Inverse spectral results in Sobolev spaces for the AKNS operator with partial informations on the potentials

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Abstract

We consider the AKNS (Ablowitz-Kaup-Newell-Segur) operator on the unit interval with potentials belonging to Sobolev spaces in the framework of inverse spectral theory. Precise sets of eigenvalues are given in order that it together with the knowledge of the potentials on the side $(a, 1)$ and partial informations on the potential on $(a - \varepsilon, a)$ for some arbitrary small $\varepsilon > 0$ determine the potentials entirely on $(0, 1)$. Naturally, the smaller is a and the more partial informations are known, the less is the number of the needed eigenvalues.

1 Introduction and statement of the result.

In this short paper we consider the following operator acting in $L^2(0,1) \times L^2(0,1)$,

$$\mathcal{H}(p,q) \;=\; \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \frac{d}{dx} \;+\; \begin{pmatrix} -q & p \\ p & q \end{pmatrix}, \tag{1}$$

for x in $(0, 1)$ and for real-valued square integrable potentials p and q defined on $(0, 1)$.

This operator is called the AKNS (Ablowitz-Kaup-Newell-Segur) operator. Let us recall that it is unitarily equivalent to the Zakharov-Shabat operator. Moreover, the AKNS operator is related to the first operator appearing in the decomposition as a direct sum of the Dirac operator in $\mathbb{R}^3$ with a radial potential and it may be also named Dirac operator. Note that following [LS, Chapter 7.1] operators with symmetric matrix-valued potentials can be transformed into operators with symmetric matrix-valued potentials with vanishing traces. The AKNS operator is related to QCD (Quantum Chromodynamic) as model for hadrons (see [S, Section 1]). Let us also mention that the AKNS operator is the self-adjoint operator of the Lax pair associated to the one dimensional nonlinear cubic defocusing Schrödinger equation $iu_t + u_{xx} - 2|u|^2u = 0$. The vanishing trace property of the matrix-valued potential in (1) implies a negative factor for the nonlinear term $|u|^2u$ and the corresponding Schrödinger equation is defocusing.

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A vector-valued function of $L^2(0,1)\times L^2(0,1)$ is here denoted $\begin{pmatrix} Y \\ Z \end{pmatrix}$. The AKNS operator is associated with the Dirichlet boundary conditions $Y(0)=Y(1)=0$. One may also choose without loss (see Section 2) of generality the boundary conditions $Z(0)=Z(1)=0$ or more generally the separated limit boundary conditions,

$$\left\{ \begin{array}{lcl} \cos\alpha\, Y(0) \;+\; \sin\alpha\, Z(0) & = 0 \\ \cos\beta\, Y(1) \;+\; \sin\beta\, Z(1) & = 0 \end{array} \right. \tag{2}$$

for $(\alpha,\beta)\in\mathbb{R}^2$.

The operator $H(\alpha,\beta,p,q)$ shall denote the following self-adjoint operator

$$\begin{array}{l} D(H(\alpha,\beta,p,q)) \;=\; \left\{ F=\begin{pmatrix} Y \\ Z \end{pmatrix} \text{ belongs to } H^1(0,1)\times H^1(0,1) \text{ and satisfies (2)} \right\} \\ H(\alpha,\beta,p,q)F=\mathcal{H}(p,q)F,\ \forall\, F\in D(H(\alpha,\beta,p,q)) \end{array}.$$

The spectrum of $H(\alpha,\beta,p,q)$ is a strictly increasing sequence of eigenvalues $(\lambda_k(\alpha,\beta,p,q))_{k\in\mathbb{Z}}$. Each eigenvalue $\lambda_k(\alpha,\beta,p,q)$ is simple. The spectrum is not bounded from below and the asymptotic expansion of the eigenvalues is given by (see [A1]),

$$(\lambda_k(\alpha,\beta,p,q)-k\pi-\alpha+\beta)_{k\in\mathbb{Z}}\in\ell^2(\mathbb{Z}). \tag{3}$$

Note that the eigenvalues are properly labeled according to (3).

The main result of the paper is Theorem 1.1 below. This result gives the number of eigenvalues that are sufficient in order to determine uniquely the pair of potentials when the following informations on the potentials are already given. Firstly, the two pair of potentials are equal on the part $(a,1)$ of the interval $(0,1)$. Secondly, the the pair of potentials are close enough in some precise sense locally inside the other part $(0,a)$ (see hypothesis (H') or (H'') in Theorem 1.1).

For any real-valued function u defined on a subset of $\mathbb{R}$, $u^{(j)}(x)$ denotes if it exists the derivative of order j of the function u at the point x and $(u_1,u_2)^{(j)}=(u_1^{(j)},u_2^{(j)})$ if u_1 and u_2 are real-valued functions defined on a subset of $\mathbb{R}$, for any $j\in\mathbb{N}$.

For any set of complex-valued numbers E and for all $t\geq 0$, we set

$$n_E(t)=\sharp\{e\in E \quad | \quad |e|\leq t\}. \tag{4}$$

Theorem 1.1. *Set* $(p_1,q_1),(p_2,q_2)\in L^2(0,1)\times L^2(0,1)$. *Fix* $a\in(0,\frac{1}{2}]$ *and suppose that* $(p_1,q_1)=(p_2,q_2)$ *a.e. on* $(a,1)$.

Let $\alpha,\beta\in\mathbb{R}^2$. *Let* S *be a set of common eigenvalues of* $H(\alpha,\beta,p_1,q_1)$ *and* $H(\alpha,\beta,p_2,q_2)$, *i.e.,* $S\subseteq\sigma(H(\alpha,\beta,p_1,q_1))\cap\sigma(H(\alpha,\beta,p_2,q_2))$. *Set* $k\in\mathbb{N}\cup\{0\}$, $r\in[2,+\infty]$ *and assume that* S *is large enough in the following sense,*

$$\exists M\geq 0,\ n_S(t)\ \geq\ 2a\, n_{\sigma(A)}(t)-k-1+\frac{1}{r},\ t\in\sigma(A),\ t\geq M, \quad (H)$$

where in (H) *the operator* A *denotes either* $H(\alpha,\beta,p_1,q_1)$ *or* $H(\alpha,\beta,p_2,q_2)$*.*

Suppose also that,

$$x \mapsto (a-x)^{-k}((p_1-p_2)(x),(q_1-q_2)(x)) \in L^r(a-\varepsilon,a)\times L^r(a-\varepsilon,a), \quad (H')$$

for some arbitrary small $\varepsilon>0$*.*

Then $(p_1,q_1)=(p_2,q_2)$ *all over* $(0,1)$*.*

Remark 1.2.

1. *In Theorem 1.1, hypothesis* (H') *may be replaced by the following assumption,*

$$(p_1-p_2,q_1-q_2)\in W^{k,r}(a-\varepsilon,a)\times W^{k,r}(a-\varepsilon,a),\ (p_1,q_1)^{(j)}(a^-)=(p_2,q_2)^{(j)}(a^-),\ j=0,\dots,k-1, \quad (H'')$$

for some arbitrary small $\varepsilon>0$*. This shall be underlined in the next Section by proving that* (H'') *implies* (H')*.*

2. *Let us emphasize here that the case* $r=+\infty$ *is considered in Theorem 1.1. In that case, the term* $\frac{1}{r}$ *in the hypothesis* (H) *is suppressed.*

3. *Only the difference of the pair of potentials is considered in assumptions* (H') *in Theorem 1.1 or* (H'') *in Remark 1.2 and not the pair of potentials themselves.*

4. *The partial informations* (H') *or* (H'') *on the potentials are additional informations that allows to remove* k *(or* $k+1$*) eigenvalues from the known spectrum.*

5. *One may also suppose that one of the parameters* α *or* β *is not known and recover it from* (3)*.*

For the case of AKNS operators, related results are already obtained in [DG]. In [DG] the known spectra S have the special form $S=\{\lambda_{j_0 j}(p,q,\alpha,\beta),\ j\in\mathbb{Z}\}$ for some given $j_0\in\mathbb{N}$ and the potentials are supposed to be L^2, that is to say, the particular case $k=0$ and $r=2$ is considered there instead of $k\in\mathbb{N}\cup\{0\}$ and $r\in[2,+\infty]$ as in Theorem 1.1.

This type of results have been initiated in [Ha], [HL] and [GS] for the Schrödinger operators. In 1978, the case $a=\frac{1}{2}$ with L^1 potentials is considered in [HL]. In 1980, it is proved that continuity assumption of the potentials allows to remove an eigenvalue from the known spectrum (see [Ha]). In 2000, the results in [Ha] and [HL] are largely extended in [GS] by considering any $a\in(0,\frac{1}{2}]$ and C^{2k} potentials allowing to remove $k+1$ eigenvalues from the known spectrum. See also related results established in 1997 in [DGS1] and [DGS2].

The steps of the proof of Theorem 1.1 are borrowed to [DG] (see also [L] for the proof that two spectra determine the potential in the case of Schrödinger operators on the unit interval). Namely, we start from the same entire function f (c.f. (5)) in Section 2 as in [DG] depending on the pair of potentials (p_1, q_1), (p_2, q_2) and vanishing on S. 1) The first step consists into proving that f is actually entirely vanishing. This point is the main point and it becomes the purpose of this paper. Let us mention that our approach of that step is different to the one in [DG] even if there are similarities, that is to say, when setting $k = 0$, $r = 2$ and $S = \{\lambda_{j_0 j}(p, q),\ j \in \mathbb{Z}\}$ for some $j_0 \in \mathbb{N}$ our proof is not becoming the proof appearing in [DG]. 2) The second step is to derive that f equals zero implies that $(p_1, q_1) = (p_2, q_2)$. This step does not depend on the assumptions (H') or (H'') in Theorem 1.1, it is effectuated in [DG] and it is therefore not appearing here.

The proof of the first step described above relies here on [AFR] and [AF] where the case of Schrödinger operators are analyzed in $W^{k,p}$ spaces. There are also similarities with [A2] where the map $\mu \times \kappa$ is proved to be one to one for the AKNS systems with L^2 potentials (see [A2]). Let us also mention that still in the case of the Schrödinger operators these type of results has been obtained in [GS] [DGS1] [DGS2] for C^{2k} potentials by different methods involving Weyl-Tichmarsch's functions.

Let us outline the proof of step 1 described above. On one side, the assumptions (H') or (H'') on the potentials (p_1, q_1) and (p_2, q_2) allows us to derive a bound on the entire function f in the complex plane which provides a first estimate of $N_{f^{-1}(0)}$ a density repartition of the zeroes of f according to Jensen's Theorem under the assumption that f is not entirely vanishing. On the other side, the assumption (H) in Theorem 1.1 implies an estimate of density repartition of the common eigenvalues. Since the common eigenvalues also are zeroes of f the last point provides a second estimate on $N_{f^{-1}(0)}$ the density repartition of the zeroes of f. These two estimates are in contradiction. Therefore f is entirely vanishing.

The next Section is concerned with the proof of Theorem 1.1.

2 Proof of Theorem 1.1.

Let us first remark that we may suppose in Theorem 1.1 without loss of generality that $\alpha = \beta = 0$. Indeed from [A2, Remark 2.1], there is a unitary map in $L^2(0,1) \times L^2(0,1)$ denoted here $\mathcal{U}_{\alpha\beta}$ such that $\sigma(H(\alpha, \beta, p, q)) = \sigma(H(0, 0, \mathcal{U}_{\alpha\beta}(p, q)) + \alpha - \beta$ for any $(p, q) \in L^2(0,1) \times L^2(0,1)$ and each $(\alpha, \beta) \in \mathbb{R}^2$. Namely, $\mathcal{U}_{\alpha\beta}(p, q) = (\cos\theta p + \sin\theta q, -\sin\theta p, \cos\theta q)$ a.e. on $(0,1)$ with $\theta(x) = -2(\alpha - \beta)x + 2\alpha$, for $x \in (0,1)$. Therefore if S, (p_1, q_1), (p_2, q_2), (α, β), a, k, r are given as in Theorem 1.1 then one may apply Theorem 1.1 with $S' = S - \alpha + \beta$, $\mathcal{U}_{\alpha\beta}(p_1, q_1)$, $\mathcal{U}_{\alpha\beta}(p_2, q_2)$, $(\alpha, \beta) = (0, 0)$, a, k, r to obtain that $\mathcal{U}_{\alpha\beta}(p_1, q_1)$=$\mathcal{U}_{\alpha\beta}(p_2, q_2)$.

Next it is noticed that the assumption (H'') implies assumption (H') in Theorem 1.1. This point follows from a Hardy type inequality. Namely, there exists a constant C depending only on $k \in \mathbb{N}$ and $r \in (1, +\infty]$

such that,

$$||u/d^k||_{L^r(0,1)} \leq C||u||_{W^{k,r}(0,1)}$$

for any $u \in W^{k,r}(0,1)$ satisfying $u^{(j)}(0)$ for $j = 0, \dots, k-1$ where $d(x) = x$ for a.e. $x \in (0,1)$. Let us mention that, when $r \in (1, +\infty)$ this inequality may be proved by iteration on k using the same argument appearing in the usual proof in the case $k = 1$. When $r = +\infty$ the proof of this inequality is simpler as a consequence of the Taylor formula, starting from $u(x) - u(0) = \int_0^x u'(t)dt$ for $u \in W^{1,1}(0,1)$.

We now define the entire function f. Let us also recall that $F(\cdot, z, p, q)$ is defined on $[0,1]$ as the solution to $\mathcal{H}(p,q)F(\cdot, z, p, q) = zF(\cdot, z, p, q)$, $F(0, z, p, q) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ for any $(p,q) \in L^2(0,1) \times L^2(0,1)$ and for any z in $\mathbb{C}$ and that u and v are the the two bilinear forms on $\mathbb{R}^2 \times \mathbb{R}^2$ defined by $u(a,b) = a_1 b_1 - a_2 b_2, \quad v(a,b) = a_1 b_2 + a_2 b_1$ for any $a = (a_1, a_2) \in \mathbb{R}^2$ and $b = (b_1, b_2) \in \mathbb{R}^2$. It is already known (see [LS]) that each component of $z \mapsto F(x, z, p, q)$ is an entire function for all $x \in [0,1]$ and $(p,q) \in L^2(0,1) \times L^2(0,1)$.

Let us define

$$f(z) = \int_0^a \left\langle \begin{pmatrix} v(F(x,z,p_1,q_1), F(x,z,p_2,q_2)) \\ -u(F(x,z,p_1,q_1), F(x,z,p_2,q_2)) \end{pmatrix} ; \begin{pmatrix} p_1(x) - p_2(x) \\ q_1(x) - q_2(x) \end{pmatrix} \right\rangle dx, \tag{5}$$

for all $z \in \mathbb{C}$.

The first property of f is to vanish on any common eigenvalue of $H(0,0,p_1,q_1)$ and $H(0,0,p_2,q_2)$. This point is proved in [DG] and we recall the proof for the sake of completeness: the derivative with respect to x of the scalar product appearing in the r.h.s. of (5) equals $-[F(x,z,p_1,q_1), F(x,z,p_2,q_2)]$ for any $z \in \mathbb{C}$ where the Wronskian $\left[\begin{pmatrix} G_1 \\ G_2 \end{pmatrix}, \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \right] = G_1 H_2 - G_2 H_1$. Integrating this equality gives $f(z) = [F(x,z,p_1,q_1), F(x,z,p_2,q_2)]|_{x=0}^{x=1}$ which vanishes when assuming that $z \in \sigma(H(0,0,p_1,q_1)) \cap \sigma(H(0,0,p_2,q_2))$.

Proposition 2.1. *Let $(p_1,q_1), (p_2,q_2) \in L^2(0,1) \times L^2(0,1)$ Fix $k \in \mathbb{N} \cup \{0\}$ and let $r \in [2, +\infty]$. Fix any arbitrary small $\varepsilon > 0$ and assume that, $x \mapsto (a-x)^{-k}((p_1 - p_2)(x), (q_1 - q_2)(x))$ belongs to $L^r(a - \varepsilon, a) \times L^r(a - \varepsilon, a)$. Then there is a real positive number C independent on $z \in \mathbb{C}$ such that,*

$$|f(z)| \leq C \frac{e^{2|\Im z|a}}{|\Im z|^{k+1-\frac{1}{r}}} (e^{-\varepsilon'|\Im z|} + o(1)) \tag{6}$$

as $\varepsilon' \to 0^+$ uniformly in $z \in \mathbb{C} \backslash \mathbb{R}$.

Proof of Proposition 2.1: Fix $0 < \varepsilon' < \varepsilon < a$. The function f is decomposed as,

$$f(z) = f_1(z) + f_2(z) + f_3(z), \tag{7}$$

for all $z \in \mathbb{C}$ where the domain of integration of f_1, f_2, f_3 are respectively $(0, a - \varepsilon)$, $(a - \varepsilon, a - \varepsilon')$, $(a - \varepsilon', a)$. Since $F(x,z,p,q) = O(e^{|\Im z|x})$ uniformly on $x \in (0,1)$ and $z \in \mathbb{C}$ then $f_1(z) = O(e^{2|\Im z|(a - \varepsilon)})$.

In particular,

$$|f_1(z)| \leq C\frac{e^{2a|\Im z|}}{|\Im z|^l}, \tag{8}$$

for any $l \in \mathbb{N}$ and $z \in \mathbb{C}\backslash\mathbb{R}$. In order to estimate f_2 and f_3 one verifies that, for any $a > \varepsilon_1 > \varepsilon_2 > 0$, for any $\xi > 0$, for any $k \in \mathbb{N}$ and $u \in L^r(a-\varepsilon_1, a-\varepsilon_2)$ with $r \in [1, +\infty]$, one has $(\frac{1}{r'} + \frac{1}{r'} = 1)$,

$$\left|\int_{a-\varepsilon_1}^{a-\varepsilon_2} e^{2\xi x}(a-x)^k u(x)\,dx\right| \leq \Gamma(kr'+1)^{\frac{1}{r'}}||u||_{L^r(a-\varepsilon_1,a-\varepsilon_2)}\frac{e^{(2a-\varepsilon_2)\xi}}{\xi^{k+\frac{1}{r'}}}. \tag{9}$$

This inequality follows by first applying Hölder inequality and then using the change of variable $t = \xi r'(a-x)$.

Set $u(x) = (a-x)^{-k}(|(p_1-p_2)(x)|+|(q_1-q_2)(x)|)$ for a.e. $x \in (a-\varepsilon, a)$. By assumption, $u \in L^r(a-\varepsilon, a)$. Using (9) with $\varepsilon_1 = \varepsilon$, $\varepsilon_2 = \varepsilon'$ and $\xi = |\Im z|$, one has that,

$$|f_2(z)| \leq C\frac{e^{(2a-\varepsilon')|\Im z|}}{|\Im z|^{k+\frac{1}{r'}}}, \tag{10}$$

and using (9) with $\varepsilon_1 = \varepsilon'$, $\varepsilon_2 = 0$, $\xi = |\Im z|$, one sees that,

$$|f_3(z)| \leq C||u||_{L^r(a-\varepsilon',a)}\frac{e^{2a|\Im z|}}{|\Im z|^{k+\frac{1}{r'}}}, \tag{11}$$

for all $z \in \mathbb{C}\backslash\mathbb{R}$.

In particular, (7) (8) (10) (11) prove (6) in Proposition 2.1 with $o(1) = ||u||_{L^r(a-\varepsilon',a)}$. $\square$

Let $E = (e_j)$ be a family of real (or complex) numbers. For any $t > 0$ let $n_E(t)$ denotes the number of elements of E counted with their multiplicity inside the closed ball of radius t and centered at the origin. Define

$$N_E(R) \;=\; \int_0^R \frac{n_E(t)}{t}\,dt, \tag{12}$$

for any $R > 0$.

As a corollary of the preceding Proposition we have the following result.

Corollary 2.2. *Let a, k, r, (p_1, q_1) and (p_2, q_2) be as in Theorem 1.1. Assume that f is not entirely vanishing. Then*

$$\lim_{R\to+\infty} N_{f^{-1}(0)}(R) - \frac{4a}{\pi}R + \left(k+1-\frac{1}{r}\right)\ln R = -\infty. \tag{13}$$

Proof of Corollary 2.2: it is a consequence of Jensen's Theorem (see also [A2] for similar computations). $\square$

We shall now derive a second estimate on $N_{f^{-1}(0)}$ starting from hypothesis (H) in Theorem 1.1 contradicting (13) unless the function f is entirely vanishing.

Proposition 2.3. *Let $(p,q) \in L^2(0,1) \times L^2(0,1)$ and let $\sigma(H(0,0,p,q)) = (\lambda_j(0,0,p,q))_{j\in\mathbb{Z}}$ be the spectrum of $H(0,0,p,q)$. The $\lambda_j(0,0,p,q)$ are indexed in such a way that $(\lambda_j(p,q) - j\pi) \in \ell^2(\mathbb{Z})$. Let S be any subset of $\sigma(H(0,0,p,q))$ verifying,*

$$n_S(t) \geq 2an_{\sigma(H(0,0,p,q))}(t) + b, \quad \forall\, t \in \sigma(H(p,q)), \tag{14}$$

for some $a \in (0, \frac{1}{2}]$ and $b \in \mathbb{R}$. Define

$$R_j = \min(-\lambda_{-j}(0,0,p,q), \lambda_j(0,0,p,q)), \quad \forall\, j \in \mathbb{N}.$$

Then the sequence $(N_S(R_j) - \frac{4a}{\pi}R_j - b\ln R_j)_{j\in\mathbb{N}}$ is bounded from below.

Proof of Proposition 2.3: the proof is similar to [A2, Proposition 2.3]. Define $r_j = \max(-\lambda_{-j}(p,q), \lambda_j(p,q))$, $\forall\, j \in \mathbb{N}$. Let $k_0 \in \mathbb{N}$ be such that $R_k > 0$ for all $k \geq k_0$. According to hypothesis (14),

$$\begin{array}{rcll} n_S(t) & \geq & 4ak + b, & \forall\, t \in [R_k, r_k[\\ n_S(t) & \geq & 4ak + 2a + b, & \forall\, t \in [r_k, R_{k+1}[, \quad \forall\, k \geq k_0 \end{array} \tag{15}$$

and

$$\begin{aligned} N_S(R_j) &\geq \sum_{k=k_0}^{j-1} \int_{R_k}^{r_k} \frac{4ak}{t}\,dt + \sum_{k=k_0}^{j-1} \int_{r_k}^{R_{k+1}} \frac{4ak+2a}{t}\,dt + b\ln R_j + O(1) \\ &\geq \sum_{k=k_0}^{j-1} \int_{R_k}^{R_{k+1}} \frac{4ak}{t}\,dt + \sum_{k=k_0}^{j-1} \int_{r_k}^{R_{k+1}} \frac{2a}{t}\,dt + b\ln R_j + O(1), \end{aligned} \tag{16}$$

as $j \to +\infty$.

We denote by T_1, T_2 and T_3 the first, second and third term in (16) above. One has,

$$\begin{aligned} T_1 &= 4a \sum_{k=k_0}^{j-1} k(\ln R_{k+1} - \ln R_k) \\ &= 4a \sum_{k=k_0}^{j-1} ((k+1)\ln R_{k+1} - k\ln R_k) - 4a \sum_{k=k_0}^{j-1} \ln R_{k+1} \\ &= 4aj\ln R_j - 4a \sum_{k=k_0}^{j-1} \ln R_{k+1} + O(1), \end{aligned} \tag{17}$$

and

$$\begin{aligned} T_2 &= \sum_{k=k_0}^{j-1} \left(\int_{r_k}^{R_k} \frac{2a}{t}\,dt + \int_{R_k}^{R_{k+1}} \frac{2a}{t}\,dt \right) \\ &= 2a \sum_{k=k_0}^{j-1} \ln \frac{R_k}{r_k} + 2a\ln R_j + O(1), \end{aligned} \tag{18}$$

as $j \to +\infty$. The first term in (18) is $O(1)$ as $j \to +\infty$ since $(R_k - k\pi)$ and $(r_k - k\pi)$ belong to $\ell^2(\mathbb{N})$. Thus, (16) − (18) gives,

$$N_S(R_j) \geq 4aj \ln R_j - 4a \sum_{k=k_0}^{j} \ln R_k + (2a+b) \ln R_j + O(1), \tag{19}$$

as $j \to +\infty$. Since $\ln R_j = \ln j + O(1)$, $j \ln R_j = j \ln j + j \ln \pi + O(1)$ and $\sum_{k=k_0}^{j} \ln R_k = (j+1) \ln \pi + \left(j + \frac{1}{2}\right) \ln j - j + O(1)$ by Stirling's formula, as $j \to +\infty$, one sees that $N_S(R_j) \geq 4aj + b \ln j + O(1) \geq \frac{4a}{\pi} R_j + b \ln R_j + O(1)$ as $j \to +\infty$. This completes the proof. □

Proof of Theorem 1.1: Set

$$b = -\left(k + 1 - \frac{1}{r}\right). \tag{20}$$

Again the case $r = +\infty$ is allowed. It is learned from Proposition 2.3 that $(N_S(R_j) - \frac{4a}{\pi} R_j - b \ln j)_{j \in \mathbb{N}}$ is bounded from below. If f is not vanishing then Corollary 2.2 implies that $(N_S(R_j) - \frac{4a}{\pi} R_j - b \ln j)_{j \in \mathbb{N}}$ goes to $-\infty$ as j tends to $+\infty$ with b defined in (20). This leads to a contradiction and the function f is entirely vanishing. Using the same argument as in [DG], this fact implies that $(p_1, q_1) = (p_2, q_2)$ on $(0, 1)$ and Theorem 1.1 is derived. □.

References

[A1] Amour L., *Inverse spectral theory for the AKNS system with separated boundary conditions*, Inv. Problems 5 (1993), 507-523.

[A2] L. Amour, The coordinates system $\kappa \times \mu$ on $L^2(0,1) \times L^2(0,1)$ for the AKNS operators on the unit interval, preprint.

[AF] Amour L., Faupin, Inverse spectral results for Schrödinger operators in Sobolev spaces, Int. Math. Res. Notes, 2010, 4319-4333.

[AFR] L. Amour, J. Faupin and T. Raoux, *Inverse Spectral Results for Schrödinger Operators on the Unit Interval with Partial Informations Given on the Potentials*, Journal of Mathematical Physics, no.50, 033505(2009).

[DGS1] R. del Rio, F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential, III. Updating boundary conditions*, Int. Math. Res. Notices 1997, No. 15, 751-758.

[DGS2] R. del Rio, F. Gesztesy, and B. Simon, *Corrections and addendum to Inverse spectral analysis with partial information on the potential, III. Updating boundary conditions*, Int. Math. Res. Notices

1999, No. 11, 623-625.

[DG] R. del Rio and B. Grébert, *Inverse spectral results for the AKNS systems with partial information on the potentials*, Math. Phys. Anal. Geom., 4 (2001), no.3, 229-244.

[GS] F. Gesztesy and B. Simon, *Inverse spectral analysis with partial information on the potential. II. The case of discrete spectrum*, Trans. Amer. Math. Soc. 352 (2000), no.6, 2765-2787.

[Ha] O. H. Hald, *Inverse eigenvalue problem for the mantle*, Geophys. J. R. Astr. Soc. 62 (1980), 41-48.

[HL] H. Hochstadt and B. Lieberman, *An inverse Sturm-Liouville problem with mixed given data*, SIAM J. Appl. Math. 34 (1978), 676-680.

[L] B. J. Levin, *Distribution of zeros of entire functions*, Trans. math. Mon. AMS 5 (1964).

[LS] B. M. Levitan and I. S. Sargsjan, *Sturm-Liouville and Dirac operators*, Mathematics and its Applications (Soviet Series) 59, Kluwer Academic Publishers Group, Dordrecht, (1991).

[S] F. Sérier, *Inverse spectral problem for singular Ablowitz-Kaup-Newell-Segur operators*, Inverse Problems 22 (2006), no.4, 1457-1484.