the decay of the gauge bosons W^- into an electron and its associated neutrino. The first part contains a detailed construction of the Fock Hilbert spaces and the mathematical Hamiltonian for the decay. The second part of Section 2 deals with the central results of the spectral analysis for this Hamiltonian, as well as some steps of the proof for the limiting absorption principle. All details can be found in [4]. Section 3 is concerned with the decay of the gauge bosons Z^0 into electrons and positrons. There, we also give a detailed description of the Hilbert spaces, notably different than in the previous section due to the writing of the Dirac fields with spherical waves. We also write the construction of the Hamiltonian for the decay of the Z^0 boson. We subsequently present the main theorems on spectral and dynamical properties, with some hints concerning the proof for the limiting absorption principle. All details can be found in [9]. Section 4 is devoted to a short presentation of open questions and ongoing work; whenever it is possible we point out the mathematical difficulties for these new problems.

2 Interaction of the Gauge boson $W^{\pm}$ with an electron and a massless neutrino

According to the Standard Model, the weak decay of the intermediate bosons W^+ and W^- involves the full family of leptons: electrons, muons, tauons, their associated neutrinos and the corresponding antiparticles (see [30, Formula (4.139)] and [50]). In the Standard Model, neutrinos and antineutrinos are assumed to be massless. Despite experimental evidences [20] that in fact neutrinos have a mass, an extended version of the Standard Model to account for this mass is beyond the scope of this article.

Neutrinos and antineutrinos are particles with helicity $-1/2$ and $+1/2$, respectively. Here we shall assume that both neutrinos and antineutrinos have helicity $\pm 1/2$.

As already mentioned in the introduction, without loss of generality, we restrict ourselves to the decay of the gauge boson W^- into an electron and an antineutrino,

$$W^- \to e^- + \bar{\nu}_e. \tag{2.1}$$

However, all results remain true if we consider instead the decay of the $W^{\pm}$ into the full family of leptons.

If we include the corresponding antiparticles in the process (2.1), the interaction described in the Schrödinger representation is formally given by (see [30, (4.139)] and [50, (21.3.20)])

$$I_{W^{\pm}} = \int_{\mathbb{R}^3} \overline{\Psi_e}(x)\gamma^{\alpha}(1-\gamma_5)\Psi_{\nu_e}(x)W_{\alpha}(x)\mathrm{d}x + \int_{\mathbb{R}^3} \overline{\Psi_{\nu_e}}(x)\gamma^{\alpha}(1-\gamma_5)\Psi_e(x)W_{\alpha}(x)^*\mathrm{d}x,$$

where γ^{α}, $\alpha = 0,1,2,3$, and γ_5 are the Dirac matrices, $\Psi_{\cdot}(x)$ and $\overline{\Psi_{\cdot}}(x)$ are the Dirac fields for $e_{\pm}$, ν_e, and $\bar{\nu}_e$, and W_{α} are the boson fields (see [49, §5.3]) given respectively by

$$\begin{aligned}
\Psi_e(x) =& (2\pi)^{-\frac{3}{2}} \sum_{s=\pm\frac{1}{2}} \int_{\mathbb{R}^3} \Big(\frac{u(p,s)}{(2(|p|^2+\mathrm{m}_e{}^2)^{\frac{1}{2}})^{\frac{1}{2}}}\, b_+(p,s)\mathrm{e}^{ip.x} \\
&+ \frac{v(p,s)}{(2(|p|^2+\mathrm{m}_e{}^2)^{\frac{1}{2}})^{\frac{1}{2}}}\, b_-^*(p,s)\mathrm{e}^{-ip.x}\Big)\mathrm{d}p, \\
\Psi_{\nu_e}(x) =& (2\pi)^{-\frac{3}{2}} \sum_{s=\pm\frac{1}{2}} \int_{\mathbb{R}^3} \Big(\frac{u(p,s)}{(2|p|)^{\frac{1}{2}}}\, c_+(p,s)\mathrm{e}^{ip.x} + \frac{v(p,s)}{(2|p|)^{\frac{1}{2}}}\, c_-^*(p,s)\mathrm{e}^{-ip.x}\Big)\mathrm{d}p\,, \\
\overline{\Psi_e}(x) =& \Psi_e(x)^{\dagger}\gamma^0\,, \quad \overline{\Psi_{\nu_e}}(x) = \Psi_{\nu_e}(x)^{\dagger}\gamma^0\,,
\end{aligned}$$

and

$$\begin{aligned}
W_{\alpha}(x) = (2\pi)^{-\frac{3}{2}} \sum_{\lambda=-1,0,1} \int_{\mathbb{R}^3} \Big(& \frac{\epsilon_{\alpha}(k,\lambda)}{(2(|k|^2+\mathrm{m}_W{}^2)^{\frac{1}{2}})^{\frac{1}{2}}} a_+(k,\lambda)\mathrm{e}^{ik.x} \\
& + \frac{\epsilon_{\alpha}^*(k,\lambda)}{(2(|k|^2+\mathrm{m}_W{}^2)^{\frac{1}{2}})^{\frac{1}{2}}} a_-^*(k,\lambda)\mathrm{e}^{-ik.x}\Big)\mathrm{d}k\,.
\end{aligned}$$

Here $\mathrm{m}_e > 0$ is the mass of the electron and $u(p,s)/(2(|p|^2+\mathrm{m}_e{}^2)^{1/2})^{1/2}$ and $v(p,s)/(2(|p|^2+\mathrm{m}_e{}^2)^{1/2})^{1/2}$ are the normalized solutions to the Dirac equation (see for example [30, Appendix]), where $p \in \mathbb{R}^3$ is the momentum variable of the electron, or its antiparticle, and s is its spin. The mass of the bosons $W^{\pm}$ is denoted by m_W, and fulfills $\mathrm{m}_W > \mathrm{m}_e$ ($\mathrm{m}_W/\mathrm{m}_e \approx 1.57 \times 10^5$). The vectors $\epsilon_{\alpha}(k,\lambda)$ are the polarizations of the massive spin 1 bosons (see [49, Section 5.2]), and as follows from the Standard Model, neutrinos and antineutrinos are considered here to be massless particles.

The operators $b_+(p,s)$ and $b_+^*(p,s)$ (respectively $c_+(p,s)$ and $c_+^*(p,s)$), are the annihilation and creation operators for the electrons (respectively for the neutrinos associated with the electrons), satisfying the anticommutation relations. The index $-$ in $b_-(p,s)$, $b_-^*(p,s)$, $c_-(p,s)$ and $c_-^*(p,s)$ are used to denote the annihilation and creation operators of the corresponding antiparticles. The operators $a_+(k,\lambda)$ and $a_+^*(k,\lambda)$ (respectively $a_-(k,\lambda)$ and $a_-^*(k,\lambda)$) are the annihilation and creation operators for the bosons W^- (respectively W^+) satisfying the canonical commutation relations. The definition of these operators is very standard (see e.g. [49] or [12]).

2.1 Rigorous definition of the model

The mathematical model for the weak decay of the vector bosons $W^{\pm}$ is defined as follows.

Let $\xi_1 = (p_1,\ s_1)$ be the quantum variable of a massive lepton, electron or positron, where $p_1 \in \mathbb{R}^3$ is the momentum and $s_1 \in \{-1/2,\ 1/2\}$ is the spin. Let $\xi_2 = (p_2,\ s_2)$ be the quantum variables of a massless neutrino or antineutrino, where $p_2 \in \mathbb{R}^3$ and $s_2 \in \{-1/2,\ 1/2\}$ is the helicity of particles and antiparticles, and, finally, let $\xi_3 = (k,\ \lambda)$ be the quantum variables of the spin 1 bosons W^+ and W^-, with momenta $k \in \mathbb{R}^3$ and where $\lambda \in \{-1,\ 0,\ 1\}$ accounts for the polarization of the vector bosons (see [49, section 5.2]).

We define $\Sigma_1 = \mathbb{R}^3 \times \{-1/2,\ 1/2\}$ for the configuration space of the leptons and $\Sigma_2 = \mathbb{R}^3 \times \{-1,\ 0,\ 1\}$ for the bosons. Thus $L^2(\Sigma_1)$ is the one particle Hilbert space of each lepton of this process (electron, positron, neutrino and antineutrino of the electron) and $L^2(\Sigma_2)$ is the one particle Hilbert space of each boson. In the sequel, we shall use the notations $\int_{\Sigma_1} \mathrm{d}\xi := \sum_{s=+\frac{1}{2},-\frac{1}{2}} \int \mathrm{d}p$ and $\int_{\Sigma_2} \mathrm{d}\xi := \sum_{\lambda=0,1,-1} \int \mathrm{d}k$.

The Hilbert space for the weak decay of the vector bosons $W^{\pm}$ is the Fock space for leptons and bosons describing the set of states with indefinite number of particles or antiparticles which we define below.

The space $\mathfrak{F}_L$ is the fermionic Fock space for the massive electron and positron with the associated neutrino and antineutrino, i.e.

$$\mathfrak{F}_L = \bigotimes^4 \mathfrak{F}_a(L^2(\Sigma_1)) = \bigotimes^4 \left(\oplus_{n=0}^{\infty} \otimes_a^n L^2(\Sigma_1)\right) ,$$

where $\otimes_a^n$ denotes the antisymmetric n-th tensor product and $\otimes_a^0 L^2(\Sigma_1) := \mathbb{C}$.

The bosonic Fock space $\mathfrak{F}_W$ for the vector bosons W^+ and W^- reads

$$\mathfrak{F}_W = \bigotimes^2 \mathfrak{F}_s(L^2(\Sigma_2)) = \bigotimes^2 \left(\oplus_{n=0}^{\infty} \otimes_s^n L^2(\Sigma_2)\right) ,$$

where $\otimes_s^n$ denotes the symmetric n-th tensor product and $\otimes_s^0 L^2(\Sigma_2) := \mathbb{C}$.

The Fock space for the weak decay of the vector bosons W^+ and W^- is thus

$$\mathfrak{F} = \mathfrak{F}_L \otimes \mathfrak{F}_W \ .$$

Furthermore, $b_\epsilon(\xi_1)$ (resp. $b_\epsilon^*(\xi_1)$) is the annihilation (resp. creation) operator for the corresponding species of massive particle if $\epsilon = +$ and for the corresponding species of massive antiparticle if $\epsilon = -$. Similarly, $c_\epsilon(\xi_2)$ (resp. $c_\epsilon^*(\xi_2)$) is the annihilation (resp. creation) operator for the corresponding species of neutrino if $\epsilon = +$ and for the corresponding species of antineutrino if $\epsilon = -$. Finally, the operator $a_\epsilon(\xi_3)$ (resp. $a_\epsilon^*(\xi_3)$) is the annihilation (resp. creation) operator for the boson W^- if $\epsilon = +$, and for the boson W^+ if $\epsilon = -$. The operators $b_\epsilon(\xi_1)$, $b_\epsilon^*(\xi_1)$, $c_\epsilon(\xi_2)$, and $c_\epsilon^*(\xi_2)$ fulfil the usual canonical anticommutation relations (CAR), whereas $a_\epsilon(\xi_3)$ and $a_\epsilon^*(\xi_3)$ fulfil the canonical commutation relation (CCR), see e.g. [49]. Moreover, the a's commute with the b's and the

c's. In addition, following the convention described in [49, section 4.1] and [49, section 4.2], we will assume that fermionic creation and annihilation operators of different species of leptons anticommute (see e.g. [12] for an explicit definition involving this additional requirement). Therefore, the following canonical anticommutation and commutation relations hold,

$$\begin{aligned}
&\{b_\epsilon(\xi_1), b^*_{\epsilon'}(\xi'_1)\} = \delta_{\epsilon\epsilon'}\delta(\xi_1 - \xi'_1) \ , \quad \{c_\epsilon(\xi_2), c^*_{\epsilon'}(\xi'_2)\} = \delta_{\epsilon\epsilon'}\delta(\xi_2 - \xi'_2) \ , \\
&[a_\epsilon(\xi_3), a^*_{\epsilon'}(\xi'_3)] = \delta_{\epsilon\epsilon'}\delta(\xi_3 - \xi'_3) \ , \\
&\{b_\epsilon(\xi_1), b_{\epsilon'}(\xi'_1)\} = \{c_\epsilon(\xi_2), c_{\epsilon'}(\xi'_2)\} = 0 \ , \\
&[a_\epsilon(\xi_3), a_{\epsilon'}(\xi'_3)] = 0 \ , \\
&\{b_\epsilon(\xi_1), c_{\epsilon'}(\xi_2)\} = \{b_\epsilon(\xi_1), c^*_{\epsilon'}(\xi_2)\} = 0 \ , \\
&[b_\epsilon(\xi_1), a_{\epsilon'}(\xi_3)] = [b_\epsilon(\xi_1), a^*_{\epsilon'}(\xi_3)] = [c_\epsilon(\xi_2), a_{\epsilon'}(\xi_3)] = [c_\epsilon(\xi_2), a^*_{\epsilon'}(\xi_3)] = 0 \ ,
\end{aligned}$$

where $\{b, b'\} = bb' + b'b$ and $[a, a'] = aa' - a'a$. For $\varphi \in L^2(\Sigma_1)$, the operators

$$\begin{aligned}
b_\epsilon(\varphi) &= \int_{\Sigma_1} b_\epsilon(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi, \quad c_\epsilon(\varphi) = \int_{\Sigma_1} c_\epsilon(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi \ , \\
b^*_\epsilon(\varphi) &= \int_{\Sigma_1} b^*_\epsilon(\xi)\varphi(\xi)\mathrm{d}\xi, \quad c^*_\epsilon(\varphi) = \int_{\Sigma_1} c^*_\epsilon(\xi)\varphi(\xi)\mathrm{d}\xi \ ,
\end{aligned}$$

are bounded operators on $\mathfrak{F}$ satisfying $\|b^\sharp_\epsilon(\varphi)\| = \|c^\sharp_\epsilon(\varphi)\| = \|\varphi\|_{L^2}$, where $b^\sharp$ (resp. $c^\sharp$) is b (resp. c) or b^* (resp. c^*).

The free Hamiltonian $H_{W,0}$ is given by

$$\begin{aligned}
H_{W,0} = &\sum_{\epsilon=\pm} \int w^{(1)}(\xi_1) b^*_\epsilon(\xi_1) b_\epsilon(\xi_1)\mathrm{d}\xi_1 + \sum_{\epsilon=\pm} \int w^{(2)}(\xi_2) c^*_\epsilon(\xi_2) c_\epsilon(\xi_2)\mathrm{d}\xi_2 \\
&+ \sum_{\epsilon=\pm} \int w^{(3)}(\xi_3) a^*_\epsilon(\xi_3) a_\epsilon(\xi_3)\mathrm{d}\xi_3 \ ,
\end{aligned} \tag{2.2}$$

where the free relativistic energy of the massive leptons, the neutrinos, and the bosons are respectively given by

$$w^{(1)}(\xi_1) = (|p_1|^2 + {\mathrm{m}_e}^2)^{\frac{1}{2}}, \ w^{(2)}(\xi_2) = |p_2|, \text{ and } w^{(3)}(\xi_3) = (|k|^2 + {\mathrm{m}_W}^2)^{\frac{1}{2}} \ .$$

The interaction $H_{W,I}$ is described in terms of annihilation and creation operators together with kernels $G^{(\alpha)}_{\epsilon,\epsilon'}(.,.,.)$ $(\alpha = 1, 2)$.

As emphasized in the introduction, each kernel $G^{(\alpha)}_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3)$, computed in theoretical physics, contains a δ-distribution because of the conservation of the momentum (see [30], [49, section 4.4]). Here, we approximate the singular kernels by square integrable functions. Therefore, we assume the following

Hypothesis 2.1. *For $\alpha = 1, 2$, $\epsilon, \epsilon' = \pm$, we assume*

$$G^{(\alpha)}_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) \in L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2) \ . \tag{2.3}$$

Based on [30, p159, (4.139)] and [50, p308, (21.3.20)], we define the interaction as

$$H_{W,I} = H^{(1)}_{W,I} + H^{(2)}_{W,I} \ , \tag{2.4}$$

where

$$\begin{aligned} H^{(1)}_{W,I} = &\sum_{\epsilon\neq\epsilon'} \int G^{(1)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) b^*_\epsilon(\xi_1) c^*_{\epsilon'}(\xi_2) a_\epsilon(\xi_3)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \\ &+ \sum_{\epsilon\neq\epsilon'} \int \overline{G^{(1)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} a^*_\epsilon(\xi_3) c_{\epsilon'}(\xi_2) b_\epsilon(\xi_1)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \ , \end{aligned} \tag{2.5}$$

$$\begin{aligned} H^{(2)}_{W,I} = &\sum_{\epsilon\neq\epsilon'} \int G^{(2)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) b^*_\epsilon(\xi_1) c^*_{\epsilon'}(\xi_2) a^*_\epsilon(\xi_3) \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \\ &+ \sum_{\epsilon\neq\epsilon'} \int \overline{G^{(2)}_{\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} a_\epsilon(\xi_3) c_{\epsilon'}(\xi_2) b_\epsilon(\xi_1)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \ . \end{aligned} \tag{2.6}$$

The operator $H^{(1)}_{W,I}$ describes the decay of the bosons W^+ and W^- into leptons, and $H^{(2)}_{W,I}$ is responsible for the fact that the bare vacuum will not be an eigenvector of the total Hamiltonian, as expected from physics.

All terms in $H^{(1)}_{W,I}$ and $H^{(2)}_{W,I}$ are well defined as quadratic forms on the set of finite particle states consisting of smooth wave functions. According to [41, Theorem X.24] (see details in [13]), one can construct a closed operator associated with the quadratic form defined by (2.4)-(2.6).

The total Hamiltonian is thus ($g \in \mathbb{R}$ is a coupling constant),

$$H_W = H_{W,0} + g H_{W,I}.$$

2.2 Limiting absorption principle and spectral properties

We begin with a basic self-adjointness property.

Theorem 2.2 (Self-adjointness)**.** *Let $g_1 > 0$ be such that*

$$\frac{6g_1^2}{\mathrm{m}_W}\left(\frac{1}{\mathrm{m}_e{}^2}+1\right)\sum_{\alpha=1,2}\sum_{\epsilon\neq\epsilon'} \|G^{(\alpha)}_{\epsilon,\epsilon'}\|^2_{L^2(\Sigma_1\times\Sigma_1\times\Sigma_2)} < 1 \ .$$

Then, for every g satisfying $|g| \leq g_1$, H_W is a self-adjoint operator in $\mathfrak{F}$ with domain $\mathcal{D}(H_W) = \mathcal{D}(H_{W,0})$.

Ideas of the proof. The proof of this result is a trivial consequence of the following norm relative boundedness of $H_{W,I}$ with respect to $H_{W,0}$.

Lemma 2.3. *For any $\eta > 0$, $\beta > 0$, and $\psi \in \mathcal{D}(H_{W,0})$, we have*

$$\begin{aligned}
&\|H_{W,I}\psi\| \\
&\leq 6 \sum_{\alpha=1,2} \sum_{\epsilon,\epsilon'} \|G^{(\alpha)}_{\epsilon,\epsilon'}\|^2 \left(\frac{1}{2\mathrm{m}_W} \left(\frac{1}{{\mathrm{m}_e}^2} + 1 \right) + \frac{\beta}{2\mathrm{m}_W {\mathrm{m}_e}^2} + \frac{2\eta}{{\mathrm{m}_e}^2}(1+\beta) \right) \|H_{W,0}\psi\|^2 \\
&+ \left(\frac{1}{2\mathrm{m}_W} \left(1 + \frac{1}{4\beta} \right) + 2\eta \left(1 + \frac{1}{4\beta} \right) + \frac{1}{2\eta} \right) \|\psi\|^2 .
\end{aligned} \tag{2.7}$$

Such a relative bound is obtained by using N_τ estimates of [21]. Details can be found in [13] and [4]. □

For the sequel, we shall make some of the following additional assumptions on the kernels $G^{(\alpha)}_{\epsilon,\epsilon'}$.

Hypothesis 2.4. *There exists $\tilde{K}(G) < \infty$ and $\widetilde{\tilde{K}}(G) < \infty$ such that for $\alpha = 1, 2$, $\epsilon, \epsilon' = \pm$, $i, j = 1, 2, 3$, and $\sigma \geq 0$,*

$$\textit{(i)} \quad \int_{\Sigma_1 \times \Sigma_1 \times \Sigma_2} \frac{|G^{(\alpha)}_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3)|^2}{|p_2|^2} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 < \infty \ ,$$

$$\textit{(ii)} \quad \left(\int_{\Sigma_1 \times \left(\{|p_2| \leq \sigma\} \times \{-\frac{1}{2}, \frac{1}{2}\} \right) \times \Sigma_2} |G^{(\alpha)}_{\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \right)^{\frac{1}{2}} \leq \tilde{K}(G)\, \sigma \ ,$$

(iii-a) $(p_2 \cdot \nabla_{p_2}) G^{(\alpha)}_{\epsilon,\epsilon'}(.,.,.) \in L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2)$ *and*

$$\int_{\Sigma_1 \times \left(\{|p_2| \leq \sigma\} \times \{-\frac{1}{2}, \frac{1}{2}\} \right) \times \Sigma_2} \left| [(p_2 \cdot \nabla_{p_2}) G^{(\alpha)}_{\epsilon,\epsilon'}](\xi_1, \xi_2, \xi_3) \right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 < \widetilde{\tilde{K}}(G)\, \sigma,$$

$$\textit{(iii-b)} \quad \int_{\Sigma_1 \times \Sigma_1 \times \Sigma_2} p_{2,i}^2\, p_{2,j}^2 \left| \frac{\partial^2 G^{(\alpha)}_{\epsilon,\epsilon'}}{\partial p_{2,i} \partial p_{2,j}}(\xi_1, \xi_2, \xi_3) \right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 < \infty \ .$$

Remark 2.5. *Note that obviously, Hypothesis 2.4 (i) is stronger than Hypothesis 2.4 (ii).*

Our first main result is the existence of a ground state for H_W together with the location of the spectrum of H_W.

Theorem 2.6 (Existence of a ground state and location of the spectrum)**.**

Assume that the kernels $G^{(\alpha)}_{\epsilon,\epsilon'}$ satisfy Hypothesis 2.1 and 2.4(i). Then, there exists $g_2 \in (0,\, g_1]$ such that H_W has a unique ground state for $|g| < g_2$. Moreover, for

$$E = \inf \mathrm{Spec}(H_W) \,,$$

we have $E \leq 0$ and the spectrum of H_W fulfils

$$\mathrm{Spec}(H_W) = [E,\, \infty).$$

Ideas of the proof. The main ingredients of the proof of the existence of a ground state are the construction of infrared-cutoff operators and the existence of a gap above the ground state energy for these operators (see [13, Proposition 3.5]). This is an adaptation to our case of techniques due to Pizzo [40] and Bach, Fröhlich and Pizzo [5]. The details can be found in [13]. A different proof of the existence of a ground state can also be achieved by mimicking the proof given in [8].

The location of the spectrum follows from the existence of asymptotic Fock representations for the CAR associated with the neutrino creation and annihilation operators (see [34], [46] and [13]). □

Our next main result deals with the absolute continuity of the spectrum and local energy decay. Such a result is established using *standard Mourre theory*, and is a consequence of a limiting absorption principle. To state this result, we need to introduce the definition of the neutrino position operator B.

Let b be the operator in $L^2(\Sigma_1)$ accounting for the position of the neutrino

$$b = i\nabla_{p_2} \ ,$$

and let

$$\langle b\rangle = (1+|b|^2)^{\frac{1}{2}} \ .$$

Its second quantized version $\mathrm{d}\Gamma(\langle b\rangle)$ is self-adjoint in $\mathfrak{F}_a(L^2(\Sigma_1))$. We thus define on $\mathfrak{F} = \mathfrak{F}_L \otimes \mathfrak{F}_W$ the position operator B for neutrinos and antineutrinos by

$$B = (\mathbb{1}\otimes\mathbb{1}\otimes \mathrm{d}\Gamma(\langle b\rangle)\otimes\mathbb{1})\otimes \mathbb{1}_{\mathfrak{F}_W} \ + \ (\mathbb{1}\otimes\mathbb{1}\otimes\mathbb{1}\otimes \mathrm{d}\Gamma(\langle b\rangle))\otimes \mathbb{1}_{\mathfrak{F}_W} .$$

We are now ready to state the main result concerning spectral and dynamical properties of H_W above the ground state energy. Note that the main achievement of Theorem 2.7 is to be able to prove absolute continuity of the spectrum and local energy decay down to the ground state energy *without* assuming any infrared regularization.

Theorem 2.7 (Absolutely continuous spectrum, Limiting Absorption Principle and Local Energy Decay)**.** *Assume that the kernels $G^{(\alpha)}_{\epsilon,\epsilon'}$ satisfy Hypothesis 2.1 and 2.4 (ii)-(iii). For any $\delta > 0$ satisfying $0 < \delta < \mathrm{m}_e$, there exists $g_\delta > 0$ such that for $0 < |g| < g_\delta$:*

(i) The spectrum of H_W in $(E,\, E+\mathrm{m}_e-\delta]$ is purely absolutely continuous.

(ii) For $s > 1/2$, $\varphi \in \mathfrak{F}$, and $\psi \in \mathfrak{F}$, the limits

$$\lim_{\epsilon\to 0}(\varphi,\, \langle B\rangle^{-s}(H_W - \lambda \pm i\epsilon)\langle B\rangle^{-s}\psi)$$

exist uniformly for λ in every compact subset of $(E,\, E+\mathrm{m}_e-\delta)$.

(iii) For $s \in (1/2,\, 1)$ and $f \in C_0^\infty\left((E,\, E+\mathrm{m}_e-\delta)\right)$, we have

$$\left\|(B+1)^{-s}\mathrm{e}^{-itH_W} f(H_W)(B+1)^{-s}\right\| = \mathcal{O}\big(t^{-(s-1/2)}\big) .$$

Ideas of the proof. The main problem we face is that the bottom of the spectrum E is a threshold of the total Hamiltonian H_W by our choice of the conjugate operator. This renders the analysis of the spectrum and of the dynamics close to E difficult. To overcome this difficulty, it is not possible to adapt the proof of Fröhlich, Griesemer and Sigal [19] used in the context of nonrelativistic QED, since in [19] it is possible to regularize the infrared behavior of the interaction by using a unitary Pauli-Fierz transformation that has no equivalent for our model. Instead, to circumvent infrared difficulties, and to avoid infrared regularization of [13], we adapt to our context the proof of [14] established for a model of non-relativistic QED for a free electron at fixed total momentum. Due to the complicated structure of their interaction operator, the authors in [14] used some Feshbach-Schur map before proving a Mourre estimate for an effective Hamiltonian. Here, thanks to some specific estimates that we can derive for our model, we do not need to apply such a map, and we obtain a Mourre estimate directly for H_W.

The main steps of the proof are as follows (details can be found in [4]):

The regularity assumptions Hypothesis 2.4 (iii-a) and (iii-b) on the kernels allow us to establish a Mourre estimate (Theorem 2.8) and a limiting absorption principle (Theorem 2.10) for any spectral interval down to the energy of the ground state and below the mass of the electron. Hence, the whole spectrum between the ground state and the first threshold is purely absolutely continuous.

To prove Theorems 2.10 and 2.8, we first approximate the total Hamiltonian H_W by a cutoff Hamiltonian $H_{W,\sigma}$ with the property that the interaction between the massive particles and the neutrinos or antineutrinos of energies $\leq \sigma$ has been suppressed. We denote by H_W^σ the restriction of $H_{W,\sigma}$ to the Fock space for the massive particles together with the neutrinos and antineutrinos of energies $\geq \sigma$. Then, as in [13], adapting the method of [5], we prove that for some suitable sequence $\sigma_n \to 0$, the Hamiltonian $H_W^{\sigma_n}$ has a gap of size $\sim \sigma_n$ in its spectrum above its ground state energy for all $n \in \mathbb{N}$.

Thus, we use this gap property in combination with the conjugate operator method developed in [3] and [44] in order to establish a Mourre estimate for a sequence of energy intervals $(\Delta_n)_{n\geq 0}$ smaller and smaller, accumulating at the ground state energy of H_W, and covering the interval $(E,\, E + \mathrm{m}_e - \delta)$. This requires to build up a sequence $(A_n^{(\tau)})_{n\geq 0}$ of generators that only affects the massless particles of low energies. For each n, the *self-adjoint conjugate operators* $A_n^{(\tau)}$ is the generator of dilatations in the Fock space for neutrinos and antineutrinos *with a cutoff in the momentum variable*, and is defined as follows.

Set $\tau := 1 - \delta/(2(2\mathrm{m}_e - \delta))$, $\gamma := 1 - \delta/(2\mathrm{m}_e - \delta)$ and define $\chi^{(\tau)} \in C^\infty(\mathbb{R},\, [0,1])$ as

$$\chi^{(\tau)}(\lambda) = \begin{cases} 1 & \text{for } \lambda \in (-\infty,\, \tau]\,, \\ 0 & \text{for } \lambda \in [1,\, \infty)\,. \end{cases}$$

For the sequence of small neutrino momentum cutoffs $(\sigma_n)_{n\geq 0}$ given by $\sigma_0 = 2\mathrm{m}_e + 1$, $\sigma_1 = \mathrm{m}_e - \delta/2$ and for $n \geq 1$, $\sigma_{n+1} = \gamma\sigma_n$, we define, for

all $p_2 \in \mathbb{R}^3$ and $n \geq 1$,

$$\chi_n^{(\tau)}(p_2) = \chi^{(\tau)}\left(\frac{|p_2|}{\sigma_n}\right).$$

The one-particle (neutrino) conjugate operator is

$$a_n^{(\tau)} = \chi_n^{(\tau)}(p_2)\frac{1}{2}\left(p_2 \cdot i\nabla_{p_2} + i\nabla_{p_2} \cdot p_2\right)\chi_n^{(\tau)}(p_2),$$

and its second quantized version is

$$A_n^{(\tau)} = \mathbb{1} \otimes \mathrm{d}\Gamma(a_n^{(\tau)}) \otimes \mathbb{1}, \tag{2.8}$$

where, as above, $\mathrm{d}\Gamma(\cdot)$ refers to the usual second quantization of one particle operators. We also set

$$\langle A_n^{(\tau)} \rangle = (1 + (A_n^{(\tau)})^2)^{\frac{1}{2}} \ .$$

The operators $a_n^{(\tau)}$ and $A_n^{(\tau)}$ are self-adjoint.

Let $(\Delta_n)_{n\geq 0}$ be a sequence of open sets covering any compact subset of $(\inf \mathrm{Spec}(H_W), \mathrm{m}_e - \delta)$ be defined as $\Delta_n := [(\gamma - \epsilon_\gamma)^2\sigma_n, (\gamma + \epsilon_\gamma)\sigma_n]$, where $\epsilon_\gamma > 0$ is fixed and small enough.

Using the spectral gap result for H^{σ_n}, relative bounds as in Lemma 2.3 and Helffer-Sjöstrand calculus (see details in [4, § 5]), we obtain

Theorem 2.8 (Mourre inequality)**.** *Suppose that the kernels $G^{(\alpha)}_{\epsilon,\epsilon'}$ satisfy Hypothesis 2.1, 2.4(ii), and 2.4(iii.a). Then, there exists $C_\delta > 0$ and $g_\delta > 0$ such that, for $|g| < g_\delta$ and $n \geq 1$,*

$$E_{\Delta_n}(H_W - E)\,[H_W, iA_n^{(\tau)}]\,E_{\Delta_n}(H_W - E) \geq C_\delta \frac{\gamma^2}{N^2}\,\sigma_n\,E_{\Delta_n}(H_W - E)\,. \tag{2.9}$$

Then we establish a regularity result of H_W with respect to the conjugate operator $A_n^{(\tau)}$.

Theorem 2.9 ($C^2(A_n^{(\tau)})$-regularity)**.** *Suppose that the kernels $G^{(\alpha)}_{\epsilon,\epsilon'}$ satisfy Hypothesis 2.1 and Hypothesis 2.4(iii). Then, H_W is locally of class $C^2(A_n^{(\tau)})$ in $(-\infty, \mathrm{m}_e - \delta/2)$ for every $n \geq 1$.*

The proof of this result is a straightforward adaptation of [13, Theorem 3.7], substituting there A by $A_n^{(\tau)}$.

Now, according to Theorems 0.1 and 0.2 in [44] (see also [28], [25], and [19]), the $C^2(A_n^{(\tau)})$-regularity in Theorem 2.9 and the Mourre inequality in Theorem 2.8 imply the following limiting absorption principle for sufficiently small coupling constants.

Theorem 2.10 (Limiting absorption principle)**.** *Suppose that the kernels $G^{(\alpha)}_{\epsilon,\epsilon'}$ satisfy Hypothesis 2.1, 2.4 (ii), and 2.4 (iii). Then, for any $\delta > 0$ satisfying $0 < \delta < \mathrm{m}_e/2$, there exists $g_\delta > 0$ such that, for $|g| < g_\delta$, for $s > 1/2$, φ, $\psi \in \mathfrak{F}$ and for $n \geq 1$, the limits*

$$\lim_{\epsilon\to 0}(\varphi,\, \langle A_n^{(\tau)}\rangle^{-s}(H_W - \lambda \pm i\epsilon)\langle A_n^{(\tau)}\rangle^{-s}\psi)$$

exist uniformly for $\lambda \in \Delta_n$. *Moreover, for* $1/2 < s < 1$, *the map*

$$\lambda \mapsto \langle A_n^{(\tau)} \rangle^{-s} (H_W - \lambda \pm i0)^{-1} \langle A_n^{(\tau)} \rangle^{-s}$$

is Hölder continuous of order $s - 1/2$ *in* Δ_n.

Eventually, the proof of Theorem 2.7 is a direct consequence of the limiting absorption principle. The absolutely continuous spectrum is deduced from [44, Theorem 0.1 and Theorem 0.2], and the dynamical properties are derived in the usual way. □

3 Interaction of the gauge boson Z^0 with an electron and a positron

In this section, we do the spectral analysis for the Hamiltonian associated to the decay of the vector boson Z^0 into electrons and positrons,

$$Z^0 \to e^- + e^+ .$$

The interaction between the electrons/positrons and the vector bosons Z^0, in the Schrödinger representation is given, up to coupling constant, by (see [30, (4.139)] and [50, (21.3.20)])

$$I_{Z^0} = \int \overline{\Psi_e}(x) \gamma^\alpha (g'_V - \gamma_5) \Psi_e(x) Z_\alpha(x) \, \mathrm{d}x \, + h.c., \tag{3.1}$$

where γ^α, $\alpha = 0, 1, 2, 3$, and γ_5 are the Dirac matrices, g'_V is a real parameter such that $g'_V \simeq 0,074$ (see e.g [30]), $\Psi_e(x)$ and $\overline{\Psi_e}(x)$ are the Dirac fields for the electron e_- and the positron e_+ of mass m_e, and Z_α is the massive boson field for Z^0.

The field $\Psi_e(x)$ is formally defined by

$$\Psi_e(x) = \int \psi_+(\xi, x) b_+(\xi) + \widetilde{\psi}_-(\xi, x) b_-^*(\xi) \, \mathrm{d}\xi,$$

with

$$\widetilde{\psi}_-(\xi, x) = \widetilde{\psi}_-((p, \gamma), x) = \psi_-((p, (j, -m_j, -\kappa_j)), x) \, . \tag{3.2}$$

and where $\psi_\pm(\xi, x)$ are the generalized eigenfunctions associated with the continuous spectrum of the free Dirac operator labeled by the total angular momentum quantum numbers j and m_j, and the quantum numbers κ_j.

The boson field Z_α is formally defined by (see e.g. [49, Eq. (5.3.34)]),

$$\begin{aligned} &Z_\alpha(x) \\ &= (2\pi)^{-\frac{3}{2}} \int \frac{\mathrm{d}\xi_3}{(2(|k|^2 + \mathrm{m}_{Z^0}{}^2)^{\frac{1}{2}})^{\frac{1}{2}}} \Big(\epsilon_\alpha(k, \lambda) a(\xi_3) \mathrm{e}^{ik.x} + \epsilon_\alpha^*(k, \lambda) a^*(\xi_3) \mathrm{e}^{-ik.x} \Big), \end{aligned}$$

where the vectors $\epsilon_\alpha(k,\lambda)$ are the polarizations vectors of the massive spin 1 bosons (see [49, Section 5.3]), and with $\xi_3 = (k,\, \lambda)$, where $k \in \mathbb{R}^3$ is the momentum variable of the boson and $\lambda \in \{-1, 0, 1\}$ is its polarization.

If one considers, as mentioned in the introduction, the full interaction I_{Z^0} in (3.1) describing the decay of the gauge boson Z^0 into massive leptons and if one formally expands this interaction with respect to products of creation and annihilation operators, we are left with a finite sum of terms with kernels yielding singular operators which cannot be defined as closed operators. Therefore, in order to obtain a well-defined Hamiltonian (see e.g [21, 7, 8, 13, 4]), we replace these kernels by square integrable functions $G^{(\alpha)}$. In particular, this implies large momentum cutoffs for the electrons, positrons and Z^0 bosons. Moreover, we confine in space the interaction between the electrons/positrons and the bosons by adding a localization function $f(|x|)$, with $f \in \mathrm{C}_0^\infty([0,\infty))$.

3.1 Rigorous definition of the model

3.1.1 The Fock spaces for electrons, positrons and Z^0 bosons

In order to properly define the interaction I_{Z^0} formally introduced above, since we use a spectral representation of the free Dirac operator generated by the sequence of spherical waves, we first recall a few facts about solutions of the free Dirac equation.

The energy of a free relativistic electron of mass m_e is described by the self-adjoint Dirac Hamiltonian

$$H_D = \boldsymbol{\alpha} \cdot \frac{1}{i}\nabla + \boldsymbol{\beta}\, \mathrm{m}_e,$$

(see [42, 47] and references therein) acting on the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^3; \mathbb{C}^4)$, with domain $\mathfrak{D}(H_D) = H^1(\mathbb{R}^3; \mathbb{C}^4)$. We use the system of units $\hbar = c = 1$. Here $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ and $\boldsymbol{\beta}$ are the Dirac matrices in the standard form.

The generalized eigenfunctions associated with the continuous spectrum of the Dirac operator H_D are labeled by the total angular momentum quantum numbers

$$j \in \big\{\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \dots\big\}, \qquad m_j \in \{-j, -j+1, \dots, j-1, j\}, \tag{3.3}$$

and by the quantum numbers

$$\kappa_j \in \big\{ \pm (j + \frac{1}{2})\big\}. \tag{3.4}$$

In the sequel, we will drop the index j and set

$$\gamma = (j,\, m_j,\, \kappa_j)\,, \tag{3.5}$$

and a sum over γ will thus denote a sum over j, m_j and κ_j. We denote by Γ the set $\{(j,\, m_j,\, \kappa_j),\, j \in \mathbb{N} + \frac{1}{2},\, m_j \in \{-j, -j+1, \dots, j-1, j\},\, \kappa_j \in \{\pm(j+\frac{1}{2})\}\}$.

For $\mathbf{p} \in \mathbb{R}^3$ being the momentum of the electron, and $p := |\mathbf{p}|$, the continuum energy levels are given by $\pm\,\omega(p)$, where

$$\omega(p) := (\mathrm{m_e}^2 + p^2)^{\frac{1}{2}}\,. \tag{3.6}$$

We introduce the notation

$$\xi = (p,\,\gamma)\ \in \mathbb{R}_+ \times \Gamma. \tag{3.7}$$

The continuum eigenstates of H_D are denoted by

$$\psi_\pm(\xi, x) = \psi_\pm((p,\gamma), x)\,.$$

We then have

$$H_D\ \psi_\pm((p,\gamma), x) = \pm\,\omega(p)\ \psi_\pm((p,\gamma), x).$$

The generalized eigenstates $\psi_\pm$ are normalized in such a way that

$$\begin{aligned}
\int_{\mathbb{R}^3} \psi_\pm^\dagger((p,\gamma), x)\,\psi_\pm((p',\gamma'), x)\,\mathrm{d}x &= \delta_{\gamma\gamma'}\delta(p-p'),\\
\int_{\mathbb{R}^3} \psi_\pm^\dagger((p,\gamma), x)\,\psi_\mp((p',\gamma'), x)\,\mathrm{d}x &= 0\,.
\end{aligned}$$

Here $\psi_\pm^\dagger((p,\gamma),\,x)$ is the adjoint spinor of $\psi_\pm((p,\gamma),\,x)$.

According to the hole theory [42, 43, 47, 49], the absence in the Dirac theory of an electron with energy $E < 0$ and charge e is equivalent to the presence of a positron with energy $-E > 0$ and charge $-e$.

Let us split the Hilbert space $\mathfrak{H} = L^2(\mathbb{R}^3; \mathbb{C}^4)$ into

$$\mathfrak{H}_{c^-} = P_{(-\infty,-\mathrm{m}_e]}(H_D)\mathfrak{H} \quad \text{and} \qquad \mathfrak{H}_{c^+} = P_{[\mathrm{m}_e,+\infty)}(H_D)\mathfrak{H}.$$

Here $P_I(H_D)$ denotes the spectral projection of H_D corresponding to the interval I.

Let $\Sigma := \mathbb{R}_+ \times \Gamma$. We can identify the Hilbert spaces $\mathfrak{H}_{c^\pm}$ with

$$\mathfrak{H}_c := L^2(\Sigma; \mathbb{C}^4) \simeq \oplus_\gamma L^2(\mathbb{R}_+; \mathbb{C}^4)\,,$$

by using the unitary operators $U_{c^\pm}$ defined from $\mathfrak{H}_{c^\pm}$ to $\mathfrak{H}_c$ via the identities in the L^2 sense

$$(U_{c^\pm}\phi)(p,\gamma) = \int\ \psi_\pm^\dagger((p,\gamma)\,, x)\,\phi(x)\,\mathrm{d}x\,. \tag{3.8}$$

On $\mathfrak{H}_c$, we define the scalar products

$$(g,\,h) = \int \overline{g(\xi)}h(\xi)\mathrm{d}\xi = \sum_{\gamma\in\Gamma}\int_{\mathbb{R}^+} \overline{g(p,\gamma)}h(p,\gamma)\,\mathrm{d}p\,. \tag{3.9}$$

In the sequel, we shall denote the variable $(p,\,\gamma)$ by $\xi_1 = (p_1, \gamma_1)$ in the case of electrons, and $\xi_2 = (p_2, \gamma_2)$ in the case of positrons, respectively.

We next introduce the Fock space for electrons and positrons.

Let

$$\mathfrak{F}_a := \mathfrak{F}_a(\mathfrak{H}_c) = \bigoplus_{n=0}^{\infty} \otimes_a^n \mathfrak{H}_c,$$

be the Fermi-Fock space over $\mathfrak{H}_c$, and let

$$\mathfrak{F}_D := \mathfrak{F}_a \otimes \mathfrak{F}_a$$

be the Fermi-Fock space for electrons and positrons, with vacuum Ω_D.

The creation and annihilation operators for electrons and positrons are defined as follows

We set, for every $g \in \mathfrak{H}_c$,

$$b_{\gamma,+}(g) = b_+(P_\gamma g)\,, \quad b^*_{\gamma,+}(g) = b^*_+(P_\gamma g)\,,$$

where P_γ is the projection of $\mathfrak{H}_c$ onto the γ-th component defined according to (3.8), and $b_+(P_\gamma g)$ and $b^*_+(P_\gamma g)$ are respectively the annihilation and creation operator for an electron.

As above, we set, for every $h \in \mathfrak{H}_c$,

$$\begin{aligned} b_{\gamma,-}(h) &= b_-(P_\gamma h)\,, \\ b^*_{\gamma,-}(h) &= b^*_-(P_\gamma h)\,, \end{aligned}$$

where $b_-(P_\gamma g)$ and $b^*_-(P_\gamma g)$ are respectively the annihilation and creation operator for a positron.

As in [41, Chapter X], we introduce operator-valued distributions $b_\pm(\xi)$ and $b^*_\pm(\xi)$ that fulfill for $g \in \mathfrak{H}_c$,

$$\begin{aligned} b_\pm(g) &= \int b_\pm(\xi)\, \overline{(P_\gamma g)\,(p)}\, \mathrm{d}\xi \\ b^*_\pm(g) &= \int b^*_{\gamma,\pm}(p)\, (P_\gamma g)\,(p)\, \mathrm{d}\xi \end{aligned}$$

where we used the notation of (3.9).

We give here the construction of the Fock space for the Z^0 boson.

Let

$$\Sigma_3 := \mathbb{R}^3 \times \{-1,\, 0,\, 1\}\,.$$

The one-particle Hilbert space for the particle Z^0 is $L^2(\Sigma_3)$ with scalar product

$$(f,\, g) = \int_{\Sigma_3} \overline{f(\xi_3)} g(\xi_3) \mathrm{d}\xi_3\,, \tag{3.10}$$

with the notations

$$\xi_3 = (k,\, \lambda) \quad \text{and} \quad \int_{\Sigma_3} \mathrm{d}\xi_3 = \sum_{\lambda=-1,0,1} \int_{\mathbb{R}^3} \mathrm{d}k\,, \tag{3.11}$$

where $\xi_3 = (k,\, \lambda) \in \Sigma_3$.

The bosonic Fock space for the vector boson Z^0, denoted by $\mathfrak{F}_{Z^0}$, is thus the symmetric Fock space

$$\mathfrak{F}_{Z^0} = \mathfrak{F}_s(L^2(\Sigma_3))\,. \tag{3.12}$$

For $f \in L^2(\Sigma_3)$, we define the annihilation and creation operators, denoted by $a(f)$ and $a^*(f)$ by

$$a(f) = \int_{\Sigma_3} \overline{f(\xi_3)} a(\xi_3) \mathrm{d}\xi_3 \tag{3.13}$$

and

$$a^*(f) = \int_{\Sigma_3} f(\xi_3) a^*(\xi_3) \mathrm{d}\xi_3 \tag{3.14}$$

where the operators $a(\xi_3)$ (respectively $a^*(\xi_3)$) are the bosonic annihilation (respectively bosonic creation) operator for the boson Z^0 (see e.g [36, 12, 13]).

3.1.2 The Hamiltonian

The quantization of the Dirac Hamiltonian H_D, acting on $\mathfrak{F}_D$, is given by

$$T_D = \int \omega(p)\, b_+^*(\xi_1)\, b_+(\xi_1) \mathrm{d}\xi_1 \;+\; \int \omega(p)\, b_-^*(\xi_2)\, b_-(\xi_2) \mathrm{d}\xi_2,$$

with $\omega(p)$ given in (3.6). The operator T_D is the Hamiltonian of the quantized Dirac field.

Let $\mathfrak{D}_D$ denote the set of vectors $\Phi \in \mathfrak{F}_D$ for which $\Phi^{(r,s)}$ is smooth and has a compact support and $\Phi^{(r,s)} = 0$ for all but finitely many (r, s). Then T_D is well-defined on the dense subset $\mathfrak{D}_D$ and it is essentially self-adjoint on $\mathfrak{D}_D$. The self-adjoint extension will be denoted by the same symbol T_D, with domain $\mathfrak{D}(T_D)$.

The operators number of electrons and number of positrons, denoted respectively by N_+ and N_-, are given by

$$N_+ = \int b_+^*(\xi_1)\, b_+(\xi_1) \mathrm{d}\xi_1 \quad \text{and} \quad N_- = \int b_-^*(\xi_2)\, b_-(\xi_2) \mathrm{d}\xi_2\,. \tag{3.15}$$

They are essentially self-adjoint on $\mathfrak{D}_D$.

We have

$$\mathrm{Spec}(T_D) = \{0\} \cup [\mathrm{m}_e, \infty).$$

The set $[\mathrm{m}_e, \infty)$ is the absolutely continuous spectrum of T_D.

The Hamiltonian of the bosonic field, acting on $\mathfrak{F}_{Z^0}$, is

$$T_Z := \int \omega_3(k)\, a^*(\xi_3) a(\xi_3)\, \mathrm{d}\xi_3$$

where

$$\omega_3(k) = \sqrt{|k|^2 + {\mathrm{m}_{Z^0}}^2}. \tag{3.16}$$

The operator T_Z is essentially self-adjoint on the set of vectors $\Phi \in \mathfrak{F}_{Z^0}$ such that $\Phi^{(n)}$ is smooth and has compact support and $\Phi^{(n)} = 0$ for all but finitely many n. Its self-adjoint extension is denoted by the same symbol.

The spectrum of T_Z consists of an absolutely continuous spectrum covering $[\mathrm{m}_{Z^0}, \infty)$ and a simple eigenvalue, equal to zero, whose corresponding eigenvector is the vacuum state $\Omega_s \in \mathfrak{F}_{Z^0}$.

The free Hamiltonian is defined on $\mathcal{H} := \mathfrak{F}_D \otimes \mathfrak{F}_{Z^0}$ by

$$H_{Z,0} = T_D \otimes \mathbb{1} + \mathbb{1} \otimes T_Z \,. \tag{3.17}$$

The operator $H_{Z,0}$ is essentially self-adjoint on $\mathfrak{D}(T_D) \otimes \mathfrak{D}(T_Z)$. Since $\mathrm{m}_e < \mathrm{m}_{Z^0}$, the spectrum of $H_{Z,0}$ is given by

$$\mathrm{Spec}(H_{Z,0}) = \{0\} \cup [\mathrm{m}_e, \infty) \,.$$

More precisely,

$$\mathrm{Spec}_{\mathrm{pp}}(H_{Z,0}) = \{0\}, \quad \mathrm{Spec}_{\mathrm{sc}}(H_{Z,0}) = \emptyset, \quad \mathrm{Spec}_{\mathrm{ac}}(H_{Z,0}) = [\mathrm{m}_e, \infty), \tag{3.18}$$

where $\mathrm{Spec}_{\mathrm{pp}}$, $\mathrm{Spec}_{\mathrm{sc}}$, $\mathrm{Spec}_{\mathrm{ac}}$ denote the pure point, singular continuous and absolutely continuous spectra, respectively. Furthermore, 0 is a non-degenerate eigenvalue associated to the vacuum $\Omega_D \otimes \Omega_s$.

The interaction Hamiltonian is defined on $\mathcal{H} = \mathfrak{F}_D \otimes \mathfrak{F}_{Z^0}$ by

$$H_{Z,I} = H_{Z,I}^{(1)} + H_{Z,I}^{(1)^*} + H_{Z,I}^{(2)} + H_{Z,I}^{(2)^*} \,, \tag{3.19}$$

with

$$\begin{aligned} H_{Z,I}^{(1)} = & \int \left(\int_{\mathbb{R}^3} f(|x|) \overline{\psi_+(\xi_1,\, x)} \gamma^\mu (g'_V - \gamma_5) \widetilde{\psi}_-(\xi_2,\, x) \frac{\epsilon_\mu(\xi_3)}{\sqrt{2\omega_3(k)}} \, \mathrm{e}^{ik\cdot x} \, \mathrm{d}x \right) \\ & \times G^{(1)}(\xi_1,\, \xi_2,\, \xi_3) b_+^*(\xi_1) b_-^*(\xi_2) a(\xi_3) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \,, \end{aligned} \tag{3.20}$$

$$\begin{aligned} H_{Z,I}^{(1)^*} = & \int \left(\int_{\mathbb{R}^3} f(|x|) \overline{\widetilde{\psi}_-(\xi_2,\, x)} \gamma^\mu (g'_V - \gamma_5) \psi_+(\xi_1,\, x) \frac{\epsilon_\mu^*(\xi_3)}{\sqrt{2\omega_3(k)}} \, \mathrm{e}^{-ik\cdot x} \, \mathrm{d}x \right) \\ & \times \overline{G^{(1)}(\xi_1,\, \xi_2,\, \xi_3)} a^*(\xi_3) b_-(\xi_2) b_+(\xi_1) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \,, \end{aligned} \tag{3.21}$$

$$\begin{aligned} H_{Z,I}^{(2)} = & \int \left(\int_{\mathbb{R}^3} f(|x|) \overline{\psi_+(\xi_1,\, x)} \gamma^\mu (g'_V - \gamma_5) \widetilde{\psi}_-(\xi_2,\, x) \frac{\epsilon_\mu^*(\xi_3)}{\sqrt{2\omega_3(k)}} \, \mathrm{e}^{-ik\cdot x} \, \mathrm{d}x \right) \\ & \times G^{(2)}(\xi_1,\, \xi_2,\, \xi_3) b_+^*(\xi_1) b_-^*(\xi_2) a^*(\xi_3) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \,, \end{aligned} \tag{3.22}$$

and

$$\begin{aligned} H_{Z,I}^{(2)^*} = & \int \left(\int_{\mathbb{R}^3} f(|x|) \overline{\widetilde{\psi}_-(\xi_2,\, x)} \gamma^\mu (g'_V - \gamma_5) \psi_+(\xi_1,\, x) \frac{\epsilon_\mu(\xi_3)}{\sqrt{2\omega_3(k)}} \, \mathrm{e}^{ik\cdot x} \, \mathrm{d}x \right) \\ & \times \overline{G^{(2)}(\xi_1,\, \xi_2,\, \xi_3)} a(\xi_3) b_-(\xi_2) b_+(\xi_1) \, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \,. \end{aligned} \tag{3.23}$$

Performing the integration with respect to x in the expressions above, we see that $H^{(1)}_{Z,I}$ and $H^{(2)}_{Z,I}$ can be written under the form

$$\begin{aligned} H^{(1)}_{Z,I} &:= H^{(1)}_{Z,I}(F^{(1)}) := \int F^{(1)}(\xi_1,\, \xi_2,\, \xi_3) b^*_+(\xi_1) b^*_-(\xi_2) a(\xi_3)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\,, \\ H^{(2)}_{Z,I} &:= H^{(2)}_{Z,I}(F^{(2)}) := \int F^{(2)}(\xi_1,\, \xi_2,\, \xi_3) b^*_+(\xi_1) b^*_-(\xi_2) a^*(\xi_3)\, \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3\,, \end{aligned} \tag{3.24}$$

where, for $\alpha = 1, 2$,

$$F^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3) := h^{(\alpha)}(\xi_1, \xi_2, \xi_3) G^{(\alpha)}(\xi_1, \xi_2, \xi_3), \tag{3.25}$$

and $h^{(1)}(\xi_1, \xi_2, \xi_3)$, $h^{(2)}(\xi_1, \xi_2, \xi_3)$ are given by the integral over x in (3.20) and (3.22), respectively.

Our main result, Theorem 3.5 below, requires the functions $F^{(\alpha)}$ to be sufficiently regular near $p_1 = 0$ and $p_2 = 0$ (where, recall, $\xi_l = (p_l, \gamma_l)$ for $l = 1, 2$).

Note that this regularity is required for applying the conjugate operator method. In practice, starting from the physical (ill-defined) Hamiltonian, applying UV cutoffs $G^{(\alpha)}_{\epsilon,\epsilon'}$ and a space localization $f(|x|)$ to the interaction $H_{Z,I}$ as done above, this regularity is fulfilled, except, solely, for the part of the field corresponding to quantum number $j = 1/2$. This is a consequence of a careful analysis of the behavior for momenta p close to zero of the generalized eigenstates $\psi_+(\xi,\, x) = \psi_+((p, (j, m_j, \kappa_j); x)$ and their derivatives have a too singular behavior at $\xi = 0$. This analysis is done in [9, Appendix A])

The total Hamiltonian of the decay of the boson Z^0 into an electron and a positron is

$$H_Z := H_{Z,0} + g\, H_{Z,I},$$

where g is a real coupling constant.

3.2 Limiting absorbtion principle and spectral properties

For $p \in \mathbb{R}_+$, $j \in \{\frac{1}{2},\, \frac{3}{2},\, \cdots\}$, $\gamma = (j,\, m_j,\, \kappa_j)$ and $\gamma_j = j + \frac{1}{2}$, we define

$$A(\xi) = A(p, \gamma) := \frac{(2p)^{\gamma_j + 1}}{\Gamma(\gamma_j)} \left(\frac{\omega(p) + \mathrm{m}_e}{\omega(p)} \right)^{\frac{1}{2}} \left(\int_0^\infty |f(r)| r^{2\gamma_j} (1 + r^2) \mathrm{d}r \right)^{\frac{1}{2}}, \tag{3.26}$$

where Γ denotes Euler's Gamma function, and $f \in \mathrm{C}^\infty_0([0, \infty))$ is the localization function appearing in (3.20)–(3.23). We make the following hypothesis on the kernels $G^{(\alpha)}$.

Hypothesis 3.1. *For $\alpha = 1, 2$,*

$$\int A(\xi_1)^2 A(\xi_2)^2 (|k|^2 + \mathrm{m}_{Z^0}{}^2)^{\frac{1}{2}} \left| G^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3) \right|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 < \infty. \tag{3.27}$$

Note that up to universal constants, the functions $A(\xi)$ in (3.26) are upper bounds for the integrals with respect to x that occur in (3.20). These bounds are derived using the inequality (see [49, Eq.(5.3.23)-(5.3.25)])

$$\left| \frac{\epsilon_\mu(\xi_3)}{\sqrt{2\omega_3(k)}} \right| \leq C_{\mathrm{m}_{Z^0}} (1+|k|^2)^{\frac{1}{4}} \,. \tag{3.28}$$

For $C_Z := 156\, C_{\mathrm{m}_{Z^0}}$, let us define

$$\begin{aligned} K_1(G^{(\alpha)})^2 &:= {C_Z}^2 \left(\int A(\xi_1)^2 A(\xi_2)^2 \, |G^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3)|^2 \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \right), \\ K_2(G^{(\alpha)})^2 &:= {C_Z}^2 \left(\int A(\xi_1)^2 A(\xi_2)^2 \, |G^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3)|^2 (|k|^2+1)^{\frac{1}{2}} \mathrm{d}\xi_1 \mathrm{d}\xi_2 \mathrm{d}\xi_3 \right). \end{aligned} \tag{3.29}$$

Our fist result is a basic result on self-adjointness.

Theorem 3.2 (Self-adjointness)**.** *Assume that Hypothesis 3.1 holds. Let $g_0 > 0$ be such that*

$$ {g_0}^2 \left(\sum_{\alpha=1,2} K_1(G^{(\alpha)})^2 \right) \left(\frac{1}{{\mathrm{m}_e}^2} + 1 \right) < 1 \,. \tag{3.30}$$

Then for any real g such that $|g| \leq g_0$, the operator $H_Z = H_{Z,0} + g H_{Z,I}$ is self-adjoint with domain $\mathfrak{D}(H_{Z,0})$. Moreover, any core for $H_{Z,0}$ is a core for H_Z.

Notice that combining (3.18), relative boundedness of $H_{Z,I}$ with respect to $H_{Z,0}$ and standard perturbation theory of isolated eigenvalues (see e.g. [37]), we deduce that, for $|g| \ll \mathrm{m}_e$, $\inf \mathrm{Spec}(H_Z)$ is a non-degenerate eigenvalue of H_Z. In other words, H_Z admits a unique ground state.

Theorem 3.2 follows from the Kato-Rellich Theorem together with standard estimates of creation and annihilation operators in Fock space, showing that the interaction Hamiltonian $H_{Z,I}$ is relatively bounded with respect to $H_{Z,0}$.

To establish our next theorems, we need to strengthen the conditions on the kernels $G^{(\alpha)}$.

Hypothesis 3.3. *For $\alpha = 1, 2$, the kernels $G^{(\alpha)} \in L^2(\Sigma \times \Sigma \times \Sigma_3)$ satisfy*

(i) There exists a compact set $K \subset \mathbb{R}_+ \times \mathbb{R}_+ \times \mathbb{R}^3$ such that $G^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$ if $(p_1, p_2, k) \notin K$.

(ii) There exists $\varepsilon \geq 0$ such that

$$\sum_{\gamma_1, \gamma_2, \lambda} \int (1 + x_1^2 + x_2^2)^{1+\varepsilon} \left| \hat{G}^{(\alpha)}(x_1, \gamma_1, x_2, \gamma_2, k, \lambda) \right|^2 \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}k < \infty,$$

where $\hat{G}^{(\alpha)}$ denotes the Fourier transform of $G^{(\alpha)}$ with respect to the variables (p_1, p_2), and x_j is the variable dual to p_j.

(iii) If $\gamma_{1j} = 1$ *or* $\gamma_{2j} = 1$*, where for* $l = 1, 2$*,* $\gamma_{lj} = |\kappa_{j_l}|$ *(with* $\gamma_l = (j_l, m_{j_l}, \kappa_{j_l})$*), and if* $p_1 = 0$ *or* $p_2 = 0$*, then* $G^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda) = 0$*.*

Remark. *1) The assumption that* $G^{(\alpha)}$ *is compactly supported in the variables* (p_1, p_2, k) *is an "ultraviolet" constraint that is made for convenience. It could be replaced by the weaker assumption that* $G^{(\alpha)}$ *decays sufficiently fast at infinity.*

2) Hypothesis 3.3 (ii) *comes from the fact that the coupling functions* $G^{(\alpha)}$ *must satisfy some "minimal" regularity for our method to be applied. In fact, Hypothesis* (ii) *could be slightly improved with a refined choice of interpolation spaces in our proof. In Hypothesis 3.3* (iii)*, we need in addition an "infrared" regularization. We remark in particular that Hypotheses* (ii) *and* (iii) *imply that, for* $0 \leq \varepsilon < 1/2$*,*

$$\left|G^{(\alpha)}(p_1, \gamma_1, p_2, \gamma_2, k, \lambda)\right| \lesssim |p_l|^{\frac{1}{2}+\varepsilon}, \quad l = 1, 2.$$

We emphasize, however, that this infrared assumption is required only in the case $\gamma_{lj} = 1$*, that is, for* $j = 1/2$*. For all other* $j \in \mathbb{N} + \frac{1}{2}$*, we do* not *need to impose any infrared regularization on the generalized eigenstates* $\psi_\pm((p, \gamma), x)$*; They are already regular enough.*

3) One verifies that Hypotheses 3.3(i) and 3.3(ii) imply Hypothesis 3.1.

Theorem 3.4 (Location of the spectrum)**.** *Assume that Hypothesis 3.3 holds. There exists* $g_1 > 0$ *such that, for all* $|g| \leq g_1$*,*

$$\mathrm{Spec}(H_Z) = \{\inf \mathrm{Spec}(H_Z)\} \cup [\inf \mathrm{Spec}(H_Z) + \mathrm{m}_e, \infty).$$

In particular, H_Z *has no eigenvalue below its essential spectrum except for the ground state energy,* $\inf \mathrm{Spec}(H_Z)$*, which is an isolated simple eigenvalue.*

We use the Dereziński-Gérard partition of unity [16] in a version that accommodates the Fermi-Dirac statistics and the CAR. Such a partition of unity was used previously in [1] (see [9] for details).

Theorem 3.5 (Absolutely continuous spectrum)**.** *Assume that Hypothesis 3.3 holds with* $\varepsilon > 0$ *in Hypothesis 3.3(ii). For all* $\delta > 0$*, there exists* $g_\delta > 0$ *such that, for all* $|g| \leq g_\delta$*, the spectrum of* H_Z *in the interval*

$$[\inf \mathrm{Spec}(H_Z) + \mathrm{m}_e, \inf \mathrm{Spec}(H_Z) + \mathrm{m}_{Z^0} - \delta]$$

is purely absolutely continuous.

Ideas of the proof. The proof of Theorem 3.5 relies on Mourre positive commutator method. Though, the standard choice of a conjugate operator as the second quantized version of the one electron operator $\frac{1}{2}((\nabla_p \omega \cdot i\nabla_p + i\nabla_p \cdot (\nabla_p \omega))$ fails to give a Mourre estimate near thresholds, already for the free Hamiltonian $H_{Z,0}$.

Hence, we construct a conjugate operator A by following the idea of Hübner and Spohn [35] (see also [23, 24]). As in [35], the operator A is only maximal symmetric, and generates a C_0-semigroup of isometries. Therefore, we need to use Singular Mourre theory with non self-adjoint conjugate operator. Such extensions of the usual conjugate operator theory [38, 3] considered in [35] were later extended in [45] and in [23, 24].

The general strategy remains similar to the one using regular Mourre Theory. We prove regularity of the total Hamiltonian $H_{Z,I}$ with respect to the conjugate operator A. For this sake, we use here real interpolation theory together with a version of the Mourre theory requiring only low regularity of the Hamiltonian with respect to the conjugate operator (see [18] and [9, Appendix B]).

We then establish a Mourre estimate. Formally, our choice of the conjugate operator A yields $[H_{Z,0}, iA] = N_+ + N_-$, where $N_\pm$ are the number operators for electrons and positrons. Since $N_\pm \geq 1$ away from the vacuum, to obtain a strict Mourre inequality, it suffices to control $g[H_{Z,I}, iA]$ for g small enough. This is possible using general relative bounds with respect to $H_{Z,0}$ for perturbations of the form $H_{Z,I}(-iaF^{(\alpha)})$ (see (3.24)), for a denoting the one-particle conjugate operator, and $F^{(\alpha)}$ being the kernels given by (3.25).

Combining the Mourre estimate with a regularity property of the Hamiltonian with respect to the conjugate operator allow us to deduce a Virial theorem and a limiting absorption principle, from which we obtain Theorem 3.5.

Our main achievement consists in proving that the physical interaction Hamiltonian $H_{Z,I}$ is regular enough for the Mourre theory to be applied, except for the terms associated to the "first" generalized eigenstates ($j = 1/2$). For the latter, unfortunately, we need to impose a non-physical infrared condition. □

4 Prospectives

Despite the number of results concerning spectral and dynamical properties for weak interaction Hamiltonians or similar models, [7, 8, 2, 11, 13, 26, 4, 9, 10, 32, 33], the study of weak interactions from a rigorous point of view still requires to be investigated.

We mention here some open problems.

- *Spectral study above the boson thresholds.* To complete the spectral study of the above two models, it remains to prove that the spectrum above the massive bosons ($W^\pm$ or Z^0) thresholds is purely absolutely continuous, as expected for weak interactions models for which there should be no bound states except for the vacuum. Picking a conjugate operator including the massive bosons, i.e., a conjugate operator similar to the one we picked, with an additional term acting on the Bosonic Fock space, the general strategy adopted above is expected to give purely absolutely continuous spectrum away from bosonic thresholds. Near bosonic thresholds, like for instance near $(\inf \mathrm{Spec}(H_Z)) + \mathrm{m}_{Z^0}$ or $(\inf \mathrm{Spec}(H_W)) + \mathrm{m}_W$, we face some infrared problems. To obtain a limiting absorption principle near bosonic

thresholds, it is expected, in the case of Z^0 decay, that one first has to derive local properties of the solutions of the Proca equation for massive spin 1 particles.

- *Weak decay of the intermediate boson Z into neutrinos and antineutrinos* The decay of the Z^0,
$$Z^0 \to \nu_e + \bar{\nu}_e,$$
is apparently very similar to the model studied in Section 3. However, the two fermionic particles created in this process are massless, as described by the Standard Model. From a technical point of view, using conjugate operator theory with non self-adjoint conjugate operator as in Section 3 to prove absolutely continuity of the spectrum of the Hamiltonian H, yields additional difficulties in that case since, unlike for the model treated in Section 3, the commutator $[H, iA]$ is not comparable with H.

- *Decay of muonic atoms.* The decay of a free muon or of a muon in the electromagnetic field of a nucleus always produces more than three particles
$$\mu^- \to \nu_\mu + \bar{\nu}_e + e^-.$$
A natural way to describe this decay in muonic atoms, is to restrict the Fock space for muons to bound states of Dirac-Coulomb. Moreover, to account for high energies involved in this decay, it is sufficient to consider only free electron/positron states.

  The inherent mathematical difficulty is that we have to deal with a process with four fermionic particles, two of which are massless as given by the Standard Model. For this model, technical difficulties arise already for getting a relative bound with respect to the free Hamiltonian for the interaction. Without such a bound, it remains illusory with the current techniques to derive any interesting spectral properties.

- *Model with neutrino mass.* As mentioned in the introduction of Section 2, neutrinos (of the electrons, muons or tauons) have a mass. To account for this, one can add a mass to the neutrino in the model of Section 2. This model already gives interesting mathematical challenges, since the massive fermions "create" thresholds in the spectrum, but the masses of the neutrino are so tiny, that relative bounds can not be used as in Section 2 in the context of usual perturbative theory, unless dealing with interaction with irrelevant coupling constant $g \ll 1$.

  A physically more relevant way to take into account the neutrino mass is the study of Hamiltonians of post Standard Models.

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