SPECTRAL THEORY FOR A MATHEMATICAL MODEL OF THE WEAK INTERACTION: THE DECAY OF THE INTERMEDIATE VECTOR BOSONS W+/-. II

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In memory of Pierre Duclos.

Abstract. We do the spectral analysis of the Hamiltonian for the weak leptonic decay of the gauge bosons $W^{\pm}$. Using Mourre theory, it is shown that the spectrum between the unique ground state and the first threshold is purely absolutely continuous. Neither sharp neutrino high energy cutoff nor infrared regularization are assumed.

1. Introduction

We study a mathematical model for the weak decay of the intermediate vector bosons $W^{\pm}$ into the full family of leptons. The full family of leptons involves the electron e^- and the positron e^+, together with the associated neutrino ν_e and antineutrino $\bar{\nu}_e$, the muons μ^- and μ^+ together with the associated neutrino ν_μ and antineutrino $\bar{\nu}_\mu$ and the tau leptons τ^- and τ^+ together with the associated neutrino ν_τ and antineutrino $\bar{\nu}_\tau$.

The model is patterned according to the Standard Model in Quantum Field Theory (see [21, 30]).

A representative and well-known example of this general process is the decay of the gauge boson W^- into an electron and an antineutrino of the electron, that occurs in β-decay,

$$W^- \to e^- + \bar{\nu}_e. \tag{1.1}$$

In the process (1.1), if we include the corresponding antiparticles, the interaction described in the Schrödinger representation is formally given by (see [21, (4.139)] and [30, (21.3.20)])

$$I = \int \mathrm{d}^3x\, \overline{\Psi_e}(x)\gamma^\alpha(1-\gamma_5)\Psi_{\nu_e}(x)W_\alpha(x) + \int \mathrm{d}^3x\, \overline{\Psi_{\nu_e}}(x)\gamma^\alpha(1-\gamma_5)\Psi_e(x)W_\alpha(x)^*, \tag{1.2}$$

where γ^α, $\alpha = 0,1,2,3$ and γ_5 are the Dirac matrices, $\Psi_{\cdot}(x)$ and $\overline{\Psi_{\cdot}}(x)$ are the Dirac fields for $e_\pm$, ν_e and $\bar{\nu}_e$ and W_α are the boson fields (see [29, §5.3]) given by

$$\begin{aligned}\Psi_e(x) =&(2\pi)^{-\frac{3}{2}}\sum_{s=\pm\frac{1}{2}}\int \mathrm{d}^3p\Big(b_{e,+}(p,s)\frac{u(p,s)}{(2(|p|^2+m_e^2)^{\frac{1}{2}})^{\frac{1}{2}}}\mathrm{e}^{ip.x}\\ &+b^*_{e,-}(p,s)\frac{v(p,s)}{(2(|p|^2+m_e^2)^{\frac{1}{2}})^{\frac{1}{2}}}\mathrm{e}^{-ip.x}\Big),\\ \Psi_{\nu_e}(x) =&(2\pi)^{-\frac{3}{2}}\sum_{s=\pm\frac{1}{2}}\int \mathrm{d}^3p\,(c_{e,+}(p,s)\frac{u(p,s)}{(2|p|)^{\frac{1}{2}}}\mathrm{e}^{ip.x}+c^*_{e,-}(p,s)\frac{v(p,s)}{(2|p|)^{\frac{1}{2}}}\mathrm{e}^{-ip.x})\,,\\ \overline{\Psi_e}(x) =&\Psi_e(x)^\dagger\gamma^0\,,\quad \overline{\Psi_{\nu_e}}(x)=\Psi_{\nu_e}(x)^\dagger\gamma^0\,,\end{aligned}$$

and

$$\begin{aligned}W_\alpha(x)=(2\pi)^{-\frac{3}{2}}\sum_{\lambda=-1,0,1}\int\frac{\mathrm{d}^3k}{(2(|k|^2+m_W^2)^{\frac{1}{2}})^{\frac{1}{2}}}\Big(&\epsilon_\alpha(k,\lambda)a_+(k,\lambda)\mathrm{e}^{ik.x}\\ &+\epsilon^*_\alpha(k,\lambda)a^*_-(k,\lambda)\mathrm{e}^{-ik.x}\Big)\,.\end{aligned}$$

Here $m_e > 0$ is the mass of the electron and $u(p,s)/(2(|p|^2+m_e^2)^{1/2})^{1/2}$ and $v(p,s)/(2(|p|^2+m_e^2)^{1/2})^{1/2}$ are the normalized solutions to the Dirac equation (see [21, Appendix]), $m_W > 0$ is the mass of the bosons $W^\pm$ and the vectors $\epsilon_\alpha(k,\lambda)$ are the polarizations of the massive spin 1 bosons (see [29, Section 5.2]), and as follows from the Standard Model, neutrinos and antineutrinos are considered massless particles.

The operators $b_{e,+}(p,s)$ and $b^*_{e,+}(p,s)$ (respectively $c_{\nu_e,+}(p,s)$ and $c^*_{\nu_e,+}(p,s)$), are the annihilation and creation operators for the electrons (respectively for the neutrinos associated with the electrons), satisfying the anticommutation relations. The index $-$ in $b_{e,-}(p,s)$, $b^*_{e,-}(p,s)$, $c_{\nu_e,-}(p,s)$ and $c^*_{\nu_e,-}(p,s)$ are used to denote the annihilation and creation operators of the corresponding antiparticles.

The operators $a_+(k,\lambda)$ and $a^*_+(k,\lambda)$ (respectively $a_-(k,\lambda)$ and $a^*_-(k,\lambda)$) are the annihilation and creation operators for the bosons W^- (respectively W^+) satisfying the canonical commutation relations.

If one considers the full interaction describing the decay of the gauge bosons $W^\pm$ into leptons ([21, (4.139)]) and if one formally expands this interaction with respect to products of creation and annihilation operators, we are left with a finite sum of terms associated with kernels of the form

$$\delta(p_1+p_2-k)g(p_1,\ p_2,\ k)\ .$$

The δ-distributions that occur here shall be approximated by square integrable functions. Therefore, in this article, the interaction for the weak decay of $W^\pm$ into the full family of leptons will be described in terms of annihilation and creation operators together with kernels which are square integrable with respect to momenta (see (2.7) and (2.8)-(2.10)).

Under this assumption, the total Hamiltonian, which is the sum of the free energy of the particles (see (2.6)) and of the interaction, is a well-defined self-adjoint operator in the Fock space for the leptons and the vector bosons (Theorem 2.2). This allows us to study its spectral properties.

Among the four fundamental interactions known up to now, the weak interaction does not generate bound states, which is not the case for the strong, electromagnetic and gravitational interactions. Thus we are expecting that the spectrum of the Hamiltonian associated with every model of weak decays is purely absolutely continuous above the ground state energy.

With additional assumptions on the kernels that are fulfilled by the model described in theoretical physics, we can prove (Theorem 3.2; see also [10, Theorem 3.3]) that the total Hamitonian has a unique ground state in the Fock space for a sufficiently small coupling constant, corresponding to the dressed vacuum. The strategy for proving existence of a unique ground state dates back to the early works of Bach, Fröhlich, and Sigal [5] and Griesemer, Lieb and Loss [22], for the Pauli-Fierz model of non relativistic QED. Our proofs follow these techniques as they were adapted to a model of quantum electrodynamics [7, 8, 15] and a model of the Fermi weak interactions [2].

Moreover, under natural regularity assumptions on the kernels, we establish a Mourre estimate (Theorem 5.1) and a limiting absorption principle (Theorem 7.1) for any spectral interval above the energy of the ground state and below the mass of the electron, for small enough coupling constants. As a consequence, the spectrum between the unique ground state and the first threshold is shown to be purely absolutely continuous (Theorem 3.3).

To achieve the spectral analysis above the ground state energy, our methods are taken largely from [4], [16], and [12]. More precisely, we begin with approximating the total Hamiltonian H by a cutoff Hamiltonian H_σ which has the property that the interaction between the massive particles and the neutrinos or antineutrinos of energies $\leq \sigma$ has been suppressed. The restriction of H_σ to the Fock space for the massive particles together with the neutrinos and antineutrinos of energies $\geq \sigma$ is in this paper denoted by H^σ. Adapting the method of [4], we prove that, for some suitable sequence $\sigma_n \to 0$, the Hamiltonian H^{σ_n} has a gap of size $O(\sigma_n)$ in its spectrum above its ground state energy, for all $n \in \mathbb{N}$. In contrast to [10], we do not require a sharp neutrino high energy cutoff here.

Next, as in [16], [10] and [12], we use the gap property in combination with the conjugate operator method developed in [3] and [28] in order to establish a limiting absorption principle near the ground state energy of H. In [10], the chosen conjugate operator is the generator of dilatations in the Fock space for neutrinos and antineutrinos. As a consequence, an infrared regularization is assumed in [10] in order to be able to implement the strategy of [16]. Let us mention that no infrared regularization is required in [16], because for the model of non-relativistic QED with a fixed nucleus which is studied in that paper, a unitary Pauli-Fierz transformation can be applied with the effect of regularizing the infrared behavior of the interaction.

In the present paper, we choose a conjugate operator which is the generator of dilatations in the Fock space for neutrinos and antineutrinos *with a cutoff in the momentum variable*. Hence our conjugate operator only affects the massless particles of low energies. A similar choice is made in [12], where the Pauli-Fierz model of non-relativistic QED for a free electron at a fixed total momentum is studied. Due to the complicated structure of the interaction operator in this context, the authors in [12] make use of some Feshbach-Schur map before proving a Mourre estimate for an effective Hamiltonian. Here we do not need to apply such a map, and we prove a

Mourre estimate directly for H. Compared with [16], our method involves further estimates, which allows us to avoid any infrared regularization.

As mentioned before, some of the basic results of this article have been previously stated and proved, under stronger assumptions, in [9, 10]. The main achievement of this paper in comparison with [10] is that no sharp neutrino high energy cutoff and no infrared regularization are assumed here.

The nature of the spectrum above the first threshold and the scattering theory of this model remain to be studied elsewhere.

The paper is organized as follows. In the next section, we give a precise definition of the Hamiltonian. Section 3 is devoted to the statements of the main spectral properties. In sections 4-7, we establish the results necessary to apply Mourre theory, namely, we derive a gap condition, a Mourre estimate, local C^2-regularity of the Hamiltonian, and a limiting absorption principle. Section 8 details the proof of Theorem 3.3 on absolutely continuity of the spectrum. Eventually, in Appendix A, we state and prove several technical lemmata.

For the sake of clarity, all proofs in sections 4 to 8 and in appendix A are given for the particular process depicted in (1.1). The general situation can be recovered by a straightforward generalization.

2. Definition of the model

The weak decay of the intermediate bosons W^+ and W^- involves the full family of leptons together with the bosons themselves, according to the Standard Model (see [21, Formula (4.139)] and [30]).

The full family of leptons consists of the electron e^-, the muon μ^-, the tau lepton τ^-, their associated neutrinos ν_e, ν_μ, ν_τ and all their antiparticles e^+, μ^+, τ^+, $\bar{\nu}_e$, $\bar{\nu}_\mu$, and $\bar{\nu}_\tau$. In the Standard Model, neutrinos and antineutrinos are massless particles, with helicity $-1/2$ and $+1/2$ respectively. Here we shall assume that both neutrinos and antineutrinos have helicity $\pm 1/2$.

The mathematical model for the weak decay of the vector bosons $W^\pm$ is defined as follows.

The index $\ell \in \{1, 2, 3\}$ denotes each species of leptons: $\ell = 1$ denotes the electron e^-, the positron e^+ and the associated neutrinos ν_e, $\bar{\nu}_e$; $\ell = 2$ denotes the muons μ^-, μ^+ and the associated neutrinos ν_μ and $\bar{\nu}_\mu$; and $\ell = 3$ denotes the tau-leptons and the associated neutrinos ν_τ and $\bar{\nu}_\tau$.

Let $\xi_1 = (p_1,\ s_1)$ be the quantum variables of a massive lepton, where $p_1 \in \mathbb{R}^3$ and $s_1 \in \{-1/2,\ 1/2\}$ is the spin polarization of particles and antiparticles. Let $\xi_2 = (p_2,\ s_2)$ be the quantum variables of a massless lepton where $p_2 \in \mathbb{R}^3$ and $s_2 \in \{-1/2,\ 1/2\}$ is the helicity of particles and antiparticles and, finally, let $\xi_3 = (k,\ \lambda)$ be the quantum variables of the spin 1 bosons W^+ and W^- where $k \in \mathbb{R}^3$ and $\lambda \in \{-1,\ 0,\ 1\}$ accounts for the polarization of the vector bosons (see [29, section 5.2]).

We set $\Sigma_1 = \mathbb{R}^3 \times \{-1/2,\ 1/2\}$ for the configuration space of the leptons and $\Sigma_2 = \mathbb{R}^3 \times \{-1,\ 0,\ 1\}$ for the bosons. Thus $L^2(\Sigma_1)$ is the Hilbert space of each lepton and $L^2(\Sigma_2)$ is the Hilbert space of each boson. In the sequel, we shall use the notations $\int_{\Sigma_1} \mathrm{d}\xi := \sum_{s=+\frac{1}{2},-\frac{1}{2}} \int \mathrm{d}p$ and $\int_{\Sigma_2} \mathrm{d}\xi := \sum_{\lambda=0,1,-1} \int \mathrm{d}k$.

The Hilbert space for the weak decay of the vector bosons $W^\pm$ is the Fock space for leptons and bosons describing the set of states with indefinite number of particles or antiparticles, which we define below.

For every ℓ, $\mathfrak{F}_\ell$ is the fermionic Fock space for the corresponding species of leptons including the massive particle and antiparticle together with the associated neutrino and antineutrino, i.e.,

$$\mathfrak{F}_\ell = \bigotimes^4 \mathfrak{F}_a(L^2(\Sigma_1)) = \bigotimes^4 \left(\oplus_{n=0}^\infty \otimes_a^n L^2(\Sigma_1)\right) , \quad \ell = 1,2,3 . \tag{2.1}$$

where $\otimes_a^n$ denotes the antisymmetric n-th tensor product and $\otimes_a^0 L^2(\Sigma_1) := \mathbb{C}$.

The fermionic Fock space $\mathfrak{F}_L$ for the leptons is

$$\mathfrak{F}_L = \otimes_{\ell=1}^3 \mathfrak{F}_\ell . \tag{2.2}$$

The bosonic Fock space $\mathfrak{F}_W$ for the vector bosons W^+ and W^- is

$$\mathfrak{F}_W = \bigotimes^2 \mathfrak{F}_s(L^2(\Sigma_2)) = \bigotimes^2 \left(\oplus_{n=0}^\infty \otimes_s^n L^2(\Sigma_2)\right) , \tag{2.3}$$

where $\otimes_s^n$ denotes the symmetric n-th tensor product and $\otimes_s^0 L^2(\Sigma_2) := \mathbb{C}$.

The Fock space for the weak decay of the vector bosons W^+ and W^- is thus

$$\mathfrak{F} = \mathfrak{F}_L \otimes \mathfrak{F}_W . \tag{2.4}$$

For each $\ell = 1,2,3$, $b_{\ell,\epsilon}(\xi_1)$ (resp. $b^*_{\ell,\epsilon}(\xi_1)$) is the annihilation (resp. creation) operator for the corresponding species of massive particle when $\epsilon = +$ and for the corresponding species of massive antiparticle when $\epsilon = -$. Similarly, for each $\ell = 1,2,3$, $c_{\ell,\epsilon}(\xi_2)$ (resp. $c^*_{\ell,\epsilon}(\xi_2)$) is the annihilation (resp. creation) operator for the corresponding species of neutrino when $\epsilon = +$ and for the corresponding species of antineutrino when $\epsilon = -$. Finally, the operator $a_\epsilon(\xi_3)$ (resp. $a^*_\epsilon(\xi_3)$) is the annihilation (resp. creation) operator for the boson W^- when $\epsilon = +$, and for the boson W^+ when $\epsilon = -$. The operators $b_{\ell,\epsilon}(\xi_1)$, $b^*_{\ell,\epsilon}(\xi_1)$, $c_{\ell,\epsilon}(\xi_2)$ and $c^*_{\ell,\epsilon}(\xi_2)$ fulfil the usual canonical anticommutation relations (CAR), whereas $a_\epsilon(\xi_3)$ and $a^*_\epsilon(\xi_3)$ fulfil the canonical commutation relation (CCR), see e.g. [29]. Moreover, the a's commute with the b's and the c's.

In addition, following the convention described in [29, section 4.1] and [29, section 4.2], we shall assume that fermionic creation and annihilation operators of different species of leptons will always anticommute (see e.g. [11] for explicit definitions).

Therefore, the following canonical anticommutation and commutation relations hold.

$$\begin{aligned}
&\{b_{\ell,\epsilon}(\xi_1), b^*_{\ell',\epsilon'}(\xi'_1)\} = \delta_{\ell\ell'}\delta_{\epsilon\epsilon'}\delta(\xi_1 - \xi'_1) , \\
&\{c_{\ell,\epsilon}(\xi_2), c^*_{\ell',\epsilon'}(\xi'_2)\} = \delta_{\ell\ell'}\delta_{\epsilon\epsilon'}\delta(\xi_2 - \xi'_2) , \\
&[a_\epsilon(\xi_3), a^*_{\epsilon'}(\xi'_3)] = \delta_{\epsilon\epsilon'}\delta(\xi_3 - \xi'_3) , \\
&\{b_{\ell,\epsilon}(\xi_1), b_{\ell',\epsilon'}(\xi'_1)\} = \{c_{\ell,\epsilon}(\xi_2), c_{\ell',\epsilon'}(\xi'_2)\} = 0 , \\
&[a_\epsilon(\xi_3), a_{\epsilon'}(\xi'_3)] = 0 , \\
&\{b_{\ell,\epsilon}(\xi_1), c_{\ell',\epsilon'}(\xi_2)\} = \{b_{\ell,\epsilon}(\xi_1), c^*_{\ell',\epsilon'}(\xi_2)\} = 0 , \\
&[b_{\ell,\epsilon}(\xi_1), a_{\epsilon'}(\xi_3)] = [b_{\ell,\epsilon}(\xi_1), a^*_{\epsilon'}(\xi_3)] = [c_{\ell,\epsilon}(\xi_2), a_{\epsilon'}(\xi_3)] = [c_{\ell,\epsilon}(\xi_2), a^*_{\epsilon'}(\xi_3)] = 0 .
\end{aligned}$$

Here, $\{b, b'\} = bb' + b'b$, $[a, a'] = aa' - a'a$.

We recall that the following operators, with $\varphi \in L^2(\Sigma_1)$,

$$b_{\ell,\epsilon}(\varphi) = \int_{\Sigma_1} b_{\ell,\epsilon}(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi, \quad c_{\ell,\epsilon}(\varphi) = \int_{\Sigma_1} c_{\ell,\epsilon}(\xi)\overline{\varphi(\xi)}\mathrm{d}\xi \ ,$$
$$b^*_{\ell,\epsilon}(\varphi) = \int_{\Sigma_1} b^*_{\ell,\epsilon}(\xi)\varphi(\xi)\mathrm{d}\xi, \quad c^*_{\ell,\epsilon}(\varphi) = \int_{\Sigma_1} c^*_{\ell,\epsilon}(\xi)\varphi(\xi)\mathrm{d}\xi$$

are bounded operators in $\mathfrak{F}$ such that

$$\|b^\sharp_{\ell,\epsilon}(\varphi)\| = \|c^\sharp_{\ell,\epsilon}(\varphi)\| = \|\varphi\|_{L^2} \ , \tag{2.5}$$

where $b^\sharp$ (resp. $c^\sharp$) is b (resp. c) or b^* (resp. c^*).

The free Hamiltonian H_0 is given by

$$\begin{aligned} H_0 = &\sum_{\ell=1}^3 \sum_{\epsilon=\pm} \int w^{(1)}_\ell(\xi_1) b^*_{\ell,\epsilon}(\xi_1) b_{\ell,\epsilon}(\xi_1)\mathrm{d}\xi_1 + \sum_{\ell=1}^3 \sum_{\epsilon=\pm} \int w^{(2)}_\ell(\xi_2) c^*_{\ell,\epsilon}(\xi_2) c_{\ell,\epsilon}(\xi_2)\mathrm{d}\xi_2 \\ &+ \sum_{\epsilon=\pm} \int w^{(3)}(\xi_3) a^*_\epsilon(\xi_3) a_\epsilon(\xi_3)\mathrm{d}\xi_3 \ , \end{aligned} \tag{2.6}$$

where the free relativistic energy of the massive leptons, the neutrinos, and the bosons are respectively

$$w^{(1)}_\ell(\xi_1) = (|p_1|^2 + m_\ell^2)^{\frac{1}{2}}, \ w^{(2)}_\ell(\xi_2) = |p_2| \text{ and } w^{(3)}(\xi_3) = (|k|^2 + m_W^2)^{\frac{1}{2}} \ .$$

Here m_ℓ is the mass of the lepton ℓ and m_W is the mass of the bosons, with $m_1 < m_2 < m_3 < m_W$.

The interaction H_I is described in terms of annihilation and creation operators together with kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(.,.,.)$ ($\alpha = 1, 2$).

As emphasized previously each kernel $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3)$, computed in theoretical physics, contains a δ-distribution because of the conservation of the momentum ([21], [29, section 4.4]). Here, we approximate the singular kernels by square integrable functions. Therefore, we assume the following

Hypothesis 2.1. *For $\alpha = 1, 2$, $\ell = 1, 2, 3$, $\epsilon, \epsilon' = \pm$, we assume*

$$G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) \in L^2(\Sigma_1 \times \Sigma_1 \times \Sigma_2) \ . \tag{2.7}$$

Based on [21, p159, (4.139)] and [30, p308, (21.3.20)] we define the interaction terms as

$$H_I = H^{(1)}_I + H^{(2)}_I \ , \tag{2.8}$$

with

$$\begin{aligned} H^{(1)}_I = &\sum_{\ell=1}^3 \sum_{\epsilon\neq\epsilon'} \int G^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3) b^*_{\ell,\epsilon}(\xi_1) c^*_{\ell,\epsilon'}(\xi_2) a_\epsilon(\xi_3)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \\ &+ \sum_{\ell=1}^3 \sum_{\epsilon\neq\epsilon'} \int \overline{G^{(1)}_{\ell,\epsilon,\epsilon'}(\xi_1, \xi_2, \xi_3)} a^*_\epsilon(\xi_3) c_{\ell,\epsilon'}(\xi_2) b_{\ell,\epsilon}(\xi_1)\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \ , \end{aligned} \tag{2.9}$$

$$H_I^{(2)} = \sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\int G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3) b^*_{\ell,\epsilon}(\xi_1) c^*_{\ell,\epsilon'}(\xi_2) a^*_{\epsilon}(\xi_3) \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3$$
$$+ \sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\int \overline{G^{(2)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)} a_{\epsilon}(\xi_3) c_{\ell,\epsilon'}(\xi_2) b_{\ell,\epsilon}(\xi_1) \mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 \ . \tag{2.10}$$

The operator $H_I^{(1)}$ describes the decay of the bosons W^+ and W^- into leptons, and $H_I^{(2)}$ is responsible for the fact that the bare vacuum will not be an eigenvector of the total Hamiltonian, as expected from the physics.

For $\ell = 1, 2, 3$, all terms in $H_I^{(1)}$ and $H_I^{(2)}$ are well defined as quadratic forms on the set of finite particle states consisting of smooth wave functions. According to [27, Theorem X.24] (see details in [10]) one can construct a closed operator associated with the quadratic form defined by (2.8)-(2.10).

The total Hamiltonian is thus (g is a coupling constant),

$$H = H_0 + gH_I, \quad g > 0 \ . \tag{2.11}$$

Theorem 2.2. *Let $g_1 > 0$ be such that*

$$\frac{6g_1^2}{m_W}\Big(\frac{1}{m_1^2}+1\Big)\sum_{\alpha=1,2}\sum_{\ell=1}^{3}\sum_{\epsilon\neq\epsilon'}\|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}\|^2_{L^2(\Sigma_1\times\Sigma_1\times\Sigma_2)} < 1 \ .$$

Then for every g satisfying $g \leq g_1$, H is a self-adjoint operator in $\mathfrak{F}$ with domain $\mathcal{D}(H) = \mathcal{D}(H_0)$.

Under the same assumption as here, this result was proved in [10, Theorem 2.6], with a prefactor 2 missing.

3. Location of the spectrum, existence of a ground state, absolutely continuous spectrum, and dynamical properties

In the sequel, we shall make some of the following additional assumptions on the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$.

Hypothesis 3.1. *There exists $\tilde{K}(G)$ and $\tilde{\tilde{K}}(G)$ such that for $\alpha = 1, 2$, $\ell = 1, 2, 3$, $\epsilon, \epsilon' = \pm$, $i, j = 1, 2, 3$, and $\sigma \geq 0$,*

(i) $$\int_{\Sigma_1\times\Sigma_1\times\Sigma_2}\frac{|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2}{|p_2|^2}\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 < \infty \ ,$$

(ii) $$\left(\int_{\Sigma_1\times\{|p_2|\leq\sigma\}\times\Sigma_2}|G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(\xi_1,\xi_2,\xi_3)|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3\right)^{\frac{1}{2}} \leq \tilde{K}(G)\,\sigma \ ,$$

(iii-a) $(p_2\cdot\nabla_{p_2})G^{(\alpha)}_{\ell,\epsilon,\epsilon'}(.,.,.) \in L^2(\Sigma_1\times\Sigma_1\times\Sigma_2)$ *and*
$$\int_{\Sigma_1\times\{|p_2|\leq\sigma\}\times\Sigma_2}\Big|[(p_2\cdot\nabla_{p_2})G^{(\alpha)}_{\ell,\epsilon,\epsilon'}](\xi_1,\xi_2,\xi_3)\Big|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 < \tilde{\tilde{K}}(G)\,\sigma,$$

(iii-b) $$\int_{\Sigma_1\times\Sigma_1\times\Sigma_2}p_{2,i}^2\,p_{2,j}^2\left|\frac{\partial^2 G^{(\alpha)}_{\ell,\epsilon,\epsilon'}}{\partial p_{2,i}\partial p_{2,j}}(\xi_1,\xi_2,\xi_3)\right|^2\mathrm{d}\xi_1\mathrm{d}\xi_2\mathrm{d}\xi_3 < \infty \ .$$

Our first main result is devoted to the existence of a ground state for H together with the location of the spectrum of H and of its absolutely continuous spectrum.

Theorem 3.2. *Assume that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypothesis 2.1 and 3.1(i), then there exists $g_2 \in (0, g_1]$ such that H has a unique ground state for $g \leq g_2$.*

Moreover, for

$$E := \inf \mathrm{Spec}(H)\,,$$

the spectrum of H fulfils

$$\mathrm{Spec}(H) = \mathrm{Spec}_{\mathrm{ac}}(H) = [E,\,\infty)\ ,$$

with $E \leq 0$.

Proof. The proof of Theorem 3.2 is done in [10]. The main ingredients of this proof are the cutoff operators and the existence of a gap above the ground state energy for these operators (see Proposition 4.1 below and [10, Proposition 3.5]).

Note that a more general proof of the existence of a ground state can also be achieved by mimicking the proof given in [8]. □

Let b be the operator in $L^2(\Sigma_1)$ accounting for the position of the neutrino

$$b = i\nabla_{p_2}\ ,$$

and let

$$\langle b\rangle = (1+|b|^2)^{\frac{1}{2}}\,.$$

Its second quantized version $\mathrm{d}\Gamma(\langle b\rangle)$ is self-adjoint in $\mathfrak{F}_a(L^2(\Sigma_1))$. We thus define the "total position" operator B for all neutrinos and antineutrinos by

$$\begin{aligned} &B_\ell = \mathbf{1}\otimes\mathbf{1}\otimes\mathrm{d}\Gamma(\langle b\rangle)\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}\otimes\mathrm{d}\Gamma(\langle b\rangle)\,, \quad \text{in } \mathfrak{F}_\ell \\ &B = (B_1\otimes\mathbf{1}\otimes\mathbf{1} + \mathbf{1}\otimes B_2\otimes\mathbf{1} + \mathbf{1}\otimes\mathbf{1}\otimes B_3)\otimes(\mathbf{1}\otimes\mathbf{1}) \quad \text{in } \mathfrak{F}\,. \end{aligned} \tag{3.1}$$

Theorem 3.3. *Assume that the kernels $G^{(\alpha)}_{\ell,\epsilon,\epsilon'}$ satisfy Hypothesis 2.1 and 3.1 (ii)-(iii). For any $\delta > 0$ satisfying $0 < \delta < m_1$, there exists $g_\delta > 0$ such that for $0 < g \leq g_\delta$:*

(i) *The spectrum of H in $(E,\, E + m_1 - \delta]$ is purely absolutely continuous.*

(ii) *For $s > 1/2$, $\varphi \in \mathfrak{F}$, and $\psi \in \mathfrak{F}$, the limits*

$$\lim_{\epsilon\to 0}(\varphi,\,\langle B\rangle^{-s}(H-\lambda\pm i\epsilon)\langle B\rangle^{-s}\psi)$$

exist uniformly for λ in every compact subset of $(E,\, E + m_1 - \delta)$.

(iii) *For $s \in (1/2,\, 1)$ and $f \in C_0^\infty\left((E,\, E + m_1 - \delta)\right)$, we have*

$$\left\|(B+1)^{-s}\mathrm{e}^{-itH}f(H)(B+1)^{-s}\right\| = \mathcal{O}\left(t^{-(s-1/2)}\right)\,.$$

The assertions (i), (ii) and (iii) of Theorem 3.3 are based on a limiting absorption principle stated in Section 7, obtained by a positive commutator estimate, called Mourre estimate (Section 5), and a regularity property of H (Section 6).

The proof of Theorem 3.3 is detailed in Section 8.

Remark 3.1. *As a representative example of the general process described above, one can consider for example the decay (1.1) of the intermediate vector boson W^- into an electron and an electron antineutrino*

All Theorems stated in Sections 2 and 3 will obviously remain true for this simplified model, as well as for any other reduced model involving only one species of leptons, i.e., for a fixed value of $\ell \in \{1,\, 2,\, 3\}$, and with or without the inclusion of their corresponding antiparticles ($\epsilon = \pm$ and $\epsilon' = \pm$).

Moreover, the proofs of these results, based on the theorems stated in Sections 4, 5, 6 and 7, follow exactly the same arguments and estimates in the general case as in the case we fix ℓ, ϵ and ϵ'.

For this reason, and for the sake of clarity, we shall fix $\ell = 1$, $\epsilon = +$ and $\epsilon' = -$ in the next sections, and we shall adopt the following obvious notations

(3.2)
$$b^\sharp_{1,+}(\xi_1) =: b^\sharp(\xi_1), \quad c^\sharp_{1,-}(\xi_2) =: c^\sharp(\xi_2), \quad a^\sharp_+(\xi_3) =: a^\sharp(\xi_3), \text{ and } \quad G^{(\alpha)}_{\ell,\epsilon,\epsilon'} =: G^{(\alpha)} .$$

4. Spectral gap for auxiliary operators

A key ingredient to the proof of Theorem 3.2 and Theorem 3.3 is the study of auxiliary operators associated with infrared cutoff Hamiltonians with respect to the momenta of the neutrinos.

The main result of this section is Proposition 4.1 where we prove that auxiliary operators have gap in their spectrum above the ground state energy. This property was already derived in [10] in the case of a sharp ultraviolet cutoff. We show here that this result remains true in the present case where no sharp ultraviolet cutoff assumption is made.

According to Remark 3.1, for the sake of clarity, we will consider only the case of one species $\ell = 1$ of leptons, and pick $\epsilon = +$, and $\epsilon' = -$. We thus use the notations (3.2).

Let us first define the auxiliary operators which are the Hamiltonians with infrared cutoff with respect to the momenta of the neutrinos.

Let $\chi_0(.) \in C^\infty(\mathbb{R}, [0,1])$, with $\chi_0 = 1$ on $(-\infty, 1]$.

For $\sigma > 0$ we set, for $p \in \mathbb{R}^3$,

$$\begin{aligned} \chi_\sigma(p) &= \chi_0(|p|/\sigma) , \\ \tilde{\chi}^\sigma(p) &= 1 - \chi_\sigma(p) . \end{aligned} \tag{4.1}$$

The operator $H_{I,\sigma}$ is the interaction given by (2.8), (2.9) and (2.10) and associated with the kernels $\tilde{\chi}^\sigma(p_2)G^{(\alpha)}(\xi_1, \xi_2, \xi_3)$. We then set

$$H_\sigma := H_0 + gH_{I,\sigma} . \tag{4.2}$$

Let

$$\begin{aligned} &\Sigma_{1,\sigma} = \Sigma_1 \cap \{(p_2, s_2);\ |p_2| < \sigma\} , \quad {\Sigma_1}^\sigma = \Sigma_1 \cap \{(p_2, s_2);\ |p_2| \geq \sigma\} \\ &\mathfrak{F}_{2,\sigma} = \mathfrak{F}_a(L^2(\Sigma_{1,\sigma})) , \quad {\mathfrak{F}_2}^\sigma = \mathfrak{F}_a(L^2({\Sigma_1}^\sigma)) , \end{aligned}$$

The space $\mathfrak{F}_a(L^2(\Sigma_1))$ is the Fock space for the massive leptons and $(\mathfrak{F}_{2,\sigma} \otimes {\mathfrak{F}_2}^\sigma)$ is the Fock space for the antineutrinos.

Set

$${\mathfrak{F}_L}^\sigma = \mathfrak{F}_a(L^2(\Sigma_1)) \otimes {\mathfrak{F}_2}^\sigma , \text{ and } \quad \mathfrak{F}_{L,\sigma} = \mathfrak{F}_{2,\sigma} .$$

We thus have

$$\mathfrak{F}_L \simeq {\mathfrak{F}_L}^\sigma \otimes \mathfrak{F}_{L,\sigma} .$$

Set

$$\mathfrak{F}^\sigma = \mathfrak{F}^\sigma_L \otimes \mathfrak{F}_W , \quad \text{and} \quad \mathfrak{F}_\sigma = \mathfrak{F}_{L,\sigma} . \tag{4.3}$$

We have

$$\mathfrak{F} \simeq \mathfrak{F}^\sigma \otimes \mathfrak{F}_\sigma .$$

Set

$$H_0^{(1)} = \int w^{(1)}(\xi_1)\, b^*(\xi_1) b(\xi_1) \mathrm{d}\xi_1 \ ,$$
$$H_0^{(2)} = \int w^{(2)}(\xi_2)\, c^*(\xi_2) c(\xi_2) \mathrm{d}\xi_2 \ ,$$
$$H_0^{(3)} = \int w^{(3)}(\xi_3) a^*(\xi_3) a(\xi_3) \mathrm{d}\xi_3 \ ,$$

and

$$\begin{aligned} H_0^{(2)\sigma} &= \int_{|p_2|>\sigma} w^{(2)}(\xi_2)\, c^*(\xi_2) c(\xi_2) \mathrm{d}\xi_2 \ , \\ H_{0,\sigma}^{(2)} &= \int_{|p_2|\le\sigma} w^{(2)}(\xi_2)\, c^*(\xi_2) c(\xi_2) \mathrm{d}\xi_2 \ . \end{aligned} \tag{4.4}$$

We have on $\mathfrak{F}^\sigma \otimes \mathfrak{F}_\sigma$

$$H_0^{(2)} = H_0^{(2)\sigma} \otimes \mathbf{1}_\sigma + \mathbf{1}^\sigma \otimes H_{0,\sigma}^{(2)} \ .$$

Here, $\mathbf{1}^\sigma$ (resp. $\mathbf{1}_\sigma$) is the identity operator on $\mathfrak{F}^\sigma$ (resp. $\mathfrak{F}_\sigma$).

Define

$$H^\sigma = H_\sigma|_{\mathfrak{F}^\sigma} \quad \text{and} \quad H_0^\sigma = H_0|_{\mathfrak{F}^\sigma} \ . \tag{4.5}$$

We get

$$H^\sigma = H_0^{(1)} + H_0^{(2)\,\sigma} + H_0^{(3)} + gH_{I,\sigma} \quad \text{on } \mathfrak{F}^\sigma \ , \tag{4.6}$$

and

$$H_\sigma = H^\sigma \otimes \mathbf{1}_\sigma + \mathbf{1}^\sigma \otimes H_{0,\sigma}^{(2)} \quad \text{on } \mathfrak{F}^\sigma \otimes \mathfrak{F}_\sigma \ . \tag{4.7}$$

Let $\delta \in \mathbb{R}$ be such that

$$0 < \delta < m_1 \ . \tag{4.8}$$

Define the sequence $(\sigma_n)_{n\ge 0}$ by

$$\begin{aligned} \sigma_0 &= 2m_1 + 1 \ , \\ \sigma_1 &= m_1 - \frac{\delta}{2} \ , \\ \sigma_{n+1} &= \gamma \sigma_n, \ n \ge 1 \ , \end{aligned} \tag{4.9}$$

where

$$\gamma = 1 - \frac{\delta}{2m_1 - \delta} \ . \tag{4.10}$$

For $n \ge 0$, we then define the auxiliary operators on $\mathfrak{F}^n := \mathfrak{F}^{\sigma_n}$ by

$$H^n := H^{\sigma_n}, \quad H_0^n := H_0^{\sigma_n}, \tag{4.11}$$

and we denote for $n \ge 0$

$$E^n = \inf \mathrm{Spec}(H^n) \,. \tag{4.12}$$

We set

$$\tilde{D}_\delta(G) = \max\left\{ \frac{4(2m_1+1)\gamma}{2m_1 - \delta}, \ 2 \right\} \tilde{K}(G)(2m_1 \tilde{C}_{\beta\eta} + \tilde{B}_{\beta\eta}) \tag{4.13}$$

where $\tilde{K}(G)$ is given by Hypothesis 3.1 *(iii-a)* and $\tilde{B}_{\beta\eta}$ and $\tilde{C}_{\beta\eta}$ are defined for given $\eta > 0$ and $\beta > 0$ as in [10, (3.29)] by the following relations

$$\begin{aligned} C_{\beta\eta} &= \left(\frac{3}{m_W}(1+\frac{1}{{m_1}^2}) + \frac{3\beta}{m_W {m_1}^2} + \frac{12\,\eta}{{m_1}^2}(1+\beta) \right)^{\frac{1}{2}} , \\ B_{\beta\eta} &= \left(\frac{3}{m_W}(1+\frac{1}{4\beta}) + 12(\,\eta(1+\frac{1}{4\beta}) + \frac{1}{4\eta}\,) \right)^{\frac{1}{2}} . \end{aligned} \tag{4.14}$$

$$\tilde{C}_{\beta\eta} = C_{\beta\eta}\left(1 + \frac{g_1 K(G) C_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\right),$$

$$\tilde{B}_{\beta\eta} = \Big(\,1 + \frac{g_1\,K(G) C_{\beta\eta}}{1 - g_1\,K(G)\,C_{\beta\eta}}(\,2 + \frac{g_1 K(G) B_{\beta\eta} C_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\,)\Big) B_{\beta\eta}\ ,$$

and

$$K(G) = \left(\sum_{\alpha=1,2} \left\| G^{(\alpha)} \right\|^2 \right)^{\frac{1}{2}} . \tag{4.15}$$

Let $g_\delta^{(1)}$ be such that

$$0 < g_\delta^{(1)} < \min\left\{1,\ g_1,\ \frac{\gamma - \gamma^2}{3\tilde{D}_\delta(G)}\right\} . \tag{4.16}$$

We then have

Proposition 4.1. *Suppose that the kernels $G^{(\alpha)}$ satisfy Hypothesis 2.1 and 3.1(ii). Then there exists $\tilde{g}_\delta^{(2)} > 0$ such that $\tilde{g}_\delta^{(2)} \le g_\delta^{(1)}$ and for $g \le \tilde{g}_\delta^{(2)}$, E^n is a simple eigenvalue of H^n for $n \ge 1$, and H^n does not have spectrum in $(E^n, E^n + (1 - \frac{3g\tilde{D}_\delta(G)}{\gamma})\sigma_n)$.*

Proof. The proof of Proposition 4.1 is a slight modification of the proof of [10, proposition 3.5] which was based on the method develop in [4]. The only difference here to the proof of [10, proposition 3.5] is that we have to deal with the absence of the sharp ultraviolet cutoff.

For $n \ge 0$ we define

$$\Sigma_n^{n+1} = \Sigma_1 \cap \{(p_2,\, s_2);\, \sigma_{n+1} \le |p_2| < \sigma_n\} \tag{4.17}$$

and

$$\mathfrak{F}_n^{n+1} = \mathfrak{F}_a\left(L^2(\Sigma_n^{n+1})\right) . \tag{4.18}$$

We thus have

$$\mathfrak{F}^{n+1} \simeq \mathfrak{F}^n \otimes \mathfrak{F}_n^{n+1} . \tag{4.19}$$

Let Ω^n (respectively, Ω_n^{n+1}) be the vacuum state in $\mathfrak{F}^n$ (respectively in $\mathfrak{F}_n^{n+1}$). We set

$$H_{0,n}^{n+1} = \int_{\sigma_{n+1} \le |p_2| < \sigma_n} w^{(2)}(\xi_2) c^*(\xi_2) c(\xi_2) \mathrm{d}\xi_2 \, . \tag{4.20}$$

The operator $H_{0,n}^{n+1}$ is self-adjoint in $\mathfrak{F}_n^{n+1}$.

We denote respectively by H_I^n and $H_{I,n}^{n+1}$ the operator defined as the interaction H_I given by (2.8)-(2.10) (for $\ell = 1$, $\epsilon = +$ and $\epsilon' = -$), but associated respectively with the kernels

$$\tilde{\chi}^{\sigma_n}(p_2) G^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3) \tag{4.21}$$

and

$$\left(\tilde{\chi}^{\sigma_{n+1}}(p_2) - \tilde{\chi}^{\sigma_n}(p_2)\right) G^{(\alpha)}(\xi_1,\, \xi_2,\, \xi_3) \tag{4.22}$$

where $\tilde{\chi}^{\sigma_n}$ and $\tilde{\chi}^{\sigma_{n+1}}$ are defined as in (4.1).

We also consider

$$\begin{aligned} H_+^n &= H^n - E^n\,, \\ \widetilde{H_+^n} &= H_+^n \otimes \mathbf{1}_n^{n+1} + \mathbf{1}_n \otimes H_{0,n}^{n+1}\,, \end{aligned} \tag{4.23}$$

which are self-adjoint operators in $\mathfrak{F}^n$ and $\mathfrak{F}^{n+1}$ respectively.

Combining (A.27) of Lemma A.5 with (4.14) and (4.15), we obtain for $n \geq 0$ and $\psi \in \mathcal{D}(H_0^n) \subset \mathfrak{F}^n$,

$$g\|H_I^n \psi\| \leq g K(G) \left(C_{\beta\eta}\|H_0\psi\| + B_{\beta\eta}\|\psi\|\right)\,. \tag{4.24}$$

It follows from [25, Section V, Theorem 4.11]

$$\begin{aligned} H^n &\geq -\frac{gK(G)B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}} \geq -\frac{g_1 K(G)B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\,, \\ E^n &\geq -\frac{gK(G)B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\,. \end{aligned} \tag{4.25}$$

We have

$$(\Omega^n,\, H^n \Omega^n) = 0. \tag{4.26}$$

Therefore

$$E^n \leq 0\,, \quad \text{and} \quad |E^n| \leq \frac{gK(G)B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\,. \tag{4.27}$$

Let

$$K_n^{n+1}(G) = K(\mathbf{1}_{\sigma_{n+1} \leq |p_2| \leq 2\sigma_n} G)\,. \tag{4.28}$$

Combining (A.27) with (4.14) and (4.28), we obtain, for $n \geq 0$,

$$g\|H_{I,n}^{n+1}\psi\| \leq g K_n^{n+1}(G) \left(C_{\beta\eta}\|H_0^{n+1}\psi\| + B_{\beta\eta}\|\psi\|\right)\,, \tag{4.29}$$

for $\psi \in \mathcal{D}(H_0^{n+1}) \subset \mathfrak{F}^{n+1}$. We also have

$$H_0^{n+1}\psi = \widetilde{H_+^n}\psi + E^n\psi - g(H_I^n \otimes \mathbf{1}_n^{n+1})\psi\,, \tag{4.30}$$

and by (4.24),

$$g\|(H_I^n \otimes \mathbf{1}_n^{n+1})\psi\| \leq gK(G)(C_{\beta\eta}\|H_0^{n+1}\psi\| + B_{\beta\eta}\|\psi\|)\,. \tag{4.31}$$

In view of (4.27) and (4.30), it follows from (4.31) that

$$\begin{aligned} &g\left\|\left(H_I^n \otimes \mathbf{1}_n^{n+1}\right)\psi\right\| \\ &\leq \frac{gK(G)C_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\|\widetilde{H_+^n}\psi\| + \frac{gK(G)B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\left(1 + \frac{gK(G)B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\right)\|\psi\|\,. \end{aligned} \tag{4.32}$$

Therefore, we obtain

$$g\|H_{I,n}^{n+1}\psi\| \leq gK_n^{n+1}(G)\left(\tilde{C}_{\beta\eta}\|\widetilde{H_+^n}\psi\| + \tilde{B}_{\beta\eta}\|\psi\|\right) . \tag{4.33}$$

Due to Hypothesis 3.1*(ii)*, we have

$$K_n^{n+1}(G) \leq 2\,\sigma_n \tilde{K}(G) . \tag{4.34}$$

Recall that for $n \geq 0$,

$$\sigma_{n+1} < m_1 . \tag{4.35}$$

Therefore, by (4.33) and (4.34) we get

$$g\,\|H_{I,n}^{n+1}\psi\| \leq g\,K_n^{n+1}(G)\left(\tilde{C}_{\beta\eta}\|(\tilde{H}_+^n + \sigma_{n+1})\psi\| + (\tilde{C}_{\beta\eta}\,m_1 + \tilde{B}_{\beta\eta})\|\psi\|\right) ,$$

and for $\phi \in \mathfrak{F}$,

$$\begin{aligned} g\|H_{I,n}^{n+1}(\tilde{H}_+^n + \sigma_{n+1})^{-1}\phi\| &\leq g\,K_n^{n+1}(G)\left(\tilde{C}_{\beta\eta} + \frac{m_1\tilde{C}_{\beta\eta} + \tilde{B}_{\beta\eta}}{\sigma_{n+1}}\right)\|\phi\| \\ &\leq \frac{g}{\gamma}\,\tilde{K}(G)(2m_1\tilde{C}_{\beta\eta} + \tilde{B}_{\beta\eta})\|\phi\| . \end{aligned} \tag{4.36}$$

Thus, by (4.36), the operator $H_{I,n}^{n+1}(\tilde{H}_+^n + \sigma_{n+1})^{-1}$ is bounded and

$$g\|H_{I\,n}^{\ n+1}(\tilde{H}_+^n + \sigma_{n+1})^{-1}\| \leq g\frac{\tilde{D}_\delta(G)}{\gamma} ,$$

where $\tilde{D}_\delta(G)$ is given by (4.13). This yields, for $\psi \in \mathcal{D}(\tilde{H}_+^n)$,

$$g\|H_{I,n}^{n+1}\psi\| \leq g\frac{\tilde{D}_\delta(G)}{\gamma}\|(\tilde{H}_+^n + \sigma_{n+1})\psi\| .$$

Hence it follows from [25, §V, Theorems 4.11 and 4.12] that

$$g|(H_{I,n}^{n+1}\psi,\, \psi)| \leq g\frac{\tilde{D}_\delta(G)}{\gamma}(\,(\tilde{H}_+^n + \sigma_{n+1})\psi,\, \psi\,) . \tag{4.37}$$

For $g_\delta^{(1)}$ given by (4.16), let $g_\delta^{(2)} > 0$ be such that

$$g_\delta^{(2)}\frac{\tilde{D}_\delta(G)}{\gamma} < 1 \quad \text{and} \quad g_\delta^{(2)} \leq g_\delta^{(1)} .$$

By (4.37) we get, for $g \leq g_\delta^{(2)}$,

$$H^{n+1} = \tilde{H}_+^n + E^n + gH_{I,n}^{n+1} \geq E^n - \frac{g\,\tilde{D}_\delta(G)}{\gamma}\,\sigma_{n+1} + \Big(1 - \frac{g\,\tilde{D}_\delta(G)}{\gamma}\Big)\tilde{H}_+^n . \tag{4.38}$$

Because $(1 - g\tilde{D}_\delta(G)/\gamma)\tilde{H}_+^n \geq 0$, we get from (4.38), for $n \geq 0$,

$$E^{n+1} \geq E^n - \frac{g\,\tilde{D}_\delta(G)}{\gamma}\,\sigma_{n+1}. \tag{4.39}$$

Suppose that $\psi^n \in \mathfrak{F}^n$ satisfies $\|\psi^n\| = 1$ and for $\epsilon > 0$,

$$E^n \leq (\psi^n,\, H^n\psi^n) \leq E^n + \epsilon . \tag{4.40}$$

Let

$$\tilde{\psi}^{n+1} = \psi^n \otimes \Omega_n^{n+1} \in \mathfrak{F}^{n+1} . \tag{4.41}$$

We obtain

$$E^{n+1} \leq (\tilde{\psi}^{n+1}, \, H^{n+1}\tilde{\psi}^{n+1}) \leq E^n + \epsilon + g(\tilde{\psi}^{n+1}, \; H_{I,n}^{n+1}\,\tilde{\psi}^{n+1})\,. \tag{4.42}$$

By (4.37), (4.40), (4.41) and (4.42) we get, for every $\epsilon > 0$,

$$E^{n+1} \leq E^n + \epsilon(1 + \frac{g\,\tilde{D}_\delta(G)}{\gamma}) + \frac{g\,\tilde{D}_\delta(G)}{\gamma}\,\sigma_{n+1}\;,$$

where $g \leq g_\delta^{(2)}$. This yields

$$E^{n+1} \leq E^n + \frac{g\,\tilde{D}_\delta(G)}{\gamma}\,\sigma_{n+1}\;, \tag{4.43}$$

and by (4.39), we obtain

$$|E^n - E^{n+1}| \leq \frac{g\,\tilde{D}_\delta(G)}{\gamma}\,\sigma_{n+1}\;.$$

Let us first check that, for σ_0 given by (4.9), $E^0 := E^{\sigma_0}$ is a simple isolated eigenvalue of H^{σ_0} with

$$\inf \mathrm{Spec}(H^{\sigma_0}) \setminus \{E^0\} \geq m_1\,. \tag{4.44}$$

We have

$$\begin{aligned} g\|H_I(\mathbf{1}_{\sigma_0\leq|p_2|}G)\psi\| &\leq gK(G)(C_{\beta\eta}\|H_0^{\sigma_0}\psi\| + B_{\beta\eta}\|\psi\|) \\ &\leq gK(G)(C_{\beta\eta}\|(H_0^{\sigma_0}+1)\psi\| + (C_{\beta\eta}+B_{\beta\eta})\|\psi\|)\,. \end{aligned} \tag{4.45}$$

By (4.45) we get

$$g\|H_I(\mathbf{1}_{\sigma_0\leq|p_2|}G)\psi\| \leq gK(G)(2C_{\beta\eta} + B_{\beta\eta})\|(H_0^{\sigma_0}+1)\psi\| \tag{4.46}$$

and

$$g\left|(\psi, \, H_I(\mathbf{1}_{\sigma_0\leq|p_2|}G)\psi)\right| \leq gK(G)(2C_{\beta\eta} + B_{\beta\eta})(\psi, \, (H_0^{\sigma_0}+1)\psi)\,. \tag{4.47}$$

Set

$$\mu_2 = \sup_{\substack{\phi\in\mathfrak{F}^{\sigma_0}\\ \phi\neq 0}} \; \inf_{\substack{\psi\in\mathcal{D}(H^{\sigma_0})\\ (\psi,\phi)=0\\ \|\psi\|=1}} (\psi, \, H^{\sigma_0}\psi)\,. \tag{4.48}$$

By (4.47) and (4.48), we have, for Ω^0 being the vacuum state in $\mathfrak{F}^0 = \mathfrak{F}^{\sigma_0}$,

$$\mu_2 \geq \inf_{\substack{\psi\in\mathcal{D}(H^{\sigma_0})\\ (\psi,\Omega^0)=0\\ \|\psi\|=1}} (\psi, \, H^{\sigma_0}\psi) \geq \sigma_0 - gK(G)(2C_{\beta\eta}+B_{\beta\eta})(\sigma_0+1)\,. \tag{4.49}$$

Using the definition

$$g_3 := \frac{1}{2K(G)(2C_{\beta\eta}+B_{\beta\eta})}\,, \tag{4.50}$$

we get, for $g \leq g_3$,

$$\mu_2 \geq \frac{\sigma_0 - 1}{2} \geq E^{\sigma_0} + m_1 \tag{4.51}$$

since $\sigma_0 = 2m_1 + 1$ and $E^{\sigma_0} \leq 0$. Therefore, by the min-max principle, E^{σ_0} is a simple eigenvalue of H^{σ_0} such that (4.44) holds true.

We now conclude the proof of Proposition 4.1 by induction in $n \in \mathbb{N}$. Suppose that E^n is a simple isolated eigenvalue of H^n such that, for $n \geq 1$,

$$\inf \big(\mathrm{Spec}(H_+^n) \setminus \{0\}\big) \geq \Big(1 - \frac{3g\tilde{D}_\delta(G)}{\gamma}\Big)\sigma_n \, .$$

Due to (4.9)-(4.16), we have, for $0 < g \leq g_\delta^{(1)}$ and $n \geq 1$,

$$0 < \sigma_{n+1} < \Big(1 - \frac{3g\tilde{D}_\delta(G)}{\gamma}\Big)\sigma_n \, . \tag{4.52}$$

Therefore, for $g \leq g_\delta^{(2)}$, 0 is also a simple isolated eigenvalue of $\tilde{H}_+^n$ such that

$$\inf \Big(\mathrm{Spec}(\tilde{H}_+^n) \setminus \{0\}\Big) \geq \sigma_{n+1} \ . \tag{4.53}$$

We must now prove that E^{n+1} is a simple isolated eigenvalue of H^{n+1} such that

$$\inf \big(\mathrm{Spec}(H_+^{n+1}) \setminus \{0\}\big) \geq \Big(1 - \frac{3g\tilde{D}_\delta(G)}{\gamma}\Big)\sigma_{n+1} \ .$$

Let

$$\lambda^{(n+1)} = \sup_{\substack{\psi \in \mathfrak{F}^{n+1} \\ \psi \neq 0}} \ \inf_{\substack{(\phi,\psi)=0 \\ \phi \in \mathcal{D}(H^{n+1}) \\ \|\phi\|=1}} (\phi, \, H_+^{n+1}\phi) \ .$$

By (4.38) and (4.43), we obtain, in $\mathfrak{F}^{n+1}$,

$$\begin{aligned} H_+^{n+1} &\geq E^n - E^{n+1} - \frac{g\tilde{D}_\delta(G)}{\gamma}\sigma_{n+1} + \Big(1 - \frac{g\tilde{D}_\delta(G)}{\gamma}\Big)\tilde{H}_+^n \\ &\geq \Big(1 - \frac{g\tilde{D}_\delta(G)}{\gamma}\Big)\tilde{H}_+^n - \frac{2g\tilde{D}_\delta(G)}{\gamma}\sigma_{n+1} \ . \end{aligned} \tag{4.54}$$

By (4.41), $\tilde{\psi}^{n+1}$ is the unique ground state of $\tilde{H}_+^n$ and by (4.53) and (4.54), we have, for $g \leq g_\delta^{(2)}$,

$$\begin{aligned} \lambda^{(n+1)} &\geq \inf_{\substack{(\phi,\tilde{\psi}^{n+1})=0 \\ \phi \in \mathcal{D}(H^{n+1}) \\ \|\phi\|=1}} (\phi, H_+^{n+1}\phi) \\ &\geq \Big(1 - \frac{g\tilde{D}_\delta(G)}{\gamma}\Big)\sigma_{n+1} - \frac{2g\tilde{D}_\delta(G)}{\gamma}\sigma_{n+1} = \Big(1 - \frac{3g\tilde{D}_\delta(G)}{\gamma}\Big)\sigma_{n+1} > 0 \ . \end{aligned}$$

This concludes the proof of Proposition 4.1, if one proves that for

$$\tilde{g}_\delta^{(2)} := \min\{g_\delta^{(2)}, \, g_3\} \, , \tag{4.55}$$

the operator H^1 satisfies the gap condition

$$\inf \big(\mathrm{Spec}(H_+^1) \setminus \{0\}\big) \geq \Big(1 - \frac{3g\tilde{D}_\delta(G)}{\gamma}\Big)\sigma_1 \, . \tag{4.56}$$

By noting that 0 is a simple isolated eigenvalue of $\tilde{H}_+^0$ such that $\inf(\mathrm{Spec}(\tilde{H}_+^0) \setminus \{0\}) \geq \sigma_1$, we prove that E^1 is indeed an isolated simple eigenvalue of H^1 such that (4.56) holds, by mimicking the proof given above for H_+^{n+1}. □

5. Mourre inequality

Set

$$\tau := 1 - \frac{\delta}{2(2m_1 - \delta)} \ . \tag{5.1}$$

We have, according to (4.10)

$$0 < \gamma < \tau < 1 \quad \text{and} \quad \frac{\tau - \gamma}{2} < \gamma \ . \tag{5.2}$$

Let $\chi^{(\tau)} \in C^\infty(\mathbb{R}, [0,1])$ be such that

$$\chi^{(\tau)}(\lambda) = \begin{cases} 1 & \text{for } \lambda \in (-\infty, \tau] \,, \\ 0 & \text{for } \lambda \in [1, \infty) \,. \end{cases} \tag{5.3}$$

With the definition (4.9) of $(\sigma_n)_{n\geq 0}$ we set for all $p_2 \in \mathbb{R}^3$ and $n \geq 1$,

$$\chi_n^{(\tau)}(p_2) = \chi^{(\tau)}\left(\frac{|p_2|}{\sigma_n}\right) , \tag{5.4}$$

$$a_n^{(\tau)} = \chi_n^{(\tau)}(p_2)\frac{1}{2}\left(p_2 \cdot i\nabla_{p_2} + i\nabla_{p_2} \cdot p_2\right)\chi_n^{(\tau)}(p_2) \ , \tag{5.5}$$

and

$$A_n^{(\tau)} = \mathbf{1} \otimes \mathrm{d}\Gamma(a_n^{(\tau)}) \otimes \mathbf{1} \,, \tag{5.6}$$

where $\mathrm{d}\Gamma(.)$ refers to the usual second quantization of one particle operators. The operators $a_n^{(\tau)}$ and $A_n^{(\tau)}$ are self-adjoint. We also have

$$a_n^{(\tau)} = \frac{1}{2}\left(\chi_n^{(\tau)}(p_2)^2 p_2 \cdot i\nabla_{p_2} + i\nabla_{p_2} \cdot p_2\, \chi_n^{(\tau)}(p_2)^2\right) \ . \tag{5.7}$$

Let N be the smallest integer such that

$$N\gamma \geq 1 \ . \tag{5.8}$$

Due to (4.9)-(4.16), we have, for $0 < g \leq g_\delta^{(1)}$,

$$0 < \gamma < \left(1 - \frac{3g\tilde{D}_\delta(G)}{\gamma}\right) , \tag{5.9}$$

Therefore, according to (5.9) and (5.8), we have

$$\gamma < \gamma + \frac{1}{N}\left(1 - \frac{3g\tilde{D}_\delta(G)}{\gamma} - \gamma\right) < 1 - \frac{3g\tilde{D}_\delta}{\gamma} \ , \tag{5.10}$$

and

$$\frac{\gamma}{N} \leq \gamma - \frac{1}{N}\left(1 - \frac{3g\tilde{D}_\delta(G)}{\gamma} - \gamma\right) < \gamma \ . \tag{5.11}$$

Let

$$\epsilon_\gamma := \min\left\{\frac{1}{2N}\left(1 - \frac{3g\tilde{D}_\delta}{\gamma} - \gamma\right), \frac{\tau - \gamma}{4}\right\} \ . \tag{5.12}$$

We choose $f \in C_0^\infty(\mathbb{R})$ such that $0 \leq f \leq 1$ and

$$f(\lambda) := \begin{cases} 1 & \text{if } \lambda \in [(\gamma - \epsilon_\gamma)^2, \gamma + \epsilon_\gamma] \,, \\ 0 & \text{if } \lambda > \gamma + 2\epsilon_\gamma \,, \\ 0 & \text{if } \lambda < (\gamma - 2\epsilon_\gamma)^2 \,. \end{cases} \tag{5.13}$$

Note that using the definition (5.12) of ϵ_γ and (5.10), (5.2) and (5.11), we have, for $g \leq g_\delta^{(1)}$,

$$\gamma + 2\epsilon_\gamma < 1 - \frac{3g\tilde{D}_\delta}{\gamma} \,, \tag{5.14}$$

where $g_\delta^{(1)}$ is defined by (4.16). We also have $\gamma + 2\epsilon_\gamma < \tau$, and

$$\gamma - \epsilon_\gamma > \frac{\gamma}{N} \,. \tag{5.15}$$

Let us next define, for $n \geq 1$,

$$f_n(\lambda) := f\left(\frac{\lambda}{\sigma_n}\right) \,. \tag{5.16}$$

and

$$H_n = H_{\sigma_n} \,, \quad E_n = \inf \operatorname{Spec}(H_n) \quad \text{and } H_{0,n}^{(2)} = H_{0,\sigma_n}^{(2)} \,, \tag{5.17}$$

where we used the definitions (4.4) and (4.7) for $H_{0,\sigma_n}^{(2)}$ and H_{σ_n}. Note that $E_n = E^n$, where E^n is defined by (4.12). Let P^n denote the ground state projection of H^n. It follows from Proposition 4.1 that for $n \geq 1$ and $g \leq \tilde{g}_\delta^{(2)}$,

$$f_n(H_n - E_n) = P^n \otimes f_n(H_{0,n}^{(2)}) \,. \tag{5.18}$$

For $E = \inf \operatorname{Spec}(H)$ being the ground state energy of H defined in Theorem 3.2, and any interval Δ, let $E_\Delta(H - E)$ be the spectral projection for the operator $(H - E)$ onto Δ. Consider, for $n \geq 1$,

$$\Delta_n := [(\gamma - \epsilon_\gamma)^2 \sigma_n, \, (\gamma + \epsilon_\gamma)\sigma_n] \,. \tag{5.19}$$

We are now ready to state the Mourre inequality.

Theorem 5.1 (Mourre inequality)**.** *Suppose that the kernels $G^{(\alpha)}$ satisfy Hypothesis 2.1, 3.1(ii) and 3.1(iii.a).*

Then there exists $C_\delta > 0$ and $\tilde{g}_\delta^{(3)} > 0$ such that $\tilde{g}_\delta^{(3)} < \tilde{g}_\delta^{(2)}$ and

$$E_{\Delta_n}(H - E)\,[H, \, iA_n^{(\tau)}]\,E_{\Delta_n}(H - E) \geq C_\delta \frac{\gamma^2}{N^2}\,\sigma_n\,E_{\Delta_n}(H - E) \,, \tag{5.20}$$

for $g < \tilde{g}_\delta^{(3)}$ and $n \geq 1$.

Proof. Let

(5.21)

$$\begin{aligned}
\mathfrak{D}_1 &:= \{\psi \in \mathfrak{F}_a(L^2(\Sigma_1)) \mid \psi^{(n)} \in C_0^\infty \text{ for all } n \in \mathbb{N}, \text{ and } \psi^{(n)} = 0 \text{ for almost all } n\} \,, \\
\mathfrak{D}_2 &:= \mathfrak{D}_1 \,, \\
\mathfrak{D}_W &:= \{\psi \in \mathfrak{F}_W \mid \psi^{(n)} \in C_0^\infty \text{ for all } n \in \mathbb{N} \,, \text{ and } \psi^{(n)} = 0 \text{ for almost all } n\} \,,
\end{aligned}$$

and consider the algebraic tensor product

$$\mathfrak{D} = \mathfrak{D}_1 \hat{\otimes} \mathfrak{D}_2 \hat{\otimes} \mathfrak{D}_W \,. \tag{5.22}$$

According to [13, Lemma 28] and [14, Theorem 13] (see also [1, Proposition 2.11]), one easily shows that the sesquilinear form defined on $\mathfrak{D} \times \mathfrak{D}$ by

$$(\varphi, \, \psi) \to (H\varphi, \, iA_n^{(\tau)}\psi) - (A_n^{(\tau)}\varphi, \, iH\psi) \,, \tag{5.23}$$

is the one associated with the following symmetric operator denoted by $[H,\, iA_n^{(\tau)}]$,

$$[H,\, iA_n^{(\tau)}]\psi = \Big(\mathrm{d}\Gamma((\chi_n^{(\tau)})^2 w^{(2)}) \ + \ g\, H_I(-i(a_n^{(\tau)}G))\Big)\,\psi\,. \tag{5.24}$$

Let us prove that $[H,\, iA_n^{(\tau)}]$ is continuous for the graph topology of H. Combining (A.27) of Lemma A.5 with (4.14) and (4.15) we get, for $g \le g_1$, $n \ge 1$ and for $\psi \in \mathfrak{D}$

$$g\|H_I(-i(a_n^{(\tau)}G))\psi\| \le gK(-ia_n^{(\tau)}G)\,(C_{\beta\eta}\|H_0\psi\| + B_{\beta\eta}\|\psi\|)\ . \tag{5.25}$$

It follows from Hypothesis 3.1*(iii-a)* that there exists a constant $\tilde{C}(G)$ such that, for $n \ge 1$,

$$K(-i(a_n^{(\tau)}G)) \le \tilde{C}(G)\sigma_n\,. \tag{5.26}$$

We have, for $g \le g_1$,

$$\|H_0\psi\| \le \|H\psi\| + g\,\|H_I(G)\psi\| \le \|H\psi\| + gK(G)\,(C_{\beta\eta}\|H_0\psi\| + B_{\beta\eta}\|\psi\|)\ . \tag{5.27}$$

By definition of g_1 we have

$$g_1K(G)C_{\beta\eta} < 1\ . \tag{5.28}$$

By (5.27) and (5.28) we get

$$\|H_0\psi\| \le \frac{1}{1-g_1K(G)C_{\beta\eta}}\,(\|H\psi\| + g_1K(G)B_{\beta\eta}\|\Psi\|)\ . \tag{5.29}$$

Therefore, for $\psi \in \mathfrak{D}$,

$$\|\mathrm{d}\Gamma((\chi_n^{(\tau)})^2 w^{(2)})\psi\| \le \|H_0\psi\| \le \frac{1}{1-g_1K(G)C_{\beta\eta}}\,(\|H\psi\| + g_1K(G)B_{\beta\eta}\|\Psi\|)\ . \tag{5.30}$$

By (5.25), (5.26) and (5.29) we get, for $g \le g_1$, $n \ge 1$ and $\psi \in \mathfrak{D}$

$$\begin{aligned} &g\|H_I(-i(a_n^{(\tau)}G))\psi\| \\ &\le g\tilde{C}(G)\sigma_n\left(\frac{C_{\beta\eta}}{1-g_1K(G)C_{\beta\eta}}\|H\psi\| + \Big(\frac{g_1K(G)C_{\beta\eta}}{1-g_1K(G)C_{\beta\eta}}+1\Big)B_{\beta\eta}\|\psi\|\right)\,. \end{aligned} \tag{5.31}$$

Since $\mathfrak{D}$ is a core for H and $[H,\, iA_n^{(\tau)}]$, then (5.30) and (5.31) are fulfilled for $\psi \in \mathcal{D}(H)$. Therefore, (5.24) holds for $\psi \in \mathcal{D}(H)$. Moreover, it follows from [10, Proposition 3.6(iii)] that H is of class $C^1(A_n^{(\tau)})$ (see [3, Theorem 6.3.4] and condition (M') in [17]) for $g \le g_1$ and $n \ge 1$.

Recall from (5.16) that $f_n(\lambda) = f(\lambda/\sigma_n)$, where f is given by (5.13). Let $\tilde{f}(.)$ be an almost analytic extension of $f(.)$ satisfying

$$\left|\frac{\partial\tilde{f}}{\partial\bar{z}}(x+iy)\right| \le C\,y^2\,. \tag{5.32}$$

Note that

$$\tilde{f}(x+iy) \in C_0^\infty(\mathbb{R}^2)\ . \tag{5.33}$$

We have

$$f(s) = \int\frac{\mathrm{d}\tilde{f}(z)}{z-s}, \qquad \mathrm{d}\tilde{f}(z) = -\frac{1}{\pi}\frac{\partial\tilde{f}}{\partial\bar{z}}\mathrm{d}x\mathrm{d}y\ . \tag{5.34}$$

It follows from (5.18) that, for $g \leq \tilde{g}_\delta^{(2)}$,

$$\begin{aligned}\|\mathrm{d}\Gamma\big((\chi_n^{(\tau)})^2 w^{(2)}\big) f_n(H_n - E_n)\| &= \|P^n \otimes \mathrm{d}\Gamma\big((\chi_n^{(\tau)})^2 w^{(2)}\big) f_n(H_{0,n}^{(2)})\| \\ &\leq \|H_{0,n}^{(2)} f_n(H_{0,n}^{(2)})\| \ .\end{aligned} \tag{5.35}$$

Therefore, there exists $C_1^f > 0$, depending on f, such that for $g \leq \tilde{g}_\delta^{(2)}$,

$$\Big\|\mathrm{d}\Gamma\big((\chi_n^{(\tau)})^2 w^{(2)}\big) f_n(H_n - E_n)\Big\| \leq C_1^f \, \sigma_n \ . \tag{5.36}$$

Recall that (see (4.2) and (5.17)) $H_n = H_0 + gH_{I,n}$, where $H_{I,n} := H_{I,\sigma_n}$ is the interaction given by (2.8), (2.9) and (2.10) and associated with the kernels $\tilde{\chi}^{\sigma_n}(p_2) G^{(\alpha)}(\xi_1, \xi_2, \xi_3)$.

In (4.27), it is stated

$$|E^n| \leq \frac{g\, K(G)\, B_{\beta\eta}}{1 - g_1 K(G)\, C_{\beta\eta}} \ . \tag{5.37}$$

For $z \in \mathrm{supp}(\tilde{f})$, we have

$$\begin{aligned}(H_0 + 1)(H_n - E_n - z\sigma_n)^{-1} &= 1 + (E_n + z\sigma_n)(H_n - E_n - z\sigma_n)^{-1} \\ &\quad - gH_{I,n}(H_n - E_n - z\sigma_n)^{-1} + (H_n - E_n - z\sigma_n)^{-1}.\end{aligned} \tag{5.38}$$

Mimicking the proof of (5.29) and (5.31) and using (5.37), we get for $g \leq g_1$,

$$\begin{aligned}&g\|H_{I,n}(H_n - E_n - z\sigma_n)^{-1}\| \\ &\leq \frac{g_1 K(G) C_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}} \left(1 + \left(\frac{g_1 K(G) B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}} + |z|\sigma_n + \frac{g_1 K(G) B_{\beta\eta}}{1 - g_1 K(G) C_{\beta\eta}}\right) \frac{1}{|\mathrm{Im} z|\sigma_n}\right) \\ &\quad + \frac{g_1 K(G) B_{\beta\eta}}{|\mathrm{Im} z|\sigma_n} \ .\end{aligned} \tag{5.39}$$

It follows from (5.37), (5.38), and (5.39) that there exists $\tilde{C}_2(G) > 0$ such that, for $g \leq g_1$ and $n \geq 1$,

$$\|(H_0 + 1)(H_n - E_n - z\sigma_n)^{-1}\| \leq \tilde{C}_2(G)\, \frac{1 + |z|\sigma_n}{|\mathrm{Im} z|\sigma_n} \ . \tag{5.40}$$

Mimicking the proof of (5.40), we show that there exists $\tilde{C}_3(G) > 0$ such that, for $g \leq g_1$ and $n \geq 1$,

$$\|(H_0 + 1)(H - E - z\sigma_n)^{-1}\| \leq \tilde{C}_3(G)\, \frac{1 + |z|\sigma_n}{|\mathrm{Im} z|\sigma_n} \ . \tag{5.41}$$

We have

$$\begin{aligned}&gH_I(-i(a_n^{(\tau)} G)) f_n(H_n - E_n) \\ &= -\sigma_n \int \mathrm{d}\tilde{f}(z) H_I(-i(a_n^{(\tau)} G))(H_0 + 1)^{-1}(H_0 + 1)(H_n - E_n - z\sigma_n)^{-1} \ .\end{aligned} \tag{5.42}$$

By (5.27), (5.31), (5.40), and (5.42), there exists $\tilde{C}_4^f(G) > 0$ depending on f, such that for $g \leq g_1$,

$$g\, \Big\|H_I(-i(a_n^{(\tau)} G)) f_n(H_n - E_n)\Big\| \leq g\, \tilde{C}_4^f(G)\, \sigma_n \ . \tag{5.43}$$

Similarly, by (5.41), we easily show that there exists $\tilde{C}_5^f(G) > 0$, depending on f, such that for $g \le g_1$

$$g \left\| H_I(-i(a_n^{(\tau)}G)) f_n(H-E) \right\| \le g\, \tilde{C}_5^f(G)\, \sigma_n \ . \tag{5.44}$$

By (5.18), we have, for $g \le \tilde{g}_\delta^{(2)}$,

$$f_n(H_n-E_n)\mathrm{d}\Gamma((\chi_n^{(\tau)})^2 w^{(2)}) f_n(H_n-E_n) = P^n \otimes f_n(H_{0,n}^{(2)})\mathrm{d}\Gamma((\chi_n^{(\tau)})^2 w^{(2)}) f_n(H_{0,n}^{(2)}). \tag{5.45}$$

Since $\chi_n^{(\tau)}(\lambda) = 1$ if $\lambda \le (\gamma + 2\epsilon_\gamma)\sigma_n$, we have

$$f_n(H_{0,n}^{(2)})\, \mathrm{d}\Gamma\Big((\chi_n^{(\tau)})^2 w^{(2)}\Big)\, f_n(H_{0,n}^{(2)}) = f_n(H_{0,n}^{(2)})\, H_{0,n}^{(2)}\, f_n(H_{0,n}^{(2)}) \ . \tag{5.46}$$

Now, by (5.13), (5.15), (5.45), and (5.46), we obtain, with $g \le \tilde{g}_\delta^{(2)}$ and $n \ge 1$,

$$\begin{aligned} f_n(H_n-E_n)\, \mathrm{d}\Gamma\Big((\chi_n^{(\tau)})^2 w^{(2)}\Big)\, f_n(H_n-E_n) &\ge (\inf \operatorname{supp}(f_n)) f_n(H_n-E_n)^2 \\ &\ge \frac{\gamma^2}{N^2}\sigma_n f_n(H_n-E_n)^2 \ . \end{aligned} \tag{5.47}$$

Note that

$$\|f_n(H_n-E_n)\| = \|f_n(H-E)\| = \sup_\lambda |f_n(\lambda)| = 1 \,. \tag{5.48}$$

By (5.43) and (5.48) we get, for $g \le g_1$,

$$f_n(H_n-E_n) g H_I(-i(a_n^{(\tau)})G) f_n(H_n-E_n) \ge -g\tilde{C}_4^f(G)\sigma_n \,. \tag{5.49}$$

Thus we get, for $g \le \tilde{g}_\delta^{(2)}$, using (5.47) and (5.49),

$$f_n(H_n-E_n)[H,\, iA_n^{(\tau)}] f_n(H_n-E_n) \ge \frac{\gamma^2}{N^2}\sigma_n f_n(H_n-E_n)^2 - g\tilde{C}_4^f(G)\sigma_n \,. \tag{5.50}$$

We have

$$\begin{aligned} & f_n(H-E)[H,\, iA_n^{(\tau)}] f_n(H-E) \\ &= f_n(H_n-E_n)[H,\, iA_n^{(\tau)}] f_n(H_n-E_n) \\ &+ \left(f_n(H-E) - f_n(H_n-E_n)\right)[H,\, iA_n^{(\tau)}] f_n(H_n-E_n) \\ &+ f_n(H-E)[H,\, iA_n^{(\tau)}] \left(f_n(H-E) - f_n(H_n-E_n)\right) \ . \end{aligned} \tag{5.51}$$

Using (5.36) and Lemma A.3, we get, for $g \le \tilde{g}_\delta^{(2)}$

$$\left(f_n(H-E) - f_n(H_n-E_n)\right) \mathrm{d}\Gamma\left((\chi_n^{(\tau)})^2 w^{(2)}\right) f_n(H_n-E_n) \ge -gC_1^f\tilde{C}_6^f(G)\sigma_n \,. \tag{5.52}$$

By (5.43) and Lemma A.3, we obtain, for $g \le g_2$

$$g\left(f_n(H-E) - f_n(H_n-E_n)\right) H_I(-i(a_n^{(\tau)}G)) f_n(H_n-E_n) \ge -gg_2\tilde{C}_4^f(G)\tilde{C}_6^f(G)\sigma_n \,. \tag{5.53}$$

Thus it follows from (5.52) and (5.53) that

$$\begin{aligned} & \left(f_n(H-E) - f_n(H_n-E_n)\right)[H,\, iA_n^{(\tau)}] f_n(H_n-E_n) \\ & \ge -g\tilde{C}_6^f(G)\left(C_1^f + g_2\tilde{C}_4^f(G)\right)\sigma_n \,, \end{aligned} \tag{5.54}$$

for $g \le \inf(g_2,\, \tilde{g}_\delta^{(2)})$.

Similarly, by Lemma A.4 and (5.47), we obtain, for $g \leq \inf(g_2, \tilde{g}_\delta^{(2)})$

$$f_n(H-E)\mathrm{d}\Gamma\left((\chi_n^{(\tau)})^2 w^{(2)}\right)(f_n(H-E)-f_n(H_n-E_n)) \geq -g\tilde{C}_7^f(G)\sigma_n\,. \tag{5.55}$$

Moreover by (5.44) and Lemma A.3 we get, for $g \leq g_2$

$$gf_n(H-E)H_I(-i(a_n^{(\tau)}G))(f_n(H-E)-f_n(H_n-E_n)) \geq -gg_1\tilde{C}_5^f(G)\tilde{C}_6^f(G)\sigma_n\,. \tag{5.56}$$

Thus, it follows from (5.55) and (5.56) that

$$\begin{aligned} &f_n(H-E)[H,\, iA_n^{(\tau)}](f_n(H-E)-f_n(H_n-E_n)) \\ &\geq -g\left(\tilde{C}_7^f(G)+g_1\tilde{C}_5^f(G)\tilde{C}_6^f(G)\right)\sigma_n\,, \end{aligned} \tag{5.57}$$

for $g \leq \inf(g_2, \tilde{g}_\delta^{(2)})$.

By Lemma A.3 and (5.48) we easily get, for $g \leq g_2$

$$\begin{aligned} f_n(H_n-E_n)^2 =& f_n(H-E)^2 + (f_n(H_n-E_n)-f_n(H-E))^2 \\ &+ f_n(H-E)(f_n(H_n-E_n)-f_n(H-E)) \\ &+ (f_n(H_n-E_n)-f_n(H-E))f_n(H-E) \\ &\geq f_n(H-E)^2 - g\tilde{C}_6^f(G)(g_2\tilde{C}_6^f(G)+2)\,. \end{aligned} \tag{5.58}$$

It then follows from (5.50) and (5.58) that

$$\begin{aligned} &f_n(H_n-E_n)[H,\, iA_n^{(\tau)}]f_n(H_n-E_n) \\ &\geq \frac{\gamma^2}{N^2}\sigma_n f_n(H-E)^2 - g\sigma_n\left(\tilde{C}_4^f(G)+\frac{\gamma^2}{N^2}\tilde{C}_6^f(G)\left(g_2\tilde{C}_6^f(G)+2\right)\right)\,, \end{aligned} \tag{5.59}$$

for $g \leq \inf(g_2, \tilde{g}_\delta^{(2)})$.

Combining (5.51) with (5.54), (5.57) and (5.59), we obtain, for $g \leq \inf(g_2, \tilde{g}_\delta^{(2)})$

$$f_n(H-E)[H,\, iA_n^{(\tau)}]f_n(H-E) \geq \frac{\gamma^2}{N^2}\sigma_n f_n(H-E)^2 - g\sigma_n\tilde{C}_\delta\,, \tag{5.60}$$

where $\tilde{C}_\delta = \tilde{C}_6^f(G)(C_1^f + g_1\tilde{C}_4^f(G)) + \tilde{C}_7^f(G) + g_1\tilde{C}_5{}^f(G)\tilde{C}_6^f(G) + \tilde{C}_4^f(G) + \frac{\gamma^2}{N^2}\tilde{C}_6^f(G)(g_1\tilde{C}_6^f(G)+2)$. Multiplying both sides of (5.60) with $E_{\Delta_n}(H-E)$ we then get

$$E_{\Delta_n}(H-E)[H,\, iA_n^{(\tau)}]E_{\Delta_n}(H-E) \geq \left(\frac{\gamma^2}{N^2}-g\tilde{C}_\delta\right)\sigma_n E_{\Delta_n}(H-E)\,. \tag{5.61}$$

Picking a constant $\tilde{g}_\delta^{(3)}$ such that

$$\tilde{g}_\delta^{(3)} < \min\left\{g_2,\, \tilde{g}_\delta^{(2)},\, \frac{\gamma^2}{N^2}\frac{1}{\tilde{C}_\delta}\right\}, \tag{5.62}$$

Theorem 5.1 is proved, for $g \leq \tilde{g}_\delta^{(3)}$ and $n \geq 1$, with $C_\delta = \frac{\gamma^2}{N^2}\left(1-\frac{N^2}{\gamma^2}\tilde{C}_\delta\tilde{g}_\delta^{(3)}\right)$. □

6. $C^2(A_n^{(\tau)})$-regularity

Theorem 6.1. *Suppose that the kernels $G^{(\alpha)}$ satisfy Hypothesis 2.1 and Hypothesis 3.1(iii). Then H is locally of class $C^2(A_n^{(\tau)})$ in $(-\infty, m_1 - \frac{\delta}{2})$ for every $n \geq 1$.*

Proof. The proof is achieved by substituting $A_n^{(\tau)}$ for A_σ in the proof of Theorem 3.7 in [10]. □

Remark 6.1. *It is likely that the operator H is of class $C^2(A_n^{(\tau)})$, i.e., not only locally.*

7. Limiting Absorption Principle

For $A_n^{(\tau)}$ defined by (5.6), we set

$$\langle A_n^{(\tau)} \rangle = (1 + A_n^{(\tau)2})^{\frac{1}{2}} \ . \tag{7.1}$$

Recall that $[\sigma_{n+2}, \sigma_{n+1}] \subset \Delta_n = [\,(\gamma - \epsilon_\gamma)^2 \sigma_n,\, (\gamma + \epsilon_\gamma)\sigma_n], \quad n \geq 1.$

Theorem 7.1 (Limiting Absorption Principle)**.** *Suppose that the kernels $G^{(\alpha)}$ satisfy Hypothesis 2.1, 3.1 (ii), and 3.1 (iii). Then for any $\delta > 0$ satisfying $0 < \delta < m_1$, there exists $g_\delta > 0$ such that, for $0 < g \leq g_\delta$, for $s > 1/2$, φ, $\psi \in \mathfrak{F}$ and for $n \geq 1$, the limits*

$$\lim_{\epsilon \to 0} (\varphi,\, \langle A_n^{(\tau)} \rangle^{-s} (H - \lambda \pm i\epsilon) \langle A_n^{(\tau)} \rangle^{-s} \psi) \tag{7.2}$$

exist uniformly for $\lambda \in \Delta_n$.

Moreover, for $1/2 < s < 1$, the map

$$\lambda \mapsto \langle A_n^{(\tau)} \rangle^{-s} (H - \lambda \pm i0)^{-1} \langle A_n^{(\tau)} \rangle^{-s} \tag{7.3}$$

is Hölder continuous of degree $s - 1/2$ in Δ_n.

Proof. Theorem 7.1 follows from the $C^2(A_n^{(\tau)})$-regularity in Theorems 6.1 and the Mourre inequality in Theorem 5.1 with $g_\delta = \tilde{g}_\delta^{(3)}$ (defined by (5.62)), according to Theorems 0.1 and 0.2 in [28] (see also [20], [18] and [16]). □

8. Proof of Theorem 3.3

• We first prove (i) of Theorem 3.3.
According to (4.9) we have

$$[\sigma_{n+2}, \sigma_{n+1}] \subset [(\gamma - \epsilon_\gamma)^2 \sigma_n,\, (\gamma + \epsilon_\gamma)\sigma_n] = \Delta_n \,,$$

thus $\bigcup_n \Delta_n$ is a covering by open sets of any compact subset of $(\inf \mathrm{Spec}(H),\, m_1 - \delta)$. Therefore, [28, Theorem 0.1 and Theorem 0.2] together with the Mourre inequality (5.20) in Theorem 5.1 and the local $C^2(A_n^{(\tau)})$ regularity in Theorem 6.1 imply that (i) of Theorem 3.3 holds true.

• For the proof of (ii) of Theorem 3.3, let us first note that since $\bigcup_n \Delta_n$ is a covering by intervals of $(\inf \mathrm{Spec}(H),\, m_1 - \delta)$, using subadditivity, it suffices to prove the result for any $n \geq 1$ and $f \in C_0^\infty(\Delta_n)$.

For $a_n^{(\tau)} = \chi_n^{(\tau)}(p_2)\frac{1}{2}\left(p_2 \cdot i\nabla_{p_2} + i\nabla_{p_2}\cdot p_2\right)\chi_n^{(\tau)}(p_2)$, as given by (5.5), and $b = i\nabla_{p_2}$, we have

$$\begin{aligned} \|a_n^{(\tau)}\varphi\| &= \|\chi_n^{(\tau)}(p_2)\frac{1}{2}\left(p_2 \cdot i\nabla_{p_2} + i\nabla_{p_2}\cdot p_2\right)\chi_n^{(\tau)}(p_2)\varphi\| \\ &\leq \frac{1}{2}(\|\chi_n^{(\tau)}(p_2)\,p_2\| \;+\|p_2\chi_n^{(\tau)}(p_2)\|)\|i\nabla_{p_2}\varphi\| + \frac{1}{2}\|i\nabla_{p_2}\,p_2\chi_n^{(\tau)}\|\,\|\varphi\|\,, \end{aligned} \tag{8.1}$$

for all $\varphi \in \mathcal{D}(b)$. Therefore, there exists $c_n > 1$ such that

$$|a_n^{(\tau)}|^2 \leq c_n\langle b\rangle^2\,. \tag{8.2}$$

Since $\langle b\rangle$ is a nonnegative operator, [19, Proposition 3.4 ii)] implies

$$\mathrm{d}\Gamma(a_n^{(\tau)})^2 \leq c_n \mathrm{d}\Gamma(\langle b\rangle)^2\,, \tag{8.3}$$

and thus

$$\left(A_n^{(\tau)}\right)^2 \leq c_n B^2\ . \tag{8.4}$$

This implies

$$\|(B+1)^{-1}\langle A_n^{(\tau)}\rangle\| < \infty \quad \text{and} \quad \|\langle A_n^{(\tau)}\rangle(B+1)^{-1}\| < \infty\,. \tag{8.5}$$

The map

$$F(z) := \mathrm{e}^{-z\ln(B+1)}\mathrm{e}^{z\ln\langle A_n^{(\tau)}\rangle}\phi$$

is analytic on the strip $S := \{z\in\mathbb{C} \mid 0 < \mathrm{Re}\, z < 1\}$, for all $\phi\in\mathcal{D}(B)\subset\mathcal{D}(\langle A_n^{(\tau)}\rangle)$. For $\mathrm{Re}\, z = 0$, the operator $F(z)$ is bounded by $\|\phi\|$ and, for $\mathrm{Re}\, z = 1$, according to (8.5), $F(z)$ is bounded by $\|(B+1)^{-1}\langle A_n^{(\tau)}\rangle\|\,\|\phi\|$. Therefore, due to Hadamard's three-line theorem, $F(z)$ is a bounded operator on the strip S. In particular, for all $s\in(0,1)$, we obtain

$$\|(B+1)^{-s}\langle A_n^{(\tau)}\rangle^s\| < \infty \quad \text{and} \quad \|\langle A_n^{(\tau)}\rangle^s(B+1)^{-s}\| < \infty\,. \tag{8.6}$$

Using (8.6), we can write

$$\begin{aligned} &(\varphi,\, \langle B+1\rangle^{-s}(H-\lambda\pm\epsilon)^{-1}\langle B+1\rangle^{-s}\psi) \\ &(\langle A_n^{(\tau)}\rangle^s\langle B+1\rangle^{-s}\varphi,\, \langle A_n^{(\tau)}\rangle^{-s}(H-\lambda\pm\epsilon)^{-1}\langle A_n^{(\tau)}\rangle^{-s}\langle A_n^{(\tau)}\rangle^{s}\langle B+1\rangle^{-s}\psi)\,. \end{aligned}$$

We thus conclude the proof of Theorem 3.3 (ii) by using Theorem 7.1.

• We finally prove (iii) of Theorem 3.3. For that sake, we first need to establish the following lemma.

Lemma 8.1. *Suppose that $s\in(1/2,\,1)$ and that for some n, $f\in C_0^\infty(\Delta_n)$. Then*

$$\left\|\langle A_n^{(\tau)}\rangle^{-s}\mathrm{e}^{-itH}f(H)\langle A_n^{(\tau)}\rangle^{-s}\right\| = \mathcal{O}\left(t^{-(s-\frac{1}{2})}\right)\ . \tag{8.7}$$

Proof. The proof is the same as the one done in [16, Theorem 25] for the Pauli-Fierz model of non-relativistic QED. It makes use of the local Hölder continuity stated in Theorem 7.1. □

We eventually prove (iii) of Theorem 3.3 by using (8.6), Lemma 8.1, and writing

$$\begin{aligned} &\|(B+1)^{-s}\mathrm{e}^{itH}f(H)(B+1)^{-s}\| \\ &\leq \|(B+1)^{-s}\langle A_n^{(\tau)}\rangle^s\|\,\|\langle A_n^{(\tau)}\rangle^{-s}\mathrm{e}^{itH}f(H)\langle A_n^{(\tau)}\rangle^{-s}\|\,\|\langle A_n^{(\tau)}\rangle^s(B+1)^{-s}\|\,. \end{aligned}$$

Appendix A

In this section, we establish several lemmata that are useful for the proof of the Mourre estimate in Section 5.

Lemma A.1. *Suppose that the kernels $G^{(\alpha)}$ satisfy Hypothesis 2.1 and 3.1(ii). Then there exists a constant $D_1(G)$ such that, for $g \le g_2$ and $n \ge 1$,*

$$|E - E_n| \le g D_1(G) \sigma_n \, . \tag{A.8}$$

Proof. For $g \le g_2$, where $0 < g_2$ is given by [10, Theorem 3.3], we consider ϕ (respectively ϕ_n), the unique normalized ground state of H (respectively H_n) (see again [10, Theorem 3.3]). We thus have

$$\begin{aligned} E - E_n &\le (\phi_n, (H - H_n)\phi_n) \, , \\ E_n - E &\le (\phi, (H_n - H)\phi) \, , \end{aligned} \tag{A.9}$$

with

$$H - H_n = g H_I(\chi_{\sigma_n/2}(p_2) G) \tag{A.10}$$

where

$$\chi_{\sigma_n/2}(p) = \chi_0 \left(\frac{|p|}{\sigma_n/2} \right) \, , \tag{A.11}$$

and $\chi_0(.) \in C^\infty(\mathbb{R}, [0,1])$ is fixed.

Combining (A.27) of Lemma A.5 with (4.14) and (4.15), we get, together with (A.10), for $g \le g_2$,

$$\|(H - H_n)\phi_n\| \le g K(\chi_{\sigma_n/2}(p_2)G) \left(C_{\beta\eta} \|H_0 \phi_n\| + B_{\beta\eta} \right) \, , \tag{A.12}$$

and

$$\|(H - H_n)\phi\| \le g K(\chi_{\sigma_n/2}(p_2)G) \left(C_{\beta\eta} \|H_0 \phi\| + B_{\beta\eta} \right) \, . \tag{A.13}$$

It follows from Hypothesis 3.1*(ii)*, [10, (4.9)], and with (5.37) that there exists a constant $D_1(G) > 0$ depending on G, such that

$$\sup \left(\|(H - H_n)\phi_n\|, \ \|(H - H_n)\phi\| \right) \le g D_1(G) \sigma_n \, , \tag{A.14}$$

for $n \ge 1$ and $g \le g_2$.

By (A.9), this proves Lemma A.1. □

Lemma A.2. *We have*

$$\begin{aligned} &\|d\Gamma((\chi_n^{(\tau)})^2 w^{(2)}) \, (H_n - E_n - z\sigma_n)^{-1} \| \\ &\le \|(H_n - E_n)(H_n - E_n - z\sigma_n)^{-1}\| \le 1 + \frac{|z|}{|\mathrm{Im} z|} \, . \end{aligned} \tag{A.15}$$

Proof. We have

$$\mathbf{1} \otimes d\Gamma((\chi_n^{(\tau)})^2 w^{(2)}) \le \mathbf{1} \otimes H_{0,n}^{(2)} \le H_n - E_n \, . \tag{A.16}$$

Set

$$M_1 = \mathbf{1} \otimes H_{0,n}^{(2)}, \quad M_2 = (H^n - E^n) \otimes \mathbf{1}, \quad \text{and} \quad M = M_1 + M_2 = H_n - E_n \, .$$

Let ψ be in the algebraic tensor product $\mathfrak{D}(M_1)\hat{\otimes}\mathfrak{D}(M_2)$. We obtain

$$\begin{aligned}
&\|(M_1\otimes\mathbf{1}+\mathbf{1}\otimes M_2)\psi\|^2\\
&=\|(M_1\otimes\mathbf{1})\psi\|^2+\|(\mathbf{1}\otimes M_2)\psi\|^2+2\,\mathrm{Re}(\psi,\,(M_1\otimes\mathbf{1})(\mathbf{1}\otimes M_2)\psi)\\
&=\|(M_1\otimes\mathbf{1})\psi\|^2+\|(\mathbf{1}\otimes M_2)\psi\|^2+2((M_1{}^{\frac12}\otimes\mathbf{1})\psi,\,(\mathbf{1}\otimes M_2)\,(M_1{}^{\frac12}\otimes\mathbf{1})\psi)\\
&\geq\|(M_1\otimes\mathbf{1})\psi\|^2\,.
\end{aligned}$$

Thus we obtain

$$\left\|\mathrm{d}\Gamma\big((\chi_n^{(\tau)})^2w^{(2)}\big)\psi\right\|\leq\|(H_n-E_n)\psi\|\,. \tag{A.17}$$

The set $\mathfrak{D}(M_1)\hat{\otimes}\mathfrak{D}(M_2)$ is a core for M, thus (A.17) is satisfied for every $\psi\in\mathcal{D}(H_n-E_n)=\mathfrak{D}(H_0)$. Setting

$$\psi=(H_n-E_n-z\sigma_n)^{-1}\phi\,,$$

in (A.17), we immediately get (A.15). □

Lemma A.3. *Suppose that the kernels $G^{(\alpha)}$ verify Hypothesis 2.1 and 3.1(ii). Then there exists a constant $\tilde{C}_6^f(G)>0$ such that*

$$\|f_n(H_n-E_n)-f_n(H-E)\|\leq g\,\tilde{C}_6^f(G)\,, \tag{A.18}$$

for $g\leq g_2$ and $n\geq1$.

Proof. We have

$$f_n(H_n-E_n)-f_n(H-E)=\sigma_n\int\frac{1}{H_n-E_n-z\sigma_n}(H_n-H+E-E_n)\frac{1}{H-E-z\sigma_n}\mathrm{d}\tilde{f}(z)\,. \tag{A.19}$$

Combining (A.27) of Lemma A.5, (4.14), (4.15), and Hypothesis 3.1*(ii)*, we obtain, for every $\psi\in\mathcal{D}(H_0)$ and for $g\leq g_2$,

$$g\|H_I(\chi_{\sigma_n/2}G)\psi\|\leq g\sigma_n\tilde{K}(G)(C_{\beta\eta}\|(H_0+1)\psi\|+(C_{\beta\eta}+B_{\beta\eta})\|\psi\|)\,. \tag{A.20}$$

This yields

$$g\|H_I(\chi_{\sigma_n/2}G)(H_0+1)^{-1}\|\leq gD_2(G)\sigma_n\,, \tag{A.21}$$

for some constant $D_2(G)$ and for $g\leq g_2$. Combining Lemma A.1 with (5.41) and (A.19)-(A.21), we obtain

$$\|f_n(H_n-E_n)-f_n(H-E)\|\leq gD_2(G)\tilde{C}_3(G)\int\frac{\left|\frac{\partial\tilde{f}}{\partial\bar{z}}(x+iy)\right|}{y^2}(1+|z|m_1)\mathrm{d}x\mathrm{d}y\,, \tag{A.22}$$

for $g\leq g_2$.

Using (5.32) and (5.33), we conclude the proof of Lemma A.3 with

$$\tilde{C}_6^f(G)=D_2(G)\tilde{C}_3(G)\int\frac{\left|\frac{\partial\tilde{f}}{\partial\bar{z}}(x+iy)\right|}{y^2}(1+|z|m_1)\mathrm{d}x\mathrm{d}y\,.$$

□

Lemma A.4. *Suppose that the kernels $G^{(\alpha)}$ satisfy Hypothesis 2.1 and 3.1(ii). Then there exists a constant $\tilde{C}_7^f(G)>0$ such that, for $g\leq g_2$ and $n\geq1$,*

$$\left\|\mathrm{d}\Gamma\big((\chi_n^{(\tau)})^2w^{(2)}\big)\left(f_n(H_n-E_n)-f_n(H-E)\right)\right\|\leq g\,\tilde{C}_7^f\sigma_n\,. \tag{A.23}$$

Proof. We have

(A.24)
$$\mathrm{d}\Gamma\left((\chi_n^{(\tau)})^2 w^{(2)}\right)(f_n(H_n-E_n)-f_n(H-E))$$
$$=\sigma_n\int \mathrm{d}\Gamma\left((\chi_n^{(\tau)})^2 w^{(2)}\right)\frac{1}{H_n-E_n-z\sigma_n}(H_n-H+E_n-E)\frac{1}{H-E-z\sigma_n}\mathrm{d}\tilde{f}(z)\,.$$

Combining Lemmata A.1 and A.2 with (5.41) and (A.19)-(A.21), we obtain

$$\begin{aligned}&\|\mathrm{d}\Gamma\left((\chi_n^{(\tau)})^2 w^{(2)}\right)(f_n(H_n-E_n)-f_n(H-E))\,\|\\ &\le gD_2(G)\tilde{C}_3(G)\sigma_n\int\left|\frac{\partial\tilde{f}}{\partial\bar{z}}(x+iy)\right|\left(1+\frac{|z|}{y}\right)\left(\frac{1+|z|m_1}{y}\right)\mathrm{d}x\mathrm{d}y\,.\end{aligned}\tag{A.25}$$

Using (5.32) and (5.33), we conclude the proof of Lemma A.4 with

$$\tilde{C}_7^f(G)=D_2(G)\tilde{C}_3(G)\int\left|\frac{\partial\tilde{f}}{\partial\bar{z}}(x+iy)\right|\left(1+\frac{|z|}{y}\right)\left(\frac{1+|z|m_1}{y}\right)\mathrm{d}x\mathrm{d}y\,.\tag{A.26}$$

□

The following lemma was proved in [10, (2.53)-(2.54)], and gives explicitly the relative bound for H_I with respect to H_0. Note that this bound holds for any interaction operator H_I of the form (2.8)-(2.10), as soon as the kernels $G^{(\alpha)}$ fulfil Hypothesis 2.1.

Lemma A.5. *For any $\eta>0$, $\beta>0$, and $\psi\in\mathcal{D}(H_0)$, we have*

$$\begin{aligned}&\|H_I\psi\|\\ &\le 6\sum_{\alpha=1,2}\|G^{(\alpha)}\|^2\left(\frac{1}{2m_W}\left(\frac{1}{m_1^2}+1\right)+\frac{\beta}{2m_W m_1^2}+\frac{2\eta}{{m_1}^2}(1+\beta)\right)\|H_0\psi\|^2\\ &+\left(\frac{1}{2m_W}\left(1+\frac{1}{4\beta}\right)+2\eta\left(1+\frac{1}{4\beta}\right)+\frac{1}{2\eta}\right)\|\psi\|^2\,.\end{aligned}\tag{A.27}$$

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