

MAXIMAL SPEED OF QUANTUM PROPAGATION FOR THE HARTREE EQUATION

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ABSTRACT. We prove maximal speed estimates for nonlinear quantum propagation in the context of the Hartree equation. More precisely, under some regularity and integrability assumptions on the pair (convolution) potential, we construct a set of energy and space localized initial conditions such that, up to time-decaying tails, solutions starting in this set stay within the light cone of the corresponding initial datum. We quantify precisely the light cone speed, and hence the speed of nonlinear propagation, in terms of the momentum of the initial state.

1. THE PROBLEM AND RESULTS

In contrast to the key principle of relativity, in Quantum Mechanics, a local change of initial conditions effects the solutions everywhere instantaneously. As Quantum Mechanics is the theory of the quantum matter in non-extreme conditions, it is important to understand the limitations this imposes and to investigate universal properties of the Quantum Mechanical (QM) propagation in some depth.

QM predictions are probabilistic and therefore so is the characterization of quantum evolution. It was shown in [39] that, assuming the energy is bounded initially, the supports of solutions of the Schrödinger equation, up to vanishing in time probability tails, spread with a finite speed. This result was improved in [22, 41, 1] and extended in [5] to photon interacting with an atomic or molecular system, while [14, 15] developed a related approach in condensed matter physics.

For an electron in an atom the maximal velocity of propagation obtained in [1] is close to the one computed heuristically and to the one observed: of the order of 10^5 m/s which is less than 0.1% of the speed of light. Hence, in a few body case, the QM predictions here are fairly reliable.

Though bounds obtained in [39, 22, 41] and especially in [1] are valid for rather general class of potentials, for many-particles systems, their dependence on the number of particles (which in applications could vary from a few to 10^{20}) is rather poor.

In this paper, we prove bounds on the speed of propagation for bosonic many-body systems in the mean-field approximation, i.e. for the Hartree equation (HE),

$$i \frac{\partial}{\partial t} \psi_t = \left(-\frac{1}{2} \Delta + V\right) \psi_t + v * |\psi_t|^2 \psi_t, \quad (1.1)$$

in the physical space \mathbb{R}^d , $d \geq 3$. Here V and v are real functions, external (‘one-body’) and internal (interparticle, or ‘pair’) potentials.

We make the following two sets of assumptions on the pair convolution potential v , and the external potential V :

- We assume that $v : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies ¹

$$v \in W^{\gamma, (q, \infty)} \text{ if } 1 < q < d/2, \quad \text{for } d/(2q) < \gamma, \quad (1.2)$$
or $v \in W^{\gamma, 1}$ with $\gamma > d/2$,

where $W^{\gamma, q}$ is the standard Sobolev space ($W^{\gamma, 2} \equiv H^\gamma$), and $W^{\gamma, (q, \infty)}$ denotes the Lorentz-Sobolev space; see (1.10) below.

- We assume that $V : \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies Yajima-type conditions² (cf. [43, 17, 35], [19, Eq. (2.10)], and references therein):

$$|\langle x \rangle^\sigma \langle \nabla \rangle^\alpha V| \lesssim 1, \quad \sigma \geq \frac{3}{2}d + 3, \quad \alpha \leq \gamma + \lfloor \frac{d-1}{2} \rfloor, \quad (1.3)$$

where $\langle x \rangle := \sqrt{1 + |x|^2}$.

- Furthermore, we let $H := -\frac{1}{2}\Delta + V$ and assume that

$$H \text{ has neither nonpositive eigenvalues nor a resonance at } 0. \quad (1.4)$$

We recall that $H := -\frac{1}{2}\Delta + V$ is said to have a resonance at 0 if the equation $Hu = 0$ has a distributional solution $u \in \langle x \rangle^\nu L^2 = \{f = \langle \cdot \rangle^\nu g \mid g \in L^2\}$, for any $\nu > \frac{1}{2}$. One can show that there are no zero energy resonances in dimensions $d \geq 5$. In Proposition 1.8 below we give explicit restrictions on V for which condition (1.4) is satisfied.

The assumptions (1.3)-(1.4) are more than enough to guarantee that the propagator e^{-itH} is bounded on the Sobolev space H^γ , and satisfies the same dispersive estimates as the propagator $e^{it\Delta/2}$; see (B.2) and (B.11).

Denote $\chi = \chi(r)$ for a smooth bump function supported in $[0, 1]$ and such that $\chi = 1$ on $[0, 1/2]$. Let χ_A denote the characteristic function of a set A . For $I \subset \mathbb{R}$ a bounded open interval, we define the upper speed (or momentum, as the mass of particles is set equal to 1) bound given that the energy of the initial state ψ_0 is supported in I as

$$k_I := \|\nabla|\chi_I(H)\|. \quad (1.5)$$

Finally, let $\mathcal{B}^s(\varepsilon)$ denote the ball in H^s centered at the origin and of radius $\varepsilon > 0$. With this, we formulate our main result.

Theorem 1.1 (Maximal propagation speed for HE). *Assume Conditions (1.2), (1.3) and (1.4). Let I be a bounded open interval, $g \in C_0^\infty(I; \mathbb{R})$, $b > 0$ and $s \geq \gamma$. Then, there exist $\varepsilon > 0$ and a set*

$$S_{g,b} \subset \mathcal{B}^s(C\varepsilon), \quad (1.6)$$

¹The regularity assumptions on v are probably not optimal and can be weakened. We however decided to state things in this way for technical convenience.

²Note that these conditions are not optimal, and we opted to state them in a simple and concise form that applies to all dimensions $d \geq 3$. We refer to the cited papers for more precise assumptions.

for some absolute $C > 0$ (see (3.12)), such that the following hold true:

- (i) $S_{g,b}$ is in one-to-one correspondence with the set $g(H)\chi(|x|/b)\mathcal{B}^s(\varepsilon)$;
- (ii) For any initial condition $\psi_0 \in S_{g,b}$, the Hartree equation (1.1) has a global solution in H^s and this solution satisfies the estimate

$$\|\chi_{\{|x| \geq c|t|+a\}}\psi_t\| \lesssim \varepsilon \langle t \rangle^{-1/2}, \tag{1.7}$$

for any constants c and a satisfying $c > k_I$ and $a > b$.

Remark 1.2 (On the decay rate). The algebraic decay rate of $-1/2$ in (1.7) can probably be improved. The restriction is coming from the presence of the nonlinearity, as one can see by comparing with the arbitrary decay rate obtained in the linear (time-independent) case in [1].

Remark 1.3 (Scaling). In the quantum mechanical context, one would like to keep $\|\psi_0\|^2$, interpreted as the initial number of particles, at least of $O(1)$. To reflect this interpretation, one may phrase the main result of Theorem 1.1 without assuming smallness on the data, but assuming instead that the pair interaction potential is small. In other words, we may let $\varepsilon = 1$ in the statement of Theorem 1.1 and change v to $\varepsilon^2 v$ in the Hartree equation (1.1).

By definition (3.12) below, the set $S_{g,b}$ consists of H^s -functions ‘localized’ in the position x and energy H (essentially to $|x| \leq b$ and I , respectively). Estimate (1.7) shows that (up to time-decaying tails), a solution to the Hartree equation starting in the set

$$S := \bigcup_{g \in C_0^\infty, b > 0} S_{g,b}$$

stays inside the c -light cone of an initial condition. After [39], we call such a result a *maximal (propagation) speed bound* (MSB).

The infimum of all c ’s for which (1.7) holds is called the *maximal speed of propagation*, c_{\max} . Theorem 1.1 implies the bound $c_{\max} \leq k_I$.

As was mentioned above, for the (linear) Schrödinger equation ($v = 0$), such an estimate was found in [39], improved in [22, 41, 1].

The MSB is conceptually close to the celebrated Lieb-Robinson bound in Quantum Statistical Mechanics proved in [28] and improved by many authors and extended to various areas of quantum physics with many important applications (see [7, 20, 31] for reviews and references).³

For certain models of quantum many-body systems without the Hartree or Hartree-Fock approximation, the Lieb-Robinson and maximal velocity bounds were obtained in a number of papers, see [13, 18, 29, 14, 15, 16, 30, 7, 37, 42, 27, 44] and references therein.

Our approach follows that of [1] (originating in turn in [39]) for the linear Schrödinger equation and is based on constructing nearly monotonic (for bounded energy

³The Lieb-Robinson bound does not involve an explicit energy cut-off, but there is an implicit one in setting up the problem on a lattice, rather than a continuous space.

intervals) ‘propagation observables’ providing quantitative information about the quantum evolution in question. One of our key contributions is the extension of the quantum energy localization method, which proved to be exceptionally effective in the linear context, to nonlinear problems.

Remark 1.4 (MSB-intuition). The MSB is determined by the kinetic energy term and the main effort is directed at controlling the influence of the effective time-dependent potential, due to the nonlinearity (time-dependent self-consistent potential). Such a control is subtle as, a priori, one cannot rule out that the time-dependent potential does not pump energy onto the systems leading to an acceleration.

Remark 1.5 (Finiteness of k_I). Since V is bounded (in fact, Δ -boundedness with a relative bound < 1 (see (4.1) below) suffices), Δ is H -bounded and therefore k_I in (1.5) is finite.

Remark 1.6 (Sharpness of the MSB). Under our assumptions, the bound $c_{\max} \leq k_I$ is sharp. To illustrate this, we consider the special case of $v = 0$. In this case, we can take $S_{g,b} = g(H)\chi(|x|/b)L^2(\mathbb{R}^d)$. Then, one can show (see e.g. [38]) that solutions of (1.1) with $v = 0$ (the Schrödinger equation) with initial conditions in $S_{g,b}$ concentrate on the trajectories $x = pt + x_0$, where $p = -i\nabla$ is the quantum momentum operator and $x_0 \in \mathbb{R}^d$, and consequently $g(H) \approx g(\frac{1}{2}|p|^2 + V(pt + x_0)) \approx g(\frac{1}{2}|p|^2)$, as $t \rightarrow \infty$. Choosing I to be a small interval around an energy $E > 0$ and using that $g(H)$ is supported in I , we see that the support of the solution ψ_t expands at the rate $\sqrt{E}t$. On the other hand $k_I = \||p|\chi_I(H)\| \approx \sqrt{E}$. (Of course, one can simplify the arguments by taking $V = 0$.)

Remark 1.7 (Dependence on number of particles). The bound k_I is of the same form as in the (linear) one-particle case and gives a similar magnitude for the speed as in that case. However, the constant (subsumed in the symbol \lesssim) on the r.h.s. of (1.7) depends on the size of the pair potential v . (In the mean-field derivation of the HE, one considers the pair potential w in the many-body SE to be of the form $w = \frac{1}{n}v$, with $v = O(1)$ in the particle number n (see [4]). This v enters the HE.)

Open problems. (a) Prove MSB (1.7) with the implicit constant on the r.h.s. independent of the size of the pair potential v .

(b) Prove the MSB (1.7) for n -particle systems with n -independent bounds.

(c) Prove the MSB (1.7) for the Hartree-von Neumann (‘mixed state’ Hartree) equation, see [33] for some results on the long-time dynamics of this equation.

(d) Investigate a possibility of an acceleration due to strong nonlinearities.

The following proposition gives explicit restrictions on V guaranteeing that Condition (1.4) holds.

Proposition 1.8. *Assume that the positive and negative parts of the potential V , $V_+ = \max(V, 0)$, $V_- = \max(-V, 0)$, satisfy*

$$V_+(x) \leq C\langle x \rangle^{-\alpha}, \quad V_-(x) \leq \delta\langle x \rangle^{-\alpha}, \quad \alpha > 2, \quad (1.8)$$

for some $C > 0$ and for $\delta > 0$ small enough. Then the operator $H = -\frac{1}{2}\Delta + V$ has no eigenvalues and 0 is not a resonance of H .

Proposition 1.8, or a similar statement, might be known. For the convenience of the reader, we provide its proof in Appendix A.

Theorem 1.1 and Proposition 1.8 imply

Corollary 1.9. *Assume that Conditions (1.2), (1.3) and (1.8) hold. Then the conclusions of Theorem 1.1 hold.*

Remark 1.10 (The case of pure power NLS). Our main result, Theorem 1.1, can be extended also to the case of the *pure power NLS equation*

$$i \frac{\partial}{\partial t} \psi_t = \left(-\frac{1}{2}\Delta + V\right) \psi_t + |\psi_t|^{2\sigma} \psi_t, \quad \|\psi_0\|_{H^s \cap L^{p'}(\mathbb{R}^d)} \leq \varepsilon, \quad (1.9)$$

see Remark 4.4 below for a detailed discussion.

Organization of the paper. In Section 2, we state a standard global existence result with time decay for small solutions. In Section 3, we describe the class of admissible data, $S = \bigcup_{g \in C_0^\infty, b > 0} S_{g,b}$, for our MPS bounds. The main result of this section is proven in Section 6, after we state in Section 5 some preliminary estimates. In Section 4, we prove Theorem 1.1. Technical proofs are collected in the appendices.

Notation. As mentioned above, L^q and $W^{s,q}$ denote the usual Lebesgue and Sobolev spaces, $L^{q,\infty}$ denotes the weak- L^q space, and $W^{s,(q,\infty)}$ ($q > 1$) stands for Lorentz-Sobolev space,

$$W^{\gamma,(q,\infty)} = \{f \in L^{q,\infty}, \langle \nabla \rangle^\alpha f \in L^{q,\infty}, \alpha \leq \gamma\}. \quad (1.10)$$

To simplify the statements below, we will also write, abusing notations in the case where $q = 1$,

$$L^{1,\infty} \equiv L^1, \quad W^{s,(1,\infty)} \equiv W^{s,1}.$$

We let $\langle x \rangle := \sqrt{1 + |x|^2}$ for $x \in \mathbb{R}^d$, and denote by $L_\gamma^2 := \langle x \rangle^{-\gamma} L^2$ the usual weighted L^2 space.

We use standard notation for norms, and will often let $\|\cdot\| = \|\cdot\|_{L^2}$. We will sometimes also use $\|\cdot\|$ to denote the operator norm when there is no risk of confusion.

As usual, p' denotes the Hölder conjugate exponent of $p \in [1, \infty]$. The Lebesgue indices q and p appearing in the statements of the results (see for example (2.1) and (2.2) in Theorem 2.1) are related as follows:

$$\frac{1}{p} = \frac{1}{2} - \frac{1}{2q}, \quad \frac{1}{p} + \frac{1}{p'} = 1. \quad (1.11)$$

We write $A \lesssim B$ if $A \leq CB$ for some absolute constant $C > 0$ independent of A and B . We write $A \approx B$ if $A \lesssim B$ and $B \lesssim A$.

More notation will be introduced in the course of the proofs.

Acknowledgements. We are grateful to the anonymous referees for their many constructive remarks and suggestions. In particular, Remarks 1.3, 1.4, 1.6 and a part of 1.7 and Open Problems (c) and (d) were inserted in response to questions and comments by the referees.

The research of JA and IMS was supported in part by NSERC grant 7901. FP was supported in part by a start-up grant from the University of Toronto, and NSERC grant RGPIN-2018-06487.

2. GLOBAL SOLUTIONS

We begin with a basic theorem on the existence of global small solutions with time decay.

Theorem 2.1. *Let $1 \leq q < d/2$ and $s > d/(2q)$. Assume that V satisfies Conditions (1.3), (1.4) and that*

$$v \in L^{q,\infty} \quad \text{if } 1 < q < d/2, \quad v \in L^1 \quad \text{if } q = 1. \quad (2.1)$$

Then, there exists $\bar{\varepsilon} > 0$ such that, for any $\varepsilon \leq \bar{\varepsilon}$, Eq. (1.1) with an initial condition ψ_0 satisfying

$$\psi_0 \in H^s \cap L^{p'}, \quad \|\psi_0\|_{H^s \cap L^{p'}} \leq \varepsilon, \quad (2.2)$$

where p' is given by (1.11), i.e. $p' = \frac{2q}{q+1}$ for $q \geq 1$, has a unique global solution $\psi_t \in C(\mathbb{R}, H^s(\mathbb{R}^d))$ and this solution satisfies, for all $t \in \mathbb{R}$,

$$\|\psi_t\|_{L^2} = \|\psi_0\|_{L^2}, \quad (2.3)$$

$$\|\psi_t\|_{H^s} \lesssim \|\psi_0\|_{H^s}, \quad (2.4)$$

$$\|\psi_t\|_{L^p} \lesssim \varepsilon \langle t \rangle^{-d/(2q)}. \quad (2.5)$$

Theorem 2.1 is proven using standard energy and decay estimates. For $q = 1$ ($v \in L^1$), it is proven in [12]. In Appendix B, we present a proof for the full range of q 's.

3. SET OF INITIAL CONDITIONS AND ASYMPTOTIC ENERGY CUT-OFF

In this section, we describe the set of initial conditions, $S = \bigcup_{g \in C_0^\infty, b > 0} S_{g,b}$, used in Theorem 1.1. To do this, we introduce *asymptotic energy cut-offs*.

Let $f(|\psi|^2) := v * |\psi|^2$ and let H_t^ψ be the ψ_t -dependent Schrödinger operator on $L^2(\mathbb{R}^d)$ defined as

$$H_t^\psi := -\frac{1}{2}\Delta + V + f(|\psi_t|^2). \quad (3.1)$$

If $\psi_t \in L^p$, with p as in Theorem 2.1, by Young's inequality we have $f(|\psi_t|^2) \in L^\infty$. Then, by Kato's result (see e.g. [8]), the operator H_t^ψ is self-adjoint on the domain

of Δ . We let $U_t^\psi \equiv U_{t,0}^\psi$ be the propagator generated by H_t^ψ . Then a solution ψ_t of (1.1) satisfies the fixed point equation $\psi_t = U_t^\psi \psi_0$. We also denote

$$W_t^\psi := f(|\psi_t|^2) = v * |\psi_t|^2, \quad H_t^\psi = H + W_t^\psi. \quad (3.2)$$

For real-valued $g \in C_0^\infty(\mathbb{R})$ we define the asymptotic energy cut-offs (cf. [1, 39]) as the operator norm limit

$$g_+^\psi(H) := \lim_{\tau \rightarrow \infty} (U_\tau^\psi)^{-1} g(H) U_\tau^\psi. \quad (3.3)$$

The next proposition will imply that this limit exists under our assumptions.

Proposition 3.1. *Let $W_t(x) = W(x, t)$ be a real, time-dependent bounded potential satisfying*

$$\int_0^\infty \|\partial_x^\alpha W_t\|_{L^\infty} dt < \infty, \quad \text{where either } \alpha = 0 \quad \text{or} \quad 1 \leq |\alpha| \leq 2. \quad (3.4)$$

Let $U_t := U(t, 0)$ be the evolution generated by $H_t := H + W_t$. Then, for all $g \in C_0^\infty(\mathbb{R}; \mathbb{R})$, the following operator-norm limit exists

$$g_+(H) := \lim_{t \rightarrow \infty} U_t^{-1} g(H) U_t. \quad (3.5)$$

Proof. Define $g_t(H) := U_t^{-1} g(H) U_t$, and write $g_t(H)$ as the integral of the derivative and use that $\partial_r g_r(H) = -i U_r^{-1} [g(H), W_r] U_r$ to obtain

$$g_t(H) = g(H) - i \int_0^t U_r^{-1} [g(H), W_r] U_r dr. \quad (3.6)$$

If (3.4) holds with $\alpha = 0$, using the trivial estimate

$$\|[g(H), W_r]\| \lesssim \|g(H)\| \|W_r\|_{L^\infty} \lesssim \|W_r\|_{L^\infty},$$

shows that (3.5) exists.

If (3.4) holds with $1 \leq |\alpha| \leq 2$, then we use Lemma E.3, which shows that

$$\|[g(H), W_r]\| \lesssim \max_{1 \leq |\alpha| \leq 2} \|\partial_x^\alpha W_r\|_{L^\infty}.$$

Hence (3.5) exists. □

Remark 3.2 (Bound (3.4)). For either the Hartree case (3.2) or the NLS one (4.7), the bound (3.4) follows from the decay of $\|\psi_t\|_{L^p}$ at a sufficiently fast rate in t ; see for example Lemma 4.2.

Corollary 3.3 (Existence of the asymptotic cutoff (3.3)). *Under the conditions of Theorem 2.1, if ψ_t is a solution of (1.1) then, for all $g \in C_0^\infty(\mathbb{R}; \mathbb{R})$, the limit (3.5) exists and*

$$g_+^\psi(H) = g(H) - i \int_0^\infty (U_r^\psi)^{-1} [g(H), W_r^\psi] U_r^\psi dr. \quad (3.7)$$

Proof. Under the conditions of Theorem 2.1, Young's inequality implies that

$$\|W_t^\psi\|_{L^\infty} \lesssim \langle t \rangle^{-d/q}, \quad (3.8)$$

see Lemma 4.2 below. Hence since $q < d$, Proposition 3.1 implies the existence of the limit (3.3) and the formula (3.7). \square

To proceed, recall that $\mathcal{B}^s(\varepsilon)$ denotes the ball of radius ε in H^s ,

$$\mathcal{B}^s(\varepsilon) := \{f \in H^s, \|f\|_{H^s} < \varepsilon\}. \quad (3.9)$$

Recall also that $L_\gamma^2 := \langle x \rangle^{-\gamma} L^2$ and let $\|f\|_{L_\gamma^2} := \|\langle x \rangle^\gamma f\|_{L^2}$ be the corresponding norm. Note that if $\gamma > d/(2q)$, then $L_\gamma^2 \subset L^{p'}$ where p' is given by (1.11). For $\varepsilon > 0$, $\mathcal{B}_\gamma^s(\varepsilon)$ denotes the ball of radius ε in $L_\gamma^2 \cap H^s$:

$$\mathcal{B}_\gamma^s(\varepsilon) := \{f \in L_\gamma^2 \cap H^s, \|f\|_{L_\gamma^2} + \|f\|_{H^s} < \varepsilon\}. \quad (3.10)$$

The next proposition is the main result of this section.

Proposition 3.4. *Let $1 < q < d/2$, $d/(2q) < \gamma < d/q - 1$, and $\gamma \leq s$. Let $g \in C_0^\infty(\mathbb{R}; \mathbb{R})$. Let $\varepsilon, \tilde{\varepsilon} > 0$ be such that $C_0\varepsilon = \tilde{\varepsilon} \ll 1$ with C_0 sufficiently large. Then assuming Conditions (1.2), (1.3) and (1.4) on the potentials V and v ,*

- *For all $\phi \in \mathcal{B}_\gamma^s(\varepsilon)$ there exists a unique $\psi_0 \in \mathcal{B}_\gamma^s(\tilde{\varepsilon})$ solving the equation*

$$\psi_0 = g_+^{\Psi(\psi_0)}(H)\phi, \quad (3.11)$$

where $g_+^\psi(H)$ is defined in (3.3) and $\Psi(\psi_0) : t \rightarrow$ the solution ψ_t of (1.1) with the datum ψ_0 (which exists globally since $\psi_0 \in \mathcal{B}_\gamma^s(\tilde{\varepsilon}) \subset \tilde{\varepsilon}(H^s \cap L^{p'})$ and satisfies the properties in Theorem 2.1).

- *The map $\Phi_g : \phi \mapsto \psi_0$ restricted to the domain $g(H)\mathcal{B}_\gamma^s(\varepsilon)$ is injective (note that $g(H)\mathcal{B}_\gamma^s(\varepsilon) \subset \mathcal{B}_\gamma^s(C\varepsilon)$ for some $C > 0$).*

Eq. (3.11) is a fixed point problem depending on ϕ (or an implicit function equation). We prove Proposition 3.4 in Section 6.

Definition 3.5 (Set $S_{g,b}$). Let Φ_g be the injective maps defined in the second statement of Proposition 3.4. For $g \in C_0^\infty(\mathbb{R}; \mathbb{R})$ and $b > 0$, we define the set $S_{g,b}$ of initial conditions, ψ_0 , appearing in Theorem 1.1, Eq. (1.6), as

$$S_{g,b} := \Phi_g(g(H)\chi(|x|/b)\mathcal{B}^s(\varepsilon)). \quad (3.12)$$

Proposition 3.4 shows that the class of initial data $S_{g,b}$ is in one-to-one correspondence with the set $g(H)\chi(|x|/b)\mathcal{B}^s(\varepsilon)$ (note that $\chi(|x|/b)\mathcal{B}^s(\varepsilon) \subset \mathcal{B}_\gamma^s(C\varepsilon)$ for some $C > 0$). Proposition 3.4 and Theorem 2.1 imply that

Corollary 3.6. *Under the conditions of Proposition 3.4, the set $S_{g,b}$ of initial conditions, ψ_0 , constructed in (3.12), has properties (i) and the first part of (ii) (that for any initial condition $\psi_0 \in S_{g,b}$, the Hartree equation (1.1) has a global solution in H^s) of Theorem 1.1.*

4. PROOF OF THEOREM 1.1

We will use the following result about the linear propagators proven in [1] under stronger assumptions.

Theorem 4.1 (Maximal propagation speed for t -dependent potentials). *Suppose that $H_t = -\frac{1}{2}\Delta + V + W_t$, with $V(x)$ satisfying the inequality*

$$\|Vu\| \leq \frac{a_1}{2}\|\Delta u\| + a_2\|u\|, \quad (4.1)$$

for some $0 \leq a_1 < 1$, $a_2 > 0$, and $W_t(x)$, a real, time-dependent, bounded potential such that

$$\begin{aligned} \text{either } \int_0^\infty w_t dt < \infty \quad \text{or} \quad \int_0^\infty w'_t dt < \infty, \\ \text{where } w_t := \int_t^\infty \|W_r\|_{W^{1,\infty}} dr \quad \text{and} \quad w'_t := \max_{1 \leq |\alpha| \leq 2} \int_t^\infty \|\partial_x^\alpha W_r\|_{L^\infty} dr. \end{aligned} \quad (4.2)$$

Let I be a bounded open interval, $g \in C_0^\infty(I; \mathbb{R})$ and let k_I be as in (1.5). If $c > k_I$ and $a > b$, then, for all $0 < \beta < 1$, the evolution $U_t = U(t, 0)$ generated by H_t satisfies the estimate

$$\|\chi_{\{|x| \geq ct+a\}} U_t g_+(H) \chi_{\{|x| \leq b\}}\| \lesssim t^{-\min(\frac{1}{2}, 1-\beta)} + w_{t\beta}^\# \quad (4.3)$$

for $t \geq 1$. Here $w_{t\beta}^\#$ is either $w_{t\beta}$ or $w'_{t\beta}$, depending on the condition in (4.2).

A proof of Theorem 4.1 is given in Appendix D. Note that in our context, it is preferable to use the condition $\int_0^\infty \int_t^\infty \|W_r^\psi\|_{W^{1,\infty}} dr dt < \infty$ since it requires less regularity on v than the condition $\int_0^\infty \int_t^\infty \|\partial_\alpha W_r^\psi\|_{L^\infty} dr dt < \infty$, $1 \leq |\alpha| \leq 2$.

Proof of Theorem 1.1. Let $\psi_t \in C(\mathbb{R}, H^s(\mathbb{R}^d))$ be the unique global solution of Eq. (1.1) with an initial condition ψ_0 in the set $S_{g,b}$ (see (3.12)), given in Theorem 2.1. Theorem 1.1 follows from Theorem 4.1 by letting (see (3.2))

$$W_t \mapsto W_t^\psi := f(|\psi_t|^2), \quad U_t \mapsto U_t^\psi, \quad \beta = 1/2, \quad (4.4)$$

provided we verify Condition (4.2), that $w_{t\beta} \lesssim \langle t \rangle^{-1/2}$, and Proposition 3.4 which shows that the class of initial data $S_{g,b}$ is in one-to-one correspondence with the set $g(H)\chi(|x|/b)\mathcal{B}^s(\varepsilon) \subset g(H)\mathcal{B}_\gamma^s(C\varepsilon)$ for some $C > 0$ (see Corollary 3.6).

For Condition (4.2), we need the following direct consequence of Theorem 2.1:

Lemma 4.2. *Under the conditions of Theorem 2.1, we have that*

$$\|W_t^\psi\|_{L^\infty} \lesssim \varepsilon^2 \langle t \rangle^{-\frac{d}{q}}. \quad (4.5)$$

Suppose in addition that v satisfies (1.2). Then

$$\|W_t^\psi\|_{W^{\gamma,\infty}} \lesssim \varepsilon^2 \langle t \rangle^{-\frac{d}{q}}. \quad (4.6)$$

Proof of Lemma 4.2. First consider the L^∞ -norm of W_t^ψ . By Young's inequality,

$$\|W_t^\psi\|_{L^\infty} \lesssim \|v\|_{L^{q,\infty}} \|\psi_t\|_{L^{q'}}^2 \lesssim \|v\|_{L^{q,\infty}} \|\psi_t\|_{L^p}^2,$$

since $p = 2q'$. Hence the first inequality in (4.5) follows from (2.5) in Theorem 2.1.

If we suppose in addition that v belongs to $W^{\gamma,(q,\infty)}$, then we can write

$$\|W_t^\psi\|_{W^{\gamma,\infty}} \lesssim \|v\|_{W^{\gamma,(q,\infty)}} \|\psi_t\|_{L^{q'}}^2 \lesssim \|v\|_{W^{\gamma,(q,\infty)}} \|\psi_t\|_{L^p}^2,$$

and hence (4.6) follows again from (2.5). \square

Now, condition (4.2) follows from Lemma 4.2 since, using $\gamma \geq 1$, we have $\|W_t^\psi\|_{W^{1,\infty}} \lesssim \langle t \rangle^{-d/q}$ with $q < d/2$. Hence, $w_t \lesssim \langle t \rangle^{-d/q+1}$ and Theorem 1.1 follows. \square

Remark 4.3 (Integrability assumptions). In dimension 3, the endpoint case in Theorem 2.1 for our integrability conditions is $v \in L^{3/2}$ and $\psi_0 \in L^{6/5}$. This would correspond to sharp decay in L_x^6 that implies that $\|W_t\|_{L^\infty} \lesssim t^{-2}$, which is the borderline rate for the current argument; see (4.2).

Remark 4.4 (The case of pure power NLS). To extend Theorem 1.1 to the NLS (1.9) (see Remark 1.10), we take

$$W_t^\psi := |\psi_t|^{2\sigma} \tag{4.7}$$

instead of $v * |\psi_t|^2$. It is not hard to show that there exist numbers $\sigma_0 = \sigma_0(d)$ and $s_0 = s_0(\sigma, d)$ sufficiently large, such that assuming that $\sigma \geq \sigma_0(d)$ and $s \geq s_0(\sigma, d)$, a global existence result analogous to Theorem 2.1 holds for (1.9), and implies bounds on W_t that can be used to verify condition (4.2) and apply Theorem 4.1. The analogue of Proposition 3.4 needed to construct a suitable class of initial data, can also be proven following the same arguments we give in Section 6, and the analogous results from Section 5 for the flow of (1.9).

For example, in dimension $d = 3$, consider $\sigma = 3/2$ (quartic NLS) and let $p' \in [1, 6/5)$. Then one can show (as in the proof of Theorem 2.1) that the same estimate (2.4)-(2.5) hold for solution of (1.9):

$$\|\psi_t\|_{L^p} \lesssim \varepsilon \langle t \rangle^{-3(1/2-1/p)}, \quad \|\psi_t\|_{H^s} \lesssim \varepsilon. \tag{4.8}$$

Note that $3(1/2 - 1/p) > 1$ (since $p \in (6, \infty]$).

Then, let us first fix, for the sake of explanation, $p' = 1$. It follows that

$$\|W_t^\psi\|_{W^{1,\infty}} = \|\psi_t\|_{W^{1,\infty}}^2 \lesssim \|\psi_t\|_{W^{1,\infty}} \|\psi_t\|_{L^\infty}^2 \lesssim \varepsilon^3 \langle t \rangle^{-3}, \tag{4.9}$$

provided $s > 5/2$, and, for all $\gamma \leq s$,

$$\|W_t^\psi\|_{H^\gamma} = \|\psi_t\|_{H^\gamma}^2 \lesssim \|\psi_t\|_{H^\gamma} \|\psi_t\|_{L^\infty}^2 \lesssim \varepsilon^3 \langle t \rangle^{-3}. \tag{4.10}$$

In particular, the condition in Proposition 3.1, and the stronger condition (4.2) in Theorem 4.1 hold. Analogues of Lemma 4.2 and 5.2 also hold (note that the only relevant thing is that the exponent $d/q > 2$, so we can fix $q = 3/2 - \epsilon$ when comparing to the rates in (4.9)-(4.10)). The mapping properties in Lemma 5.4 and 5.5 can also be proved using (4.8), and the same goes for the estimates on the differences from Lemmas 5.6-5.9, since these only rely on (4.8)-(4.10) and the above

mentioned lemmas. Theorem 1.1 then follows for solutions of (1.9) with $\sigma = 3/2$ and $p' = 1$.

5. MAPPING PROPERTIES OF W_t^ψ AND U_t^ψ

In this section, we state several properties of W_t^ψ and the flow U_t^ψ that will be essential ingredients in the proofs in the next section. Proofs are deferred to Appendix C.

5.1. Mapping properties of W_t^ψ . Recall that the norms of W_t^ψ in the spaces L^∞ and $W^{\gamma,\infty}$ have been estimated in Lemma 5.1. We also need to estimate the $W^{s,2q}$ -norm of W_t^ψ .

Lemma 5.1. *Under the conditions of Theorem 2.1, we have that*

$$\|W_t^\psi\|_{W^{s,2q}} \lesssim \varepsilon^2 \langle t \rangle^{-\frac{d}{2q}}. \quad (5.1)$$

Identifying W_t^ψ with a multiplication operator, Lemmas 4.2 and 5.1 imply the following

Lemma 5.2. *Under the conditions of Theorem 2.1, we have that*

$$\|W_t^\psi\|_{H^s \rightarrow H^s} \lesssim \varepsilon^2 \langle t \rangle^{-\frac{d}{2q}}. \quad (5.2)$$

Suppose in addition that v satisfies (1.2). Then

$$\|W_t^\psi\|_{H^\gamma \rightarrow H^\gamma} \lesssim \varepsilon^2 \langle t \rangle^{-\frac{d}{q}}. \quad (5.3)$$

Lemmas 4.2 and 5.2 show that by imposing stronger regularity conditions on v , namely $v \in W^{s,(q,\infty)}$ (see Condition (1.2)) instead of $v \in L^{q,\infty}$, one improves the decay rate of W_t^ψ . This can also be achieved by assuming more regularity on the initial data ψ_0 .

Lemma 5.3. *Let $1 \leq q < d/2$ and $s > \gamma$. Let $\sigma = \sigma_1 + \sigma_2$ with $\sigma_1, \sigma_2 \geq 0$ and*

$$s > \frac{d}{2q} + \frac{\sigma_2 d}{d - 2q}.$$

Assume that V satisfies Conditions (1.3), (1.4) and that

$$v \in W^{\sigma_1,(q,\infty)} \quad \text{if } 1 < q < d/2, \quad v \in L^{\sigma_1,1} \quad \text{if } q = 1. \quad (5.4)$$

Then, there exists $\bar{\varepsilon} > 0$ such that, for any $\varepsilon \leq \bar{\varepsilon}$, for any

$$\psi_0 \in H^s \cap L^{p'}, \quad \|\psi_0\|_{H^s \cap L^{p'}} \leq \varepsilon, \quad (5.5)$$

we have

$$\|W_t^\psi\|_{W^{\sigma,\infty}} \lesssim \varepsilon^{2-2\varepsilon'q/d} \langle t \rangle^{-d/q+\varepsilon'},$$

with $d/q - \varepsilon' > 2$. In particular,

$$\|W_t^\psi\|_{H^s \rightarrow H^s} \lesssim \varepsilon^{2-2\varepsilon'q/d} \langle t \rangle^{-d/q+\varepsilon'}.$$

In the next lemmas of this section, one can replace the regularity assumption on v (i.e. v satisfies Condition 1.2) by the assumptions of Lemma 5.3. For simplicity, and since in our application we need that v satisfies (1.2), we do not elaborate.

5.2. Mapping properties of U_t^ψ . We now prove some mapping properties for U_t^ψ , where ψ is a global solution of (1.1) as in Theorem 2.1. Our first lemma shows that U_t^ψ is bounded as an operator in H^s .

Lemma 5.4. *Under the conditions of Theorem 2.1, there exists an absolute constant $C > 0$ such that*

$$\|U_t^\psi\|_{H^s \rightarrow H^s} \leq C. \quad (5.6)$$

In order to prove Proposition 3.4, we also need to estimate the norm of U_t^ψ as an operator from $H^\gamma \cap L_\gamma^2$ to L_γ^2 . Note that here we need to impose stronger regularity conditions on v than in the previous lemma.

Lemma 5.5. *Under the conditions of Theorem 1.1 and with $\gamma < d/q - 1$, for all $\varphi \in H^\gamma \cap L_\gamma^2$, we have*

$$\|U_t^\psi \varphi\|_{L_\gamma^2} \lesssim \langle t \rangle^\gamma \|\varphi\|_{H^\gamma} + \|\varphi\|_{L_\gamma^2}. \quad (5.7)$$

We mention that our proof of Lemma 5.5 actually establishes a stronger result than (5.7), as we estimate $\|\langle H \rangle^{\frac{\ell}{2}} \langle x \rangle^{\gamma'} U_t^\psi \varphi\|$ for suitable values of ℓ and γ' , see (C.21). The estimate (5.7) is however sufficient for our purpose.

5.3. Estimates on differences. The result of this subsection are needed to prove the contraction property in the fixed point argument used to establish Proposition 3.4. The first lemma estimates the differences between two solutions of (1.1).

Lemma 5.6. *Under the conditions of Theorem 2.1, consider ψ_t and φ_t two global solutions of (1.1) as in Theorem 2.1. We have*

$$\|\psi_t - \varphi_t\|_{L^p} \lesssim \langle t \rangle^{-d/(2q)} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}. \quad (5.8)$$

Using Lemma 5.6, it is not difficult to prove the following lemma.

Lemma 5.7. *Under the conditions of Theorem 2.1, consider ψ_t and φ_t two global solutions of (1.1) as in Theorem 2.1. We have*

$$\|W_t^\psi - W_t^\varphi\|_{L^\infty} \lesssim \varepsilon \langle t \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}, \quad (5.9)$$

$$\|W_t^\psi - W_t^\varphi\|_{W^{s,2q}} \lesssim \langle t \rangle^{-d/(2q)} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}, \quad (5.10)$$

and

$$\|W_t^\psi - W_t^\varphi\|_{H^s \rightarrow H^s} \lesssim \langle t \rangle^{-d/(2q)} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}. \quad (5.11)$$

If in addition v satisfies (1.2), then

$$\|W_t^\psi - W_t^\varphi\|_{W^{\gamma,\infty}} \lesssim \varepsilon \langle t \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}, \quad (5.12)$$

and

$$\|W_t^\psi - W_t^\varphi\|_{H^\gamma \mapsto H^\gamma} \lesssim \varepsilon \langle t \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}. \quad (5.13)$$

Finally, we estimate the norms of the differences of the flows $U_t^\psi - U_t^\varphi$.

Lemma 5.8. *Under the conditions of Theorem 2.1, consider ψ_t and φ_t two global solutions of (1.1) as in Theorem 2.1. Then we have*

$$\|U_t^\psi - U_t^\varphi\|_{H^s \mapsto H^s} \lesssim \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}. \quad (5.14)$$

Lemma 5.9. *Under the conditions of Theorem 1.1 and with $\gamma \leq d/q - 1$, consider ψ_t and φ_t two global solutions of (1.1) as in Theorem 2.1. For all $f \in H^\gamma \cap L_\gamma^2$, we have*

$$\|(U_t^\psi - U_t^\varphi)f\|_{L_\gamma^2} \lesssim \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} (\langle t \rangle^\gamma \|f\|_{H^\gamma} + \|f\|_{L_\gamma^2}). \quad (5.15)$$

6. PROOF OF PROPOSITION 3.4

In the first part of this proof, we will omit the superindex ψ for W_s^ψ , U_s^ψ and g_+^ψ and so on, when there is no risk of confusion.

Let $0 < \varepsilon \ll 1$ and $\phi \in \mathcal{B}_\gamma^s(\varepsilon)$. Fix $\tilde{\varepsilon} > 0$ such that $C_0\varepsilon \leq \tilde{\varepsilon} \ll 1$ for some absolute $C_0 > 1$ to be determined. We will show that the map

$$\psi_0 \mapsto F_\phi(\psi_0) := g_+^{\Psi(\psi_0)}(H)\phi$$

is a contraction in $\mathcal{B}_\gamma^s(\tilde{\varepsilon})$. Let $\psi_0 \in \mathcal{B}_\gamma^s(\tilde{\varepsilon})$. From (3.7) we have

$$F_\phi(\psi_0) = g(H)\phi - i \int_0^\infty U_r^{-1} W_r' U_r \phi \, dr, \quad W_r' := [g(H), W_r]. \quad (6.1)$$

With p as in (1.11), by Hölder's inequality we have $\|f\|_{L^{p'}} \leq \|\langle x \rangle^{-\gamma}\|_{L^{2q}} \|\langle x \rangle^\gamma f\|_{L^2}$, and, since $\gamma > d/(2q)$, $L_\gamma^2 \subset L^{p'}$. Therefore, using Theorem 2.1, for any given $\psi_0 \in \mathcal{B}_\gamma^s(\tilde{\varepsilon})$ we can construct a unique global solution ψ_t to (1.1) satisfying (2.4)-(2.5).

Boundedness on H^s . We begin by proving the bound on H^s . We want to show

$$\|g_+(H)\phi\|_{H^s} \lesssim \|\phi\|_{H^s}. \quad (6.2)$$

Since (6.2) obviously holds true with g instead of g_+ , by (6.1) it suffices to prove that

$$\|U_r^{-1} g(H) W_r U_r\|_{H^s} \lesssim \langle r \rangle^{-d/(2q)} \|\phi\|_{H^s}, \quad (6.3)$$

and use that $d/(2q) > 1$. Note that we are writing $g(H)W_r$ instead of the full commutator W_r' from (6.1). We will adopt a similar convention in the rest of the proofs in this section. The estimate (6.3) exchanging the position of $g(H)$ and W_r can be obtained in the same way (since, in particular, we will not make use of the fact that the projection $g(H)$ is bounded from H^s to L^2 in what follows).

By Lemma 5.4, we have $\|U_t^\psi\|_{H^s \rightarrow H^s} \lesssim 1$, while Lemma 5.2 gives $\|W_r^\psi\|_{H^s \rightarrow H^s} \lesssim \tilde{\varepsilon} \langle r \rangle^{-d/(2q)}$. This implies (6.3).

Boundedness on L_γ^2 . Next, we prove boundedness on L_γ^2 , that is,

$$\|g_+(H)\phi\|_{L_\gamma^2} \lesssim \|\phi\|_{L_\gamma^2 \cap H^\gamma}, \quad \phi \in L_\gamma^2 \cap H^\gamma. \quad (6.4)$$

Since it is not difficult to show the necessary estimate for $g(H)$, by (6.1) it suffices to prove that, for any $\phi \in L_\gamma^2 \cap H^\gamma$, we have

$$\|U_r^{-1}g(H)W_r U_r \phi\|_{L_\gamma^2} \lesssim \langle r \rangle^{\gamma-d/q} \|\phi\|_{L_\gamma^2 \cap H^\gamma}, \quad (6.5)$$

and then use $\gamma < d/q - 1$ so that the above bound is integrable. Note that we are once again just working with $g(H)W_r$ instead of the commutator.

First, using Lemma 5.5, we obtain

$$\|U_r^{-1}g(H)W_r U_r \phi\|_{L_\gamma^2} \lesssim \langle r \rangle^\gamma \|g(H)W_r U_r \phi\|_{H^\gamma} + \|g(H)W_r U_r \phi\|_{L_\gamma^2}. \quad (6.6)$$

For the first term, we use Lemmas 5.2 and 5.4, which yield

$$\|g(H)W_r U_r \phi\|_{H^\gamma} \lesssim \tilde{\varepsilon}^2 \langle r \rangle^{-d/q} \|\phi\|_{H^\gamma}. \quad (6.7)$$

For the second term, since $g(H) : L_\gamma^2 \rightarrow L_\gamma^2$ is bounded, we can write

$$\begin{aligned} \|g(H)W_r U_r \phi\|_{L_\gamma^2} &\lesssim \|W_r\|_{L^\infty} \|U_r \phi\|_{L_\gamma^2} \\ &\lesssim \tilde{\varepsilon}^2 \langle r \rangle^{-d/q} (\langle r \rangle^\gamma \|\phi\|_{H^\gamma} + \|\phi\|_{L_\gamma^2}), \end{aligned} \quad (6.8)$$

having used Lemmas 4.2 and 5.5 in the second inequality. Equations (6.6), (6.7) and (6.8) imply (6.5) and therefore (6.4).

Contraction. To prove that F_ϕ is contractive, we use arguments that are similar to those above, but we now need to apply them to the difference $F_\phi(\varphi_0) - F_\phi(\psi_0)$, for data $\psi_0, \varphi_0 \in L_\gamma^2 \cap H^s$. Let us denote by ψ_t and φ_t the respective global solutions guaranteed by Theorem 2.1.

We skip the estimate for the Sobolev norm since it is easier, and concentrate on estimating the L_γ^2 norm. We restore the superindex ψ for $H_t^\psi, W_t^\psi, U_t^\psi$ and so on. For $\psi_0, \varphi_0 \in \mathcal{B}_\gamma^s(\tilde{\varepsilon}) \subset L_\gamma^2 \cap H^s$, and $\phi \in \mathcal{B}_\gamma^s(\varepsilon)$, we estimate first

$$\|F_\phi(\varphi_0) - F_\phi(\psi_0)\|_{L_\gamma^2} \leq D_1 + D_2 + D_3 + \text{'similar'}, \quad (6.9)$$

$$D_1 := \int_0^\infty \|(U_r^\varphi)^{-1} g(H)W_r^\varphi (U_r^\varphi - U_r^\psi)\phi\|_{L_\gamma^2} dr, \quad (6.10)$$

$$D_2 := \int_0^\infty \|(U_r^\varphi)^{-1} g(H)(W_r^\varphi - W_r^\psi)U_r^\psi \phi\|_{L_\gamma^2} dr, \quad (6.11)$$

$$D_3 := \int_0^\infty \|((U_r^\varphi)^{-1} - (U_r^\psi)^{-1})g(H)W_r^\psi U_r \phi\|_{L_\gamma^2} dr, \quad (6.12)$$

where we are again only looking at terms with $g(H)W_r$ and can disregard the ‘similar’ ones with $W_r g(H)$. We then want to prove

$$D_1, D_2, D_3 \lesssim \varepsilon \tilde{\varepsilon} \|\psi_0 - \varphi_0\|_{L^2_\gamma \cap H^s}. \quad (6.13)$$

The terms D_1 and D_3 can be estimated similarly so we just focus on the first. Using Lemma 5.5, we obtain

$$\begin{aligned} & \| (U_r^\varphi)^{-1} g(H) W_r^\varphi (U_r^\varphi - U_r^\psi) \phi \|_{L^2_\gamma} \\ & \lesssim \langle r \rangle^\gamma \| g(H) W_r^\varphi (U_r^\varphi - U_r^\psi) \phi \|_{H^\gamma} + \| g(H) W_r^\varphi (U_r^\varphi - U_r^\psi) \phi \|_{L^2_\gamma}. \end{aligned} \quad (6.14)$$

To estimate the first term in the rhs of (6.14), we use Lemmas 5.2 and 5.8 yielding

$$\| g(H) W_r^\varphi (U_r^\varphi - U_r^\psi) \phi \|_{H^\gamma} \lesssim \tilde{\varepsilon}^2 \langle r \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \|\phi\|_{H^\gamma}. \quad (6.15)$$

Since $g(H) : L^2_\gamma \rightarrow L^2_\gamma$ is bounded, the second term in the rhs of (6.14) can be estimated by

$$\begin{aligned} & \| g(H) W_r^\varphi (U_r^\varphi - U_r^\psi) \phi \|_{L^2_\gamma} \\ & \lesssim \| W_r^\varphi \|_{L^\infty} \| (U_r^\varphi - U_r^\psi) \phi \|_{L^2_\gamma} \\ & \lesssim \langle r \rangle^{-d/q} \tilde{\varepsilon}^2 \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} (\langle r \rangle^\gamma \|\phi\|_{H^\gamma} + \|\phi\|_{L^2_\gamma}), \end{aligned} \quad (6.16)$$

the second inequality being a consequence of Lemmas 4.2 and 5.9. Inserting (6.15) and (6.16) into (6.14) gives

$$\begin{aligned} & \| (U_r^\varphi)^{-1} g(H) W_r^\varphi (U_r^\varphi - U_r^\psi) \phi \|_{L^2_\gamma} \\ & \lesssim \langle r \rangle^{-d/q+\gamma} \tilde{\varepsilon}^2 \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \|\phi\|_{L^2_\gamma \cap H^\gamma}. \end{aligned} \quad (6.17)$$

Therefore, since $\gamma < d/q - 1$ and $\|\phi\|_{L^2_\gamma \cap H^\gamma} \leq \varepsilon$, we have shown that

$$D_1 \lesssim \varepsilon \tilde{\varepsilon}^2 \|\psi_0 - \varphi_0\|_{L^2_\gamma \cap H^\gamma}. \quad (6.18)$$

The same bound holds for D_3 .

To estimate D_2 , we write using Lemma 5.5

$$\begin{aligned} & \| (U_r^\varphi)^{-1} g(H) (W_r^\varphi - W_r^\psi) U_r^\psi \phi \|_{L^2_\gamma} \\ & \lesssim \langle r \rangle^\gamma \| (W_r^\varphi - W_r^\psi) U_r^\psi \phi \|_{H^\gamma} + \| (W_r^\varphi - W_r^\psi) U_r^\psi \phi \|_{L^2_\gamma}. \end{aligned} \quad (6.19)$$

The first term is estimated using Lemma 5.7, which gives

$$\begin{aligned} \| (W_r^\varphi - W_r^\psi) U_r^\psi \phi \|_{H^\gamma} & \lesssim \tilde{\varepsilon} \langle r \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \| U_r^\psi \phi \|_{H^\gamma} \\ & \lesssim \tilde{\varepsilon} \langle r \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \|\phi\|_{H^\gamma}, \end{aligned} \quad (6.20)$$

the second inequality following from Lemma 5.4. The second term in the rhs of (6.19) is estimated as

$$\begin{aligned} & \|(W_r^\varphi - W_r^\psi)U_r^\psi\phi\|_{L_\gamma^2} \\ & \lesssim \|W_r^\varphi - W_r^\psi\|_{L^\infty} \|U_r^\psi\phi\|_{L_\gamma^2} \\ & \lesssim \tilde{\varepsilon}\langle r \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} (\langle r \rangle^\gamma \|\phi\|_{H^\gamma} + \|\phi\|_{L_\gamma^2}), \end{aligned} \quad (6.21)$$

where we have used Lemmas 5.7 and 5.5 to obtain the second inequality. Plugging (6.20) and (6.21) into (6.19) gives

$$\|(U_r^\varphi)^{-1}g(H)(W_r^\varphi - W_r^\psi)U_r^\psi\phi\|_{L_\gamma^2} \lesssim \varepsilon\tilde{\varepsilon}\langle r \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma}, \quad (6.22)$$

since $\|\phi\|_{L_\gamma^2 \cap H^\gamma} \leq \varepsilon$, and therefore

$$D_2 \lesssim \varepsilon\tilde{\varepsilon} \|\psi_0 - \varphi_0\|_{L_\gamma^2 \cap H^\gamma}. \quad (6.23)$$

Hence, since $s \geq \gamma$, we have proven (6.13), which implies

$$\|F_\phi(\varphi_0) - F_\phi(\psi_0)\|_{L_\gamma^2} \lesssim \tilde{\varepsilon}\varepsilon \|\varphi_0 - \psi_0\|_{L_\gamma^2 \cap H^s}. \quad (6.24)$$

The analogous estimate for the Sobolev norm, that is,

$$\|F_\phi(\varphi_0) - F_\phi(\psi_0)\|_{H^s} \lesssim \tilde{\varepsilon}\varepsilon \|\varphi_0 - \psi_0\|_{L_\gamma^2 \cap H^s}, \quad (6.25)$$

can be obtained similarly, and is in fact easier to show. Since $\tilde{\varepsilon} \ll 1$, (6.24)-(6.25) imply that F_ϕ is a contraction.

Injectivity. Finally, we verify that the map $\phi \mapsto \psi_0 = g_+^\psi(H)\phi$ is injective on $g(H)\mathcal{B}_\gamma^s(\varepsilon)$. Indeed, assume that for $\ell = 1, 2$ we have $\phi_\ell \in \mathcal{B}_\gamma^s(\varepsilon)$ with $g(H)\phi_1 \neq g(H)\phi_2$, and let $\psi_{0,\ell} \in \mathcal{B}_\gamma^s(C_0\varepsilon)$ be the (unique) solutions of $\psi_{0,\ell} = g_+^{\psi_\ell}(H)\phi_\ell$ with $\psi_\ell = U_t^{\psi_\ell}\psi_{0,\ell}$. Then, from (6.1) and arguments similar to those above, we can estimate

$$\begin{aligned} & \|g_+^{\psi_1}(H)\phi_1 - g_+^{\psi_2}(H)\phi_2\|_{H^s} \\ & \geq \|g(H)(\phi_1 - \phi_2)\|_{H^s} - C\varepsilon^2 [\|\phi_1 - \phi_2\|_{H^s} + \|\psi_1 - \psi_2\|_{H^s \cap L_\gamma^2}]. \end{aligned}$$

This concludes the proof. \square

APPENDIX A. PROOF OF PROPOSITION 1.8

Proof of Proposition 1.8. Let $V = V_+ - V_-$ be such that (1.8) holds. We first prove that $-\frac{1}{2}\Delta + V_+$ does not have nonpositive eigenvalues nor a resonance at 0. Clearly, since $-\frac{1}{2}\Delta + V_+ \geq 0$, its spectrum is contained in \mathbb{R}_+ . We show that 0 is not an eigenvalue nor a resonance of $-\frac{1}{2}\Delta + V_+$.

Suppose that $\phi \in \cap_{\gamma > \frac{1}{2}} L_{-\gamma}^2$ is a solution to $(-\frac{1}{2}\Delta + V_+)\phi = 0$. Since $V_+^{\frac{1}{2}}(-\Delta)^{-1}V_+^{\frac{1}{2}}$ is a bounded operator in L^2 by the assumption (1.8), this implies that

$$V_+^{\frac{1}{2}}\phi = -2V_+^{\frac{1}{2}}(-\Delta)^{-1}V_+\phi.$$

Taking the scalar product with $V_+^{\frac{1}{2}}\phi$ gives

$$\|V_+^{\frac{1}{2}}\phi\|_{L^2}^2 = -2\langle\phi, V_+(-\Delta)^{-1}V_+\phi\rangle_{L^2}.$$

Since in addition $V_+^{\frac{1}{2}}(-\Delta)^{-1}V_+^{\frac{1}{2}}$ is nonnegative, this shows that $V_+^{\frac{1}{2}}\phi = 0$. Hence $(-\frac{1}{2}\Delta)\phi = 0$, which implies that $\phi = 0$. Thus $-\frac{1}{2}\Delta + V_+$ does not have nonpositive eigenvalues nor a resonance at 0.

Next we show that $-\frac{1}{2}\Delta + V$ does not have negative eigenvalues. Let $\lambda > 0$. Let $\phi \in L^2$ be such that $(-\frac{1}{2}\Delta + V + \lambda)\phi = 0$. As above, since $(-\frac{1}{2}\Delta + V_+ + \lambda)$ is invertible, this implies that

$$\|V_-^{\frac{1}{2}}\phi\|_{L^2}^2 = \langle\phi, V_-(-\frac{1}{2}\Delta + V_+ + \lambda)^{-1}V_-\phi\rangle_{L^2}. \quad (\text{A.1})$$

We have $-\frac{1}{2}\Delta + V_+ + \lambda \geq -\frac{1}{2}\Delta + \lambda$ and hence, since both operators are invertible and $-\frac{1}{2}\Delta + \lambda$ is positive,

$$\left(-\frac{1}{2}\Delta + V_+ + \lambda\right)^{-1} \leq \left(-\frac{1}{2}\Delta + \lambda\right)^{-1}. \quad (\text{A.2})$$

Indeed, for any self-adjoint, invertible operators A and B having the same domain and satisfying $A \leq B$ and $A > 0$, we have $B^{-1/2}AB^{-1/2} \leq 1$, which implies $B^{1/2}A^{-1}B^{1/2} \geq 1$ giving $A^{-1} \geq B^{-1}$. Relation (A.2) together with (A.1), implies

$$\|V_-^{\frac{1}{2}}\phi\|_{L^2}^2 \leq \langle\phi, V_-(-\frac{1}{2}\Delta + \lambda)^{-1}V_-\phi\rangle_{L^2}. \quad (\text{A.3})$$

Due to the assumption $\langle x \rangle^\alpha V_-(x) \leq \delta$, this yields

$$\|V_-^{\frac{1}{2}}\phi\|_{L^2}^2 \leq \langle\langle x \rangle^{\frac{\alpha}{2}}V_-\phi, \langle x \rangle^{-\frac{\alpha}{2}}(-\frac{1}{2}\Delta + \lambda)^{-1}\langle x \rangle^{-\frac{\alpha}{2}}\langle x \rangle^{\frac{\alpha}{2}}V_-\phi\rangle_{L^2} \quad (\text{A.4})$$

$$\leq \delta\|V_-^{\frac{1}{2}}\phi\|_{L^2}^2\|\langle x \rangle^{-\frac{\alpha}{2}}(-\frac{1}{2}\Delta + \lambda)^{-1}\langle x \rangle^{-\frac{\alpha}{2}}\|. \quad (\text{A.5})$$

Since $\alpha > 2$, the operator $\langle x \rangle^{-\frac{\alpha}{2}}(-\frac{1}{2}\Delta + \lambda)^{-1}\langle x \rangle^{-\frac{\alpha}{2}} : L^2 \rightarrow L^2$ is bounded uniformly in $\lambda \geq 0$. Hence, for δ small enough, we deduce that $V_-\phi = 0$. Therefore $(-\frac{1}{2}\Delta + V_+ + \lambda)\phi = 0$ which yields $\phi = 0$ since we know that $-\frac{1}{2}\Delta + V_+$ does not have negative eigenvalues.

Now we show that 0 is neither an eigenvalue nor a resonance of $-\frac{1}{2}\Delta + V$. Suppose that $\phi \in \cap_{\gamma > \frac{1}{2}} L_{-\gamma}^2$ is a solution to $(-\frac{1}{2}\Delta + V)\phi = 0$. Letting $\lambda \rightarrow 0$ in (A.2) shows that $(-\frac{1}{2}\Delta + V_+)^{-1} : L_\gamma^2 \rightarrow L_{-\gamma}^2$ is bounded for $\gamma > 1$. Hence, using (1.8), we see that the equation $(-\frac{1}{2}\Delta + V)\phi = 0$ implies $V_-^{\frac{1}{2}}\phi = V_-^{\frac{1}{2}}(-\frac{1}{2}\Delta + V_+)^{-1}V_-\phi$. Taking the scalar product with $V_-^{\frac{1}{2}}\phi$ and using $V_-(x) \leq \delta\langle x \rangle^{-\alpha}$ we obtain

$$\|V_-^{\frac{1}{2}}\phi\|_{L^2}^2 \leq \delta^2\|V_-^{\frac{1}{2}}\phi\|_{L^2}^2\|\langle x \rangle^{-\frac{\alpha}{2}}(-\frac{1}{2}\Delta + V_+)^{-1}\langle x \rangle^{-\frac{\alpha}{2}}\|.$$

For δ small enough, we can conclude that $V_-\phi = 0$. Therefore $(-\frac{1}{2}\Delta + V_+)\phi = 0$, which yields that $\phi = 0$ since we know that 0 is not a resonance of $-\frac{1}{2}\Delta + V_+$.

It remains to prove that $-\frac{1}{2}\Delta + V$ does not have positive eigenvalues. This is a standard result given the condition (1.8) (see e.g. [34, Theorem XIII.58]). \square

APPENDIX B. PROOF OF THEOREM 2.1

To prove Theorem 2.1, we use standard arguments, combining energy and decay estimates. Recall that global existence in $L^2(\mathbb{R}^d)$ of solutions satisfying (2.3) is standard (see e.g. [6]). Local existence in $H^s(\mathbb{R}^d)$ is also standard, see e.g. [6, Theorem 4.10.1], for $s > d/(2q)$, an integer. We prove it here for convenience of the reader as the proof under our conditions is simpler than that of [6] which is done for fairly general nonlinearities. We then bootstrap it to the global existence. We also use some of the estimates, or variants of them, in Section C.

As is standard in the local existence proofs, we use the Duhamel principle to rewrite the Hartree equation (1.1) as a fixed point problem

$$\psi_t = G_t(\psi), \quad G_t(\psi) := e^{-iHt}\psi_0 - i \int_0^t e^{-iH(t-r)} W_r^\psi \psi_r dr, \quad (\text{B.1})$$

(recall the definition of the nonlinear potential W_t^ψ from (3.2)) and then use the contraction mapping principle to prove the existence of a unique fixed point in a ball in H^s .

By time-reversal symmetry we may assume $t \geq 0$. Elementary estimates under Condition (1.3) show the equivalence of the norms $\|\psi\|_{H^s}$ and $\|(H + C)^{s/2}\psi\|_{L^2}$, where $C \geq -\inf H + 1$, which yields the bound

$$\|e^{itH} f\|_{H^s} \lesssim \|f\|_{H^s}. \quad (\text{B.2})$$

Using definition (B.1) of G and estimate (B.2), we find right away for all $t \in [0, T]$:

$$\|G_t(\psi)\|_{H^s} \leq \|\psi_0\|_{H^s} + \|W_t^\psi \psi_t\|_{L_t^1([0, T])_{H_x^s}}. \quad (\text{B.3})$$

Applying the Kato-Ponce inequality (or fractional Leibniz rule) and the weak Young's inequality, recalling that $p = 2q' = 2q/(q - 1)$, and observing that

$$1/2 = 1/(2q) + 1/p, \quad 1 + 1/(2q) = 1/q + 1/p_1, \quad (\text{B.4})$$

where $1/p_1 = 1/2 + 1/p$, we estimate the H^s -norms of the last term in (B.6) for fixed t as follows

$$\begin{aligned} \|W_t^\psi \psi_t\|_{H^s} &\lesssim \|W_t^\psi\|_{L^\infty} \|\psi_t\|_{H^s} + \|W_t^\psi\|_{W^{s, 2q}} \|\psi_t\|_{L^p} \\ &\lesssim \|v\|_{L^{q, \infty}} \left(\|\psi_t\|_{L^{q'}} \|\psi_t\|_{H^s} + \|\psi_t\|_{W^{s, p_1}} \|\psi_t\|_{L^p} \right) \\ &\lesssim \|v\|_{L^{q, \infty}} \|\psi_t\|_{L^p}^2 \|\psi_t\|_{H^s}. \end{aligned} \quad (\text{B.5})$$

Now, consider the Banach spaces $H_T^s := L^\infty([0, T], H^s)$ and $L_T^p := L^\infty([0, T], L^p)$, with the norms $\|f\|_{H_T^s} := \sup_{0 \leq t \leq T} \|f(t)\|_{H^s}$ and $\|f\|_{L_T^p} := \sup_{0 \leq t \leq T} \|f(t)\|_{L^p}$, and

let $\psi_t \in H_T^s$ such that $\psi_0 \in H^s$. Then the last two inequalities give, after taking the supremum in t over $[0, T]$,

$$\|G_t(\psi)\|_{H_T^s} \lesssim \|\psi_0\|_{H^s} + T\|v\|_{L^{q,\infty}} \|\psi_t\|_{L_T^p}^2 \|\psi_t\|_{H_T^s}. \quad (\text{B.6})$$

Hence, since $H^s \hookrightarrow L^p$ (as $s > d/(2q) = d(1/2 - 1/p)$), the map G takes the ball $H_{T,R}^s$ in H_T^s of the radius R centred at the origin into itself, provided R satisfies $R \geq C(\|\psi_0\|_{H^s} + T\|v\|_{L^{q,\infty}} R^3)$ with C large enough.

Similarly, we estimate the difference $\|G_t(\psi) - G_t(\phi)\|_{H_T^s}$:

$$\|G_t(\psi) - G_t(\phi)\|_{H_T^s} \lesssim T\|v\|_{L^{q,\infty}} (\|\psi_t\|_{H_T^s} + \|\phi_t\|_{H_T^s})^2 \|\psi_t - \phi_t\|_{H_T^s}.$$

Hence G is a contraction on $H_{T,R}^s$ provided R and T satisfy $R \geq C(\|\psi_0\|_{H^s} + T\|v\|_{L^{q,\infty}} R^3)$ and $CT\|v\|_{L^{q,\infty}} R^2 < 1$ for some constant $C > 1$ independent of R and T . This implies local well-posedness in H_T^s , provided the local time of existence $T > 0$ is sufficiently small.

The local existence proven above implies that the bounds in (2.4) and (2.5) hold for some finite time. We now bootstrap the local existence and these bounds to the global existence and the global bounds.

More precisely, we assume, for some $T > 0$ and D large enough, that the solution ψ_t of (1.1), with $\|\psi_0\|_{L^{p'} \cap H^s} \leq \varepsilon$, satisfies

$$\sup_{t \in [0, T]} \|\psi_t\|_{H^s} \leq 2D\|\psi_0\|_{H^s}, \quad (\text{B.7})$$

$$\sup_{t \in [0, T]} (\langle t \rangle^{d/2q} \|\psi_t\|_{L^p}) \leq 2D\|\psi_0\|_{L^{p'} \cap H^s}, \quad (\text{B.8})$$

and then show that

$$\sup_{t \in [0, T]} \|\psi_t\|_{H^s} \leq D\|\psi_0\|_{H^s}, \quad (\text{B.9})$$

$$\sup_{t \in [0, T]} (\langle t \rangle^{d/2q} \|\psi_t\|_{L^p}) \leq D\|\psi_0\|_{L^{p'} \cap H^s}. \quad (\text{B.10})$$

To begin with, we mention first that under Conditions (1.3) and (1.4), the unitary evolution of the linear part e^{-iHt} of (1.1) is bounded from $L^{p'}$ to L^p and satisfies the dispersive estimate

$$\|e^{-iHt} f\|_{L^p} \lesssim t^{-d(\frac{1}{2} - \frac{1}{p})} \|f\|_{L^{p'}}, \quad t > 0, \quad (\text{B.11})$$

(see for example [43, 3], the introduction of [19] and the recent survey [36]). This estimate, together with estimate (B.2) and with the Sobolev embedding $H^s \hookrightarrow L^p$ (as $s > d(1/2 - 1/p)$) and the relation $1/2 - 1/p = 1/(2q)$ yield the bound

$$\|e^{-iHt} f\|_{L^p} \lesssim \langle t \rangle^{-d/(2q)} \|f\|_{H^s \cap L^{p'}}. \quad (\text{B.12})$$

Now, applying estimates (B.6), (B.7) and (B.8) to the fixed point equation (B.1) gives

$$\begin{aligned} \|\psi_t\|_{H^s} &\leq C\|\psi_0\|_{H^s} + 4CD^3\|v\|_{L^{q,\infty}}\|\psi_0\|_{L^{p'}\cap H^s}^2\|\psi_0\|_{H^s}\int_0^t\langle r\rangle^{-d/q}dr \\ &\leq C\varepsilon + D^3\tilde{C}\varepsilon^3, \end{aligned}$$

where the integral converges since $d/q > 2$. Altogether, this implies (B.9), provided $C + D^3\tilde{C}\varepsilon^2 \leq D$, for D sufficiently large. This bound implies also that $\psi_t \in C([0, T], H^s)$.

Let us now prove (B.10). Applying the L^p -norm to the fixed point equation (B.1) and using estimate (B.12), we obtain, for all $t \in [0, T]$,

$$\begin{aligned} \|\psi_t\|_{L^p} &\leq \|e^{-iHt}\psi_0\|_{L^p} + \left\| \int_0^t e^{-iH(t-r)}W_r^\psi\psi_r dr \right\|_{L^p} \\ &\leq \langle t\rangle^{-d/(2q)}\|\psi_0\|_{H^s\cap L^{p'}} + \int_0^t \langle t-r\rangle^{-d/(2q)}\|W_r^\psi\psi_r\|_{H^s\cap L^{p'}} dr. \end{aligned} \quad (\text{B.13})$$

Now, observing that $1/p' = 1/q + 1/p$, using the Hölder estimate $\|W_t^\psi\psi_t\|_{L^{p'}} \lesssim \|W_t^\psi\|_{L^q}\|\psi_t\|_{L^p}$ and then the weak Young one $\|W_t^\psi\|_{L^q} \lesssim \|v\|_{L^{q,\infty}}\|\psi_t\|_{L^1}$, together with (B.5), (B.13), (B.7), (B.8) and $\|\psi_0\|_{L^{p'}\cap H^s} \leq \varepsilon$, gives

$$\|\psi_t\|_{L^p} \lesssim \langle t\rangle^{-d/2q}\varepsilon + \varepsilon^3D^3\int_0^t \langle t-r\rangle^{-d/2q}\langle r\rangle^{-d/2q}dr.$$

Since $d/2q > 1$, the integral above is bounded by $C\langle t\rangle^{-d/2q}$ and hence (B.10) follows provided we choose ε so that $1 + \varepsilon^2D^3 \ll D$.

Thus, we have shown (B.9) and (B.10) which allows us to iterate the local existence result by a standard continuation argument to complete the proof of the theorem. \square

APPENDIX C. PROOF OF LEMMAS 5.1–5.9

In this section we prove the results stated in Section 5 which were used in Section 6. Some of the arguments used below are similar to those in the proof of Theorem 2.1 just given above, so we will skip some details.

Notation: As above, we will use in this section U_t^ψ (and similarly U_t^φ) to denote the flow of $H_t^\psi = H + f(|\psi|^2) = -\frac{1}{2}\Delta + V + v*|\psi|^2$, see (3.2), where V satisfies the conditions (1.3)-(1.4) and $\psi = \psi_t$ is the unique global H^s ($s > d/(2q)$) solution of (1.1) under the conditions of Theorem 2.1; in particular, we are assuming that the initial data ψ_0 satisfies (2.2), and ψ_t satisfies (2.4) and (2.5). Also, the indexes p, q and p_1 satisfy the same relations used so far:

$$1/p = 1/2 - 1/(2q) \quad \text{and} \quad 1/p_1 = 1/2 + 1/p \quad (\text{C.1})$$

($p = \infty$ for $q = 1$). Recall that we write $L^{1,\infty} \equiv L^1$, $W^{\gamma,(q,\infty)} \equiv W^{\gamma,q}$ in the case where $q = 1$. We also recall that Condition (1.3) implies the equivalence of the norms $\|\psi\|_{H^s}$ and $\|(H+C)^{s/2}\psi\|_{L^2}$, where $C \geq -\inf H + 1$.

Proof of Lemma 5.1. Using Young's and Hölder's inequalities, recalling that $1/p_1 = 1 - 1/(2q) = 1/2 + 1/p$, we obtain

$$\|W_t^\psi\|_{W^{s,2q}} \lesssim \|v\|_{L^{q,\infty}} \|\psi_t\|_{W^{s,p_1}}^2 \lesssim \|v\|_{L^{q,\infty}} \|\psi_t\|_{L^p} \|\psi_t\|_{H^s}.$$

Hence (5.1) follows from (2.4)–(2.5) in Theorem 2.1. \square

Proof of Lemma 5.2. Let $f \in H^s$, with $s > d(1/2 - 1/p) = d/(2q)$. Applying the Kato-Ponce inequality (or fractional Leibniz rule), we have

$$\|W_t^\psi f\|_{H^s} \lesssim \|W_t^\psi\|_{L^\infty} \|f\|_{H^s} + \|W_t^\psi\|_{W^{s,2q}} \|f\|_{L^p}. \quad (\text{C.2})$$

Using Sobolev's embedding $H^s(\mathbb{R}^d) \hookrightarrow L^p(\mathbb{R}^d)$, for $s > d/(2q)$, together with Lemmas 4.2 and 5.1, we obtain (5.2). To prove (5.3), we write similarly,

$$\|W_t^\psi f\|_{H^\gamma} \lesssim \|W_t^\psi\|_{L^\infty} \|f\|_{H^\gamma} + \|W_t^\psi\|_{W^{\gamma,\infty}} \|f\|_{L^2},$$

and hence the result follows from (4.5)–(4.6) of Lemma 4.2 applied to the last two inequalities. \square

Proof of Lemma 5.3. Using Young's inequality, we write

$$\begin{aligned} \|W_t^\psi\|_{W^{\sigma,\infty}} &\lesssim \|(\langle \nabla \rangle^{\sigma_1} v) * (\langle \nabla \rangle^{\sigma_2} |\psi_t|^2)\|_{L^\infty} \\ &\lesssim \|\langle \nabla \rangle^{\sigma_1} v\|_{L^{q,\infty}} \|\langle \nabla \rangle^{\sigma_2} \psi_t\|_{L^{2q'}} \|\psi_t\|_{L^{2q'}} \\ &= \|\langle \nabla \rangle^{\sigma_1} v\|_{L^{q,\infty}} \|\psi_t\|_{W^{\sigma_2,p}} \|\psi_t\|_{L^p}, \end{aligned} \quad (\text{C.3})$$

since $p = 2q'$. Next, the Gagliardo-Nirenberg-Sobolev inequality gives

$$\|\psi_t\|_{W^{\sigma_2,p}} \lesssim \|\psi_t\|_{H^s}^{1-\beta} \|\psi_t\|_{L^p}^\beta, \quad (\text{C.4})$$

provided that

$$\frac{1}{p} = \frac{\sigma_2}{d} + \left(\frac{1}{2} - \frac{s}{d}\right)(1-\beta) + \frac{\beta}{p}, \quad 0 < \beta < 1. \quad (\text{C.5})$$

Now, given $\varepsilon' > 0$ such that $d/(2q) - \varepsilon' > 0$, we choose β such that $\frac{d\beta}{2q} = \frac{d}{2q} - \varepsilon'$. The condition (C.5) then yields

$$s = \frac{d}{2q} + \frac{\sigma_2}{1-\beta} > \frac{d}{2q} + \frac{2d}{d-2q}.$$

Equations (C.3), (C.4) together with Theorem 2.1 imply

$$\|W_t^\psi\|_{W^{\sigma,\infty}} \lesssim \|\langle \nabla \rangle^{\sigma_1} v\|_{L^{q,\infty}} \|\psi_t\|_{L^p}^{1+\beta} \lesssim \|\langle \nabla \rangle^{\sigma_1} v\|_{L^{q,\infty}} \langle t \rangle^{(1+\beta)\frac{d}{2q}}.$$

This proves the lemma. \square

Proof of Lemma 5.4. For any $f_0 \in H^s$ we let $f = f_t := U_t^\psi f_0$ and write

$$f = e^{-iHt} f_0 - i \int_0^t e^{-iH(t-r)} W_r^\psi f dr. \quad (\text{C.6})$$

For the integrated term, using $1/p = 1/2 - 1/(2q)$, we estimate

$$\begin{aligned} \left\| \int_0^t e^{-iH(t-r)} W_r^\psi f dr \right\|_{H^s} &\lesssim \int_0^t \left(\|W_r^\psi\|_{L^\infty} \|f\|_{H^s} + \|W_r^\psi\|_{W^{s,2q}} \|f\|_{L^p} \right) dr \\ &\lesssim \varepsilon \int_0^t \langle r \rangle^{-\frac{d}{2q}} \|f\|_{H^s} dr, \end{aligned} \quad (\text{C.7})$$

having used Sobolev embedding and Lemmas 4.2 and 5.1. Using (C.7) and $\|e^{-iHt} f_0\|_{H^s} \lesssim \|f_0\|_{H^s}$ (see (B.2)) in (C.6) we can then obtain (5.6) by Gronwall's inequality since $d/(2q) > 1$. \square

Proof of Lemma 5.5. Let n be a nonnegative integer such that $n \leq \gamma$ and suppose that $v \in W^{n,(q,\infty)}$. We first prove by induction that for all $k \in \{0, \dots, n\}$ and $\ell \in \{0, \dots, n-k\}$,

$$\| \langle H \rangle^{\frac{\ell}{2}} \langle x \rangle^k U_t^\psi \varphi \| \leq C \sum_{j=0}^k \langle t \rangle^j \| \langle H \rangle^{\frac{\ell+j}{2}} \langle x \rangle^{k-j} \varphi \|. \quad (\mathcal{H}_{k,\ell})$$

Note that $(\mathcal{H}_{k,\ell})$ is a natural statement in view of the dispersion relation and the consequent localization property “ $|x|^2 \approx t^2 H$ ” for (linear) Schrödinger flows.

For $k = 0$, $(\mathcal{H}_{0,\ell})$ holds for any $\ell \in \{0, \dots, n\}$ as follows from Lemma 5.4. Let $k \in \{0, \dots, n-1\}$. Suppose that $(\mathcal{H}_{k',\ell})$ holds for all $k' \leq k$ and all $\ell \in \{0, \dots, n-k'\}$. First we show that $(\mathcal{H}_{k+1,0})$. Using the relation

$$[A, U_r] = -iU_r \int_0^r U_\tau^{-1} [A, H_\tau] U_\tau d\tau, \quad (\text{C.8})$$

we write

$$\begin{aligned} &\| \langle x \rangle^{k+1} U_t^\psi \varphi \| \\ &\lesssim \| \langle x \rangle^{k+1} \varphi \| + \int_0^t \| [\langle x \rangle^{k+1}, \Delta] U_r^\psi \varphi \| dr \\ &\lesssim \| \langle x \rangle^{k+1} \varphi \| + \int_0^t \left(\| \langle \nabla \rangle \langle x \rangle^k U_r^\psi \varphi \| + \| \langle x \rangle^{k-1} U_r^\psi \varphi \| \right) dr \\ &\lesssim \| \langle x \rangle^{k+1} \varphi \| + \int_0^t \left(\| \langle H \rangle^{\frac{1}{2}} \langle x \rangle^k U_r^\psi \varphi \| + \| \langle x \rangle^{k-1} U_r^\psi \varphi \| \right) dr \\ &\lesssim \| \langle x \rangle^{k+1} \varphi \| + \sum_{j=0}^k \int_0^t \langle r \rangle^j \| \langle H \rangle^{\frac{1+j}{2}} \langle x \rangle^{k-j} \varphi \| dr \\ &\lesssim \| \langle x \rangle^{k+1} \varphi \| + \sum_{j=0}^k \langle r \rangle^{j+1} \| \langle H \rangle^{\frac{1+j}{2}} \langle x \rangle^{k-j} \varphi \|, \end{aligned} \quad (\text{C.9})$$

where we used the induction hypothesis in the inequality before last. This easily implies that $(\mathcal{H}_{k+1,0})$ holds. Next, let $\ell \in \{0, \dots, n - (k+2)\}$. Assuming in addition that $(\mathcal{H}_{k+1,\ell'})$ holds for all $\ell' \leq \ell$, we show that $(\mathcal{H}_{k+1,\ell+1})$ holds. As above, we write

$$\begin{aligned}
 & \left\| \langle H \rangle^{\frac{\ell+1}{2}} \langle x \rangle^{k+1} U_t^\psi \varphi \right\| \\
 & \lesssim \left\| \langle H \rangle^{\frac{\ell+1}{2}} \langle x \rangle^{k+1} \varphi \right\| + \int_0^t \left\| [\langle H \rangle^{\frac{\ell+1}{2}} \langle x \rangle^{k+1}, H_r^\psi] U_r^\psi \varphi \right\| dr \\
 & \lesssim \left\| \langle H \rangle^{\frac{\ell+1}{2}} \langle x \rangle^{k+1} \varphi \right\| + \int_0^t \left\| \langle H \rangle^{\frac{\ell+1}{2}} [\langle x \rangle^{k+1}, \Delta] U_r^\psi \varphi \right\| dr \\
 & \quad + \int_0^t \left\| [\langle H \rangle^{\frac{\ell+1}{2}}, W_r^\psi] \langle x \rangle^{k+1} U_r^\psi \varphi \right\| dr.
 \end{aligned} \tag{C.10}$$

For the first integrated term, one verifies that

$$\begin{aligned}
 & \left\| \langle H \rangle^{\frac{\ell+1}{2}} [\langle x \rangle^{k+1}, \Delta] U_r^\psi \varphi \right\| \\
 & \lesssim \left\| \langle H \rangle^{\frac{\ell+2}{2}} \langle x \rangle^k U_r^\psi \varphi \right\| + \left\| \langle H \rangle^{\frac{\ell+1}{2}} \langle x \rangle^{k-1} U_r^\psi \varphi \right\|.
 \end{aligned} \tag{C.11}$$

The second one can be estimated by

$$\begin{aligned}
 \left\| [\langle H \rangle^{\frac{\ell+1}{2}}, W_r^\psi] \langle x \rangle^{k+1} U_r^\psi \varphi \right\| & \lesssim \sum_{\ell'=0}^{\ell} \|W_r^\psi\|_{W^{\ell-\ell'+1,\infty}} \left\| \langle H \rangle^{\frac{\ell'}{2}} \langle x \rangle^{k+1} U_r^\psi \varphi \right\| \\
 & \lesssim \langle r \rangle^{-\frac{d}{q}} \sum_{\ell'=0}^{\ell} \left\| \langle H \rangle^{\frac{\ell'}{2}} \langle x \rangle^{k+1} U_r^\psi \varphi \right\|,
 \end{aligned} \tag{C.12}$$

where the last inequality follows from Lemma 4.2. Inserting (C.11) and (C.12) into (C.10) and using the induction hypothesis, we obtain that

$$\begin{aligned}
 & \left\| \langle H \rangle^{\frac{\ell+1}{2}} \langle x \rangle^{k+1} U_t^\psi \varphi \right\| \\
 & \lesssim \left\| \langle H \rangle^{\frac{\ell+1}{2}} \langle x \rangle^{k+1} \varphi \right\| + \sum_{j=0}^k \int_0^t \langle r \rangle^j \left\| \langle H \rangle^{\frac{n-k+j}{2}} \langle x \rangle^{k-j} \varphi \right\| dr \\
 & \quad + \sum_{j=0}^k \int_0^t \langle r \rangle^{-\frac{d}{q}} \langle r \rangle^j \left\| \langle H \rangle^{\frac{n-k+j}{2}} \langle x \rangle^{k+1-j} \varphi \right\| dr.
 \end{aligned} \tag{C.13}$$

Since $\langle r \rangle^{-d/q+j}$ is integrable for all $j \leq n$ (since $n \leq \gamma < d/q - 1$), one deduces from the previous estimate that $(\mathcal{H}_{k+1,\ell+1})$ holds.

Thus, we have proven that $(\mathcal{H}_{k,\ell})$ holds for all $k \in \{0, \dots, n\}$ and $\ell \in \{0, \dots, n - k\}$.

Next, we claim that, for all $\ell \in \{0, \dots, n - k\}$ we have

$$\left\| \langle H \rangle^{\frac{\ell}{2}} \langle x \rangle^k U_t^\psi \varphi \right\| \lesssim \langle t \rangle^k \left\| \langle H \rangle^{\frac{k+\ell}{2}} \varphi \right\| + \left\| \langle H \rangle^{\frac{\ell}{2}} \langle x \rangle^k \varphi \right\|, \tag{C.14}$$

for all nonnegative integers ℓ, k such that $v \in W^{\ell+k,(q,\infty)}$. It suffices to bound each term in the sum on the right-hand side of $(\mathcal{H}_{k,\ell})$ by the right-hand side of (C.14)

(which is equivalent to the first and last terms in the sum in $(\mathcal{H}_{k,\ell})$); that is, it suffices to prove that for all $j = 0, \dots, k$

$$\langle t \rangle^j \|\langle H \rangle^{\frac{\ell+j}{2}} \langle x \rangle^{k-j} \varphi\| \lesssim \langle t \rangle^k \|\langle H \rangle^{\frac{k+\ell}{2}} \varphi\| + \|\langle H \rangle^{\frac{\ell}{2}} \langle x \rangle^k \varphi\|. \quad (\text{C.15})$$

Recall that, under our assumptions on V , we have the equivalence of the Sobolev norms

$$\|\langle H \rangle^{s/2} f\|_{L^2} \approx \|\langle \nabla \rangle^s f\|_{L^2} = \|f\|_{H^s}. \quad (\text{C.16})$$

Then, we square (C.15) to see that it is equivalent to

$$\langle t \rangle^{2j} \|\langle x \rangle^{k-j} \varphi\|_{H^{\ell+j}}^2 \lesssim \langle t \rangle^{2k} \|\varphi\|_{H^{\ell+k}}^2 + \|\langle x \rangle^k \varphi\|_{H^\ell}^2, \quad 0 \leq j \leq k. \quad (\text{C.17})$$

On the standard H^s space we can now use the (inhomogeneous) Littlewood-Paley decomposition $f = \sum_{N \geq 0} P_N f$, with $\widehat{P_N f}(\xi) := \chi_N(\xi) \widehat{f}(\xi)$ where χ_N is a bump function supported on $|\xi| \in [2^{N-1}, 2^{N+1}]$ for $N > 0$, and compactly supported in $[-2, 2]$ for $N = 0$. Recall that

$$\|f\|_{H^s}^2 \approx \sum_{N \geq 0} 2^{2Ns} \|P_N f\|_{L^2}^2; \quad (\text{C.18})$$

moreover, by commuting P_N and $\langle x \rangle$, we can see that

$$\|P_N \langle x \rangle f\|_{L^2} \approx \|\langle x \rangle P_N f\|_{L^2}, \quad (\text{C.19})$$

having slightly abused notation by disregarding similar terms with P_{N-1} and P_{N+1} instead of P_N on the right-hand side. Then, we write

$$\begin{aligned} \langle t \rangle^{2j} \|\langle x \rangle^{k-j} \varphi\|_{H^{\ell+j}}^2 &\approx \langle t \rangle^{2j} \sum_{N \geq 0} 2^{2N(\ell+j)} \|P_N \langle x \rangle^{k-j} \varphi\|_{L^2}^2 \\ &\lesssim \langle t \rangle^{2j} \sum_{N \geq 0} 2^{2N(\ell+j)} \|\langle x \rangle^{k-j} P_N \varphi\|_{L^2}^2. \end{aligned} \quad (\text{C.20})$$

In (C.20) we then distinguish the cases $\langle x \rangle \leq \langle t \rangle 2^N$ and $\langle x \rangle > \langle t \rangle 2^N$. When $\langle x \rangle \leq \langle t \rangle 2^N$ we bound

$$\begin{aligned} &\langle t \rangle^{2j} \sum_{N \geq 0} 2^{2N(\ell+j)} \|\langle x \rangle^{k-j} P_N \varphi\|_{L^2(\langle x \rangle \leq \langle t \rangle 2^N)}^2 \\ &\lesssim \langle t \rangle^{2j} \sum_{N \geq 0} 2^{2N(\ell+j)} \|(\langle t \rangle 2^N)^{k-j} P_N \varphi\|_{L^2}^2 \\ &= \langle t \rangle^{2k} \sum_{N \geq 0} 2^{2N(\ell+k)} \|P_N \varphi\|_{L^2}^2 \approx \langle t \rangle^{2k} \|\varphi\|_{H^{\ell+k}}^2; \end{aligned}$$

this last term is accounted for in the right-hand side of (C.17). Similarly, we bound the contribution from (C.20) in the region $\langle x \rangle > \langle t \rangle 2^N$ by

$$\begin{aligned} & \langle t \rangle^{2j} \sum_{N \geq 0} 2^{2N(\ell+j)} \|\langle x \rangle^{k-j} P_N \varphi\|_{L^2(\langle x \rangle > \langle t \rangle 2^N)}^2 \\ & \lesssim \sum_{N \geq 0} 2^{2N\ell} \|\langle x \rangle^k (\langle x \rangle^{-1} \langle t \rangle 2^N)^j P_N \varphi\|_{L^2(\langle x \rangle > \langle t \rangle 2^N)}^2 \\ & = \sum_{N \geq 0} 2^{2N\ell} \|\langle x \rangle^k P_N \varphi\|_{L^2}^2 \approx \|\langle x \rangle^k \varphi\|_{H^\ell}^2, \end{aligned}$$

having used (C.19) and (C.18) for the last equivalence. This gives us (C.17) and therefore (C.14).

Now, let $[\gamma]$ be the integer part of $\gamma \geq 1$. We claim that, by interpolation, one can deduce from (C.14) that, for all $\gamma' \geq 0$ and ℓ an integer such that $\gamma' + \ell \leq [\gamma]$,

$$\|\langle H \rangle^{\frac{\ell}{2}} \langle x \rangle^{\gamma'} U_t^\psi \varphi\| \lesssim \langle t \rangle^{\gamma'} \|\langle H \rangle^{\frac{\gamma'+\ell}{2}} \varphi\| + \|\langle H \rangle^{\frac{\ell}{2}} \langle x \rangle^{\gamma'} \varphi\|. \quad (\text{C.21})$$

To see this, consider the linear operator $T_\ell : \varphi \rightarrow \langle H \rangle^{\ell/2} U_t^\psi \varphi$ for $\ell = 0, \dots, n-k$ and $v \in W^{\ell+k, (q, \infty)}$ as above. Then, using the equivalence of Sobolev norms (C.16) and commuting (standard) derivatives and weights, the inequality (C.14) says that T maps $H^k(\langle t \rangle^k dx) \cap L_k^2$ into L_k^2 (recall the definition above (3.10)). Standard interpolation between Sobolev spaces and between weighted L^2 spaces then gives that T maps $H^{\gamma'}(\langle t \rangle^{\gamma'} dx) \cap L_{\gamma'}^2$ into $L_{\gamma'}^2$, that is, using again the equivalence (C.16), inequality (C.21).

Taking $\ell = 0$ in (C.21), we obtain

$$\|\langle x \rangle^{\gamma'} U_t^\psi \varphi\| \lesssim \langle t \rangle^{\gamma'} \|\langle H \rangle^{\frac{\gamma'}{2}} \varphi\| + \|\langle x \rangle^{\gamma'} \varphi\|, \quad (\text{C.22})$$

for all $\gamma' \leq [\gamma]$. We then write, similarly as in (C.9),

$$\begin{aligned} & \|\langle x \rangle^\gamma U_t^\psi \varphi\| \\ & \lesssim \|\langle x \rangle^\gamma \varphi\| + \int_0^t (\|\langle H \rangle^{\frac{1}{2}} \langle x \rangle^{\gamma-1} U_r^\psi \varphi\| + \|\langle x \rangle^{\gamma-2} U_r^\psi \varphi\|) dr. \end{aligned} \quad (\text{C.23})$$

Since $\gamma - 1 \leq [\gamma]$, we can apply (C.21), which yields

$$\|\langle H \rangle^{\frac{1}{2}} \langle x \rangle^{\gamma-1} U_r^\psi \varphi\| \lesssim \langle r \rangle^{\gamma-1} \|\langle H \rangle^{\frac{\gamma}{2}} \varphi\| + \|\langle H \rangle^{\frac{1}{2}} \langle x \rangle^{\gamma-1} \varphi\|.$$

Inserting this into (C.23), and then applying (C.21) again (if $\gamma > 2$) gives

$$\begin{aligned} & \|\langle x \rangle^\gamma U_t^\psi \varphi\| \\ & \lesssim \|\langle x \rangle^\gamma \varphi\| + \langle t \rangle^\gamma \|\langle H \rangle^{\frac{\gamma}{2}} \varphi\| + \langle t \rangle \|\langle H \rangle^{\frac{1}{2}} \langle x \rangle^{\gamma-1} \varphi\| + \int_0^t \|\langle x \rangle^{\gamma-2} U_r^\psi \varphi\| dr \\ & \lesssim \|\langle x \rangle^\gamma \varphi\| + \langle t \rangle^\gamma \|\langle H \rangle^{\frac{\gamma}{2}} \varphi\| + \langle t \rangle \|\langle H \rangle^{\frac{1}{2}} \langle x \rangle^{\gamma-1} \varphi\|. \end{aligned} \quad (\text{C.24})$$

Finally, we recall (C.15) which, for $\ell = 0, j = 1$ reads

$$\langle t \rangle \| \langle H \rangle^{\frac{1}{2}} \langle x \rangle^{k-1} \varphi \| \lesssim \langle t \rangle^k \| \langle H \rangle^{\frac{k}{2}} \varphi \| + \| \langle x \rangle^k \varphi \|,$$

and using interpolation for the weighted Sobolev spaces we obtain the same inequality above with γ replacing k ; plugging it into (C.24) we get

$$\| \langle x \rangle^\gamma U_t^\psi \varphi \| \lesssim \langle t \rangle^\gamma \| \langle H \rangle^{\frac{\gamma}{2}} \varphi \| + \| \langle x \rangle^\gamma \varphi \|. \quad (\text{C.25})$$

This concludes the proof of the lemma. \square

Proof of Lemma 5.6. The proof of (5.8) uses similar estimates to (B.13) applied to the difference of two solutions. From Duhamel's representation we have

$$u_t = e^{-iHt} u_0 - i \int_0^t e^{-iH(t-r)} W_r^u u_r dr, \quad u \in \{\psi, \varphi\}.$$

A bound by the right-hand side of (5.8) for the difference of the linear flows, $e^{-iHt} \psi_0 - e^{-iHt} \varphi_0$, follows directly from the linear dispersive estimate (B.12). For the difference of the nonlinear parts we have

$$\begin{aligned} & \left\| \int_0^t e^{-iH(t-r)} (W_r^\psi \psi_r - W_r^\varphi \varphi_r) dr \right\|_{L^p} \\ & \lesssim \int_0^t \langle t-r \rangle^{-d/(2q)} \left(\| (W_r^\psi - W_r^\varphi) \psi_r \|_{L^{p'} \cap H^s} + \| W_r^\varphi (\psi_r - \varphi_r) \|_{L^{p'} \cap H^s} \right) dr. \end{aligned} \quad (\text{C.26})$$

The estimates for the H^s norms are the same used for the contraction argument in the proof of Theorem 2.1 so we only show the bound for the $L^{p'}$ norm.

To estimate the first term in (C.26) we note that, in view of (C.1), $1/p' = 1/2 + 1/(2q)$ and $1 + 1/(2q) = 1/q + 1/p_1$, with $1/p_1 = 1/2 + 1/p$. Then, using Hölder's and Young's inequalities, and (2.3), we get

$$\begin{aligned} \| (W_r^\psi - W_r^\varphi) \psi_r \|_{L^{p'}} & \lesssim \| v * (|\psi_r|^2 - |\varphi_r|^2) \|_{L^{2q}} \| \psi_r \|_{L^2} \\ & \lesssim \| |\psi_r|^2 - |\varphi_r|^2 \|_{L^{p_1}} \| \psi_r \|_{L^2} \\ & \lesssim \varepsilon^2 \| \psi_r - \varphi_r \|_{L^p}. \end{aligned} \quad (\text{C.27})$$

For the second term in (C.26), using again Hölder with (C.1), Young's inequality and (2.3), we have

$$\| W_r^\varphi (\psi_r - \varphi_r) \|_{L^{p'}} \lesssim \| \psi_r - \varphi_r \|_{L^p} \| v * |\varphi_r|^2 \|_{L^q} \lesssim \varepsilon^2 \| \psi_r - \varphi_r \|_{L^p}. \quad (\text{C.28})$$

Plugging (C.27) and (C.28) into (C.26) we have obtained

$$\| \psi_t - \varphi_t \|_{L^p} \leq C \langle t \rangle^{-d/(2q)} \| \psi_0 - \varphi_0 \|_{L^{p'}} + C \varepsilon^2 \int_0^t \langle t-r \rangle^{-d/(2q)} \| \psi_r - \varphi_r \|_{L^p} dr.$$

From this we can obtain the conclusion (5.8) by a bootstrap argument (such as the one for the quantity in (B.8) in the proof of Theorem 2.1). \square

Proof of Lemma 5.7. (5.9) follows from the Hölder and Young inequalities, (2.5) and Lemma 5.6:

$$\begin{aligned} \|W_t^\psi - W_t^\varphi\|_{L^\infty} &\lesssim \|v\|_{L^{q,\infty}} \left\| |\psi_t|^2 - |\varphi_t|^2 \right\|_{L^{q'}} \\ &\lesssim \varepsilon \|v\|_{L^{q,\infty}} \langle t \rangle^{-d/(2q)} \|\psi_t - \varphi_t\|_{L^p} \\ &\lesssim \varepsilon \|v\|_{L^{q,\infty}} \langle t \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}. \end{aligned}$$

Similarly, using (2.4),

$$\begin{aligned} \|W_t^\psi - W_t^\varphi\|_{W^{s,2q}} &\lesssim \|v\|_{L^{q,\infty}} \left\| |\psi_t|^2 - |\varphi_t|^2 \right\|_{W^{s,p_1}} \\ &\lesssim \|v\|_{L^{q,\infty}} \|\psi_t - \varphi_t\|_{L^p} \\ &\lesssim \|v\|_{L^{q,\infty}} \langle t \rangle^{-d/(2q)} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}. \end{aligned}$$

This proves (5.10). Assuming in addition that v satisfies (1.2) with $\gamma \leq s$, (5.12) can be proven in the same way.

To prove (5.11), we proceed similarly as in the proof of Lemmas 4.2 and 5.2, writing for $f \in H^s(\mathbb{R}^d)$,

$$\begin{aligned} &\|(W_t^\psi - W_t^\varphi)f\|_{H^s} \\ &\lesssim \|W_t^\psi - W_t^\varphi\|_{L^\infty} \|f\|_{H^s} + \|W_t^\psi - W_t^\varphi\|_{W^{s,2q}} \|f\|_{L^p} \\ &\lesssim \varepsilon \langle t \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s} \|f\|_{H^s} + \langle t \rangle^{-d/(2q)} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s} \|f\|_{L^p}, \end{aligned}$$

where we used (5.9) and (5.10) in the last inequality. Together with Sobolev's embedding and Lemma 5.6, this proves (5.11). If in addition v satisfies (1.2) with $\gamma \leq s$, then, using (5.9) and (5.12) gives

$$\begin{aligned} &\|(W_t^\psi - W_t^\varphi)f\|_{H^\gamma} \\ &\lesssim \|W_t^\psi - W_t^\varphi\|_{L^\infty} \|f\|_{H^\gamma} + \|W_t^\psi - W_t^\varphi\|_{W^{\gamma,\infty}} \|f\|_{L^2} \\ &\lesssim \varepsilon \|v\|_{W^{\gamma,(q,\infty)}} \langle t \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s} \|f\|_{H^\gamma}. \end{aligned}$$

This establishes (5.13). □

Proof of Lemma 5.8. For a given $\phi \in H^s$, $\|\phi\|_{H^s} \leq 1$, let us define

$$(u_1)_t = u_1 := U_t^\psi \phi, \quad (u_2)_t = u_2 := U_t^\varphi \phi.$$

We want to estimate $\|u_1 - u_2\|_{H^s} \lesssim \varepsilon \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}$. From Duhamel's formula we have

$$\begin{aligned} \|u_1 - u_2\|_{H^s} &\lesssim \int_0^t \|W_r^\psi(u_1)_r - W_r^\varphi(u_2)_r\|_{H^s} dr \\ &\lesssim \int_0^t \left(\|(W_r^\psi - W_r^\varphi)(u_1)_r\|_{H^s} + \|W_r^\varphi((u_1)_r - (u_2)_r)\|_{H^s} \right) dr. \end{aligned} \tag{C.29}$$

For the first of the two quantities in the integral in (C.29) we use Lemma 5.7, which gives

$$\begin{aligned} \|(W_r^\psi - W_r^\varphi)(u_1)_r\|_{H^s} &\lesssim \langle r \rangle^{-d/(2q)} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s} \|(u_1)_r\|_{H^s} \\ &\lesssim \langle r \rangle^{-d/(2q)} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s}. \end{aligned} \quad (\text{C.30})$$

having also used (5.6).

For the second term in the integral in (C.29), we use Lemma 5.2, which gives

$$\|W_r^\varphi((u_1)_r - (u_2)_r)\|_{H^s} \lesssim \varepsilon \langle r \rangle^{-d/(2q)} \|(u_1)_r - (u_2)_r\|_{H^s}. \quad (\text{C.31})$$

Putting together (C.29)-(C.31) we have obtained

$$\|(u_1)_t - (u_2)_t\|_{H^s} \lesssim \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^s} + \int_0^t \langle r \rangle^{-d/(2q)} \|(u_1)_r - (u_2)_r\|_{H^s} dr,$$

which implies (5.14) via Gronwall's inequality since $d/(2q) > 1$. \square

Proof of Lemma 5.9. Let $\phi \in L_\gamma^2 \cap H^\gamma$. We use the notations of the proof of Lemma 5.8 and proceed similarly. As in (C.29), we have

$$\begin{aligned} \|u_1 - u_2\|_{L_\gamma^2} &\lesssim \int_0^t \|e^{-i(t-r)H} (W_r^\psi(u_1)_r - W_r^\varphi(u_2)_r)\|_{L_\gamma^2} dr \\ &\lesssim \int_0^t \left(\langle t-r \rangle^\gamma \|W_r^\psi(u_1)_r - W_r^\varphi(u_2)_r\|_{H^\gamma} \right. \\ &\quad \left. + \|W_r^\psi(u_1)_r - W_r^\varphi(u_2)_r\|_{L_\gamma^2} \right) dr, \end{aligned} \quad (\text{C.32})$$

where we used Lemma 5.5 (with $v = 0$) in the second inequality. It follows from (C.30)–(C.31) that

$$\begin{aligned} &\|W_r^\psi(u_1)_r - W_r^\varphi(u_2)_r\|_{H^\gamma} \\ &\lesssim \langle r \rangle^{-d/(2q)} (\|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \|\phi\|_{H^\gamma} + \|(u_1)_r - (u_2)_r\|_{H^\gamma}). \end{aligned}$$

Applying Lemma 5.8, this gives

$$\|W_r^\psi(u_1)_r - W_r^\varphi(u_2)_r\|_{H^\gamma} \lesssim \langle r \rangle^{-d/(2q)} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \|\phi\|_{H^\gamma}. \quad (\text{C.33})$$

For the L_γ^2 -norm, using Lemma 5.5 and Lemma 5.7, we have

$$\begin{aligned} &\|(W_r^\psi - W_r^\varphi)(u_1)_r\|_{L_\gamma^2} \\ &\lesssim \|(W_r^\psi - W_r^\varphi)\|_{L^\infty} \|(u_1)_r\|_{L_\gamma^2} \\ &\lesssim \varepsilon \langle r \rangle^{-\frac{d}{q} + \gamma} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \|\phi\|_{H^\gamma} + \varepsilon \langle r \rangle^{-\frac{d}{q}} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \|\phi\|_{L_\gamma^2}. \end{aligned} \quad (\text{C.34})$$

Next, we obtain from Lemma 4.2 that

$$\begin{aligned} \|W_r^\varphi((u_1)_r - (u_2)_r)\|_{L_\gamma^2} &\lesssim \|W_r^\varphi\|_{L^\infty} \|(u_1)_r - (u_2)_r\|_{L_\gamma^2} \\ &\lesssim \varepsilon^2 \langle r \rangle^{-d/q} \|(u_1)_r - (u_2)_r\|_{L_\gamma^2}. \end{aligned} \quad (\text{C.35})$$

Putting together (C.32)-(C.35) we have obtained

$$\begin{aligned}
 \|u_1 - u_2\|_{L^2_\gamma} &\lesssim \langle t \rangle^\gamma \int_0^t (\langle r \rangle^{-d/(2q)} + \langle r \rangle^{-\frac{d}{q} + \gamma}) \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \|\phi\|_{H^\gamma} dr \\
 &\quad + \varepsilon \int_0^t \langle r \rangle^{-d/q} \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} \|\phi\|_{L^2_\gamma} dr \\
 &\quad + \varepsilon^2 \int_0^t \langle r \rangle^{-d/q} \|(u_1)_r - (u_2)_r\|_{L^2_\gamma} dr \\
 &\lesssim \|\psi_0 - \varphi_0\|_{L^{p'} \cap H^\gamma} (\langle t \rangle^\gamma \|\phi\|_{H^\gamma} + \|\phi\|_{L^2_\gamma}) \\
 &\quad + \varepsilon^2 \int_0^t \langle r \rangle^{-d/q} \|(u_1)_r - (u_2)_r\|_{L^2_\gamma} dr, \tag{C.36}
 \end{aligned}$$

since $d/(2q)$ and $d/q - \gamma > 1$. Using again $d/q - \gamma > 1$, Eq. (C.36) implies (5.15) via Gronwall's lemma. \square

APPENDIX D. PROOF OF THEOREM 4.1

In this appendix we give the proof of Theorem 4.1 which is an improved version of the maximal velocity bounds for linear time-dependent potentials in [1]; see Theorem 3.3 there. We consider time-dependent hamiltonians of the form

$$H_t := H + W_t, \tag{D.1}$$

where $H = -\frac{1}{2}\Delta + V(x)$, with V real and satisfying (4.1) and $W_t(x) = W(x, t)$, a real, time-dependent bounded potential satisfying

$$\int_0^\infty \int_t^\infty \|\partial_x^\alpha W_r\|_{L^\infty} dr dt < \infty \text{ with either } 0 \leq |\alpha| \leq 1 \text{ or } 1 \leq |\alpha| \leq 2. \tag{D.2}$$

The main difference with respect to [1] is the quantification of the dependence of the bounds on the norm in (D.2). The assumption (D.2) in fact is weaker than the assumption made in [1] and, while this may still not be optimal, such an improvement over [1] is necessary to obtain the results in the present paper.

We prove Theorem 4.1 in the case where

$$w_t := \int_t^\infty \|W_r\|_{W^{1,\infty}} dr \tag{D.3}$$

is integrable and explain next how to modify the proof in the case where

$$w'_t := \max_{1 \leq |\alpha| \leq 2} \int_t^\infty \|\partial_x^\alpha W_r\|_\infty dr \tag{D.4}$$

is integrable.

D.1. Preliminary estimates. We will use the following notation:

$$A_\rho^\pm := \{x \in \mathbb{R}^d : \pm|x| \geq \pm\rho\}, \quad \chi_b^- := \chi_{A_b^-}, \quad (\text{D.5})$$

and

$$x_{ts} := s^{-1}(\langle x \rangle - a - vt),$$

and the convention that $A \dot{\leq} B$ and $A \dot{\lesssim} B$ mean that for any integer $n > 0$, there is $C_n > 0$ s.t. $A \leq B + C_n s^{-n}$ and $A \lesssim B + C_n s^{-n}$, respectively. Recall that k_I has been defined in (1.5). Given I a bounded open interval, we fix $c > v > k_I$. and let $\mathcal{F} \subset C^\infty(\mathbb{R}; \mathbb{R})$ be the set of functions $f \geq 0$, supported in \mathbb{R}^+ and satisfying $f(\lambda) = 1$ for $\lambda \geq c - v$, and $f' \geq 0$, with $\sqrt{f'} \in C^\infty$. We say that u is admissible if u is a smooth function such that $\text{supp}(u) \subset (0, c - v)$ and $\sqrt{u} \in C^\infty$.

In what follows we use the notation $p := -i\nabla$.

Lemma D.1. *Let I be a bounded open interval, $g \in C_0^\infty(I; \mathbb{R})$, $f \in \mathcal{F}$ and $u^2 = f'$. Then there is \tilde{u} , with \tilde{u}^2 admissible, s.t.*

$$\|pu(x_{ts})g(H)\psi\| \dot{\leq} k_I \|u(x_{ts})g(H)\psi\| + s^{-1} \|\tilde{u}(x_{ts})g(H)\psi\|. \quad (\text{D.6})$$

Proof. We write $g(H) = \tilde{g}(H)g(H)$, with $\tilde{g} \in C_0^\infty(I; \mathbb{R})$ and $\tilde{g} = 1$ on $\text{supp } g$. Commuting $\tilde{g}(H)$ to the left, we find

$$pu(x_{ts})g(H) = p\tilde{g}(H)u(x_{ts})g(H) + p[u(x_{ts}), \tilde{g}(H)]g(H). \quad (\text{D.7})$$

Now, it follows from (E.3) in Appendix E that

$$p[u(x_{ts}), \tilde{g}(H)] = \sum_{k=1}^{n-1} \frac{s^{-k}}{k!} pB_k u^{(k)}(x_{ts}) + O(s^{-n}),$$

for any n , with pB_k bounded. Taking the norm of (D.7) applied to ψ and using $\|p\tilde{g}(H)\| \lesssim k_I$, we obtain

$$\|pu(x_{ts})g(H)\psi\| \dot{\leq} k_I \|u(x_{ts})g(H)\psi\| + s^{-1} \sum_{k=1}^{n-1} \|u^{(k)}(x_{ts})g(H)\psi\|.$$

Since $u^{(k)}$ are smooth and $\text{supp } u^{(k)} \subset (0, c - v)$, one easily verifies that (D.6) holds for some admissible \tilde{u} . \square

D.2. Proof of Theorem 4.1. Note that Eq. (3.6) implies that

$$g_+(H) - g_t(H) = -i \int_t^\infty U_r^{-1} [g(H), W_r] U_r dr \quad (\text{D.8})$$

and, therefore,

$$\|g_+(H) - g_t(H)\| \lesssim w_t, \quad (\text{D.9})$$

which, together with $U_t g_t(H) = U_t U_t^{-1} g(H) U_t = g(H) U_t$, implies

$$\|U_t g_+(H) - g(H) U_t\| \lesssim w_t. \quad (\text{D.10})$$

Proof of Theorem 4.1. Let

$$\psi_t := U_t g_+(H) \phi_0, \quad \phi_0 := \chi_b^- \phi, \quad (\text{D.11})$$

and

$$\Phi_{ts} = f(x_{ts}), \quad (\text{D.12})$$

for some $f \in \mathcal{F}$, with $0 \leq t \leq s$. Let $\langle \Phi_{ts} \rangle_t := \langle \psi_t, \Phi_{ts} \psi_t \rangle$. We have

$$\partial_t \langle \Phi_{ts} \rangle_t = \langle \psi_t, D\Phi_{ts} \psi_t \rangle, \quad D\Phi_{ts} := i[H, \Phi_{ts}] + \frac{\partial}{\partial t} \Phi_{ts}. \quad (\text{D.13})$$

We compute $D\Phi_{ts}$. First, we have

$$\frac{\partial}{\partial t} \Phi_{ts} = -s^{-1} v f'(x_{ts}). \quad (\text{D.14})$$

Then, letting $A := \frac{1}{2}(p \cdot (\nabla \langle x \rangle) + (\nabla \langle x \rangle) \cdot p)$, factorizing $f' = u^2$ and using that $[H_t, \langle x \rangle] = A$ and $[[A, u], u] = 0$, one verifies that

$$\begin{aligned} i[H_t, \Phi_{ts}] &= \frac{i}{2}[p^2, \Phi_{ts}] = \frac{1}{2}s^{-1}(A f'(x_{ts}) + f'(x_{ts})A) \\ &= s^{-1} u(x_{ts}) A u(x_{ts}). \end{aligned} \quad (\text{D.15})$$

Together with (D.14), this yields:

$$D\Phi_{ts} = s^{-1} u(x_{ts}) (A - v) u(x_{ts}). \quad (\text{D.16})$$

For convenience, let $R := \psi_t - g(H)U_t \phi_0$. Note that, by (D.10)-(D.11) we have

$$R = O(w_t) \phi_0. \quad (\text{D.17})$$

Moreover, we can also see that

$$\|pR\| = O(w_t) \quad (\text{D.18})$$

as follows: from (D.11) we have $pR = p(U_t g_+(H) - g(H)U_t) \phi_0$, and from (D.8) we get

$$p(U_t g_+(H) - g(H)U_t) = -ip U_t \int_t^\infty U_r^{-1} [g(H), W_r] U_r dr; \quad (\text{D.19})$$

then, using the same notation in (E.8) (that is, $S := (H + c)^{1/2}$ with $c := \inf H + 1$, and $B' := p(H + c)^{-1/2}$) we can control $p = B'S$ by S , apply the commutator bound (E.10) and its analogue for the inverse U_r^{-1} , and, recalling the definition of w_t in (4.2), we arrive at (D.18).

Using the above notation for R , from the formula (D.13), we can write

$$\begin{aligned} \partial_t \langle \Phi_{ts} \rangle_t &= \langle g(H)U_t \phi_0, D\Phi_{ts} g(H)U_t \phi_0 \rangle \\ &+ \langle R, D\Phi_{ts} g(H)U_t \phi_0 \rangle + \langle g(H)U_t \phi_0, D\Phi_{ts} R \rangle + \langle R, D\Phi_{ts} R \rangle. \end{aligned} \quad (\text{D.20})$$

Now, we claim that, with k_I defined in (1.5), there is $C > 0$ s.t.

$$\begin{aligned} g(H)u(x_{ts}) A u(x_{ts})g(H) &\leq k_I g(H)u(x_{ts})^2 g(H) \\ &+ Cs^{-1} g(H) \tilde{u}(x_{ts})^2 g(H), \end{aligned} \quad (\text{D.21})$$

where \tilde{u} is an admissible function. To see this, we first estimate

$$\begin{aligned} & |\langle \psi, g(H)u(x_{ts}) Au(x_{ts})g(H)\psi \rangle| \\ & \leq \|\nabla \langle x \rangle u(x_{ts})g(H)\psi\| \|pu(x_{ts})g(H)\psi\|. \end{aligned} \quad (\text{D.22})$$

This inequality, together with (D.6), gives

$$\begin{aligned} & |\langle \psi, g(H)u(x_{ts}) Au(x_{ts})g(H)\psi \rangle| \\ & \leq \|u(x_{ts})g(H)\psi\| \left(k_I \|u(x_{ts})g(H)\psi\| + s^{-1} \|\tilde{u}(x_{ts})g(H)\psi\| \right), \end{aligned} \quad (\text{D.23})$$

which implies (D.21). Now, using (D.21), together with (D.16) and the definitions $u(x_{ts})^2 = f'(x_{ts})$ and $h(x_{ts}) := \tilde{u}(x_{ts})^2$, we obtain

$$\begin{aligned} & g(H)D\Phi_s(t)g(H) \\ & \leq (k_I - v)s^{-1}g(H)f'(x_{ts})g(H) + Cs^{-2}g(H)h(x_{ts})g(H). \end{aligned} \quad (\text{D.24})$$

Next, we claim that the three terms in the second line of (D.20) are all at least $O(s^{-1}w_t)$. For the first term this bound follows using (D.16), (D.17) and (D.6):

$$\begin{aligned} & |\langle R, D\Phi_{ts}g(H)U_t\phi_0 \rangle| \\ & = s^{-1} |\langle R, u(x_{ts})Au(x_{ts})g(H)U_t\phi_0 \rangle - v \langle R, u^2(x_{ts})g(H)U_t\phi_0 \rangle| \\ & \lesssim s^{-1} \|R\| (\|\phi_0\| + \|pu(x_{ts})g(H)U_t\phi_0\|) = s^{-1}O(w_t). \end{aligned}$$

The second term can be treated identically. For the last term on the right-hand side of (D.20) we can use (D.18) to get an even better bound of $O(s^{-1}w_t^2)$:

$$|\langle R, D\Phi_{ts}R \rangle| \lesssim s^{-1} \|R\| (\|R\| + \|pR\|) = s^{-1}O(w_t^2).$$

Going back to (D.20), using also (D.24) to bound the first terms on the r.h.s., we get, for some admissible function \tilde{f} ,

$$\begin{aligned} \partial_t \langle \Phi_{ts} \rangle_t & = \langle g(H)U_t\phi_0, D\Phi_{ts}g(H)U_t\phi_0 \rangle + O(s^{-1}w_t) \\ & \leq (k_I - v)s^{-1} \langle g(H)U_t\phi_0, f'(x_{ts})g(H)U_t\phi_0 \rangle \\ & \quad + Cs^{-2} \langle g(H)U_t\phi_0, \tilde{f}(x_{ts})g(H)U_t\phi_0 \rangle + O(s^{-1}w_t). \end{aligned} \quad (\text{D.25})$$

Next, passing back to ψ_t by using the pull-through relations in the opposite direction and the fact that \tilde{f} is bounded, we obtain

$$\partial_t \langle \Phi_{ts} \rangle_t \leq (k_I - v)s^{-1} \langle f'(x_{ts}) \rangle_t + Cs^{-2} + Cs^{-1}w_t. \quad (\text{D.26})$$

Since $v > k_I$, we can drop the first term on the r.h.s. Using the definition $\Phi_{ts} := f(x_{ts})$, the conditions (4.2) and $s \geq t$, we find

$$\langle f(x_{ts}) \rangle_t \leq \langle f(x_{0s}) \rangle_0 + Cs^{-1}. \quad (\text{D.27})$$

For the first term on the r.h.s., we claim that, for $0 < \beta < 1$,

$$\langle f(x_{0s}) \rangle_0 = O(s^{2\beta-2}) + O(w_{s^\beta}^2). \quad (\text{D.28})$$

To prove this estimate, we recall $\psi_t := U_t g_+(H) \chi_b^- \phi$, note that

$$\langle f(x_{0s}) \rangle_0 = \|\chi(x_{0s})\psi_0\|^2,$$

with $\chi^2 = f$, and pass from $g_+(H)$ to $g_{s^\beta}(H) := U_{s^\beta}^{-1}g(H)U_{s^\beta}$, with $\beta < 1$, paying with the error $O(w_{s^\beta})$ (see (D.10)):

$$\chi(x_{0s})\psi_0 = \chi(x_{0s})g_+(H)\chi_b^-\phi = \chi(x_{0s})g_{s^\beta}(H)\chi_b^-\phi + O(w_{s^\beta}).$$

In Lemma E.2 of Appendix E we show that

$$\chi(x_{0s})g_{s^\beta}(H)\chi_b^- = O(s^{\beta-1}).$$

This, together with the previous estimate yields (after squaring up) (D.28). Finally, (D.28) and (D.27) imply

$$\langle f(x_{ts}) \rangle_t \leq Cs^{-1} + Cs^{2\beta-2} + Cw_{s^\beta}^2$$

which, in view of the definition of f , gives, after setting $s = t$, Theorem 4.1 in the case of integrable $w_t = \int_t^\infty \|W_r\|_{W^{1,\infty}} dr$.

The proof in the case of integrable $w'_t = \max_{1 \leq |\alpha| \leq 2} \int_t^\infty \|\partial_x^\alpha W_r\|_\infty dr$ is identical, the only difference being that the estimate

$$\|[g(H), W_r]\| \lesssim \|W_r\|_{L^\infty}$$

used to prove (D.9) is replaced by

$$\|[g(H), W_r]\| \lesssim \max_{1 \leq |\alpha| \leq 2} \|\partial_x^\alpha W_r\|_{L^\infty},$$

see Lemma E.3 below. □

APPENDIX E. COMMUTATOR EXPANSIONS AND LOCALIZATION ESTIMATE

First, we state commutator expansions and estimates, first derived in [39] and then improved in [41, 23, 24] (see also [1, Appendix B]). We follow [23] and refer to this paper as well as to [1] for details and references. To begin with, we mention that, by the Helffer-Sjöstrand formula, a function $f(A)$ of a self-adjoint operator A can be written as

$$f(A) = \int d\tilde{f}(z)(z - A)^{-1}, \tag{E.1}$$

where $\tilde{f}(z)$ is an almost analytic extension of f to \mathbb{C} supported in a complex neighborhood of $\text{supp } f$. For $f \in C^{n+2}(\mathbb{R})$, we can choose \tilde{f} satisfying the estimates (see (B.8) of [23]):

$$\int |d\tilde{f}(z)| |\text{Im}(z)|^{-p-1} \lesssim \sum_{k=0}^{n+2} \|f^{(k)}\|_{k-p-1}, \tag{E.2}$$

where $\|f\|_m := \int \langle x \rangle^m |f(x)|$.

The essential commutator estimates are incorporated in the following lemma. We refer the reader to e.g. [1, Lemma A.1] for a proof.

Lemma E.1. *Let $f \in C^\infty(\mathbb{R})$ be bounded, with $\sum_{k=0}^{n+2} \|f^{(k)}\|_{k-2} < \infty$, for some $n \geq 1$. Let $x_s = s^{-1}(\langle x \rangle - a)$ for $a > 0$ and $1 \leq s < \infty$. Suppose that H satisfies (1.2) and let $g \in C_0^\infty(\mathbb{R})$. Then, for any $n \geq 1$,*

$$[g(H), f(x_s)] = \sum_{k=1}^{n-1} \frac{s^{-k}}{k!} B_k f^{(k)}(x_s) + O(s^{-n}), \quad (\text{E.3})$$

uniformly in $a \in \mathbb{R}$, where $H^j B_k$, $j = 0, 1, k = 1, \dots, n-1$, are bounded operators and $\|H^j O(s^{-n})\| \lesssim s^{-n}$, $j = 0, 1$. For $n = 1$, the sum on the r.h.s. is omitted.

Now, we prove a localization estimate used at the of the proof of Theorem 4.1. Recall the definition $g_{s^\beta}(H) := U_{s^\beta}^{-1} g(H) U_{s^\beta}$.

Lemma E.2. *Suppose that*

$$\int_0^\infty w_t^{(1)} dt < +\infty, \quad w_t^{(1)} := \max_{|\alpha|=1} \int_t^\infty \|\partial_x^\alpha W_r\|_\infty dr. \quad (\text{E.4})$$

With the notations of Eq. (D.5), we have

$$\|\chi(x_{0s}) g_{s^\beta}(H) \chi_b^-\| = O(s^{\beta-1}). \quad (\text{E.5})$$

Proof. Let $\chi \equiv \chi(x_{0s})$. Using $\chi \chi_b^- = 0$, we write

$$\begin{aligned} \chi g_{s^\beta}(H) \chi_b^- &= [\chi, U_{s^\beta}^{-1}] g(H) U_{s^\beta} \chi_b^- + U_{s^\beta}^{-1} [\chi, g(H)] U_{s^\beta} \chi_b^- \\ &\quad + U_{s^\beta}^{-1} g(H) [\chi, U_{s^\beta}] \chi_b^-. \end{aligned} \quad (\text{E.6})$$

Since $[\chi, U_{s^\beta}] = U_{s^\beta} (U_{s^\beta}^{-1} \chi U_{s^\beta} - \chi) = U_{s^\beta} \int_0^{s^\beta} \partial_r (U_r^{-1} \chi U_r) dr$ and $\partial_r (U_r^{-1} \chi U_r) = i U_r^{-1} [H_r, \chi] U_r$, we have

$$[\chi, U_{s^\beta}] = i U_{s^\beta} \int_0^{s^\beta} U_r^{-1} [H_r, \chi] U_r dr. \quad (\text{E.7})$$

Note that $[H_r, \chi] = -ip \nabla \chi + \frac{1}{2} (\Delta \chi)$. We control p by $S^{-1} = (H + c)^{-1/2}$, with $S = (H + c)^{1/2}$ and $c := \inf H + 1$:

$$p = SB = B'S, \quad (\text{E.8})$$

where $B := (H + c)^{-1/2} p$ and $B' := p(H + c)^{-1/2}$, bounded operators. Eq. (E.7), together with the last two relations, gives

$$[\chi, U_{s^\beta}] = U_{s^\beta} \int_0^{s^\beta} U_r^{-1} (SB \nabla \chi + i \frac{1}{2} (\Delta \chi)) U_r dr. \quad (\text{E.9})$$

Next, we commute $(H + c)^{1/2}$ to the left. To this end, we apply the equation

$$[S, U_r] = O(1), \quad (\text{E.10})$$

which we now prove. First, we write $S = (H + c)^{1/2} = (H + c)(H + c)^{-1/2}$ and use the explicit formula $(H + c)^{-s} := c' \int_0^\infty (H + c + \omega)^{-1} d\omega / \omega^s$, where $s \in (0, 1)$ and $c' := [\int_0^\infty (1 + \omega)^{-1} d\omega / \omega^s]^{-1}$, to obtain $[W_r, (H + c)^{1/2}] = O(w_r^{(1)})$. This implies the

estimate $[H_r, (H+c)^{1/2}] = [W_r, (H+c)^{1/2}] = O(w_r^{(1)})$, which, together with the fact that $w_r^{(1)}$ is integrable and the relation

$$[S, U_r] = U_r \int_0^r iU_{r'}^{-1}[H_{r'}, S]U_{r'} dr', \quad (\text{E.11})$$

yields (E.10).

Commuting S in Eq. (E.9) to the left (either twice through U_r^{-1} and U_{s^β} , or once through $U_{s^\beta}U_r^{-1} = U(s^\beta, r)$) and using (E.10), $\nabla\chi = O(s^{-1})$ and $\Delta\chi = O(s^{-2})$, gives

$$\begin{aligned} [\chi, U_{s^\beta}] &= SO(s^{\beta-1}) + \int_0^{s^\beta} (O(s^{-1}) + O(s^{-2}))U_r dr \\ &= SO(s^{\beta-1}) + O(s^{\beta-1}). \end{aligned} \quad (\text{E.12})$$

A similar estimate holds for $[\chi, U_{s^\beta}^{-1}] = -[\chi, U_{s^\beta}]^*$:

$$[\chi, U_{s^\beta}^{-1}] = O(s^{\beta-1})S + O(s^{\beta-1}). \quad (\text{E.13})$$

Now, the second term on the r.h.s. of the above relation produces the right bound, $O(s^{-1+\beta})$ and so does the first term multiplied by $g(H)$, as $(H+1)^{1/2}g(H)$ is a bounded operator. This shows that the first term on the r.h.s. of (E.6) is of the order $O(s^{-1+\beta})$. The same estimates apply to the third term on the r.h.s. of (E.6) giving $O(s^{-1+\beta})$. For the second term on the r.h.s. of (E.6), we use (E.3) to obtain $[\chi, g(H)] = O(s^{-1})$. This proves the lemma. \square

Finally we prove a lemma used in the proof of Theorem 4.1 in the case of integrable $w'_t = \max_{1 \leq |\alpha| \leq 2} \int_t^\infty \|\partial_x^\alpha W_r\|_\infty dr$.

Lemma E.3. *We have*

$$\| [g(H), W_r] \| \lesssim \max_{1 \leq |\alpha| \leq 2} \|\partial_x^\alpha W_r\|_{L^\infty}. \quad (\text{E.14})$$

Proof. Using (E.1), we have

$$[W_r, g(H)] = \int d\tilde{g}(z)(z-H)^{-1}[W_r, H](z-H)^{-1}. \quad (\text{E.15})$$

Using this, estimate (E.2) for $\tilde{g}(z)$ and the fact that $[W_r, H] = \nabla W_r \cdot \nabla + \frac{1}{2}\Delta W_r$ is $(H+c)^{1/2}$ -bounded, where $c := \inf H + 1$ and the estimate

$$\|(H+c)^{1/2}R(z)\| \lesssim |\operatorname{Re} z|^{1/2}/|\operatorname{Im} z|,$$

we arrive at (E.14). \square

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