Inverse spectral results for Schrödinger operators on the unit interval with partial informations given on the potentials

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Abstract

We pursue the analysis of the Schrödinger operator on the unit interval in inverse spectral theory initiated in [AR]. Whereas the potentials in [AR] belong to L^1 with their difference in L^p $(1 \leq p < \infty)$ we consider here potentials in $W^{k,1}$ spaces having their difference in $W^{k,p}$ where $1 \leq p \leq +\infty$, $k \in \{0, 1, 2\}$. It is proved that two potentials in $W^{k,1}([0,1])$ being equal on $[a, 1]$ are also equal on $[0, 1]$ if their difference belongs to $W^{k,p}([0, a])$ and if the number of their common eigenvalues is sufficiently high. Naturally, this number decreases as the parameter a decreases and as the parameters k and p are increasing.

1 Introduction and statement of the results

In this paper we consider the Schrödinger operator

$$A_{q,h,H} = -\frac{d^2}{dx^2} + q \tag{1}$$

defined on $[0, 1]$ associated with the following boundary conditions,

$$u'(0) + hu(0) = 0, \qquad u'(1) + Hu(1) = 0. \tag{2}$$

In (2) and throughout the paper we use the abbreviated notation $'$ for the derivative with respect to x and h, H are real numbers. In (1) the potential q is a real-valued function belonging to $L^1([0, 1])$. For each $(q, h, H) \in L^1([0, 1]) \times \mathbb{R}^2$ it is known that the operator $A_{q,h,H}$ is self-adjoint in $L^2([0, 1])$ and we denote by $\sigma(A_{q,h,H})$ the spectrum of this operator. Moreover, $\sigma(A_{q,h,H})$ is an increasing sequence of eigenvalues $(\lambda_j(q, h, H))_{j\in\mathbb{N}\cup\{0\}}$, each eigenvalue being of multiplicity one. The asymptotic expansion of the eigenvalues is as follows ([LG]),

$$\lambda_j(q, h, H) = j^2\pi^2 \, + \, 2(H - h) \, + \, \int_0^1 q(x)dx \, + \, o(1) \quad \text{as } j \to +\infty. \tag{3}$$

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For any sequence $\alpha = (\alpha_j)_{j\in\mathbb{N}\cup\{0\}}$, with $\alpha_j \in \mathbb{C}$, and for any $t \geq 0$, let $n_\alpha(t)$ denotes

$$n_\alpha(t) = \sharp\{j \in \mathbb{N}\cup\{0\} \quad | \quad |\alpha_j| \leq t\}. \tag{4}$$

The main result of the paper is the following.

Theorem 1.1. *Let $k \in \{0,1,2\}$. Fix $q_1, q_2 \in W^{k,1}([0,1])$ and $h_1, h_2, H \in \mathbb{R}$. Consider an infinite set S*

$$S \subseteq \sigma(A_{q_1,h_1,H}) \cap \sigma(A_{q_2,h_2,H}). \tag{5}$$

Fix $a \in (0, \frac{1}{2}]$ and $p \in [1, +\infty]$. Suppose that $q_1 = q_2$ on $[a,1]$ and $q_1 - q_2 \in W^{k,p}([0,a])$. Assume that

$$n_S(t) \;\geq\; 2a\, n_{\sigma(A)}(t) - \frac{k}{2} + \frac{1}{2p} - \frac{1}{2} - a,\ t \in \sigma(A),\ t \text{ large enough}, \quad (H)$$

where the operator A denotes either $A_{q_1,h_1,H}$ or $A_{q_2,h_2,H}$. Then $h_1 = h_2$ and $q_1 = q_2$.

In the case $p = +\infty$ the term $\frac{1}{p}$ in the hypothesis (H) is omitted.

Remark 1.2. *Theorem 1.1 remains true when replacing the assumption (H) by the hypothesis:*

there exists a real number C such that

$$2a\, n_{\sigma(A)}(t) + C \;\geq\; n_S(t) \;\geq\; 2a\, n_{\sigma(A)}(t) - \frac{k}{2} + \frac{1}{2p} - 2a,\ t \in S,\ t \text{ large enough}, \quad (H').$$

For $a = \frac{1}{2}$ the lower bounds in the assumptions (H) and (H') are the same. Nevertheless, when $a < \frac{1}{2}$ and if the known spectrum is in some sense regularly spaced then (H') becomes useful. For example, in [AR] we show that in the L^1 case the even (respectively odd) spectrum determines the potential on $[0, \frac{1}{4}]$ using (H') (with $k = 0$ and $p = 1$) whereas it is not possible with (H).

One recovers Theorem 1.1 in [AR] from Theorem 1.1 above with the assumption (H') setting $k = 0$ and $1 \leq p < +\infty$. We remark that the case $p = +\infty$ is excluded in [AR] whereas it is allowed here, the reason being that the details of the two proofs are different.

Let us mention some already known results related to Theorem 1.1. In 1978, [HL] proved that if $\sigma(A_{q_1,h,H}) = \sigma(A_{q_2,h,H})$ and if $q_1 = q_2$ on $[\frac{1}{2}, 1]$ then $q_1 = q_2$ on $[0,1]$ (for L^1 potentials). This is Theorem 1.1 setting $(a,k,p) = (\frac{1}{2}, 0, 1)$ in the particular case $h_1 = h_2 = h$. In 1980 it is derived in [H] when q_1 and q_2 are continuous near $x = \frac{1}{2}$ that $q_1 = q_2$ on $[0,1]$ and $h_2 = h_1$ under the assumptions $\sigma(A_{q_1,h_1,H}) = \sigma(A_{q_2,h_2,H})$ excepted for at most one eigenvalue and $q_1 = q_2$ on $[\frac{1}{2}, 1]$. In 2000, these two results were largely extended in [GS].

Theorem 1.1 with $(k,p) = (0,1)$ (which is also Theorem 1.1 in [AR] setting $p = 1$) is related to Theorem 1.3 in [GS]. See [AR] for comparisons between these two results. In another result it is proved in [GS] that $q_1 = q_2$ on $[0,1]$ and $h_1 = h_2$ if q_1 and q_2 are C^{2k} near $x = a$, if $q_1 = q_2$ on $[a,1]$ and assuming that $n_S(t) \geq\ 2an_{\sigma(A)}(t) - (k+1) + \frac{1}{2} - a$, $t \in \mathbb{R}$ large enough. Note that instead of $t \in \mathbb{R}$, in (H) and (H') it suffices to consider $t \in \sigma(A)$ and S respectively, which can be useful (see the proof of corollary 1.2 in [AR]). In particular ($a = \frac{1}{2}$), the potential already known on one half of the interval together with its spectrum except possibly $k+1$ eigenvalues determine uniquely the potential on the other half of the interval when the potential is C^{2k} near the middle of the interval.

We now emphasize that Theorem 1.1 admits the following corollary.

Corollary 1.3. *Fix $H, h_1, h_2 \in \mathbb{R}$. Suppose that q_1 and q_2 belongs to $L^1([0,1])$ and are equal on $\left[\frac{1}{2}, 1\right]$. If $\sigma(A_{q_1,h_1,H}) = \sigma(A_{q_2,h_2,H})$ excepted for at most one eigenvalue and if the difference $q_1 - q_2$ belongs to $L^\infty(\left[0, \frac{1}{2}\right])$ then $h_1 = h_2$ and $q_1 = q_2$.*

It is an immediate consequence of Theorem 1.1 in the particular case $a = \frac{1}{2}$, $k = 0$, $p = +\infty$.

The above corollary is already known ([H]) when q_1 and q_2 are continuous near $x = \frac{1}{2}$ (see above) whereas the condition here is $q_1 - q_2 \in L^\infty([0, \frac{1}{2}])$.

Similarly, Theorem 1.1 implies that if $\sigma(A_{q_1,h_1,H}) = \sigma(A_{q_2,h_2,H})$ excepted for $k+1$ eigenvalues, if q_1 and q_2 belong to $W^{2k,1}([0,1])$ and are equal on $\left[\frac{1}{2}, 1\right]$, if the difference $q_1 - q_2$ is in $W^{2k,\infty}(\left[0, \frac{1}{2}\right])$, then $h_1 = h_2$ and $q_1 = q_2$. This holds here for $k = 0, 1$ and we believe that it should be valid for all $k \in \mathbb{N} \cup \{0\}$. This type of results ($k+1$ eigenvalues missing from the known part of the spectrum) appears in [GS] for potentials being in C^{2k} near $x = \frac{1}{2}$ (see above). More generally (for any $a \in (0, \frac{1}{2}]$ and not only for $a = \frac{1}{2}$) one also notes that Theorem 1.1 with $p = +\infty$ and the result in [GS] have exactly the same assumption on $n_S(t)$. This common assumption is $n_S(t) \geq\ 2an_{\sigma(A)}(t) - k - \frac{1}{2} - a$ ($t \in \sigma(A)$ in Theorem 1.1 and $t \in \mathbb{R}$ in [GS]). They differ from the hypotheses on the potential: $q_1, q_2 \in W^{2k,1}$, $q_1 - q_2 \in W^{2k,\infty}$ in Theorem 1.1 and $q_1, q_2 \in C^{2k}$ near $x = a$ in [GS]. Therefore Theorem 1.1 in the particular case $p = +\infty$ and the result in [GS] are close but different.

At this point we note that our proof for Theorem 1.1 is different from the proofs of the results in [GS].

As it is explained in [H], Theorem 1.1 with $a = \frac{1}{2}$ and $k = 0$ is closely related to inverse problem for the Earth which consists in the determination of the density, the incompressibility and the rigidity in the lower mantle, the upper mantle and the crust. However it is supposed in [H] that the density, the incompressibility and the rigidity are twice differentiable. Then one may think that Theorem 1.1 with $a = \frac{1}{2}$ and $k = 2$ could be used for further analysis of this problem and in particular to remove some eigenvalues of the torsional spectrum, as it is mentioned in [H].

The starting point of the proof of Theorem 1.1 is related to the proof found in [L] that two spectra determine the potential, that is to say $\sigma(A_{q_1,h_1,H}) = \sigma(A_{q_2,h_2,H'})$ and $\sigma(A_{q_1,h'_1,H}) = \sigma(A_{q_2,h'_2,H'})$ imply $q_1 = q_2$, $h_1 = h_2$, $h'_1 = h'_2$ and $H = H'$. Let us recall the main steps of the proof of this result. First an entire function f is introduced having the property to vanish on the known part of the common spectra. Next it is derived that f is identically vanishing from the maximum modulus principle. Then it is proved that $f \equiv 0$ imply that $q_1 = q_2$ (see also step 5 in [AR] for a quick proof of this point). Here we start similarly and introduce the entire function f (see (7)). Then the goal of the paper is to derive that if f vanishes on S then f is identically vanishing. This is established thanks to some precise estimates on the growth of f proved in Proposition 3.1. These estimates are related to the Paley-Wiener Theorem. In other words, the entire function f of a given growth vanishes if it has a sufficiently large number of zeros. This point is effectuated using Jensen formula.

This article is organized as follows. In Section 2 we recall elementary results to be used in Section 3. Section 3 is concerned with the proof of a global estimate of the entire function f. In Section 4 we derive Theorem 1.1.

2 Preliminaries

The asymptotic expansion (3) of the eigenvalues for $q \in L^1([0,1])$ gives the spectral invariant $\int_0^1 q(x)dx + 2(H-h)$. In particular, since S is an infinite set,

$$\int_0^1 q_1(x)dx - \int_0^1 q_2(x)dx = 2(h_1 - h_2). \tag{6}$$

Let $q \in L^1([0,1])$ and $H, h \in \mathbb{R}$. For $z \in \mathbb{C}$, let $\psi(\cdot, z, q, h)$ defined on $[0,1]$ as the solution to $-\frac{d^2\psi}{dx^2} + q\psi = z^2\psi$, $\psi(0) = 1$, $\psi'(0) = -h$. It is known that $\psi(x, \cdot, q, h)$ is an entire function ([LG]).

For all $z \in \mathbb{C}$ set

$$f(z) = \int_0^a \left(\psi(x, z, q_1, h_1)\psi(x, z, q_2, h_2) - \frac{1}{2} \right) (q_1(x) - q_2(x))dx. \tag{7}$$

In order to globally estimate the entire function f on $\mathbb{C}$ let us define $y_1(x, z, q)$ and $y_2(x, z, q)$ as the solutions to

$$-\frac{d^2 y_1}{dx^2}(x, z, q) + q(x)y_1(x, z, q) = z^2 y_1(x, z, q), \quad x \in [0,1]$$
$$y_1(0, z, q) = 1, \qquad y_1'(0, z, q) = 0$$

and

$$-\frac{d^2 y_2}{dx^2}(x, z, q) + q(x)y_2(x, z, q) = z^2 y_2(x, z, q), \quad x \in [0,1]$$
$$y_2(0, z, q) = 0, \qquad y_2'(0, z, q) = 1$$

for $q \in L^1([0,1])$, $z \in \mathbb{C}$. It is known that $y_l(\cdot, \cdot, q)$ are analytic on $[0,1] \times \mathbb{C}$ for $l = 1, 2$.

In particular, one observes that

$$\psi(x,z,q,h) = y_1(x,z,q) - h y_2(x,z,q) \tag{8}$$

for all $(x,z) \in [0,1] \times \mathbb{C}$.

Moreover it is clear that z^2 is an eigenvalue of $A_{q,h,H}$ if and only if $\psi'(1,z,q,h) + H\psi(1,z,q,h) = 0$. Furthermore, if z^2 is an eigenvalue of $A_{q,h,H}$ then $\psi(\cdot,z,q,h)$ is up to a normalization coefficient the corresponding eigenfunction.

It is known (see [PT] for similar computations with potentials in $L^2([0,1])$) that there exist $c_j^{(l)}(x,z,q)$'s such that

$$y_l(x,z,q) = \sum_{j=0}^{+\infty} c_j^{(l)}(x,z,q)$$

for $l = 1,2$ and

$$y_l(x,z,q) = \sum_{j=0}^{k} c_j^{(l)}(x,z,q) + O\left(\frac{e^{|\Im z|x}}{|z|^{k+l}}\right) \tag{9}$$

for $l = 1,2$, $k \in \mathbb{N}$ and all $(x,z,q) \in [0,1] \times \mathbb{C} \times L^1([0,1])$. Let us mention that

$$c_0^{(1)}(x,z,q) = c_0^{(1)}(x,z) = \cos zx, \qquad c_0^{(2)}(x,z,q) = c_0^{(2)}(x,z) = \frac{\sin zx}{z}$$

and

$$c_j^{(l)}(x,z,q) = \int_{0 \le t_1 \le \cdots \le t_{j+1} = x} c_0^{(l)}(t_1,z) \prod_{m=1}^{j} \left(c_0^{(2)}(t_{m+1}-t_m,z) q(t_m)\right) dt_1 \ldots dt_j.$$

The coefficients $c_j^{(l)}(x,z,q)$ above satisfy

$$|c_0^{(1)}(x,z)| \le e^{|\Im z|x}, \qquad |c_0^{(2)}(x,z)| \le e^{|\Im z|x}$$

and for $j \ge 1$

$$|c_j^{(l)}(x,z,q)| \le \frac{||q||^j_{L^1([0,1])} e^{|\Im z|x}}{j!|z|^{j+l-1}}.$$

Note that the factor $\frac{1}{|z|^{j+l-1}}$ is not explicitly mentioned in [PT] but it will be useful for our purpose.

We shall modify (9) for $k = 1,2$ in order to prove Theorem 1.1. Integrating by parts the $c_j^{(l)}(x,z,q)$ for $l = 1,2$ one deduce (see also section 1 in [PT]) the expansions of y_1 and y_2

$$y_1(x,z,q) = \sum_{j=0}^{k} C_j(x,z,q) + O\left(\frac{e^{|\Im z|x}}{|z|^{k+1}}\right) \tag{10}$$

and

$$y_2(x,z,q) = \sum_{j=0}^{k-1} D_j(x,z,q) + O\left(\frac{e^{|\Im z|x}}{|z|^{k+1}}\right) \tag{11}$$

uniformly in $x \in \mathbb{R}$ and $z \in \mathbb{C}$ under the assumption $q \in W^{k,1}([0,1])$. The coefficients $C_j(x,z,q)$ and $D_j(x,z,q)$ are defined below.

Let us first define the following functions Q (resp. R) by

$$Q(x) = \int_0^x q(t)\,dt, \qquad \text{(resp. } R(x) = q(x) - q(0) - \frac{1}{2}Q^2(x)),$$

for $x \in [0,1]$ provided that the potential $q \in L^1$ (resp. $q \in W^{1,1}$).

The coefficients $C_j(x,z,q)$ and $D_j(x,z,q)$ for $j \le k$ when $q \in W^{k,1}$ $(k = 0,1,2)$ are defined by

$$\begin{aligned} C_0(x,z,q) = C_0(x,z) &= \cos zx \\ C_1(x,z,q) &= \frac{\sin zx}{2z}\,Q(x) \\ C_2(x,z,q) &= \frac{\cos zx}{4z^2}\,R(x) \end{aligned} \tag{12}$$

and

$$\begin{aligned} D_0(x,z,q) = D_0(x,z) &= \frac{\sin zx}{z} \\ D_1(x,z,q) &= -\frac{\cos zx}{2z^2}\,Q(x) \end{aligned} \tag{13}$$

for $x \in \mathbb{R}$ and $z \in \mathbb{C}$. Using the inequalities

$$|\cos zx| \le e^{|\Im z|x}, \qquad |\sin zx| \le e^{|\Im z|x}$$

for $x \in \mathbb{R}$ and $z \in \mathbb{C}$ we verify that

$$C_j(x,z,q) = O\left(\frac{e^{|\Im z|x}}{|z|^j}\right) \tag{14}$$

for $j = 0,1,2$ and

$$D_j(x,z,q) = O\left(\frac{e^{|\Im z|x}}{|z|^{j+1}}\right) \tag{15}$$

for $j = 0,1$ and uniformly in $x \in \mathbb{R}$ and $z \in \mathbb{C}$.

The computations (integrations by parts) in order to get useful expressions for the coefficients $C_3(x,z,q)$'s and $D_2(x,z,q)$'s when $q \in W^{3,1}$ would be rather complicated. Furthermore the expression for the corresponding k_3 would be even more complicated (see next section for the definitions of k_0, k_1, k_2). Then we restrict ourselves to the three cases $k = 0,1,2$.

3 Estimation of the entire function f

This section is devoted to the proof of the next proposition which states a global estimate related to the parameters k, p, a on the function f. Let us recall that throughout this section $\frac{1}{p} + \frac{1}{p'} = 1$ including the cases p or p' being infinite.

Proposition 3.1. *Fix $k \in \{0, 1, 2\}$ and let $p \in [1, +\infty]$. Fix $q_1, q_2 \in W^{k,1}([0, 1])$ such that $q_1 - q_2 \in W^{k,p}([0, a])$ and $h_1, h_2, H \in \mathbb{R}$. In the case $k = 2$ we assume that $\sigma(A_{q_1,h_1,H}) \cap \sigma(A_{q_2,h_2,H})$ is infinite. Also suppose that $q_1 = q_2$ on $[a, 1]$. There is a real positive number C depending only on p and $||q_1 - q_2||_{W^{k,p}([0,a])}$ such that for any $\varepsilon > 0$ there exists a real positive number δ_ε depending only on ε, p, a and $||q_1 - q_2||_{W^{k,p}([0,a])}$ verifying*

$$\lim_{\varepsilon \to 0} \delta_\varepsilon = 0$$

and

$$|f(z)| \leq C \frac{e^{2|\Im z|a}}{|\Im z|^{k+\frac{1}{p'}}} (e^{-\varepsilon|\Im z|} + \delta_\varepsilon)$$

for all $z \in \mathbb{C}$ and where $\frac{1}{p} + \frac{1}{p'} = 1$.

Let us mention here that the above inequality is also valid when $\varepsilon = 0$ and that the role of $\varepsilon > 0$ will appear in next section.

The fact that $\operatorname{supp}(q_1 - q_2) \subset [0, a]$ appears in the term $e^{2|\Im z|a}$ above. The regularity of the potentials $q_1, q_2 \in W^{k,1}$ gives the power k in denominator and the regularity of the difference $q_1 - q_2 \in W^{k,p}$ provides an additional power $\frac{1}{p'}$. This estimate is in some sense related to the Paley-Wiener theorem. Nevertheless the factor $e^{2|\Im z|a}$ will not be sufficient for our purpose and an improved estimate is needed. Therefore the factor $e^{-\varepsilon|\Im z|}$ is introduced.

Set

$$\begin{aligned}
k_0(x, z) &= C_0(x, z)C_0(x, z) - \frac{1}{2} \\
k_1(x, z, q_1, h_1, q_2, h_2) &= C_0(x, z)C_1(x, z, q_2) + C_1(x, z, q_1)C_0(x, z) \\
&\quad - h_1 D_0(x, z)C_0(x, z) - h_2 C_0(x, z)D_0(x, z) \\
k_2(x, z, q_1, h_1, q_2, h_2) &= C_0(x, z)C_2(x, z, q_2) + C_2(x, z, q_1)C_0(x, z) \\
&\quad + C_1(x, z, q_1)C_1(x, z, q_2) \\
&\quad - h_2 C_0(x, z)D_1(x, z, q_2) - h_2 D_1(x, z, q_1)C_0(x, z) \\
&\quad - h_2 C_1(x, z, q_1)D_0(x, z) - h_1 D_0(x, z)C_1(x, z, q_2) \\
&\quad + h_1 h_2 D_0(x, z)D_0(x, z),
\end{aligned} \tag{16}$$

for all $(x,z) \in [0,1] \times \mathbb{C}$.

Then we define

$$K_l(z) = \int_0^a k_l(x,z)(q_1(x) - q_2(x))dx$$

for all $z \in \mathbb{C}$.

From (8) together with (10) (11) and (14) (15) (16), one can easily check that

$$\psi(x,z,q_1,h_1)\psi(x,z,q_2,h_2) - \frac{1}{2} = \sum_{l=0}^{k} k_l(x,z,q_1,h_1,q_2,h_2) + O\left(\frac{e^{2|\Im z|x}}{|z|^{k+1}}\right). \tag{17}$$

We shall use the result below appearing in the proof of Theorem A.III.1.3 in [L].

Lemma 3.2. *Let $a \in (0,1]$. Suppose that the function u defined on $[0,1]\times\mathbb{C}$ satisfy $|u(x,z)| = O\left(e^{2|\Im z|x}\right)$ and let $v \in L^p([0,1])$ with $1 \le p \le +\infty$. Set $w(z) = \int_0^a u(x,z)v(x)dx$. There is a real positive number C depending only on p and $||v||_{L^p([0,a])}$ such that for any $\varepsilon > 0$ there is a real positive number δ_ε depending only on ε, p, a and $||v||_{L^p([0,a])}$ verifying*

$$\lim_{\varepsilon\to 0} \delta_\varepsilon = 0$$

and

$$|w(z)| \le C\frac{e^{2|\Im z|a}}{|\Im z|^{\frac{1}{p'}}}(e^{-\varepsilon|\Im z|} + \delta_\varepsilon).$$

This result follows directly from Hölder inequality (including the case $p = \infty$). Indeed, $|w(z)| \le C'\int_I e^{2|\Im z|x}|v(x)|dx + C'\int_J e^{2|\Im z|x}|v(x)|dx \le C'(||v||_{L^p([0,a])}\frac{e^{2|\Im z|(a-\varepsilon)}}{|\Im z|^{\frac{1}{p'}}} + ||v||_{L^p(J)}\frac{e^{2|\Im z|a}}{|\Im z|^{\frac{1}{p'}}})$ where $I = [0, a-\varepsilon]$ and $J = [a-\varepsilon, a]$ with $0 < \varepsilon < a$. Here C' is a real number which may vary from line to line. Thus, if v is not identically vanishing then δ_ε equals $||v||_{L^p(J)}$ up to a numerical multiplicative factor.

We easily deduce the following proposition.

Proposition 3.3. *Fix $a \in (0, \frac{1}{2}]$, $k = 0,1,2$ and suppose that $q_1, q_2 \in W^{k,1}([0,1])$. Then there is C depending only on $||q_1 - q_2||_{L^1}$ such that for any $\varepsilon > 0$ there is δ_ε depending only on ε and $||q_1 - q_2||_{L^1}$ satisfying $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$ and*

$$\left|f(z) - \sum_{l=0}^{k} K_l(z)\right| \le C\frac{e^{2|\Im z|a}}{|\Im z|^{k+1}}(e^{-\varepsilon|\Im z|} + \delta_\varepsilon) \tag{18}$$

for all $z \in \mathbb{C}$.

Proof of Proposition 3.3:

Following the definition of f together with equality (17) we have

$$f(z) = \int_0^a \left\{ \sum_{l=0}^{k} k_l(x, z, q_1, h_1, q_2, h_2) + O\left(\frac{e^{2|\Im z|x}}{|z|^{k+1}} \right) \right\} \{q_1(x) - q_2(x)\} \, dx$$

for all $z \in \mathbb{C}$. Applying Lemma 3.2 with $v = q_1 - q_2$ we finish the proof (Note that Lemma 3.2 is only used to get a factor $e^{-\varepsilon|\Im z|}$). $\square$

Proof of Proposition 3.1. The case $k = 0$:

Following Proposition 3.3 it remains to check that $|K_0(z)| \le C \frac{e^{2|\Im z|a}}{|\Im z|^{\frac{1}{p'}}} (e^{-\varepsilon|\Im z|} + \delta_\varepsilon)$ when $q_1, q_2 \in L^1([0,1])$ and $q_1 - q_2 \in L^p([0,a])$. This clearly follows from $k_0(x,z) = \frac{\cos 2zx}{2}$, the definition of K_0 together with Lemma 3.2 applied with $u(x,z) = \cos 2zx$ and $v = q_1 - q_2$. $\square$

Proof of Proposition 3.1. The case $k = 1$:

From Proposition 3.3 it suffices to verify that $|K_l(z)| \le C \frac{e^{2|\Im z|a}}{|\Im z|^{1+\frac{1}{p'}}} (e^{-\varepsilon|\Im z|} + \delta_\varepsilon)$ when $q_1, q_2 \in W^{1,1}([0,1])$ and $q_1 - q_2 \in W^{1,p}([0,a])$ for $l = 0, 1$. Since $q_1 - q_2 \in W^{1,1}([0,1])$ we may integrate by parts $K_0(z)$ to obtain

$$K_0(z) = \frac{\sin 2zx}{4z}(q_1(x) - q_2(x))|_0^a - \int_0^a \frac{\sin 2zx}{4z}(q_1'(x) - q_2'(x))dx.$$

Since $q_1(a) = q_2(a)$ then the first term in the r.h.s. of the above equality vanishes and one concludes using Lemma 3.2 with $v = q_1' - q_2' \in L^p([0,a])$ that $|K_0(z)| \le C \frac{e^{2|\Im z|a}}{|\Im z|^{1+\frac{1}{p'}}} (e^{-\varepsilon|\Im z|} + \delta_\varepsilon)$.

Set $Q_j(x) = \int_0^x q_j(t)dt$, $j = 1, 2$. Similarly

$$K_1(z) = \int_0^a \frac{\sin 2zx}{4z}(Q_1(x) + Q_2(x) - 2(h_1 + h_2))(q_1(x) - q_2(x))dx$$

also verify $|K_1(z)| \le C \frac{e^{2|\Im z|a}}{|\Im z|^{1+\frac{1}{p'}}} (e^{-\varepsilon|\Im z|} + \delta_\varepsilon)$ using again Lemma 3.2 with $v = (Q_1 + Q_2 - 2(h_1 + h_2))(q_1 - q_2) \in L^p$ being the product of function in L^∞ with a function in L^p. $\square$

It remains to consider the case $k = 2$ which is much longer.

We shall first prove the following proposition.

Proposition 3.4. *Under the assumptions of Proposition 3.1 (with $k = 2$) there exists a real number $L(q_1, h_1, q_2, h_2)$ and there is a real positive number C depending only on p and $||q_1 - q_2||_{W^{2,p}([0,a])}$ such that for any $\varepsilon > 0$ there exists a real positive number δ_ε depending only on ε, p, a and $||q_1 - q_2||_{W^{2,p}([0,a])}$ verifying* $\lim_{\varepsilon \to 0} \delta_\varepsilon = 0$ *and*

$$\left| f(z) - \frac{L(q_1, h_1, q_2, h_2)}{z^2} \right| \le C \frac{e^{2|\Im z|a}}{|\Im z|^{2+\frac{1}{p'}}} (e^{-\varepsilon|\Im z|} + \delta_\varepsilon).$$

Proposition 3.4 follows from Propositions 3.3, 3.5, 3.7 and 3.9 below.

In order to derive proposition 3.4 let us estimate K_l for $l = 0, 1, 2$ in the next three propositions.

Proposition 3.5. *Let $p \in [1, +\infty]$. Fix $q_1, q_2 \in W^{2,1}([0,1])$ with $q_1 - q_2 \in W^{2,p}([0,a])$ and $h_1, h_2, H \in \mathbb{R}$. Suppose that $q_1 = q_2$ on $[a, 1]$. Set $L_0(q_1, q_2) = -\frac{1}{8}(q_1'(0) - q_2'(0))$. There is a real positive number C depending only on p and $||q_1'' - q_2''||_{L^p([0,a])}$ such that for any $\varepsilon > 0$ there is a real positive number δ_ε depending only on ε, p, a and $||q_1'' - q_2''||_{L^p([0,a])}$ satisfying $\lim_{\varepsilon \to 0} \delta_\varepsilon = 0$ and*

$$\left| K_0(z) - \frac{L_0(q_1, q_2)}{z^2} \right| \le C \frac{e^{2|\Im z| a}}{|\Im z|^{2+\frac{1}{p'}}} (e^{-\varepsilon |\Im z|} + \delta_\varepsilon).$$

Note that since $q_j \in W^{2,1}([0,1])$ then q_j' has a continuous representant for all $j = 1, 2$. Therefore $L_0(q_1, q_2)$ is well-defined.

Proof of Proposition 3.5:

Since $q_j \in W^{2,1}([0,1])$ we may integrate by parts twice $K_0(z)$ for each $z \in \mathbb{C}$. Using $q_1(a) = q_2(a)$ and $q_1'(a) = q_2'(a)$ we remark using two integrations by parts that

$$K_0(z) = -\frac{q_1'(0) - q_2'(0)}{8z^2} - \int_0^a \frac{\cos 2zx}{8z^2} (q_1''(x) - q_2''(x)) dx.$$

Using Lemma 3.2 and $q_1 - q_2 \in W^{2,p}([0,a])$ for each $\varepsilon > 0$ there is δ_ε ($\delta_\varepsilon \to 0$ as $\varepsilon \to 0$) such that the second term in the r.h.s. of the above equality is now bounded by $\dfrac{e^{2|\Im z| a}}{|\Im z|^{2+\frac{1}{p'}}} (e^{-\varepsilon |\Im z|} + \delta_\varepsilon)$. □.

In the proof of Proposition 3.7 we shall need the following lemma.

Lemma 3.6. *Suppose $q_1, q_2 \in W^{2,1}([0,1])$ with their difference $q_1 - q_2 \in W^{2,p}([0,1])$ and assume that $q_1(a) = q_2(a)$ ($a \in [0,1]$). Then $(Q_1 + Q_2)(q_1 - q_2) \in W^{1,p}([0,1])$.*

Here $Q_j(x) = \int_0^x q_j(t) dt$, $j = 1, 2$.

Proof of Lemma 3.6:

Since $(Q_1 + Q_2)$ and $(q_1 - q_2)$ belongs to $W^{1,1}([0,1])$ then $(Q_1 + Q_2)(q_1 - q_2) \in W^{1,1}([0,1])$ and

$$((Q_1 + Q_2)(q_1 - q_2))' = (q_1 + q_2)(q_1 - q_2) + (Q_1 + Q_2)(q_1 - q_2)'.$$

Using $Q_1 + Q_2 \in L^\infty$ and $(q_1 - q_2)' \in W^{1,1} \subset L^p$ we deduce that the second term above is in $L^p([0,1])$. Let us check that the first term also belongs to $L^p([0,1])$. □.

We have for all $x \in [0,1]$,

$$q_1^2(x) - q_2^2(x) = 2\int_a^x q_1(t)q_1'(t) - q_2(t)q_2'(t)dt \quad (19)$$

$$= \int_a^x (q_1(t)-q_2(t))(q_1'(t)+q_2'(t)) + (q_1(t)+q_2(t))(q_1'(t)-q_2'(t))dt, \quad (20)$$

since $q_1^2(a) = q_2^2(a)$.

Since $q_j \in W^{2,1}$ then $q_j \in W^{1,\infty}$ for $j = 1,2$ and we have

$$|q_1^2(x) - q_2^2(x)| \leq C\int_a^x |q_1(t) - q_2(t)| + |q_1'(t) - q_2'(t)|dt$$

for all $x \in [0,1]$.

Here C is a positive real number which may vary from line to line. Then Hölder inequality shows

$$|q_1^2(x) - q_2^2(x)| \leq C(||q_1 - q_2||_{L^p([0,1])} + ||q_1' - q_2'||_{L^p([0,1])}),$$

for all $x \in [0,1]$.

Thus, a convexity inequality shows that

$$|q_1^2(x) - q_2^2(x)|^p \leq C(||q_1 - q_2||^p_{L^p([0,1])} + ||q_1' - q_2'||^p_{L^p([0,1])})$$

for all $x \in [0,1]$ and integrate this inequality over $[0,1]$ to obtain Lemma 3.6. □

Proposition 3.7. *Let $p \in [1,+\infty]$. Fix $q_1, q_2 \in W^{2,1}([0,1])$ with $q_1 - q_2 \in W^{2,p}([0,a])$ and $h_1, h_2, H \in \mathbb{R}$. Set $L_1(q_1,q_2,h_1,h_2) = -\frac{1}{4}(h_1+h_2)(q_1(0)-q_2(0))$. There is a real positive number C depending only on p and $||q_1 - q_2||_{W^{1,p}([0,a])}$ such that for any $\varepsilon > 0$ there exists a real positive number δ_ε depending only on ε, p, a and $||q_1 - q_2||_{W^{1,p}([0,a])}$ satisfying $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$ and*

$$\left|K_1(z) - \frac{L_1(q_1,q_2,h_1,h_2)}{z^2}\right| \leq C\frac{e^{2|\Im z|a}}{|\Im z|^{2+\frac{1}{p'}}}(e^{-\varepsilon|\Im z|} + \delta_\varepsilon).$$

Proof of Proposition 3.7:

Following (12)(13) and (16)

$$k_1(x,z) = \frac{\sin 2zx}{2z}A(x)$$

where

$$A(x) = \frac{Q_1(x)+Q_2(x)}{2} - (h_1+h_2),$$

for all $x \in [0,1]$ and all $z \in \mathbb{C}$. From Lemma 3.6 we have $A(q_1 - q_2) \in W^{1,p}$. Since $A(q_1 - q_2) \in W^{1,1}$ we may integrate by parts $K_1(z)$ for each $z \in \mathbb{C}$ to obtain,

$$K_1(z) = -\frac{1}{4z^2}(h_1+h_2)(q_1(0)-q_2(0)) + \int_0^a \frac{\cos 2zx}{4z^2}(A(x)(q_1(x)-q_2(x)))'dx,$$

for all $(x, z) \in [0,1] \times \mathbb{C}$.

Since $A(q_1 - q_2) \in W^{1,p}([0,a])$ we derive from Lemma 3.2 that for each $\varepsilon > 0$ there is δ_ε going to zero as $\varepsilon \to 0$ such that the second term above is now bounded by $\dfrac{e^{2|\Im z|a}}{|\Im z|^{2+\frac{1}{p'}}}(e^{-\varepsilon|\Im z|} + \delta_\varepsilon)$. □.

Set

$$\begin{aligned} B_1 &= R_1 + R_2 + 2h_1Q_1 + 2h_2Q_2 \\ B_2 &= Q_1Q_2 - 2h_2Q_1 - 2h_1Q_2 + 4h_1h_2 \\ B &= B_1 + B_2 \end{aligned} \tag{21}$$

on $[0,1]$, where

$$R_j(x) = q_j(x) - q_j(0) - \frac{1}{2}Q_j^2(x),$$

for $x \in [0,1]$ and $j = 1,2$.

Lemma 3.8. *Suppose that $q_1, q_2 \in W^{2,1}([0,1])$ with $q_1 - q_2 \in W^{2,p}([0,1])$. Then we have $B_j(q_1 - q_2) \in L^p([0,1])$ for $j = 1,2$.*

Proof of Lemma 3.8:

Since $Q_1, Q_2 \in L^\infty([0,1])$ it suffice to check that $(R_1 + R_2)(q_1 - q_2) \in L^p([0,1])$.

From

$$\begin{aligned}(R_1(x) + R_2(x))(q_1(x) - q_2(x)) &= q_1^2(x) - q_2^2(x) - (q_1(0) + q_2(0))(q_1(x) - q_2(x)) \\ &\quad - \frac{1}{2}(Q_1^2(x) + Q_2^2(x))(q_1(x) - q_2(x))\end{aligned}$$

and similarly as in the proof of Lemma 3.6 we derive Lemma 3.8. □

Proposition 3.9. *Let $p \in [1, +\infty]$. Fix $q_1, q_2 \in W^{2,1}([0,1])$ with $q_1 - q_2 \in W^{2,p}([0,a])$ and $h_1, h_2, H \in \mathbb{R}$. Set $L_2(q_1, q_2, h_1, h_2) = \frac{1}{8}\int_0^a B(x)(q_1(x) - q_2(x))dx$. There is a real positive number C depending only on p and $||q_1 - q_2||_{L^p([0,a])}$ such that for any $\varepsilon > 0$ there exists a real positive number δ_ε depending only on ε, p, a and $||q_1 - q_2||_{L^p([0,a])}$ satisfying $\lim_{\varepsilon\to 0}\delta_\varepsilon = 0$ and*

$$\left|K_2(z) - \frac{L_2(q_1, q_2, h_1, h_2)}{z^2}\right| \leq C\frac{e^{2|\Im z|a}}{|\Im z|^{2+\frac{1}{p'}}}(e^{-\varepsilon|\Im z|} + \delta_\varepsilon).$$

Proof of Proposition 3.9:

From (16) and (21)

$$k_2(x,z) = \frac{\cos^2 zx}{4z^2}B_1(x) + \frac{\sin^2 zx}{4z^2}B_2(x)$$

$$= \frac{B_1(x)+B_2(x)}{8z^2} + \frac{\cos 2zx}{8z^2}B_1(x) - \frac{\sin 2zx}{8z^2}B_2(x).$$

From Lemma 3.6 and Lemma 3.8

$$\left|\int_0^a \frac{\cos 2zx}{8z^2}B_1(x)(q_1(x)-q_2(x))dx\right| \leq C\frac{e^{2|\Im z|a}}{|\Im z|^{2+\frac{1}{p'}}}(e^{-\varepsilon|\Im z|}+\delta_\varepsilon)$$

and

$$\left|\int_0^a \frac{\sin 2zx}{8z^2}B_2(x)(q_1(x)-q_2(x))dx\right| \leq C\frac{e^{2|\Im z|a}}{|\Im z|^{2+\frac{1}{p'}}}(e^{-\varepsilon|\Im z|}+\delta_\varepsilon)$$

for all $z \in \mathbb{C}$ and where C and δ_ε are defined in the statement of the proposition.

This and the definition of L_2 prove Proposition 3.9. □

In particular Proposition 3.5, 3.7 and 3.9 imply Proposition 3.4 with

$$L(q_1,h_1,q_2,h_2) = L_0(q_1,q_2) + L_1(q_1,h_1,q_2,h_2) + L_2(q_1,h_1,q_2,h_2).$$

In order to get Proposition 3.1 in the case $k=2$ from Proposition 3.4 it suffices to verify the next result which follows directly from Riemann-Lebesgue lemma.

Proposition 3.10. *If $\sigma(A_{q_1,h_1,H}) \cap \sigma(A_{q_2,h_2,H})$ is infinite then*

$$L(q_1,h_1,q_2,h_2) = 0.$$

Proof of Proposition 3.10:

We first remark that the following estimates holds (for $z \in \mathbb{R}$)

$$f(z) = \frac{L(q_1,h_1,q_2,h_2)}{z^2} + o\left(\frac{1}{z^2}\right) \qquad \text{as } z \to +\infty. \tag{22}$$

Indeed, in order to get (22) one may use Riemann-Lebesgue lemma instead of using Lemma 3.2 in the proofs of Proposition 3.5, 3.7 and 3.9.

Let $(t_j)_{j\in\mathbb{N}}$ be an infinite increasing sequence in $\sigma(A_{q_1,h_1,H}) \cap \sigma(A_{q_2,h_2,H})$. It is known that

$$\forall j \in \mathbb{N}, \qquad f(t_j) = 0. \tag{23}$$

Indeed let $z^2 = t_j$ and $\psi(x,z,q_1,h_1)$ and $\psi(x,z,q_2,h_2)$ be the corresponding eigenfunctions, multiply $\left(-\frac{d^2}{dx^2}+q_1(x)-z^2\right)\psi(x,z,q_1,h_1)=0$ by $\psi(x,z,q_2,h_2)$, multiply $\left(-\frac{d^2}{dx^2}+q_2(x)-z^2\right)\psi(x,z,q_2,h_2)=0$ by $\psi(x,z,q_1,h_1)$ and integrate their difference on $[0,1]$ to obtain that $f(z)$ equals $2(h_1-h_2)+\int_0^1 q_2(x)-q_1(x)dx$. This term is zero from (6). Consequently, (6) shows (23). Therefore (22) and (23) prove Proposition 3.10. $\square$.

Proof of Proposition 3.1. The case $k=2$:

It now follows from Proposition 3.4 and Proposition 3.10. $\square$

As a complementary result we have

Proposition 3.11. *Let $q \in W^{2,1}([0,1])$ and $h \in \mathbb{R}$. Define*

$$L^\sharp(q,h) = -q'(0) - 4hq(0) + \int_0^1 q^2(t)dt + 4h^2\int_0^1 q(t)dt - 2h\left(\int_0^1 q(t)dt\right)^2 + \frac{1}{3}\left(\int_0^1 q(t)dt\right)^3.$$

Then

$$8L(q_1,h_1,q_2,h_2) = L^\sharp(q_1,h_1) - L^\sharp(q_2,h_2).$$

In particular $L^\sharp(q,h)$ is a spectral invariant.

This means that if $q_1,q_2 \in W^{2,1}([0,1])$, $H,h_1,h_2 \in \mathbb{R}$ and if $\sigma(A_{q_1,h_1,H}) = \sigma(A_{q_2,h_2,H})$ then $L^\sharp(q_1,h_1) = L^\sharp(q_2,h_2)$. This spectral invariant is very probably related to the spectral invariant $c_3(q,h,H)$ appearing in the asymptotic expansion of the eigenvalues $\lambda_j(q,h,H)$ when $q \in W^{2,1}([0,1])$

$$\sqrt{\lambda_j(q,h,H)} = j\pi + \frac{c_1(q,h,H)}{j\pi} + \frac{c_3(q,h,H)}{j^3\pi^3} + o\left(\frac{1}{j^3}\right)$$

as $j \to +\infty$ in the sense that $c_3(q_1,h_1,H) = c_3(q_2,h_2,H) \Rightarrow L^\sharp(q_1,h_1) = L^\sharp(q_2,h_2)$ in the same way that $c_1(q,h,H) = H - h + \frac{1}{2}\int_0^1 q(x)dx$ is a spectral invariant and that $c_1(q_1,h_1,H) = c_1(q_2,h_2,H) \Rightarrow \int_0^1 q_1(x)dx + 2h_1 = \int_0^1 q_2(x)dx + 2h_2$.

The spectral invariant $L^\sharp(q,h)$ is probably well-known but we are not able to indicate an exact reference. In particular, we have not seen the coefficient c_3 written down explicitly for the boundary conditions (2) (for the existence of c_3 and more generally for the existence of c_{2j+1} when the potential is sufficiently regular, see Section 5.6.1 in [LS], Problems in Section 5 of [M], or [LG]. See also [KP] and the references therein for the spectral invariants in the periodic case related to the KdV hierarchy). The proof of Proposition 3.11 is obtained by direct computations and we omit it. Furthermore, if the coefficient c_3 was known, Proposition 3.11 would provide an alternative proof (instead of the use of Riemann-Lebesgue lemma) of the equality $L(q_1,h_1,q_2,h_2) = 0$ in Proposition 3.10.

4 Proof of Theorem 1.1

The purpose of this section is to deduce Theorem 1.1 from Proposition 3.1 and the hypothesis (H) or the hypothesis (H').

Define the s_j, $j \in \mathbb{N}$ as the strictly increasing sequence being in S. For any $c \in \mathbb{R}$ the map $q \to q + c$ acts as $\lambda_j(q, h, H) \to \lambda_j(q, h, H) + c$ for each $j \in \mathbb{N}$. Therefore we suppose without loss of generality that all the s_j are strictly positive numbers. Therefore we may define the sets

$$S^{\frac{1}{2}} = \{\pm\sqrt{s_j},\ j \in \mathbb{N}\}, \quad S^{\frac{1}{2},+} = \{\sqrt{s_j},\ j \in \mathbb{N}\}.$$

We also set for any sequence of numbers α,

$$N_\alpha(R) = \int_0^R \frac{n_\alpha(t)}{t} dt,$$

for any $R > 0$ and where $n_\alpha(t)$ is defined in (4).

Proposition 4.1. *The hypothesis (H') implies that the sequence*

$$\left(N_{S^{\frac{1}{2}}}(\sqrt{s_j}) - \frac{4a}{\pi}\sqrt{s_j} + \left(k + 1 - \frac{1}{p}\right) \ln \sqrt{s_j}\right)_{j \in \mathbb{N}}$$

is bounded from below.

Proof of Proposition 4.1:

This is a straightforward modification of the arguments found in [AR], where $k = 0$. It suffice to replace $\frac{1}{p'}$ with $k + \frac{1}{p'}$ in (H_1), (H_2) and (H_L) in [AR] (where $\frac{1}{p} + \frac{1}{p'} = 1$). We also note that $p = +\infty$ is allowed. □

In the following, A is the operator chosen in Theorem 1.1. We also denote by (λ_j), $j \in \mathbb{N}$ the increasing sequence of its eigenvalues and $\sigma(A)^{\frac{1}{2},+}$ denote the sequence $(\sqrt{\lambda_j})$, $j \in \mathbb{N}$.

Proposition 4.2. *The assumption (H) implies that the sequence*

$$\left(N_{S^{\frac{1}{2}}}(\sqrt{\lambda_j}) - \frac{4a}{\pi}\sqrt{\lambda_j} + \left(k + 1 - \frac{1}{p}\right) \ln \sqrt{\lambda_j}\right)_{j \in \mathbb{N}}$$

is bounded from below.

In the proof of the Proposition 4.1 the assumption (H') is mainly used in order to derive an asymptotic expansion of the sequence $(\sqrt{s_j})$ (see (H_2) in [AR]). In particular, it is at this point that we need an estimate from above for the s_j's provided by the left inequality in (H'). Then $N_{S^{\frac{1}{2}}}$ is evaluated on the

$\sqrt{s_j}$'s. In the proof of the Proposition 4.2 we proceed slightly differently even if the computations are similar. We first use (H) to minorize $N_{S^{\frac{1}{2}}}(R)$ for all $R > 0$. We evaluate $N_{S^{\frac{1}{2}}}$ on the $\sqrt{\lambda_j}$'s. The point being that asymptotic expansion of the $\sqrt{\lambda_j}$'s is available without any supplementary assumption. Consequently, we do not need any estimate from above for the s_j's and there is no estimation from above in (H) for $n_S(t)$.

Proof of Proposition 4.2:

We have for any $R > 0$,

$$\begin{aligned}
N_{S^{\frac{1}{2}}}(R) &= \int_0^R \frac{n_{S^{\frac{1}{2}}}(t)}{t} dt \\
&= 2\int_0^R \frac{n_{S^{\frac{1}{2}},+}(t)}{t} dt \\
&\geq 2\int_0^R \frac{2an_{\sigma(A)^{\frac{1}{2}},+}(t) + \frac{1}{2p} - \frac{1}{2} - a - \frac{k}{2}}{t} dt,
\end{aligned}$$

where the last inequality follows from (H).

Consequently,

$$\begin{aligned}
N_{S^{\frac{1}{2}}}(\sqrt{\lambda_j}) &\geq 2\sum_{k=0}^{j-1} \int_{\sqrt{\lambda_k}}^{\sqrt{\lambda_{k+1}}} \frac{2a(k+1) + \frac{1}{2p} - \frac{1}{2} - a - \frac{k}{2}}{t} dt + O(1) \\
&= 4a\sum_{k=0}^{j-1}(k+1)(\ln\sqrt{\lambda_{k+1}} - \ln\sqrt{\lambda_k}) + \left(\frac{1}{p} - 1 - 2a - k\right)(\ln\sqrt{\lambda_{k+1}} - \ln\sqrt{\lambda_k}) + O(1) \\
&= 4a\sum_{k=0}^{j-1}((k+1)\ln\sqrt{\lambda_{k+1}} - k\ln\sqrt{\lambda_k}) - 4a\sum_{k=0}^{j-1}\ln\sqrt{\lambda_k} + O(1) \\
&+ \left(\frac{1}{p} - 1 - 2a - k\right)\ln\sqrt{\lambda_j} + O(1) \\
&= 4aj\ln\sqrt{\lambda_j} - 4a\sum_{k=0}^{j-1}\ln\sqrt{\lambda_k} + \left(\frac{1}{p} - 1 - 2a - k\right)\ln\sqrt{\lambda_j} + O(1),
\end{aligned} \tag{24}$$

for $j \in \mathbb{N}$. From (3) we see that there exist a positive real number C satisfying

$$j\pi - \frac{C}{j} \leq \sqrt{\lambda_j} \leq j\pi + \frac{C}{j}, \quad j \in \mathbb{N}. \tag{25}$$

Therefore,

$$j\ln\sqrt{\lambda_j} = j(\ln j + \ln\pi) + O(1), \quad \ln\sqrt{\lambda_j} = \ln j + O(1), \tag{26}$$

as $j \to +\infty$.

Moreover, using Stirling asymptotic expansion $\ln j! = \left(j+\frac{1}{2}\right)\ln j - j + O(1)$ as $j \to +\infty$ we obtain,

$$\begin{aligned}
\sum_{k=0}^{j-1} \ln\sqrt{\lambda_k} &\leq \sum_{k=1}^{j-1} \ln\left(k\pi + \frac{C}{k}\right) + O(1) \\
&\leq \sum_{k=1}^{j-1} \ln\pi + \ln k + \ln\left(1 + \frac{C}{k^2}\right)) + O(1) \\
&\leq j\ln\pi + \ln(j-1)! + O(1) \\
&= j\ln\pi + \left(j - \frac{1}{2}\right)\ln j - j + O(1),
\end{aligned} \tag{27}$$

as $j \to +\infty$.

Combining (24) with (26) and (27) we obtain the following estimate

$$N_{S^{\frac{1}{2}}}(\sqrt{\lambda_j}) \geq 4aj + \left(\frac{1}{p} - 1 - k\right)\ln j + O(1), \tag{28}$$

as $j \to +\infty$.

Turning back to the $\sqrt{\lambda_j}$'s, (28) reads as

$$N_{S^{\frac{1}{2}}}(\sqrt{\lambda_j}) \geq \frac{4a}{\pi}\sqrt{\lambda_j} + \left(\frac{1}{p} - 1 - k\right)\ln\sqrt{\lambda_j} + O(1),$$

as $j \to +\infty$. This concludes the proof. □

Proposition 4.3. *Suppose that the function f is not identically vanishing on $\mathbb{C}$. Then*

$$\lim_{R\to+\infty} N_{S^{\frac{1}{2}}}(R) - \frac{4a}{\pi}R + \left(k + 1 - \frac{1}{p}\right)\ln R = -\infty.$$

Proof of Proposition 4.3:

This is a consequence of Proposition 3.1 with Jensen's Theorem. This argument is borrowed to [L]. Let $n_f(t)$ be the number of zeros of the entire function f in the closed ball centered at the origin with radius $t > 0$. In the proof of Proposition 3.10 we recall that if z^2 belongs to S then $f(z) = 0$. Thus $n_{S^{\frac{1}{2}}}(t) \leq n_f(t)$ for all $t > 0$. Moreover, using the estimates in Proposition 3.1 in Jensen's Theorem we obtain

$$\int_0^R \frac{n_f(t)}{t}\,dt = \frac{4a}{\pi}R - \left(k + 1 - \frac{1}{p}\right)\ln R + \frac{1}{2\pi}\int_0^{2\pi} \ln(e^{-\varepsilon R|\sin\theta|} + \delta_\varepsilon)d\theta + O(1). \tag{29}$$

If R is large enough and ε is small enough (> 0) the third term in the r.h.s. of (29) is smaller than any negative number. □

Proof of Theorem 1.1:

If f is not identically vanishing then Proposition 4.1 or Proposition 4.2 together with Proposition 4.3 lead to a contradiction. Thus $f(z) = 0$ for all $z \in \mathbb{C}$. This implies using the short argument in [AR, step 5] that $q_1 = q_2$ and $h_1 = h_2$. $\square$

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