Existence of a ground state for the confined hydrogen atom in non-relativistic QED

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Abstract. We consider a system of a hydrogen atom interacting with the quantized electromagnetic field. Instead of fixing the nucleus, we assume that the system is confined by its center of mass. This model is used in theoretical physics to explain the Lamb-Dicke effect. After a brief review of the literature, we explain how to verify some properly chosen binding conditions which, by [25], lead to the existence of a ground state for our model, and for all values of the fine-structure constant.

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EXISTENCE OF GROUND STATES IN NON-RELATIVISTIC QED

In quantum theories, the problem of the existence of stable systems is related, mathematically, to the existence of ground states for the Hamiltonians associated with the physical systems under consideration.

In the framework of quantum mechanics, the energy of an atom, a molecule or an ion can be described by a Schrödinger operator acting in a Hilbert space $\mathscr{H}_{part}$. If, for instance, the number of electrons N is less than the total number of protons Z, it is well-known that the bottom of the spectrum of the associated Schrödinger operator is an eigenvalue. Actually, under this assumption $N \leq Z$, it is proved that the spectrum of the Schrödinger operator consists of an infinite sequence of negative eigenvalues and of the essential spectrum $[0,\infty[$ (see [41]).

When considering an *atomic system* coupled to a quantized *massless Bose field*, the question whether a ground state exists or not is generally more subtle. Let us describe such a model by a Hamiltonian H acting in a Hilbert space $\mathscr{H}_{part} \otimes \mathscr{F}_s$ in the following way:

$$H = H_{part} \otimes I + I \otimes H_f + H_I(\alpha) = H_0 + H_I(\alpha). \tag{1}$$

The operator H_{part} acts in the Hilbert space $\mathscr{H}_{part}$ and corresponds to the energy of the atomic system; the operator H_f acts in the *Bosonic Fock space* $\mathscr{F}_s$ and corresponds to the energy of the quantized free bosons field; finally, the operator $H_I(\alpha)$ corresponds to the coupling between the atomic system and the field. Here α is the coupling constant.

Usually, for such a model, the bottom of the spectrum of the Hamiltonian is also the bottom of its essential spectrum. This is due to the fact that, in the case of massless fields, states with arbitrary low energy bosons can occur. Thus, even if the coupling parameter α is small, the ground state energy of the total Hamiltonian H is not isolated from the rest of the spectrum. This is a characteristic of non-relativistic quantum electrodynamics which makes difficult the problem of the existence of a ground state.

Actually, when considering *massive Bose fields*, a gap appears between the bottom and the rest of the spectrum of the unperturbed Hamiltonian H_0. Then for small coupling constants, the existence of a ground state for H follows from the usual perturbation theory. However, even in the massive case, for arbitrary values of α, the problem of the existence of a ground state remains difficult.

Recently a large number of papers have been devoted to the study of such matter. In [30], M. Hübner and H. Spohn proved the existence of a ground state for a *spin-boson model*, that is a two levels system *linearly coupled* to a *scalar* Bose field. In [4], A. Arai and M. Hirokawa got the result for what they called a generalized spin-boson model. These two results were obtained for small values of the coupling constant. In [16], J. Derezinski and C. Gérard got the existence of a ground state for some *confined* atomic system linearly coupled to a *massive* and scalar Bose field; no restriction on the fine structure constant was imposed. We used some of the tools developed in [16] in order to get the main result exposed in this paper. In [23], C. Gérard proved that, for all values of the coupling constant, a ground state exists for a system of one confined quantum particle linearly coupled to a scalar Bose field; this model is sometimes called *the Nelson model*. For all of these results with a linear coupling (see also [5], [38]), an *infrared regularization* was imposed on the interaction between the small system and the field. It is proved, indeed, that these models (with a linear coupling) are infrared divergent in the sense that no ground state exists in $\mathscr{H}_{part} \otimes \mathscr{F}_s$ if no infrared regularization is imposed (see [6], [32], [26]).

As for *the standard model of non-relativistic QED*, sometimes called *the Pauli-Fierz model*, it is proved that ground states can exist without an infrared regularization. The Pauli-Fierz model describes a quantum atomic system (a finite number of non-relativistic quantum particles) *minimally coupled* to the quantized radiation field. In [8, 10], V. Bach, J. Fröhlich and I. M. Sigal proved the existence of a ground state for the Pauli-Fierz model without an infrared regularization. They only needed to assume that $\inf \sigma(H_{part}) < \inf \sigma_{\text{ess}}(H_{part})$, where H_{part} denotes here the Schrödinger operator associated with the atomic system. Besides, their results hold for small values of the fine-structure constant and for *static* nuclei. Note that in [27], using different tools, F. Hiroshima also proved the existence of a ground state for the Pauli-Fierz model, with an infrared regularization; in [28], using functional integral methods, he studied the multiplicity of the ground state provided it exists (and provided that the non-relativistic particles under consideration are *spinless*). In [25], M. Griesemer, E. H. Lieb and M. Loss succeeded in removing the smallness condition on the fine-structure constant. To do this, they assumed that some *binding conditions* were satisfied. The nuclei were still treated as static in their work.

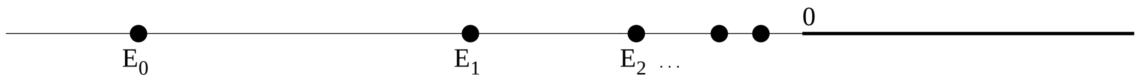

FIGURE 1. Spectrum of the hydrogen atom in quantum mechanics for a static nucleus

Finally in [31], the authors proved that these binding conditions are satisfied for all values of the fine-structure constant (and provided that $N \leq Z$).

Considering now *moving* nuclei, if no other restriction is imposed, the Hamiltonian is translation invariant in the sense that it commutes with the operator of the total momentum. Then no ground state can exist. However, the Hamiltonian admits a decomposition as a direct integral $H \simeq \int_{\mathbb{R}^3}^{\oplus} H(P) dP$ and one can ask wether, for a fixed total momentum P, $H(P)$ has a ground state or not. For some responses to this question in the case of Nelson's model or Pauli-Fierz model, we mention among other papers: [21], [22], [12], [34], [20], [3], [33].

THE LAMB-DICKE EFFECT

Let us consider a system of one nucleus and one electron (that is a hydrogen atom or, more generally, a hydrogenoïd ion) in quantum mechanics. The spins of the nucleus and the electron are neglected. We denote by x_j, p_j, q_j, m_j respectively the position, the momentum, the charge and the mass of the particle j (the electron or the nucleus). In particular, we set $q_1 = q$, $q_2 = -Zq$ where Z is the number of protons in the nucleus. The interaction between the two particles is described by the Coulomb potential that we write as:

$$V(x_1 - x_2) = -Zq^2 \frac{\mathrm{C}}{|x_1 - x_2|}, \tag{2}$$

where C is a positive constant.

First assume that the nucleus is treated as static. Then the spectrum of the Schrödinger operator associated with the hydrogen atom takes the form pictured in figure 1.

Consider now a more realistic model with a dynamical nucleus. Assume moreover that the center of mass motion of the atom is free (so that the system is translation invariant). Since both the total energy and the total momentum are conserved, if the atom is initially in some state of internal energy E_i, it may fall into a state of lower internal energy E_f by emitting a photon of energy

$$|k| = E_i - E_f - \frac{k^2}{2(m_1 + m_2)} + \frac{k.P}{m_1 + m_2}. \tag{3}$$

Here k denotes the momentum of the emitted photon and P denotes the momentum of the center of mass before the emission process. The units are such that the Planck constant

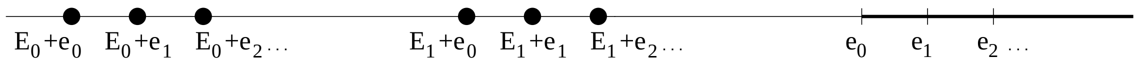

FIGURE 2. Spectrum of the confined hydrogen atom in quantum mechanics

$\hbar = h/2\pi$ and the velocity of light c are put equal to 1. The term $k^2/2(m_1+m_2)$ is a *recoil energy* whereas the term $k.P/(m_1+m_2)$ is due to the *Doppler effect*.

Suppose finally that the center of mass of the atom is confined by some potential U. Then the Schrödinger operator corresponding to the system acts on $\mathrm{L}^2(\mathbb{R}^6)$ and can be written as

$$H_{part} := \sum_{j=1,2} \frac{p_j^2}{2m_j} + U + V. \tag{4}$$

We define the variables R, P associated with the center of mass and the internal variables r, p by:

$$\begin{aligned} R &:= \frac{m_1x_1 + m_2x_2}{M} \quad , \quad P := p_1 + p_2, \\ r &:= x_1 - x_2 \quad , \quad \frac{p}{\mu} := \frac{p_1}{m_1} - \frac{p_2}{m_2}, \end{aligned} \tag{5}$$

where we have set $M := m_1 + m_2$ and $\mu := m_1m_2/(m_1+m_2)$. Then we can write H_{part} as

$$H_{part} \simeq \left(\frac{p^2}{2\mu} + V \right) \otimes I + I \otimes \left(\frac{P^2}{2M} + U \right) \tag{6}$$

as an operator on $\mathrm{L}^2(\mathbb{R}^3) \otimes \mathrm{L}^2(\mathbb{R}^3)$. Assuming to simplify that the spectrum of $P^2/2M + U$ is purely discrete (this is the case, for instance, if $U(R) \to \infty$ as $|R| \to \infty$), we get the spectrum of H_{part} as described in figure 2.

Let us consider, for instance, the system in an electronic excited state of energy E_1, and the energy associated with the center of mass motion in its lowest value e_0. Assuming that $e_1 - e_0 < E_1 - E_0$, we see that the initial energy of the atomic system $E_1 + e_0$ can decay into $E_0 + e_1$ by spontaneous emission of a photon of energy $E_1 - E_0 + e_0 - e_1$. Thus, in the scattering spectrum of the physical system, there are not only the rays corresponding to the electronic transitions of energies $E_{l_1} - E_{l_2}$, as it is the case if one assumes that the nucleus is static; new intense rays corresponding to the emission of photons of energies $E_{l_1} - E_{l_2} + e_{n_1} - e_{n_2}$ appear.

In the "Lamb-Dicke regime" (in particular, for a sufficiently "strong" confinement), it is proved that if $l_1 \neq l_2$, then the probability of emission of one photon with energy $E_{l_1} - E_{l_2} + e_{n_1} - e_{n_2}$ is negligible as $n_1 \neq n_2$. This means that the rays corresponding to the electronic transitions are much more intense than the other ones. As compared to the scattering spectrum of the free atom, the Doppler effect and the recoil energy have disappeared; this is what is called *the Lamb-Dicke effect*.

This effect has been discussed for the first time by R. H. Dicke in [17], where the reduction of the Doppler width of the light emitted by a dense gas is studied: it is assumed that the effect of collisions between the molecules of the gas is to confine the centers of mass of the emitters. Let us also mention that the Lamb-Dicke effect is used in theoretical physics to study the cooling of atoms or ions by lasers (see e.g. [40]). In [15, 14], a Pauli-Fierz Hamiltonian describing a confined hydrogen atom in non-relativistic QED is used to explain the Lamb-Dicke effect. This is this model that we consider in this paper.

THE MODEL OF THE CONFINED HYDROGEN ATOM

As mentioned above, we consider a moving hydrogen atom (or more generally one moving nucleus interacting with one electron) minimally coupled to the quantized radiation field. We assume that the nucleus and the electron are spinless, and the units are chosen so that $\hbar = c = 1$. Instead of fixing the nucleus, we make the more realistic assumption that the center of mass of the hydrogen atom is confined by some potential U.

The Hilbert space for the electron, the nucleus and the polarized photons is

$$\mathscr{H} := \mathrm{L}^2(\mathbb{R}^6) \otimes \mathscr{F}_s \simeq \int_{\mathbb{R}^6}^{\oplus} \mathscr{F}_s dX, \tag{7}$$

where $\mathscr{F}_s := \mathscr{F}_s(\mathrm{L}^2(\mathbb{R}^3;\mathbb{C}^2))$ is the Bosonic Fock space of transversally polarized photons, constructed over $\mathrm{L}^2(\mathbb{R}^3;\mathbb{C}^2)$, that is:

$$\mathscr{F}_s = \mathbb{C} \oplus \bigoplus_{n\geq 1} S_n \otimes_{k=1}^n \mathrm{L}^2(\mathbb{R}^3;\mathbb{C}^2). \tag{8}$$

Here S_n denotes the symetrization in the tensor product $\otimes_{k=1}^n \mathrm{L}^2(\mathbb{R}^3;\mathbb{C}^2)$.

The Hamiltonian that describes the system acts in $\mathscr{H}$ and is written formally as

$$H_U^V := \sum_{j=1,2} \frac{1}{2m_j}(p_j - q_j A_j)^2 + H_f + U + V. \tag{9}$$

The attractive Coulomb potential V acts on the internal variable r and is defined in (2). The confining potential U acts on the center of mass R; we require the following hypothesis (where $U^- = \max(-U,0)$ denotes the negative part of U):

$$(\mathscr{H}_0)\begin{cases} (i) & U \in \mathrm{L}^2_{\mathrm{loc}}(\mathbb{R}^3), \\ (ii) & \inf(U) > -\infty \text{ and } U^- \text{ is compactly supported}, \\ (iii) & P^2/2M + U \text{ has a ground state } \phi_U > 0 \text{ such that } \phi_U, \nabla\phi_U \in \mathrm{L}^\infty(\mathbb{R}^3). \end{cases}$$

The operator corresponding to the energy of the free radiation field, H_f, is written as

$$H_f = \sum_{\lambda=1,2} \int_{\mathbb{R}^3} |k| \widehat{a}^*_\lambda(k) \widehat{a}_\lambda(k) dk. \tag{10}$$

For $j = 1,2$, A_j denotes the quantized electromagnetic vector potential in the Coulomb gauge associated with the particle j. It is defined by:

$$A_j := \int^{\oplus}_{\mathbb{R}^6} A(x_j) dX, \tag{11}$$

with $X = (x_1, x_2)$ and

$$A(x) := \frac{1}{2\pi} \sum_{\lambda=1,2} \int_{\mathbb{R}^3} \frac{\widehat{\chi}_\Lambda(k)}{\sqrt{|k|}} \varepsilon_\lambda(k) \left(\widehat{a}^*_\lambda(k) e^{-ik.x} + \widehat{a}_\lambda(k) e^{ik.x} \right) dk. \tag{12}$$

In (10) and (12), $\widehat{a}^*_\lambda(k)$ and $\widehat{a}_\lambda(k)$ denote the usual creation and annihilation operators satisfying the Canonical Commutation Rules (in the sense of operator-valued distributions):

$$\begin{aligned} [\widehat{a}_\lambda(k), \widehat{a}^*_{\lambda'}(k')] &= \delta_{\lambda\lambda'} \delta(k-k'), \\ [\widehat{a}_\lambda(k), \widehat{a}_{\lambda'}(k')] &= [\widehat{a}^*_\lambda(k), \widehat{a}^*_{\lambda'}(k')] = 0. \end{aligned} \tag{13}$$

Moreover $\varepsilon_1(k)$ and $\varepsilon_2(k)$ are the orthonormal polarization vectors that we choose as

$$\varepsilon_1(k) := \frac{(k_2, -k_1, 0)}{\sqrt{k_1^2 + k_2^2}} \quad , \quad \varepsilon_2(k) := \frac{k}{|k|} \wedge \varepsilon_1(k). \tag{14}$$

Finally, Λ is the parameter of the ultraviolet cutoff, and $\widehat{\chi}_\Lambda$ is a real smooth function depending only on $|k|$, which is equal to 1 in the ball $B(0, \Lambda/2)$ and which vanishes outside the ball $B(0, \Lambda)$.

Denote by $Q(A)$ the domain of the quadratic form associated with the self-adjoint operator A. Then we can prove that:

Proposition 1 *For all values of q, Λ, H_U^V is a well-defined self-adjoint operator with form domain*

$$Q(H_U^V) = Q(p_1^2 + p_2^2) \cap Q(U^+) \cap Q(H_f), \tag{15}$$

where $U^+ := \max(U, 0)$ denotes the positive part of U.

RESULTS AND COMMENTS

The main results that we describe here are theorems 2 and 5 that lead to the existence of a ground state for H_U^V. A detailed proof is given in [2]; we propose here a slightly different sketch. Let us also mention that, mathematically, the Lamb-Dicke effect briefly described above is related to the problem of the existence of resonances for H_U^V. More precisely, one expects that the unperturbed eigenvalues $E_l + e_n$ turn into resonances when the coupling with the photons is added. We have been able to prove this in [18] by using *renormalization group* methods developed in [8, 9, 11]. Contrary to the existence of a ground state which is obtained for all values of q and Λ, these results about resonances hold for sufficiently small values of the fine-structure constant.

Let us write the ground state energy of H_U^V as

$$E_U^V := \inf_{\Phi \in Q(H_U^V), \|\Phi\|=1} (\Phi, H_U^V \Phi). \tag{16}$$

Moreover, we set $D_T := \left\{ \Phi \in Q(H_U^V), \Phi(X) = 0 \text{ if } |X| < T \right\}$ and

$$\Sigma_U^V := \lim_{T \to \infty} \left(\inf_{\Phi \in D_T, \|\Phi\|=1} (\Phi, H_U^V \Phi) \right). \tag{17}$$

By the general strategy of [25],

$$\left(E_U^V < \Sigma_U^V \right) \Rightarrow H_U^V \text{ has a ground state.} \tag{18}$$

Recall that, since V defined in (2) is the attractive Coulomb potential, we have:

$$V_\infty := \inf \sigma_{\text{ess}}(p^2/2\mu + V) = 0. \tag{19}$$

We define in the same way:

$$U_\infty := \inf \sigma_{\text{ess}}(P^2/2M + U). \tag{20}$$

Note that, if the spectrum of $P^2/2M + U$ is purely discrete, we have $U_\infty = \infty$. Moreover, by Persson's theorem, we have:

$$U_\infty = \lim_{T \to \infty} \left(\inf_{\phi \in D_{T,U}, \|\phi\|=1} (\phi, [P^2/2M + U]\phi) \right), \tag{21}$$

where $D_{T,U} := \left\{ \phi \in Q(P^2/2M + U), \phi(R) = 0 \text{ if } |R| < T \right\}$. Now, we define H_U^0 (respectively $H_{U_\infty}^V$ for $U_\infty < \infty$) as the Hamiltonian obtained by replacing V with $V_\infty = 0$ (respectively by replacing U with U_∞) in the definition of H_U^V. Moreover, E_U^0 (respectively $E_{U_\infty}^V$)

denotes the corresponding ground state energy. If $U_\infty = \infty$, we set $E^V_{U_\infty} = \infty$. It is easy to see that

$$\min\left(E^0_U, E^V_{U_\infty}\right) \leq \Sigma^V_U. \tag{22}$$

Then, together with (18), (22) yields:

$$\left(E^V_U < \min\left(E^0_U, E^V_{U_\infty}\right)\right) \Rightarrow H^V_U \text{ has a ground state,} \tag{23}$$

so that the point is to prove the following *binding conditions*:

$$\text{(B.C.)} \begin{cases} (i) & E^V_U < E^V_{U_\infty}, \\ (ii) & E^V_U < E^0_U. \end{cases}$$

The easiest part is condition $(\text{B.C.})(i)$:

Theorem 2 *We have*

$$E^V_U < E^V_{U_\infty}. \tag{24}$$

Sketch of the proof
Note that if $U_\infty = \infty$, the result is obvious. Now, if $U_\infty < \infty$, following [25, Theorem 3.2], we are actually able to prove a stronger result than (24), namely:

$$E^V_U \leq E^V_{U_\infty} + \inf \sigma(P^2/2M + U - U_\infty), \tag{25}$$

with $\inf \sigma(P^2/2M + U - U_\infty) < 0$ by hypothesis $(\mathscr{H}_0)$. The key point to derive (25) is to use the translation invariance of $H^V_{U_\infty}$ in the sense that for all $y \in \mathbb{R}^3$:

$$\left[H^V_{U_\infty}, e^{-iy.(p_1+p_2+d\Gamma(k))}\right] = 0. \tag{26}$$

□

In order to prove the second condition $(\text{B.C.})(ii)$, we can not use the same argument, since H^0_U is not translation invariant. Let us see where is the difficulty. Let Φ_j be a minimizing sequence for H^0_U, that is $\Phi_j \in Q(H^0_U)$, $\|\Phi_j\| = 1$ and

$$(\Phi_j, H^0_U \Phi_j) \underset{j\to\infty}{\longrightarrow} E^0_U. \tag{27}$$

Obviously, we can write

$$E^V_U \leq (\Phi_j, H^V_U \Phi_j) = (\Phi_j, H^0_U \Phi_j) + (\Phi_j, V\Phi_j), \tag{28}$$

so that the condition $(\text{B.C.})(ii)$ is not trivial only if $(\Phi_j, V\Phi_j) \to 0$. Indeed, we have:

Proposition 3 *Assume that*

$$\exists\rho > 0, \exists a > 0, \forall j, \int_{B(0,\rho)} \left[\int_{\mathbb{R}^3} \|\Phi_j(R,r)\|^2 dR \right] dr \geq a. \tag{29}$$

Then $E_U^V \leq E_U^0 - Zq^2 \mathrm{C} a/\rho$.

Sketch of the proof
The hypothesis (29) means that the probability of finding the nucleus and the electron in the ball $B(0,\rho)$ is positive uniformly in j. Thus $(\Phi_j, V\Phi_j)$ is negative uniformly in j:

$$(\Phi_j, V\Phi_j) \leq -Zq^2 \mathrm{C} a/\rho. \tag{30}$$

□

Now, if $(\Phi_j, V\Phi_j) \to 0$, that is, if the probability of finding the two particles in every ball of fixed radius goes to 0 as j goes to ∞, then the condition (B.C.)(ii) becomes much more subtle. Let us define a new Hamiltonian acting in $\mathrm{L}^2(\mathbb{R}^6; \mathscr{F}_s \otimes \mathscr{F}_s)$ by

$$\widetilde{H}_U^0 := \frac{1}{2m_1}(p_1 - q_1 A_1)^2 \otimes I + \frac{1}{2m_2} I \otimes (p_2 - q_2 A_2)^2 + H_f \otimes I + I \otimes H_f + U. \tag{31}$$

This operator corresponds to the energy of virtual states describing a nucleus and an electron interacting *independently* with a photon field. In other words, in such states, the cloud of photons which interacts with the nucleus does not interact with the electron and vice versa. Besides, the center of mass of the atomic system is still confined by the potential U. We denote by $\widetilde{E}_U^0$ the ground state energy of $\widetilde{H}_U^0$. Then we have:

Proposition 4 *Assume that*

$$\forall n \in \mathbb{N}^*, \exists j_n, \int_{B(0,n)} \int_{\mathbb{R}^3} \|\Phi_{j_n}(X)\|^2 dR dr \leq \frac{1}{n}. \tag{32}$$

Then $E_U^V < \widetilde{E}_U^0 \leq E_U^0$.

Sketch of the proof
Note that the inequality $E_U^V < \widetilde{E}_U^0$ is always satisfied. The assumption (32) becomes necessary to prove $\widetilde{E}_U^0 \leq E_U^0$. Furthermore, the proofs of the two inequalities $E_U^V < \widetilde{E}_U^0$ and $\widetilde{E}_U^0 \leq E_U^0$ appeal to quite similar arguments. The tools used to derive them are mainly borrowed to [31].
Let us, for instance, sketch the proof of $E_U^V < \widetilde{E}_U^0$. Let $\varepsilon > 0$. Pick an approximation $\widetilde{\Phi}$ of the ground state of $\widetilde{H}_U^0$, that is $\widetilde{\Phi} \in Q(\widetilde{H}_U^0)$, $\|\widetilde{\Phi}\| = 1$ and

$$(\widetilde{\Phi}, \widetilde{H}_U^0 \widetilde{\Phi}) \leq \widetilde{E}_U^0 + \varepsilon. \tag{33}$$

From this state $\widetilde{\Phi}$, we construct a new normalized state $\widetilde{\Phi}_{loc}$ where the nucleus, the electron and the photons are localized in the following sense: for all $L > 0$, there exists $y_1, y_2 \in \mathbb{R}^3$ such that

1. $\widetilde{\Phi}_{loc}(X) = 0$ on $\{X = (x_1, x_2), x_1 \notin B(y_1, L^\gamma)$ or $x_2 \notin B(y_2, L^\gamma)\}$ for some $1/2 < \gamma < 1$.
2. The cloud of photons interacting with x_1 is localized in $B(y_1, L)$.
3. The cloud of photons interacting with x_2 is localized in $B(y_2, L)$.
4. $(\widetilde{\Phi}_{loc}, \widetilde{H}_U^0 \widetilde{\Phi}_{loc}) \leq \widetilde{E}_U^0 + \varepsilon + o(L^{-1})$.

The next step in to construct, from $\widetilde{\Phi}_{loc} \in \mathrm{L}^2(\mathbb{R}^6; \mathscr{F}_s \otimes \mathscr{F}_s)$, a suitable state $\Phi_{loc} \in \mathrm{L}^2(\mathbb{R}^6; \mathscr{F}_s)$ which allows to compare E_U^V and $\widetilde{E}_U^0$. This is done by decomposing $\widetilde{\Phi}_{loc}(X)$ in a suitable basis of $\mathscr{F}_s \otimes \mathscr{F}_s$. More precisely, we consider an orthornormal basis $(f_i)_{i \geq 0}$ of $\mathrm{L}^2(\mathbb{R}^3; \mathbb{C}^2)$ and we write $\widetilde{\Phi}_{loc}(X)$ as

$$\widetilde{\Phi}_{loc}(X) = \sum_{\substack{n \geq 0 \\ n' \geq 0}} \sum_{\substack{i_1 < i_2 < \cdots < i_n \\ i'_1 < i'_2 < \cdots < i'_{n'}}} \sum_{\substack{p_1, \ldots, p_n \\ p'_1, \ldots, p'_{n'}}} \Phi_{\substack{i_1, p_1; \ldots; i_n, p_n \\ i'_1, p'_1; \ldots; i'_{n'}, p'_{n'}}}(X) |i_1, p_1; \ldots; i_n, p_n\rangle \otimes |i'_1, p'_1; \ldots; i'_{n'}, p'_{n'}\rangle,$$

where

$$|i_1, p_1; \ldots; i_n, p_n\rangle = \frac{1}{\sqrt{p_1! \ldots p_n!}} a^*(f_{i_1})^{p_1} \ldots a^*(f_{i_n})^{p_n} \Omega. \tag{34}$$

Here $a^*(f_{i_j})$ is the creation operator in $\mathscr{F}_s$ associated with the one-photon state f_{i_j} and $\Omega := (1, 0, 0, \ldots)$ is the vacuum vector in $\mathscr{F}_s$. Then we define

$$\Phi_{loc}(X) = \sum_{\substack{n \geq 0 \\ n' \geq 0}} \sum_{\substack{i_1 < i_2 < \cdots < i_n \\ i'_1 < i'_2 < \cdots < i'_{n'}}} \sum_{\substack{p_1, \ldots, p_n \\ p'_1, \ldots, p'_{n'}}} \Phi_{\substack{i_1, p_1; \ldots; i_n, p_n \\ i'_1, p'_1; \ldots; i'_{n'}, p'_{n'}}}(X) |i_1, p_1; \ldots; i_n, p_n\rangle \hat{\otimes} |i'_1, p'_1; \ldots; i'_{n'}, p'_{n'}\rangle,$$

with

$$\begin{aligned} &|i_1, p_1; \ldots; i_n, p_n\rangle \hat{\otimes} |i'_1, p'_1; \ldots; i'_{n'}, p'_{n'}\rangle \\ &= \frac{1}{\sqrt{p_1! \ldots p_n!}} \frac{1}{\sqrt{p'_1! \ldots p'_{n'}!}} a^*(f_{i_1})^{p_1} \ldots a^*(f_{i_n})^{p_n} a^*(f_{i'_1})^{p'_1} \ldots a^*(f_{i'_{n'}})^{p'_{n'}} \Omega. \end{aligned} \tag{35}$$

Using these decompositions, we can show:

$$(\Phi_{loc}, H_U^V \Phi_{loc}) \leq (\widetilde{\Phi}_{loc}, \widetilde{H}_U^0 \widetilde{\Phi}_{loc}) + (\widetilde{\Phi}_{loc}, V \widetilde{\Phi}_{loc}) + \varepsilon. \tag{36}$$

Finally, the last crucial step is to use the fact that $\widetilde{H}_U^0$ is translation invariant in the sense that for all $y \in \mathbb{R}^3$:

$$\left[\widetilde{H}_U^0, e^{iy.\frac{m_2}{M}(p_1 + d\Gamma(k))} \otimes e^{-iy.\frac{m_1}{M}(p_2 + d\Gamma(k))} \right] = 0. \tag{37}$$

In other words, if one translates on one hand the nucleus with its cloud of photons in some direction, and on the other hand the electron with its cloud of photons in another direction, the energy of the state under consideration does not change provided that the position of the center of mass stays fixed. This allows to assume that in the state $\widetilde{\Phi}_{loc}$, x_1 and x_2 are localized in some balls $B(y_1,L)$ and $B(y_2,L)$ such that, for instance,

$$\mathrm{dist}(B(y_1,L),B(y_2,L)) = L. \tag{38}$$

This implies

$$(\widetilde{\Phi}_{loc}, V\widetilde{\Phi}_{loc}) \leq -Zq^2\frac{\mathrm{C}}{5L}, \tag{39}$$

and together with (36), this leads to $E_U^V < \widetilde{E}_U^0$. □

To conclude, propositions 3 and 4 prove that the second binding condition $(\mathrm{B.C.})(ii)$ is satisfied:

Theorem 5 *We have*

$$E_U^V < E_U^0. \tag{40}$$

REFERENCES

1. L. Amour and J. Faupin. L'ion hydrogénoïde confiné en électrodynamique quantique non relativiste. *C. R. Acad. Sc. Paris*, Ser. I 343, 2006.
2. L. Amour and J. Faupin. The confined hydrogenoïd ion in non-relativistic quantum electrodynamics. *Cubo Math. J.*, to appear.
3. L. Amour, B. Grébert and J. C. Guillot. The dressed mobile atoms and ions. *J. Math. Pures et Appl.*, **86**:177-200, 2006.
4. A. Arai and M. Hirokawa. On the existence and uniqueness of ground states of a generalized spin-boson model. *J. Funct. Annal.*, **151**:455-503, 1997.
5. A. Arai and M. Hirokawa. Ground states of a general class of qunatum field Hamiltonians. *Rev. Math. Phys.*, **12**(8):1085-1135, 2000.
6. A. Arai, M. Hirokawa and F. Hiroshima. On the absence of eigenvectors of Hamiltonians in a class of massless quantum field models without infrared cutoffs. *J. Funct. Annal.*, **168**:470-497, 1999.
11. V. Bach, T. Chen, J. Fröhlich and I. M. Sigal. Smooth Feshbach map and operator-theoretic renormalization group methods. *J. Funct. Anal.*, **203**:44-92, 2002.
8. V. Bach, J. Fröhlich and I. M. Sigal. Quantum electrodynamics of confined non-relativistic particles. *Adv. in Math.*, **137**:299-395, 1998.
9. V. Bach, J. Fröhlich and I. M. Sigal. Renormalization group analysis of spectral problems in quantum field theory. *Adv. in Math.*, **137**:205-298, 1998.
10. V. Bach, J. Fröhlich and I. M. Sigal. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation fields. *Comm. Math. Phys.*, **207**(2):249-290, 1999.
11. V. Bach, J. Fröhlich, I. M. Sigal and A. Soffer. Positive commutators and spectrum of Pauli-Fierz Hamiltonians of atoms and molecules. *Comm. Math. Phys.*, **207**(4):557-587, 1999.
12. T. Chen. Operator-theoretic infrared renormalization and construction of dressed 1-particle states in non-relativistic QED. PhD thesis, ETH Zürich, 2002.

13. H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon. *Schrödinger Operators with application to quantum mechanics and global geometry*, Springer-Verlag, 1986.
14. C. Cohen-Tannoudji. Cours au collège de France 2003-2004, *www.phys.ens/cours/college-de-france*.
15. C. Cohen-Tannoudji, J. Dupont-Roc and G. Grynberg. *Processus d'interaction entre photons et atomes*. Edition du CNRS, 2001.
16. J. Derezinski and C. Gérard. Asymptotic completeness in quantum field theory. Massive Pauli-Fierz Hamiltonians. *Rev. Math. Phys.*, **11**(4):383-450, 1999.
17. R. H. Dicke. The Effect of Collisions upon the Doppler Width of Spectral Lines. *Phys. Rev.*, **89**(2), 472-473.
18. J. Faupin. Resonances of the confined hydrogenoïd ion and the Dicke effect in non-relativistic quantum electrodynamics, *preprint mp_arc*, 06-344, 2006.
19. J. Fröhlich, M. Griesemer and B. Schlein. Asymptotic completeness for Rayleigh scattering. *Ann. Henri Poincaré*, **3**: 107-170, 2002.
20. J. Fröhlich, M. Griesemer and B. Schlein. Asymptotic completeness for Compton scattering. *Comm. Math. Phys.*, **252**: 415-476, 2004.
21. J. Fröhlich. On the infrared problem in a model of scalar electrons and massless scalar bosons. *Ann. Henri Poincaré*, **19**:1-103, 1973.
22. J. Fröhlich. Existence of dressed one-electron states in a class of persistent models. *Fortschr. Phys.*, **22**:159-198, 1974.
23. C. Gérard. On the existence of ground states for massless Pauli-Fierz Hamiltonians. *Ann. Henri Poincaré*, **1**:443-459, 2000.
24. M. Griesemer. Exponential decay and ionization thresholds in non-relativistic quantum electrodynamics, *J. Funct. Annal.*, **210**:321-340, 2002.
25. M. Griesemer, E. H. Lieb and M. Loss. Ground states in non-relativistic quantum electrodynamics. *Invent. Math.*, **145**:557-595, 2001.
26. M. Hirokawa. Infrared Catastrophe for Nelson's Model. *Publications of RIMS*, **42**(4):897-922, 2006.
27. F. Hiroshima. Ground states of a model in nonrelativistic quantum electrodynamics. I. *J. math. phys.*, **40**(12):6209-6222, 1999.
28. F. Hiroshima. Ground states of a model in nonrelativistic quantum electrodynamics. II. *J. math. phys.*, **41**(2):661-674, 2000.
29. F. Hiroshima. Analysis of ground states of atoms interacting with a quantized radiation field. *Topics in the theory of Schrödinger Operators, ed. H. Araki and H. Ezawa*, 145-272, 2004.
30. M. Hübner and H. Spohn. Spectral properties of the spin-boson Hamiltonian. *Ann. Inst. H. Poincaré Phys. Théor.*, **62**(3):289-323, 1995.
31. E. Lieb and M. Loss. Existence of atoms and molecules in non-relativstic quantum electrodynamics. *Adv. Theor. Math. Phys.*, **7**:667-710, 2003.
32. J. Lorinczi, R. A. Minlos and H. Spohn. The infrared behaviour in Nelson's model of a quantum particle coupled to a massless scalar field. *Ann. Henri Poincaré*, **3**(2):269-295, 2002.
33. M. Loss, T. Myao and H. Spohn. Lowest energy states in nonrelativistic QED: atoms and ions in motion, *preprint mp_arc*, 06-136 ,2006.
34. J. S. Møller. The translation invariant massive Nelson model: I. The bottom of the spectrum. *Ann. Henri Poincaré*, **6**:1091-1135, 2005.
35. M. Reed and B. Simon. *Methods of modern mathematical physics, vol. I, Functional Analysis*, second edition, Academic Press, New York, 1980.
36. M. Reed and B. Simon. *Methods of modern mathematical physics, vol. II, Fourier Analysis, Self-adjointness*, Academic Press, New York, 1975.
37. M. Reed and B. Simon. *Methods of modern mathematical physics, vol. IV, Analysis of operators*, Academic Press, New York, 1978.
38. H. Spohn. Ground state of a quantum particle coupled to a scalar Bose field. *Lett. Math. Phys.*, **44**(1):9-16, 1998.

39. H. Spohn. *Dynamics of charged particles and their radiation field*, Cambridge university press, 2004.
40. D. J. Wineland and W. M. Itano. Laser cooling of atoms. *Phys. Rev. A*, **20**(4), 1521-1540, 1979.
41. G. M. Zhislin. Discussion of the spectrum of the Schrödinger Operator. *Tr. Mosk. Math. Obs.*, **9**:81-128, 1960.