Inverse spectral results for the Schrödinger operator in Sobolev spaces

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Abstract

We provide sufficiently large sets of eigenvalues that determine the potential of a Schrödinger operator on the unit interval $[0,1]$ partially known on $[a,1]$ and belonging to $W^{k,p}$ in a neighbourhood of a ($k \in \mathbb{N} \cup \{0\}$, $p \in [1,+\infty]$). The number of these given eigenvalues depends on (a,k,p).

1 Introduction and statement of the results

This paper is concerned with the Schrödinger operator

$$A_{q,h,H} = -\frac{d^2}{dx^2} + q \tag{1}$$

defined on the unit interval with real-valued potentials q belonging to $L^1((0,1))$. This operator is associated with the boundary conditions

$$u'(0) + hu(0) = 0, \qquad u'(1) + Hu(1) = 0 \tag{2}$$

where h, H are real numbers and where the notation $'$ stands for the derivative with respect to the variable x. It is well-known that, for each $(q,h,H) \in L^1([0,1]) \times \mathbb{R}^2$ the operator $A_{q,h,H}$ is a self-adjoint operator in $L^2([0,1])$. Its spectrum $\sigma(A_{q,h,H})$ is an increasing and non-bounded sequence of non degenerate eigenvalues denoted by $(\lambda_j(q,h,H))_{j\in\mathbb{N}\cup\{0\}}$.

Our purpose here is to provide sets of eigenvalues sufficiently large in order to determine a potential that is already known on $[a,1]$ (for some given $a \in (0,\frac{1}{2}]$) when it belongs to some $W^{k,p}$ space. This problem has been initiated in 1978 by [HL] in the special case $a = \frac{1}{2}$ and for potentials in $L^1([0,1])$. In 2000, the problem is studied in [GS] for any a and for potentials in C^{2k} near a ($k \in \mathbb{N} \cup \{0\}$). Results like one spectrum and half of another one added to the knowledge of the potential on $[\frac{3}{4},1]$ uniquely determine the potential are derived in [DGS1] and [DGS2]. Potentials in L^p spaces are considered in [Ho] and [AR].

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Recently, potentials in $W^{k,p}([0,a])$ are considered in [AFR] for any a ($p \in [1,+\infty]$), with however the restriction $k \in \{0,1,2\}$. We have conjectured in [AFR] that the result in [AFR] should be valid for all $k \in \mathbb{N} \cup \{0\}$. This is one of our aim here to get rid of this condition on k and to consider *all* $k \in \mathbb{N} \cup \{0\}$. Our second goal is to replace regularity hypotheses of q_1, q_2 on $[0,a]$ by regularity hypotheses on q_1, q_2 only on an arbitrary small neighborhood of a (as in [GS]).

The following function is involved in the statement of the main theorem (Theorem 1.1). For any complex-valued sequence $\alpha = (\alpha_j)_{j \in \mathbb{N} \cup \{0\}}$ and for all $t \geq 0$, we set

$$n_\alpha(t) = \sharp\{j \in \mathbb{N} \cup \{0\} \quad | \quad |\alpha_j| \leq t\}. \tag{3}$$

The main result of the paper is the following.

Theorem 1.1.

Set $q_1, q_2 \in L^1((0,1))$. Fix $a \in (0, \frac{1}{2}]$ and suppose that $q_1 = q_2$ on $[a,1]$.

Let $k \in \mathbb{N} \cup \{0\}$ and $p \in [1,+\infty]$. Assume that $q_1, q_2 \in W^{k,1}((a-\varepsilon, a))$ with $q_1 - q_2 \in W^{k,p}((a-\varepsilon,a))$ for some arbitrary small $\varepsilon \in (0,a)$. If $k \geq 1$ assume in addition that $q_1 - q_2 \in C^{k-1}((a-\varepsilon, a+\varepsilon))$ with any arbitrary small $\varepsilon \in (0,a)$.

Fix the real numbers h_1, h_2 and H. Assume that a set of common eigenvalues $S \subseteq \sigma(A_{q_1,h_1,H}) \cap \sigma(A_{q_2,h_2,H})$ verifies either

$$n_S(t) \ \geq \ 2a\, n_{\sigma(A)}(t) - \frac{k}{2} + \frac{1}{2p} - \frac{1}{2} - a, \ t \in \sigma(A), \ t \text{ large enough}, \quad (H)$$

or assume that there exists a real number C such that

$$2a\, n_{\sigma(A)}(t) + C \ \geq \ n_S(t) \ \geq \ 2a\, n_{\sigma(A)}(t) - \frac{k}{2} + \frac{1}{2p} - 2a, \ t \in S, \ t \text{ large enough}, \quad (H')$$

where in (H) and (H') the operator A denotes either $A_{q_1,h_1,H}$ or $A_{q_2,h_2,H}$.

Then $h_1 = h_2$ and $q_1 = q_2$.

Let us emphasize here that the case $p = +\infty$ is considered in Theorem 1.1. In that case, the term $\frac{1}{p}$ in the hypotheses (H) or (H') is suppressed. Also note that only the difference of the two potentials needs to be in $W^{k,p}$ and C^{k-1} near a.

One may replace the assumptions on q_1 and q_2 in Theorem 1.1 by the more concise (but stronger) hypotheses: $q_1, q_2 \in L^1((0,1)) \cap W^{k,1}((a-\varepsilon,a))$ with $q_1 - q_2 \in W^{k,p}((a-\varepsilon, a+\varepsilon))$ (since $W^{k,p}((a-\varepsilon, a+\varepsilon)) \subset C^{k-1}([a-\varepsilon, a+\varepsilon])$ for $k \geq 1$).

Let us gives two corollaries of Theorem 1.1. The first one concerns the particular case $k = 0$, $p = 1$ and $a = \frac{1}{4}$. It is already given in [AR] (where $k = 0$), it is however recalled here in order to emphasize on the role of (H') in Theorem 1.1. Namely, this corollary may be proved using the assumption (H') while it is not be derived assuming (H) (see [AR]). It is written in a short way.

Corollary 1.2. *Suppose that q belongs to $L^1((0,1))$ and $H \in \mathbb{R}$. Then the even (resp. odd) spectrum $(\lambda_{2j}(q,h,H))_{j\geq 0}$ (resp. $(\lambda_{2j+1}(q,h,H))_{j\geq 0}$), $q|_{[0,\frac{1}{4}]}$ and H uniquely determine h and the potential q on all of $[0,1]$.*

The second corollary is Theorem 1.1 in the particular case $p = +\infty$ and $a = \frac{1}{2}$ using hypothesis (H). It allows us to remove a precise number of eigenvalues when the potentials (and their difference) are sufficiently regular. It slightly improves one of the results established in [GS]. The result in [GS] is the same as Corollary 1.3 but the potentials satisfy $q_1, q_2 \in L^1((0,1)) \cap C^{2k}((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon))$ for some small positive ε.

Corollary 1.3. *Let $k \in \mathbb{N} \cup \{0\}$. Suppose that q_1 and q_2 belong to $L^1((0,1)) \cap W^{2k,\infty}((\frac{1}{2} - \varepsilon, \frac{1}{2}))$ and if $k \geq 1$ also assume that the difference $q_1 - q_2$ is in $C^{2k-1}((\frac{1}{2} - \varepsilon, \frac{1}{2} + \varepsilon))$ for some $\varepsilon > 0$. Suppose that $q_1 = q_2$ on $\left[\frac{1}{2}, 1\right]$. Let $h_1, h_2, H \in \mathbb{R}$.*

If $\sigma(A_{q_1,h_1,H}) = \sigma(A_{q_2,h_2,H})$ excepted for at most $k+1$ eigenvalues, then $h_1 = h_2$ and $q_1 = q_2$.

Also note that this implies that, if $q \in L^1((0,1))$ is L^∞ near $x = \frac{1}{2}$ then q on $[0, \frac{1}{2}]$, H and all the eigenvalues of $\sigma(A_{q,h,H})$ excepted one, uniquely determine h and q on $[0,1]$.

The proof of Theorem 1.1 relies on the same strategy as in [AFR] excepted that [AFR, Proposition 3.1] is replaced by Proposition 1.4 below. Let us also mention that our proof is different from the result in [GS] which and is based on Weyl-Titchmarsh functions.

The estimate in Proposition 1.4 is the same as the one in [AFR, Proposition 3.1] but the assumption on k and on the regularity on $q_1 - q_2$ are largely weakened. Firstly, $k \in \{0,1,2\}$ in [AFR] is replaced here with $k \in \mathbb{N} \cup \{0\}$. Secondly, the hypotheses $q_1, q_1 \in W^{k,1}([0,1])$ and $q_1 - q_2 \in W^{k,p}([0,a])$ in [AFR] is now replaced by the hypotheses on q_1, q_2 in Theorem 1.1, namely, $q_1, q_2 \in L^1((0,1)) \cap W^{k,1}((a-\varepsilon, a))$ with $q_1 - q_2 \in W^{k,p}((a-\varepsilon, a))$ added when $k \geq 1$ to $q_1 - q_2 \in C^{k-1}((a-\varepsilon, a+\varepsilon))$ (for any arbitrary small $\varepsilon \in (0,a)$). Note that Sobolev's imbedding implies that $W^{k,p}((a-\varepsilon, a)) \subset C^{k-1}([a-\varepsilon, a])$ when $k \geq 1$.

We now define the entire function f which is involved in Proposition 1.4. Fix $q \in L^1((0,1))$ and fix $H, h \in \mathbb{R}$. For any z in $\mathbb{C}$, let $\psi(\cdot, z, q, h)$ be defined on $[0,1]$ as the solution to $-\frac{d^2\psi}{dx^2} + q\psi = z^2\psi$, $\psi(0) = 1$, $\psi'(0) = -h$. It is known that $\psi(x, \cdot, q, h)$ is an entire function ([LG]).

For all $z \in \mathbb{C}$, let us define

$$f(z) = \int_0^a \left(\psi(x,z,q_1,h_1)\psi(x,z,q_2,h_2) - \frac{1}{2} \right) (q_1(x) - q_2(x))dx. \tag{4}$$

Proposition 1.4. *Set $a \in (0, \frac{1}{2}]$, fix $k \in \mathbb{N} \cup \{0\}$ and let $p \in [1, +\infty]$. Fix $q_1, q_2 \in L^1((0,1)) \cap W^{k,1}((a-\varepsilon, a))$ such that $q_1 - q_2 \in W^{k,p}((a-\varepsilon, a))$ and assume furthermore that $q_1 - q_2 \in C^{k-1}((a-\varepsilon, a+\varepsilon))$ when $k \geq 1$ for some arbitrary small $\varepsilon \in (0,a)$. Then there is a real positive number C independent of $z \in \mathbb{C}$ and $\varepsilon' > 0$ such that $|f(z)| \leq C \frac{e^{2|\Im z|a}}{|\Im z|^{k+1-\frac{1}{p}}}(e^{-\varepsilon'|\Im z|} + o(1))$ as $\varepsilon' \to 0^+$ uniformly in $z \in \mathbb{C}\backslash\mathbb{R}$.*

Proof of Theorem 1.1: It is the same as the one of [AFR, Theorem 1] when replacing [AFR, Proposition 3.1] by Proposition 1.4 above. For the sake of completeness let us recall very briefly here the main steps (see [AFR] for more details). Suppose that $a, k, p, q_1, q_2, h_1, h_2$ satisfy the same assumptions as the ones in Theorem 1.1. Define the s_j, $j \in \mathbb{N}$ as the strictly increasing sequence being in S and define the set $S^{\frac{1}{2}} = \{\pm\sqrt{s_j},\ j \in \mathbb{N}\}$. We also define for any set of complex numbers α, $N_\alpha(R) = \int_0^R \frac{n_\alpha(t)}{t}dt$, for any $R > 0$ and where $n_\alpha(t)$ is given in (3). On one hand, the hypothesis (H) or (H') implies that the sequence $\left(N_{S^{\frac{1}{2}}}(\sqrt{s_j}) - \frac{4a}{\pi}\sqrt{s_j} + \left(k+1-\frac{1}{p}\right)\ln\sqrt{s_j}\right)_{j\in\mathbb{N}}$ is bounded from below ([AFR, Prop. 4.1 and 4.2]). On the other side, using Proposition 1.4 above, using [AFR, Prop. 4.3] and assuming that f is not entirely vanishing in order to use Jensen's Theorem, we deduce that $\lim_{R\to+\infty} N_{f^{-1}(0)}(R) - \frac{4a}{\pi}R + \left(k+1-\frac{1}{p}\right)\ln R = -\infty$. The last two points combined to $N_{f^{-1}(0)} \geq N_{S^{\frac{1}{2}}}$ (see (23) in [AFR]) lead to a contradiction if f is not entirely vanishing. The fact that $f \equiv 0$ implies that $(q_1, h_1) = (q_2, h_2)$ is already proved in [L]. $\square$

The rest of this paper is therefore concerned with the proof of Proposition 1.4. The main difference here is that we imply the transformation operators [L] (see also references therein and see [Le],[LS],[M],...) instead of using expansions of the fundamental solutions to $A_{q,h,H}y = zy$

Proposition 1.4 is derived in the next section. The case of Dirichlet boundary conditions is considered in Appendix A.

2 Proof of Proposition 1.4

The proof of Proposition 1.4 shall follow from Lemmata 2.1-2.7 below.

We first start with the definition of the transformation operators (see [L],[Le],[LS],[M] and references therein). We shall use in the following the kernel $\tilde{L}$ (see (11) below) computed in [L]. This kernel is expressed in terms of the kernel L (see (7) below). Its properties are taken from [M].

We first recall the definition of L given by [M]. To do this we first define the kernel K.

Suppose $q \in L^1_{loc}((0,1))$. There exists a kernel $K \equiv K(x,t)$ for $0 \le x \le 1$ and $-x \le t \le x$ (see [M]) such that, for each $z \in \mathbb{C}$, the solution $\alpha \equiv \alpha(x,z)$ to

$$-\alpha'' + q\alpha = z^2\alpha \quad [0,1], \quad \alpha(0,z) = 1, \ \alpha'(0,z) = iz \tag{5}$$

may be expressed as

$$\alpha(x,z) = e^{izx} + \int_0^x K(x,t)e^{izt}\,dt, \quad x \in [0,1]. \tag{6}$$

Let

$$L(x,t) = -h + K(x,t) + K(x,-t) - h\int_t^x (K(x,\xi) - K(x,-\xi))\,d\xi, \tag{7}$$

for $0 \le t \le x \le 1$. One then obtains that, for each $z \in \mathbb{C}$, the solution $\beta \equiv \beta(x,z)$ to

$$-\beta'' + q\beta = z^2\beta \quad [0,1], \quad \beta(0,z) = 1, \ \beta'(0,z) = -h \tag{8}$$

may be expressed as

$$\beta(x,z) = \cos zx + \int_0^x L(x,t)\cos zt\,dt, \quad x \in [0,1]. \tag{9}$$

Let us denote respectively by I and T_L the identity operator and Volterra operator with kernel $L(x,t)$. Let $\beta_0(x,z) = \cos zx$ ($z \in \mathbb{C}$). With these notations (9) is also written as $\beta(\cdot,z) = (I + T_L)\beta_0(\cdot,z)$ for any $z \in \mathbb{C}$. Fix $q \in L^1((0,1))$ and $h \in \mathbb{R}$. The main point is that the operator $(I + T_L)$ maps the solution to (8) with q identically vanishing and $h = 0$ to solution to (8) with the potential q and the parameter h.

Fix q_j in $L^1((0,1))$ and $h_j \in \mathbb{R}$ for $j = 1,2$. Set L_j the function defined in (7) associated to $q = q_j$ and $h = h_j$, $j = 1,2$ and extended for $t \in [-x,0]$ by setting $L(x,t) = L(x,-t)$. With these notations one obtains (see [L, Appendix IV]),

$$\psi(x,z,q_1,h_1)\psi(x,z,q_2,h_2) - \frac{1}{2} = \frac{1}{2}\cos 2zx + \frac{1}{2}\int_{-x}^x \tilde{L}(x,\tau)\cos 2z\tau\,d\tau, \tag{10}$$

with $x \in [0,1]$ and where

$$\tilde{L}(x,\tau) = \begin{cases} 2(L_1(x,x-2\tau) + L_2(x,x-2\tau)) + \int_{-x+2\tau}^x L_1(x,s)L_2(x,s-2\tau)\,ds \text{ if } \tau > 0, \\ \\ \int_{-x}^{x+2\tau} L_1(x,s)L_2(x,s-2\tau)\,ds \text{ if } \tau < 0. \end{cases} \tag{11}$$

Throughout the paper we suppose that a and ε are fixed in $\left(0, \frac{1}{2}\right]$ and $(0,a)$ respectively. Let us first decompose f as

$$f(z) = f_{a-\varepsilon}(z) + f_a(z)$$

with

$$f_{a-\varepsilon}(z) = \int_0^{a-\varepsilon} \left(\psi(x,z,q_1,h_1)\psi(x,z,q_2,h_2) - \frac{1}{2}\right)(q_1(x) - q_2(x))dx$$

and

$$f_a(z) = \int_{a-\varepsilon}^a \left(\psi(x,z,q_1,h_1)\psi(x,z,q_2,h_2) - \frac{1}{2}\right)(q_1(x) - q_2(x))dx,$$

for all $z \in \mathbb{C}$. The function $f_{a-\varepsilon}$ is easily estimated.

Lemma 2.1. *For q_1 and q_2 in $L^1((0,1))$ we have*

$$f_{a-\varepsilon}(z) = O(e^{2|Im\, z|(a-\varepsilon)})$$

uniformly in $z \in \mathbb{C}$.

Proof of Lemma 2.1: It follows from the asymptotic expansions of the function ψ.
Namely, $\psi(x,z,q,h) = O(e^{|Im\, z|x})$ uniformly for $(z,x) \in \mathbb{C} \times [0,1]$ (see [LG]) and $q_1 - q_2 \in L^1((0,1))$ directly implies the stated estimate on $f_{a-\varepsilon}(z)$. $\square$

In view of (10) the function f_a is split as

$$f_a(z) = f_0(z) + \tilde{f}(z),\ \forall\, z \in \mathbb{C},$$

where

$$f_0(z) = \frac{1}{2}\int_{a-\varepsilon}^{a} \cos 2zx(q_1(x) - q_2(x))\, dx,\ \forall\, z \in \mathbb{C} \tag{12}$$

and

$$\tilde{f}(z) = \frac{1}{2}\int_{a-\varepsilon}^{a} \left(\int_{-x}^{x} \tilde{L}(x,\tau)\cos 2z\tau\, d\tau\right)(q_1(x) - q_2(x))\, dx,\ \forall\, z \in \mathbb{C}. \tag{13}$$

We shall first estimate the function f_0. For any $k \in \mathbb{N}$, let c_k be the k^{th} integral of the cosine function verifying $c_k^{(l)}(0) = 0$, $l = 0, \ldots, k-1$, that is to say, $c_k(x) = \int_0^x \int_0^{t_k} \cdots \int_0^{t_2} \cos(t_1)\, dt_1 \ldots dt_k$. For any $l \in \mathbb{N}$ and for any sufficiently smooth function g depending only on one variable, $g^{(l)}$ denotes its l^{th} derivative.

Lemma 2.2. *Fix $k \in \mathbb{N} \cup \{0\}$. Let $q_1, q_2 \in W^{k,1}((a-\varepsilon, a))$ with $q_1 - q_2 \in C^{k-1}((a-\varepsilon, a+\varepsilon))$ if $k \geq 1$. There exist k complex numbers $L_{0,l}(z)$ ($l = 1, \ldots, k$) satisfying*

$$L_{0,l}(z) = O\left(e^{2|\Im\, z|(a-\varepsilon)}\right) \tag{14}$$

uniformly in $z \in \mathbb{C}$ and such that

$$f_0(z) = \sum_{l=1}^{k} \frac{L_{0,l}(z)}{(2z)^l} + \int_{a-\varepsilon}^{a} \frac{c_k(2zx)}{(-2z)^k}(q_1 - q_2)^{(k)}(x)\, dx, \tag{15}$$

for all $z \in \mathbb{C}\backslash\{0\}$.

Proof of Lemma 2.2: Clearly one may suppose that $k \geq 1$. Then one can integrate by parts k times the r.h.s. of (12) since $q_1 - q_2 \in W^{k,1}((a-\varepsilon, a))$. Since $q_1 - q_2 \in C^{k-1}$ near a and using $q_1 - q_2 \equiv 0$ on $[a,1]$ we see that $(q_1 - q_2)^{(l)}(a) = 0$, $l = 0, \ldots k-1$. This shows that the k boundary terms at $x = a$ are vanishing. It remains k boundary terms at $x = a - \varepsilon$. These terms lead to $\sum_{l=1}^{k} \frac{L_{0,l}(z)}{2z^l}$ with the $L_{0,l}(z) = (-1)^{l-1} c_l(2z(a-\varepsilon))(q_1 - q_2)^{(l-1)}(a-\varepsilon)$. Using $|\sin z| \leq e^{|\Im z|}$ and $|\cos z| \leq e^{|\Im z|}$ for all $z \in \mathbb{C}$ one clearly gets (14) and (15). $\square$

Next and in order to deal with $\tilde{f}$ we write using Fubini's theorem that

$$\tilde{f} = f_1 + f_2 + f_3$$

with

$$f_1(z) = \int_{a-\varepsilon}^{a} \int_{\tau}^{a} \tilde{L}(x,\tau)(q_1(x) - q_2(x)) \cos 2z\tau \, dx \, d\tau, \tag{16}$$

$$f_2(z) = \int_{-a}^{-a+\varepsilon} \int_{-\tau}^{a} \tilde{L}(x,\tau)(q_1(x) - q_2(x)) \cos 2z\tau \, dx \, d\tau, \tag{17}$$

and

$$f_3(z) = \int_{a-\varepsilon}^{a} \int_{-a+\varepsilon}^{a-\varepsilon} \tilde{L}(x,\tau)(q_1(x) - q_2(x)) \cos 2z\tau \, d\tau \, dx, \tag{18}$$

for all $z \in \mathbb{C}$. Consequently, we shall only consider f_1 and f_3 in the sequel since the treatment of f_2 would be similar to f_1 making the change of variables $\tau \mapsto -\tau$ in f_2 .

Set

$$w(\tau) = \int_{\tau}^{a} \tilde{L}(x,\tau)(q_1(x) - q_2(x)) \, dx, \tag{19}$$

for any $\tau \in (a - \varepsilon, a)$. That is to say,

$$f_1(z) = \int_{a-\varepsilon}^{a} w(\tau) \cos 2z\tau \, d\tau, \tag{20}$$

for all $z \in \mathbb{C}$. In order to integrate by parts the r.h.s. of (20), we need that w defined in (19) belongs to $W^{k,1}((a-\varepsilon,a))$. It is actually in $W^{k,\infty}([a-\varepsilon,a])$. This is precisely the purpose of Lemma 2.4 below with the help of Lemma 2.3.

In the sequel, for any sufficiently smooth function g depending on the variables $(x_1,\dots,x_n)$, $\partial_{j_1,\dots,j_l} g$ stands for the derivative of order l of g with respect the variables $x_{j_1},\dots,x_{j_l}$ (with $j_1,\dots,j_l \in \{1,\dots,n\}$, $l \in \mathbb{N}$) and $\partial_j^m g$ denotes the derivative of order m of g with respect the variable x_j (where $j \in \{1,\dots,n\}$, $m \in \mathbb{N}$).

Let us recall that the kernel $\tilde{L}$ is written in terms of the two kernels L_1 and L_2 and these two kernels L_j ($j = 1, 2$) are expressed in (7) with the functions K_j corresponding to $q = q_j$.

Set $T_{a,\varepsilon}$ be the triangle $\{a - \varepsilon \le t \le x \le a\}$ and let $D_{a,\varepsilon}$ be the diagonal $D_{a,\varepsilon} = \{(\tau,\tau) \mid \tau \in [a-\varepsilon,a]\}$. Let us recall here that in this section ε is fixed in $(0,a)$.

Lemma 2.3. *(i) Fix $k \in \mathbb{N} \cup \{0\}$ and $q \in L^1((0,1)) \cap W^{k,1}((a-\varepsilon,a))$. Then, the kernel K associated to q belongs to $C^k(T_{a,\varepsilon})$.*

(ii) Suppose that $q \in L^1((0,1)) \cap W^{k,1}((a-\varepsilon,a))$ for some $k \in \mathbb{N} \cup \{0\}$. Then, the kernel L defined in (7) corresponding to q is in $C^k(T_{a,\varepsilon})$.

(iii) Assume that $q_1, q_2 \in L^1((0,1)) \cap W^{k,1}((a-2\varepsilon, a))$ for $k \geq 0$ (with $0 < \varepsilon < \frac{a}{2}$). If $k = 0$ then the kernel $\tilde{L}$ given by (11) is in $C^0(T_{a,\varepsilon})$. When $k \geq 1$ then $\partial_2^j \tilde{L} \in C^0(T_{a,\varepsilon})$ for all $0 \leq j \leq k$ and $[(\partial_2^l \tilde{L})|_{D_{a,\varepsilon}}]^{(\alpha)} \in C^0(D_{a,\varepsilon})$ for $l + \alpha \leq k$ (with $l \geq 0$ and $\alpha \geq 0$).

Proof of Lemma 2.3:

(i) It is proved in Theorem 1.2.1 in [M] (see also Problem 1 in [M]) that if $q \in L^1_{loc}((0,1))$ then the kernel K belongs to $C^0(T)$ where $T = \{0 \leq t \leq x \leq 1\}$. When $k \geq 1$, if $q \in W^{k,1}((0,1))$ then $q \in C^{k-1}([0,1])$ and it is derived in Theorem 1.2.2 ([M]) that $K \in C^k(T)$.

Here $q \in L^1((0,1))$ then K exists and is continuous on T and the same arguments as in ([M]) show that $K \in C^k(T_{a,\varepsilon})$ when $q \in W^{k,1}((a-\varepsilon, a))$ $(k \geq 1)$.

(ii) From the definition of L (see (7)) and (i) we only have to check that I defined by $I(x,t) = \int_t^x K(x,\xi)\, d\xi$ verifies $I \in C^k(T_{a,\varepsilon})$ when $q \in W^{k,1}((a-\varepsilon, a))$ $(k \geq 0)$.

If $k = 0$ then $K \in C^0(T)$ and $I \in C^0(T_{a,\varepsilon})$.

If $k \geq l_1 \geq 1$ then

$$\partial_1^{l_1} I(x,t) = \sum_{i,j \geq 0, i+j = l_1 - 1} [(\partial_1^j K)|_{D_{a,\varepsilon}}]^{(i)}(x) + \int_t^x \partial_1^{l_1} K(x,\xi)\, d\xi, \tag{21}$$

for all $(x,t) \in T_{a,\varepsilon}$.

If $k \geq l_2 \geq 1$ then

$$\partial_2^{l_2} I(x,t) = -\partial_2^{l_2 - 1} K(x,t) \tag{22}$$

for all $(x,t) \in T_{a,\varepsilon}$.

Thus, if $l_1 \geq 1$, $l_2 \geq 1$ with $l_1 + l_2 \leq k$ then,

$$\partial_1^{l_1} \partial_2^{l_2} I(x,t) = -\partial_1^{l_1} \partial_2^{l_2 - 1} K(x,t) \tag{23}$$

for any (x,t) in $T_{a,\varepsilon}$. In view of (21) (22) (23) and according to (i) we see that $I \in C^k(T_{a,\varepsilon})$ when $k \geq 1$.

(iii) From the definition (11) and following the point (ii) above it is sufficient to verify that J satisfies $\partial_2^k J \in C^0(T_{a,\varepsilon})$ and $[\partial_2^l J|_{D_{a,\varepsilon}}]^{(\alpha)} \in C^0(D_{a,\varepsilon})$ when $l + \alpha \leq k$ where the function J is defined by

$$J(x,\tau) = \int_{-x+2\tau}^{x} L_1(x,s) L_2(x, s - 2\tau)\, ds \tag{24}$$

for all $(x,\tau) \in T_{a,\varepsilon}$.

If $k = 0$ then L_1 and L_2 are continuous on $T_{a,2\varepsilon}$ and $J \in C^0(T_{a,\varepsilon})$.

Suppose $k \geq 1$. One may differentiate the r.h.s of (24) k times with respect to the second variable. Indeed, one gets

$$\partial_2^k J(x,\tau) = \sum_{i,j\geq 0, i+j=k-1} 2^i(-2)^{j+1}\partial_2^i L_1(x,-x+2\tau)\partial_2^j L_2(x,-x)+$$

$$(-2)^k \int_{-x+2\tau}^{x} L_1(x,s)\partial_2^k L_2(x,s-2\tau)\,ds, \tag{25}$$

for all (x,τ) in $T_{a,\varepsilon}$. According to (ii), this implies that $\partial_2^k J \in C^0(T_{a,\varepsilon})$. Moreover, on the diagonal $D_{a,\varepsilon}$ the last integral in (25) vanishes and we obtain after differentiating α times that,

$$[(\partial_2^l J)|_{D_{a,\varepsilon}}]^{(\alpha)}(x) = \sum_{i+j=l-1} \sum_{\alpha_1+\alpha_2=\alpha} c_{ij\alpha_1\alpha_2}\partial_2^i(\partial_1+\partial_2)^{\alpha_1} L_1(x,x)\partial_2^j(\partial_1+\partial_2)^{\alpha_2} L_2(x,-x),$$

for some numerical real number $c_{ij\alpha_1\alpha_2}$, for any $x \in [a-\varepsilon,a]$ and for all $l+\alpha \leq k$. Since $i+\alpha_1 \leq l+\alpha$, $j+\alpha_2 \leq l+\alpha$, $l+\alpha \leq k$ and since L_1 and L_2 are $C^k(T_{a,\varepsilon})$ then $[(\partial_2^k J)|_{D_{a,\varepsilon}}]^{(\alpha)}$ is continuous on $[a-\varepsilon,a]$. $\square$

Lemma 2.4. *Set $k \in \mathbb{N}\cup\{0\}$ and let $q_1, q_2 \in L^1((0,1)) \cap W^{k,1}((a-2\varepsilon,a))$. Then the function w defined in* (19) *belongs to $W^{k,\infty}((a-\varepsilon,a))$.*

Proof of Lemma 2.4: From (19) it is clear that

$$w^{(j)}(\tau) = \sum_{l,m\geq 0, l+m=j-1} \sum_{\alpha,\beta\geq 0,\alpha+\beta=m} c_{jlm\alpha\beta}[(\partial_2^l \tilde{L})|_{D_{a,\varepsilon}}]^{(\alpha)}(\tau)(q_1-q_2)^{(\beta)}(\tau)+\int_\tau^a \partial_2^j\tilde{L}(x,\tau)(q_1-q_2)(x)\,dx, \tag{26}$$

for all $\tau \in [a-\varepsilon,a]$ and for some numerical coefficients $c_{jlm\alpha\beta}$ provided that the r.h.s. is well-defined. If $j=0$ the first term in the r.h.s. of the equality above is omitted. Let us verify that $w^{(j)} \in L^\infty((a-\varepsilon,a))$ for $j \leq k$. Since $l+\alpha \leq l+m \leq k-1$ then $\tau \mapsto [(\partial_2^l\tilde{L})|_{D_{a,\varepsilon}}]^{(\alpha)}(\tau) \in L^\infty((a-\varepsilon,a))$ by Lemma 2.3 (iii). Since $\beta \leq m \leq k-1$ then $(q_1-q_2)^{(\beta)} \in L^\infty((a-\varepsilon,a))$. Thus, the first term in the r.h.s. of the above equality is in $L^\infty([a-\varepsilon,a])$ as a function of the variable τ. Furthermore, $(q_1-q_2) \in L^1((a-\varepsilon,a))$ and $\partial_2^j\tilde{L} \in L^\infty(T_{a,\varepsilon})$ by Lemma 2.3 (iii) imply that the second term in the r.h.s. of the above equality for all $j \in \mathbb{N}\cup\{0\}$ with $j \leq k$ is also in $L^\infty((a-\varepsilon,a))$ as a function of the variable τ. $\square$

With this Lemma, we are now able to integrate by parts k times the function in the r.h.s. of (20). We recall that the functions c_k are defined before Lemma 2.2.

Lemma 2.5. *Let $k \in \mathbb{N}\cup\{0\}$. Set $q_1, q_2 \in L^1((0,1)) \cap W^{k,1}((a-2\varepsilon,a))$ and if $k \geq 1$ assume in addition that $q_1-q_2 \in C^{k-1}((a-\varepsilon,a+\varepsilon))$. One has*

$$f_1(z) = \sum_{l=1}^{k} \frac{L_{1,l}(z)}{(2z)^l} + \frac{1}{(-2z)^k}\int_{a-\varepsilon}^{a} w^{(k)}(\tau)c_k(2z\tau)\,d\tau$$

for all $z \in \mathbb{C}$, for $i = 1,2$ where the $L_{1,l}(z)$ are k real numbers satisfying

$$L_{1,l}(z) = O\left(e^{2|\Im z|(a-\varepsilon)}\right) \tag{27}$$

for all $z \in \mathbb{C}\backslash\{0\}$.

Proof of Lemma 2.5: It suffices to suppose $k \geq 1$. As in Lemma 2.2, the proof follows from k integrations by parts. These are justified by the regularity of w provided by Lemma 2.4. Note also that all the boundary terms at $\tau = a$ are vanishing. Indeed, in view of (26) one sees that $w^{(\beta)}(a) = 0$, $\beta = 0, \ldots, k-1$ since $q_1 - q_2 \in C^{k-1}((a-\varepsilon, a+\varepsilon))$ and $q_1 - q_2 = 0$ on $[a,1]$ and since the last integral vanishes. Therefore $L_{1,l}(z) = (-1)^{l-1} c_l(2z(a-\varepsilon)) w^{(l-1)}(a-\varepsilon)$. From Lemma 2.6 the function $w^{(k)} \in C^0([a-\varepsilon, a])$ and using again $|c_k(2z\tau)| \leq e^{2|\Im z|\tau}$ one gets the estimate (27). $\square$

Finally we consider $f_3(z)$ defined in (18).

Lemma 2.6. *Let $k \in \mathbb{N} \cup \{0\}$. Let $q_1, q_2 \in L^1((0,1))$. One has*

$$f_3(z) = O(e^{2|\Im z|(a-\varepsilon)})$$

for all $z \in \mathbb{C}$.

Proof of Lemma 2.6: it follows directly from (18) with $\tilde{L} \in C^0(T)$ and $|\cos 2z\tau| \leq e^{2|\Im z|\tau}$ for all $z \in \mathbb{C}$ and all $\tau \in \mathbb{R}$ together with $q_1 - q_2 \in L^1((0,1))$. $\square$

We are now ready to derive Proposition 1.4. Let us first recall the following result (see [L] and see also Lemma 3.2 in [AFR] for a short proof replacing 0 by b).

Lemma 2.7. *Let $a, b \in (0,1]$ with $b < a$. Suppose that the function u defined on $[0,1] \times \mathbb{C}$ satisfies $|u(x,z)| = O\left(e^{2|\Im z|x}\right)$ and let $v \in L^p([0,1])$ with $1 \leq p \leq +\infty$. Set $g(z) = \int_b^a u(x,z)v(x)dx$. There is a real positive number C depending only on p and $||v||_{L^p([b,a])}$ such that for any $\varepsilon' > 0$ there is a real positive number $\delta_{\varepsilon'}$ depending only on ε', p, a, b and $||v||_{L^p([b,a])}$ verifying*

$$\lim_{\varepsilon' \to 0} \delta_{\varepsilon'} = 0$$

and

$$|g(z)| \leq C \frac{e^{2|\Im z|a}}{|\Im z|^{1-\frac{1}{p}}} (e^{-\varepsilon'|\Im z|} + \delta_{\varepsilon'}).$$

Proof of Proposition 1.4: Without loss of generality we suppose that q_1 and q_2 are in $W^{k,1}((a-2\varepsilon, a))$ instead of $W^{k,1}((a-\varepsilon, a))$. Let us denote by w_1 the preceding function w defined in (19) associated to f_1 and by w_2 the similar one corresponding to f_2. Using the estimates for $f_{a-\varepsilon}$, f_0, f_1 (and the analogous one for f_2) and f_3 in Lemma 2.1, 2.2 2.5 and 2.6 respectively, one has

$$f(z) = \sum_{l=0}^{k} O\left(\frac{e^{2|\Im z|(a-\varepsilon)}}{|z|^l}\right) + \frac{1}{(-2z)^k} \int_{a-\varepsilon}^{a} c_k(2zx) \left(q_1 - q_2 + w_1 + w_2\right)^{(k)}(x)\, dx. \tag{28}$$

Since $(q_1 - q_2) \in W^{k,p}((a-\varepsilon, a))$ and $w_1, w_2 \in W^{k,\infty}((a-\varepsilon, a))$ one concludes with Lemma 2.7 that the integral term in the r.h.s. of (28) is bounded by $\frac{e^{2|\Im z|a}}{|\Im z|^{k+1-\frac{1}{p}}}(e^{-\varepsilon'|\Im z|} + o(1))$ as $\varepsilon' \to 0^+$ uniformly in $z \in \mathbb{C}$.

In the sum in the r.h.s. of (28), the functions $f_{a-\varepsilon}$ and f_3 are contributing for $l = 0$. Writing

$$\frac{e^{2|\Im z|(a-\varepsilon)}}{|\Im z|^l} \leq \frac{e^{2|\Im z|a}}{|\Im z|^{k+1-\frac{1}{p}}} e^{-|\Im z|\varepsilon'} e^{-|\Im z|\varepsilon} |\Im z|^{k+1-\frac{1}{p}-l}$$

for all $\varepsilon' \leq \varepsilon$ and since $e^{-|\Im z|\varepsilon}|\Im z|^{k+1-\frac{1}{p}-l} \leq C_l$ for all $z \in \mathbb{C}$ and for some C_l depending on l (and ε) one sees that the sum in the r.h.s. of (28) is $o\left(\frac{e^{2|\Im z|a}}{|\Im z|^{k+1-\frac{1}{p}}}\right)$ as $\varepsilon' \to 0^+$ uniformly in $z \in \mathbb{C}$. These two points complete the proof of Proposition 1.4. □

3 Appendix A

The case of Dirichlet boundary conditions

$$u(0) = 0, \quad u(1) = 0$$

corresponding to $h = H = \infty$ may be considered similarly to the case of finite h and H. Let us also mention that the cases $(h = \infty, H < \infty)$ and $(h < \infty, H = \infty)$ may be not treated analogously entirely, the reason being that the leading term in the asymptotic expansions of the square roots of the sequences of eigenvalues is (up to the π factor) an half-integer whereas it is an integer for the cases $(h < \infty, H < \infty)$ and $(h = \infty, H = \infty)$ and one cannot follow [AFR, Section 4].

Theorem 3.1. *Under the hypotheses of Theorem 1.1 with $H = \infty$ one concludes that, $h_1 = h_2 = \infty$ and $q_1 = q_2$ on* $[0, 1]$.

Proof of Theorem 3.1: it is a direct modification of the proof of Theorem 1.1 when setting

$$f(z) = z^2 \int_0^a (\psi(x, z, q_1)\psi(x, z, q_2))(q_1(x) - q_2(x))dx$$

where $\psi(\cdot, z, q)$ defined on $[0, 1]$ is the solution to $-\frac{d^2\psi}{dx^2} + q\psi = z^2\psi$ with the initial conditions $\psi(0, z, q) = 0$, $\psi'(0, z, q) = 1$.

Therefore we emphasize here on the main changes comparing to the case $(h < \infty, H < \infty)$. Note that the missing factor $\frac{1}{2}$ in the definition of f comes from the fact that $\int_0^1 q(x)dx$ is a spectral invariant in the Dirichlet case. The sequence of eigenvalues is denoted by $(\lambda_j(q))_{j \geq 1}$.

The point of adding the z^2 factor in the definition of f is the following. On one side, 0 is now a supplementary zero of order two for the function f compensating the missing eigenvalue $\lambda_0(q)$. Furthermore the leading term in the asymptotic expansion of $(\sqrt{\lambda_j(q)})$ is the same. In particular, when setting

$$S^{\frac{1}{2}} = \left\{ \ 0, \ 0, \ \pm\sqrt{s_j}, \ j \geq 1 \right\}$$

the same analysis as in [AFR, Section 4] holds and one obtains exactly the same estimates on $N_{S^{\frac{1}{2}}}$ as the ones in [AFR, Section 4]. On the other side, the functions $\psi(\cdot, z, q_1)$ and $\psi(\cdot, z, q_2)$ are written using a transformation operator with kernel L_1 and L_2 starting from $\frac{\sin zx}{z}$ instead of $\cos zx$. These two factors $\frac{1}{z}$ vanish with the added z^2 factor.

Moreover, the kernel L_j $(j = 1, 2)$ is simpler since it is essentially (up to a change of sign) the same kernel as in the case of finite h_j and H with $h_j = 0$ (see [M]). Therefore the results concerning the regularity properties of the kernels involved in Section 2 are unchanged and Proposition 1.4 holds for the function f defined above. □

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