INTERIOR COMPONENTS OF A TILE ASSOCIATED TO A QUADRATIC CANONICAL NUMBER SYSTEM - PART II

JULIEN BERNAT, BENOÎT LORIDANT, AND JÖRG M. THUSWALDNER

Abstract. Let $\alpha = -2+\sqrt{-1}$ be a root of the polynomial $p(x) = x^2+4x+5$. It is well-known that the pair $(\alpha, \{0,1,2,3,4\})$ forms a *canonical number system*, *i.e.*, that each $\gamma \in \mathbb{Z}[\alpha]$ admits a finite representation of the shape $\gamma = a_0 + a_1\alpha + \cdots + a_\ell \alpha^\ell$ with $a_i \in \{0,1,2,3,4\}$. The set $\mathcal{T}$ of points with integer part 0 in this number system

$$\mathcal{T} := \left\{ \sum_{i=1}^{\infty} a_i \alpha^{-i},\ a_i \in \{0,1,2,3,4\} \right\}$$

is called the *fundamental domain* of this canonical number system. It is a plane continuum with nonempty interior which induces a tiling of $\mathbb{C}$. Moreover, it has a disconnected interior $\mathcal{T}^o$. In the first paper of this series we described the closures C_0, C_1, C_2 and C_3 of the four largest components of $\mathcal{T}^o$ as attractors of graph-directed self-similar sets. Each of these four sets is a translation of C_0. We conjectured that the closures of the other components are images of C_0 by similarity transformations. In this article we prove this conjecture. Moreover, we provide a graph from which the suitable similarities can be read off.

1. Introduction and statement of the main result

We study topological properties of a plane self-similar tile with disconnected interior. In particular, we describe the connected components of its interior (see Figure 1). In a first paper [18], the closure C_0 of the component containing the origin was described by a graph-directed construction. We now obtain all the other components: they are images of C_0 by similarity transformations that can be read off from a graph. Although we deal with a single example in the present work, we are confident that our method may be used for whole families of self-similar sets, such as classes of tiles associated to quadratic canonical number systems (the example in this paper is a tile related to this kind of number systems).

1.1. **The tile $\mathcal{T}$.** We first give the definition of the tile $\mathcal{T}$ that forms the main object of this paper. It is known (see [10, 15, 16]) that the root $\alpha = -2+\sqrt{-1}$ of the polynomial x^2+4x+5 together with $\mathcal{N} := \{0,1,2,3,4\}$ gives rise to a *canonical number system* (or *CNS*) $(\alpha, \mathcal{N})$, *i.e.*, each element $\gamma \in \mathbb{Z}[\alpha]$ has a unique representation

$$\gamma = \sum_{i=0}^{\ell(\gamma)} a_i \alpha^i$$

for some non-negative integer $\ell(\gamma)$ and $a_i \in \mathcal{N}$ with $a_{\ell(\gamma)} \neq 0$ for $\gamma \neq 0$. In the natural embedding

$$\begin{array}{rcl} \Phi : \mathbb{C} & \to & \mathbb{R}^2 \\ \gamma & \mapsto & (\Re(\gamma), \Im(\gamma)) \end{array}$$

the multiplication by α can be represented by the 2×2 matrix

$$\mathbf{A} := \begin{pmatrix} -2 & -1 \\ 1 & -2 \end{pmatrix},$$

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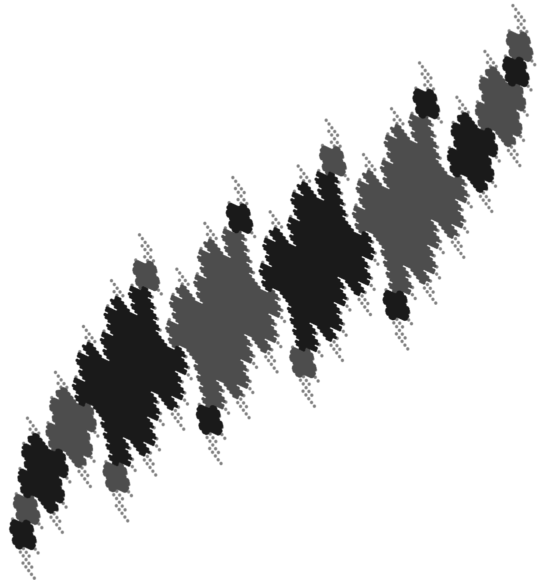

FIGURE 1. Closure of some inner components of the self similar tile $\mathcal{T}$.

i.e., for every $z \in \mathbb{C}$,

$$\Phi(\alpha z) = \mathbf{A}\Phi(z).$$

In these notations, the contractions

$$\psi_i(z) = \mathbf{A}^{-1}\left(z + \Phi(i)\right),\ z \in \mathbb{R}^2 \qquad (0 \le i \le 4) \tag{1.1}$$

form an iterated function system whose attractor

$$\mathcal{T} = \bigcup_{i=0}^{4} \psi_i(\mathcal{T}) \tag{1.2}$$

is a *tile*. Indeed, it is a self-similar connected compact set (or *continuum*) which is equal to the closure of its interior (see [14]). It induces a *tiling* of the plane by its translates. Remember that a *tiling* (*cf.* [12, 27]) of the plane is a decomposition of $\mathbb{R}^2$ into sets whose interiors are pairwise disjoint (or *non-overlapping sets*), each set being the closure of its interior and having a boundary of Lebesgue measure zero. Properties of tiles and tilings can be found for instance in [4, 8, 17, 25, 28]. It was shown in [14] that the family of sets

$$\{\mathcal{T} + \Phi(\omega),\ \omega \in \mathbb{Z}[\alpha]\} \tag{1.3}$$

is a tiling of the plane. For the special case of tiles related to canonical number systems, we refer to [6, 7, 15, 16] and to the survey [1]. Finally, among many topological results, it is shown in [2] that our tile $\mathcal{T}$ has disconnected interior. Ngai and Tang proved in [22] that the closure of each of its interior components is homeomorphic to a closed disk.

Our purpose is to describe the closure of all the connected components of $\mathcal{T}^o$ in terms of the natural subdivisions of $\mathcal{T}$, which are defined as follows. Let w be a *finite digit string*, that is $w = (a_1, \ldots, a_n)$ with $n \in \mathbb{N}$ and $a_i \in \mathcal{N}$. If w is empty, we write $w = \epsilon$. The integer n is then called the *length* of the string w (we write $|w| = n$, $|\epsilon| = 0$). For a finite string $w = (a_1, \ldots, a_n)$ we define the mapping ψ_w by

$$\psi_w(z) := \psi_{a_1} \circ \ldots \circ \psi_{a_n}(z) = \mathbf{A}^{-n}z + \sum_{i=1}^{n} \mathbf{A}^{-i}\Phi(a_i), \qquad z \in \mathbb{R}^2. \tag{1.4}$$

We add the convention $\psi_\epsilon = \mathrm{id}$, the identity mapping of $\mathbb{R}^2$. If $|w| = n$ the set $\psi_w(\mathcal{T})$ is called an *n-th level subpiece of* $\mathcal{T}$.

Iterating (1.2) we have for every $n \geq 1$ the *subdivision principle*

$$\mathcal{T} = \bigcup_{w,|w|=n} \psi_w(\mathcal{T}). \tag{1.5}$$

We finally remark that the set $\mathcal{T}$ consists of the points of integer part zero in the canonical number system $(\alpha, \mathcal{N})$ embedded into the plane:

$$\mathcal{T} := \left\{ \sum_{i=1}^{\infty} \Phi(\alpha^{-i} a_i),\ (a_i)_{i\in\mathbb{N}} \in \mathcal{N}^{\mathbb{N}} \right\} = \left\{ \sum_{i=1}^{\infty} \mathbf{A}^{-i} \Phi(a_i),\ (a_i)_{i\in\mathbb{N}} \in \mathcal{N}^{\mathbb{N}} \right\}. \tag{1.6}$$

Thus each point of this set can be represented by an *infinite string* $w = (a_1, a_2, a_3, \dots)$ with $a_i \in \mathcal{N}$. This representation is not always unique.

1.2. **Graphs $\mathcal{G}_d$ and pregraph $\mathcal{P}$.** The closures of the four biggest components of $\mathcal{T}^o$, *i.e.*, the sets C_0, C_1, C_2 and C_3 which can be seen in Figure 1 were computed in [18]. Their description relies on the graphs $\mathcal{G}_d$ ($d = 0, 1, 2, 3$) depicted in Figure 2. For the so-called *accepting state* $\circ$ in $\mathcal{G}_d$, there is by convention an edge $\circ \xrightarrow{a} \circ$ for every $a \in \mathcal{N}$. In particular, C_d is the set of all $x = \sum_{i\geq 1} \mathbf{A}^{-i} a_i$ such that $w = (a_1, a_2, \dots)$ is a labeling of a walk in $\mathcal{G}_d$ starting at F_d. In other words, C_d is the attractor of the graph-directed self-similar set associated to the state F_d of $\mathcal{G}_d$.

The sets C_0, C_1, C_2 and C_3 (which are translates of each other) can be used to describe all the components of $\mathcal{T}^o$. Indeed, we will show that the closure of any component is of the form $\psi_w(C_d)$, where w is a certain finite walk in the *pregraph* $\mathcal{P}$. This pregraph is depicted in Figure 3. To avoid too many crossings in the drawing of the graph, the node GG' at the top was duplicated (somewhat shaded) at the bottom of the figure. Thus GG' can be also reached from N_1 and N_1' by an edge with the label 2. The states S, G_1, G_1', I_1, I_1' are called *transition states*. They will connect the pregraph $\mathcal{P}$ to the graphs $\mathcal{G}_d$ ($d = 0, 1, 2, 3$). We will dwell on this more precisely later.

The graphs $\mathcal{G}_d$ ($d = 0, 1, 2, 3$) and $\mathcal{P}$ are *right resolving*, *i.e.*, each walk is uniquely defined by its starting state together with its labeling. Note that the graph $\mathcal{P}$ and the graphs $\mathcal{G}_d$ have a similar structure. The main differences are the following. In $\mathcal{P}$ outgoing edges have been added to those states which correspond to states in $\mathcal{G}_d$ that do not accept all letters $\{0, 1, 2, 3, 4\}$ on their outgoing edges. Moreover the accepting state was removed in $\mathcal{P}$. Thus, in some way, $\mathcal{P}$ and $\mathcal{G}_d$ ($0 \leq d \leq 3$) complement each other.

1.3. **Graph notations.** We will need the following notations related to our graphs. Let $\mathcal{H} \in \{\mathcal{P}, \mathcal{G}_0, \mathcal{G}_1, \mathcal{G}_2, \mathcal{G}_3\}$. A *walk* w in $\mathcal{H}$ is a sequence of edges

$$Z_1 \xrightarrow{a_1} Z_2 \xrightarrow{a_2} \dots \xrightarrow{a_n} Z_{n+1},$$

where n is an integer, and, for each i, Z_i is a state of $\mathcal{H}$ and $a_i \in \mathcal{N}$. We say that w starts at Z_1 and *ends* at Z_{n+1}. Since $\mathcal{H}$ is right resolving, we will write $w = (Z; a_1, \dots, a_n)$ for a walk w starting in Z with labeling $(a_1, \dots, a_n)$. If we emphasize on the labeling $(a_1, \dots, a_n)$ of a walk w we will just write $w = (a_1, \dots, a_n)$. For subsets of the walks in $\mathcal{H}$ we adopt the following notations:

$\mathcal{H}(Z_1)$	set of walks in $\mathcal{H}$ starting at node Z_1,
$\mathcal{H}(Z_1, Z_2)$	set of walks in $\mathcal{H}(Z_1)$ ending at node Z_2.

If w is a walk in $\mathcal{H}$ with labeling $(a_1, \dots, a_n)$, then we denote the walk which corresponds to w in the transposed graph $\mathcal{H}^T$ by w^T (backwards walk). Its labeling is obviously $(a_n, \dots, a_1)$. If w_1 and w_2 are two walks in $\mathcal{H}$ and w_2 starts at the ending state of w_1, then we write $w_1 \& w_2$ for the concatenation of these two walks. If we concatenate $w_1 = (Z_1; a_1, \dots, a_n)$ and $w_2 = (Z_2; b_1, \dots, b_m)$ we will often write $(Z_1, a_1, \dots a_n)\&(b_1, \dots, b_m)$ because the starting state of w_2 is defined via w_1. For a walk w of length n and $k \leq n$ we denote by $w|_k$ the prefix of w consisting of the first k edges of w, *i.e.*, $(a_1, \dots, a_n)|_k = (a_1, \dots, a_k)$.

If Z is a state of $\mathcal{H}$, we call Z' its *dual*. By convention the states S, GG', F_d, and $\circ$ are equal to

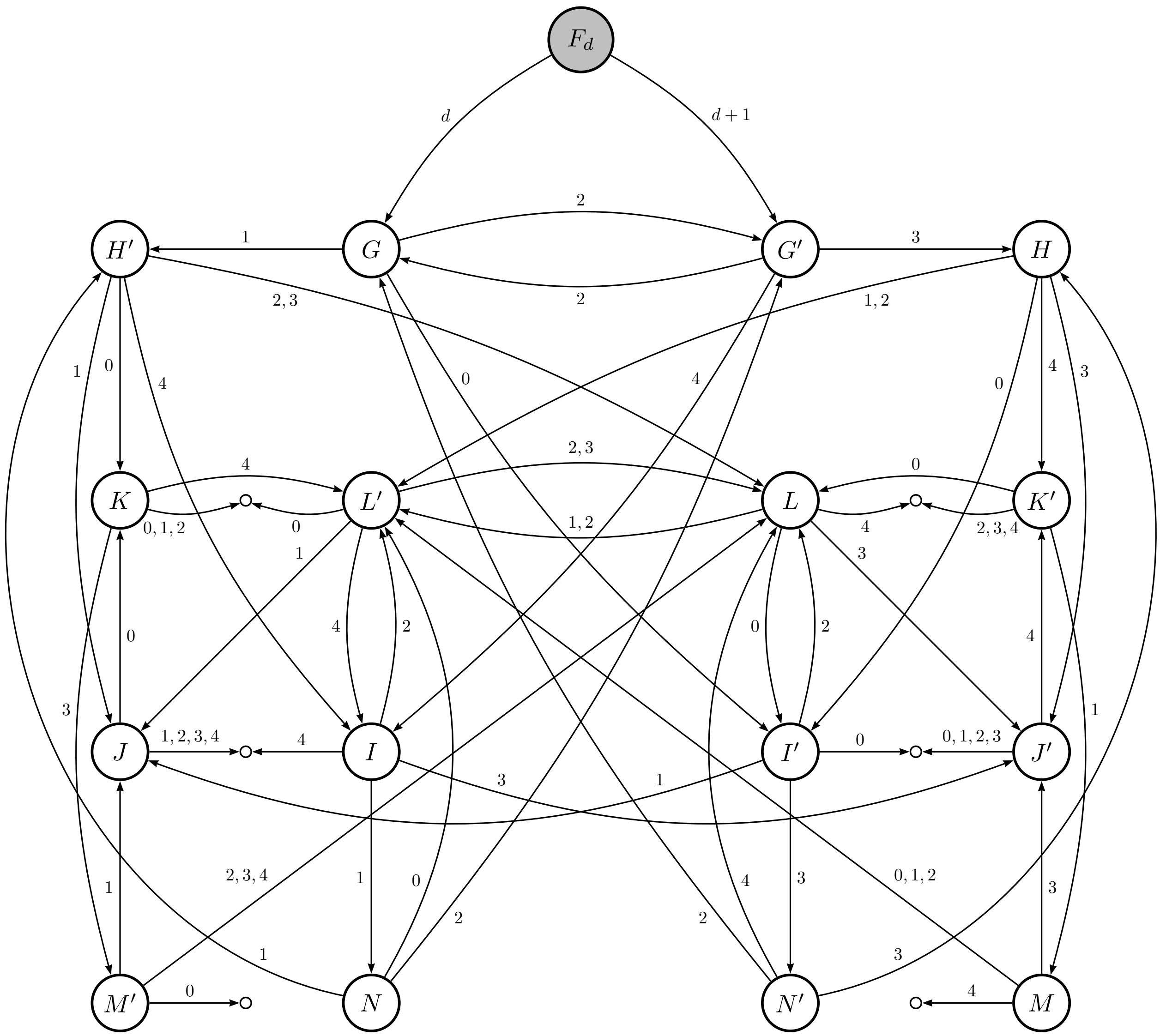

FIGURE 2. Graph $\mathcal{G}_d$ of the closure of the interior component of $\mathcal{T}$ containing $d\alpha^{-1}$.

their dual, and $Z'' = Z$ for all the other states of $\mathcal{H}$. Note that in $\mathcal{H}$ every edge $Z_1 \xrightarrow{a} Z_2$ with $Z_1 \neq F_d$ has a *dual edge* $Z_1' \xrightarrow{4-a} Z_2'$.

1.4. **Main result.** As mentioned above, in [18] it was proved that the graph $\mathcal{G}_d$ ($d = 0,1,2,3$) describes the closure of the inner component containing $d\alpha^{-1}$ in the following sense:

$$C_d = \overline{\bigcup_{w \in \mathcal{G}_d(F_d,\circ)} \psi_w(\mathcal{T})}.$$

The component C_d differs from C_0 by the translation $\Phi(d\alpha^{-1})$. Remember that in [22] it is shown that they are topological disks. We use these four sets to describe all the other components, as stated in Theorem 1.2 below.

Definition 1.1. In $\mathcal{P}$, we call

- S a *d-ending state* for $d \in \{0,1,2,3\}$,
- G_1 a *d-ending state* for $d \in \{2,3\}$,

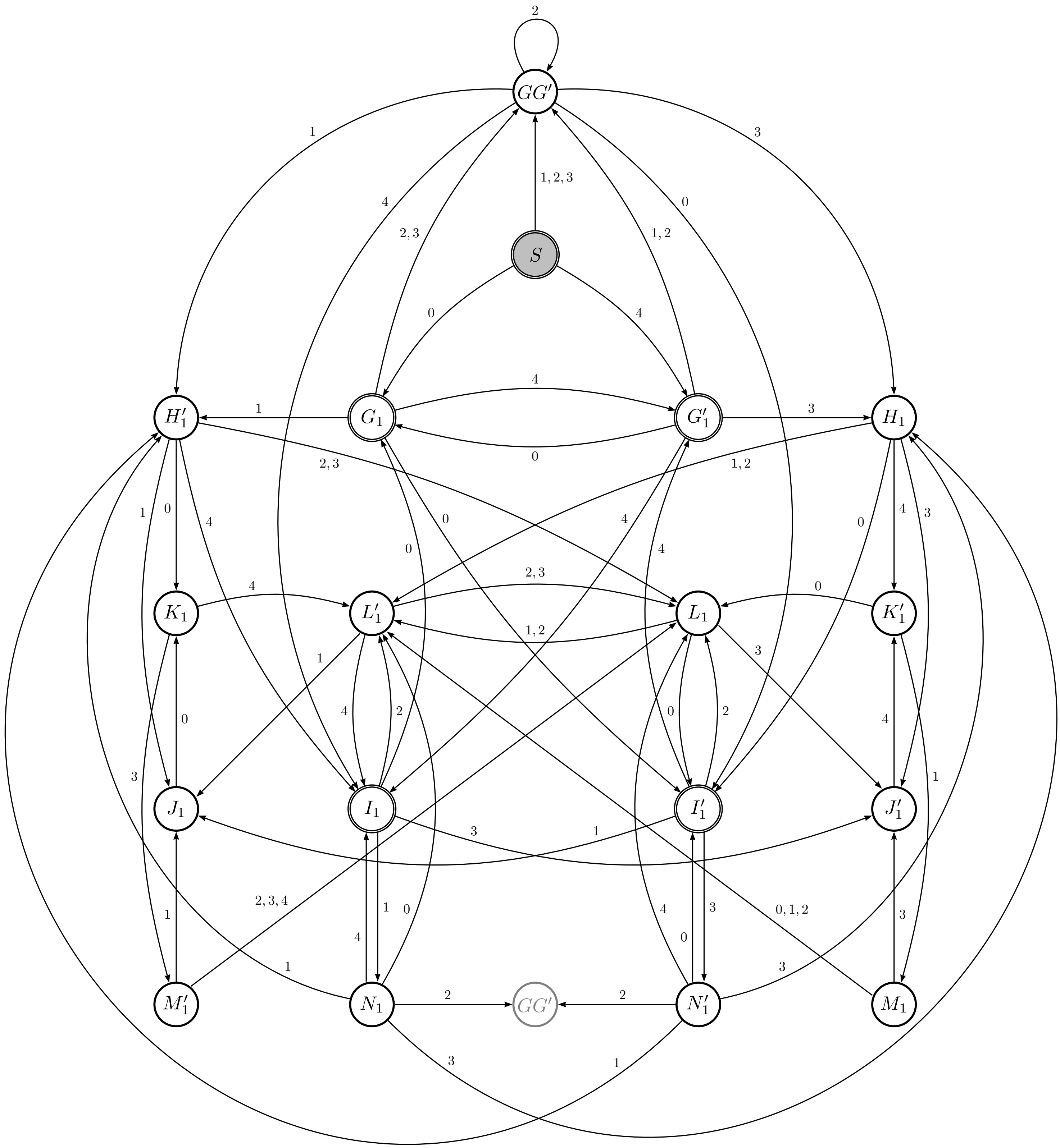

FIGURE 3. Pregraph $\mathcal{P}$ for the closure of the interior components of $\mathcal{T}$.

- G_1' a *d-ending state* for $d \in \{0,1\}$,
- I_1 a *d-ending state* for $d \in \{0\}$,
- I_1' a *d-ending state* for $d \in \{3\}$.

Moreover, if a walk $w \in \mathcal{P}(S)$ ends at a d-ending state, we call it a *d-ending walk.*

Hence each transition state is a d-ending state for one, two or four values of d. Also, the empty walk $w = \epsilon$ is a d-ending walk for all values of d.

Theorem 1.2. *Let w be a finite string of digits and $d \in \{0,1,2,3\}$. Then C is the closure of a component of $\mathcal{T}^o$ if and only if it is of the form $C = \psi_w(C_d)$ such that $w \in \mathcal{P}(S)$ is a d-ending walk. In particular, the closure of each component of $\mathcal{T}^o$ is a similar image of C_0.*

We will divide the proof in two parts. The result of Section 2 (Proposition 2.1) will be used to prove that the tile $\mathcal{T}$ is covered by the closure of the union of the sets $\psi_w(C_d)$ with the property that w is a d-ending walk. In Section 3, we will prove that each piece $\psi_w(C_d)$ is indeed the closure of an inner component of $\mathcal{T}$. To this matter, we will show that the boundary of such a piece is a subset of the boundary of $\mathcal{T}$. In the last section, we finish the proof of Theorem 1.2.

2. A cover of the tile $\mathcal{T}$

The result of this section will be used to show that the closure of the union of the sets $\psi_w(C_d)$ such that w is a d-ending walk equals the tile $\mathcal{T}$, *i.e.*,

$$\mathcal{T} = \overline{\bigcup_{\substack{w \text{ is a} \\ d\text{-ending walk}}} \psi_w(C_d)}.$$

Obviously, $\mathcal{P}(S)$ does not recognize the full shift. We denote by $\mathcal{P}_\circ$ the extension of $\mathcal{P}$ obtained as follows. The states of $\mathcal{P}_\circ$ are the states of $\mathcal{P}$ together with the accepting state $\circ$. The edges of $\mathcal{P}_\circ$ are the edges of $\mathcal{P}$ together with the edges $Z \xrightarrow{a} \circ$ if $Z \xrightarrow{a}$ is not in $\mathcal{P}$. Also, as before, there is an edge $\circ \xrightarrow{a} \circ$ in $\mathcal{P}_\circ$ for all $a \in \mathcal{N}$. Now $\mathcal{P}_\circ(S)$ recognizes the full shift.

In the sequel, the notations of Subsection 1.3 will be used for $\mathcal{H} = \mathcal{P}_\circ$ also. Moreover, we will need the set of walks $(\mathcal{P}\&\mathcal{G}_d)(S,\circ)$, which we define to be the set of walks $w\&w'$, where w is d-ending walk of $\mathcal{P}(S)$ and w' is a walk of $\mathcal{G}_d(F_d,\circ)$.

We will prove the following proposition.

Proposition 2.1. *Let w be a walk in $\mathcal{P}_\circ(S,\circ)$. Then $w \in (\mathcal{P}\&\mathcal{G}_d)(S,\circ)$ for some $d \in \{0,1,2,3\}$.*

Before we start with the proof we give an example in order to illustrate what is going on in the proof.

Example 2.2. Consider $w = (S;1,1,4,1,2,2,4,0,4,4,4)$, which is a walk in $\mathcal{P}_\circ(S,\circ)$:

$$S \xrightarrow{1} GG' \xrightarrow{1} H_1' \xrightarrow{4} I_1 \xrightarrow{1} N_1 \xrightarrow{2} GG' \xrightarrow{2} GG' \xrightarrow{4} I_1 \xrightarrow{0} G_1 \xrightarrow{4} G_1' \xrightarrow{4} I_1 \xrightarrow{4} \circ.$$

We choose d such that $\mathcal{G}_d(F_d)$ reads the maximal number of digits of w before it gets stuck, if ever. Trying out the four possibilities shows that there is exactly one choice, which is $d = 1$:

$$F_1 \xrightarrow{(1,1,4,1,2,2)} G$$

exists in $\mathcal{G}_1(F_1)$ after which the walk gets stuck as there is no edge $G \xrightarrow{4}$ in $\mathcal{G}_1(F_1)$. We look back for the last transition state passed through in $\mathcal{P}_\circ$:

$$S \xrightarrow{(1,1,4)} I_1,$$

and write $w = (1,1,4)\&(1,2,2,4,0,4,4,4)$. The last transition state passed through, I_1, is an e-ending state for the value $e = 0$ only. Hence, now we check if the remaining walk $w^1 = (1,2,2,4,0,4,4,4)$ is accepted by $\mathcal{G}_0(F_0)$. We see that this walk leads from F_0 to I , in particular

$$F_0 \xrightarrow{(1,2,2,4)} I.$$

where w^1 gets stuck, but one more digit (the digit 4) could be read than in our first trial. Moreover, one more transition state was passed through in $\mathcal{P}_\circ$. It is again I_1 :

$$S \xrightarrow{(1,1,4)} I_1 \xrightarrow{(1,2,2,4)} I_1,$$

an e-ending state for $e = 0$ only. Writing $w = (1,1,4,1,2,2,4)\&(0,4,4,4)$, we feed $\mathcal{G}_0(F_0)$ with $w^2 = (0,4,4,4)$:

$$F_0 \xrightarrow{0} G,$$

and w^2 gets stuck, but again one more digit (the digit 0) could be read than in the situation before. In $\mathcal{P}_\circ$, we now reached one more transition state; it is G_1:

$$S \xrightarrow{(1,1,4)} I_1 \xrightarrow{(1,2,2,4)} I_1 \xrightarrow{0} G_1,$$

which is an e-ending state for $e \in \{2,3\}$. We choose between $\mathcal{G}_2(F_2)$ and $\mathcal{G}_3(F_3)$ the graph that reads the maximal number of digits of $w^3 = (4,4,4)$: it is $\mathcal{G}_3(F_3)$ (in this example, $\mathcal{G}_2(F_2)$ reads in fact no digit of w^3). Now $\mathcal{G}_3(F_3)$ accepts w^3. Indeed

$$F_3 \xrightarrow{(4,4,4)} \circ$$

is a walk in $\mathcal{G}_3(F_3)$ and we are done : $w = v\&v'$ with $v = (1,1,4,1,2,2,4,0)$ and $v' = (4,4,4)$, where v is a 3-ending walk in $\mathcal{P}$ and v' is a walk of $\mathcal{G}_3(F_3,\circ)$. Thus $w \in (\mathcal{P}\&\mathcal{G}_3)(S,\circ)$.

The following proof of Proposition 2.1 follows the lines of the example. We start with a walk w in $\mathcal{P}_\circ(S,\circ)$ and scan its transition states as long as we arrive at a point where this transition state leads to a decomposition $w = v\&v'$ such that v' is readable by a suitable graph $\mathcal{G}_d(F_d)$. To this matter we need to show that the algorithm outlined in the example always terminates properly.

Proof of Proposition 2.1. Let $w = (a_1, a_2, \ldots, a_n)$ be a walk in $\mathcal{P}_\circ(S,\circ)$: $S \xrightarrow{w} \circ$. Two cases can occur.

(1) w is accepted by $\mathcal{G}_d(F_d)$ for some $d \in \{0,1,2,3\}$.
(2) For each $d \in \{0,1,2,3\}$, w is not accepted by $\mathcal{G}_d(F_d)$. Each $\mathcal{G}_d(F_d)$ accepts a prefix of w of length $p_d - 1$ say, *i.e.*, the letter with index p_d is the first one which is not accepted by $\mathcal{G}_d(F_d)$. We choose a d with $p_d = \max_{0\le e\le 3} p_e$ (we will see that this choice is unique).

In case (1), w is accepted by $\mathcal{G}_d(F_d,\circ)$, and taking $v = \epsilon$, $v' = w$, we have $w = v\&v'$ where v is a d-ending walk of $\mathcal{P}(S)$ and v' is a walk of $\mathcal{G}_d(F_d,\circ)$. Thus $w \in (\mathcal{P}\&\mathcal{G}_d)(S,\circ)$.

In case (2), let us write w in the form

$$w = (a_1, 2, 2, \ldots, 2, a_{k_0+2}, \ldots, a_n)$$

with $a_{k_0+2} \neq 2$ ($k_0 = 0$ is not excluded). If $a_1 = 0$ (or 4), then the unique choice will be $d = 0$ (or 3), and $p_d \geq k_0+2$. If $a_1 = 1, 2$ or 3, then the unique choice for d (leading to $p_d \geq k_0+3$) will be made according to the parity of k_0. This is because in $\mathcal{G}_d$ we have the edges $F_{a_1} \xrightarrow{a_1} G \leftrightarrow^2 G' \xleftarrow{a_1} F_{a_1-1}$. Moreover, the outgoing edges of G and G' are labelled with $0,1,2$ and $2,3,4$, respectively. Hence, the element $d \in \{0,1,2,3\}$ with $p_d = \max_{0\le e\le 3} p_e$ is uniquely determined also in this case. We set $p := p_d$.

The walk w gets stuck in $\mathcal{G}_d(F_d)$ after reading the prefix $w|_{p-1}$. At this stage the graph $\mathcal{G}_d(F_d)$ must rest in a state which does not have an outgoing edge for some $a \in \mathcal{N}$. Thus it must rest in one of the states contained in the set $\{G, G', I, I', N, N'\}$. We want to know what happens in

$\mathcal{P}_\circ(S)$ when reading $w|_p$. As the structures of $\mathcal{G}_d(F_d)$ and $\mathcal{P}_\circ(S)$ are very similar to each other, by inspecting both graphs we easily see that one of the following possibilities must hold :

$$\begin{array}{rcl}
S \xrightarrow{w_{|p-1}} & G_1 & \xrightarrow{w_p\in\{3,4\}} \dots \\
S \xrightarrow{w_{|p-1}} & G_1' & \xrightarrow{w_p\in\{0,1\}} \dots \\
S \xrightarrow{w_{|p-1}} & I_1 & \xrightarrow{w_p=0} \dots \\
S \xrightarrow{w_{|p-1}} & I_1' & \xrightarrow{w_p=4} \dots \\
S \xrightarrow{w_{|p-2}} I_1 \xrightarrow{1} & N_1 & \xrightarrow{w_p\in\{3,4\}} \dots \\
S \xrightarrow{w_{|p-2}} I_1' \xrightarrow{3} & N_1' & \xrightarrow{w_p\in\{0,1\}} \dots
\end{array}$$

As $\mathcal{P}_\circ(S)$ also contains the state GG' the following additional possibilities can occur

$$\begin{array}{rcl}
S \xrightarrow{w_{|p-k_1-1}} G_1 \xrightarrow{2} & GG' \xrightarrow{2} \dots \xrightarrow{2} GG' & \xrightarrow{w_p\neq 2} \dots, \\
S \xrightarrow{w_{|p-k_1-2}} I_1 \xrightarrow{1} N_1 \xrightarrow{2} & GG' \xrightarrow{2} \dots \xrightarrow{2} GG' & \xrightarrow{w_p\neq 2} \dots,
\end{array}$$

where either k_1 is even and $w_p \in \{0,1\}$, or k_1 is odd and $w_p \in \{3,4\}$, and finally

$$\begin{array}{rcl}
S \xrightarrow{w_{|p-k_1-1}} G_1' \xrightarrow{2} & GG' \xrightarrow{2} \dots \xrightarrow{2} GG' & \xrightarrow{w_p\neq 2} \dots, \\
S \xrightarrow{w_{|p-k_1-2}} I_1' \xrightarrow{3} N_1' \xrightarrow{2} & GG' \xrightarrow{2} \dots \xrightarrow{2} GG' & \xrightarrow{w_p\neq 2} \dots,
\end{array}$$

where k_1 is odd and $w_p \in \{0,1\}$, or k_1 is even and $w_p \in \{3,4\}$. We call this set of possibilities $\mathrm{Poss}(p)$.

In all these cases, $F_d \xrightarrow{w_{|p-1}} \dots$ is a walk in $\mathcal{G}_d(F_d)$ and the walk $S \xrightarrow{w_{|p-1}} \dots$ in $\mathcal{P}_\circ(S)$ passes through some transition state. We are interested in the last transition state passed in $\mathcal{P}_\circ(S)$ when reading $w_{|p-1}$. We call it T. Thus there is a $q < p$ such that

$$S \xrightarrow{w_{|q}} T.$$

Depending on the state of $\mathcal{G}_d(F_d)$ where w got stuck, we have

$$q = p-1,\ q = p-2,\ q = p-k_1-1 \text{ or } q = p-k_1-2.$$

As in the example given before the proof, we write $w = v^1 \& w^1$ with $v^1 = w_{|q}$. The state T is a e-ending state for one or two values of e. Let us say that it is an e-ending and e'-ending state (maybe $e = e'$).

We claim that there are a $p' \geq p$, a unique $d' \in \{e, e'\}$ and states $U_1 \in \mathcal{P}_\circ$ and $U \in \mathcal{G}_{d'}$ such that the following walks exist :

$$\begin{array}{rcl}
S \xrightarrow{w_{|q}} T & \xrightarrow{(w_{q+1},\dots,w_p,\dots,w_{p'})} & U_1 \in \mathcal{P}_\circ(S), \\
F_{d'} & \xrightarrow{(w_{q+1},\dots,w_p,\dots,w_{p'})} & U \in \mathcal{G}_{d'}(F_{d'}),
\end{array}$$

where the following simple relation holds : $U = H$ if $U_1 = H_1$, $U = G$ if $U_1 = G_1$, and so on.

This has to be checked for all the possibilities of $\mathrm{Poss}(p)$ listed above. We provide the details only for two cases, all the other cases can be treated likewise. First consider

$$S \xrightarrow{w_{|p-1}} G_1 \xrightarrow{w_p\in\{3,4\}} \dots.$$

Then $q = p-1$, $S \xrightarrow{w_{|q}} G_1$ ($T = G_1$) and G_1 is 3- and 4-ending state ($e = 3$ and $e' = 4$).
If $w_p = 4$, then

$$\begin{array}{rcl}
S \xrightarrow{w_{|q}} G_1 & \xrightarrow{w_{q+1}=w_p=4} & G_1' = U_1, \\
F_3 & \xrightarrow{w_{q+1}=w_p=4} & G' = U,
\end{array}$$

where $d' = 3$ is the unique possible choice. Setting $p' = p$ the claim is proved in this instance. If $w_p = 3$, we have four possibilities in $\mathcal{P}_\circ(S)$ (always $T = G_1$ holds)

$$\begin{array}{l}
S \xrightarrow{w_{|q_0}} G_1 \xrightarrow{w_{q_0+1}=w_{p_0}=3} GG' \underbrace{\xrightarrow{2} \dots \xrightarrow{2}}_{k\text{ times}} GG' \xrightarrow{w_{p_0+k}=0} I_1',\\
S \xrightarrow{w_{|q_0}} G_1 \xrightarrow{w_{q_0+1}=w_{p_0}=3} GG' \underbrace{\xrightarrow{2} \dots \xrightarrow{2}}_{k\text{ times}} GG' \xrightarrow{w_{p_0+k}=1} H_1',\\
S \xrightarrow{w_{|q_0}} G_1 \xrightarrow{w_{q_0+1}=w_{p_0}=3} GG' \underbrace{\xrightarrow{2} \dots \xrightarrow{2}}_{k\text{ times}} GG' \xrightarrow{w_{p_0+k}=3} H_1,\\
S \xrightarrow{w_{|q_0}} G_1 \xrightarrow{w_{q_0+1}=w_{p_0}=3} GG' \underbrace{\xrightarrow{2} \dots \xrightarrow{2}}_{k\text{ times}} GG' \xrightarrow{w_{p_0+k}=4} I_1,
\end{array}$$

where $k \geq 0$. The choice of d' will depend on the parity of k together with the value of w_{p+k}. Suppose k is even and $w_{p+k} \in \{0,1\}$, the only choice is $d' = 3$:

$$\begin{array}{l}
F_3 \xrightarrow{w_{q+1}=w_p=3} G \underbrace{\xrightarrow{2} G' \dots \xrightarrow{2}}_{k\text{ times}} G \xrightarrow{w_{p+k}=0} I',\\
F_3 \xrightarrow{w_{q+1}=w_p=3} G \underbrace{\xrightarrow{2} G' \dots \xrightarrow{2}}_{k\text{ times}} G \xrightarrow{w_{p+k}=1} H',
\end{array}$$

hence, setting $p' = p + k$ proves the claim. Suppose k is even but $w_{p+k} \in \{3,4\}$, then the only choice is $d' = 2$:

$$\begin{array}{l}
F_2 \xrightarrow{w_{q+1}=w_p=3} G' \underbrace{\xrightarrow{2} G \dots \xrightarrow{2}}_{k\text{ times}} G' \xrightarrow{w_{p+k}=3} H\\
F_2 \xrightarrow{w_{q+1}=w_p=3} G' \underbrace{\xrightarrow{2} G \dots \xrightarrow{2}}_{k\text{ times}} G' \xrightarrow{w_{p+k}=4} I,
\end{array}$$

and taking $p' = p + k$ proves the claim. Everything runs along similar lines if k is odd.

As a second instance consider now the case

$$S \xrightarrow{w_{|p-k_1-2}} I_1 \xrightarrow{1} N_1 \xrightarrow{2} \quad GG' \xrightarrow{2} \dots \xrightarrow{2} GG' \xrightarrow{w_p \neq 2} \dots$$

with k_1 even and $w_p \in \{0,1\}$. Then $q = p - k_1 - 2$, $S \xrightarrow{w_{|q}} I_1$ (here $T = I_1$) and I_1 is a 0-ending state ($e = e' = 0$). We have the following walks in $\mathcal{P}_\circ(S)$ and $\mathcal{G}_0(F_0)$, respectively:

$$\begin{array}{rl}
S \xrightarrow{w_{|p-k_1-2}} I_1 \xrightarrow{1} N_1 \xrightarrow{2} & GG' \xrightarrow{2} \dots \xrightarrow{2} GG' \xrightarrow{w_p=0\text{ (or 1)}} I_1' \text{ (or } H_1')\\
F_0 \xrightarrow{w_{q+1}=1} G' \xrightarrow{2} & G \xrightarrow{2} G' \dots \xrightarrow{2} G \xrightarrow{w_p=0\text{ (or 1)}} I' \text{ (or } H'),
\end{array}$$

hence, taking $p' = p$ proves the claim.

As all the other cases can be treated likewise, our claim follows.

By this claim, we came to a similar situation as at the beginning of the proof. Indeed, two cases can occur.

(1′) w^1 is accepted by $\mathcal{G}_{d'}(F_{d'})$.
(2′) w^1 is not accepted by $\mathcal{G}_{d'}(F_{d'})$. $\mathcal{G}_{d'}(F_{d'})$ accepts a prefix of w^1 of maximal length $p' - 1$, *i.e.*, $\mathcal{G}_{d'}(F_{d'})$ accepts the $p' - 1$ first letters of w^1 and can not read the p'-th letter.

In case (1′), w^1 even belongs to $\mathcal{G}_{d'}(F_{d'}, \circ)$. Taking $v = v^1$, $v' = w^1$, we have $w = v\&v'$ where v is a d'-ending walk and v' is a walk of $\mathcal{G}_{d'}(F_{d'}, \circ)$. Thus $w \in (\mathcal{P}\&\mathcal{G}_{d'})(S, \circ)$ and we are done.

In case (2′), we consider the first moment w^1 gets stuck in $\mathcal{G}_{d'}(F_{d'})$:

$$p' = \min\{k; w^1_{|k} \notin \mathcal{G}_{d'}(F_{d'})\}.$$

The claim assures that $p'-1 \geq 1$ and that one of the possibilities of $\mathrm{Poss}(p+p'-1)$ occurs when the walk w^1 gets stuck in $\mathcal{G}_{d'}(F_{d'})$. In all the cases, $F_{d'} \xrightarrow{(w_p,\dots,w_{p+p'-1})} \in \mathcal{G}_{d'}(F_{d'})$ and $T \xrightarrow{w^1_{|p'}}$ passes through some transition state. We look back for the last transition state passed through in $\mathcal{P}_\circ(S)$ and call it T' :

$$S \xrightarrow{v^1} T \xrightarrow{w^1_{|q'}} T'$$

for some $q' \geq 1$. Hence we are in the former situation and we can write $w = v^1\&v^2\&w^2$ with $v^2 = w^1_{|q'}$. The preceding claim holds in a similar way and gives some value d'' with the required properties (in particular, T' is a d''-ending state).

Iterating this procedure like in the example given before this proof, since w is finite, one eventually gets $w = v^1\&\dots\&v^r\&w^r$ and some $d^{(r)} \in \{0,1,2,3\}$ such that $v^1\&\dots\&v^r$ is a $d^{(r)}$-ending walk and w^r is a walk of $\mathcal{G}_{d^{(r)}}(F_{d^{(r)}},\circ)$. Hence, $w \in (\mathcal{P}\&\mathcal{G}_{d^{(r)}})(S,\circ)$ and the proposition is proved.

□

3. Boundary inclusions

In this section we prove that if w is a d-ending walk of $\mathcal{P}(S)$, then the boundary of the piece $\psi_w(C_d)$ is contained in the boundary of $\mathcal{T}$.

Let us recall a way to identify points of the boundary of $\mathcal{T}$. Let $\mathcal{B}$ be the graph of Figure 4. It is given in [24, 26] for bases of quadratic canonical number systems in general and reproduced here for $\alpha = -2+\sqrt{-1}$.

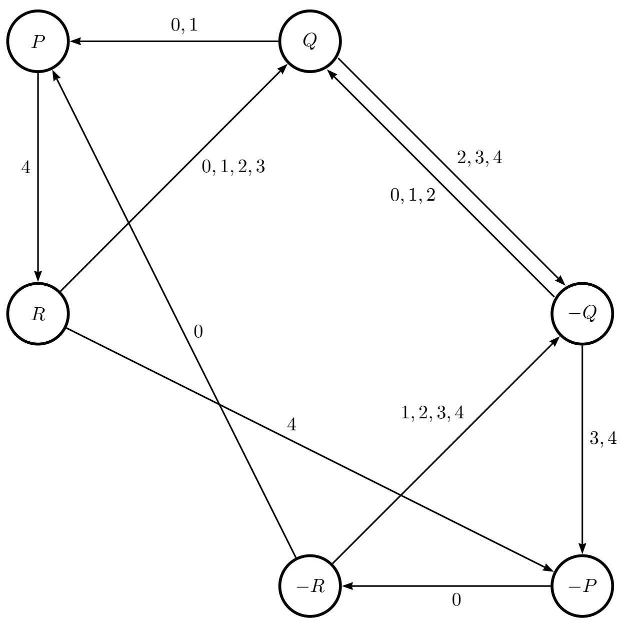

Figure 4. Graph $\mathcal{B}$.

The following proposition gives a way to check whether an infinite walk of digits represents a point of the boundary.

Proposition 3.1 (Müller *et al.* [20]). *If there exists an infinite walk*

$$V \xleftarrow{a_1} V_1 \xleftarrow{a_2} \dots$$

in $\mathcal{B}$ and $x = \sum_{i\geq 1} \mathbf{A}^{-i}\Phi(a_i)$, then $x \in \partial\mathcal{T}$.

Lemma 3.2. *Let $d \in \{0,1,2,3\}$. Furthermore, let $n \in \mathbb{N}$ and w with $|w| = n$ be a walk in $\mathcal{P}(S)$ with*

$$S \xrightarrow{w} \in \mathcal{P}(S).$$

Then, for each $V \in \mathcal{V}(Z)$ of Table 1, there is a walk in $\mathcal{B}$ of the form

$$V \xrightarrow{w^T} W$$

for some W. This also holds for the duals with[1] *$\mathcal{V}(Z') := \{B \in \mathcal{B}, -B \in \mathcal{V}(Z)\} = -\mathcal{V}(Z)$.*

Proof. This is proved by induction on the length of w. For $n = 1$, $w \in \{(0),(1),(2),(3),(4)\}$ with the following edges:

$$\begin{array}{ccc} S & \xrightarrow{0} & G_1, \\ S & \xrightarrow{1,2,3} & GG', \\ S & \xrightarrow{4} & G_1', \end{array}$$

and one checks the existence in $\mathcal{B}$ of the edges

$$\begin{array}{rcl} -P, \pm Q, \pm R & \xrightarrow{0} & W, \\ \pm Q, \pm R & \xrightarrow{1,2,3} & W, \\ P, \pm Q, \pm R & \xrightarrow{4} & W, \end{array}$$

for some W.

Assume now that the lemma is true for the walks of length n ($n \geq 1$). Consider a walk w of length $n+1$:

$$S \xrightarrow{w} Z$$

with $w = (a_1, \ldots, a_n, a_{n+1})$. Thus we wonder if, for all $V_0 \in \mathcal{V}(Z)$, a walk

$$V_0 \xrightarrow{a_{n+1}} V \xrightarrow{a_n} \ldots \xrightarrow{a_1} W$$

exists in $\mathcal{B}$.

Suppose that w ends up in $Z = G_1$. Then $w_{|n}$ ends up in G_1' or I_1, because $G_1' \xrightarrow{0} G_1$ and $I_1 \xrightarrow{0} G_1$ are the only edges in $\mathcal{P}$ leading to G_1. S need not be considered, because $n \geq 1$. Also, $a_{n+1} = 0$. If $w_{|n}$ ends up in G_1', then by assumption there are walks

$$V \xrightarrow{w|_n^T} W$$

in $\mathcal{B}$ for all $V \in \{P, Q, -Q, R, -R\} = \mathcal{V}(G_1')$. Hence, for each $V_0 \in \mathcal{V}(G_1)$, one just needs to check whether

$$V_0 \xrightarrow{a_{n+1}=0} V$$

[1]Note that for the self dual states Z of $\mathcal{P}$ we have $\mathcal{V}(Z) = -\mathcal{V}(Z)$.

Z	$\mathcal{V}(Z)$
G_1	$\{-P, Q, -Q, R, -R\}$
H_1	$\{Q, R, -R\}$
I_1	$\{P, Q, -R\}$
J_1	$\{-R\}$
K_1	$\{-P\}$
L_1	$\{Q, -R\}$
M_1	$\{Q\}$
$G'N$	$\{R, Q, -Q\}$
GG'	$\{Q, -Q, R, -R\}$

Table 1. Table for Lemma 3.2.

Z	Z_1, end of $w\|_n$	a_{n+1}	$V \in \mathcal{V}(Z_1)$	$V_0 \in \mathcal{V}(Z)$	$V_0 \xrightarrow{a_{n+1}} V$
G_1	G_1' I_1	0	$\pm R, P, \pm Q$ $P, Q, -R$	$\pm R, -P, \pm Q$	$-P \to -R; R, -Q \to Q; -R, Q \to P$ $-P \to -R; Q, R \to Q; Q \to P$
GG'	G_1 GG' G_1' N_1 N_1'	$2,3$ 2 $1,2$ 2 2	$-P, \pm Q, \pm R$ $\pm Q, \pm R$ $P, \pm Q, \pm R$ $R, \pm Q$ $-R, \pm Q$	$\pm R, \pm Q$	$Q, -R \xrightarrow{2,3} -Q; -Q \xrightarrow{2} Q; -Q \xrightarrow{3} -P; R \xrightarrow{2,3} Q$ $\pm R \to \pm Q; \pm Q \to \mp Q$ $Q, -R \xrightarrow{2,3} -Q; -Q \xrightarrow{2} Q; -Q \xrightarrow{3} -P; R \xrightarrow{2,3} Q$ $\{\pm R, \pm Q\} \to \{\pm Q\}$ $\{\pm R, \pm Q\} \to \{\pm Q\}$
I_1	H_1' GG' L_1' G_1' N_1	4	$-Q, \pm R$ $\pm Q, \pm R$ $-Q, R$ $P, \pm Q, \pm R$ $R \pm Q$	$P, Q, -R$	$\{P, Q, -R\} \to \{R, -Q\}$
N_1	I_1	1	$P, Q, -R$	$R, \pm Q$	$R, -Q \to Q; Q \to P$
L_1	H_1' L_1' M_1' N_1' I_1' K_1'	$2,3$ $2,3$ $2,3,4$ 4 2 0	$-Q, \pm R$ $-Q, R$ $-Q$ $R, \pm Q$ $-P, -Q, -R$ P	$Q, -R$	$Q, -R \xrightarrow{2,3,4} -Q$ $Q, -R \xrightarrow{0} P$
M_1	K_1'	P	Q	$Q \to P$	
J_1	H_1' M_1' L_1' I_1'	1	P $-Q, \pm R$ $-Q$ $-Q, R$ $-P, -Q, R$	$-R$	$-R \to -Q$
H_1	GG' G_1 N_1' N_1	1	$\pm Q, \pm R$ $-P, \pm Q, \pm R$ $-R, \pm Q$ $R, \pm Q$	$-Q, \pm R$	$-Q, \pm R \to \pm Q$
K_1	H_1' J_1	0	$-Q, \pm R$	$-P$	$-P \to -R$

TABLE 2. Proof of Lemma 3.2.

exists in $\mathcal{B}$ for at least one $V \in \{P, Q, -Q, R, -R\}$. This is true since the edges

$$-P \xrightarrow{0} -R,\ Q \xrightarrow{0} P,\ -Q \xrightarrow{0} Q,\ R \xrightarrow{0} Q,\ -R \xrightarrow{0} P$$

all exist in $\mathcal{B}$. The case where w ends up in G_1' is treated likewise. The proof runs along similar lines for the other possible endings Z of w.

All the results are summed up in Table 2 (the duals can be treated likewise).

□

Let us define, for $d = 0, 1, 2, 3$, the graph $\mathcal{G}_d'$ emerging from $\mathcal{G}_d$ when the accepting state $\circ$ is removed.

Remark 3.3. If $(a_1, a_2, a_3, \ldots)$ is accepted by $\mathcal{G}_d'(F_d)$ then it is also accepted by $\mathcal{P}(S)$.

Lemma 3.4. *Let w be a d-ending walk of $\mathcal{P}(S)$ and $w' \in \mathcal{G}_d'(F_d)$ an infinite walk. Denote by l the labeling of w'. Then $w\&l \in \mathcal{P}(S)$.*

Proof. All the possible ending states of w can be considered. We only treat two examples.

First suppose that $w = \epsilon$, that is w ends at S, and w is a d-ending state for all values of d. Take an infinite walk $w' \in \mathcal{G}_d(F_d)$ ($d = 0, 1, 2$ or 3). Denote by l the labeling of w'. Then, by Remark 3.3, $l \in \mathcal{P}(S)$, hence $w\&l \in \mathcal{P}(S)$ holds trivially. But this remark is also helpful for the other cases.

Indeed, suppose now that w ends in the state G_1:

$$S \xrightarrow{w} G_1,$$

which is d-ending for $d = 2$ and $d = 3$. Suppose here that $w' \in \mathcal{G}_3(F_3)$. We write $l = (a_1, a_2, \dots)$ for its labeling. As we mentioned, we have the property that $l \in \mathcal{P}(S)$:

$$S \xrightarrow{a_1} Z^{(1)} \xrightarrow{a_2} Z^{(2)} \xrightarrow{a_3} \dots \in \mathcal{P}(S).$$

If $a_1 = 4$, then

$$S \xrightarrow{a_1=4} Z^{(1)} = G_1'$$

is the prefix of l. On the other side,

$$S \xrightarrow{w} G_1 \xrightarrow{a_1=4} G_1'$$

is the starting $w\&a_1$ of $w\&l$. It belongs to $\mathcal{P}(S)$. Thus the concatenation with $(a_2, a_3, \dots)$ remains in $\mathcal{P}(S)$:

$$S \xrightarrow{w} G_1 \xrightarrow{a_1=4} G_1' = Z^{(1)} \xrightarrow{a_2} Z^{(2)} \xrightarrow{a_3} \dots \in \mathcal{P}(S),$$

that is, $w\&l = w\&(a_1, a_2, \dots)$ belongs to $\mathcal{P}(S)$.
If on the contrary $a_1 = 3$, then

$$S \xrightarrow{a_1=3} Z^{(1)} = GG',$$

and also here

$$S \xrightarrow{w} G_1 \xrightarrow{a_1=3} GG'.$$

Hence $w\&l = w\&(a_1, a_2, \dots)$ belongs to $\mathcal{P}(S)$:

$$S \xrightarrow{w} G_1 \xrightarrow{a_1=3} GG' = Z^{(1)} \xrightarrow{a_2} Z^{(2)} \xrightarrow{a_3} \dots \in \mathcal{P}(S).$$

If w ends in the state G_1 but $w' \in \mathcal{G}_2(F_2)$, we have again

$$S \xrightarrow{a_1\in\{2,3\}} Z^{(1)} = GG'$$

and

$$S \xrightarrow{w} G_1 \xrightarrow{a_1\in\{2,3\}} GG',$$

thus $w\&l = w\&(a_1, a_2, \dots)$ belongs to $\mathcal{P}(S)$:

$$S \xrightarrow{w} G_1 \xrightarrow{a_1\in\{2,3\}} GG' = Z^{(1)} \xrightarrow{a_2} Z^{(2)} \xrightarrow{a_3} \dots \in \mathcal{P}(S).$$

The cases where w ends in G_1', N_1 and N_1' are treated similarly.

□

Proposition 3.5. *If w is a d-ending walk of $\mathcal{P}(S)$, then $\partial\psi_w(C_d) \subset \partial\mathcal{T}$.*

Proof. For each d, $\mathcal{G}_d'$ defines a system of graph directed sets $\delta\mathbf{M}_d(Z)$, where Z runs through the states of $\mathcal{G}_d'$. Let us write $\delta\mathbf{M}_d(F_d) =: \delta\mathbf{M}_d$. It was shown in [18] that $\delta\mathbf{M}_d \subset \partial C_d$ and $\partial C_d \setminus \delta\mathbf{M}_d$ is countable. Consequently, for any finite sequence of digits w and any digit d,

$$\psi_w(\delta\mathbf{M}_d) \subset \psi_w(\partial C_d) = \partial(\psi_w(C_d))$$

and equality holds up to countably many points. We show that this implies equality. Indeed, suppose that $z \in \partial C_d \setminus \delta\mathbf{M}_d$. Then, as $\delta\mathbf{M}_d$ is compact, there is an $\varepsilon > 0$ such that the ball $B_\varepsilon(z)$ of radius ε around z has empty intersection with $\delta\mathbf{M}_d$. However al C_d is a closed disk, ∂C_d is a simple closed curve. Thus $\partial C_d \cap B_\varepsilon(z)$ is uncountable. This yields uncountably many elements of $\partial C_d \setminus \delta\mathbf{M}_d$, a contradiction. Thus $\partial C_d \setminus \delta\mathbf{M}_d = \emptyset$ which means that

$$\psi_w(\delta\mathbf{M}_d) = \partial(\psi_w(C_d)).$$

Hence, it remains to prove that $\psi_w(\delta\mathbf{M}_d) \subset \partial\mathcal{T}$ in the case that w is a d-ending walk.

Let w be a d-ending walk and $z \in \psi_w(\delta\mathbf{M}_d)$. Then there is a $v = (a_1, a_2, \ldots)$ satisfying $v = w\&l$ for some labeling l of a walk of $\mathcal{G}'_d(F_d)$ and $z = \sum_{i=1}^{\infty} \mathbf{A}^{-i}\Phi(a_i)$. By Lemma 3.4, $v \in \mathcal{P}(S)$. Therefore, each prefix $v|_n$ of v belongs to $\mathcal{P}(S, Z_n)$ for some Z_n. Note that $z \in \psi_{v|_n}(\mathcal{T})$ for all n. Let $\varepsilon > 0$. Take $n \geq 1$ such that $\operatorname{diam}(\psi_{v|_n}(\mathcal{T})) \leq \varepsilon$. By Lemma 3.2, choosing a $V \in \mathcal{V}(Z_n)$, there is a W such that

$$W \xleftarrow{a_1} \ldots \xleftarrow{a_n} V$$

is a (backwards) walk in $\mathcal{B}$. Let $(a'_{n+1}, a'_{n+2}, \ldots)$ any sequence of digits such that

$$V \xleftarrow{a'_{n+1}} V_{n+1} \xleftarrow{a'_{n+2}} \ldots$$

is an infinite (backwards) walk in $\mathcal{B}$. Hence $(a_1, \ldots, a_n, a'_{n+1}, a'_{n+2}, \ldots)$ is an infinite backwards walk of $\mathcal{B}$ that ends at W. By Proposition 3.1,

$$z' = \sum_{i=1}^{n} \mathbf{A}^{-i}\Phi(a_i) \ + \sum_{i \geq n+1} \mathbf{A}^{-i}\Phi(a'_i) \ \in \partial\mathcal{T}.$$

But $z' \in \psi_{v|_n}(\mathcal{T})$. Consequently, we proved that the intersection $B_\varepsilon(z) \cap \partial\mathcal{T}$ is nonempty. This is true for all $\varepsilon > 0$, and $\partial\mathcal{T}$ is closed, thus we obtain that $z \in \partial\mathcal{T}$.

□

4. Proof of the main result

Proof of Theorem 1.2. From Proposition 3.5 we conclude that if w is a d-ending walk, then $\partial\psi_w(C_d) \subset \partial\mathcal{T}$. Moreover, since the set $\psi_w(C_d)$ is the image of C_d under the (bijective) similarity transformation ψ_w, its interior is a connected subset of $\mathcal{T}^o$. Thus $\psi_w(C_d)$ is the closure of a component of $\mathcal{T}^o$.

It remains to prove that

$$\mathcal{T} = \overline{\bigcup_{\substack{w \text{ is a} \\ d\text{-ending walk}}} \psi_w(C_d)}.$$

Consider a point $z \in \mathcal{T}$ and an associated sequence of digits $v = (a_1, a_2, \ldots)$ satisfying $z = \sum_{i\geq 1} \mathbf{A}^{-i}\Phi(a_i)$. In Section 2, we mentioned that $\mathcal{P}_\circ(S)$ accepts every infinite sequence of digits.

Suppose that v eventually ends up in $\circ$ in this graph. Then v has a prefix $v' \in \mathcal{P}_\circ(S, \circ)$. By Proposition 2.1, $v' \in (\mathcal{P}\&\mathcal{G}_d)(S, \circ)$ for some $d \in \{0, 1, 2, 3\}$. Thus v' has a prefix v'' which is a d-ending walk of $\mathcal{P}(S)$ and such that $z \in \psi_{v''}(C_d)$.

Otherwise, v is an infinite walk in $\mathcal{P}(S)$. We show that $\{z\}$ is a limit set of the family

$$\{\psi_w(C_d)\ ;\ w \text{ is a } d\text{-ending walk}\}.$$

Let $\varepsilon > 0$. Then there is an n such that $\operatorname{diam}(\psi_{v|_n}(\mathcal{T})) \leq \varepsilon$. It is easy to see that each state of $\mathcal{P}$ is at a distance of at most three edges of a d-ending state. Thus there is a w_n having at most three digits such that $w := v|_n\&w_n$ is a d-ending walk of $\mathcal{P}(S)$ for some d. Note that $z \in \psi_{v|_n}(\mathcal{T}) \supset \psi_w(\mathcal{T})$. Hence the ball $B_\varepsilon(z)$ intersects the set $\psi_w(C_d)$ where w is a d-ending walk. Since such a set can be found for all $\varepsilon > 0$, we are done.

□

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Elie Cartan Institute, Nancy (Mathematics), University Henri Poincaré Nancy 1. B.P. 70239, F-54506 Vandoeuvre-les-Nancy Cedex, FRANCE

Department of mathematics, Faculty of science, Niigata university, Ikarashi 2 8050 Niigata, 950 2181, JAPAN

Department of Mathematics and Statistics, Leoben University, Franz-Josef-Strasse 18, A-8700 Leoben, AUSTRIA

E-mail address: julien.bernat@lorraine.iufm.fr

E-mail address: loridant@dmg.tuwien.ac.at

E-mail address: joerg.thuswaldner@mu-leoben.at