One-Boson Scattering Processes in the massive Spin-Boson Model

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Abstract

The Spin-Boson model describes a two-level quantum system that interacts with a second-quantized boson scalar field. Recently the relation between the integral kernel of the scattering matrix and the resonance in this model has been established in [18] for the case of massless bosons. In the present work, we treat the massive case. On the one hand, one might rightfully expect that the massive case is easier to handle since, in contrast to the massless case, the corresponding Hamiltonian features a spectral gap. On the other hand, it turns out that the non-zero boson mass introduces a new complication as the spectrum of the complex dilated, free Hamiltonian exhibits lines of spectrum attached to every multiple of the boson rest mass energy starting from the ground and excited state energies. This leads to an absence of decay of the corresponding complex dilated resolvent close to the real line, which, in [18], was a crucial ingredient to control the time evolution in the scattering regime. With the new strategy presented here, we provide a proof of an analogous formula for the scattering kernel as compared to the massless case and use the opportunity to provide the required spectral information by a Mourre theory argument combined with a suitable application of the Feshbach-Schur map instead of complex dilation.

1 Introduction

The Spin-Boson model is a widely employed model in quantum field theory that describes the interaction between a two-level quantum system and a second-quantized scalar field. The model is interesting as it shares many important features of, e.g., quantum electrodynamics or the Yukawa theory, such as the absence of a gap in the massless case, the appearance of a resonance, and the ultraviolet divergence, which can be studied with mathematical rigor without being obstructed by additional complications, such as dispersion of the sources or additionally spin degrees of freedom of the fields. In the case of a massless scalar field, the Spin-Boson model describes a two-level atom that interacts

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with a photon field and is therefore frequently employed in quantum optics. The unperturbed energies of the two-level system shall be denoted by real numbers $0 = e_0 < e_1$. It is well-known that after switching on the interaction with the second-quantized scalar field that may induce transitions between the two levels, the free ground state energy e_0 is shifted to the interacting ground state energy $e_0 > \lambda_0 \in \mathbb{R}$ on the real line while the free excited state with energy e_1 turns into a resonance with an "energy" $\lambda_1 \in \mathbb{C}$ situated in the lower complex plane. In a recent work [18], a formula revealing the relation between the resonance λ_1 and the integral kernel of the scattering matrix was derived for the case of a massless scalar field. It was proven that the scattering matrix coefficients of one-boson scattering processes, excluding forward scattering, feature the expected Lorenzian shape in leading order in the neighborhood of the real part of the resonance λ_1. More precisely, it was shown that the leading order in the coupling constant g (for small g) of the integral kernel of the transition matrix T fulfills

$$T(k,k') \sim 4\pi i g^2 \left\|\Psi_{\lambda_0}\right\|^{-2} f(k)^2 \delta(|k|-|k'|) \frac{\operatorname{Re}\lambda_1 - \lambda_0}{(|k|+\lambda_0-\lambda_1)(|k|-\lambda_0+\overline{\lambda_1})}. \tag{1.1}$$

Here, Ψ_{λ_0} denotes the (due to the construction, unnormalized) ground state corresponding to λ_0 and δ the Dirac delta distribution. Due to the absence of a spectral gap, a subtle study by means of multi-scale perturbation analysis was necessary to construct the ground state and resonance and control the required spectral estimates [17]. To the best of our knowledge this is one of the first results towards a clarification of the relation between resonances and scattering theory in quantum field theory in the same vain as it was done in quantum mechanics, see [60] and references therein. In contrast, it has to be emphasized that the relation between the imaginary value of the resonance and the decay rate of the unstable excited state has been established rigorously in various models of quantum field theory in several articles [1, 48, 58, 16]. The result in [18] and also the one provided here, hence, naturally draw from many existing results: Resonance theory for models of quantum field theory has been developed in many works mainly studying the massless case of various models of quantum field theory with methods of renormalization group, see, e.g., [11, 13, 12, 9, 14, 6, 42, 46, 59, 34, 19], as well as with methods of multi-scale perturbation analysis, see, e.g., [55, 56, 7, 8]. Scattering theory has also been developed for various models of non-relativistic quantum electrodynamics, see, e.g., [37, 36, 20, 40, 39, 22, 24, 23, 15, 10, 47, 30, 5, 31, 61, 33, 32, 44, 45], and in particular for the massless Spin-Boson model, see, e.g., [27, 28, 29, 15, 49].

In the previous work [18], the main tool used to control the time evolution in the scattering regime, and hence, the scattering matrix coefficients, was the Laplace transform representation of the unitary time evolution generated by the corresponding Hamiltonian H, i.e.,

$$\left\langle \phi, e^{-itH}\psi \right\rangle = \lim_{\epsilon \downarrow 0} \frac{1}{2\pi i} \int_{\mathbb{R}+i\epsilon} \mathrm{d}z\, e^{-itz} \left\langle \phi, (H-z)^{-1}\psi \right\rangle. \tag{1.2}$$

In oder to justify this identity in a rigorous sense, precise control of the resolvent close to the real axis is needed to infer sufficient decay for the integral to converge. For this purpose, the Hamiltonian was studied with the help of a conveniently chosen complex dilation in which it exhibits a spectrum consisting of the ground state energy λ_0, a resonance λ_1 having negative imaginary part, and the rest of the spectrum being localized in cones in the lower complex plane attached to λ_0 and λ_1, respectively. Thanks to this fact, a well-defined meaning can be given to (1.2) by deforming the integration contour $\mathbb{R} + i\epsilon$ at $-\infty$ and $+\infty$ towards the lower complex plane.

In the case of a scalar field with mass $m > 0$ as discussed in this work, this strategy fails. The reason is that the spectrum of the corresponding dilated unperturbed Hamiltonian contains the points

$$\{e_0 + km\}_{k\in\mathbb{N}_0} \cup \{e_1 + km\}_{k\in\mathbb{N}_0}, \qquad \text{where} \qquad \mathbb{N}_0 := \mathbb{N} \cup \{0\}. \tag{1.3}$$

This leads to an absence of decay of the corresponding complex dilated resolvent close to the real line, which, in [18], was a crucial ingredient to control the time evolution in the scattering regime. Therefore, compared to [18], a different strategy to control the time evolution has to be developed which is the content of this paper. As discussed in Section 2 below, we use Mourre theory to obtain the required spectral control. In particular, we combine Mourre theory with perturbation theory and the Feshbach-Schur map. In Section 2 we compare this approach to the method of complex dilation which was employed in [18, 54].

We point out to the reader that, in general, Mourre theory has been studied in a variety of models (see, e.g., [4, 3, 25, 43]). We emphasize, however, that our application of this theory is non-standard. In the spirit of [2, 35], we prove a "reduced" limiting absorption principle for the unperturbed Hamiltonian at the excited energy e_1 and we apply perturbation theory – see Lemma 5.3 and Proposition 3.8 (iii). One of the main achievements of the present paper is then to combine the obtained limiting absorption principle with a suitable application of the Feshbach-Schur map. Using in addition Fermi's Golden Rule, we then manage to obtain the required control of the time evolution.

The paper is structured as follows: In Section 1.1 we define the massive Spin-Boson model and recall its properties relevant to this work, in Sections 1.3 and 1.2 we review the required results from scattering theory and the constructions of the ground state, and in Section 2 we present our main result, i.e., Theorem 2.2. The remaining sections consist of the main technical ingredient given in Section 3 and its proof in Section 5, the proof of the main result in Section 4, and an Appendix for the reasons of self-containedness. We lay out a roadmap for these sections in the end of Section 2.

1.1 Definition of the Spin-Boson model

In this section we introduce the considered model and state preliminary definitions and well-known tools and facts from which we start our analysis. Most parts of this section are drawn from [18, Section 1.1]. If the reader is already familiar with [18], this section can be skipped – except for Assumption 1.1.

The non-interacting Spin-Boson Hamiltonian is defined as

$$H_0 \equiv H_0(\omega) := K + H_f, \qquad K := \begin{pmatrix} e_1 & 0 \\ 0 & e_0 \end{pmatrix}, \qquad H_f \equiv H_f(\omega) := \int \mathrm{d}^3k \, \omega(k) a(k)^* a(k). \tag{1.4}$$

We regard K as an idealized free Hamiltonian of a two-level atom, where $0 = e_0 < e_1$ denote its two energy levels. Moreover, the annihilation and creation operators a, a^* are defined on the standard Fock space in (1.12) below and H_f is the free Hamiltonian of scalar field having dispersion relation $\omega(k) = \sqrt{k^2 + m^2}$. In this work we only consider massive scalar fields, i.e., $m > 0$. In the remainder of this work we sometimes refer to K as the atomic part and to H_f as the free field part of the Hamiltonian. Furthermore, the sum of those operators, H_0, we simply call "free Hamiltonian". The interaction term reads

$$V \equiv V(f) := \sigma_1 \otimes \Phi(f), \quad \text{where} \quad \sigma_1 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Phi(f) := a(f) + a(f)^*. \tag{1.5}$$

We point out to the reader that our proofs require that the boson form factor f satisfies $f, Df, D^2 f, (\frac{k^2}{\omega(k)})^{-1/2} Df(k) \in L^2(\mathbb{R}^3)$, where D is the generator of dilations introduced in Definition 3.1 (ii) below. We also suppose that f is spherically symmetric and use this in order to simplify our notation (a minor modification in some of our calculations would be necessary in order to drop this assumption). We identify $f(k) \equiv f(|k|)$ and assume that

$$f(\sqrt{e_1^2 - m^2}) > 0. \tag{1.6}$$

In particular, f does not have to be analytic and the infrared singularity is not an issue here. This being said, for concreteness, we consider a particular choice that meets the conditions above in the remainder of the paper, i.e.,

$$f : \mathbb{R}^3 \setminus \{0\} \to \mathbb{R}, \qquad k \mapsto e^{-k^2/\Lambda^2} \omega(k)^{-\frac{1}{2}}. \tag{1.7}$$

The relativistic form factor of a scalar field would be $f(k) = (2\pi)^{-\frac{3}{2}} (2\omega(k))^{-\frac{1}{2}}$. Such an f, however, is not square integrable, and therefore, renders the model ill-defined. This is referred to as ultraviolet divergence. In our case, the gaussian factor in (1.7) acts as an ultraviolet cut-off for $\Lambda > 0$ being the ultraviolet cut-off parameter. For the sake of simplicity, we absorb the missing factor $2^{-\frac{1}{2}} (2\pi)^{-\frac{3}{2}}$ in the coupling constant g.

The full Spin-Boson Hamiltonian is then defined as

$$H \equiv H(\omega, f) := H_0(\omega) + gV(f) \equiv H_0 + gV \tag{1.8}$$

for some coupling constant $g \geq 0$, on the Hilbert space

$$\mathcal{H} := \mathcal{K} \otimes \mathcal{F}[\mathfrak{h}], \qquad \mathcal{K} := \mathbb{C}^2, \tag{1.9}$$

where

$$\mathcal{F}[\mathfrak{h}] := \bigoplus_{n=0}^{\infty} \mathcal{F}_n[\mathfrak{h}], \qquad \mathcal{F}_n[\mathfrak{h}] := \mathfrak{h}^{\odot n}, \qquad \mathfrak{h} := L^2(\mathbb{R}^3, \mathbb{C}) \tag{1.10}$$

denotes the standard bosonic Fock space, and the superscript $\odot n$ denotes the n-th symmetric tensor product, where by convention $\mathfrak{h}^{\odot 0} \equiv \mathbb{C}$. Note that we identify $K \equiv K \otimes 1_{\mathcal{F}[\mathfrak{h}]}$ and $H_f \equiv 1_{\mathcal{K}} \otimes H_f$ in our notation (see Remark 1.3 below).

Due to the direct sum, an element $\Psi \in \mathcal{F}[\mathfrak{h}]$ can be represented as a family $(\psi^{(n)})_{n\in\mathbb{N}_0}$ of wave functions $\psi^{(n)} \in \mathfrak{h}^{\odot n}$ where we recall $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. The state Ψ with $\psi^{(0)} = 1$ and $\psi^{(n)} = 0$ for all $n \geq 1$ is called the vacuum and is denoted by

$$\Omega := (1, 0, 0, \dots) \in \mathcal{F}[\mathfrak{h}]. \tag{1.11}$$

For any $h \in \mathfrak{h}$ and $\Psi = (\psi^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}[\mathfrak{h}]$, such as the vector below belongs to $\mathcal{F}[\mathfrak{h}]$, we define the creation operator

$$(a(h)^*\Psi)_{n\in\mathbb{N}_0} := \left(0, h \odot \psi^{(0)}, \sqrt{2}h \odot \psi^{(1)}, \dots\right), \quad (a(h)^*\Psi)^{(n)} = \sqrt{n}h \odot \psi^{(n-1)} \tag{1.12}$$

and the annihilation operator $a(h)$ as the respective adjoint. Occasionally, we shall also use the physics notation

$$a(h)^* = \int \mathrm{d}^3k \, h(k) a(k)^*, \tag{1.13}$$

where the action of these operators in the n boson sector of a vector $\Psi = (\psi^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}[\mathfrak{h}]$ is to be understood (only formally) as:

$$(a(k)\Psi)^{(n)}(k_1, ..., k_n) = \sqrt{n+1}\psi^{(n+1)}(k, k_1, ..., k_n), \tag{1.14}$$

$$(a(k)^*\Psi)^{(n)}(k_1, ..., k_n) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \delta^{(3)}(k - k_i)\psi^{(n-1)}(k_1, ..., \tilde{k}_i, ..., k_n). \tag{1.15}$$

Here, the notation $\tilde{\cdot}$ means that the corresponding variable is omitted and δ denotes the Dirac's delta distribution acting on Schwartz test functions. Note that a and a^* fulfill the canonical commutation relations:

$$\forall h, l \in \mathfrak{h}, \qquad [a(h), a(l)^*] = \langle h, l \rangle_2, \qquad [a(h), a(l)] = 0, \qquad [a(h)^*, a(l)^*] = 0. \tag{1.16}$$

Throughout this paper we address the case of small coupling, i.e., we assume the coupling constant g to be sufficiently small. We do this only a finite number of times which assures that there is a $\boldsymbol{g} > 0$ such that all results hold true for coupling constants $0 < g < \boldsymbol{g}$.

As mentioned above, we consider a two-level system with two distinct energy levels at $0 = e_0 < m < e_1$. Moreover, suppose that the following assumption holds true:

Assumption 1.1. *We suppose that $e_1 - e_0 \notin m\mathbb{N}$. This implies*

$$\delta := \mathrm{dist}(e_1 - e_0, m\mathbb{N}) > 0, \tag{1.17}$$

where the symbol dist *stands for the Euclidean distance. Moreover, we assume the mass of the scalar field to be smaller than the energy level e_1 in order to allow for scattering processes.*

Speaking in physical terms, this assumption excludes the possibility that a certain number of photons with zero momentum are able to flip the atom to the excited state.

Let us recall some well-known facts about the introduced model. Clearly, K is self-adjoint on $\mathcal{K}$ and its spectrum consists of two eigenvalues e_0 and e_1. The corresponding eigenvectors are

$$\varphi_0 = (0,1)^T \qquad \text{and} \qquad \varphi_1 = (1,0)^T \qquad \text{with} \qquad K\varphi_i = e_i\varphi_i, \quad i = 0,1. \tag{1.18}$$

Moreover, H_f is self-adjoint on its natural domain $\mathcal{D}(H_f) \subset \mathcal{F}[\mathfrak{h}]$ and its spectrum is given by $\sigma(H_f) = \{0\} \cup [m,\infty)$. Consequently, the spectrum of H_0 is given by $\sigma(H_0) = \{e_0\} \cup [e_0+m,\infty)$, e_0, e_1 are eigenvalues of H_0 and, asumming that $e_0+m < e_1$, the later is embedded in the absolutely continuous part of the spectrum of H_0 (see [57]).

Finally, also the self-adjointness of the full Hamiltonian H is well-known (see, e.g., [18] and [50], see also [38, Lemma 21]).

Proposition 1.2. *For every $h \in \mathfrak{h}$ and $a(h)^\# \in \{a(h)^*, a(h)\}$,*

$$\left\| a(h)^\# (H_f+1)^{-\frac{1}{2}} \right\| \le C \, \|h\|_2 \, , \tag{1.19}$$

where C is a positive constant. This implies that gV is infinitesimally bounded with respect to H_0 and, consequently, H is self-adjoint and bounded from below, on the domain

$$\mathcal{D}(H) = D(H_0) = \mathcal{D}(\mathbb{1}_K \otimes H_f), \tag{1.20}$$

and the operators

$$H_f(H+i)^{-1}, \qquad H(H_f+1)^{-1} \tag{1.21}$$

are bounded.

Remark 1.3. *In this work we omit spelling out identity operators whenever unambiguous. For every vector spaces V_1, V_2 and operators A_1 and A_2 defined on V_1 and V_2, respectively, we identify*

$$A_1 \equiv A_1 \otimes \mathbb{1}_{V_2}, \qquad\qquad A_2 \equiv \mathbb{1}_{V_1} \otimes A_2. \tag{1.22}$$

In order to simplify our notation further, and whenever unambiguous, we do not utilize specific notations for every inner product or norm that we employ.

1.2 Ground state

The existence of a unique ground state has already been proven in the more complicated situation of a massless scalar field; see e.g. [50] and [17] and for the massive model at stake it can be shown using regular perturbation theory. However, for the convenience of the reader, we provide a detailed proof in Appendix A.

Proposition 1.4 (Ground state)**.** *For any $g \geq 0$, H has a unique ground state, i.e., $\lambda_0 = \inf \sigma(H)$ is a simple eigenvalue of H. We have*

$$\lambda_0 = e_0 - g^2\Gamma_0 + R_0(g), \qquad \textit{where} \qquad \Gamma_0 := \|f/(e_1 - e_0 + \omega)\|^2, \tag{1.23}$$

and there is a constant $C > 0$ such that $|R_0(g)| \leq Cg^4$. Furthermore, denoting by Ψ_{λ_0} the (unnormalized) ground state constructed in Appendix A, we have that

$$\|\Psi_{\lambda_0} - \varphi_0 \otimes \Omega\| \leq Cg. \tag{1.24}$$

The existence of a ground state can be established for any value of g, see [30].

1.3 Scattering theory

Finally, we give a short review of scattering theory, in models of quantum field theory, which will be necessary to state the main results in Section 2. For a more detailed introduction we refer to [18, Section 1.2].

Definition 1.5 (Basic components of scattering theory)**.** *We denote the dense subspace of compactly supported, smooth, and complex-valued functions on $\mathbb{R}^3 \setminus \{0\}$ by*

$$\mathfrak{h}_0 := C_c^\infty(\mathbb{R}^3 \setminus \{0\}, \mathbb{C}) \subset \mathfrak{h}. \tag{1.25}$$

Furthermore, we define the following objects:

(i) For $h \in \mathfrak{h}_0$, the limit operators

$$a_\pm(h)\Psi := \lim_{t\to\pm\infty} a_t(h)\Psi, \quad a_t(h) := e^{itH}a(h_t)e^{-itH}, \quad h_t(k) := h(k)e^{-it\omega(k)}, \tag{1.26}$$

for all $\Psi \in \mathcal{H}$ such that the limit exists, and also their respective adjoints $a_\pm^*(h)$.

(ii) The two-body scattering matrix coefficients:

$$S(h,l) = \|\Psi_{\lambda_0}\|^{-2} \langle a_+(h)^*\Psi_{\lambda_0}, a_-(l)^*\Psi_{\lambda_0}\rangle, \qquad \forall h,l \in \mathfrak{h}_0, \tag{1.27}$$

where the factor $\|\Psi_{\lambda_0}\|^{-2}$ appears due to the fact that, in our notation, the ground state Ψ_{λ_0} is not necessarily normalized.

(iii) The two-body transition matrix coefficients given by

$$T(h,l) = S(h,l) - \langle h,l\rangle_2 \qquad \forall h,l \in \mathfrak{h}_0. \tag{1.28}$$

The operators $a_\pm$ and $a_\pm^*$ are called asymptotic outgoing/incoming annihilation and creation operators. For $\Psi \in \mathcal{D}(H_0^{1/2})$, the limits (1.26) exist. The proof of this is obtained from the fundamental theorem of calculus, i.e. we write (for example for a_-)

$$a_-(h)\Psi = a(h)\Psi + ig\int_{-\infty}^0 \mathrm{d}s\, e^{isH}\langle h_s, f\rangle_2\, \sigma_1 e^{-isH}\Psi, \tag{1.29}$$

and we apply integration by parts to show that the integral above exists (see, e.g., [37, 36, 20, 40, 39, 27, 28, 29, 30, 15]). All details of this proof are presented in [18] for the massless case. Therein, also other useful results are shown (see [18, Lemma 4.1]), e.g.,

$$a_\pm(h)\Psi_{\lambda_0} = 0. \tag{1.30}$$

The starting point for the analysis of the transition matrix is what we call preliminary scattering formula: for $h, l \in \mathfrak{h}_0$,

$$T(h,l) = -2\pi i g \left\|\Psi_{\lambda_0}\right\|^{-2} \left\langle \sigma_1 \Psi_{\lambda_0}, a_-(W)^* \Psi_{\lambda_0} \right\rangle, \;\; W(k) := |k|^2 l(k) \int \mathrm{d}\Sigma \, \overline{h(|k|,\Sigma)} f(|k|,\Sigma), \tag{1.31}$$

where we use spherical coordinates $k = (|k|, \Sigma)$. Eq. (1.31) is proven for the massless case in [18, Theorem 4.3]. In our setting, when considering massive scalar fields, the proofs of (1.29), (1.30) and (1.31) follow the same line of arguments as the proofs of Lemma 4.1 and Theorem 4.3 in [18], and therefore, we do not repeat them here.

The matrix coefficients $S(h,l)$ can be interpreted as transition amplitudes of the scattering process for the following scenario: One incoming boson with wave function l is scattered at the two-level atom into an outgoing boson with wave function h. We point out to the reader that in this work we focus on one-photon processes only, however, the matrix coefficients of multi-photon processes can be defined likewise.

2 Main result

We now come to our main result, Theorem 2.2 below, which makes precise the relation between the scattering matrix kernel and the resonance.

Definition 2.1. *Using the notation $d^3x \equiv d\Sigma r^2 dr$ for solid angles Σ and radius r in spherical coordinates, we define, for all $h, l \in \mathfrak{h}_0$,*

$$G_{h,l} : \mathbb{R} \to \mathbb{C}, \qquad r \mapsto G_{h,l}(r) := \begin{cases} \int d\Sigma d\Sigma' \, r^4 \overline{h(r,\Sigma)} l(r,\Sigma') f(r)^2 & \text{for} \quad r \geq 0 \\ 0 & \text{for} \quad r < 0 \end{cases}. \tag{2.1}$$

In the proofs below we will drop the indices h, l and write $G_{h,l} \equiv G$.

Theorem 2.2 (Scattering formula)**.** *Suppose that Assumption 1.1 holds. There exists a complex number Γ_{-0} with $\operatorname{Im}\Gamma_{-0} > 0$ such that for all $h, l \in \mathfrak{h}_0$ and $g > 0$ sufficiently small, the transition matrix coefficients* (1.28) *are given by*

$$T(h,l) = T_P(h,l) + R(h,l), \tag{2.2}$$

where

$$T_P(h,l) := 4\pi i g^2 \|\Psi_{\lambda_0}\|^{-2} \int \mathrm{d}r \, \frac{G_{h,l}(r)\,(e_1 - g^2\,\mathrm{Re}\,\Gamma_{-0} - \lambda_0)}{(\omega(r) + \lambda_0 - (e_1 - g^2\Gamma_{-0}))\left(\omega(r) - \lambda_0 + (e_1 - g^2\overline{\Gamma_{-0}})\right)}, \tag{2.3}$$

and there is a constant $C(h,l) > 0$ *such that*

$$|R(h,l)| \le C(h,l) g^2 g^{1/3} |\log(g)|. \tag{2.4}$$

In (3.55) *below we give an explicit expression of* Γ_{-0}*.*

Not surprisingly, it turns out that $\widetilde{\lambda}_1 := e_1 - g^2\Gamma_{-0}$ is the leading term of the resonance, up to order g^2. This connection can be made by the standard construction of the resonance by means of complex dilation. This computation is not carried out here since we wanted to focus on the methods of Mourre theory rather than complex dilation; see, e.g., [8] for such a construction for massless fields using the method of complex dilation. Note that, in our situation, the construction is much easier since the dilated Hamiltonian exhibits spectral gaps. For treating resonances within the realm of Mourre theory we refer to [51, 52, 21, 35].

In order to compare this formula with the massless case, see (1.1), we may rewrite (2.3) in integral kernel form which takes the form

$$T(k,k') \sim 4\pi i g^2 \|\Psi_{\lambda_0}\|^{-2} f(k)^2 \frac{|k|\delta(\omega(k) - \omega(k'))}{\omega(k)} \frac{\mathrm{Re}\,\widetilde{\lambda}_1 - \lambda_0}{(|k| + \lambda_0 - \widetilde{\lambda}_1)(|k| - \lambda_0 + \overline{\widetilde{\lambda}_1})}. \tag{2.5}$$

There are only two differences in the formulas (2.5) and (1.1). One is due to the different dispersion relations $\omega(k) = \sqrt{|k|^2 + m^2}$ and $\omega(k) = |k|$ for the massive and massless case, respectively, and the other due to the fact that, in (1.1), λ_1 figures the non-perturbative resonance while, in (2.5), the entity $\widetilde{\lambda}_1$ is only the second order perturbation in g for small g as explained above. However, the latter difference is not relevant as the rest term $R(h,l)$ in both cases is of order $g^2\, g^{1/3}|\log g|$, and thus, will swallow this difference anyway.

The difference in the order in g of the given estimates of the rest terms $R(h,l)$ between the massive, i.e., $g^2\, g^{1/3}|\log g|$ in Theorem 2.2, and the massless case, i.e., $g^2\, g|\log g|$ in Theorem 2.2 in [18], is solely due to the different techniques which were employed. While in this paper the required spectral information was inferred by Mourre theory in the paper [18] the method of complex dilation was used. If a fair comparison of both techniques is possible at all, from our experience, it turns out that Mourre theory requires less information about the model, especially, no analyticity properties, to start with, however, gives a little more imprecise estimates of the remainders. In turn, the method of complex dilation is based on these analyticity properties but, given this information, one is able to produce slightly better estimates on the remainders. Since the model features a scalar interaction, the physical perturbation processes only differ for even orders in g. Hence, the different estimates of the remainders inferred by our application of Mourre theory and the method of complex dilations can be expected to be physically insignificant. Furthermore, also technically, there seems to be room for improvement.

Compared to our previous derivation of the transition matrix formula (1.1), see [18], for the massless Spin-Boson model, there are two main innovations in the strategy of proof. First, as already explained, we do not rely on complex dilations anymore but instead use Mourre theory to infer the required spectral information. And second, as mentioned already in the introduction, we handle the problem caused by the nature of the spectrum of the free dilated Hamiltonian (see (1.3)), which is a complication due to non-zero boson mass. In previous works [18] and [17], complex dilations were used both for the construction of the resonance as well as the control of required spectral properties, in particular, the estimates on the relevant resolvents.

Formally, the main steps of our proof of Theorem 2.2 are the following. First, after some computations, we arrive at the formula $T(h,l) = 2\pi \left\|\Psi_{\lambda_0}\right\|^{-2} (T^{(1)} - T^{(2)})$, where

$$T^{(1)} = g^2 \int_0^\infty \mathrm{d}t\, \zeta(t) \left\langle \sigma_1 \Psi_{\lambda_0}, e^{-itH} \sigma_1 \Psi_{\lambda_0} \right\rangle,$$
$$T^{(2)} = g^2 \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}r\, G(r) e^{it(\omega(r)-\lambda_0)} \left\langle \sigma_1 \Psi_{\lambda_0}, e^{itH} \sigma_1 \Psi_{\lambda_0} \right\rangle,$$

and ζ, G are some functions defined in (4.5) below. Next, we study the quantity $\langle \sigma_1 \Psi_{\lambda_0}, e^{\pm itH} \sigma_1 \Psi_{\lambda_0} \rangle$. Using (1.24) and the Spectral Theorem, we rewrite

$$\left\langle \sigma_1 \Psi_{\lambda_0}, e^{\pm itH} \sigma_1 \Psi_{\lambda_0} \right\rangle = \pi^{-1} \lim_{\epsilon \to 0^+} \int_{\mathbb{R}} \mathrm{d}r\, \chi(r) e^{-itr}\, \mathrm{Im} \left\langle \Phi_1, (H - r \pm i\epsilon)^{-1} \Phi_1 \right\rangle, \qquad (2.6)$$

with $\Phi_1 = \varphi_1 \otimes \Omega$, φ_1 is an eigenstate of K associated to the excited energy e_1 (see (1.18)), and χ is a smooth function supported in a suitable small interval containing e_1. Using the Feshbach-Schur map, we write

$$\left\langle \Phi_1, (H - r \pm i\epsilon)^{-1} \Phi_1 \right\rangle = \left\langle \Phi_1, F_P(H - r \pm i\epsilon)^{-1} \Phi_1 \right\rangle,$$

where $F_P(H-z) := P(H-z)P - g^2 PV\overline{P}(H_{\overline{P}} - z)^{-1}\overline{P}VP$, with P the orthogonal projection onto the vector space spanned by Φ_1, $\overline{P} = 1 - P$ and $H_{\overline{P}} = \overline{P}H\overline{P}$. The key point then consists in computing an expansion of $(F_P(H - r \pm i0^+))^{-1}$ in g and r by means of Mourre's theory. We establish regularity properties of the boundary values of the resolvent of $H_{\overline{P}}$ near the real axis (see (3.42) below) using in particular arguments of [35], from which we deduce that $(F_P(H - r \pm i0^+))^{-1}$ is Hölder-continuous of order $\frac{1}{2}$ in *both* r and g. Together with suitable estimates on remainder terms, this allows us to prove Theorem 2.2.

To our knowledge, our approach to study (2.6) has not been used previously in the literature (see although [1, 14, 48, 51] for other expansions, with different purposes, of quantities similar to (2.6)). We believe that our argument may find applications in other contexts.

The main technical import for the proof of Theorem 2.2 is contained in the next Section 3.1. There, we provide a central Mourre estimate in Lemma 3.7 which implies a limiting absorption principle in Proposition 3.8. The latter is employed in Section 3.2, in a combination with the Feshbach-Schur map as mentioned above, to control the time evolution in the scattering regime, and hence, the transition matrix coefficient under investigation. In Section 5 we provide a proof of the limiting absorption principle, i.e., Proposition 3.8, which in parts is a self-contained review of results in the literature but also provides a non-standard result, see (3.42), which allows to conveniently apply a limiting absorption principle in the context of perturbation theory.

Remark 2.3. *In the remainder of this work we denote by C any generic, positive (indeterminate) constant which may change from line to line in the computations but does not depend on g and the parameters $z, z', \epsilon, \eta, \beta$ introduced below.*

3 Technical ingredients

In this section we derive a formula for the leading order term with respect to the coupling constant of a certain matrix element of time-evolution and estimate the error term. We rely on two main ingredients, namely, a limiting absorption principle derived from a Mourre estimate and a Feshbach-Schur map. In the first part, Section 3.1, we introduce some notation and prove technical lemmas and a Mourre estimate which allows to derive a limiting absorption principle. The latter is also stated in this section since we use it as a key tool in order to prove our main result. In the second part, Section 3.2, we introduce a Feshbach-Schur map and combine it with the limiting absorption principle in order to control a certain matrix element of time-evolution.

3.1 Limiting absorption principle

In this section we present the limiting absorption principle based on a Mourre estimate for the model at stake. We follow the construction of [25], see also [50, 35, 43]. We start with introducing some notation.

Definition 3.1. *Recall that $\mathfrak{h}_0$ has been defined in* (1.25).

(i) For any self-adjoint operator O, we define $\mathrm{d}\Gamma(O)$ as the generator of the unitary one-parameter group $\left\{\Gamma(e^{-itO})\right\}_{t\in\mathbb{R}}$, where

$$\Gamma(e^{-itO}) := \bigoplus_{n=0}^{\infty}(e^{-itO})^{\odot n}, \qquad (e^{-itO})^{\odot 0} := 1. \tag{3.1}$$

It follows from Stone's theorem that $\mathrm{d}\Gamma(O)$ is self-adjoint. Notice that $H_f = \mathrm{d}\Gamma(\omega)$.

(ii) For $\beta \in \mathbb{R}$, we define the unitary dilation operator

$$u_\beta : \mathfrak{h} \to \mathfrak{h}, \qquad \varphi(k) \mapsto \varphi_\beta(k) := e^{\frac{3}{2}\beta}\varphi(e^\beta k), \qquad \forall k \in \mathbb{R}^3. \tag{3.2}$$

We denote by D generator of dilations, which is the generator of the unitary one-parameter group $\{u_\beta\}_{\beta\in\mathbb{R}}$. Note that D is self-adjoint on $\mathcal{D}(D) \subset \mathcal{H}$ due to Stone's theorem.

Moreover, for $\varphi \in \mathfrak{h}_0$ and $\beta \in \mathbb{R}$, we observe that

$$\frac{\mathrm{d}}{\mathrm{d}\beta}\varphi_\beta(k) = \frac{1}{2}\left(\nabla_k \cdot k + k \cdot \nabla_k\right)\varphi_\beta(k), \qquad k \in \mathbb{R}^3. \tag{3.3}$$

This implies that the action of D on $\mathfrak{h}_0$ is given by $\frac{i}{2}(k \cdot \nabla_k + \nabla_k \cdot k)$.

(iii) *We introduce the function*

$$\xi : \mathbb{R}^3 \to \mathbb{R}, \qquad k \mapsto \xi(k) := k^2/\omega(k). \tag{3.4}$$

(iv) *We set*

$$\mathcal{H}_0 := \mathcal{K} \otimes \mathcal{F}_{fin}[\mathfrak{h}_0], \tag{3.5}$$

where

$$\mathcal{F}_{fin}[\mathfrak{h}_0] := \Big\{\Psi = (\psi^{(n)})_{n\in\mathbb{N}_0} \in \mathcal{F}[\mathfrak{h}] \,|\, \exists N \in \mathbb{N}_0 : \psi^{(n)} = 0\, \forall n \geq N, \tag{3.6}$$
$$\forall n \in \mathbb{N} : \psi^{(n)} \in C_c^\infty(\mathbb{R}^{3n} \setminus \{0\}, \mathbb{C})\Big\}.$$

(v) *Moreover, for every closed operator A, we denote by*

$$\|\cdot\|_A := \left(\|A\cdot\|^2 + \|\cdot\|^2\right)^{1/2}, \tag{3.7}$$

its graph norm in the domain of A.

Remark 3.2. *Note that $\mathcal{H}_0$ and $\mathcal{F}_{fin}[\mathfrak{h}_0]$ are dense subsets of the domains of H and H_f with respect to the graph norm of H and H_f, respectively. In other words, $\mathcal{H}_0$ and $\mathcal{F}_{fin}[\mathfrak{h}_0]$ are cores of H and H_f, respectively.*

The following statement is a collection of general properties of the objects introduced in Definitions 3.1 which we will use in the remainder of this work.

Lemma 3.3. *The following properties hold true:*

(i) $\mathcal{F}_{fin}[\mathfrak{h}_0] \subset \mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$.

(ii) $\mathcal{D}(H_f) \subset \mathcal{D}(\Phi(Df))$ *and* $\Phi(Df)(H_f + 1)^{-\frac{1}{2}}$ *is bounded (recall the definition of* $\Phi(f)$ *in* (1.5)*).*

(iii) $\mathcal{D}(H_f) \subset \mathcal{D}(\mathrm{d}\Gamma(\xi))$ *and* $\mathrm{d}\Gamma(\xi)(H_f + 1)^{-1}$ *is bounded.*

(iv) The operator $[H_f, i\mathrm{d}\Gamma(D)]$ *defined as a quadratic form on* $\mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$ *can be uniquely extended to a* H*-bounded operator on* $\mathcal{D}(H) = \mathcal{D}(H_0)$ *denoted by* $[H_f, i\mathrm{d}\Gamma(D)]^0$*. We have the identity:*

$$[H_f, i\mathrm{d}\Gamma(D)]^0 = \mathrm{d}\Gamma(\xi) \tag{3.8}$$

on $\mathcal{D}(H_0)$*.*

(v) The operator $[\Phi(f), i\mathrm{d}\Gamma(D)]$ *defined as a quadratic form on* $\mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$ *can be uniquely extended to a* H*-bounded operator on* $\mathcal{D}(H) = \mathcal{D}(H_0)$ *denoted by* $[\Phi(f), i\mathrm{d}\Gamma(D)]^0$*. We have the identity:*

$$[\Phi(f), i\mathrm{d}\Gamma(D)]^0 = \Phi(Df) \tag{3.9}$$

on $\mathcal{D}(H_0)$*.*

Proof. (i) Clearly, this holds by Definition 3.1.

(ii) A direct calculation shows that $Df \in \mathfrak{h}$. We conclude the claim by Proposition 1.2.

(iii) Note that, for all $k \in \mathbb{R}^3$, $\xi(k) = \frac{k^2}{\omega(k)} = \omega(k)\frac{k^2}{k^2+m^2} \leq \omega(k)$. This directly implies the desired result.

(iv) Clearly, $[H_f, i\mathrm{d}\Gamma(D)]$ can be defined as a quadratic form on $\mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$, and hence, it follows from (i) that, for $\psi \in \mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]$, we have

$$\langle \psi, [H_f, i\mathrm{d}\Gamma(D)]\psi\rangle = \langle \psi, [\mathrm{d}\Gamma(\omega), i\mathrm{d}\Gamma(D)]\psi\rangle = \langle \psi, \mathrm{d}\Gamma([\omega, iD])\psi\rangle \,. \tag{3.10}$$

Moreover, it follows from a direct calculation that

$$[\omega, iD] = \xi, \tag{3.11}$$

on $\mathfrak{h}_0$, and hence,

$$\langle \psi, [H_f, i\mathrm{d}\Gamma(D)]\psi\rangle = \langle \psi, \mathrm{d}\Gamma(\xi)\psi\rangle \qquad \forall \psi \in \mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]. \tag{3.12}$$

In order to prove that the quadratic form above extents to a H_0-bounded operator, we use [53, Proposition II.1]. We take H_0 instead of H in [53] and $\mathrm{d}\Gamma(D)$ instead of A in [53]. Hypotheses (a) and (b) of [53, Proposition II.1] follow from the definition of $\mathrm{d}\Gamma(D)$ in terms of dilations (Definition 3.1): $e^{i\beta \mathrm{d}\Gamma(D)}$ leaves the domain of H_f invariant (and hence, the domain of H_0) for every real β, and

$$e^{-i\beta \mathrm{d}\Gamma(D)} H_f e^{i\beta \mathrm{d}\Gamma(D)} = \mathrm{d}\Gamma(\omega^{(\beta)}), \qquad \omega^{(\beta)}(k) = \omega(e^{\beta} k). \tag{3.13}$$

Moreover, as $\mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]$ is a core of H_f, that is contained in $\mathcal{D}(H_f) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$, then the latter is a core of H_0 as well. Now, we verify conditions (c') in [53, Proposition II.1] taking $\mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]$ as the distinguished set in (c'): It follows from the definition of $\mathrm{d}\Gamma(D)$ (Definition 3.1 (ii)), that $e^{-i\beta \mathrm{d}\Gamma(D)}\mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0] \subset \mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]$, for every real β. This together with (3.12) and (iii) complete conditions (c') in [53, Proposition II.1]. Next, we conclude from [53, Proposition II.1] that $[H_f, i\mathrm{d}\Gamma(D)]$ uniquely extends to a H_0-bounded operator on $\mathcal{D}(H_0)$. Finally, Proposition 1.2 implies that it extends to a H-bounded operator, defined on $\mathcal{D}(H) = \mathcal{D}(H_0)$.

(v) Since $Df(k)$, $\xi^{-1/2}Df \in L^2(\mathbb{R}^3)$, it follows that the operator

$$H(\xi, Df) := \mathrm{d}\Gamma(\xi) + \Phi(Df), \tag{3.14}$$

with domain $\mathcal{D}(\mathrm{d}\Gamma(\xi))$, is self-adjoint (the proof of this is very similar to the proof of Proposition 1.2) and it is $\mathrm{d}\Gamma(\xi)$-bounded (it is also H_0-bounded, see item (iii), and H-bounded - using Proposition 1.2). At first, we show that the quadratic form

$$[H, i\mathrm{d}\Gamma(D)] \tag{3.15}$$

is represented by the operator $\mathrm{d}\Gamma(\xi) + \Phi(Df)$ on $\mathcal{D}(H) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$. Note that a simple calculation implies that this holds true on $\mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]$. We use [53, Proposition II.1] taking $\mathrm{d}\Gamma(D)$ instead of A. Hypotheses (a) and (b) of [53, Proposition II.1] follow from the definition of $\mathrm{d}\Gamma(D)$ in terms of dilations (Definition 3.1): $e^{i\beta \mathrm{d}\Gamma(D)}$ leaves the domain of H invariant (which is the domain of H_0) for every real β, and (recall (3.13))

$$e^{-i\beta \mathrm{d}\Gamma(D)} H e^{i\beta \mathrm{d}\Gamma(D)} = K + \mathrm{d}\Gamma(\omega^{(\beta)}) + \Phi(u_\beta f). \tag{3.16}$$

Now, we verify conditions (c') in [53, Proposition II.1] taking $\mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]$ as the distinguished set in (c'): It follows from the definition of $\mathrm{d}\Gamma(D)$ (Definition 3.1 (ii)), that $e^{-i\beta \mathrm{d}\Gamma(D)}\mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0] \subset \mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]$, for every real β. Conditions (c') in [53, Proposition II.1] follow from the above discussion using that $\mathcal{D}(H(\xi, Df)) \supset \mathcal{D}(H)$. Next, we conclude from [53, Proposition II.1] that $[H, i\mathrm{d}\Gamma(D)]$ is represented by $H(\xi, Df)$ on $\mathcal{D}(H) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$. Using item (iv), we finally obtain that

$$[\Phi(f), i\mathrm{d}\Gamma(D)] = [H, i\mathrm{d}\Gamma(D)] - [H_0, i\mathrm{d}\Gamma(D)] = \Phi(Df) \tag{3.17}$$

as a quadratic form on $\mathcal{D}(H) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$.

□

For the proof of our main result it suffices to control the time evolution only on a spectral subset close to the excited state. In the following we define a cut-off function with its support localized in such a subset. Recall that $\delta > 0$ has been defined in Assumption 1.1.

Definition 3.4. *We fix* $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ *such that* $\operatorname{supp} \chi \subset (e_1 - 3\delta/4, e_1 + 3\delta/4)$ *and* $\chi|_{[e_1 - \delta/2, e_1 + \delta/2]} = 1$. *Moreover, for* $0 < \kappa < 2$ *and* $g^2 \leq s \leq g^\kappa$, *we define* χ_s *by* $\chi_s(r) := \chi(e_1 + (r - e_1)/s)$ *for all* $r \in \mathbb{R}$.

Next lemma is proven in Appendix B.

Lemma 3.5. *For every $\upsilon \in C_c^\infty(\mathbb{R}, [0,1])$, there is a constant $C_\upsilon > 0$ such that*

$$\|\upsilon(H) - \upsilon(H_0)\| \leq gC_\upsilon. \tag{3.18}$$

For every $s > 0$ in a compact set there is a constant C that depends on this set such that

$$\|\chi_s(H) - \chi_s(H_0)\| \leq Cs^{-1}g. \tag{3.19}$$

In the following we derive a positive commutator estimate close to the unperturbed eigenvalue e_1. For this purpose we set (see (1.18) and (1.8))

$$H_{\overline{P}}(\omega, f) \equiv H_{\overline{P}} := \overline{P}H(\omega, f)\overline{P}, \quad H_{0,\overline{P}}(\omega) \equiv H_{0,\overline{P}} := \overline{P}H_0(\omega)\overline{P} \tag{3.20}$$

$$H_{f,\overline{P}}(\omega) \equiv H_{f,\overline{P}} := \overline{P}H_f(\omega)\overline{P}, \quad V_{\overline{P}}(f) \equiv V_{\overline{P}} := \overline{P}V(f)\overline{P}, \quad \Phi_{\overline{P}}(f) = \overline{P}\Phi(f)\overline{P},$$

where, taking P_{φ_1} and P_Ω the orthogonal projections on the spans of φ_1 and Ω, respectively, we define

$$P := P_{\varphi_1} \otimes P_\Omega, \qquad \overline{P} = \mathbb{1}_{\mathcal{H}} - P. \tag{3.21}$$

Remark 3.6. *In the proof of Lemma 3.3 (v), we show that $[H, i\mathrm{d}\Gamma(D)]$, defined as a quadratic form on $\mathcal{D}(H_0) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$, can be extended to the H-bounded operator $H(\xi, Df) := \mathrm{d}\Gamma(\xi) + \Phi(Df)$. Using this we can extend the quadratic form $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]$, on $\mathcal{D}(H_0) \cap \mathcal{D}(\mathrm{d}\Gamma(D))$, to a $H_{\overline{P}}$-bounded operator on $\mathcal{D}(H_{\overline{P}})$. We denote this extension by*

$$[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 = H_{\overline{P}}(\xi, Df). \tag{3.22}$$

Lemma 3.7 (Mourre estimate)**.** *There is a constant $\alpha > 0$ such that, for sufficiently small $g > 0$,*

$$\chi(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}}) \geq \alpha\chi(H_{\overline{P}})^2, \tag{3.23}$$

where we recall Definition 3.4.

Proof. We take a fixed function $\upsilon \in C_c^\infty\Big((e_1 - \frac{9}{10}\delta, e_1 + \frac{9}{10}\delta), [0,1]\Big)$ with $\chi\upsilon = \chi$ (since this is fixed, we identify $C \equiv C_\upsilon$ in the constants below).

Note that $\mathrm{d}\Gamma(D)$ commutes with $\overline{P} = \mathbb{1}_{\mathcal{H}} - P$. Then, Lemma 3.3 (iv) and (v) yields

$$\upsilon(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\upsilon(H_{\overline{P}}) = \upsilon(H_{\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{\overline{P}}) + g\upsilon(H_{\overline{P}})\overline{P}\sigma_1 \otimes \Phi(Df)\overline{P}\upsilon(H_{\overline{P}}). \tag{3.24}$$

It follows from Lemma 3.3 (ii) that $\upsilon(H_{\overline{P}})\overline{P}\sigma_1 \otimes \Phi(Df)(H_{0,\overline{P}}+i)^{-1}\overline{P}(H_{0,\overline{P}}+i)\upsilon(H_{\overline{P}})$ is bounded (notice that $(H_{0,\overline{P}}+i)\upsilon(H_{\overline{P}}) = (H_{0,\overline{P}}+i)(H_{\overline{P}}+i)^{-1}(H_{\overline{P}}+i)\upsilon(H_{\overline{P}})$ is bounded, which follows from Proposition 1.2). Then, we obtain

$$\Big\|g\upsilon(H_{\overline{P}})\overline{P}\sigma_1 \otimes \Phi(Df)\overline{P}\upsilon(H_{\overline{P}})\Big\| \leq Cg. \tag{3.25}$$

Similarly as above, we argue that $\upsilon(H_{\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)$ and $\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{\overline{P}})$ are bounded, using Lemma 3.3 (iii). Then, Lemma 3.5 implies that

$$\upsilon(H_{\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{\overline{P}}) \geq \upsilon(H_{0,\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{0,\overline{P}}) - gC. \tag{3.26}$$

Plugging (3.26) and (3.25) into (3.24) yields that

$$\upsilon(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\upsilon(H_{\overline{P}}) \geq \upsilon(H_{0,\overline{P}})\overline{P}\mathrm{d}\Gamma(\xi)\overline{P}\upsilon(H_{0,\overline{P}}) - gC. \tag{3.27}$$

Set $\ell \in \mathbb{N}\cup\{0\}$ be such that

$$e_1 > \ell m \qquad\qquad e_1 < (\ell+1)m. \tag{3.28}$$

Note that Assumption 1.1 implies that for all $l \in \mathbb{N}$

$$|e_1 - \ell m| \geq \delta, \tag{3.29}$$

and hence, both inequalities in (3.28) are sharp. Moreover, this together with $\upsilon \in C_c^\infty\Big((e_1 - \frac{9}{10}\delta, e_1 + \frac{9}{10}\delta), [0,1]\Big)$ yields

$$\upsilon(H_{0,\overline{P}})H_{0,\overline{P}}\upsilon(H_{0,\overline{P}}) \geq \Big(\ell m + \frac{1}{10}\delta\Big)\upsilon(H_{0,\overline{P}})^2. \tag{3.30}$$

For any self-adjoint operator O, we denote by E_O its resolution of the identity. It follows that

$$E_{H_{0,\overline{P}}}(U) = \begin{cases} \overline{P}E_{H_0}(U), & \text{if } 0 \notin U, \\ P + \overline{P}E_{H_0}(U), & \text{if } 0 \in U. \end{cases} \tag{3.31}$$

This is a consequence of the fact that the formula in the right hand side of the equation above defines a resolution of the identity and the integral of the identity function with respect to it equals $H_{0,\overline{P}}$ (notice that P commutes with $E_{H_0}(U)$). Since 0 does not belong to the support of υ, it follows that

$$\upsilon(H_{0,\overline{P}}) = \upsilon(H_0)\overline{P} = \overline{P}\upsilon(H_0)\overline{P}. \tag{3.32}$$

Set $\mathcal{N} = \mathrm{d}\Gamma(1)$ the number operator. Since $\omega(k) \geq m$, it follows that $\mathbb{1}_{\mathcal{N}>\ell}H_0 \geq (\ell+1)m$, and therefore (notice that $\mathcal{N}$ commutes with $H_{0,\overline{P}}$ and P and recall (3.32)),

$$m\upsilon(H_{0,\overline{P}})^2\mathcal{N} = m\upsilon(H_{0,\overline{P}})^2\mathbb{1}_{\mathcal{N}\leq\ell}\mathcal{N} \leq m\ell\upsilon(H_{0,\overline{P}})^2. \tag{3.33}$$

Eqs. (3.30) and (3.33) imply that

$$\upsilon(H_{0,\overline{P}})\Big(H_{0,\overline{P}} - m\mathcal{N}\Big)\upsilon(H_{0,\overline{P}}) \geq \frac{1}{10}\delta\upsilon(H_{0,\overline{P}})^2. \tag{3.34}$$

Since $\xi(k) = \frac{k^2+m^2-m^2}{\omega(k)} = \omega(k) - \frac{m^2}{\omega(k)} \geq \omega(k) - m$, we get that

$$\mathrm{d}\Gamma(\xi) \geq H_{0,\overline{P}} - m\mathcal{N}. \tag{3.35}$$

Eqs. (3.34) and (3.35) imply that

$$\upsilon(H_{0,\overline{P}})\mathrm{d}\Gamma(\xi)\upsilon(H_{0,\overline{P}}) \geq \frac{1}{10}\delta\upsilon(H_{0,\overline{P}})^2. \tag{3.36}$$

This together with Lemma 3.5 and (3.27) lead us to (see also (3.32))

$$\upsilon(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\upsilon(H_{\overline{P}}) \geq \frac{1}{10}\delta\upsilon(H_{\overline{P}})^2 - gC. \tag{3.37}$$

We multiply by $\chi(H_{\overline{P}})$ from the left and the right and use that $\chi\upsilon = \chi$ to obtain

$$\chi(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}}) \geq \frac{1}{10}\delta\chi(H_{\overline{P}})^2 - gC\chi(H_{\overline{P}})^2. \tag{3.38}$$

Our desired result follows from (3.38), taking small enough g. □

Proposition 3.8 (Limiting absorption principle)**.** *We introduce the notation*

$$\langle \mathrm{d}\Gamma(D)\rangle := \left(\left(\mathrm{d}\Gamma(D)\right)^2 + 1\right)^{1/2}. \tag{3.39}$$

For sufficiently small $g > 0$, $\epsilon \in (0, 1)$ and $z, z' \in [e_1 - \delta/4, e_1 + \delta/4]$ we have

(i) $\sigma_{pp}(H_{\overline{P}}) \cap [e_1 - \delta/4, e_1 + \delta/4] = \emptyset$, where $\sigma_{pp}(H_{\overline{P}})$ denotes the pure point spectrum of $H_{\overline{P}}$.

(ii)

$$\left\| \langle \mathrm{d}\Gamma(D)\rangle^{-1} (H_{\overline{P}} - z \pm i\epsilon)^{-1} \langle \mathrm{d}\Gamma(D)\rangle^{-1} \right\| \leq C, \tag{3.40}$$

and

$$\left\| \langle \mathrm{d}\Gamma(D)\rangle^{-1} (H_{0,\overline{P}} - z \pm i\epsilon)^{-1} \langle \mathrm{d}\Gamma(D)\rangle^{-1} \right\| \leq C, \tag{3.41}$$

(iii)

$$\left\| \langle \mathrm{d}\Gamma(D)\rangle^{-1} \left((H_{\overline{P}} - z \pm i\epsilon)^{-1} - (H_{0,\overline{P}} - z' \pm i\epsilon)^{-1}\right) \langle \mathrm{d}\Gamma(D)\rangle^{-1} \right\| \leq C\left(g^{1/2} + |z - z'|^{1/2}\right). \tag{3.42}$$

We recall that the constants above do not depend on ϵ, z, z' and g (c.f. Remark 2.3).

For the convenience of the reader, we provide a proof of statements (ii) and (iii) in Section 5 - following [25]. Notice that statement (iii) is not standard, similar results are addressed in [35]. Their work also draws from [2]. However, we present no proof for statement (i) since this is not used in the remainder of this work and it is a standard result.

3.2 Resonance and time evolution

In this section we introduce a Feshbach-Schur map, c.f. [12], in order to derive a formula for the resolvent restricted to a spectral subset. This together with the limiting absorption principle obtained in Proposition 3.8 allows then for controlling the leading order term of certain matrix elements of the time evolution (with respect to the coupling constant) and estimate the error term in Lemma 3.14 below.

Definition 3.9. *We recall Eqs.* (3.20)–(3.21)*. For all* $z \in \mathbb{C} \setminus \sigma(H)$*, we define*

$$F_P(z) \equiv F_P(H - z) := P(H - z)P - g^2 PV\overline{P}(H_{\overline{P}} - z)^{-1}\overline{P}VP, \tag{3.43}$$

as an operator on the range of P*.*

The following lemma is an application of the limiting absorption principle derived in Proposition 3.8 and allows for the control of certain term of the Feshbach-Schur map introduced in Definition 3.9.

Lemma 3.10. *For sufficiently small* g *and every* $z \in [e_1 - \delta/4, e_1 + \delta/4]$ *and* $\epsilon \in (0, 1)$*, the following estimates hold true:*

(i)

$$\left\| PV\overline{P}(H_{\overline{P}} - z \pm i\epsilon)^{-1}\overline{P}VP \right\| \leq C. \tag{3.44}$$

(ii)

$$\left\| PV\overline{P}(H_{0,\overline{P}} - z \pm i\epsilon)^{-1}\overline{P}VP \right\| \leq C. \tag{3.45}$$

(iii) if $|z - e_1| \leq r$*,*

$$\left\| PV\overline{P}\left((H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1} - (H_{\overline{P}} - z \pm i\epsilon)^{-1} \right) \overline{P}VP \right\| \leq C(g^{1/2} + r^{1/2}). \tag{3.46}$$

We recall that the constants C *do not depend on* ϵ*,* z *and* g *(c.f. Remark 2.3).*

Proof. We take $z \in [e_1 - \delta/4, e_1 + \delta/4]$ and $\epsilon \in (0, 1)$. Note that $\mathrm{d}\Gamma(D)$ commutes with P. Then, it follows from Lemma 3.3 (v) together with $\mathrm{d}\Gamma(D)P = 0$ that $\mathrm{d}\Gamma(D)\overline{P}VP = i\overline{P}\sigma_1 \otimes a(Df)^*P$, and consequently,

$$\left\| \mathrm{d}\Gamma(D)\overline{P}VP \right\| \leq \|a(Df)^*\Omega\| = \|Df\| \,. \tag{3.47}$$

Moreover, we similarly obtain

$$\left\| \overline{P}VP \right\| \leq C. \tag{3.48}$$

We recall the definition of $\langle \mathrm{d}\Gamma(D)\rangle$ in (3.39) and observe

$$\left\| \langle \mathrm{d}\Gamma(D)\rangle \,\overline{P}VP \right\|^2 = \sup_{\Psi\in\mathcal{H}, \|\Psi\|=1} \left\langle \overline{P}VP\Psi, \langle \mathrm{d}\Gamma(D)\rangle^2 \,\overline{P}VP\Psi \right\rangle \tag{3.49}$$

$$= \sup_{\Psi\in\mathcal{H}, \|\Psi\|=1} \left\langle \overline{P}VP\Psi, \left(\mathrm{d}\Gamma(D)^2+1\right) \overline{P}VP\Psi \right\rangle \leq \left\| \mathrm{d}\Gamma(D)\overline{P}VP \right\|^2 + \left\| \overline{P}VP \right\|^2 .$$

This together with (3.47) and (3.48) implies that

$$\left\| \langle \mathrm{d}\Gamma(D)\rangle \,\overline{P}VP \right\| \leq C, \tag{3.50}$$

and hence, $\langle \mathrm{d}\Gamma(D)\rangle \,\overline{P}VP$ is a bounded operator on $\mathcal{H}$. Then, it follows that also its adjoint is a bounded operator. We obtain that

$$\left\| PV\overline{P}(H_{\overline{P}} - z \pm i\epsilon)^{-1}\overline{P}VP \right\| \leq C \left\| \langle \mathrm{d}\Gamma(D)\rangle^{-1} (H_{\overline{P}} - z \pm i\epsilon)^{-1} \langle \mathrm{d}\Gamma(D)\rangle^{-1} \right\| . \tag{3.51}$$

We conclude statement (i) by Proposition 3.8 (ii). Statements (ii) and (iii) follow similarly from Proposition 3.8 (ii) and (iii). □

Next, we derive an explicit formula for the leading order of the Feshbach-Schur map with respect to the coupling constant. This allows then for an easy approximation of the resolvent restricted on a certain subset in Corollary 3.12 below.

Lemma 3.11. *For sufficiently small $r, g > 0$, $\epsilon \in (0,1)$ and $z \in \mathbb{R}$ with $|z - e_1| \leq r$, we have*

$$F_P(H - z \pm i\epsilon) = (e_1 - z - g^2\Gamma_{\pm\epsilon} \pm i\epsilon)P + R_\epsilon(g, r), \tag{3.52}$$

where $\|R_\epsilon(g,r)\| \leq Cg^2(g^{1/2} + r^{1/2})$ and

$$\Gamma_{\pm\epsilon} := \int \mathrm{d}^3k \, \frac{f(k)^2}{\omega(k) - e_1 \pm i\epsilon}. \tag{3.53}$$

Moreover, recalling $m - e_1 < 0$, we observe that the limits

$$\lim_{\epsilon\to 0} \Gamma_{\pm\epsilon} := \Gamma_{\pm 0} \tag{3.54}$$

exist (note that $\Gamma_{\pm\epsilon}$ does not depend on g, r and z) and they are given by

$$\Gamma_{\pm 0} = \mp\pi i\theta(0) + \mathcal{P}\int_{m-e_1}^{\infty} \theta(x)/x dx, \tag{3.55}$$

where, for $\tau > m - e_1$, we define

$$\theta(\tau) := 4\pi(e_1+\tau)((e_1+\tau)^2 - m^2)^{1/2} f(((e_1+\tau)^2 - m^2)^{1/2})^2. \tag{3.56}$$

Note that $\theta(0) > 0$, and hence, (see (1.6))

$$\operatorname{Im}\Gamma_{\pm 0} = \mp\pi\theta(0) \neq 0. \tag{3.57}$$

Proof. Note that $PVP = 0$ and $PH_0P = e_1P$. We take $\epsilon \in (0,1)$ and $z \in \mathbb{R}$ with $|z - e_1| \leq r$. We obtain from Definition 3.9 that

$$F_P(H - z \pm i\epsilon) = (e_1 - z \pm i\epsilon)P - g^2\hat{\Gamma}_{\pm\epsilon} + R_\epsilon(g), \tag{3.58}$$

where

$$\hat{\Gamma}_{\pm\epsilon}P := PV\overline{P}(H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1}\overline{P}VP \tag{3.59}$$

and

$$R_\epsilon(g) = g^2 PV\overline{P}\Big((H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1} - (H_{\overline{P}} - z \pm i\epsilon)^{-1}\Big)\overline{P}VP. \tag{3.60}$$

For $\kappa > 0$ and sufficiently small $g, r > 0$, Lemma 3.10 (iii) implies that $\|R_\epsilon(g)\| \leq Cg^2(g^{1/2} + r^{1/2})$. We define $\widetilde{f}_\pm(k) = \frac{f(k)}{e_0+\omega(k)-e_1\pm i\epsilon}$ and calculate

$$\begin{aligned}\hat{\Gamma}_{\pm\epsilon}P =& PV\overline{P}(H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1}\overline{P}\sigma_1 \otimes a(f)^*P = PV\overline{P}(H_{0,\overline{P}} - e_1 \pm i\epsilon)^{-1}\varphi_0 \otimes f \\ =& PV\overline{P}\varphi_0 \otimes \widetilde{f}_\pm = \int \mathrm{d}^3k \, \frac{f(k)^2}{\omega(k) - e_1 \pm i\epsilon}P,\end{aligned} \tag{3.61}$$

where we recall $e_0 = 0$. This together with the definition of $\Gamma_{\pm\epsilon}$ in (3.53) completes the first part of the proof.

In the following we compute the limits as ϵ tends to zero of $\Gamma_{\pm\epsilon}$. This is actually a consequence of the Sokhotski-Plemelj theorem, we calculate using the changes of variables $s = (r^2 + m^2)^{1/2}$ and $\tau = s - e_1$ (we recall that we identify $f(k) \equiv f(|k|)$ and we do the same with ω):

$$\begin{aligned}&\int \mathrm{d}^3k \, \frac{f(k)^2}{\omega(k) - e_1 \pm i\epsilon} = 4\pi \int_0^\infty \mathrm{d}r \, r^2 f(r)^2 \frac{1}{\omega(r) - e_1 \pm i\epsilon} \\ =& 4\pi \int_m^\infty \mathrm{d}s \, s(s^2 - m^2)^{1/2} f((s^2 - m^2)^{1/2})^2 \frac{1}{(s - e_1) \pm i\epsilon} \\ =& 4\pi \int_{m-e_1}^\infty \mathrm{d}\tau \, (e_1 + \tau)((e_1 + \tau)^2 - m^2)^{1/2} f(((e_1 + \tau)^2 - m^2)^{1/2})^2 \frac{1}{\tau \pm i\epsilon}.\end{aligned} \tag{3.62}$$

Using (3.62) and the Sokhotski-Plemelj theorem, we obtain that

$$\lim_{\epsilon \to 0} \int \mathrm{d}^3k \, \frac{f(k)^2}{\omega(k) - e_1 \pm i\epsilon} = \mp\pi i\theta(0) + \mathcal{P}\int_{m-e_1}^\infty \mathrm{d}x \, \theta(x)/x, \tag{3.63}$$

and thereby, we complete the proof. ☐

Corollary 3.12. *For sufficiently small $g, r > 0$, small enough $\epsilon > 0$ (depending on g) and $z \in \mathbb{R}$ with $|z - e_1| \leq r$, the following holds true*

$$P(H - z \pm i\epsilon)^{-1}P = (e_1 - z - g^2\Gamma_{\pm 0})^{-1}P + \widetilde{R}(\epsilon, g, r), \tag{3.64}$$

where

$$\Big\|\widetilde{R}(\epsilon, g, r)\Big\| \leq C(g^{1/2} + r^{1/2})\Big|\frac{1}{e_1 - z - g^2\Gamma_{\pm 0}}\Big|, \tag{3.65}$$

and C does not depend on ϵ, g, r and z; c.f. Remark 2.3.

Proof. It follows from [12, Eq. (IV.13)] that

$$P(H-z\pm i\epsilon)^{-1}P = F_P(H-z\pm i\epsilon)^{-1}, \tag{3.66}$$

which is invertible for small enough ϵ, r and g (this is a consequence of Lemma 3.11, we recall that $\operatorname{Im}\Gamma_{\pm 0}\neq 0$). We use Neumann series and Lemma 3.11 to get

$$\begin{aligned}\|F_P(H-z\pm i\epsilon)^{-1}-&(e_1-z-g^2\Gamma_{\pm 0})^{-1}P\| \\ \leq&\Big|\frac{1}{e_1-z-g^2\Gamma_{\pm 0}}\Big|\sum_{n=1}^{\infty}\Big\|\frac{R_\epsilon(g,r)\pm i\epsilon+g^2\Gamma_{\pm 0}-g^2\Gamma_{\pm\epsilon}}{e_1-z-g^2\Gamma_{\pm 0}}\Big\|^n \\ \leq& C(g^{1/2}+r^{1/2})\Big|\frac{1}{e_1-z-g^2\Gamma_{\pm 0}}\Big|,\end{aligned} \tag{3.67}$$

for small enough g,ϵ and r (we can take, for example, $\epsilon\leq g^{5/2}$ and so small such that $|\Gamma_{\pm 0}-\Gamma_{\pm\epsilon}|\leq g^{1/2}$). □

In addition, we present an easy formula for a certain matrix element of the time evolution restricted to a spectral subset.

Lemma 3.13. *We set $\Phi_1:=\varphi_1\otimes\Omega$. For every $s>0$, we have*

$$\Big\langle\Phi_1, e^{-itH}\chi_s(H)\Phi_1\Big\rangle = \pi^{-1}\lim_{\epsilon\to 0^+}\int_{\mathbb{R}}\mathrm{d}r\,\chi_s(r)e^{-itr}\operatorname{Im}\Big\langle\Phi_1,(H-r-i\epsilon)^{-1}\Phi_1\Big\rangle. \tag{3.68}$$

Proof. The result follows from the spectral theorem and the next calculation

$$\begin{aligned}e^{-it\lambda}\chi_s(\lambda)=&\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\mathbb{R}}dre^{-it(\lambda+\epsilon r)}\chi_s(\lambda+\epsilon r)\frac{1}{r^2+1}=\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\mathbb{R}}dre^{-itr}\chi_s(r)\frac{\epsilon}{(r-\lambda)^2+\epsilon^2}\\ =&\lim_{\epsilon\to 0}\frac{1}{\pi}\int_{\mathbb{R}}dre^{-itr}\chi_s(r)\operatorname{Im}\frac{1}{\lambda-r-i\epsilon}.\end{aligned} \tag{3.69}$$

□

The following formula strongly relies on the previous results in this section and it is a crucial ingredient for the proof of the main theorem.

Lemma 3.14. *For sufficiently small $g>0$, s as in Definition 3.4 and Lemma 3.5 sufficiently small, and all $t\in\mathbb{R}$, the following holds true*

$$\Big\langle\Phi_1, e^{-itH}\Phi_1\Big\rangle = \pi^{-1}\int_{\mathbb{R}}\mathrm{d}z\,e^{-itz}\operatorname{Im}(e_1-z-g^2\Gamma_{-0})^{-1}+r_0(g,s), \tag{3.70}$$

where

$$|r_0(g,s)|\leq C\left((g^{1/2}+s^{1/2})|\log(g)|+gs^{-1}\right), \tag{3.71}$$

and we recall $\Phi_1=\varphi_1\otimes\Omega$. The constant C does not depend on g, s and t.

Proof. The spectral calculus implies $\chi(H_0)\Phi_1 = \Phi_1$, and hence, it follows from Lemma 3.5 that

$$\Big\langle \Phi_1, e^{-itH}\Phi_1 \Big\rangle = \Big\langle \Phi_1, e^{-itH}\chi_s(H)\Phi_1 \Big\rangle + r_1(g,s), \quad \text{where} \quad |r_1(g,s)| \le Cgs^{-1}. \tag{3.72}$$

Lemma 3.13 yields

$$\Big\langle \Phi_1, e^{-itH}\chi_s(H)\Phi_1 \Big\rangle = \pi^{-1} \lim_{\epsilon\to 0^+} \int_{\mathbb{R}} \mathrm{d}z\, \chi_s(z) e^{-itz}\, \mathrm{Im} \Big\langle \Phi_1, P(H - z - i\epsilon)^{-1} P\Phi_1 \Big\rangle . \tag{3.73}$$

We calculate:

$$\Big\langle \Phi_1, e^{-itH}\chi_s(H)\Phi_1 \Big\rangle = \pi^{-1} \lim_{\epsilon\to 0^+} \left(\int_{\mathbb{R}} \mathrm{d}z\, e^{-itz}\, \mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1} + r_2(g,\epsilon,s) + r_3(g,s) \right), \tag{3.74}$$

where

$$r_2(g,\epsilon,s) = \pi^{-1} \int_{\mathbb{R}} \mathrm{d}z\, \chi_s(z) e^{-itz}\, \mathrm{Im} \Big\langle \Phi_1, \Big(P(H - z - i\epsilon)^{-1} P - (e_1 - z - g^2\Gamma_{-0})^{-1} \Big) \Phi_1 \Big\rangle \tag{3.75}$$

and

$$r_3(g,s) = \pi^{-1} \int_{\mathbb{R}} \mathrm{d}z\, (1 - \chi_s(z)) e^{-itz}\, \mathrm{Im}(e_1 - z - g^2\Gamma_{+0})^{-1}. \tag{3.76}$$

Now, we use Corollary 3.12, for sufficiently small s, to get

$$\begin{aligned} &\Big| \chi_s(z) e^{-itz}\, \mathrm{Im} \Big\langle \Phi_1, \Big(P(H - z - i\epsilon)^{-1} P - (e_1 - z - g^2\Gamma_{-0})^{-1} \Big) \Phi_1 \Big\rangle \Big| \\ &\qquad\qquad \le C(g^{1/2} + s^{1/2}) \Big| e_1 - z - g^2\Gamma_{-0} \Big|^{-1} \chi_s(z). \end{aligned} \tag{3.77}$$

This together with (3.75) and Definition 3.4 yields then that

$$\begin{aligned} |r_2(g,\epsilon,s)| \le& C(g^{1/2} + s^{1/2}) \int \mathrm{d}z\, \chi_s(z) \Big| e_1 - z - g^2\Gamma_{-0} \Big|^{-1} \\ =& C(g^{1/2} + s^{1/2}) \int \mathrm{d}z\, \chi((z - e_1)/s + e_1) \Big((e_1 - z - g^2\, \mathrm{Re}\,\Gamma_{-0})^2 + g^4 (\mathrm{Im}\,\Gamma_{-0})^2 \Big)^{-1/2} \\ \le& C(g^{1/2} + s^{1/2}) \int_{-\frac{3}{4}\delta s - g^2\, \mathrm{Re}\,\Gamma_{-0}}^{\frac{3}{4}\delta s - g^2\, \mathrm{Re}\,\Gamma_{-0}} dr \frac{1}{g^2} \frac{1}{((\frac{r}{g^2})^2 + (\mathrm{Im}\,\Gamma_{-0})^2)^{1/2}} \\ \le& C(g^{1/2} + s^{1/2}) \int_{|r| \le csg^{-2}} dr \frac{1}{(r^2 + (\mathrm{Im}\,\Gamma_{-0})^2)^{1/2}}, \end{aligned} \tag{3.78}$$

where the last step follows for $g > 0$ sufficiently small and some constant $c > 0$. Here, we recall from Definition 3.4 that $g^2 \le s \le g^\kappa$ for some $0 < \kappa < 2$. Employing that $2\sqrt{x^2 + y^2} \ge |x| + |y|$, we find a constant $C > 0$ such that

$$|r_2(g,\epsilon,s)| \le C(g^{1/2} + s^{1/2}) |\log(g)|. \tag{3.79}$$

Moreover, it follows from (3.76) together with the definition of χ and $0 \le \chi \le 1$ that there is a constant $c > 0$ such that

$$\begin{aligned}|r_3(g,s)| \le &\pi^{-1}\int \mathrm{d}z\,(1-\chi_s(z))\left|\mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1}\right| \\ \le &\pi^{-1}g^2\,\mathrm{Im}\,\Gamma_{+0}\int \mathrm{d}z\,(1-\chi_s(z))\frac{1}{(e_1 - z - g^2\,\mathrm{Re}\,\Gamma_{-0})^2 + g^4\,\mathrm{Im}\,\Gamma_{-0}^2} \\ \le &Cg^2\int_{|r|\ge cs}\mathrm{d}r\frac{1}{g^4}\frac{1}{(\frac{r}{g^2})^2 + \mathrm{Im}\,\Gamma_{-0}^2} = C\int_{|x|\ge cs/g^2}\mathrm{d}x\frac{1}{x^2+\mathrm{Im}\,\Gamma_{-0}^2} \le Cg^2 s^{-1}.\end{aligned} \tag{3.80}$$

□

4 Proof of the main result

In this section we provide a proof of the main result; c.f. Theorem 2.2.

Proof of Theorem 2.2. We start with (1.31) and use (1.30):

$$T(h,l) = -2\pi i g\,\|\Psi_{\lambda_0}\|^{-2}\,\langle a_-(W)\sigma_1\Psi_{\lambda_0},\Psi_{\lambda_0}\rangle = -2\pi i g\,\|\Psi_{\lambda_0}\|^{-2}\,\langle [a_-(W),\sigma_1]\Psi_{\lambda_0},\Psi_{\lambda_0}\rangle\,, \tag{4.1}$$

It follows from (1.29) that

$$\begin{aligned}T(h,l) &= 2\pi(ig)^2\,\|\Psi_{\lambda_0}\|^{-2}\int_{-\infty}^0 \mathrm{d}t\,\overline{\langle W_t, f\rangle_2}\left\langle\left[e^{itH}\sigma_1 e^{-itH},\sigma_1\right]\Psi_{\lambda_0},\Psi_{\lambda_0}\right\rangle \\ &= 2\pi g^2\,\|\Psi_{\lambda_0}\|^{-2}\int_0^\infty \mathrm{d}t\,\langle f, W_{-t}\rangle_2\left\langle\left[e^{-itH}\sigma_1 e^{itH},\sigma_1\right]\Psi_{\lambda_0},\Psi_{\lambda_0}\right\rangle \\ &= 2\pi\,\|\Psi_{\lambda_0}\|^{-2}\left(T^{(1)} - T^{(2)}\right),\end{aligned} \tag{4.2}$$

where we recall the notation $W_s(k) = e^{-is\omega(k)}W(k)$ and use the abbreviations

$$\begin{aligned}T^{(1)} :&= g^2\int_0^\infty \mathrm{d}t\int \mathrm{d}^3k\,W(k)f(k)e^{it(\omega(k)+\lambda_0)}\left\langle\sigma_1\Psi_{\lambda_0}, e^{-itH}\sigma_1\Psi_{\lambda_0}\right\rangle \\ &= g^2\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}r\,G(r)e^{it(\omega(r)+\lambda_0)}\left\langle\sigma_1\Psi_{\lambda_0}, e^{-itH}\sigma_1\Psi_{\lambda_0}\right\rangle \\ &= g^2\int_0^\infty \mathrm{d}t\,\zeta(t)\left\langle\sigma_1\Psi_{\lambda_0}, e^{-itH}\sigma_1\Psi_{\lambda_0}\right\rangle\end{aligned} \tag{4.3}$$

and

$$T^{(2)} := g^2\int_0^\infty \mathrm{d}t\int_0^\infty \mathrm{d}r\,G(r)e^{it(\omega(r)-\lambda_0)}\left\langle\sigma_1\Psi_{\lambda_0}, e^{itH}\sigma_1\Psi_{\lambda_0}\right\rangle. \tag{4.4}$$

Here, we changed to spherical coordinates $k = (r,\Sigma)$ and take:

$$G(r) = \int \mathrm{d}\Sigma\mathrm{d}\Sigma'\,r^4\overline{h(r,\Sigma)}l(r,\Sigma')f(r)^2, \qquad \zeta(t) := \int_0^\infty \mathrm{d}r\,G(r)e^{it(\omega(r)+\lambda_0)}. \tag{4.5}$$

Moreover, we observe that $G \in C_c^\infty(\mathbb{R} \setminus \{0\}, \mathbb{C})$. Notice that an integration by parts (using that $e^{i\theta\omega(r)} = \frac{\partial}{\partial r}\Big(e^{i\theta\omega(r)} \frac{1}{i\theta \frac{\partial}{\partial\theta}\omega(r)}\Big) - e^{i\theta\omega(r)} \frac{\partial}{\partial r}\Big(\frac{1}{i\theta\frac{\partial}{\partial\theta}\omega(r)}\Big)$) ensures that

$$|\zeta(t)| \le C/(1+t^2), \qquad \forall t \in \mathbb{R}, \tag{4.6}$$

which guarantees the existence of the integrals in (4.3) and (4.4).

Recall $\Phi_1 = \varphi_1 \otimes \Omega$ (see Lemma 3.13). It follows from Proposition 1.4 that

$$\Big\langle \sigma_1 \Psi_{\lambda_0}, e^{-isH} \sigma_1 \Psi_{\lambda_0} \Big\rangle = \Big\langle \Phi_1, e^{-isH} \Phi_1 \Big\rangle + \rho_1(g), \tag{4.7}$$

where $|\rho_1(g)| \le Cg$. Moreover, we recall that Lemma 3.14 states that

$$\Big\langle \Phi_1, e^{-itH} \Phi_1 \Big\rangle = \pi^{-1} \int_{\mathbb{R}} \mathrm{d}z \, e^{-itz} \, \mathrm{Im}(e_1 - z - g^2 \Gamma_{-0})^{-1} + r_0(g,s), \tag{4.8}$$

where

$$|r_0(g,s)| \le C\Big(\Big(g^{1/2} + s^{1/2}\Big) |\log(g)| + gs^{-1}\Big). \tag{4.9}$$

Note that [18, Remark 4.8] implies that the first term in (4.8) is bounded by a constant as $g \to 0^+$ (this actually follows from computing the integral). Then, (4.3) together with (4.7) and (4.8) yields

$$T^{(1)} = T_0^{(1)} + R_1(g,s), \tag{4.10}$$

where

$$T_0^{(1)} := \pi^{-1} g^2 \int_0^\infty \mathrm{d}t \, \zeta(t) \int_{\mathbb{R}} \mathrm{d}z \, e^{-itz} \, \mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1}, \tag{4.11}$$

and $|R_1(g,s)| \le Cg^2((g^{1/2}+s^{1/2})|\log(g)|+gs^{-1})$ for some constant $C > 0$. As $\mathrm{Im}\,\Gamma_{-0} > 0$ and $\mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1}$ decays as $|z|^{-2}$ at infinity, (4.11) is absolutely integrable, and consequently, Fubini's theorem allows for interchanging the order of integration. Similarly, we argue that we can apply the dominated convergence theorem and conclude

$$T_0^{(1)} = \lim_{\eta \to 0^+} T_0^{(1)}(\eta), \tag{4.12}$$

where

$$\begin{aligned} T_0^{(1)}(\eta) =& \pi^{-1} g^2 \int_{\mathbb{R}} \mathrm{d}z \ \mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1} \int_0^\infty \mathrm{d}t \int_0^\infty \mathrm{d}r \, G(r) e^{it(\omega(r)+\lambda_0 - z + i\eta)} \\ =& \pi^{-1} g^2 \int_{\mathbb{R}} \mathrm{d}z \ \mathrm{Im}(e_1 - z - g^2\Gamma_{-0})^{-1} \int_0^\infty \mathrm{d}t \, \zeta(t) e^{-t\eta} e^{-itz}. \end{aligned} \tag{4.13}$$

Again, Fubini's theorem yields for $Q > 0$

$$\int_0^Q \mathrm{d}t \int_0^\infty \mathrm{d}r \, G(r) e^{it(\omega(r)+\lambda_0-z+i\eta)} = i \int_0^\infty \mathrm{d}r \, \frac{G(r)}{\omega(r)+\lambda_0 - z + i\eta} \Big(1 - e^{iQ(\omega(r)+\lambda_0-z+i\eta)}\Big). \tag{4.14}$$

Moreover, for all $\eta > 0$, we obtain by the integration by parts formula (see above (4.6)) together with $G \in C_c^\infty(\mathbb{R} \setminus \{0\}, \mathbb{C})$ that there is a constant $C(\eta, g) > 0$ such that

$$\left| \int_0^\infty \mathrm{d}r \, \frac{G(r)}{\omega(r) + \lambda_0 - z + i\eta} e^{iQ(\omega(r)+\lambda_0 - z + i\eta)} \right| \leq C(\eta, g) Q^{-1}, \tag{4.15}$$

and consequently, (4.14) implies that

$$\int_0^\infty \mathrm{d}s \int_0^\infty \mathrm{d}r \, G(r) e^{is(\omega(r)+\lambda_0 - z + i\eta)} = i \int_0^\infty \mathrm{d}r \, \frac{G(r)}{\omega(r) + \lambda_0 - z + i\eta}. \tag{4.16}$$

This together with Fubini's theorem yields that

$$T_0^{(1)}(\eta) = i\pi^{-1} g^2 \int_0^\infty \mathrm{d}r \, G(r) \int_{\mathbb{R}} \mathrm{d}z \, \operatorname{Im}(e_1 - z - g^2 \Gamma_{-0})^{-1} \frac{1}{\omega(r) + \lambda_0 - z + i\eta}. \tag{4.17}$$

For $a > e_1$, we define $\mathcal{Q}_a := [-a, a] \cup \{ae^{-i\varphi} : \varphi \in [0, \pi]\} \subset \overline{\mathbb{C}^-}$ to be a closed contour with mathematical negative orientation. Note that, for real z, as in (4.17),

$$\operatorname{Im}(e_1 - z - g^2 \Gamma_{-0})^{-1} = \frac{1}{2i} \Big((e_1 - z - g^2 \Gamma_{-0})^{-1} - (e_1 - z - g^2 \overline{\Gamma_{-0}})^{-1} \Big), \tag{4.18}$$

i.e. we do not conjugate z. We extend the formula above, in a meromorphic way, to the lower half of the complex plane. We obtain, for small enough η, using the residue theorem that

$$\begin{aligned} &\int_{\mathbb{R}} \mathrm{d}z \, \operatorname{Im}(e_1 - z - g^2 \Gamma_{-0})^{-1} \frac{1}{\omega(r) + \lambda_0 - z + i\eta} \\ &= (2i)^{-1} \lim_{a \to \infty} \int_{\mathcal{Q}_a} \mathrm{d}z \, \frac{1}{\omega(r) + \lambda_0 - z + i\eta} \Big((e_1 - z - g^2 \Gamma_{-0})^{-1} - (e_1 - z - g^2 \overline{\Gamma_{-0}})^{-1} \Big) \\ &= \frac{\pi}{\omega(r) + \lambda_0 - (e_1 - g^2 \Gamma_{-0}) + i\eta}. \end{aligned} \tag{4.19}$$

This together with (4.17) yields that

$$\begin{aligned} \lim_{\eta \to 0^+} T_0^{(1)}(\eta) &= \lim_{\eta \to 0^+} \int_0^\infty \mathrm{d}r \, \frac{ig^2 G(r)}{\omega(r) + \lambda_0 - (e_1 - g^2 \Gamma_{-0}) + i\eta} \\ &= \int_0^\infty \mathrm{d}r \, \frac{ig^2 G(r)}{\omega(r) + \lambda_0 - (e_1 - g^2 \Gamma_{-0})}, \end{aligned} \tag{4.20}$$

where in the last step we applied the dominated convergence theorem which is justified because $G \in C_c^\infty(\mathbb{R} \setminus \{0\}, \mathbb{C})$. Consequently, it follows from (4.10) and (4.12) that

$$T^{(1)} = ig^2 \int_0^\infty \mathrm{d}r \, \frac{G(r)}{\omega(r) + \lambda_0 - (e_1 - g^2 \Gamma_{-0})} + R_1(g, s), \tag{4.21}$$

where we recall that $|R_1(g, s)| \leq C g^2 ((s^{1/2} + g^{1/2}) |\log(g)| + g s^{-1})$. Analogously, we obtain

$$T^{(2)} = ig^2 \int \mathrm{d}r \, \frac{G(r)}{\omega(r) - \lambda_0 + (e_1 - g^2 \overline{\Gamma_{-0}})} + R_2(g, s), \tag{4.22}$$

and $|R_2(g,s)| \leq Cg^2((s^{1/2}+g^{1/2})|\log(g)| + gs^{-1})$ for some constant C. Finally, we conclude from (4.21) and (4.22) together with (4.2) that

$$\begin{aligned} T(h,l) =& 2\pi i g^2 \left\|\Psi_{\lambda_0}\right\|^{-2} \int \mathrm{d}r \left(\frac{G(r)}{\omega(r)+\lambda_0-(e_1-g^2\Gamma_{-0})} - \frac{G(r)}{\omega(r)-\lambda_0+(e_1-g^2\overline{\Gamma_{-0}})} \right) \\ &+ R(g,s) \\ =& 4\pi i g^2 \left\|\Psi_{\lambda_0}\right\|^{-2} \int \mathrm{d}r \frac{G(r)\left(e_1-g^2 \operatorname{Re}\Gamma_{+0}-\lambda_0\right)}{\left(\omega(r)+\lambda_0-(e_1-g^2\Gamma_{-0})\right)\left(\omega(r)-\lambda_0+(e_1-g^2\overline{\Gamma_{-0}})\right)} \\ &+ R(g,s), \end{aligned} \tag{4.23}$$

where $R(g,s) := R_1(g,s) + R_2(g,s)$. Hence, there is a constant $C > 0$ such that $|R(g,s)| \leq Cg^2((s^{1/2}+g^{1/2})|\log(g)| + gs^{-1})$. We take $s = g^{2/3}$ and obtain that $|R(g,s)| \leq Cg^2 g^{1/3}|\log(g)|$. This completes the proof. □

5 Mourre Theory and the Limiting Absorption Principle

In this section we present a proof of Proposition 3.8 (ii) and (iii). Although Mourre theory is a standard tool to prove limiting absorption principles, in this section we do not address the usual procedures because we prove perturbative results in the spirit of [2, 35] (see Proposition 3.8 (iii)). Note that in [35] an abstract family of Hamiltonians is studied.

The main result of this section is Proposition 3.8 (iii). Despite the fact that Proposition 3.8 (ii) is standard, we also prove it because we need it to prove Proposition 3.8 (iii). Some other well-known estimates in the context of Mourre theory are not proven in this section – we will give instead proper references.

We also mention that we do not employ the original techniques of Mourre to study domain problems and commutators (see [53, 25]). Instead, we directly dilate the operators at stake: our approach is close to the usual one based on the theory of operators of class C^k with respect to a self-adjoint conjugate operator (see [4, 3]), but, in our paper, given the explicit form of the operators at stake, we do not need to rely on this theory and we give a more transparent presentation.

In this section we address the limiting absorption principle, i.e. we study the behavior of the resolvent operator $(H_{\overline{P}} - (z \pm i\epsilon))^{-1}$ as $\epsilon > 0$ tends to 0 and z belongs to the interval

$$I := [e_1 - \delta/4, e_1 + \delta/4]. \tag{5.1}$$

Of course, the norm of $(H_{\overline{P}} - (z \pm i\epsilon))^{-1}$ tends to infinity as ϵ tends to zero. Then, controlling its behavior requires restricting its domain, and this is achieved by multiplying by the operator

$$\rho := \langle \mathrm{d}\Gamma(D) \rangle^{-1}. \tag{5.2}$$

Our goal is to obtain uniform norm-bounds for $\rho(H_{\overline{P}} - (z \pm i\epsilon))^{-1}\rho$ and regularity properties with respect to g (this is what we call above perturbative Mourre theory) and z.

Intuitively, one might consider the operator $H_{\overline{P}} - z$ as a real quantity because it is self adjoint. One of the clever ideas of Mourre is to add to $H_{\overline{P}} - (z \pm i\epsilon)$ a non-zero imaginary part of size $\eta > 0$ and sign $\pm$ (according to $\pm i\epsilon$). Then, the resulting operator ($H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ – see (5.4) below) can be intuitively regarded as a real quantity plus $\mp i(\epsilon+\eta)$. It is, therefore, invertible and the norm of its inverse is uniformly bounded with respect to ϵ. Our goal is to study the behavior of the resolvent operator associated to $H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ as ϵ and η tend to zero. More precisely, the imaginary part that we refer above is given by the operator $\mp i\eta M^2$, where η is a strictly positive small enough real number and (see Lemma 3.7)

$$M^2 := \chi(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}}) \geq \alpha\chi(H_{\overline{P}})^2, \tag{5.3}$$

which is a bounded operator (see Remark 3.6). We properly select ρ as a function of $\mathrm{d}\Gamma(D)$ because $\rho\mathrm{d}\Gamma(D)$ is bounded. This allows us to control the unbounded operator $\mathrm{d}\Gamma(D)$ in the above commutator. The other operator in this commutator is chosen in order to cancel resolvents (see (5.23) and (5.25) below for the limiting absorption principle, and (5.61) for perturbative results).

We define the operators (for $z \in I$)

$$H_{\overline{P}}^{\pm\eta} := H_{\overline{P}} \mp i\eta M^2, \qquad R^{\pm\eta}(z_{\pm\epsilon}) = \left(H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}\right)^{-1}, \quad z_{\pm\epsilon} := z \pm i\epsilon. \tag{5.4}$$

It is a standard result that $H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}$ is invertible (with bounded inverse) – see [25] – and that $R^{\pm\eta}(z_{\pm\epsilon})$ is continuous at $\eta = 0$ and derivable with respect to η, for $\eta > 0$ small enough. Its derivative is given by

$$\mathrm{d}/\mathrm{d}\eta\, R^{\pm\eta}(z_{\pm\epsilon}) = \pm iR^{\pm\eta}(z_{\pm\epsilon})M^2R^{\pm\eta}(z_{\pm\epsilon}), \qquad \forall\eta \in (0, \boldsymbol{\eta}). \tag{5.5}$$

For the convenience of the reader we give a proof of this in Appendix C below (see also [25]). Moreover, if we multiply $R^{\pm\eta}(z_{\pm\epsilon})$ by an operator that localizes the spectral region of $H_{\overline{P}}$ far away from z, we get a bounded operator which satisfies:

$$\|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\| \leq C, \tag{5.6}$$

where $\overline{\chi} = 1 - \chi$. This is proven in Appendix C (see also [25]).

As announced above, it follows that the norm of $R^{\pm\eta}(z_{\pm\epsilon})$ can be uniformly bounded (with respect to ϵ). Actually, the following estimate holds:

$$\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\| \leq C/\eta, \tag{5.7}$$

where C does not depend on z, ϵ and g (see [25] and Section C).

Estimate (5.7) itself is not enough because we still have the singularity C/η and we need to consider the operator ρ, otherwise we cannot expect to have a limiting absorption principle - this is explained above. For this reason, we define

$$F^{\pm\eta}(z_{\pm\epsilon}) := \rho R^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.8}$$

and get a better estimate which is a key ingredient of Mourre theory. Note that this is the only place where the Mourre estimate (see (5.3)) is used:

$$\|(H_{\overline{P}}+i)R^{\pm\eta}(z_{\pm\epsilon})\rho\| \leq C\left(1+\eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right). \tag{5.9}$$

Eq. (5.9) is a standard result (see, e.g., [25]), but we prove it in Appendix C. Looking at (5.7) and (5.9), it seams that we get again the unsatisfactory bound

$$\|F^{\pm\eta}(z_{\pm\epsilon})\| \leq C/\eta. \tag{5.10}$$

At this point, the line of reasoning becomes more subtle. Actually, in the lines above we never use that M^2 is defined in terms of the commutator $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0$. The only thing we utilize about M^2 is that it satisfies the Mourre estimate (5.3). All the material presented above in this section is standard and it can be directly deduced from the proofs in [25]. Therefore, we do not include proofs of this in the present section. For the convenience of the reader we provide proofs in Appendix C.

In this section we use all estimates and statements presented above (without proofs) and provide a detailed proof of the limiting absorption principle (Proposition 3.8-(ii)) and its perturbative version (Proposition 3.8-(iii)). The idea of the proof of Proposition 3.8-(ii) (which amounts to bound $\|F^{\pm\eta}(z_{\pm\epsilon})\|$ by a constant) is quite simple, we just write $F^{\pm\eta}(z_{\pm\epsilon})$ as the integral of its derivative. Then, the difficult part is to estimate the referred derivative (Lemma 5.2 below). This derivative consists of a sum of several terms and each of them is separately estimated. The most singular term is $Q_{1,1}$ defined in (5.23) below. The analysis of $Q_{1,1}$ is the only part of the proof of Proposition 3.8-(ii) that requires that M^2 is defined in terms of the commutator $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0$: we control the unbounded operator $\mathrm{d}\Gamma(D)$ using that $\rho\mathrm{d}\Gamma(D)$ is bounded and $H_{\overline{P}}$ is important to cancel resolvent operators (see (5.25) below).

As we mention above, the main result of this section is Proposition 3.8-(iii). The proof of it follows the same strategy of the proof of item (ii), but it is substantially more complicated. Again, we study the terms we are interested in using that they are integrals of their derivatives. The difficult part is to estimate the derivatives, which consist on several terms that must be analyzed separately. This is achieved in Lemma 5.3 below.

Before we start with the proofs, we state two last results that we use in this section and prove in Appendix D: the operator $R^{\pm\eta}(z_{\pm\epsilon})$ leaves the domain of $\mathrm{d}\Gamma(D)$ invariant. Moreover, there is a bounded operator that we denote by

$$[\mathrm{d}\Gamma(D), M^2]^0 \tag{5.11}$$

that represents the quadratic form $[\mathrm{d}\Gamma(D), M^2]$. These results can be proved as in [53, 25] (defining a scale of Hilbert spaces and regularizing the generator of dilations) or [4, 3] (using that the Hamiltonian is of class C^k with respect to the generator of dilations). We provide a more direct proof in Appendix D.

Remark 5.1. *The definitions and estimates introduced above in this section are also valid for the case* $g = 0$*. We distinguish this case by adding everywhere in our notations a subscript* 0*. For example:*

$$M_0^2 := M^2|_{g=0}, \qquad H_{0,\overline{P}} := H_{\overline{P}}|_{g=0}.$$

Lemma 5.2. *For $g \geq 0, \boldsymbol{\eta} > 0$ sufficiently small, $\eta \in (0, \boldsymbol{\eta})$, $\epsilon \in (0,1)$, $z, z' \in I$ and $z_{\pm\epsilon} := z \pm i\epsilon$,*

$$\left\| \mathrm{d}/\mathrm{d}\eta \, F^{\pm\eta}(z_{\pm\epsilon}) \right\| \leq C \left(1 + \eta^{-1/2} \left\| F^{\pm\eta}(z_{\pm\epsilon}) \right\|^{1/2} + \left\| F^{\pm\eta}(z_{\pm\epsilon}) \right\| \right). \tag{5.12}$$

Proof. It follows from (5.3), (5.5) and (5.8) that

$$\pm i \mathrm{d}/\mathrm{d}\eta \, F^{\pm\eta}(z_{\pm\epsilon}) = Q_1 + Q_2 + Q_3 + Q_4, \tag{5.13}$$

where

$$Q_1 := -\rho R^{\pm\eta}(z_{\pm\epsilon})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.14}$$

$$Q_2 := -\rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.15}$$

$$Q_3 := \rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.16}$$

$$Q_4 := \rho R^{\pm\eta}(z_{\pm\epsilon})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\rho. \tag{5.17}$$

Remark 3.6 and (5.6) imply that

$$\left\| [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon}) \right\| \leq C. \tag{5.18}$$

This yields that

$$\|Q_2\| \leq C \left\| \rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}}) \right\| \leq C, \tag{5.19}$$

where we use again (5.6) (taking the adjoint). Taking the adjoint in (5.18), it follows that

$$\|Q_3\| \leq C \left\| R^{\pm\eta}(z_{\pm\epsilon})\rho \right\| \leq C \left(1 + \eta^{-1/2} \left\| F^{\pm\eta}(z_{\pm\epsilon}) \right\|^{1/2} \right), \tag{5.20}$$

where we use (5.9). Similarly, taking the adjoint in (5.9) we obtain that

$$\|Q_4\| \leq C \left(1 + \eta^{-1/2} \left\| F^{\pm\eta}(z_{\pm\epsilon}) \right\|^{1/2} \right). \tag{5.21}$$

In the remainder of the proof, we estimate Q_1. For $\phi, \psi \in \mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$, Remark 3.6 and the fact that $R^{\pm\eta}(z_{\pm\epsilon})$ leaves the domain of $\mathrm{d}\Gamma(D)$ invariant (see above (5.11)) allows us to write

$$\langle \phi, Q_1 \psi \rangle = \langle \phi, Q_{11}\psi \rangle + \langle \phi, Q_{12}\psi \rangle, \tag{5.22}$$

where

$$\begin{aligned} \langle \phi, Q_{11}\psi \rangle := &\left\langle \left(H_{\overline{P}} \pm i\eta M^2 - z_{\mp\epsilon} \right) R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D) R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ &- \left\langle (-i\mathrm{d}\Gamma(D)) R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, \left(H_{\overline{P}} \mp i\eta M^2 - z_{\pm\epsilon} \right) R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle, \end{aligned} \tag{5.23}$$

$$\begin{aligned} \langle \phi, Q_{12}\psi \rangle := \pm i\eta \Big(&\left\langle M^2 R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D) R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \\ &- \left\langle (-i\mathrm{d}\Gamma(D)) R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, M^2 R^{\pm\eta}(z_{\pm\epsilon})\rho\psi \right\rangle \Big). \end{aligned} \tag{5.24}$$

Employing that $\|\mathrm{d}\Gamma(D)\rho\| \leq 1$, we find

$$
\begin{aligned}
|\langle\phi, Q_{11}\psi\rangle| &= \left|\left\langle\left(-i\mathrm{d}\Gamma(D)\right)\rho\phi, R^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle - \left\langle R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D)\rho\psi\right\rangle\right| \\
&\leq \|\phi\|\,\|\psi\|\left(\left\|R^{\mp\eta}(z_{\mp\epsilon})\rho\right\| + \left\|R^{\pm\eta}(z_{\pm\epsilon})\rho\right\|\right). \qquad (5.25)
\end{aligned}
$$

It follows again from (5.9) that

$$
|\langle\phi, Q_{11}\psi\rangle| \leq C\,\|\phi\|\,\|\psi\|\left(1 + \eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right). \qquad (5.26)
$$

Furthermore, we estimate (using again that $R^{\pm\eta}(z_{\pm\epsilon})$ leaves the domain of $\mathrm{d}\Gamma(D)$ invariant and the text around (5.11))

$$
\begin{aligned}
|\langle\phi, Q_{12}\psi\rangle| &\leq \eta\,\|\phi\|\,\|\psi\|\,\left\|R^{\mp\eta}(z_{\mp\epsilon})\rho\right\|\,\left\|R^{\pm\eta}(z_{\pm\epsilon})\rho\right\|\,\left\|[M^2, \mathrm{d}\Gamma(D)]^0\right\| \qquad (5.27) \\
&\leq C\eta\,\|\phi\|\,\|\psi\|\left(1 + \eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right)^2 \\
&\leq C\,\|\phi\|\,\|\psi\|\left(1 + \left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|\right),
\end{aligned}
$$

where we use (5.9). It follows from (5.26) together with (5.27), (5.22) and the density of $\mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$ in $\mathcal{H}$ that

$$
\|Q_1\| \leq C\left(1 + \eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2} + \left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|\right). \qquad (5.28)
$$

This together with (5.13), (5.19), (5.20) and (5.21) completes the proof. □

Proof of Proposition 3.8 (ii). Let $\eta \in (0, \boldsymbol{\eta})$ (and $\boldsymbol{\eta}$ is sufficiently small). We use the fundamental theorem of calculus

$$
F^{\pm\eta}(z_{\pm\epsilon}) = F^{\pm\boldsymbol{\eta}}(z_{\pm\epsilon}) + \int_{\pm\boldsymbol{\eta}}^{\pm\eta} \mathrm{d}\tilde{\eta}\,\mathrm{d}/\mathrm{d}\tilde{\eta}\,F^{\pm\tilde{\eta}}(z_{\pm\epsilon}), \qquad (5.29)
$$

and (5.10) to obtain that there is a constant $C(\boldsymbol{\eta}) > 0$ such that

$$
\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\| \leq \left\|F^{\pm\boldsymbol{\eta}}(z_{\pm\epsilon})\right\| + C\left|\int_{\pm\boldsymbol{\eta}}^{\pm\eta} \mathrm{d}\tilde{\eta}/\tilde{\eta}\right| \leq C(\boldsymbol{\eta})\,|\log\eta|. \qquad (5.30)
$$

Inserting this in Lemma 5.2, we obtain

$$
\left\|\mathrm{d}/\mathrm{d}\eta\,F^{\pm\eta}(z_{\pm\epsilon})\right\| \leq C(\boldsymbol{\eta})\eta^{-1/2}\,|\log\eta|, \qquad (5.31)
$$

and similarly as above, we find

$$
\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\| \leq \left\|F^{\pm\boldsymbol{\eta}}(z_{\pm\epsilon})\right\| + C(\boldsymbol{\eta})\left|\int_{\pm\boldsymbol{\eta}}^{\pm\eta} \mathrm{d}\tilde{\eta}\,\eta^{-1/2}\,|\log\eta|\right|. \qquad (5.32)
$$

We conclude that there is a constant $C(\boldsymbol{\eta}) > 0$ such that

$$
\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\| \leq C(\boldsymbol{\eta}). \qquad (5.33)
$$

Now we use the text below (5.4) and take the limit $\eta \to 0^+$ in (5.33). We conclude that (3.40) holds true (also (3.41), taking $g = 0$). Analogously, we show (3.41). □

In the remainder of this section we prove Proposition 3.8 (iii). The spirit of the proof is similar to the proof of statement (ii), however, we need additional estimates which are collected in the lemma below.

Lemma 5.3. *For $g \geq 0, \boldsymbol{\eta} > 0$ sufficiently small, $\eta \in (0, \boldsymbol{\eta})$, $\epsilon \in (0,1)$, $z, z' \in I$ and $z_{\pm\epsilon} := z \pm i\epsilon$, the following estimates hold true*

(i)

$$\left\| \mathrm{d}/\mathrm{d}\eta \left(F^{\pm\eta}(z_{\pm\epsilon}) - F^{\pm\eta}(z'_{\pm\epsilon}) \right) \right\| \leq C\eta^{-1/2} \tag{5.34}$$

(ii)

$$\left\| \mathrm{d}/\mathrm{d}\eta \left(F^{\pm\eta}(z_{\pm\epsilon}) - F^{\pm\eta}(z'_{\pm\epsilon}) \right) \right\| \leq C\eta^{-3/2}|z - z'| \tag{5.35}$$

(iii)

$$\left\| \mathrm{d}/\mathrm{d}\eta \left(F^{\pm\eta}(z_{\pm\epsilon}) - F_0^{\pm\eta}(z_{\pm\epsilon}) \right) \right\| \leq C\eta^{-3/2} g, \tag{5.36}$$

see Remark 5.1.

Proof. (i) It follows from Lemma 5.2 and (5.33).

(iii) Using the second resolvent identity, Remark 5.1, Remark 3.6 and (5.3), we get

$$\mp i \frac{\mathrm{d}}{\mathrm{d}\eta} \left(F^{\pm\eta}(z_{\pm\epsilon}) - F_0^{\pm\eta}(z_{\pm\epsilon}) \right) = \pm ig \frac{\mathrm{d}}{\mathrm{d}\eta} \left(\rho R^{\pm\eta}(z_{\pm\epsilon}) \tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon}) \rho \right), \tag{5.37}$$

where (see (3.20))

$$\tilde{V}_\eta := \sigma_1 \left(\Phi_{\overline{P}}(f) \mp i\eta \chi(H_{\overline{P}}) \Phi_{\overline{P}}(Df) \chi(H_{\overline{P}}) \right). \tag{5.38}$$

We write

$$\mp i \frac{\mathrm{d}}{\mathrm{d}\eta} \left(F^{\pm\eta}(z_{\pm\epsilon}) - F_0^{\pm\eta}(z_{\pm\epsilon}) \right) = g \left(W^{(1)} + W^{(2)} + W^{(3)} \right), \tag{5.39}$$

where

$$W^{(1)} := \rho \left(\pm i \frac{\mathrm{d}}{\mathrm{d}\eta} R^{\pm\eta}(z_{\pm\epsilon}) \right) \tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon}) \rho, \tag{5.40}$$

$$W^{(2)} := \rho R^{\pm\eta}(z_{\pm\epsilon}) \tilde{V}_\eta \left(\pm i \frac{\mathrm{d}}{\mathrm{d}\eta} R_0^{\pm\eta}(z_{\pm\epsilon}) \right) \rho, \tag{5.41}$$

$$W^{(3)} := \rho R^{\pm\eta}(z_{\pm\epsilon}) \chi(H_{\overline{P}}) \Phi_{\overline{P}}(Df) \chi(H_{\overline{P}}) R_0^{\pm\eta}(z_{\pm\epsilon}) \rho. \tag{5.42}$$

Eqs. (5.5) and (5.3) yield that

$$W := W^{(1)} + W^{(2)} = \sum_{i=1}^{4} (W_i^{(1)} + W_i^{(2)}), \tag{5.43}$$

where

$$W_1^{(1)} := -\rho R^{\pm\eta}(z_{\pm\epsilon})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.44}$$

$$W_2^{(1)} := \rho R^{\pm\eta}(z_{\pm\epsilon})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.45}$$

$$W_3^{(1)} := \rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.46}$$

$$W_4^{(1)} := -\rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.47}$$

$$W_1^{(2)} := -\rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})[H_{0,\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.48}$$

$$W_2^{(2)} := \rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})[H_{0,\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{0,\overline{P}}) R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.49}$$

$$W_3^{(2)} := \rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{0,\overline{P}})[H_{0,\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R_0^{\pm\eta}(z_{\pm\epsilon})\rho \tag{5.50}$$

$$W_4^{(2)} := -\rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{0,\overline{P}})[H_{0,\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{0,\overline{P}}) R_0^{\pm\eta}(z_{\pm\epsilon})\rho. \tag{5.51}$$

We observe from (5.39) that in order to complete the proof of statement (iii) it suffices to show that

$$\|W\| \le C\eta^{-3/2} \qquad \text{and} \qquad \left\|W^{(3)}\right\| \le C\eta^{-3/2}. \tag{5.52}$$

It follows from Proposition 1.2, (5.9) and similar estimates that that

$$\begin{aligned}\left\|\tilde{V}_\eta R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| &\le \left\|\tilde{V}_\eta (H_{f,\overline{P}}+i)^{-1}\right\| \left\|(H_{f,\overline{P}}+i)(H_{\overline{P}}+i)^{-1}\right\| \left\|(H_{\overline{P}}+i)R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \\ &\le C\left(1+\eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right)\end{aligned} \tag{5.53}$$

and similarly, using the adjoint operator, we find

$$\left\|\rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta\right\| \le C\left(1+\eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right). \tag{5.54}$$

This and (5.33) imply that

$$\left\|\tilde{V}_\eta R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \le C\eta^{-1/2} \qquad \text{and} \qquad \left\|\rho R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta\right\| \le C\eta^{-1/2}. \tag{5.55}$$

Using additionally (5.18), we get

$$\left\|W_2^{(1)}\right\| \le \left\|\rho R^{\pm\eta}(z_{\pm\epsilon})\right\| \left\|[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 \overline{\chi}(H_{\overline{P}}) R^{\pm\eta}(z_{\pm\epsilon})\right\| \left\|\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \le C\eta^{-1}. \tag{5.56}$$

Eqs. (5.55), (5.6) and (5.7), and the fact that $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0$ is $H_{\overline{P}}$-bounded (see Remark 3.6) imply that

$$\left\|W_3^{(1)}\right\| \le \left\|R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})\right\| \left\|[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0 R^{\pm\eta}(z_{\pm\epsilon})\right\| \left\|\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \le C\eta^{-3/2}. \tag{5.57}$$

Moreover, we obtain from (5.55), (5.6) and (5.18) that

$$\begin{aligned}\left\|W_4^{(1)}\right\| &\leq \left\|\rho R^{\pm\eta}(z_{\pm\epsilon})\overline{\chi}(H_{\overline{P}})\right\| \left\|[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\right\| \left\|\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \\ &\leq C\eta^{-1/2}. \end{aligned} \tag{5.58}$$

Analogously, we deduce that

$$\left\|W_2^{(2)}\right\|, \left\|W_3^{(2)}\right\|, \left\|W_4^{(2)}\right\| \leq C\eta^{-3/2}. \tag{5.59}$$

Next, we estimate the terms $W_1^{(1)}$ and $W_1^{(2)}$. For $\phi, \psi \in \mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$, we find

$$\left\langle \phi, \left(W_1^{(1)} + W_1^{(2)}\right)\psi\right\rangle = A_1 + A_2 + A_3 + A_4, \tag{5.60}$$

where

$$\begin{aligned} A_1 :=& -\left\langle \left(H_{\overline{P}}^{\mp\eta} - z_{\mp\epsilon}\right) R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D)R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle \\ &+ \left\langle (-i\mathrm{d}\Gamma(D))\, R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, \left(H_{\overline{P}}^{\pm\eta} - z_{\pm\epsilon}\right) R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle, \\ A_2 :=& \mp i\eta\Big(\left\langle M^2 R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D)R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle \\ &- \left\langle (-i\mathrm{d}\Gamma(D))\, R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, M^2 R^{\pm\eta}(z_{\pm\epsilon})\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle \Big), \\ A_3 :=& -\left\langle \left(H_{0,\overline{P}}^{\mp\eta} - z_{\mp\epsilon}\right) R_0^{\mp\eta}(z_{\mp\epsilon})(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D)R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle \\ &+ \left\langle (-i\mathrm{d}\Gamma(D))\, R_0^{\mp\eta}(z_{\mp\epsilon})(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, \left(H_{0,\overline{P}}^{\pm\eta} - z_{\pm\epsilon}\right) R_0^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle, \\ A_4 :=& \mp i\eta\Big(\left\langle M^2 R_0^{\mp\eta}(z_{\mp\epsilon})(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, i\mathrm{d}\Gamma(D)R^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle \\ &- \left\langle (-i\mathrm{d}\Gamma(D))\, R_0^{\mp\eta}(z_{\mp\epsilon})(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})\rho\phi, M^2 R^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\rangle \Big). \end{aligned} \tag{5.61}$$

This is possible because ρ maps the Hilbert space $\mathcal{H}$ into the domain of $\mathrm{d}\Gamma(D)$ and – by Lemma D.3 – $R^{\pm}(z_{\pm}\epsilon)$, $(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})$ and $V_\eta R^{\pm\eta}(z_{\pm\epsilon})$ preserve the domain of $\mathrm{d}\Gamma(D)$ (see above (5.11) – this holds true also for $g = 0$, see Remark 5.1). We estimate

$$|A_2| \leq \eta \|\phi\| \|\psi\| \left\|\tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \left\|R^{\pm\eta}(z_{\pm\epsilon})\right\| \left\|R^{\mp\eta}(z_{\mp\epsilon})\rho\right\| \left\|[M^2, \mathrm{d}\Gamma(D)]^0\right\|. \tag{5.62}$$

Eqs. (5.55), (5.33), (5.9) and (5.7) imply that

$$|A_2| \leq C \|\phi\| \|\psi\| \eta^{-1}, \tag{5.63}$$

and analogously, we find

$$|A_4| \leq C \|\phi\| \|\psi\| \eta^{-1}. \tag{5.64}$$

As we argue above, Lemma D.3 implies that $R^{\pm}(z_{\pm}\epsilon)$, $(\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon})$ and $V_\eta R^{\pm\eta}(z_{\pm\epsilon})$ preserve the domain of $\mathrm{d}\Gamma(D)$ (see above (5.11) - this holds true also for $g = 0$, see Remark 5.1). Moreover, the quadratic form $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]$ is represented by a $H_{\overline{P}}$-bounded operator that we denote by $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]^0$ (see Lemma D.5). We obtain that

$$\begin{aligned} A_1 + A_3 = &- \left\langle (-i\mathrm{d}\Gamma(D))\, \rho\phi, R^{\pm\eta}(z_{\pm\epsilon}) \tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon}) \rho\psi \right\rangle \\ &+ \left\langle R^{\mp\eta}(z_{\mp\epsilon}) \rho\phi, [(i\mathrm{d}\Gamma(D)) \,, \tilde{V}_\eta]^0 R_0^{\pm\eta}(z_{\pm\epsilon}) \rho\psi \right\rangle \\ &+ \left\langle R_0^{\mp\eta}(z_{\mp\epsilon}) (\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon}) \rho\phi, i\mathrm{d}\Gamma(D) \rho\psi \right\rangle. \end{aligned} \tag{5.65}$$

It follows from (5.9), (5.33) and the fact that $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]^0$ is $H_{\overline{P}}$-bounded (see Lemma D.5) that

$$\left| \left\langle R^{\mp\eta}(z_{\mp\epsilon}) \rho\phi, [(i\mathrm{d}\Gamma(D)) \,, \tilde{V}_\eta]^0 R_0^{\pm\eta}(z_{\pm\epsilon}) \rho\psi \right\rangle \right| \leq \|\phi\| \|\psi\| \eta^{-1}. \tag{5.66}$$

We obtain from (5.55) and (5.7) that

$$\left| \left\langle (-i\mathrm{d}\Gamma(D))\, \rho\phi, R^{\pm\eta}(z_{\pm\epsilon}) \tilde{V}_\eta R_0^{\pm\eta}(z_{\pm\epsilon}) \rho\psi \right\rangle \right| \leq C \|\phi\| \|\psi\| \eta^{-3/2}, \tag{5.67}$$

$$\left| \left\langle R_0^{\mp\eta}(z_{\mp\epsilon}) \tilde{V}_\eta^* R^{\mp\eta}(z_{\mp\epsilon}) \rho\phi, i\mathrm{d}\Gamma(D) \rho\psi \right\rangle \right| \leq C \|\phi\| \|\psi\| \eta^{-3/2}. \tag{5.68}$$

This together with (5.65) and (5.66) yield that

$$|A_1 + A_3| \leq C \|\phi\| \|\psi\| \eta^{-3/2}. \tag{5.69}$$

It follows from (5.69), (5.63), (5.64) and (5.60) that

$$\left\| W_1^{(1)} + W_1^{(2)} \right\| \leq C \eta^{-3/2}. \tag{5.70}$$

Collecting (5.43), (5.56), (5.57), (5.58), (5.59) and (5.70), we deduce that

$$\|W\| \leq C \eta^{-3/2}. \tag{5.71}$$

Eqs. (5.33) and (5.9) together with the $H_{0,\overline{P}}$-boundedness of $\Phi_{\overline{P}}(Df)$ yield that

$$\left\| W^{(3)} \right\| \leq C \eta^{-1}. \tag{5.72}$$

This together with (5.71) imply that (5.52) holds true and, thereby, we complete the proof of Item (iii).

(ii) The proof of Item (ii) follows the same line of arguments as the proof of Item (iii). In fact, it is simpler since the term $\tilde{V}_\eta$ does not appear.

□

Proof of Proposition 3.8 (iii). We estimate, for $z, z' \in I$,

$$\left\| F^0(z'_{\pm\epsilon}) - F^0_0(z_{\pm\epsilon}) \right\| \leq \left\| F^0(z'_{\pm\epsilon}) - F^0(z_{\pm\epsilon}) \right\| + \left\| F^0(z_{\pm\epsilon}) - F^0_0(z_{\pm\epsilon}) \right\|. \tag{5.73}$$

Hence, it suffices to show that

$$\left\| F^0(z'_{\pm\epsilon}) - F^0(z_{\pm\epsilon}) \right\| \leq C|z - z'|^{1/2}, \tag{5.74}$$

and

$$\left\| F^0(z_{\pm\epsilon}) - F^0_0(z_{\pm\epsilon}) \right\| \leq C g^{1/2}. \tag{5.75}$$

In the remainder of the proof we show (5.74) and (5.75). We start with the first estimate and obtain for $\tilde{\eta} \in (0, \boldsymbol{\eta})$

$$\begin{aligned} F^0(z'_{\pm\epsilon}) - F^0(z_{\pm\epsilon}) &= - \int_0^{\tilde{\eta}} \mathrm{d}\eta \, \frac{d}{d\eta}(F^\eta(z'_{\pm\epsilon}) - F^\eta(z_{\pm\epsilon})) \\ - \int_{\tilde{\eta}}^{\boldsymbol{\eta}} \mathrm{d}\eta \, \frac{d}{d\eta}&(F^\eta(z'_{\pm\epsilon}) - F^\eta(z_{\pm\epsilon})) + F^{\boldsymbol{\eta}}(z'_{\pm\epsilon}) - F^{\boldsymbol{\eta}}(z_{\pm\epsilon}). \end{aligned} \tag{5.76}$$

It follows from Lemma 5.3 (i) that

$$\left\| \int_0^{\tilde{\eta}} \mathrm{d}\eta \, \frac{d}{d\eta}(F^\eta(z'_{\pm\epsilon}) - F^\eta(z_{\pm\epsilon})) \right\| \leq C \tilde{\eta}^{1/2}. \tag{5.77}$$

Moreover, it follows from Lemma 5.3 (ii) that

$$\left\| \int_{\tilde{\eta}}^{\boldsymbol{\eta}} \mathrm{d}\eta \, \frac{d}{d\eta}(F^\eta(z'_{\pm\epsilon}) - F^\eta(z_{\pm\epsilon})) \right\| \leq C|z - z'| \tilde{\eta}^{-1/2}, \tag{5.78}$$

and it follows from the resolvent identity that there is a constant $C(\boldsymbol{\eta}) > 0$ such that

$$\| F^{\boldsymbol{\eta}}(z'_{\pm\epsilon}) - F^{\boldsymbol{\eta}}(z_{\pm\epsilon}) \| \leq C(\boldsymbol{\eta})|z - z'|. \tag{5.79}$$

Note that, in principle, the constant $C(\boldsymbol{\eta})$ could depend on ϵ and z. However, this is not the case, see (5.7) Choosing $\tilde{\eta} = |z - z'|^{1/2}$, we get (5.74) from (5.76) – (5.79). Eq. (5.75) can be proven analogously employing item (iii) of Lemma 5.3 instead of item (ii). □

A Construction of the ground state

In this chapter, we construct the ground state of the Hamiltonian H and provide a proof for Proposition 1.4. First of all, for $0 < r < r' < \infty$ and $w \in \mathbb{C}$, we introduce the notation for the open annulus in the complex plane:

$$D(r, r', w) = \left\{ z \in \mathbb{C} : r < |z - w| < r' \right\}. \tag{A.1}$$

Lemma A.1. *Let $g > 0$ be small enough. Then, $H - z$ is invertible for all $z \in \overline{D(m/4, m/2, 0)}$ (defined in* (A.1)*) and*

$$\left\|(H - z)^{-1}\right\| \leq 2 \left\|(H_0 - z)^{-1}\right\| \leq 8/m \qquad \forall z \in \overline{D(m/4, m/2, 0)}. \tag{A.2}$$

Proof. First of all, note that $\sigma(H_0) = \{0\} \cup [m, \infty)$. This implies that

$$\operatorname{dist}\left(D(m/4, m/2, 0), \sigma(H_0)\right) \geq 4/m, \tag{A.3}$$

and hence, $H_0 - z$ is invertible for all $z \in D(m/4, m/2, 0)$, and for those z, we have

$$\left\|(H_0 - z)^{-1}\right\| \leq 4/m. \tag{A.4}$$

Moreover, it follows from the standard estimate in Proposition 1.2 that

$$\left\|V(H_0 + 1)^{-1}\right\| \leq C, \tag{A.5}$$

and hence, we obtain for all $z \in D(m/4, m/2, 0)$

$$\left\|V(H_0 - z)^{-1}\right\| \leq \left\|V(H_0 + 1)^{-1}\right\| \left\|\frac{H_0 + 1}{H_0 - z}\right\| \leq C \sup_{y \geq 0} \left|\frac{y + 1}{y - z}\right| \leq C(3 + 4/m). \tag{A.6}$$

Consequently, for $g > 0$ sufficiently small, we find

$$\left\|V(H_0 - z)^{-1}\right\| \leq Cg \leq 1/2, \tag{A.7}$$

and hence,

$$H - z = (1 + gV(H_0 - z)^{-1})(H_0 - z) \tag{A.8}$$

is invertible for all $z \in D(m/4, m/2, 0)$ and the resolvent fulfills

$$\left\|(H - z)^{-1}\right\| \leq 2 \left\|(H_0 - z)^{-1}\right\| \leq 8/m. \tag{A.9}$$

□

Definition A.2. *We define the contour*

$$\zeta : [0, 2\pi] \to \mathbb{C}, \qquad \varphi \mapsto \zeta(t) := m/4e^{it}. \tag{A.10}$$

Furthermore, we define the projections

$$P_{0,at} := (-2\pi i)^{-1} \oint_\zeta \mathrm{d}z \, (H_0 - z)^{-1} = P_{\varphi_0} \otimes P_\Omega \tag{A.11}$$

and

$$P_0 := (-2\pi i)^{-1} \oint_\zeta \mathrm{d}z \, (H - z)^{-1}. \tag{A.12}$$

Here, P_{φ_0} denotes the projection onto φ_0 and P_Ω the projection onto the vacuum $\Omega \in \mathcal{F}[\mathfrak{h}]$. The equality in (A.11) *can be seen by a direct calculation.*

Lemma A.3. *Let $g > 0$ be small enough and Assumption 1.1 hold true. Then, we find*

$$\|P_0 - P_{0,at}\| \le gC < 1. \tag{A.13}$$

Proof. It follows from Definition A.2 that

$$\begin{aligned} \|P_0 - P_{0,\mathrm{at}}\| &\le (2\pi)^{-1} \int_0^{2\pi} \mathrm{d}t \, \left\| (H - m/4e^{it})^{-1} - (H_0 - m/4e^{it})^{-1} \right\| \\ &\le g \sup_{t\in[0,2\pi]} \left\| (H - m/4e^{it})^{-1} \right\| \left\| V(H_0 - m/4e^{it})^{-1} \right\|, \end{aligned} \tag{A.14}$$

where we used the resolvent identity in the second step. This together with (A.7) and Lemma A.1 completes the proof. □

Proof of Proposition 1.4. Clearly, $P_{0,\mathrm{at}} = P_{\varphi_0} \otimes P_\Omega$ is a rank-one projection, and hence, it follows from Lemma A.3 that also P_0 is a rank-one projection. Consequently, the self-adjoint operator H has exactly one eigenvalue in $(-m/4, m/4)$ which we call λ_0 and $\Psi_{\lambda_0} := P_0\varphi_0 \otimes \Omega \in \mathcal{H}$ is nonzero and fulfills $H\Psi_{\lambda_0} = \lambda_0 \Psi_{\lambda_0}$.

In the remainder of the proof we compute λ_0 up to second order in g.

$$(\lambda_0 - e_0) \left|\langle \phi_0, P_0\phi_0 \rangle\right| = \langle \phi_0, (H - H_0)P_0\phi_0 \rangle = g \langle \phi_0, VP_0\phi_0 \rangle , \tag{A.15}$$

where we have introduced the notation $\phi_i = \varphi_i \otimes \Omega$ for $i = 0, 1$. Moreover, the resolvent identity yields that

$$\begin{aligned} &\left\langle \phi_0, V(H - z)^{-1}\phi_0 \right\rangle = \left\langle \phi_0, V(H_0 - z)^{-1}\phi_0 \right\rangle - g \left\langle \phi_0, V(H_0 - z)^{-1}V(H_0 - z)^{-1}\phi_0 \right\rangle \\ &+ g^2 \left\langle \phi_0, V(H_0 - z)^{-1}V(H_0 - z)^{-1}V(H_0 - z)^{-1}\phi_0 \right\rangle \\ &- g^3 \left\langle \phi_0, V(H - z)^{-1}V(H_0 - z)^{-1}V(H_0 - z)^{-1}V(H_0 - z)^{-1}\phi_0 \right\rangle . \end{aligned} \tag{A.16}$$

Note that the even orders of g vanish due to symmetry and recall from (A.4) that $\|V(H_0 - z)^{-1}\| \le C$. This implies that

$$\left\| V(H - z)^{-1} \right\| \le \left\| (H_0 - z)(H - z)^{-1} \right\| \left\| V(H_0 - z)^{-1} \right\| \le C(1 + gC). \tag{A.17}$$

Consequently, we obtain

$$\begin{aligned} \left\langle \phi_0, V(H - z)^{-1}\phi_0 \right\rangle &= -g(e_0 - z)^{-1} \left\langle a(f)^*\phi_1, (H_0 - z)^{-1}a(f)^*\phi_1 \right\rangle + \tilde{R}_0(g) \\ &= -g(e_0 - z)^{-1} \int \mathrm{d}^3k \, |f(k)|^2 (e_1 + \omega(k) - z)^{-1} + \tilde{R}_0(g), \end{aligned} \tag{A.18}$$

where $|\tilde{R}_0(g)| \le Cg^3$. Then, it follows from (A.15) together with Definition A.2 that

$$\lambda_0 = e_0 - g^2\Gamma_0 + R_0(g) \tag{A.19}$$

where $R_0(g) = g\,|\langle\phi_0, P_0\phi_0\rangle|^{-1}\,\tilde{R}_0(g)$ and

$$\Gamma_0 := (-2\pi i)^{-1} \oint_\zeta \mathrm{d}z\,(e_0 - z)^{-1} \int \mathrm{d}^3k\,|f(k)|^2 (e_1 + \omega(k) - z)^{-1}. \tag{A.20}$$

Fubini's theorem allows for interchanging the order of integration, and hence, we obtain from the Cauchy integral theorem

$$\Gamma_0 = \int \mathrm{d}^3k\,|f(k)|^2 (e_1 - e_0 + \omega(k))^{-1}. \tag{A.21}$$

This completes the proof of the first part of the proposition. The second part follows from the definition of the ground state:

$$\Psi_{\lambda_0} = P_0\varphi_0 \otimes \Omega = \varphi_0 \otimes \Omega + \widetilde{\Psi}_{\lambda_0}, \tag{A.22}$$

where $\widetilde{\Psi}_{\lambda_0} := (P_0 - P_{0,\mathrm{at}})\varphi_0 \otimes \Omega \in \mathcal{H}$ and Lemma A.3 yields that $\left\|\widetilde{\Psi}_{\lambda_0}\right\| \le Cg$. Moreover, note that $\varphi_0 \otimes \Omega$ is the unique ground state of H_0, and hence, $P_{0,\mathrm{at}}$ is a rank-one projector. We conclude the uniqueness of Ψ_{λ_0} again from Lemma A.3. □

B Spectral projections

Definition B.1. *For $v \in C^\infty(\mathbb{R}, \mathbb{C})$, we define its almost analytic extension by*

$$\tilde{v} : \mathbb{C} \to \mathbb{C}, \qquad \tilde{v}(z) = \sigma(\operatorname{Re} z, \operatorname{Im} z) \sum_{r=0}^{n} \frac{(i \operatorname{Im} z)^r}{r!} v^{(r)}(\operatorname{Re} z), \tag{B.1}$$

where $n \in \mathbb{N}$, $v^{(r)}$ denotes the r-th derivative of v and

$$\sigma(\operatorname{Re} z, \operatorname{Im} z) := \tau\left(\frac{\operatorname{Im} z}{\sqrt{(\operatorname{Re} z)^2 + 1}}\right) \tag{B.2}$$

for some $\tau \in C^\infty(\mathbb{R}, \mathbb{C})$ with $\tau(t) = 1$ for all $|t| < 1$ and $\tau(t) = 0$ for all $|t| > 2$. It follows from [26, Section 2.2] that

(i) $\tilde{v}$ is smooth as a function of $(\operatorname{Re} z, \operatorname{Im} z)$.

(ii) If v is compactly supported, $|\partial_{\bar{z}} \tilde{v}(z)| \le C|\operatorname{Im} z|^n$ (where $\frac{d}{d\bar{z}} = \frac{1}{2}(\frac{d}{dx} + i\frac{d}{dy})$, with $z = x + iy$).

Theorem B.2 (Helffer-Sjöstrand formula)**.** *For every selfadjoint operator and any $v \in C_0^\infty(\mathbb{R}, \mathbb{C})$, the next formula holds true*

$$v(O) = \pi^{-1} \int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\, \partial_{\bar{z}} \tilde{v}(z) (O - z)^{-1}, \tag{B.3}$$

where $z = x + iy$, for $x, y \in \mathbb{R}$. Eq. (B.3) *does not depend on n and σ.*

Proof of Lemma 3.5. We only prove (3.19). Since (B.3) does not depend on σ, we choose $n = 2$ and, for $s > 0$, $\sigma_s(\operatorname{Re} z, \operatorname{Im} z) := \tau\left(\frac{1}{s}\frac{\operatorname{Im} z}{\sqrt{(\operatorname{Re} z)^2+1}}\right)$. We denote by $\tilde{\chi}_s$ the corresponding almost analytic extension of χ_s. It follows form (B.3) and the resolvent equation that

$$\|\chi_s(H) - \chi_s(H_0)\| = \pi^{-1}\Big\| \int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\, \partial_{\overline{z}}\, \tilde{\chi}_s(z)(H-z)^{-1} gV(H_0 - z)^{-1}\Big\|. \tag{B.4}$$

We calculate now

$$\partial_{\overline{z}}\, \tilde{\chi}_s(z) = \frac{1}{2}\sum_{r=0}^{2} \chi_s^{(r)}(x)(iy)^r/r!(\frac{\partial}{x}\sigma_s + i\frac{\partial}{\partial y}\sigma_s) + \frac{1}{2}\chi_s^{(n+1)}(x)(iy)^n/n!\sigma. \tag{B.5}$$

Notice that $\|(H-z)^{-1}gV\frac{1}{H_f+1}(H_f+1)(H_0-z)^{-1}\| \leq Cg\frac{1}{|y|^2}$. Moreover, $|\chi_s^{(r)}(x)|\,|y|^r|\frac{\partial}{x}\sigma_s + i\frac{\partial}{\partial y}\sigma_s| \leq C\frac{1}{s}\frac{|y|^r}{s^r}$, $|\chi_s^{(n+1)}(x)||y|^n|\sigma| \leq C\frac{1}{s^3}|y|^2$. This together with (B.4) yields

$$\begin{aligned}\|\chi_s(H) - \chi_s(H_0)\| \leq{}& Cg\sum_{r=0}^{2}\int_{\operatorname{supp}(|\chi_s^{(r)}|\,|\frac{\partial}{x}\sigma_s + i\frac{\partial}{\partial y}\sigma_s|)} \mathrm{d}x\mathrm{d}y\,\frac{1}{s}\frac{|y|^{r-2}}{s^r}\\ &+ Cg\int_{\operatorname{supp}(|\chi_s^{(3)}||\sigma|)}\mathrm{d}x\mathrm{d}y\,\frac{1}{s^3}.\end{aligned} \tag{B.6}$$

For $y \in \mathbb{R}$, we observe that the diameter of the support of the functions $\mathbb{R} \ni x \mapsto |\chi_s^{(r)}(x)|\,|\frac{\partial}{x}\sigma_s(x,y) + i\frac{\partial}{\partial y}\sigma_s(x,y)|$ and $\mathbb{R} \ni x \mapsto \operatorname{supp}(|\chi_s^{(3)}(x)||\sigma(x,y)|)$ is of order s. Moreover, for $x \in \mathbb{R}$, we find that the diameter of the support of the function $\mathbb{R} \ni y \mapsto \operatorname{supp}(|\chi_s^{(3)}(x)||\sigma(x,y)|)$ is of order s. We conclude that

$$\|\chi_s(H) - \chi_s(H_0)\| \leq C\frac{g}{s}, \tag{B.7}$$

which is the desired result. □

C Standard Results from Mourre Theory

In this section we prove all assertions and estimates described at the beginning of Section 5, upto (5.9). We adapt the proofs of [25] to our model.

Lemma C.1. *Recall $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ from Definition 3.4. For $g \geq 0, \boldsymbol{\eta} > 0$ sufficiently small, $\eta \in (0, \boldsymbol{\eta})$, $\epsilon \in (0,1)$, $z \in I$ and $z_{\pm\epsilon} := z \pm i\epsilon$, the following statements hold true:*

(i) The operator $R^{\pm\eta}(z_{\pm\epsilon})$ introduced in (5.4) exists and it is in $C^1((0,\boldsymbol{\eta}))$ and $C^0([0,\boldsymbol{\eta}))$ with respect to η. Moreover, the following identity holds true:

$$\mathrm{d}/\mathrm{d}\eta\, R^{\pm\eta}(z_{\pm\epsilon}) = \pm i R^{\pm\eta}(z_{\pm\epsilon})M^2 R^{\pm\eta}(z_{\pm\epsilon}), \qquad \forall \eta \in (0, \boldsymbol{\eta}). \tag{C.1}$$

(ii)

$$\left\|(H_{\overline{P}} + i)\chi(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\psi\right\| \leq C\eta^{-1/2}\left|\langle\psi, R^{\pm\eta}(z_{\pm\epsilon})\psi\rangle\right|^{1/2}. \tag{C.2}$$

(iii)

$$\left\|(H_{\overline{P}}+i)\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\right\| \leq C, \tag{C.3}$$

where we recall that $\overline{\chi} = 1-\chi$.

(iv)

$$\left\|(H_{\overline{P}}+i)R^{\pm\eta}(z_{\pm\epsilon})\right\| \leq C/\eta. \tag{C.4}$$

(v)

$$\left\|(H_{\overline{P}}+i)R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \leq C\left(1+\eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right). \tag{C.5}$$

The constants C *above do not depend on* η, ϵ, z *and* g, *see Remark 2.3.*

Proof. (i) Recall that $H_{\overline{P}}$ is a closed operator and M^2 is bounded (see Remark 3.6). Consequently, $H_{\overline{P}}^{\pm\eta}$ is closed. For $\psi \in \mathcal{D}(H_{\overline{P}})$, we observe that

$$\left\|\left(H_{\overline{P}}^{\pm\eta}-z_{\pm\epsilon}\right)\psi\right\|^2 = \left\|\left(H_{\overline{P}}^{\pm\eta}-z\right)\psi\right\|^2 + \epsilon^2\left\|\psi\right\|^2 + 2\eta\epsilon\left\|M\psi\right\|^2, \tag{C.6}$$

and, thereby, the range of $H_{\overline{P}}^{\pm\eta}-z_{\pm\epsilon}$ is closed and $H_{\overline{P}}^{\pm\eta}-z_{\pm\epsilon}$ is injective. It also follows from the equation above that its inverse is bounded. Moreover, $\left(H_{\overline{P}}^{\pm\eta}-z_{\pm\epsilon}\right)^*$ fulfills a similar estimate and it is, therefore, injective. This implies that the range of $H_{\overline{P}}^{\pm\eta}-z_{\pm\epsilon}$ is dense and because it is also closed, $H_{\overline{P}}^{\pm\eta}-z_{\pm\epsilon}$ is surjective.
In addition, the resolvent identity yields that

$$R^{\pm\eta}(z_{\pm\epsilon}) - R^{\pm\eta_0}(z_{\pm\epsilon}) = \pm i(\eta-\eta_0)R^{\pm\eta}(z_{\pm\epsilon})M^2R^{\pm\eta_0}(z_{\pm\epsilon}). \tag{C.7}$$

It follows from (C.6) that there is a constant $C>0$ (independent of η) such that $\|R^{\pm\eta}(z_{\pm\epsilon})\| \leq C/\epsilon$. This together with (C.7) and the fact that M^2 is bounded implies that $R^{\pm\eta}(z_{\pm\epsilon})$ is continuous with respect to η, for $\eta \geq 0$, and differentiable for $\eta > 0$. Moreover, taking $\eta \to 0$ in (C.7) we get (C.1).

(ii) It follows from Lemma 3.7 that there is a constant $\alpha > 0$ such that for $\psi \in \mathcal{H}$

$$\begin{aligned}
&\left\|(H_{\overline{P}}+i)\chi(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\psi\right\|^2 = \left\langle\psi, R^{\pm\eta}(z_{\pm\epsilon})^*(H_{\overline{P}}^2+1)\chi^2(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\psi\right\rangle \\
&\leq ((e_1+\delta)^2+1)\alpha^{-1}\left\langle\psi, R^{\pm\eta}(z_{\pm\epsilon})^*\alpha\chi^2(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\psi\right\rangle \\
&\leq ((e_1+\delta)^2+1)(2\alpha\eta)^{-1}\left\langle\psi, R^{\pm\eta}(z_{\pm\epsilon})^*(2\eta M^2+2\epsilon)R^{\pm\eta}(z_{\pm\epsilon})\psi\right\rangle \\
&= ((e_1+\delta)^2+1)(2\alpha\eta)^{-1}\left\langle\psi, i(R^{\pm\eta}(z_{\pm\epsilon})^*-R^{\pm\eta}(z_{\pm\epsilon}))\psi\right\rangle \\
&\leq ((e_1+\delta)^2+1)(\alpha\eta)^{-1}\left|\left\langle\psi, R^{\pm\eta}(z_{\pm\epsilon})\psi\right\rangle\right|.
\end{aligned} \tag{C.8}$$

This implies then statement (ii).

(iii) We calculate

$$\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon}) = \overline{\chi}(H_{\overline{P}})R^0(z_{\pm\epsilon})\left(H_{\overline{P}} - z_{\pm\epsilon}\right)R^{\pm\eta}(z_{\pm\epsilon})$$
$$= \overline{\chi}(H_{\overline{P}})R^0(z_{\pm\epsilon})\left(1 \pm i\eta M^2 R^{\pm\eta}(z_{\pm\epsilon})\right). \tag{C.9}$$

It follows from Definition 3.4 and (5.1) that $\|\overline{\chi}(H_{\overline{P}})R^0(z_{\pm\epsilon})\| \le 4/\delta$. Moreover,

$$\left\|H_{\overline{P}}\overline{\chi}(H_{\overline{P}})R^0(z_{\pm\epsilon})\right\| = \left\|\overline{\chi}(H_{\overline{P}}) + z_{\pm\epsilon}\overline{\chi}(H_{\overline{P}})R^0(z_{\pm\epsilon})\right\|. \tag{C.10}$$

We obtain that

$$\left\|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}})R^0(z_{\pm\epsilon})\right\| \le C. \tag{C.11}$$

This together with (C.9) and the boundedness of M^2 yields that

$$\|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\| \le C\left(1 + \eta\left\|R^{\pm\eta}(z_{\pm\epsilon})\right\|\right). \tag{C.12}$$

Statement (iii) follows then by (iv) which is proven below.

(iv) It follows from (ii) together with (C.12) that there are constants $C, \tilde{C} > 0$ such that

$$\begin{aligned} &1 + \|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\| \\ &\le 1 + \|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\| + \|(H_{\overline{P}} + i)\chi(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\| \\ &\le 1 + \tilde{C}\left(1 + \eta\left\|R^{\pm\eta}(z_{\pm\epsilon})\right\|\right) + C\eta^{-1/2}\left\|R^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}. \end{aligned} \tag{C.13}$$

We fix $\boldsymbol{\eta} > 0$ sufficiently small such that $\tilde{C}\boldsymbol{\eta} \le 1/2$ and $\tilde{C} + 1 \le C\boldsymbol{\eta}^{-1/2}$. Then, employing $|x| + 1 \le 2\sqrt{x^2+1}$ for all $x \in \mathbb{R}$, we conclude for $\eta \in (0, \boldsymbol{\eta})$

$$\begin{aligned} &1 + \|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\| \le C\eta^{-1/2}\left(1 + \left\|R^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}\right) + \frac{1}{2}\left(1 + \left\|R^{\pm\eta}(z_{\pm\epsilon})\right\|\right) \\ &\le 2C\eta^{-1/2}\left(1 + \|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\|\right)^{1/2} + \frac{1}{2}\left(1 + \|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\|\right). \end{aligned} \tag{C.14}$$

This yields then

$$1 + \|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\| \le 4C\eta^{-1/2}\left(1 + \|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\|\right)^{1/2}, \tag{C.15}$$

and hence,

$$\|(H_{\overline{P}} + i)R^{\pm\eta}(z_{\pm\epsilon})\| \le 16C^2\eta^{-1}, \tag{C.16}$$

which implies statement (iv).

(v) For $\psi \in \mathcal{H}$, we apply statement (ii) to the vector $\rho\psi \in \mathcal{H}$ and find that there is a constant $C > 0$ such that

$$\left\|(H_{\overline{P}} + i)\chi(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\rho\psi\right\| \leq C\eta^{-1/2}\left|\langle\psi, F^{\pm\eta}(z_{\pm\epsilon})\psi\rangle\right|^{1/2}, \tag{C.17}$$

which implies

$$\left\|(H_{\overline{P}} + i)\chi(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \leq C\eta^{-1/2}\left\|F^{\pm\eta}(z_{\pm\epsilon})\right\|^{1/2}. \tag{C.18}$$

In addition, it follows from statement (iii) that

$$\left\|(H_{\overline{P}} + i)\overline{\chi}(H_{\overline{P}})R^{\pm\eta}(z_{\pm\epsilon})\rho\right\| \leq C. \tag{C.19}$$

This together with (C.18) completes the proof of statement (v).

□

D Domain Properties and Commutator Estimates in Mourre Theory

D.1 Domain Properties in Mourre Theory

In this section we prove auxiliary technical results that we need in Section 5. In particular, we prove that $R^{\pm\eta}(z_{\pm\epsilon})$ (see (5.4)) leaves the domain of $\mathrm{d}\Gamma(D)$ invariant – this (and similar results) might be regarded as the main result of this section, see Lemma D.3. In this paper we do not use the standard strategy and we believe that our method is much simpler and direct than the usual one: A novelty of our presentation is that we do not employ the usual techniques to study domain problems and commutators. The standard presentation of Mourre theory includes a scale of Hilbert spaces and a regularization of the generator of dilations in order to address domain problems (which is a technical and delicate issue – see [25]). In our case, instead of stating scales of Hilbert spaces explicitly and regularizing the generator of dilations, we directly dilate the operators at stake. We point out to the reader that the details of the arguments in this section are rarely found in the literature. A presentation of similar arguments may be found, e.g., in [41].

Definition D.1. *Let B be a closed operator, defined in $\mathcal{H}$. For every $\beta \in \mathbb{R}$, we denote its dilation by*

$$B^{(\beta)} = e^{-i\beta \mathrm{d}\Gamma(D)} B e^{i\beta \mathrm{d}\Gamma(D)}. \tag{D.1}$$

For every function $h : \mathbb{R}^3 \to \mathbb{R}$ we denote by $h^{(\beta)}(k) := h(e^{\beta}k)$. A direct calculation shows that (see Definition 3.1)

$$H_{\overline{P}}^{(\beta)} = H_{\overline{P}}(\omega^{(\beta)}, u_\beta f), \quad (M^2)^{(\beta)} = \chi(H_{\overline{P}}^{(\beta)}) H_{\overline{P}}(\xi^{(\beta)}, u_\beta Df)\chi(H_{\overline{P}}^{(\beta)}), \tag{D.2}$$

see Remark 3.6, and (see (5.4)*)*

$$(H_{\overline{P}}^{\pm\eta})^{(\beta)} := H_{\overline{P}}^{(\beta)} \mp i\eta(M^2)^{(\beta)}, \qquad (R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)} = \left((H_{\overline{P}}^{\pm\eta})^{(\beta)} - z_{\pm\epsilon}\right)^{-1}. \tag{D.3}$$

Lemma D.2. *Let B be a bounded operator in $\mathcal{H}$. Assume that the map $\beta \mapsto B^{(\beta)}$ is continuous at* 0 *and, for every $\phi \in \mathcal{D}(\mathrm{d}\Gamma(D))$, the limit*

$$\lim_{\beta \to 0} \frac{1}{\beta}(B^{(\beta)} - B)\phi \tag{D.4}$$

exists. Then, $\mathcal{D}(\mathrm{d}\Gamma(D))$ is invariant under B. In particular this holds true if the map $\beta \mapsto B^{(\beta)}$ is differentiable at 0.

Proof. We recall that $B\phi \in \mathcal{D}(\mathrm{d}\Gamma(D))$ if and only if the function $\beta \mapsto e^{-i\beta \mathrm{d}\Gamma(D)} B\phi$ is differentiable at 0. Set $\phi \in \mathcal{D}(\mathrm{d}\Gamma(D))$. We notice that the limit

$$\lim_{\beta \to 0} \frac{1}{\beta}(e^{-i\beta \mathrm{d}\Gamma(D)} - 1)B\phi = \lim_{\beta \to 0} \frac{1}{\beta}(B^{(\beta)} - B)\phi + B^{(\beta)} \frac{1}{\beta}(e^{-i\beta \mathrm{d}\Gamma(D)} - 1)\phi \tag{D.5}$$

exists because $\phi \in \mathcal{D}(\mathrm{d}\Gamma(D))$ (see (D.4) and above). □

Lemma D.3. *The derivatives (recall* (5.38)*)*

$$\frac{\partial}{\partial \beta} \frac{1}{H^{(\beta)}_{\overline{P}} - \lambda}|_{\beta=0}, \qquad \frac{d}{d\beta}\chi(H^{(\beta)}_{\overline{P}})|_{\beta=0}, \qquad \frac{\partial}{\partial \beta}(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}|_{\beta=0}, \tag{D.6}$$
$$\frac{\partial}{\partial \beta}((\tilde{V}_\eta)^* R^{\mp\eta}(z_{\mp\epsilon}))^{(\beta)}|_{\beta=0}, \qquad \frac{\partial}{\partial \beta}(\tilde{V}_\eta R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}|_{\beta=0}$$

exist, and therefore, the operators above leave $\mathcal{D}(\mathrm{d}\Gamma(D))$ invariant (see Lemma D.2).

Proof. In this proof, we denote by a dot on the top of a symbol the derivative with respect to β at zero. If it is necessary, we specify below with respect to which norm is the derivative taken. For example, the (point-wise) derivative of $u_\beta f$ with respect to β at *zero* is denoted by $(\dot{u_\beta f})$. In case that the dependence on β is written as a superscript, we sometimes omit the symbol β. For example the (point-wise) derivative of $\xi^{(\beta)}$ at zero is denoted by $\dot{\xi}$.

A simple calculation shows that

$$\left\| \beta^{-1}(f^{(\beta)} - f) - (\dot{u_\beta f}) \right\| \leq C|\beta|, \quad \left| \beta^{-1}\left(\omega^{(\beta)}(k) - \omega(k)\right) - \dot{\omega}(k) \right| \leq C|\beta|\omega(k). \tag{D.7}$$

This together with Proposition 1.2 (see also (D.2) and similar calculations) implies that

$$\left\| \Big(\frac{1}{\beta}(H_{\overline{P}}(\omega, f)^{(\beta)} - H_{\overline{P}}) - H_{\overline{P}}(\dot{\omega}, (\dot{u_\beta f})) \Big) \frac{1}{H_f + 1} \right\| \leq C|\beta|. \tag{D.8}$$

Then, the second resolvent identity and Proposition 1.2 imply that, for every $\lambda \in \mathbb{C}$ with not vanishing imaginary part (here we proceed as in (D.12) below),

$$\frac{\partial}{\partial \beta} \frac{1}{H^{(\beta)}_{\overline{P}} - \lambda}|_{\beta=0} = -\frac{1}{H_{\overline{P}} - \lambda} H_{\overline{P}}(\dot{\omega}, (\dot{u_\beta f})) \frac{1}{H_{\overline{P}} - \lambda}, \tag{D.9}$$

and therefore, we obtain that the derivative in the left term of the first line in (D.6) exists. Similar proofs (and formulas) hold for $H_f \frac{1}{H^{(\beta)}-\lambda}$ and $\frac{1}{H^{(\beta)}-\lambda} H_f$. Eq. (D.9) and the second resolvent equation (used as in (D.12) below) allows us to analyze the resolvents in the integrand in the Helffer-Sjöstrand formula ((B.3), with $n > 3$) and get (see also Proposition 1.2)

$$\frac{d}{d\beta}(H_f+1)\chi(H_{\overline{P}}^{(\beta)})|_{\beta=0} = \pi^{-1}\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\, \partial_{\overline{z}}\,\tilde{\chi}(z)\frac{\partial}{\partial\beta}(H_f+1)\frac{1}{H_{\overline{P}}^{(\beta)}-\lambda}, \tag{D.10}$$

where $z = x + iy$. This implies that the derivative in the middle term of the first line in (D.6) exists. Similarly as in (D.8), we obtain that

$$\frac{d}{d\beta}H(\xi^{(\beta)}, u_\beta Df)^{(\beta)}\frac{1}{H_f+1}|_{\beta=0} = H(\dot{\xi}, (u_\beta \dot{D}f))\Big)\frac{1}{H_f+1}. \tag{D.11}$$

Eqs. (D.10) and (D.11) imply that $(M^2)^{(\beta)}$ is differentiable with respect to β at $\beta = 0$ (see (D.2)). This and (D.8) imply that $(H_{\overline{P}}^{\pm\eta})^{(\beta)}\frac{1}{H_f+1}$ is differentiable with respect to β at $\beta = 0$. Now we calculate the derivative of $(H_f+1)(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}$ at zero using the second resolvent equation:

$$\begin{aligned}
&\frac{1}{\beta}(H_f+1)\Big((R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)} - R^{\pm\eta}(z_{\pm\epsilon})\Big) \qquad \text{(D.12)}\\
&\quad + (H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\Big[\frac{\partial}{\partial\beta}(H_{\overline{P}}^{\pm\eta})^{(\beta)}\frac{1}{H_f+1}|_{\beta=0}\Big](H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\\
&=\Big\{(H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\Big\}\Big\{\Big(\frac{1}{\beta}(H_{\overline{P}}^{\pm\eta} - (H_{\overline{P}}^{\pm\eta})^{(\beta)})\frac{1}{H_f+1} + \Big[\frac{\partial}{\partial\beta}(H_{\overline{P}}^{\pm\eta})^{(\beta)}\frac{1}{H_f+1}|_{\beta=0}\Big)\Big\}\\
&\qquad\qquad \cdot\Big\{(H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\Big\}\\
&\quad + \Big\{(H_f+1)((R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)} - R^{\pm\eta}(z_{\pm\epsilon}))\Big\}\Big\{\frac{1}{\beta}(H_{\overline{P}}^{\pm\eta} - (H_{\overline{P}}^{\pm\eta})^{(\beta)})\frac{1}{H_f+1}\Big\}\\
&\qquad\qquad \cdot\Big\{(H_f+1)R^{\pm\eta}(z_{\pm\epsilon})\Big\}.
\end{aligned}$$

It follows from Proposition 1.2 and (5.7) that $(H_f+1)R^{\pm\eta}(z_{\pm\epsilon})$ is bounded. This and the fact that $(H_{\overline{P}}^{\pm\eta})^{(\beta)}\frac{1}{H_f+1}$ is differentiable with respect to β at $\beta = 0$ imply that the first term in the right hand side side of (D.12) tends to zero as β goes to zero. The same arguments and the fact that

$$(H_f+1)(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)} = \Big((H_f+1)\frac{1}{(H_f+1)^{(\beta)}}\Big)\Big((H_f+1)(R^{\pm\eta}(z_{\pm\epsilon})\Big)^{(\beta)} \tag{D.13}$$

is uniformly bounded for small β (see Proposition 1.2 and (5.7)) imply that the second term in the right hand side of (D.12) is bounded (uniformly with respect to β). Since the second term in the left hand side of (D.12) is bounded (see arguments above), it follows that

$$\lim_{\beta\to 0}(H_f+1)\Big((R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)} - R^{\pm\eta}(z_{\pm\epsilon})\Big) = 0. \tag{D.14}$$

This in turn and the arguments above imply that the second term in the right hand side of (D.12) tends to zero as β tends to zero. We conclude that the left hand side of (D.12) tends to zero as β tends to zero and, therefore, $(H_f+1)(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}$ is differentiable at zero. This proves the existence of the derivative in the right term of the first line in (D.6). The proof that the derivative of $(\tilde{V}_\eta)^{(\beta)}\frac{1}{H_f+1}$, with respect to β, at zero exists follows exactly the same lines as the corresponding result for $(H_{\overline{P}}^{\pm\eta})^{(\beta)}\frac{1}{H_f+1}$, and therefore, we omit it. Then, using this and that $(H_f+1)(R^{\pm\eta}(z_{\pm\epsilon}))^{(\beta)}$ is differentiable at zero, we obtain that $\left(\tilde{V}_\eta\frac{1}{1+H_f}\right)\left((1+H_f)R^{\pm\eta}(z_{\pm\epsilon})^{(\beta)}\right)$ is differentiable at zero. This proves the existence of the derivative in the right term of the second line in (D.6). The proof for the left term is analogous. □

D.2 Commutator Estimates in Mourre Theory

Lemma D.4. *Recall that we introduce $\chi \in C_c^\infty(\mathbb{R},[0,1])$ in Definition 3.4. The quadratic form $[\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]$, defined in the domain of $\mathrm{d}\Gamma(D)$, extends to a bounded operator that we denote by $[\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]^0$. Additionally, $(H_{\overline{P}}+i)[\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]^0$ is bounded.*

Proof. For $\psi, \phi \in \mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$ and $z \in \mathbb{C}\setminus\mathbb{R}$, it follows from Lemma D.3 that

$$\begin{aligned}\left\langle \phi, [(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]\psi \right\rangle &= \left\langle \mathrm{d}\Gamma(D)(H_{\overline{P}}-\overline{z})^{-1}\phi, (H_{\overline{P}}-z)(H_{\overline{P}}-z)^{-1}\psi \right\rangle \\ &\quad - \left\langle (H_{\overline{P}}-\overline{z})(H_{\overline{P}}-\overline{z})^{-1}\phi, \mathrm{d}\Gamma(D)(H_{\overline{P}}-z)^{-1}\psi \right\rangle \\ &= -\left\langle \phi, (H_{\overline{P}}-\overline{z})^{-1}[H_{\overline{P}}, \mathrm{d}\Gamma(D)]^0(H_{\overline{P}}-z)^{-1}\psi \right\rangle. \end{aligned} \tag{D.15}$$

Note that

$$\left\|(H_{\overline{P}}+i)(H_{\overline{P}}-z)^{-1}\right\| \le 1 + \left\|(z+i)(H_{\overline{P}}-z)^{-1}\right\| \le C\left(1+|\operatorname{Re} z||\operatorname{Im} z|^{-1}\right). \tag{D.16}$$

Then, we observe from Remark 3.6 that

$$\left|\left\langle \phi, [(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]\psi \right\rangle\right| \le C\,\|\phi\|\,\|\psi\|\,|\operatorname{Im} z|^{-1}\left(1+|\operatorname{Re} z||\operatorname{Im} z|^{-1}\right), \tag{D.17}$$

and consequently, $[(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]$ uniquely extends to a bounded operator on $\mathcal{H}$ which we denote by $[(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]^0$ and

$$[(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]^0 = -(H_{\overline{P}}-z)^{-1}[H_{\overline{P}}, \mathrm{d}\Gamma(D)]^0(H_{\overline{P}}-z)^{-1}. \tag{D.18}$$

This together with Remark 3.6 , (D.16) and the Helffer-Sjöstrand formula (see (B.3)) yields

$$\begin{aligned}&\left\|(H_{\overline{P}}+i)[\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]^0\right\| \le \pi^{-1}\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\,|\partial_{\overline{z}}\tilde{\chi}(z)|\,\left\|(H_{\overline{P}}+i)[(H_{\overline{P}}-z)^{-1}, \mathrm{d}\Gamma(D)]^0\right\| \\ &\le \pi^{-1}\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\,|\partial_{\overline{z}}\tilde{\chi}(z)|\,\left\|(H_{\overline{P}}+i)(H_{\overline{P}}-z)^{-1}\right\|^2\left\|[H_{\overline{P}}, \mathrm{d}\Gamma(D)]^0(H_{\overline{P}}-i)^{-1}\right\| \\ &\le C\int_{\mathbb{C}} \mathrm{d}x\mathrm{d}y\,|\partial_{\overline{z}}\tilde{\chi}(z)|\left(1+|x||y|^{-1}\right)^2, \end{aligned} \tag{D.19}$$

where we take $z = x + iy$ for $x, y \in \mathbb{R}$ and $\tilde{\chi}$ is the almost analytic extension of χ (see Definition B.1). In the definition of $\tilde{\chi}$ we choose $n \geq 2$, and therefore, $|\partial_{\overline{z}}\tilde{\chi}(z)| \leq C|\operatorname{Im} z|^2$. Since χ is compactly supported, then $\tilde{\chi}$ is also compactly supported. It follows that

$$\left\|(H_{\overline{P}} + i)[\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]^0\right\| \leq C \int_{\mathrm{supp}\,(\tilde{\chi})} dxdy|y|^2 \left(1 + |x||y|^{-1}\right)^2 \leq C. \tag{D.20}$$

This completes the proof. □

Lemma D.5. *Recall that we introduce $\chi \in C_c^\infty(\mathbb{R}, [0,1])$ in Definition 3.4 and M^2 in* (5.3). *The quadratic form $[\mathrm{d}\Gamma(D), M^2]$, defined in the domain of $\mathrm{d}\Gamma(D)$, extends to a bounded operator that we denote by $[\mathrm{d}\Gamma(D), M^2]^0$. Similarly, the quadratic form $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]$ extends to a $H_{\overline{P}}$-bounded operator that we denote by $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]^0$.*

Proof. For $\phi, \psi \in \mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$, we observe from Lemma D.3 and the $H_{\overline{P}}$-boundedness of $[H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0$ that

$$\begin{aligned}
&\left\langle \mathrm{d}\Gamma(D)\phi, M^2\psi \right\rangle - \left\langle M^2\phi, \mathrm{d}\Gamma(D)\psi \right\rangle \qquad\qquad \text{(D.21)}\\
&= \left\langle [\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]\phi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\psi \right\rangle + \left\langle \mathrm{d}\Gamma(D)\chi(H_{\overline{P}})\phi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\psi \right\rangle\\
&- \left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\phi, [\chi(H_{\overline{P}}), \mathrm{d}\Gamma(D)]\psi \right\rangle - \left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\phi, \mathrm{d}\Gamma(D)\chi(H_{\overline{P}})\psi \right\rangle.
\end{aligned}$$

It follows from Lemma D.4 and Remark 3.6 that

$$\left|\left\langle [\mathrm{d}\Gamma(D), \chi(H_{\overline{P}})]\phi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\psi \right\rangle\right| \leq C\,\|\phi\|\,\|\psi\| \tag{D.22}$$

and

$$\left|\left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\phi, [\mathrm{d}\Gamma(D), \chi(H_{\overline{P}})]\psi \right\rangle\right| \leq C\,\|\phi\|\,\|\psi\|. \tag{D.23}$$

Moreover, for $\varphi, \vartheta \in \mathcal{F}_{\mathrm{fin}}[\mathfrak{h}_0]$, we obtain from Lemma 3.3 (iv) and (v) that

$$\begin{aligned}
&\left\langle \mathrm{d}\Gamma(D)\varphi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\vartheta \right\rangle - \left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\varphi, \mathrm{d}\Gamma(D)\vartheta \right\rangle\\
&= \langle \varphi, [\mathrm{d}\Gamma(D), (\mathrm{d}\Gamma_{\overline{P}}(\xi) + g\sigma_1\Phi_{\overline{P}}(Df))]\vartheta \rangle = \left\langle \varphi, \left(\mathrm{d}\Gamma_{\overline{P}}(\tilde{\xi}) - ig\sigma_1\Phi_{\overline{P}}(D^2 f)\right)\vartheta \right\rangle, \quad \text{(D.24)}
\end{aligned}$$

where $\tilde{\xi} = [D, \xi]$. Direct calculations show that $|\tilde{\xi}| \leq C\omega$ and $D^2 f \in \mathfrak{h}$. Proposition 1.2 implies that $\left(\mathrm{d}\Gamma_{\overline{P}}(\tilde{\xi}) - ig\sigma_1\Phi_{\overline{P}}(D^2 f)\right)$ is relatively bounded with respect to $H_{\overline{P}}$ and, hence, $[\mathrm{d}\Gamma(D), [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0]$ extends to a $H_{\overline{P}}$-bounded operator which we denote by $[\mathrm{d}\Gamma(D), [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0]^0 = \mathrm{d}\Gamma_{\overline{P}}(\tilde{\xi}) - ig\sigma_1\Phi_{\overline{P}}(D^2 f)$. Employing Lemma D.3, we find a constant $C > 0$ such that

$$\begin{aligned}
&\left|\left\langle \mathrm{d}\Gamma(D)\chi(H_{\overline{P}})\phi, [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\psi \right\rangle - \left\langle [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\chi(H_{\overline{P}})\phi, \mathrm{d}\Gamma(D)\chi(H_{\overline{P}})\psi \right\rangle\right|\\
&= \left|\left\langle \chi(H_{\overline{P}})\phi, \left[\mathrm{d}\Gamma(D), [H_{\overline{P}}, i\mathrm{d}\Gamma(D)]^0\right]^0 \chi(H_{\overline{P}})\psi \right\rangle\right| \leq C\,\|\phi\|\,\|\psi\|. \qquad \text{(D.25)}
\end{aligned}$$

This together with (D.21), (D.23) and (D.22) implies that there is a constant $C > 0$ such that

$$\left|\left\langle \mathrm{d}\Gamma(D)\phi, M^2\psi \right\rangle - \left\langle M^2\phi, \mathrm{d}\Gamma(D)\psi \right\rangle\right| \leq C \|\phi\| \|\psi\|, \tag{D.26}$$

and, thereby, we complete the proof, since $\mathcal{D}(\mathrm{d}\Gamma(D)) \cap \mathcal{D}(H_{\overline{P}})$ is dense in $\mathcal{H}$. The statement concerning $[i\mathrm{d}\Gamma(D), \tilde{V}_\eta]$ is proved following the same lines above. □

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