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Observability, controllability and boundary stabilization of some linear elasticity systems

AISSA GUESMIA

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Abstract. In this paper, we study the observability, the controllability and the boundary stabilizability of the linear elasticity systems. This work extends to non-isotropic systems with variable coefficients, the observability and exact controllability results for isotropic elastodynamic systems obtained by J.-L. Lions in 1988, the uniform stabilizability results for two-dimensional isotropic systems obtained by J. E. Lagnese in 1991 and the results obtained by Alabau and Komornik [1].

1. Introduction and statement of the results

Let Ω be a non-empty bounded open set in $\mathbb{R}^n$ ($n \in \mathbb{N}^*$) having a boundary Γ of class C^2 and let a_{ijkl}, $i, j, k, l = 1, \ldots, n$ be a set in $W^{1,\infty}(\Omega)$ such that

$$a_{ijkl} = a_{klij} = a_{jikl} \quad \text{in} \quad \Omega$$

and satisfying for some $\alpha > 0$ the ellipticity condition

$$a_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geq \alpha\varepsilon_{ij}\varepsilon_{ij} \quad \text{in} \quad \Omega \tag{1.1}$$

for every symmetric tensor ε_{ij}. (Here and in the sequel we shall use the summation convention for repeated indices.)

For a given function $u = (u_1, \ldots, u_n): \Omega \to \mathbb{R}^n$, we shall use the notations

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \quad \sigma_{ij} = a_{ijkl}\varepsilon_{kl} \quad \text{in} \quad \Omega$$

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where $u_{i,j} = \frac{\partial u_i}{\partial x_j}$ and $u_{j,i} = \frac{\partial u_j}{\partial x_i}$. If it is necessary to be more precise, we shall write $\varepsilon_{ij}(u)$ and $\sigma_{ij}(u)$ instead of ε_{ij}, σ_{ij}.

Consider the problem

$$(1.2) \qquad \begin{cases} u_i'' - \sigma_{ij,j} = 0 \quad \text{in} \quad \Omega \times \mathbb{R}, \\ u_i = 0 \quad \text{on} \quad \Gamma \times \mathbb{R}, \\ u_i(0) = u_i^0 \quad \text{and} \quad u_i'(0) = u_i^1 \quad \text{in} \quad \Omega, \\ i = 1, \dots, n, \end{cases}$$

where $' = \frac{\partial}{\partial t}$, $\sigma_{ij,j} = \frac{\partial \sigma_{ij}}{\partial x_j}$ and $u_i(0)$, $u_i'(0)$ denote, respectively, the functions $x \longmapsto u_i(x,0)$, $x \longmapsto u_i'(x,0)$.

This system is well-posed in the following sense (cf. Lagnese [8]):

* For every $(u^0, u^1) \in (H_0^1(\Omega))^n \times (L^2(\Omega))^n$, the system (1.2) has a unique solution (defined in a suitable weak sense) satisfying

$$u \in C(\mathbb{R}; (H_0^1(\Omega))^n) \cap C^1(\mathbb{R}; (L^2(\Omega))^n),$$

where $H_0^1(\Omega) = \{v \in H^1(\Omega) : \ v = 0 \quad \text{on} \quad \Gamma\}$.

* If $(u^0, u^1) \in (H^2(\Omega) \cap H_0^1(\Omega))^n \times (H_0^1(\Omega))^n$ then the solution (called a strong solution) is more regular :

$$u \in C(\mathbb{R}; (H^2(\Omega) \cap H_0^1(\Omega))^n) \cap C^1(\mathbb{R}; (H_0^1(\Omega))^n) \cap C^2(\mathbb{R}; (L^2(\Omega))^n).$$

* The energy of the (weak) solution, defined by the formula

$$(1.3) \qquad E = \frac{1}{2} \int_\Omega (u_i' u_i' + \sigma_{ij} \varepsilon_{ij}) dx,$$

is independent of the time t.

Fix a point $x_0 = (x_1^0, \dots, x_n^0) \in \mathbb{R}^n$ and fix a measurable partition Γ_0, Γ_1 of Γ such that $\Gamma_1 = \Gamma \setminus \Gamma_0$ we have

$$(1.4) \qquad (x - x_0).\nu(x) \leq 0 \quad \text{for all} \quad x \in \Gamma_0,$$

where ν denotes the outward unit normal vector to Γ. (For example, we may always choose $\Gamma_0 = \emptyset$ and $\Gamma_1 = \Gamma$.) Set

$$(1.5) \qquad R = \sup\{|x - x_0| : x \in \Omega\},$$

and let be γ the smaller number in $]-\infty, 2[$ satisfying

$$(1.6) \qquad (x_p - x_p^0)(\partial_p a_{ijkl}) \varepsilon_{ij} \varepsilon_{kl} \leq \gamma a_{ijkl} \varepsilon_{ij} \varepsilon_{kl} \quad \text{in} \quad \Omega,$$

where $\partial_p a_{ijkl} = \frac{\partial a_{ijkl}}{\partial x_p}$. Assume that

$$(1.7) \qquad \gamma \geq 2(1-n).$$

Then we have the following results.

Theorem 1.1. *Assume* (1.1), (1.4), (1.7) *and let* $T > \frac{4\sqrt{2/\alpha}R}{2-\gamma}$. *Then there exist two positive constants* c_1 *and* c_2 *such that every strong solution of* (1.2) *satisfies the inequalities*

$$c_1 E \leq \int_0^T \int_{\Gamma_1} \sigma_{ij}\varepsilon_{ij} d\Gamma dt \leq c_2 E. \tag{1.8}$$

Remarks. * The first inequality in (1.8) cannot hold for arbitrarily small T. The condition $T > 2\sqrt{\frac{2}{\alpha}}R$ is the best possible if the system is isotropic; i.e. when

$$a_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$$

where λ and μ are the positive Lamé constants (see Komornik [6] and note that, in this case, $\gamma = 0$).

* By a simple density argument, the second estimate in (1.8) allows us to define the trace of $\sigma_{ij}\varepsilon_{ij}$ on $\Gamma_1 \times \mathbb{R}$ as an element of $L^2_{loc}(\Gamma_1 \times \mathbb{R})$, for every weak solution of (1.2).

* For sufficiently large time T, if two solutions of (1.2) coincide in Γ_1, then the boundary integral in (1.8), for their difference, vanishes and therefore the energy of their difference is equal to zero by the first inequality in (1.8). This implies that the two solutions are identical.

Applying the Hilbert Uniqueness Method (HUM) we shall deduce from theorem 1.1 an exact controllability result for the non-homogeneous system

$$\begin{cases} y_i'' - \sigma_{ij,j}(y) = 0 & \text{in} \quad \Omega \times \mathbb{R}, \\ y_i = \vartheta_i & \text{on} \quad \Gamma \times \mathbb{R}, \\ y_i(0) = y_i^0 \quad \text{and} \quad y_i'(0) = y_i^1 & \text{in} \quad \Omega, \\ i = 1, \dots, n. \end{cases} \tag{1.9}$$

Theorem 1.2. *Assume* (1.1), (1.4), (1.7) *and fix* $T > \frac{4\sqrt{2/\alpha}R}{2-\gamma}$. *Then for any given* y^0, $\tilde{y}^0 \in (L^2(\Omega))^n$ *and* y^1, $\tilde{y}^1 \in (H^{-1}(\Omega))^n$ *there exists* $\vartheta \in L^2_{loc}(\mathbb{R}; (L^2(\Gamma))^n)$ *such that the solution of* (1.9) *satisfies*

$$y(T) = \tilde{y}^0 \quad \text{and} \quad y'(T) = \tilde{y}^1 \quad \text{in} \quad \Omega.$$

Moreover, we may assume that v *vanishes outside of* $\Gamma_1 \times (0, T)$.

In the second half of the paper we shall study the uniform stabilizability of elasticity systems by applying suitable dissipative boundary feedbacks. Consider the problem

$$
(1.10) \qquad \begin{cases} u_i'' - \sigma_{ij,j} = 0 \quad \text{in} \quad \Omega \times \mathbb{R}^+, \\ u_i = 0 \quad \text{on} \quad \Gamma_0 \times \mathbb{R}^+, \\ \sigma_{ij}\nu_j + au_i + bu_i' = 0 \quad \text{on} \quad \Gamma_1 \times \mathbb{R}^+, \\ u_i(0) = u_i^0 \quad \text{and} \quad u_i'(0) = u_i^1 \quad \text{in} \quad \Omega, \\ i = 1, \dots, n. \end{cases}
$$

Where $\mathbb{R}^+ = [0, +\infty)$, a and b are given nonnegative numbers. (It is easy to generalize our results to the case where a and b are nonnegative functions of class $C^1(\bar{\Gamma}_1)$.) Indeed, define the energy of the solutions of (1.10) by

$$
(1.11) \qquad E(t) = \frac{1}{2}\int_\Omega (u_i'u_i' + \sigma_{ij}\varepsilon_{ij})dx + \frac{1}{2}\int_{\Gamma_1} au_iu_i d\Gamma,
$$

for all $t \in \mathbb{R}^+$. The energy E is nonnegative and we have

$$
E'(t) = -\int_{\Gamma_1} bu_i'u_i' d\Gamma \le 0, \quad \forall t \ge 0.
$$

Then the energy E is non-increasing of $t \in \mathbb{R}^+$.

We shall consider the system (1.10) under the conditions (1.1), (1.4) and

$$
(1.12) \qquad |x - x_0| = R \quad \text{for all} \quad x \in \Gamma_1.
$$

For example, these conditions are satisfied if

$$
\Omega = \{x \in \mathbb{R}^n : r < |x - x_0| < R\}
$$

where $0 < r < R$ and $\Gamma_0 = \{x \in \Gamma : |x - x_0| = r\}$ or $r = 0$ and Γ_0 is empty.

Theorem 1.3. *Assume* (1.1), (1.4), (1.12) *and* $a < \frac{(2-\gamma)\alpha}{4R}$. *Then there exists a positive number* ω *such that all (weak) solution of* (1.10) *satisfy the energy estimate*

$$
(1.13) \qquad E(t) \le E(0)e^{1-\omega t}, \quad \textit{for all} \quad t \ge 0.
$$

If Γ_0 *has a positive measure, then the result holds also for* $a = 0$.

Theorem 1.4. *Assume* (1.1), (1.4). *Then every weak solution of* (1.10) *satisfies*

$$
(1.14) \qquad \lim_{t \to \infty} E(t) = 0.
$$

Remarks. * These results seem to be new even in the isotropic case.

* The proof of Theorem 1.3 will be obtained by applying a Liapunov type method based on an integral inequality applied earlier in Komornik [6, 7]. The proof of Theorem 1.4 will be based on a LaSalle invariance principle.

* If Γ_0 has a positive measure, then the result of Theorem 1.4 holds also for $a = 0$.

* Theorem 1.3 probably remains valid even if $a \geq \frac{(2-\gamma)\alpha}{4R}$. This could be proven by a compactness-uniqueness argument. Since this method does not provide explicit decay rates, we do not study this case here.

2. Stabilizability: proof of Theorem 1.3

We recall (see e.g. Lagnese [8, 9]) that this problem is well-posed in the following sense:

Theorem 2.1. *Assume* (1.1). *Then for every given* $u^0 \in V^n (:= (H^1_{\Gamma_0}(\Omega))^n)$ *and* $y^1 \in (L^2(\Omega))^n$ *the problem* (1.10) *has a unique (weak) solution satisfying*

$$u \in C(\mathbb{R}^+; V^n) \cap C^1(\mathbb{R}^+; (L^2(\Omega))^n),$$

where $H^1_{\Gamma_0}(\Omega) = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_0\}$.

Now assume also that $\bar{\Gamma}_0 \cap \bar{\Gamma}_1 = \emptyset$, *and let* $u^0 \in (H^2(\Omega) \cap V)^n$, $u^1 \in V^n$ *be such that* $\sigma_{ij}(u^0)\nu_j + au_i^0 + bu_i^1 = 0$ *on* Γ_1, $i = 1, \ldots, n$. *Then the corresponding strong solution is more regular:*

$$u \in C(\mathbb{R}^+; (H^2(\Omega) \cap V)^n) \cap C^1(\mathbb{R}^+; V^n) \cap C^2(\mathbb{R}^+; (L^2(\Omega))^n).$$

Let us turn to the proof of Theorem 1.3. All computations which follow will be justified for strong solution. Since the constant ω in (1.13) will not depend on $E(0)$, once the estimates (1.13) will be estabilished for regular solutions, they will be also satisfied for all weak solutions by an easy density argument. For this, we shall prove that $\int_0^\infty E(t)dt \leq \frac{1}{\omega}E(0)$ with ω the positive constant not depending on $E(0)$ and by [5], Th. 8.1 we deduce the estimate (1.13).

First we show the dissipativity of the problem (1.10).

Lemma 2.1. *The function* $E: \mathbb{R}^+ \to \mathbb{R}^+$ *is a non-increasing and*

$$E(0) - E(T) = \int_0^T \int_{\Gamma_1} bu_i' u_i' d\Gamma dt, \quad 0 \leq T < \infty. \tag{2.1}$$

Proof. We have

$$\begin{aligned}E' &= \int_\Omega (u_i'u_i'' + \sigma_{ij}\varepsilon_{ij}')dx + \int_{\Gamma_1} au_iu_i'd\Gamma \\ &= \int_\Omega (u_i'\sigma_{ij,j} + \sigma_{ij}u_{i,j}')dx + \int_{\Gamma_1} au_iu_i'd\Gamma \\ &= \int_{\Gamma_1} u_i'\sigma_{ij}\nu_j d\Gamma + \int_{\Gamma_1} au_iu_i'd\Gamma = -\int_{\Gamma_1} bu_i'u_i'd\Gamma \le 0;\end{aligned}$$

integrating between 0 and T we obtain (2.1).

Let $0 \le T \le \infty$ arbitrarily, we have

$$\begin{aligned}0 &= \int_0^T \int_\Omega u_i(u_i'' - \sigma_{ij,j})dxdt \\ &= \Big[\int_\Omega u_iu_i'dx\Big]_0^T - \int_0^T \int_\Gamma u_i\sigma_{ij}\nu_j d\Gamma dt + \int_0^T \int_\Omega (\sigma_{ij}\varepsilon_{ij} - u_i'u_i')dxdt,\end{aligned}$$

whence

$$\int_0^T \int_\Gamma u_i\sigma_{ij}\nu_j d\Gamma dt = \Big[\int_\Omega u_iu_i'dx\Big]_0^T + \int_0^T \int_\Omega (\sigma_{ij}\varepsilon_{ij} - u_i'u_i')dxdt. \tag{2.2}$$

Fix an arbitrary function $h \in (W^{1,\infty}(\Omega))^n$. We deduce from (1.10) that

$$\begin{aligned}0 &= \int_0^T \int_\Omega (h_m u_{i,m})(u_i'' - \sigma_{ij,j})dxdt \\ &= \Big[\int_\Omega h_m u_{i,m}u_i'dx\Big]_0^T - \int_0^T \int_\Gamma h_m u_{i,m}\sigma_{ij}\nu_j d\Gamma dt + \\ &\quad + \int_0^T \int_\Omega (h_{m,j}\sigma_{ij}u_{i,m} + h_m\sigma_{ij}u_{i,jm} - \frac{1}{2}h_m(u_i'u_i')_m)dxdt.\end{aligned}$$

Since

$$\sigma_{ij}u_{i,jm} = \sigma_{ij}\varepsilon_{ij,m} = \frac{1}{2}(\sigma_{ij}\varepsilon_{ij})_m - \frac{1}{2}(\partial_m a_{ijkl})\varepsilon_{kl}\varepsilon_{ij},$$

integrating by parts the last two terms in the last integral and then multiplying by 2, we obtain the following identity:

$$\begin{aligned}&\int_0^T \int_\Gamma (2h_m u_{i,m}\sigma_{ij}\nu_j + (h.\nu)(u_i'u_i' - \sigma_{ij}\varepsilon_{ij}))d\Gamma dt \\ &= \Big[\int_\Omega 2h_m u_{i,m}u_i'dx\Big]_0^T - \int_0^T \int_\Omega h_m(\partial_m a_{ijkl})\varepsilon_{kl}\varepsilon_{ij}dxdt + \\ &\quad + \int_0^T \int_\Omega (2h_{m,j}\sigma_{ij}u_{i,m} + (\operatorname{div} h)(u_i'u_i' - \sigma_{ij}\varepsilon_{ij}))dxdt.\end{aligned}$$

Multiplying (2.2) by $n-1+\frac{\gamma}{2}+\frac{2aR}{\alpha}$ and combine it with the preceding identity such that $h(x)=x-x_0$. Writing

$$M_i = 2(x_m - x_m^0)u_{i,m} + \Big(n-1+\frac{\gamma}{2}+\frac{2aR}{\alpha}\Big)u_i$$

for simplicity, we have

$$\Big(1-\frac{\gamma}{2}-\frac{2aR}{\alpha}\Big)\int_0^T\int_\Omega(\sigma_{ij}\varepsilon_{ij}+u_i'u_i')dxdt + \Big[\int_\Omega M_iu_i'dx\Big]_0^T$$
$$= \int_0^T\int_\Gamma(M_i\sigma_{ij}\nu_j + (h.\nu)(u_i'u_i' - \sigma_{ij}\varepsilon_{ij}))d\Gamma dt - \frac{4aR}{\alpha}\int_0^T\int_\Omega \sigma_{ij}\varepsilon_{ij}dxdt+$$
$$+\int_0^T\int_\Omega(x_m - x_m^0)(\partial_m a_{ijkl})\varepsilon_{kl}\varepsilon_{ij}dxdt - \gamma\int_0^T\int_\Omega\sigma_{ij}\varepsilon_{ij}dxdt,$$

by (1.6), the last parts of this equality is negative; taking into account the definition (1.11) of the energy, we can rewrite it in the following form:

$$\Big(2-\gamma-\frac{4aR}{\alpha}\Big)\int_0^T E(t)dt + \Big[\int_\Omega M_iu_i'dx\Big]_0^T$$
$$\le \Big(1-\frac{\gamma}{2}-\frac{2aR}{\alpha}\Big)\int_0\int_{\Gamma_1} au_iu_id\Gamma dx - \frac{4aR}{\alpha}\int_0^T\int_\Omega\sigma_{ij}\varepsilon_{ij}dxdt+$$
$$+\int_0^T\int_\Gamma(M_i\sigma_{ij}\nu_j + (h.\nu)(u_i'u_i' - \sigma_{ij}\varepsilon_{ij}))d\Gamma dt.$$

Now using the boundary conditions in (1.10) we obtain

(2.3)
$$\Big(2-\gamma-\frac{4aR}{\alpha}\Big)\int_0^T E(t)dt + \Big[\int_\Omega M_iu_i'dx\Big]_0^T$$
$$\le \int_0^T\int_{\Gamma_0}(h.\nu)\sigma_{ij}\varepsilon_{ij}d\Gamma dt - \frac{4aR}{\alpha}\int_0^T\int_\Omega\sigma_{ij}\varepsilon_{ij}dxdt+$$
$$+\int_0^T\int_{\Gamma_1}\Big(\Big(1-\frac{\gamma}{2}-\frac{2aR}{\alpha}\Big)au_iu_i - M_i(au_i+bu_i')+$$
$$+(h.\nu)(u_i'u_i' - \sigma_{ij}\varepsilon_{ij})\Big)d\Gamma dt.$$

(Note that, from the homogeneous Dirichlet boundary condition in (1.10), we have on Γ_0

$$u_i' = 0 \quad\text{and}\quad h_m u_{i,m}\sigma_{ij}\nu_j = (h.\nu)\sigma_{ij}u_{i,j} = (h.\nu)\sigma_{ij}\varepsilon_{ij}.)$$

Next we transform the integral over Γ_1. Applying twice the Green formula and using the boundary condition on Γ_0 and the relation $h = R\nu$ on Γ_1 we have (cf. [1])

$$(2.4)\quad \int_0^T\int_{\Gamma_1} -2ah_m u_{i,m}u_i d\Gamma dt = \int_0^T\int_\Omega (4aR\varepsilon_{mi}\varepsilon_{mi} - 2aRu_{i,m}u_{i,m} - 2aR|\mathrm{div}\, u|^2)dxdt + \int_0^T\int_{\Gamma_1}(2aR(\mathrm{div}\, u)(\nu.u) - 4aR\varepsilon_{mi}u_i\nu_m)d\Gamma dt.$$

$$(2.5)\quad \int_0^T\int_{\Gamma_1} -2bh_m u_{i,m}u_i' d\Gamma dt = \Big[\int_\Omega (2bR\varepsilon_{mi}\varepsilon_{mi} - bRu_{i,m}u_{i,m} - bR|\mathrm{div}\, u|^2)dx\Big]_0^T + \int_0^T\int_{\Gamma_1}(2bR(\mathrm{div}\, u)(\nu.u') - 4bR\varepsilon_{mi}u_i'\nu_m)d\Gamma dt.$$

Substituting the equalities (2.4) and (2.5) into (2.3) and using the equality $h.\nu = R$ on Γ_1 and $h.\nu \le 0$ on Γ_0 we obtain

$$(2.6)\quad \Big(2-\gamma-\frac{4aR}{\alpha}\Big)\int_0^T E(t)dt \le \Big[\int_\Omega (-M_iu_i' + 2bR\varepsilon_{mi}\varepsilon_{mi} - bRu_{i,m}u_{i,m} - bR|\mathrm{div}\, u|^2)dx\Big]_0^T + \int_0^T\int_\Omega \Big(4aR\varepsilon_{mi}\varepsilon_{mi} - 2aRu_{i,m}u_{i,m} - 2aR|\mathrm{div}\, u|^2 - \frac{4aR}{\alpha}\sigma_{ij}\varepsilon_{ij}\Big)dxdt + \int_0^T\int_{\Gamma_1}\Big(\Big(1-\frac{\gamma}{2}-\frac{2aR}{\alpha}\Big)au_iu_i - \Big(n-1+\frac{\gamma}{2}+\frac{2aR}{\alpha}\Big)u_i(au_i+bu_i') + R(u_i'u_i' - \sigma_{ij}\varepsilon_{ij}) + 2aR(\mathrm{div}\, u)(\nu.u) - 4aR\varepsilon_{mi}u_i\nu_m + 2bR(\mathrm{div}\, u)(\nu.u') - 4bR\varepsilon_{mi}u_i'\nu_m\Big)d\Gamma dt.$$

Let us majorize the right-hand side of this identity. Using the definition o the energy and the Korn inequality,

$$\Big|\int_\Omega (-M_iu_i' + 2bR\varepsilon_{mi}\varepsilon_{mi} - bRu_{i,m}u_{i,m} - bR|\mathrm{div}\, u|^2)dx\Big| \le c_1E(t)$$

and

$$\Big|-b\Big(n-1+\frac{\gamma}{2}+\frac{2aR}{\alpha}\Big)\int_0^T\int_{\Gamma_1} u_iu_i'd\Gamma dt\Big| = \Big|\frac{b}{4}\Big(2n-2+\gamma+\frac{4aR}{\alpha}\Big)\Big[\int_{\Gamma_1} u_iu_i\Big]_0^T\Big| \le c_2E(0)$$

with some constants c_1 and c_2 independent of $E(0)$ and of T.

By the condition (1.1) we have

$$\int_\Omega \Big(4aR\varepsilon_{ij}\varepsilon_{ij} - 2aRu_{i,m}u_{i,m} - 2aR|\operatorname{div} u|^2 - \frac{4aR}{\alpha}\sigma_{ij}\varepsilon_{ij}\Big)dx \le \int_\Omega \Big(\frac{4aR}{\alpha}\sigma_{ij}\varepsilon_{ij} - \frac{4aR}{\alpha}\sigma_{ij}\varepsilon_{ij}\Big)dx = 0.$$

Applying Lemma 2.1 we deduce that

$$R\int_0^T\int_{\Gamma_1} u_i'u_i'\,d\Gamma dt = \frac{R}{b}(E(0)-E(T)) \le \frac{R}{b}E(0).$$

Then we deduce from the identity (2.6) (using also (1.1) on $-R\sigma_{ij}\varepsilon_{ij}$) the following inequality:

$$(2.7)\quad \Big(2-\gamma-\frac{4aR}{\alpha}\Big)\int_0^T E(t)dt \le c_3E(0) + \Big(2-n-\gamma-\frac{4aR}{\alpha}\Big)\int_0^T\int_{\Gamma_1} au_iu_i dx + \int_0^T\int_{\Gamma_1}\Big(-R\alpha\varepsilon_{ij}\varepsilon_{ij} + 2aR(\operatorname{div} u)(u.\nu) - 4aR\varepsilon_{mi}u_i\nu_m + 2bR(\operatorname{div} u)(u'.\nu) - 4bR\varepsilon_{mi}u_i'\nu_m\Big)d\Gamma dt.$$

Here $c_3 = 2c_1 + 2c_2 + \frac{R}{b}$. For any fixed $\delta > 0$ we have

$$\begin{aligned}
2aR(\operatorname{div} u)u.\nu &\le \delta|\operatorname{div} u|^2 + a^2R^2\delta^{-1}|u|^2,\\
2bR(\operatorname{div} u)u'.\nu &\le \delta|\operatorname{div} u|^2 + b^2R^2\delta^{-1}|u'|^2,\\
-4aR\varepsilon_{mi}u_i\nu_m &\le \delta\varepsilon_{mi}\varepsilon_{mi} + 4a^2R^2\delta^{-1}|u|^2,\\
-4bR\varepsilon_{mi}u_i'\nu_m &\le \delta\varepsilon_{mi}\varepsilon_{mi} + 4b^2R^2\delta^{-1}|u'|^2.
\end{aligned}$$

Substituting them into (2.7) and using the inequality $|\operatorname{div} u|^2 \le \varepsilon_{mi}\varepsilon_{mi}$, we obtain

$$\Big(2-\gamma-\frac{4aR}{\alpha}\Big)\int_0^T E(t)dt \le c_3E(0) + \int_0^T\int_{\Gamma_1}\Big(\Big(2-n-\gamma-\frac{4aR}{\alpha}+5aR^2\delta^{-1}\Big)a|u|^2 + (4\delta-\alpha R)\varepsilon_{mi}\varepsilon_{mi} + 5bR^2\delta^{-1}b|u'|^2\Big)d\Gamma dt.$$

Using (2.1) we have

$$5bR^2\delta^{-1}\int_0^T\int_{\Gamma_1} b|u'|^2 d\Gamma dt = 5bR^2\delta^{-1}(E(0)-E(T)) \le 5bR^2\delta^{-1}E(0).$$

Substituting into the preceding inequality and choosing $\delta = \frac{\alpha R}{4}$ hence we conclude that

$$(2.8)\quad \left(2-\gamma-\frac{4aR}{\alpha}\right)\int_0^T E(t)dt \le c_4E(0) + \left(2-n-\gamma+16\frac{aR}{\alpha}\right)\int_0^T\int_{\Gamma_1} a|u|^2 d\Gamma dt$$

with $c_4 = c_3 + 20\frac{bR}{\alpha}$.

Applying a method of Conrad and Rao [3], we shall prove the following lemma (cf. [1])

Lemma 2.2. *For any given $\epsilon > 0$, there exists a constant $c_5 > 0$ such that*

$$\int_0^T\int_{\Gamma_1} |u|^2 d\Gamma dt \le c_5E(0) + \epsilon\int_0^T E(t)dt$$

for all $T \ge 0$.

Choosing $\epsilon > 0$ such that $(2-n-\gamma+\frac{16aR}{\alpha})a\epsilon < 2-\gamma-\frac{4aR}{\alpha}$ if $n < 2-\gamma+\frac{16aR}{\alpha}$ and we deduce from (2.8) the inequality

$$\int_0^T E(t)dt \le cE(0), \quad \text{for all} \quad T \ge 0$$

where c is a constant independent of $E(0)$ and of T, then we conclude that

$$\int_0^\infty E(t)dt \le cE(0)$$

and obtain (1.13) with $\omega = 1/c$.

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A. Guesmia, Institut de Recherche Mathématique Avancée, Université Louis Pasteur et CNRS, 7. rue René Descartes, 67084 Strasbourg Cédex, France; *e-mail:* guesmia@math.u-strasbg.fr