Available online at www.sciencedirect.com
ScienceDirect
Acta Mathematica Scientia 2016,**36B**(1):1–33
Mathematica Acta Scientia 数学物理学报
http://actams.wipm.ac.cn

SOME STABILITY RESULTS FOR TIMOSHENKO SYSTEMS WITH COOPERATIVE FRICTIONAL AND INFINITE-MEMORY DAMPINGS IN THE DISPLACEMENT*

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Abstract In this paper, we consider a vibrating system of Timoshenko-type in a one-dimensional bounded domain with complementary frictional damping and infinite memory acting on the transversal displacement. We show that the dissipation generated by these two complementary controls guarantees the stability of the system in case of the equal-speed propagation as well as in the opposite case. We establish in each case a general decay estimate of the solutions. In the particular case when the wave propagation speeds are different and the frictional damping is linear, we give a relationship between the smoothness of the initial data and the decay rate of the solutions. By the end of the paper, we discuss some applications to other Timoshenko-type systems.

Key words well-posedness; decay; damping; Timoshenko; thermoelasticity

2010 MR Subject Classification 35B37; 35L55; 74D05; 93D15; 93D20

1 Introduction

In this work, we are concerned with the long-time behavior of the solution of the following Timoshenko system:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x + bh(\varphi_t) + \displaystyle\int_0^{+\infty} g(s)(a\varphi_x(t-s))_x \mathrm{d}s = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) = 0, \\ \varphi(0,t) = \psi_x(0,t) = \varphi(L,t) = \psi_x(L,t) = 0, \\ \varphi(x,-t) = \varphi_0(x,t),\ \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x),\ \psi_t(x,0) = \psi_1(x), \end{cases} \tag{P}$$

*Received September 9, 2014; revised February 26, 2015.

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for $(x,t) \in]0,L[\times\mathbb{R}_+$, where $\mathbb{R}_+ = [0,+\infty[$, a, $b : [0,L] \to \mathbb{R}_+$, $g : \mathbb{R}_+ \to \mathbb{R}_+$ and $h : \mathbb{R} \to \mathbb{R}$ are given functions (to be specified later), L, ρ_i, k_i $(i = 1,2)$ are positive constants, φ_0, φ_1, ψ_0 and ψ_1 are given initial and history data, and $(\varphi,\psi) :]0,L[\times\mathbb{R}_+ \to \mathbb{R}^2$ is the state of (P).

Our aim is the study of the asymptotic behavior of the solutions of (P) in case of the equal-speed propagation

$$\frac{k_1}{\rho_1} = \frac{k_2}{\rho_2} \tag{1.1}$$

as well as in the opposite case.

Timoshenko [39], in 1921, introduced the following model to describe the transverse vibration of a beam:

$$\begin{cases} \rho u_{tt} = (K(u_x - \varphi))_x, & \text{in }]0,L[\times\mathbb{R}_+, \\ I_\rho \varphi_{tt} = (EI\varphi_x)_x + K(u_x - \varphi), & \text{in }]0,L[\times\mathbb{R}_+, \end{cases}$$

where t denotes the time variable and x is the space variable along the beam of length L, in its equilibrium configuration, u is the transverse displacement of the beam and φ is the rotation angle of the filament of the beam. The coefficients ρ, I_ρ, E, I and K are, respectively, the density (the mass per unit length), the polar moment of inertia of a cross section, Young's modulus of elasticity, the moment of inertia of a cross section, and the shear modulus. Since then, this model has had attracted the attention of many researchers and an important amount of work has been devoted to the issue of the stabilization and the search for the minimum dissipation by which the solutions decay uniformly to the stable state as time goes to infinity. To achieve this goal, diverse types of dissipative mechanisms have been used and several stability results have been obtained. We mention some of these results (for more results, we refer the reader to the list of references of this paper, which is not exhaustive, and the references therein).

In the case of presence of controls on both the rotation angle and the transverse displacement, investigations showed that the weak solutions of the (P) are stable without any restriction on the constants ρ_1, ρ_2, k_1 and k_2. In this regards, many decay estimates were obtained [14, 18, 23, 26, 34]. However, in the case of only one control on the rotation angle, the rate of decay depends heavily on the constants ρ_1, ρ_2, k_1 and k_2 and the regularity of the initial data. Precisely, if (1.1) holds, the results obtained are similar to those established for the case of the presence controls in both equations. We quote in this regard [4, 7, 12, 13, 14, 16, 17, 24, 25, 29, 30, 31, 38]. But, if (1.1) does not hold, a situation which is more interesting from the physics point of view, then it has been shown that the Timoshenko system is not exponentially stable even for exponentially decaying relaxation functions and only weak decay estimates can be obtained for regular solutions in the presence of dissipation. This has been demonstrated in [1], for the case of an internal feedback, in [7, 14, 16, 17, 27], for the case of finite and infinite memory, and in [10, 13], for complementary internal feedback and finite or infinite memory acting on the rotation angle equation.

For stabilization of Timoshenko systems via heat effect, we mention the pioneer work [28], where the following system:

$$\begin{cases} \rho_1 \varphi_{tt} - \sigma(\varphi_x, \psi)_x = 0, & \text{in }]0,L[\times\mathbb{R}_+, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi) + \gamma\theta_x = 0, & \text{in }]0,L[\times\mathbb{R}_+, \\ \rho_3 \theta_t - k\theta_{xx} + \gamma\psi_{tx} = 0, & \text{in }]0,L[\times\mathbb{R}_+ \end{cases} \tag{1.2}$$

has been considered. In their work, Rivera and Racke established, under appropriate conditions on σ, ρ_i, b, k and γ, several exponential decay results for the linearized system with several boundary conditions. They also proved a non exponential stability result for the case of different wave speeds and proved an exponential decay result for the nonlinear case. Guesmia et al. [15] discussed a linear version of (1.2) and completed the work of [28] by establishing some polynomial decay results in the case of nonequal speed of propagation.

In (1.2), the heat flux is given by Fourier's law. As a result, this theory predicts an infinite speed of heat propagation; that is, any thermal disturbance at one point has an instantaneous effect elsewhere in the body. Experiments showed that heat conduction in some dielectric crystals at low temperatures is free of this paradox and disturbances, which are almost entirely thermal, propagate in a finite speed. This phenomenon in dielectric crystals is called second sound.

To overcome this physical paradox, many theories have merged. One of which suggests that we should replace Fourier's law by Cattaneo's law. In line with this theory, (1.2), in its linear form, becomes

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x = 0, & \text{in }]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x+\psi) + \delta\theta_x = 0, & \text{in }]0,L[\times\mathbb{R}_+, \\ \rho_3\theta_t + \gamma q_x + \delta\psi_{tx} = 0, & \text{in }]0,L[\times\mathbb{R}_+, \\ \tau q_t + q + k\theta_x = 0, & \text{in }]0,L[\times\mathbb{R}_+, \end{cases} \tag{1.3}$$

where q denotes the heat flux. Fernández Sare and Racke [8] studied (1.3) and proved that the equal-speed condition $\frac{\kappa}{\rho_1} = \frac{b}{\rho_2}$ is no longer sufficient to obtain exponential stability even in the presence of an extra viscoelastic dissipation of the form $\int_0^{+\infty} g(s)\psi_{xx}(t-s)\mathrm{d}s$ in the second equation. Very recently, Santos et al. [37] considered (1.3), introduced a new stability number

$$\chi = \left(\tau - \frac{\rho_1}{\kappa\rho_3}\right)\left(\rho_2 - \frac{b\rho_1}{\kappa}\right) - \frac{\tau\rho_1\delta^2}{\kappa\rho_3}$$

and used the semigroup method to obtain an exponential decay result, for $\chi = 0$, and a polynomial decay, for $\chi \neq 0$. See, also, [14, 26, 33, 35, 36].

In all above mentioned works, the stabilization was either via both equation control or the angular rotation equation control. Very recently, Almeida Júnior et al. [2] considered the situation when the control is only on the transverse displacement equation, which is more realistic from the physics point of view. Precisely, they looked into the following system:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x + \mu\varphi_t = 0, & \text{in }]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) = 0, & \text{in }]0,L[\times\mathbb{R}_+ \end{cases} \tag{1.4}$$

and showed that the linear frictional damping in the first equation is strong enough to obtain exponential stability provided that (1.1) holds. They, also, proved some non-exponential and polynomial decay results in the case of nonequal speed situation. The same authors considered in [3]

$$\begin{cases} \rho_1\varphi_{tt} - \kappa(\varphi_x+\psi)_x + \sigma\theta_x = 0, & \text{in }]0,L[\times\mathbb{R}_+, \\ \rho_2\psi_{tt} - b\psi_{xx} + \kappa(\varphi_x+\psi) - \sigma\theta = 0, & \text{in }]0,L[\times\mathbb{R}_+, \\ \rho_3\theta_t - \gamma\theta_{xx} + \sigma(\varphi_x+\psi)_t = 0, & \text{in }]0,L[\times\mathbb{R}_+, \end{cases} \tag{1.5}$$

with various boundary conditions, and established the exponential decay stability for equal-speed case and nonexponential stability for the opposite case. In the case of lack of exponential stability, they proved some algebraic (polynomial) stability for strong solutions.

Our goal in this paper is to investigate the effect of each control on the asymptotic behavior of the solutions of (P) and on the decay rate of its energy, when both controls are acting cooperatively, allowing each control to vanish on the whole domain. We give an explicit and general characterization of the decay rate depending on the growth of g at infinity and h at zero, by considering the case when (1.1) holds and the opposite case. In the latter case, we give a general decay estimate depending on the smoothness of the initial data and the growth of g at infinity.

The proof is based on the multipliers method and an approach introduced by the first author in [9, 11], for a class of abstract hyperbolic systems of single or coupled equations with one infinite memory. In the case when (1.1) does not hold, we use also some ideas given in [10] to get a relation between the decay rate of solutions and the general growth of g at infinity characterized by the condition (2.8) below introduced in [9].

The paper is organized as follows. In Section 2, we set up the hypotheses, discuss briefly the well-posedness and present our stability results. The proofs of these stability results will be given in Section 3, for the equal-speed case, in Section 4, for the nonequal-speed case, and in Section 5, when h is linear. Finally, in Section 6, we discuss some applications to other Timoshenko-type systems.

2 Preliminaries

2.1 Hypotheses

We consider the following hypotheses:

(H1) $a,\, b : [0, L] \to \mathbb{R}_+$ are such that

$$a \in C^1([0, L]),\ b \in L^\infty([0, L]), \tag{2.1}$$

$$\inf_{x\in[0,L]}\{a(x) + b(x)\} > 0, \tag{2.2}$$

$$a \equiv 0 \text{ or } \inf_{x\in[0,L]}\{a(x)\} > 0. \tag{2.3}$$

(H2) $h : \mathbb{R} \to \mathbb{R}$ is a differentiable non-decreasing function such that there exist constants $\epsilon_1,\, c',\, c_1' > 0$, and a convex and increasing function $H : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+)\cap C^2(]0, +\infty[)$ satisfying $H(0) = 0$ and

$$H \text{ is linear on } [0, \epsilon_1]$$

or

$$H'(0) = 0 \text{ and } H'' > 0 \text{ on }]0, \epsilon_1]$$

such that

$$c'|s| \le |h(s)| \le c_1'|s| \quad \text{if } |s| \ge \epsilon_1, \tag{2.4}$$

$$s^2 + h^2(s) \le H^{-1}(sh(s)) \quad \text{if } |s| < \epsilon_1. \tag{2.5}$$

(H3) $g : \mathbb{R}_+ \to \mathbb{R}_+$ is a non-increasing differentiable function such that $g(0) > 0$ and

$$g_0 \|a\|_\infty < \frac{k_1 k_2}{k_0 k_1 + k_2}, \tag{2.6}$$

where $g_0 = \int_0^{+\infty} g(s)\mathrm{d}s$, k_0 is the smallest positive constant satisfying (Poincaré's inequality)

$$\int_0^L v^2 \mathrm{d}x \leq k_0 \int_0^L v_x^2 \mathrm{d}x, \ \forall v \in H_*^1(]0, L[)$$

and

$$H_*^1(]0, L[) = \Big\{ v \in H^1(]0, L[), \int_0^L v(x)\mathrm{d}x = 0 \Big\}.$$

(H4) There exist a positive constant c'' and an increasing strictly convex function $G : \mathbb{R}_+ \to \mathbb{R}_+$ of class $C^1(\mathbb{R}_+) \cap C^2(]0, +\infty[)$ satisfying

$$G(0) = G'(0) = 0 \quad \text{and} \quad \lim_{t \to +\infty} G'(t) = +\infty$$

such that

$$g'(t) \leq -c'' g(t), \ \forall t \geq 0 \tag{2.7}$$

or

$$\int_0^{+\infty} \frac{g(t)}{G^{-1}(-g'(t))} \mathrm{d}t + \sup_{t \in \mathbb{R}_+} \frac{g(t)}{G^{-1}(-g'(t))} < +\infty. \tag{2.8}$$

Remark 2.1 1. The hypothesis (2.8) was introduced in [9] and it allows a wider class of relaxation functions than the ones considered in [7, 27] (see examples given in [9, 14]).

2. Hypothesis (H2) (with $\epsilon_1 = 1$) was introduced and used in [20, 21] to get the asymptotic behavior of solutions of nonlinear wave equations with nonlinear boundary damping, where they obtained decay estimates depending on the solution of an explicit nonlinear ordinary differential equation.

3. Using the second equation and boundary conditions in (P), we easily verify that

$$\partial_{tt} \Big(\int_0^L \psi(x, t)\mathrm{d}x \Big) + \frac{k_1}{\rho_2} \int_0^L \psi(x, t)\mathrm{d}x = 0.$$

By solving this ordinary differential equation and using the initial data of ψ, we find

$$\int_0^L \psi(x, t)\mathrm{d}x = \Big(\int_0^L \psi_0(x)\mathrm{d}x \Big) \cos \Big(\sqrt{\frac{k_1}{\rho_2}} t \Big) + \sqrt{\frac{\rho_2}{k_1}} \Big(\int_0^L \psi_1(x)\mathrm{d}x \Big) \sin \Big(\sqrt{\frac{k_1}{\rho_2}} t \Big).$$

Let

$$\tilde{\psi}(x, t) = \psi(x, t) - \frac{1}{L} \Big(\int_0^L \psi_0(x)\mathrm{d}x \Big) \cos \Big(\sqrt{\frac{k_1}{\rho_2}} t \Big) - \frac{1}{L} \sqrt{\frac{\rho_2}{k_1}} \Big(\int_0^L \psi_1(x)\mathrm{d}x \Big) \sin \Big(\sqrt{\frac{k_1}{\rho_2}} t \Big).$$

Then, one can easily check that

$$\int_0^L \tilde{\psi}(x, t)\mathrm{d}x = 0,$$

and, hence, Poincaré's inequality is applicable for $\tilde{\psi}$. In addition, $(\varphi, \tilde{\psi})$ satisfies (P) with initial data

$$\tilde{\psi}_0(x) = \psi_0(x) - \frac{1}{L} \int_0^L \psi_0(x)\mathrm{d}x \quad \text{and} \quad \tilde{\psi}_1(x) = \psi_1(x) - \frac{1}{L} \int_0^L \psi_1(x)\mathrm{d}x$$

instead of ψ_0 and ψ_1, respectively. In the sequel, we work with $\tilde{\psi}$ instead of ψ, but, for simplicity of notation, we use ψ instead of $\tilde{\psi}$.

4. Thanks to Poincaré's inequality (applied for ψ), we have

$$k_1\int_0^L(\varphi_x+\psi)^2\mathrm{d}x \geq k_1(1-\hat{\epsilon})\int_0^L\varphi_x^2\mathrm{d}x + k_0k_1\Big(1-\frac{1}{\hat{\epsilon}}\Big)\int_0^L\psi_x^2\mathrm{d}x$$

for any $0<\hat{\epsilon}<1$. Then, thanks to (2.6), we can choose $\hat{\epsilon}>0$ such that

$$\frac{k_0k_1}{k_0k_1+k_2}<\hat{\epsilon}<\frac{1}{k_1}\big(k_1-g_0\|a\|_\infty\big)$$

and obtain

$$\hat{k}\int_0^L(\varphi_x^2+\psi_x^2)\mathrm{d}x \leq \int_0^L\big(-g_0\|a\|_\infty\varphi_x^2+k_2\psi_x^2+k_1(\varphi_x+\psi)^2\big)\mathrm{d}x, \tag{2.9}$$

where $\hat{k}=\min\Big\{k_1(1-\hat{\epsilon})-g_0\|a\|_\infty, k_2+k_0k_1(1-\frac{1}{\hat{\epsilon}})\Big\}>0$.

Because $\int_0^L\varphi_x^2\mathrm{d}x$ and $\int_0^L\psi_x^2\mathrm{d}x$ define norms, for φ and ψ on $H_0^1(]0,L[)$ and $H_*^1(]0,L[)$, respectively, then

$$\int_0^L\big(-g_0\|a\|_\infty\varphi_x^2+k_2\psi_x^2+k_1(\varphi_x+\psi)^2\big)\mathrm{d}x$$

defines a norm on $H_0^1(]0,L[)\times H_*^1(]0,L[)$, for (φ,ψ), equivalent to the one induced by $\big(H^1(]0,L[)\big)^2$.

2.2 Well-Posedness

We give here a brief idea about the existence, uniqueness and smoothness of solution of (P). Following the idea of [6], let

$$\eta(x,t,s)=\varphi(x,t)-\varphi(x,t-s),\quad \text{for } (x,t,s)\in]0,L[\times\mathbb{R}_+\times\mathbb{R}_+.$$

Then

$$\begin{cases}\eta_t+\eta_s-\varphi_t=0, & \text{in }]0,L[\times\mathbb{R}_+\times\mathbb{R}_+,\\ \eta(0,t,s)=\eta(L,t,s)=0, & \text{in } \mathbb{R}_+\times\mathbb{R}_+,\\ \eta(x,t,0)=0, & \text{in }]0,L[\times\mathbb{R}_+.\end{cases}$$

Let $\eta_0(x,s)=\eta(x,0,s)=\varphi_0(x,0)-\varphi_0(x,s)$, for $(x,s)\in]0,L[\times\mathbb{R}_+$,

$$\mathcal{H}=\begin{cases}H_0^1(]0,L[)\times H_*^1(]0,L[)\times L^2(]0,L[)\times L_*^2(]0,L[) & \text{if } a\equiv 0,\\ H_0^1(]0,L[)\times H_*^1(]0,L[)\times L^2(]0,L[)\times L_*^2(]0,L[)\times L_g & \text{if } \inf\limits_{x\in[0,L]}\{a(x)\}>0,\end{cases}$$

where

$$L_*^2(]0,L[)=\Big\{v\in L^2(]0,L[),\ \int_0^L v(x)\mathrm{d}x=0\Big\}$$

and

$$L_g=\Big\{v:\mathbb{R}_+\to H_0^1(]0,L[),\ \int_0^L a\int_0^{+\infty}g(s)v_x^2(s)\mathrm{d}s\mathrm{d}x<+\infty\Big\}$$

endowed with the inner product

$$\langle v,w\rangle_{L_g}=\int_0^L a\int_0^{+\infty}g(s)v_x(s)w_x(s)\mathrm{d}s\mathrm{d}x.$$

The space $\mathcal{H}$ is equipped with the inner product defined, if $a \equiv 0$, by

$$\langle V, W\rangle_{\mathcal{H}} = k_1 \int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2)\mathrm{d}x + \int_0^L \big(k_2 \partial_x v_2 \partial_x w_2 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4\big)\mathrm{d}x,$$

for any $V = (v_1, v_2, v_3, v_4)^T \in \mathcal{H}$ and $W = (w_1, w_2, w_3, w_4)^T \in \mathcal{H}$, and, if $\inf\limits_{x\in[0,L]}\{a(x)\} > 0$, by

$$\langle V, W\rangle_{\mathcal{H}} = \langle v_5, w_5\rangle_{L_g} + k_1 \int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2)\mathrm{d}x + \int_0^L \big(- g_0 a \partial_x v_1 \partial_x w_1 + k_2 \partial_x v_2 \partial_x w_2 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4\big)\mathrm{d}x,$$

for any $V = (v_1, v_2, v_3, v_4, v_5)^T \in \mathcal{H}$ and $W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathcal{H}$. Let

$$U = \begin{cases} (\varphi, \psi, \varphi_t, \psi_t)^T & \text{if } a \equiv 0, \\ (\varphi, \psi, \varphi_t, \psi_t, \eta)^T & \text{if } \inf\limits_{x\in[0,L]}\{a(x)\} > 0, \end{cases}$$

$$U_0 = \begin{cases} (\varphi_0, \psi_0, \varphi_1, \psi_1)^T & \text{if } a \equiv 0, \\ (\varphi_0(\cdot, 0), \psi_0, \varphi_1, \psi_1, \eta_0)^T & \text{if } \inf\limits_{x\in[0,L]}\{a(x)\} > 0 \end{cases}$$

and A is the operator defined by $A(v_1, v_2, v_3, v_4)^T = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4)^T$, for any $(v_1, v_2, v_3, v_4)^T \in D(A)$, where

$$\begin{cases} \tilde{v}_1 = -v_3, \\ \tilde{v}_2 = -v_4, \\ \tilde{v}_3 = -\dfrac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2) + \dfrac{b}{\rho_1} h(v_3), \\ \tilde{v}_4 = -\dfrac{k_2}{\rho_2}\partial_{xx} v_2 + \dfrac{k_1}{\rho_2}(\partial_x v_1 + v_2) \end{cases}$$

if $a \equiv 0$, and $A(v_1, v_2, v_3, v_4, v_5)^T = (\tilde{v}_1, \tilde{v}_2, \tilde{v}_3, \tilde{v}_4, \tilde{v}_5)^T$, for any $(v_1, v_2, v_3, v_4, v_5)^T \in D(A)$, where

$$\begin{cases} \tilde{v}_1 = -v_3, \\ \tilde{v}_2 = -v_4, \\ \tilde{v}_3 = -\dfrac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2) + \dfrac{g_0}{\rho_1}\partial_x(a\partial_x v_1) - \dfrac{1}{\rho_1}\displaystyle\int_0^{+\infty} g(s)\partial_x(a\partial_x v_5(s))\mathrm{d}s + \dfrac{b}{\rho_1} h(v_3), \\ \tilde{v}_4 = -\dfrac{k_2}{\rho_2}\partial_{xx} v_2 + \dfrac{k_1}{\rho_2}(\partial_x v_1 + v_2), \\ \tilde{v}_5 = -v_3 + \partial_s v_5 \end{cases}$$

if $\inf\limits_{x\in[0,L]}\{a(x)\} > 0$. The system (P) is equivalent to

$$\begin{cases} U'(t) + AU(t) = 0 \quad \text{on } \mathbb{R}_+, \\ U(0) = U_0. \end{cases} \tag{$\mathcal{P}$}$$

Note that, thanks to (2.4) and the fact that h is continuous, we have

$$\exists h_0 > 0: \ \ |h(s)| \le h_0(1+|s|), \quad \forall s \in \mathbb{R},$$

thus $h(v_3) \in L^2(]0,L[)$, for any $v_3 \in L^2(]0,L[)$. The domain $D(A)$ of A can be characterized by

$$D(A) = \Big\{V = (v_1, v_2, v_3, v_4)^T \in \mathcal{H}, AV \in \mathcal{H}, \ \partial_x v_2(0) = \partial_x v_2(L) = 0\Big\}$$

if $a \equiv 0$, and

$$D(A) = \Big\{V = (v_1, v_2, v_3, v_4, v_5)^T \in \mathcal{H}, AV \in \mathcal{H}, \ \partial_x v_2(0) = \partial_x v_2(L) = 0, \ v_5(0) = 0\Big\}$$

if $\inf\limits_{x\in[0,L]} \{a(x)\} > 0$. We use the classical notation $D(A^0) = \mathcal{H}$, $D(A^1) = D(A)$ and

$$D(A^n) = \Big\{V \in D(A^{n-1}), \ AV \in D(A^{n-1})\Big\}, \quad \text{for } n = 2, 3, \cdots,$$

endowed with the graph norm $\|V\|_{D(A^n)} = \sum\limits_{k=0}^{n} \|A^k V\|_{\mathcal{H}}$.

As in [10] where the frictional damping and infinite memory were considered on the second equation of (P), we can prove that the operator A is maximal monotone; that is $-A$ is dissipative and $Id + A$ is surjective. Then we deduce that A is an infinitesimal generator of a contraction semigroup on $\mathcal{H}$, which implies the following results of existence, uniqueness and smoothness of the solution of $(\mathcal{P})$ (see [19, 32]):

Theorem 2.0 1. For any $U_0 \in \mathcal{H}$, one has a unique solution

$$U \in C(\mathbb{R}_+; \mathcal{H}).$$

2. If $U_0 \in D(A)$, then the solution

$$U \in W^{1,\infty}(\mathbb{R}_+; \mathcal{H}) \cap L^\infty(\mathbb{R}_+; D(A)).$$

3. If h is linear (then A is linear) and $U_0 \in D(A^n)$ (for $n \in \mathbb{N}$), then the solution

$$U \in \bigcap_{k=0}^{n} C^{n-k}(\mathbb{R}_+; D(A^k)).$$

2.3 Stability

The energy functional associated with (P) is defined by

$$E(t) := \frac{1}{2} g \circ \varphi_x + \frac{1}{2} \int_0^L \big(\rho_1 \varphi_t^2 + \rho_2 \psi_t^2 + k_1(\varphi_x + \psi)^2 + k_2 \psi_x^2 - g_0 a \varphi_x^2\big) \mathrm{d}x, \tag{2.10}$$

where

$$\phi \circ v = \int_0^L a \int_0^{+\infty} \phi(s)(v(t) - v(t-s))^2 \mathrm{d}s \mathrm{d}x,$$

for any $v: \mathbb{R} \to L^2(]0,L[)$ and $\phi: \mathbb{R}_+ \to \mathbb{R}_+$.

Now, we give our first main stability result which concerns the case (1.1).

Theorem 2.1 Assume that (1.1) and (H1)–(H4) are satisfied and let $U_0 \in \mathcal{H}$ such that $a \equiv 0$ or (2.7) holds or

$$\sup_{t\in\mathbb{R}_+} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L \varphi_{0x}^2(x, s-t) \mathrm{d}x \mathrm{d}s < +\infty. \tag{2.11}$$

Then there exist positive constants ϵ_0, τ_0, c_1'' and c_2'', for which E satisfies

$$E(t) \le c_1'' \hat{G}^{-1}(c_2'' t), \quad \forall t \ge 0, \tag{2.12}$$

where $\hat{G}(t) = \int_t^1 \frac{1}{\hat{G}_0(s)}\mathrm{d}s$,

$$\hat{G}_0(s) = \begin{cases} H_0(s) & \text{if } a \equiv 0 \text{ or (2.7) holds,} \\ H_0(s)G'(\epsilon_0 H_0(s)) & \text{if } \inf\limits_{x\in[0,L]}\{a(x)\} > 0, \text{ (2.8) holds and (2.7) does not hold} \end{cases} \tag{2.13}$$

and

$$H_0(s) = \begin{cases} s & \text{if } H \text{ is linear on } [0, \epsilon_1], \\ sH'(\tau_0 s) & \text{otherwise.} \end{cases} \tag{2.14}$$

Remark 2.2 1. Because $\lim\limits_{t\to 0^+} G_1(t) = +\infty$, then (2.12) implies that

$$\lim_{t\to+\infty} E(t) = 0. \tag{2.15}$$

2. If $a \equiv 0$ or (2.7) holds, and $b \equiv 0$ or H is linear near zero, then

$$E(t) \le c_1'' e^{-c_2'' t}, \quad \forall t \ge 0, \tag{2.16}$$

which is the best decay rate given by (2.12). For specific examples of decay rates given by (2.12), see [10].

When (1.1) does not hold, we consider the following additional hypothesis:

(H5) Assume that (H2) is satisfied such that H is linear,

$$h \in C^1(\mathbb{R}) \quad \text{and} \quad \inf_{t\in\mathbb{R}} h'(t) > 0.$$

Theorem 2.2 Assume that (H1)–(H5) hold and $U_0 \in D(A)$ such that $a \equiv 0$ or (2.7) holds or

$$\sup_{t\in\mathbb{R}_+} \max_{k=0,1} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L \left(\frac{\partial^k \varphi_{0x}(x, s-t)}{\partial s^k}\right)^2 \mathrm{d}x\mathrm{d}s < +\infty. \tag{2.17}$$

Then there exist positive constants ϵ_0 and c_1 such that

$$E(t) \le G_0^{-1}\Big(\frac{c_1}{t}\Big), \quad \forall t > 0, \tag{2.18}$$

where

$$G_0(s) = \begin{cases} s & \text{if } a \equiv 0 \text{ or (2.7) holds,} \\ sG'(\epsilon_0 s) & \text{if } \inf\limits_{x\in[0,L]}\{a(x)\} > 0, \text{ (2.8) holds and (2.7) does not hold.} \end{cases} \tag{2.19}$$

Remark 2.3 If $a \equiv 0$ or (2.7) holds, then (2.18) becomes

$$E(t) \le \frac{c_1}{t}, \quad \forall t > 0,$$

which is the best decay rate given by (2.18).

In the particular case where h is linear and the initial data are more regular, we prove a more general stability result than (2.18).

Theorem 2.3 Assume that h is linear, and (H1)–(H4) are satisfied. Let $n \in \mathbb{N}^*$ and $U_0 \in D(A^n)$ such that $a \equiv 0$ or (2.7) holds or

$$\sup_{t\in\mathbb{R}_+} \max_{k=0,\cdots,n} \int_t^{+\infty} \frac{g(s)}{G^{-1}(-g'(s))} \int_0^L \left(\frac{\partial^k \varphi_{0x}(x, s-t)}{\partial s^k}\right)^2 \mathrm{d}x\mathrm{d}s < +\infty. \tag{2.20}$$

Then there exist positive constant ϵ_0 and c_n such that

$$E(t) \le G_n\Big(\frac{c_n}{t}\Big), \quad \forall t > 0, \tag{2.21}$$

where $G_m(s) = G_1(sG_{m-1}(s))$, for $m = 2, \cdots, n$ and $s \in \mathbb{R}_+$, $G_1 = G_0^{-1}$ and G_0 is defined in (2.19).

Remark 2.4 If $n = 1$, then (2.18) and (2.21) are the same. On the other hand, if $a \equiv 0$ or (2.7) holds, then (2.21) becomes

$$E(t) \le \frac{c_n}{t^n}, \quad \forall t > 0 \tag{2.22}$$

which is the best decay rate given by (2.21). For specific examples of decay rates given by (2.21), see [11].

3 Proof of Teorem 2.1

We will use c (sometimes c_τ which depends on some parameter τ), throughout this paper, to denote a generic positive constant. Before starting the proofs of our stability resuls, we give the following identity on the derivative of E:

Lemma 3.1 The energy functional satisfies

$$E'(t) = \frac{1}{2} g' \circ \varphi_x - \int_0^L b\varphi_t h(\varphi_t)\mathrm{d}x \le 0. \tag{3.1}$$

Proof By multiplying the first two equations in (P), respectively, by φ_t and ψ_t, integrating over $]0, L[$, and using the boundary conditions, we obtain (3.1) (note that g is non-increasing and $sh(s) \ge 0$, for all $s \in \mathbb{R}$, because h is non-decreasing and $h(0) = 0$ thanks to (2.5)). The estimate (3.1) shows that (P) is dissipative, where the entire dissipation is generated by the frictional damping and/or infinite memory. □

Lemma 3.2 The following inequalities hold:

$$\exists d_1 > 0 : \left(\int_0^L a \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\mathrm{d}x \right)^2 \le d_1 g \circ \varphi_x, \tag{3.2}$$

$$\exists d_2 > 0 : \left(\int_0^L a \int_0^{+\infty} g'(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\mathrm{d}x \right)^2 \le -d_2 g' \circ \varphi_x, \tag{3.3}$$

$$\exists d_3 > 0 : \left(\int_0^L a' \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\mathrm{d}x \right)^2 \le d_3 g \circ \varphi_x. \tag{3.4}$$

$$\left(\int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))\mathrm{d}s \right)^2 \le g_0 \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))^2\mathrm{d}s, \tag{3.5}$$

$$\left(\int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s))\mathrm{d}s \right)^2 \le -g(0) \int_0^{+\infty} g'(s)(\varphi_x(t) - \varphi_x(t-s))^2\mathrm{d}s. \tag{3.6}$$

Proof If $a \equiv 0$, (3.2)–(3.4) are trivial. If $\inf\limits_{x\in[0,L]} \{a(x)\} > 0$, we use the fact that a and a' are bounded and apply Hölder's and Poincaré's inequalities to get (3.2)–(3.4). Using again Hölder's inequality, (3.5) and (3.6) hold. □

Lemma 3.3 The functional

$$I_1(t) := -\rho_1 \int_0^L a\varphi_t \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\,\mathrm{d}x \tag{3.7}$$

satisfies, for any $\delta > 0$,

$$I_1'(t) \le -\rho_1 g_0 \int_0^L a\varphi_t^2 \mathrm{d}x + \delta \int_0^L (\varphi_t^2 + \varphi_x^2 + \psi_x^2)\mathrm{d}x$$
$$+c_\delta \int_0^L bh^2(\varphi_t)\,\mathrm{d}x + c_\delta g \circ \varphi_x - c_\delta g' \circ \varphi_x. \tag{3.8}$$

Proof First, note that

$$\begin{aligned}
&\partial_t\Big(\int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\Big)\\
&= \partial_t\Big(\int_{-\infty}^t g(t-s)(\varphi(t) - \varphi(s))\mathrm{d}s\Big)\\
&= \int_{-\infty}^t g(t-s)\varphi_t(t)\mathrm{d}s + \int_{-\infty}^t g'(t-s)(\varphi(t) - \varphi(s))\mathrm{d}s\\
&= g_0\varphi_t(t) + \int_0^{+\infty} g'(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s.
\end{aligned}$$

Then, by differentiating I_1, and using the first equation and boundary conditions in (P), we find

$$\begin{aligned}
I_1'(t) = &-\rho_1 g_0 \int_0^L a\varphi_t^2\,\mathrm{d}x - \rho_1 \int_0^L a\varphi_t \int_0^{+\infty} g'(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\mathrm{d}x\\
&+k_1 \int_0^L a(\varphi_x + \psi) \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))\mathrm{d}s\mathrm{d}x\\
&+ \int_0^L abh(\varphi_t) \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\mathrm{d}x\\
&+ \int_0^L a^2 \Big(\int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))\mathrm{d}s\Big)^2 \mathrm{d}x\\
&-g_0 \int_0^L a^2\varphi_x \int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))\mathrm{d}s\mathrm{d}x\\
&+ \int_0^L aa' \Big(\int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\Big)\Big(\int_0^{+\infty} g(s)(\varphi_x(t) - \varphi_x(t-s))\mathrm{d}s\Big)\mathrm{d}x\\
&+k_1 \int_0^L a'(\varphi_x + \psi) \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\mathrm{d}x\\
&-g_0 \int_0^L aa'\varphi_x \int_0^{+\infty} g(s)(\varphi(t) - \varphi(t-s))\mathrm{d}s\mathrm{d}x.
\end{aligned}$$

Therefore, applying Hölder's and Young's inequalities, for the last heigh terms of the above equality, and using (3.2), (3.3), (3.4), (3.5), Poincaré's inequality, for φ, and the fact that a, b and a' are bounded, we get (3.8). □

Lemma 3.4 The functional

$$I_2(t) := \int_0^L (\rho_1\varphi\varphi_t + \rho_2\psi\psi_t)\mathrm{d}x$$

satisfies, for any $\delta > 0$,

$$I_2'(t) \le \int_0^L (\rho_1\varphi_t^2 + \rho_2\psi_t^2)\mathrm{d}x - k_1 \int_0^L (\varphi_x + \psi)^2\,\mathrm{d}x - k_2 \int_0^L \psi_x^2\,\mathrm{d}x$$

$$+g_0\int_0^L a\varphi_x^2\,\mathrm{d}x+\delta\int_0^L \varphi_x^2\,\mathrm{d}x+c_\delta\int_0^L bh^2(\varphi_t)\mathrm{d}x+c_\delta g\circ\varphi_x. \tag{3.9}$$

Proof By differentiating I_2, and using the first two equations and boundary conditions in (P), we have

$$\begin{aligned}I_2'(t)=&\int_0^L(\rho_1\varphi_t^2+\rho_2\psi_t^2)\mathrm{d}x-k_1\int_0^L(\varphi_x+\psi)^2\,\mathrm{d}x-k_2\int_0^L\psi_x^2\,\mathrm{d}x\\&+g_0\int_0^L a\varphi_x^2\,\mathrm{d}x-\int_0^L b\varphi h(\varphi_t)\mathrm{d}x-\int_0^L a\varphi_x\int_0^{+\infty}g(s)(\varphi_x(t)-\varphi_x(t-s))\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Consequently, aplying Hölder's and Young's inequalities, for the last two terms of the above equality, and using (3.5), Poincaré's inequality, for φ, and the fact that a and b are bounded, we find (3.9). □

Lemma 3.5 The functional

$$I_3(t):=-\rho_2\int_0^L\psi_t(\varphi_x+\psi)\mathrm{d}x-\frac{k_2\rho_1}{k_1}\int_0^L\psi_x\varphi_t\mathrm{d}x+\frac{\rho_2}{k_1}\int_0^L a\psi_t\int_0^{+\infty}g(s)\varphi_x(t-s)\mathrm{d}s\mathrm{d}x$$

satisfies, for any δ, $\delta_1>0$,

$$\begin{aligned}I_3'(t)\le\ & k_1\int_0^L(\varphi_x+\psi)^2\,\mathrm{d}x-\rho_2\int_0^L\psi_t^2\mathrm{d}x\\&+g_0\Big(\frac{\delta_1}{2}-1\Big)\int_0^L a\varphi_x^2\,\mathrm{d}x+\frac{g_0k_0\|a\|_\infty}{2\delta_1}\int_0^L\psi_x^2\mathrm{d}x+c_\delta\int_0^L bh^2(\varphi_t)\mathrm{d}x\\&+\delta\int_0^L(\psi_t^2+\varphi_x^2+\psi_x^2)\,\mathrm{d}x+c_\delta(g\circ\varphi_x-g'\circ\varphi_x)\\&+\left(\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L\varphi_{xt}\psi_t\mathrm{d}x.\end{aligned}\tag{3.10}$$

Proof Similarly to (3.8) and using that $\lim\limits_{s\to+\infty}g(s)=0$, we see that

$$\begin{aligned}\partial_t\left(\int_0^{+\infty}g(s)\varphi_x(t-s)\mathrm{d}s\right)&=\partial_t\left(\int_{-\infty}^t g(t-s)\varphi_x(s)\mathrm{d}s\right)\\&=g(0)\varphi_x(t)+\int_{-\infty}^t g'(t-s)\varphi_x(s)\mathrm{d}s\\&=g(0)\varphi_x(t)+\int_0^{+\infty}g'(s)(\varphi_x(t-s)-\varphi_x(t)+\varphi_x(t))\mathrm{d}s\\&=-\int_0^{+\infty}g'(s)(\varphi_x(t)-\varphi_x(t-s))\mathrm{d}s.\end{aligned}$$

Therefore, exploiting the first two equations and boundary conditions in (P), we have

$$\begin{aligned}I_3'(t)=\ &k_1\int_0^L(\varphi_x+\psi)^2\mathrm{d}x-\rho_2\int_0^L\psi_t^2\mathrm{d}x+\left(\frac{k_2\rho_1}{k_1}-\rho_2\right)\int_0^L\varphi_{xt}\psi_t\mathrm{d}x\\&-g_0\int_0^L a\varphi_x^2\mathrm{d}x-g_0\int_0^L a\varphi_x\psi\,\mathrm{d}x+\int_0^L a(\varphi_x+\psi)\int_0^{+\infty}g(s)(\varphi_x(t)-\varphi_x(t-s))\mathrm{d}s\mathrm{d}x\\&-\frac{\rho_2}{k_1}\int_0^L a\psi_t\int_0^{+\infty}g'(s)(\varphi_x(t)-\varphi_x(t-s))\mathrm{d}s\mathrm{d}x+\frac{k_2}{k_1}\int_0^L b\psi_x h(\varphi_t)\mathrm{d}x.\end{aligned}$$

By applying Young's inequality, for the last four terms, Poincaré's inequality, for ψ, and using (3.5), (3.6) and the fact that a and b are bounded, (3.10) is established. □

Now, as in [4], we use a function w to get a crucial estimate.

Lemma 3.6 The function

$$w(x,t)=\int_0^x \psi(y,t)\mathrm{d}y \tag{3.11}$$

satisfies the estimates

$$\int_0^L w_x^2\mathrm{d}x=\int_0^L \psi^2\mathrm{d}x, \quad \forall t\geq 0, \tag{3.12}$$

$$\int_0^L w_t^2\mathrm{d}x\leq c\int_0^L \psi_t^2\mathrm{d}x, \quad \forall t\geq 0. \tag{3.13}$$

Proof We just have to note that $w_x=\psi$ to get (3.12). On the other hand,

$$w_t(0,t)=0 \quad \text{and} \quad w_t(L,t)=\int_0^L \psi_t(y,t)\mathrm{d}y=\partial_t\int_0^L \psi(y,t)\mathrm{d}y=0.$$

Then, applying (3.12) to w_t and using Poincaré's inequality, for w_t, we arrive at (3.13). □

Lemma 3.7 The functional

$$I_4(t):=\rho_1\int_0^L (w\varphi_t+\varphi\varphi_t)\mathrm{d}x$$

satisfies, for any δ, ϵ, $\epsilon'>0$,

$$\begin{aligned}I_4'(t)\leq{}&\Big(\rho_1+\frac{c}{\epsilon}\Big)\int_0^L \varphi_t^2\mathrm{d}x+c\epsilon\int_0^L \psi_t^2\mathrm{d}x\\&+\Big(g_0\|a\|_\infty\Big(1+\frac{\epsilon'}{2}\Big)-k_1\Big)\int_0^L(\varphi_x+\psi)^2\mathrm{d}x+\frac{g_0k_0\|a\|_\infty}{2\epsilon'}\int_0^L\psi_x^2\mathrm{d}x\\&+\delta\int_0^L(\varphi_x^2+\psi_x^2)\mathrm{d}x+c_\delta\int_0^L bh^2(\varphi_t)\mathrm{d}x+c_\delta g\circ\varphi_x.\end{aligned}\tag{3.14}$$

Proof Using the first two equations and boundary conditions in (P), and exploiting the fact that $w(0,t)=w(L,t)=0$ and $w_x=\psi$, we find

$$\begin{aligned}I_4'(t)={}&\rho_1\int_0^L\varphi_t^2\,\mathrm{d}x-k_1\int_0^L(\varphi_x+\psi)^2\,\mathrm{d}x+g_0\int_0^L a\varphi_x(\varphi_x+\psi)\,\mathrm{d}x+\rho_1\int_0^L w_t\varphi_t\,\mathrm{d}x\\&-\int_0^L b(w+\varphi)h(\varphi_t)\,\mathrm{d}x-\int_0^L a(\varphi_x+\psi)\int_0^{+\infty}g(s)(\varphi_x(t)-\varphi_x(t-s))\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Applying Young's inequality, for the last four terms, Poincaré's inequality, for φ and ψ, and exploiting (3.5), (3.12), (3.13) and the fact that a and b are bounded, we get (3.14). □

For N, $N_1,N_2,N_3>0$, let

$$I_5(t):=NE(t)+N_1I_1(t)+N_2I_2(t)+I_3(t)+N_3I_4(t). \tag{3.15}$$

Let $a_0:=\inf\limits_{x\in[0,L]}\{a(x)\}$ and $b_0:=\inf\limits_{x\in[0,L]}\{b(x)\}$. Noting that

$$-N_1\rho_1g_0a=-N_1\rho_1g_0a-N_1b+N_1b\leq -N_1(\rho_1g_0a_0+b_0)+N_1b.$$

Then, by combining (3.1), (3.8), (3.9), (3.10) and (3.14), we obtain

$$\begin{aligned}I_5'(t)\leq{}&-\int_0^L(l_0\varphi_t^2+l_1\psi_t^2+l_2(\varphi_x+\psi)^2+l_3\psi_x^2)\mathrm{d}x+l_4g_0\int_0^L a\varphi_x^2\mathrm{d}x\\&+\delta c_{N_1,N_2,N_3}\int_0^L(\varphi_t^2+\psi_t^2+\varphi_x^2+\psi_x^2)\mathrm{d}x-N\int_0^L b\varphi_th(\varphi_t)\mathrm{d}x\end{aligned}$$

$$+c_{N_1,N_2,N_3,\delta}\left(\int_0^L b(\varphi_t^2+h^2(\varphi_t))\mathrm{d}x+g\circ\varphi_x\right)$$
$$+\left(\frac{N}{2}-c_{N_1,\delta}\right)g'\circ\varphi_x+\left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L\varphi_{xt}\psi_t\mathrm{d}x, \tag{3.16}$$

where

$$l_0=N_1(\rho_1g_0a_0+b_0)-(N_2+N_3)\rho_1-\frac{c_0N_3}{\epsilon},$$
$$l_1=\rho_2(1-N_2)-c_0\epsilon N_3,\quad l_2=k_1(N_2+N_3-1)-g_0\|a\|_\infty\left(1+\frac{\epsilon'}{2}\right)N_3,$$
$$l_3=k_2N_2-\frac{g_0k_0\|a\|_\infty}{2}\left(\frac{N_3}{\epsilon'}+\frac{1}{\delta_1}\right),\quad l_4=N_2+\frac{\delta_1}{2}-1$$

and $c_0>0$, independent of N, N_i, δ, δ_1, ϵ and ϵ'. At this point, we choose carefully the constants N, N_i, δ, δ_1, ϵ and ϵ' to get desired signs of l_i.

Case 1 $a\equiv 0$: the second integral in (3.16) drops, $g\circ\varphi_x=g'\circ\varphi_x=0$ and the constants l_0, l_1, l_2 and l_3 do not depent on δ_1 and ϵ'. Therefore, we choose

$$N_3=1,\quad 0<N_2<1,\quad 0<\epsilon<\frac{\rho_2}{c_0}(1-N_2)\quad\text{and}\quad N_1>\frac{1}{b_0}(N_2+N_3)+\frac{c_0N_3}{\epsilon b_0}$$

(note that $b_0>0$ thanks to (2.2)). According to these choices, we get

$$L:=\min\left\{\frac{l_0}{\rho_1},\frac{l_1}{\rho_2},\frac{l_2}{k_1},\frac{l_3}{k_2}\right\}>0,$$

and then, using (2.9), (2.10) and (3.16),

$$I_5'(t)\le-(2L-c\delta)E(t)-N\int_0^L b\varphi_th(\varphi_t)\mathrm{d}x$$
$$+\left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L\varphi_{xt}\psi_t\mathrm{d}x+c_\delta\int_0^L b(\varphi_t^2+h^2(\varphi_t))\mathrm{d}x.$$

Case 2 $a_0>0$: we choose

$$\epsilon'=\frac{k_1-g_0\|a\|_\infty}{g_0\|a\|_\infty},\quad \delta_1=\frac{k_0g_0\|a\|_\infty}{k_2},$$
$$\frac{k_1\delta_1}{2k_1-g_0\|a\|_\infty(2+\epsilon')}<N_3<\epsilon'\left(\frac{k_2(2-\delta_1)}{g_0k_0\|a\|_\infty}-\frac{1}{\delta_1}\right),$$
$$\max\left\{1-N_3\left(1-\frac{g_0\|a\|_\infty(2+\epsilon')}{2k_1}\right),\frac{g_0k_0\|a\|_\infty}{2k_2}\left(\frac{N_3}{\epsilon'}+\frac{1}{\delta_1}\right)\right\}<N_2<1-\frac{\delta_1}{4},$$
$$0<\epsilon<\min\left\{\left(2(1-N_2)-\frac{\delta_1}{2}\right)\frac{\rho_2}{c_0N_3},\frac{\rho_2(1-N_2)}{c_0N_3}\right\}$$

and

$$N_1>\max\left\{\frac{(N_2+N_3)\rho_1+\frac{c_0N_3}{\epsilon}}{\rho_1g_0a_0+b_0},\frac{(2N_2+N_3+\frac{\delta_1}{2}-1)\rho_1+\frac{c_0N_3}{\epsilon}}{\rho_1g_0a_0+b_0}\right\}.$$

Note that ϵ' and δ_1 are positive thanks to (2.6) and $g_0\|a\|_\infty>0$, N_2 exists according to the choice of N_3, ϵ exists from the choice of N_2, and N_1 exists because $\rho_1g_0a_0+b_0>0$. On the other hand, using the definitions of ϵ' and δ_1, we see that N_3 exists if and only if

$$k_0^2k_1(g_0\|a\|_\infty)^3<k_2(k_2-k_0g_0\|a\|_\infty)(k_1-g_0\|a\|_\infty)^2.$$

Let $y_0 = \frac{k_1 k_2}{k_0 k_1 + k_2}$, $y = g_0 \|a\|_\infty \in]0, y_0[$ (see (2.6)) and

$$f(y) = k_0^2 k_1 y^3 - k_2(k_2 - k_0 y)(k_1 - y)^2.$$

We have

$$f'(y) = 3(k_0^2 k_1 + k_0 k_2) y^2 - 2(2k_0 k_1 k_2 + k_2^2) y + k_0 k_1^2 k_2 + 2k_1 k_2^2$$

and

$$f''(y) = 6(k_0^2 k_1 + k_0 k_2) y - 2(2k_0 k_1 k_2 + k_2^2).$$

Let $y_1 = \frac{2k_0 k_1 k_2 + k_2^2}{3(k_0^2 k_1 + k_0 k_2)}$. We notice that f' is decreasing on $]0, y_1[$, it is increasing on $]y_1, +\infty[$ and

$$f'(y_0) = \frac{k_0^2 k_1^3 k_2 + 2k_0 k_1^2 k_2^2}{k_0 k_1 + k_2} > 0,$$

Moreover, $y_1 \le y_0$ if and only if $k_2 \le k_0 k_1$, and, if $k_2 \le k_0 k_1$,

$$f'(y_1) = \frac{5k_0^2 k_1^2 k_2^2 - k_2^4 + 2k_0 k_1 k_2^3 + 3k_0^3 k_1^3 k_2}{3(k_0^2 k_1 + k_0 k_2)} \ge \frac{9k_2^4}{3(k_0^2 k_1 + k_0 k_2)} > 0.$$

Therefore, f' is positive on $]0, y_0[$, and then $f(y) < f(y_0)$, for any $y \in]0, y_0[$. But $f(y_0) = 0$, hence f is negative on $]0, y_0[$. This guarantees the existence of N_3.

By vertue of these choices, we notice that

$$L := \min\left\{\frac{l_0}{\rho_1}, \frac{l_1}{\rho_2}, \frac{l_2}{k_1}, \frac{l_3}{k_2}\right\} > 0 \quad \text{and} \quad l_4 \le L,$$

and then, as in case 1, using (2.9), (2.10) and (3.16), we find

$$\begin{aligned} I_5'(t) \le & -(2L - c\delta)E(t) + c_\delta g \circ \varphi_x + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_{xt}\psi_t \mathrm{d}x \\ & + c_\delta \int_0^L b(\varphi_t^2 + h^2(\varphi_t))\mathrm{d}x - N\int_0^L b\varphi_t h(\varphi_t)\mathrm{d}x + \left(\frac{N}{2} - c_\delta\right) g' \circ \varphi_x. \qquad (3.17) \end{aligned}$$

Choosing $\delta > 0$ small enough in (3.17), we deduce in both cases $a \equiv 0$ and $\inf\limits_{x \in [0,L]}\{a(x)\} > 0$ that

$$\begin{aligned} I_5'(t) \le & -cE(t) + c g \circ \varphi_x + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\int_0^L \varphi_{xt}\psi_t \mathrm{d}x \\ & + c\int_0^L b(\varphi_t^2 + h^2(\varphi_t))\mathrm{d}x - N\int_0^L b\varphi_t h(\varphi_t)\mathrm{d}x + \left(\frac{N}{2} - c\right) g' \circ \varphi_x. \qquad (3.18) \end{aligned}$$

Now, by the definitions of the functionals $I_1 - I_4$ and E, there exists a positive constant β satisfying

$$|N_1 I_1 + N_2 I_2 + I_3 + N_3 I_4| \le \beta E,$$

which implies that

$$(N - \beta)E \le I_5 \le (N + \beta)E.$$

To estimate the last two integrals of (3.18), we use some ideas from [19, 20, 22]. Let

$$\Omega_+ = \{x \in]0, L[: \ |\varphi_t| \ge \epsilon_1\} \quad \text{and} \quad \Omega_- = \{x \in]0, L[: \ |\varphi_t| < \epsilon_1\}, \qquad (3.19)$$

where ϵ_1 is defined in (H2). Using (2.4), we get (note that $sh(s) \ge 0$)

$$c\int_{\Omega_+} b(\varphi_t^2 + h^2(\varphi_t))\mathrm{d}x - N\int_0^L b\psi_t h(\varphi_t)\mathrm{d}x \le (c - N)\int_{\Omega_+} b\varphi_t h(\varphi_t)\mathrm{d}x.$$

Then we choose N large enough so that $c-N\le 0$ (so the right hand side of the above inequality is non-positive), $\frac{N}{2}-c\ge 0$ (so the last term of (3.18) is non-positive) and $N>\beta$ (that is $I_5\sim E$), we get from (3.18)

$$I_5'(t)\le -cE(t)+cg\circ\varphi_x+\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_{xt}\psi_t\mathrm{d}x+c\int_{\Omega_-} b(\varphi_t^2+h^2(\varphi_t))\mathrm{d}x. \tag{3.20}$$

Case 1 H is linear on $[0,\epsilon_1]$: then (2.4) is satisfied on $\mathbb{R}$, and therefore

$$c\int_0^L b(\varphi_t^2+h^2(\varphi_t))\mathrm{d}x-N\int_0^L b\varphi_t h(\varphi_t)\mathrm{d}x\le (c-N)\int_0^L b\varphi_t h(\varphi_t)\mathrm{d}x.$$

So, with the same choice of N, we get from (3.20), for $H_0=Id$ in this case,

$$I_5'(t)\le -cH_0(E(t))+cg\circ\varphi_x+\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_{xt}\psi_t\mathrm{d}x. \tag{3.21}$$

Case 2 $H'(0)=0$ and $H''>0$ on $]0,\epsilon_1]$: without loss of generality, we can assume that H' defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$. Let H^* denote the dual function of the convex function H given by

$$H^*(t)=\sup_{s\in\mathbb{R}_+}\{ts-H(s)\},\quad \forall t\in\mathbb{R}_+.$$

For $t\in\mathbb{R}_+$, the function $s\mapsto ts-H(s)$ reaches its maximum on $\mathbb{R}_+$ at the unique point $(H')^{-1}(t)$. Therefore

$$H^*(t)=t(H')^{-1}(t)-H((H')^{-1}(t)),\quad \forall t\in\mathbb{R}_+.$$

Because H is convex and $H(0)=0$, then, for any $s_0\in\mathbb{R}_+$,

$$H\left(\frac{b}{\max\{1,\|b\|_\infty\}}s_0\right)\le\frac{b}{\max\{1,\|b\|_\infty\}}H(s_0)+\left(1-\frac{b}{\max\{1,\|b\|_\infty\}}\right)H(0)\le bH(s_0),$$

which implis that, for $s_0=H^{-1}(\varphi_t h(\varphi_t))$,

$$bH^{-1}(\varphi_t h(\varphi_t))\mathrm{d}x\le\max\{1,\|b\|_\infty\}H^{-1}(b\varphi_t h(\varphi_t)).$$

Thus, using (2.5),

$$\int_{\Omega_-} b(\varphi_t^2+h^2(\varphi_t))\mathrm{d}x\le\int_{\Omega_-} bH^{-1}(\varphi_t h(\varphi_t))\mathrm{d}x\le c\int_{\Omega_-} H^{-1}(b\varphi_t h(\varphi_t))\mathrm{d}x.$$

Therefore, using Jensen's inequality and (3.1), we find

$$\int_{\Omega_-} b(\varphi_t^2+h^2(\varphi_t))\mathrm{d}x\le cH^{-1}\left(\int_{\Omega_-} cb\varphi_t h(\varphi_t)\mathrm{d}x\right)\le cH^{-1}(-cE'(t)).$$

Consequently, recalling (3.20), we get

$$I_5'(t)\le -cE(t)+cH^{-1}(-cE'(t))+cg\circ\varphi_x+\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_{xt}\psi_t\mathrm{d}x.$$

Let τ_0, $\tau'>0$. The fact that $E'\le 0$, $H''\ge 0$ and $I_5\ge 0$ imply that

$$\begin{aligned}
&\Big(H'(\tau_0E(t))I_5(t)+\tau'E(t)\Big)'\\
&=\tau_0E'(t)H''(\tau_0E(t))I_5(t)+H'(\tau_0E(t))I_5'(t)+\tau'E'(t)\\
&\le H'(\tau_0E(t))\Big(-cE(t)+cH^{-1}(-cE'(t))+cg\circ\varphi_x+\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_{xt}\psi_t\mathrm{d}x\Big)+\tau'E'(t).
\end{aligned}$$

Hence, Young's inequality gives

$$H^{-1}(-cE'(t))H'(\tau_0 E(t)) \leq -cE'(t) + H^*(H'(\tau_0 E(t))),$$

and the fact that $H^*(t) \leq t(H')^{-1}(t)$ and $H'(\tau_0 E)$ is non-increasing leads to

$$\begin{aligned}
&\Big(H'(\tau_0 E(t))I_5(t) + \tau' E(t)\Big)' \\
\leq& \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) H'(\tau_0 E(t)) \int_0^L \varphi_{xt}\psi_t \mathrm{d}x \\
&+cH'(\tau_0 E(0)) g \circ \varphi_x - cH'(\tau_0 E(t))E(t) + cH^*(H'(\tau_0 E(t))) + (\tau' - c)E'(t) \\
\leq& \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) H'(\tau_0 E(t)) \int_0^L \varphi_{xt}\psi_t \mathrm{d}x \\
&+cH'(\tau_0 E(0)) g \circ \varphi_x - cH_0(E(t)) + c\tau_0 H_0(E(t)) + (\tau' - c)E'(t),
\end{aligned}$$

where $H_0(t) = tH'(\tau_0 t)$ in this case. By choosing τ_0 small enough and τ' large enough, we arrive at

$$\left(\frac{H_0(E(t))}{E(t)} I_5(t) + \tau' E(t)\right)' \leq -cH_0(E(t)) + cg\circ\varphi_x + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\frac{H_0(E(t))}{E(t)} \int_0^L \varphi_{xt}\psi_t \mathrm{d}x. \quad (3.22)$$

Let

$$I_6 = \frac{H_0(E)}{E} I_5 + \tau' E,$$

where H_0 is defined by (2.14) ($I_6 = I_5$ if H is linear on $[0, \epsilon_1]$). The functional I_6 satidfies $I_6 \sim E$ (because $I_5 \sim E$ and $\frac{H_0(E)}{E}$ is non-increasing) and, using (3.21) and (3.22),

$$I_6'(t) \leq -cH_0(E(t)) + cg \circ \varphi_x + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right)\frac{H_0(E(t))}{E(t)} \int_0^L \varphi_{xt}\psi_t \mathrm{d}x. \quad (3.23)$$

Now, we estimate the term $g \circ \varphi_x$ in (3.23).

Case 1 $a \equiv 0$ or (2.7) holds: then, using (3.1),

$$g \circ \varphi_x \leq -cg' \circ \varphi_x \leq -cE'(t). \quad (3.24)$$

Case 2 $a_0 > 0$, (2.8) holds and (2.7) does not hold: we apply here the approach introduced in [9, 11] and we get this lemma.

Lemma 3.8 For any $\epsilon_0 > 0$, we have

$$G'(\epsilon_0 H_0(E(t)))g \circ \varphi_x \leq -cE'(t) + c\epsilon_0 H_0(E(t))G'(\epsilon_0 H_0(E(t))). \quad (3.25)$$

Proof Because E is non-increasing,

$$\begin{aligned}
\int_0^L a(\varphi_x(t) - \varphi_x(t-s))^2 \mathrm{d}x &\leq 2\|a\|_\infty \int_0^L \varphi_x^2(t)\mathrm{d}x + 2\|a\|_\infty \int_0^L \varphi_x^2(t-s)\mathrm{d}x \\
&\leq \begin{cases} cE(0) & \text{if } 0 \leq s < t, \\ cE(0) + 2\displaystyle\int_0^L \varphi_{0x}^2(s-t)\mathrm{d}x & \text{if } s \geq t \end{cases} \\
&:= M(t,s).
\end{aligned}$$

Let ϵ_0, $\tau_1(t,s)$, $\tau_2(t,s) > 0$ and $K(s) = \frac{s}{G^{-1}(s)}$ for $s \in \mathbb{R}_+$. The function K is non-decreasing, and therefore,

$$K\left(-\tau_2(t,s)g'(s)\int_0^L a(\varphi_x(t) - \varphi_x(t-s))^2\mathrm{d}x\right) \leq K(-M(t,s)\tau_2(t,s)g'(s)).$$

Using this inequality, we get

$$\begin{aligned} g\circ\varphi_x =& \frac{1}{G'(\epsilon_0 H_0(E(t)))}\int_0^{+\infty}\frac{1}{\tau_1(t,s)}G^{-1}\bigg(-\tau_2(t,s)g'(s)\int_0^L a(\varphi_x(t)-\varphi_x(t-s))^2\mathrm{d}x\bigg)\\ &\times\frac{\tau_1(t,s)G'(\epsilon_0 H_0(E(t)))g(s)}{-\tau_2(t,s)g'(s)}K\bigg(-\tau_2(t,s)g'(s)\int_0^L a(\varphi_x(t)-\varphi_x(t-s))^2\mathrm{d}x\bigg)\mathrm{d}s\\ \le& \frac{1}{G'(\epsilon_0 H_0(E(t)))}\int_0^{+\infty}\frac{1}{\tau_1(t,s)}G^{-1}\bigg(-\tau_2(t,s)g'(s)\int_0^L a(\varphi_x(t)-\varphi_x(t-s))^2\mathrm{d}x\bigg)\\ &\times\frac{\tau_1(t,s)G'(\epsilon_0 H_0(E(t)))g(s)}{-\tau_2(t,s)g'(s)}K(-M(t,s)\tau_2(t,s)g'(s))\mathrm{d}s\\ \le& \frac{1}{G'(\epsilon_0 H_0(E(t)))}\int_0^{+\infty}\frac{1}{\tau_1(t,s)}G^{-1}\bigg(-\tau_2(t,s)g'(s)\int_0^L a(\varphi_x(t)-\varphi_x(t-s))^2\mathrm{d}x\bigg)\\ &\times\frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 H_0(E(t)))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}\mathrm{d}s. \end{aligned}$$

Let G^* denote the dual function of G defined by

$$G^*(t)=\sup_{s\in\mathbb{R}_+}\{ts-G(s)\},\quad \forall t\in\mathbb{R}_+.$$

Thanks to (H4), G' is increasing and defines a bijection from $\mathbb{R}_+$ to $\mathbb{R}_+$, and then, for any $t\in\mathbb{R}_+$, the function $s\mapsto ts-G(s)$ reaches its maximum on $\mathbb{R}_+$ at the unique point $(G')^{-1}(t)$. Therfore

$$G^*(t)=t(G')^{-1}(t)-G((G')^{-1}(t)),\quad \forall t\in\mathbb{R}_+.$$

Using the general Young's inequality: $t_1t_2\le G(t_1)+G^*(t_2)$, for

$$t_1=G^{-1}\bigg(-\tau_2(t,s)g'(s)\int_0^L a(\varphi_x(t)-\varphi_x(t-s))^2\mathrm{d}x\bigg)$$

and

$$t_2=\frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 H_0(E(t)))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))},$$

we get

$$\begin{aligned} g\circ\psi_x \le& \frac{-1}{G'(\epsilon_0 H_0(E(t)))}\int_0^{+\infty}\frac{\tau_2(t,s)}{\tau_1(t,s)}g'(s)\int_0^L a(\varphi_x(t)-\varphi_x(t-s))^2\mathrm{d}x\mathrm{d}s\\ &+\frac{1}{G'(\epsilon_0 H_0(E(t)))}\int_0^{+\infty}\frac{1}{\tau_1(t,s)}G^*\bigg(\frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 H_0(E(t)))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}\bigg)\mathrm{d}s. \end{aligned}$$

Using the fact that $G^*(t)\le t(G')^{-1}(t)$, we get

$$\begin{aligned} g\circ\varphi_x \le& \frac{-1}{G'(\epsilon_0 H_0(E(t)))}\int_0^{+\infty}\frac{\tau_2(t,s)}{\tau_1(t,s)}g'(s)\int_0^L a(\varphi_x(t)-\varphi_x(t-s))^2\mathrm{d}x\mathrm{d}s\\ &+\int_0^{+\infty}\frac{M(t,s)g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}(G')^{-1}\bigg(\frac{M(t,s)\tau_1(t,s)G'(\epsilon_0 H_0(E(t)))g(s)}{G^{-1}(-M(t,s)\tau_2(t,s)g'(s))}\bigg)\mathrm{d}s. \end{aligned}$$

Condition (2.8) implies that

$$\sup_{s\in\mathbb{R}_+}\frac{g(s)}{G^{-1}(-g'(s))}=m_1<+\infty.$$

Then, using the fact that $(G')^{-1}$ is non-decreasing (thanks to (H4)), we get, for $\tau_2(t,s)=\frac{1}{M(t,s)}$,

$$g\circ\varphi_x\le\frac{-1}{G'(\epsilon_0 H_0(E(t)))}\int_0^{+\infty}\frac{1}{\tau_1(t,s)M(t,s)}g'(s)\int_0^L a(\varphi_x(t)-\varphi_x(t-s))^2\mathrm{d}x\mathrm{d}s$$

$$+\int_0^{+\infty}\frac{M(t,s)g(s)}{G^{-1}(-g'(s))}(G')^{-1}\Big(m_1\tau_1(t,s)M(t,s)G'(\epsilon_0 H_0(E(t)))\Big)\mathrm{d}s.$$

Choosing $\tau_1(t,s)=\frac{1}{m_1 M(t,s)}$, and using (3.1) and the fact that

$$\int_0^{+\infty}\frac{M(t,s)g(s)}{G^{-1}(-g'(s))}\mathrm{d}s=m_2<+\infty$$

(thanks to (2.8), (2.11) and the definition of $M(t,s)$), we obtain

$$g\circ\varphi_x\le\frac{-2m_1}{G'(\epsilon_0 H_0(E(t)))}E'(t)+\epsilon_0 m_2 H_0(E(t)),$$

thus (3.25) holds. □

Using (3.23), (3.24) and (3.25), we see that, in both cases,

$$\frac{\hat G_0(E(t))}{H_0(E(t))}I_6'(t)\le-(c-c\epsilon_0)\hat G_0(E(t))-cE'(t)+\Big(\frac{\rho_1 k_2}{k_1}-\rho_2\Big)\frac{\hat G_0(E(t))}{E(t)}\int_0^L\varphi_{xt}\psi_t\mathrm{d}x,$$

where $\hat G_0$ and H_0 are defined in (2.13) and (2.14), respectively. Choosing ϵ_0 small enough, we get

$$\frac{\hat G_0(E(t))}{H_0(E(t))}I_6'(t)\le-c\hat G_0(E(t))-cE'(t)+\Big(\frac{\rho_1 k_2}{k_1}-\rho_2\Big)\frac{\hat G_0(E(t))}{E(t)}\int_0^L\varphi_{xt}\psi_t\mathrm{d}x.\tag{3.26}$$

Let $\tau>0$ and

$$F=\tau\Big(\frac{\hat G_0(E)}{H_0(E)}I_6+cE\Big).\tag{3.27}$$

We have $F\sim E$ (because $I_6\sim E$ and $\frac{\hat G_0(E)}{H_0(E)}$ is non-increasing) and, using (3.26),

$$F'(t)\le-c\tau\hat G_0(E(t))+\tau\Big(\frac{\rho_1 k_2}{k_1}-\rho_2\Big)\frac{\hat G_0(E(t))}{E(t)}\int_0^L\varphi_{xt}\psi_t\mathrm{d}x.\tag{3.28}$$

Thanks to (1.1), the last term of (3.28) vanishes. Then, for $\tau>0$ such that

$$F\le E\quad\text{and}\quad F(0)\le1,\tag{3.29}$$

we get, for $c_2''=c\tau>0$ (since $\hat G_0$ is increasing),

$$F'\le-c_2''\hat G_0(F).\tag{3.30}$$

Then (3.30) implies that $(\hat G(F))'\ge c_2''$, where $\hat G(t)=\int_t^1\frac{1}{\hat G_0(s)}\mathrm{d}s$. Integrating over $[0,t]$ yields

$$\hat G(F(t))\ge c_2''t+\hat G(F(0)).$$

Because $F(0)\le1$, $\hat G(1)=0$ and $\hat G$ is decreasing, we obtain $\hat G(F(t))\ge c_2''t$ which implies that $F(t)\le\hat G^{-1}(c_2''t)$. The fact that $F\sim E$ gives (2.12). This completes the proof of Theorem 2.1.

4 Proof of Teorem 2.2

In this section, we treat the case when (1.1) does not hold which is more realistic from the physics point of view. We will estimate the last term of (3.28) using the system (P2) resulting from differentiating (P) with respect to time

$$\begin{cases}\rho_1\varphi_{ttt}-k_1(\varphi_{xt}+\psi_t)_x+\displaystyle\int_0^{+\infty}g(s)(a\varphi_{xt}(t-s))_x\mathrm{d}s+bh'(\varphi_t)\varphi_{tt}=0,\\ \rho_2\psi_{ttt}-k_2\psi_{xxt}+k_1(\varphi_{xt}+\psi_t)=0,\\ \varphi_{xt}(0,t)=\psi_t(0,t)=\varphi_{xt}(L,t)=\psi_t(L,t)=0.\end{cases}\tag{P2}$$

System (P2) is well posed for initial data $U_0 \in D(A)$. Let E_2 be the second-order energy (the energy of (P2)) defined by $E_2(t) = E_1(\varphi_t, \psi_t)(t)$, where $E_1(\varphi, \psi)(t) = E(t)$, defined by (2.10). A simple calculation (as for (3.1)) implies that

$$E_2'(t) = \frac{1}{2} g' \circ \varphi_{xt} - \int_0^L b h'(\varphi_t) \varphi_{tt}^2 \mathrm{d}x. \tag{4.1}$$

Because $\inf\limits_{t \in \mathbb{R}} h'(t) > 0$ thanks to hypothesis (H5), we have

$$E_2'(t) \leq \frac{1}{2} g' \circ \varphi_{xt} - c \int_0^L b \varphi_{tt}^2 \mathrm{d}x \leq 0. \tag{4.2}$$

Let $\tau = 1$ in (3.27). Thanks to (H5), H is linear and then (3.28) holds for $H_0 = Id$. Thus,

$$F'(t) \leq -cG_0(E(t)) + \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t \mathrm{d}x, \tag{4.3}$$

where G_0 is defined in (2.19). Now, we proceed as in [7] and we use some ideas of [10].

Lemma 4.1 For any $\epsilon > 0$, we have

$$\begin{aligned} &\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \varphi_{xt} \psi_t \mathrm{d}x \mathrm{d}t \\ \leq\ & \epsilon \int_S^T G_0(E(t)) \mathrm{d}t + c_\epsilon \int_S^T \frac{G_0(E(t))}{E(t)} (g \circ \varphi_{xt} - g' \circ \varphi_x) \mathrm{d}t \\ &+ c_\epsilon \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)),\ \forall T \geq S \geq 0. \end{aligned} \tag{4.4}$$

Proof We distinguish two cases (corresponding to hypothesis (2.3)).

Case 1 $\inf\limits_{x \in [0,L]} \{a(x)\} > 0$: we have $\inf\limits_{x \in [0,L]} \{a(x)\} := a_0 > 0$, and then

$$\begin{aligned} &\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \varphi_{xt} \psi_t \mathrm{d}x \\ =\ & \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \psi_t \int_0^{+\infty} g(s) \varphi_{xt}(t-s) \mathrm{d}s \mathrm{d}x \\ &+ \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \psi_t \int_0^{+\infty} g(s) (\varphi_{xt}(t) - \varphi_{xt}(t-s)) \mathrm{d}s \mathrm{d}x. \end{aligned} \tag{4.5}$$

Using Young's inequality and (3.5) (for φ_{xt} instead of φ_x), we get for all $\epsilon > 0$

$$\begin{aligned} &\frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \psi_t \int_0^{+\infty} g(s) (\varphi_{xt}(t) - \varphi_{xt}(t-s)) \mathrm{d}s \mathrm{d}x \\ \leq\ & c \int_0^L a |\psi_t| \int_0^{+\infty} g(s) |\varphi_{xt}(t) - \varphi_{xt}(t-s)| \mathrm{d}s \mathrm{d}x \\ \leq\ & \frac{\epsilon}{2} E(t) + c_\epsilon g \circ \varphi_{xt}. \end{aligned}$$

On the other hand, by integrating by parts and using (3.6), we obtain

$$\begin{aligned} &\frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \psi_t \int_0^{+\infty} g(s) \varphi_{xt}(t-s) \mathrm{d}s \mathrm{d}x \\ =\ & \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \psi_t \left(g(0) \varphi_x + \int_0^{+\infty} g'(s) \varphi_x(t-s) \mathrm{d}s\right) \mathrm{d}x \end{aligned}$$

$$= \frac{\frac{\rho_1 k_2}{k_1} - \rho_2}{a_0 g_0} \int_0^L a_0 \psi_t \int_0^{+\infty} (-g'(s))(\varphi_x(t) - \varphi_x(t-s)) \mathrm{d}s \mathrm{d}x$$
$$\leq \frac{\epsilon}{2} E(t) - c_\epsilon g' \circ \varphi_x.$$

Inserting these last two inequalities into (4.5), multiplying by $\frac{G_0(E)}{E}$, integrating over $[S,T]$, noting that $\frac{G_0(E)}{E}$ is non-increasing and using (3.1), we obtain (4.4).

Case 2 $a \equiv 0$: according to (2.2), we have $\inf_{x\in[0,L]} \{b(x)\} := b_0 > 0$, and then, by integration with respect to t and using the definition of $\frac{G_0(E)}{E}$, E and E_2 and their non-increasingness, we get

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \psi_t \varphi_{xt} \mathrm{d}x \mathrm{d}t$$
$$= \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \left[\frac{G_0(E(t))}{E(t)} \int_0^L \psi \varphi_{xt} \mathrm{d}x\right]_S^T - \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \left(\frac{G_0(E(t))}{E(t)}\right)' \int_0^L \psi \varphi_{xt} \mathrm{d}x \mathrm{d}t$$
$$- \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \psi \varphi_{xtt} \mathrm{d}x \mathrm{d}t.$$

Using the fact that (by vertue of Poincaré's inequality)

$$\left| \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_0^L \psi \varphi_{xt} \mathrm{d}x \right| \leq c(E(t) + E_2(t)) \leq c(E(S) + E_2(S)), \quad \forall t \geq S \geq 0.$$

Therefore, by integrating by parts the last integral with respect to x and noting that $\frac{G_0(E)}{E}$ is non-increasing, we have

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \psi_t \varphi_{xt} \mathrm{d}x \mathrm{d}t$$
$$\leq c \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)) - c(E(S) + E_2(S)) \int_S^T \left(\frac{G_0(E(t))}{E(t)}\right)' dt$$
$$+ \left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \psi_x \varphi_{tt} \mathrm{d}x \mathrm{d}t, \quad \forall T \geq S \geq 0.$$

Using the fact that $\inf_{x\in[0,L]} \{b(x)\} > 0$, we deduce that

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \psi_t \varphi_{xt} \mathrm{d}x \mathrm{d}t$$
$$\leq c \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)) + c \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L b|\psi_x||\varphi_{tt}| \mathrm{d}x \mathrm{d}t.$$

Therefore, using Young's inequality and (4.2), we estimate the last integral as follows:

$$\left(\frac{\rho_1 k_2}{k_1} - \rho_2\right) \int_S^T \frac{G_0(E(t))}{E(t)} \int_0^L \psi_t \varphi_{xt} \mathrm{d}x \mathrm{d}t$$
$$\leq c \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)) + \epsilon \int_S^T G_0(E(t)) \mathrm{d}t - c_\epsilon \frac{G_0(E(0))}{E(0)} \int_S^T E_2'(t) \mathrm{d}t$$
$$\leq c_\epsilon \frac{G_0(E(0))}{E(0)} (E(S) + E_2(S)) + \epsilon \int_S^T G_0(E(t)) \mathrm{d}t, \quad \forall T \geq S \geq 0. \tag{4.6}$$

This implies (4.4). □

Now, exploiting (4.3) and (4.4) and choosing ϵ small enough, we get

$$\int_S^T F'(t)\mathrm{d}t \le -c\int_S^T G_0(E(t))\mathrm{d}t + c\frac{G_0(E(0))}{E(0)}(E(S)+E_2(S)) \\ +c\int_S^T \frac{G_0(E(t))}{E(t)}(g\circ\varphi_{xt} - g'\circ\varphi_x)\mathrm{d}t, \quad \forall T\ge S\ge 0,$$

and recalling (3.1) and the fact that $F\sim E$ and $\frac{G_0(E)}{E}$ is non-increasing, we have

$$\int_S^T G_0(E(t))\mathrm{d}t \le c\Big(1+\frac{G_0(E(0))}{E(0)}\Big)(E(S)+E_2(S)) + c\int_S^T \frac{G_0(E(t))}{E(t)} g\circ\varphi_{xt}\mathrm{d}t. \tag{4.7}$$

To estimate the last term in (4.7), we distingish two cases.

Case 1 $a\equiv 0$ or (2.7) holds: we have $G_0 = Id$. Using (4.2), we get

$$\frac{G_0(E(t))}{E(t)} g\circ\varphi_{xt} = g\circ\varphi_{xt} \le -cg'\circ\varphi_{xt} \le -cE_2'(t).$$

Case 2 $\inf\limits_{x\in[0,L]}\{a(x)\} > 0$, (2.8) holds and (2.7) does not hold: in this case, $G_0(s) = sG'(\epsilon_0 s)$ with $\epsilon_0 > 0$. Therefore, using (2.17) and similarly to (3.25) for $g\circ\varphi_{xt}$ instead of $g\circ\varphi_x$ (here $H_0 = Id$), we get, using also (4.2),

$$\frac{G_0(E(t))}{E(t)} g\circ\varphi_{xt} \le -cE_2'(t) + c\epsilon_0 G_0(E(t)), \quad \forall \epsilon_0 > 0.$$

Then we get in both cases

$$\int_S^T \frac{G_0(E(t))}{E(t)} g\circ\varphi_{xt}\mathrm{d}t \le -c\int_S^T E_2'(t)\mathrm{d}t + c\epsilon_0\int_S^T G_0(E(t))\mathrm{d}t, \quad \forall\epsilon_0>0,\ \forall T\ge S\ge 0.$$

Inserting this inequality into (4.7) and choosing ϵ_0 small enough, we deduce that

$$\int_S^T G_0(E(t))\mathrm{d}t \le c\Big(1+\frac{G_0(E(0))}{E(0)}\Big)(E(S)+E_2(S)), \quad \forall T\ge S\ge 0. \tag{4.8}$$

Choosing $S=0$ in (4.8) and using the fact that $G_0(E)$ is non-increasing, we get

$$G_0(E(T))T \le \int_0^T G_0(E(t))\mathrm{d}t \le c\Big(1+\frac{G_0(E(0))}{E(0)}\Big)(E(0)+E_2(0)), \quad \forall T\ge 0,$$

which gives (2.18) with $c_1 = c\big(1+\frac{G_0(E(0))}{E(0)}\big)(E(0)+E_2(0))$.

5 Proof of Theorem 2.3

We prove (2.21) by induction on n. For $n=1$, condition (2.20) coincides with (2.17), and (2.21) is exactly (2.18).

Now, suppose that (2.21) holds and let $U_0\in D(A^{n+1})$ satisfying (2.20), for $n+1$ instead of n. We have $U_t(0)\in D(A^n)$ (thanks to Theorem 2.0–3), $U_t(0)$ satisfies (2.20) (because U_0 satisfies (2.20), for $n+1$) and U_t satisfies the first two equations and the boundary conditions of (P), and then the energy E_2 of (P2) (defined in Section 4) also satisfies, for some positive constant $\tilde{c}_n$,

$$E_2(t) \le G_n\Big(\frac{\tilde{c}_n}{t}\Big), \quad \forall t>0. \tag{5.1}$$

Now, choosing $S = \frac{T}{2}$ in (4.8), combining with (2.21) and (5.1), and using the fact that $G_0(E)$ is non-increasing, we deduce that

$$G_0(E(T))T \leq 2\int_{\frac{T}{2}}^{T} G_0(E(t))\mathrm{d}t \leq c\Big(1+\frac{G_0(E(0))}{E(0)}\Big)\Big(G_n\Big(\frac{2c_n}{T}\Big)+G_n\Big(\frac{2\tilde{c}_n}{T}\Big)\Big),$$

this implies that, for $c_{n+1} = \max\Big\{c\Big(1+\frac{G_0(E(0))}{E(0)}\Big), 2c_n, 2\tilde{c}_n\Big\}$,

$$E(T) \leq G_0^{-1}\Big(\frac{c_{n+1}}{T}G_n\Big(\frac{c_{n+1}}{T}\Big)\Big) = G_{n+1}\Big(\frac{c_{n+1}}{T}\Big).$$

This proves (2.21), for $n+1$. The proof of Theorem 2.3 is completed.

Remark 5.1 One important system related to (P) is the following system:

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x + b(x)h(\varphi_t) + \displaystyle\int_0^{+\infty} (a(x)g(s)\varphi_x(t-s))_x\,\mathrm{d}s = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) - \displaystyle\int_0^{+\infty} a(x)g(s)\varphi_x(t-s)\mathrm{d}s = 0, \end{cases}$$

which results from the governing equations

$$\rho_1\varphi_{tt} = S_x \quad \text{and} \quad \rho_2\psi_{tt} = M_x - S,$$

taking into account the action on two tensors

$$S = k_1(\varphi_x+\psi) - \int_0^{+\infty} a(x)g(s)\varphi_x(t-s)\mathrm{d}s \quad \text{and} \quad M = k_2\psi_x.$$

This system looks more realistic than (P) from the physics point view. However the energy given by (2.10) is not dissipative.

We believe that such a system is worth looking at and a "modified" energy needs to be defined, as well the functionals used to prove stability.

6 Applications

In this section, we give applications of our results of section 2 to some Timoshenko-type systems.

6.1 Timoshenko-heat

We start by considering coupled Timoshenko-heat system on $]0, L[$ under Fourier's law of heat conduction and in the presence of an infinite memory acting on the first equation. That is,

$$\begin{cases} \rho_1\varphi_{tt} - k_1(\varphi_x+\psi)_x - \gamma\theta_x + \displaystyle\int_0^{+\infty} (ag(s)\varphi_x(t-s))_x\,\mathrm{d}s = 0, \\ \rho_2\psi_{tt} - k_2\psi_{xx} + k_1(\varphi_x+\psi) = 0, \\ \rho_3\theta_t - \kappa\theta_{xx} + \gamma\varphi_{xt} = 0, \\ \varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0, \\ \varphi(x,-t) = \varphi_0(x,t), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \quad \theta(x,0) = \theta_0(x), \end{cases} \tag{6.1}$$

where φ, ψ and θ are functions of (x,t) and denote the transverse displacement of the beam, the rotation angle of the filament, and the difference temperature, respectively, $\rho_i, k_i, \gamma, \kappa, L$ are positive constants, and the functions a and g are as in Section 2.

From the third equation in (6.1) and the boundary conditions, we easily verify that

$$\partial_t \int_0^L \theta(x,t)\mathrm{d}x = 0.$$

By solving this ordinary differential equation and using the initial data of θ, we get

$$\int_0^L \theta(x,t)\mathrm{d}x = \int_0^L \theta_0(x)\mathrm{d}x.$$

So, we set

$$\tilde{\theta}(x,t) = \theta(x,t) - \frac{1}{L}\int_0^L \theta_0(x)\mathrm{d}x$$

to conclude that $(\varphi, \tilde{\psi}, \tilde{\theta})$ satisfies (6.1), with initial data

$$\tilde{\theta}_0(x) = \theta_0(x) - \frac{1}{L}\int_0^L \theta_0(x)\mathrm{d}x$$

instead of θ_0, and more importantly

$$\int_0^L \tilde{\theta}(x,t)\mathrm{d}x = 0;$$

which implies that Poincaré's inequality is applicable for $\tilde{\theta}$. In the sequel, we work with $\tilde{\theta}$ instead of θ, but, for simplicity of notation, we use θ instead of $\tilde{\theta}$.

6.1.1 Well-Posedness

By combining arguments from the Subsection 2.2 above and Subsection 6.1 of [14], one can easily establish the well-posedness of (6.1). For this purpose, we define η as in Subsection 2.2 and set

$$\mathcal{H} = \begin{cases} \tilde{\mathcal{H}} & \text{if } a \equiv 0, \\ \tilde{\mathcal{H}} \times L_g & \text{if } \inf\limits_{x\in[0,L]}\{a(x)\} > 0, \end{cases}$$

where L_g and its inner product are given in Subsection 2.2, and

$$\tilde{\mathcal{H}} = H_0^1(]0,L[) \times H_*^1(]0,L[) \times L^2(]0,L[) \times L_*^2(]0,L[) \times L_*^2(]0,L[).$$

If $a \equiv 0$, the space $\mathcal{H}$ is equipped with the inner product

$$\begin{aligned} \langle V, W\rangle = {}& k_1 \int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2)\mathrm{d}x \\ & + \int_0^L (k_2 \partial_x v_2 \partial_x w_2 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4 + \rho_3 v_5 w_5)\,\mathrm{d}x, \end{aligned}$$

for any $V = (v_1, v_2, v_3, v_4, v_5)^T$, $W = (w_1, w_2, w_3, w_4, w_5)^T \in \mathcal{H}$, and if $\inf\limits_{x\in[0,L]}\{a(x)\} > 0$, we equip $\mathcal{H}$ with the inner product

$$\begin{aligned} \langle V, W\rangle = {}& \langle v_6, w_6\rangle_{L_g} + k_1 \int_0^L (\partial_x v_1 + v_2)(\partial_x w_1 + w_2)\mathrm{d}x \\ & + \int_0^L (-g_0 a \partial_x v_1 \partial_x w_1 + k_2 \partial_x v_2 \partial_x w_2 + \rho_1 v_3 w_3 + \rho_2 v_4 w_4 + \rho_3 v_5 w_5)\,\mathrm{d}x, \end{aligned}$$

for any $V = (v_1, v_2, v_3, v_4, v_5, v_6)^T$, $W = (w_1, w_2, w_3, w_4, w_5, w_6)^T \in \mathcal{H}$. By letting

$$U = \begin{cases} (\varphi, \psi, \varphi_t, \psi_t, \theta)^T & \text{if}\, a \equiv 0, \\ (\varphi, \psi, \varphi_t, \psi_t, \theta, \eta)^T & \text{if} \inf\limits_{x\in[0,L]} \{a(x)\} > 0 \end{cases}$$

and

$$U_0 = \begin{cases} (\varphi_0, \psi_0, \varphi_1, \psi_1, \theta_0)^T & \text{if}\, a \equiv 0, \\ (\varphi_0(.,0), \psi_0, \varphi_1, \psi_1, \theta_0, \eta_0)^T & \text{if} \inf\limits_{x\in[0,L]} \{a(x)\} > 0, \end{cases}$$

problem (6.1) can be written as

$$\begin{cases} U' + AU = 0 \quad \text{on}\, \mathbb{R}_+, \\ U(0) = U_0, \end{cases} \tag{6.2}$$

where, if $a \equiv 0$,

$$AV = \begin{cases} -v_3, \\ -v_4, \\ -\dfrac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2) - \dfrac{\gamma}{\rho_1}\partial_x v_5, \\ -\dfrac{k_2}{\rho_2}\partial_{xx} v_2 + \dfrac{k_1}{\rho_1}(\partial_x v_1 + v_2), \\ -\dfrac{\kappa}{\rho_3}\partial_{xx} v_5 + \dfrac{\gamma}{\rho_3} v_3, \end{cases}$$

for any $V = (v_1, v_2, v_3, v_4, v_5)^T \in D(A)$ and, if $\inf\limits_{x\in[0,L]} \{a(x)\} > 0$,

$$AV = \begin{cases} -v_3, \\ -v_4, \\ -\dfrac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2) + \dfrac{g_0}{\rho_1}\partial_x(a\partial_x v_1) - \dfrac{1}{\rho_1}\displaystyle\int_0^{+\infty} g(s)\partial_x(a\partial_x v_6(s))\mathrm{d}s - \dfrac{\gamma}{\rho_1}\partial_x v_5, \\ -\dfrac{k_2}{\rho_2}\partial_{xx} v_2 + \dfrac{k_1}{\rho_1}(\partial_x v_1 + v_2), \\ -\dfrac{\kappa}{\rho_3}\partial_{xx} v_5 + \dfrac{\gamma}{\rho_3} v_3, \\ -v_3 + \partial_s v_5, \end{cases}$$

for any $V = (v_1, v_2, v_3, v_4, v_5, v_6)^T \in D(A)$. By noting that (6.2) is linear and exploiting the semigroup theory [19, 32], one can easily prove the following:

Theorem 6.1 For any $n \in \mathbb{N}$ and $U_0 \in D(A^n)$, problem (6.2) has a unique solution

$$U \in \bigcap_{k=0}^{n} C^{n-k}(\mathbb{R}_+; D(A^k)).$$

6.1.2 Stability

Similarly to (P), we establish a general stability result for solutions of (6.1), under the hypotheses (H3) and (H4). we define the first-order energy of (6.1) by

$$E(t) = \frac{1}{2} g \circ \varphi_x + \frac{1}{2}\int_0^L \left((\rho_1\varphi_t^2 + \rho_2\psi_t^2 + k_1(\varphi_x + \psi)^2 + k_2\psi_x^2 - g_0 a\varphi_x^2 + \rho_3\theta^2\right) \mathrm{d}x. \tag{6.3}$$

Straightforward computations yield

$$E'(t) = -\kappa \int_0^L \theta_x^2 \mathrm{d}x + \frac{1}{2} g' \circ \varphi_x \le 0. \tag{6.4}$$

Now, we give our first stability result.

Theorem 6.2 Assume (1.1), (2.1), (2.3), (H3) and (H4) hold, and let $U_0 \in H$ such that $a \equiv 0$ or (2.7) or (2.11) is satisfied. Then, the energy E satisfies (2.12) with $\hat{G}(t) = \int_t^1 \frac{1}{G_0(s)} \mathrm{d}s$ and G_0 is defined in (2.19).

In order to prove our main result, we adopt several functionals from section 2 and prove several lemmas.

Lemma 6.3 The functional

$$I_2(t) = \int_0^L (\rho_1 \varphi \varphi_t + \rho_2 \psi \psi_t) \mathrm{d}x$$

satisfies, for any $\delta > 0$,

$$\begin{aligned} I_2'(t) \le & \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) \mathrm{d}x - k_1 \int_0^L (\varphi_x + \psi)^2 \mathrm{d}x - k_2 \int_0^L \psi_x^2 \, \mathrm{d}x \\ & + g_0 \int_0^L a \varphi_x^2 \mathrm{d}x + \delta \int_0^L \varphi_x^2 \mathrm{d}x + c_\delta g \circ \varphi_x + c_\delta \int_0^L \theta_x^2 \, \mathrm{d}x. \end{aligned} \tag{6.5}$$

Proof By using equations (6.1), a simple integration leads to

$$\begin{aligned} I_2'(t) = & \int_0^L (\rho_1 \varphi_t^2 + \rho_2 \psi_t^2) \mathrm{d}x - k_2 \int_0^L \psi_x^2 \mathrm{d}x - k_1 \int_0^L (\varphi_x + \psi)^2 \mathrm{d}x \\ & - \gamma \int_0^L \varphi \theta_x \mathrm{d}x + \int_0^L \varphi_x \left(\int_0^{+\infty} a(x) g(s) \varphi_x(t) \mathrm{d}s \right) \mathrm{d}x \\ & + \int_0^L \varphi_x \left(\int_0^{+\infty} a g(s) (\varphi_x(t-s) - \varphi_x(t)) \mathrm{d}s \right) \mathrm{d}x. \end{aligned}$$

Exploiting Young's and Poincaré's inequalities, (6.5) follows. □

Lemma 6.4 The functional

$$I_3(t) = -\rho_2 \int_0^L \psi_t (\varphi_x + \psi) \mathrm{d}x - \frac{\rho_1 k_2}{k_1} \int_0^L \varphi_t \psi_x \mathrm{d}x + \frac{\rho_2}{k_1} \int_0^L a \psi_t \int_0^{+\infty} g(s) \varphi_x(t-s) \mathrm{d}s \mathrm{d}x$$

satisfies, for any δ, $\delta_1 > 0$,

$$\begin{aligned} I_3'(t) \le & k_1 \int_0^L (\varphi_x + \psi)^2 \mathrm{d}x - \rho_2 \int_0^L \psi_t^2 \mathrm{d}x + g_0 \Big(\frac{\delta_1}{2} - 1 \Big) \int_0^L a \varphi_x^2 \mathrm{d}x \\ & + \frac{g_0 k_0 ||a||_\infty}{2\delta_1} \int_0^L \psi_x^2 \mathrm{d}x + \delta \int_0^L (\psi_t^2 + \varphi_x^2 + \psi_x^2) \mathrm{d}x + c_\delta g \circ \varphi_x - c_\delta g' \circ \varphi_x \\ & + c_\delta \int_0^L \theta_x^2 \mathrm{d}x + \Big(\frac{\rho_1 k_2}{k_1} - \rho_2 \Big) \int_0^L \varphi_{xt} \psi_t \mathrm{d}x. \end{aligned} \tag{6.6}$$

Proof Differentiation of I_3, using equations (6.1), gives

$$\begin{aligned} I_3'(t) = & k_1 \int_0^L (\varphi_x + \psi)^2 \mathrm{d}x - \rho_2 \int_0^L \psi_t^2 \mathrm{d}x + \frac{k_2}{k_1} \gamma \int_0^L \psi_x \theta_x \mathrm{d}x \\ & - \frac{\rho_2}{k_1} \int_0^L a \psi_t \int_0^{+\infty} g'(s) \left(\varphi_x(t) - \varphi_x(t-s) \right) \mathrm{d}s \mathrm{d}x \end{aligned}$$

$$+\int_0^L a(\varphi_x+\psi)\int_0^{+\infty} g(s)\left(\varphi_x(t)-\varphi_x(t-s)\right)\mathrm{d}s\mathrm{d}x$$
$$-g_0\int_0^L a(\varphi_x+\psi)\varphi_x\mathrm{d}x+\left(\frac{\rho_1 k_2}{k_1}-\rho_2\right)\int_0^L \varphi_{xt}\psi_t\mathrm{d}x.$$

By using Young's and Poincaré's inequalities and recalling (3.2), (3.5) and (3.6), estimate (6.6) follows. □

By using w defined in (3.11) and repeating the proof of Lemma 3.7, we can easily establish this lemma.

Lemma 6.5 The functional

$$I_4(t)=\rho_1\int_0^L (w\varphi_t+\varphi\varphi_t)\mathrm{d}x$$

satisfies, for any δ, ϵ, $\epsilon'>0$,

$$\begin{aligned}I_4'(t)\le{}&(\rho_1+\frac{c}{\epsilon})\int_0^L \varphi_t^2\mathrm{d}x+c\epsilon\int_0^L\psi_t^2\mathrm{d}x\\&+\left(g_0\|a\|_\infty\left(1+\frac{\epsilon}{2}\right)-k_1\right)\int_0^L(\varphi_x+\psi)^2\mathrm{d}x+\frac{g_0k_0\|a\|_\infty}{2\epsilon'}\int_0^L\psi_x^2\mathrm{d}x\\&+\delta\int_0^L(\varphi_x^2+\psi_x^2)\mathrm{d}x+c_\delta g\circ\varphi_x+c_\delta\int_0^L\theta_x^2\mathrm{d}x.\end{aligned}\tag{6.7}$$

Proof Differentiation of I_3, using equations (6.1), leads to

$$\begin{aligned}I_4'(t)={}&\rho_1\int_0^L w_t\varphi_t\mathrm{d}x+\rho_1\int_0^L\varphi_t^2\mathrm{d}x-k_1\int_0^L(\varphi_x+\psi)^2\mathrm{d}x\\&+g_0\int_0^L\varphi_x^2\mathrm{d}x+g_0\int_0^L\varphi_x\psi\mathrm{d}x-\gamma\int_0^L w\theta_x\mathrm{d}x-\gamma\int_0^L\varphi\theta_x\mathrm{d}x\\&-\int_0^L a(\varphi_x+\psi)\int_0^{+\infty}g(s)\left(\varphi_x(t)-\varphi_x(t-s)\right)\mathrm{d}s\mathrm{d}x.\end{aligned}$$

Again, Young's and Poincaré's inequalities, (3.5), (3.12) and (3.13) give the desired result. □

Finally, we need the following lemma:

Lemma 6.6 The functional

$$I_5(t)=\rho_1\rho_3\int_0^L\varphi_t\left(\int_0^x\theta(y,t)\mathrm{d}y\right)\mathrm{d}x,$$

for any $\delta>0$,

$$I_5'(t)\le-\frac{\gamma\rho_1}{2}\int_0^L\varphi_t^2\,\mathrm{d}x+\delta\int_0^L(\varphi_x^2+\psi_x^2)\,\mathrm{d}x+c_\delta\int_0^L\theta_x^2\mathrm{d}x+c_\delta g\circ\varphi_x.\tag{6.8}$$

Proof By using equations (6.1), a simple integration keeping in mind that θ stands for $\tilde{\theta}$, leads to

$$\begin{aligned}I_5'(t)={}&\rho_3\int_0^L\left(k_1(\varphi_x+\psi)_x+\gamma\theta_x-\int_0^{+\infty}\left(a(x)g(s)\varphi_x(t-s)\right)_x\mathrm{d}s\right)\left(\int_0^x\theta(y,t)\mathrm{d}y\right)\mathrm{d}x\\&+\rho_1\int_0^L\varphi_t\left(\int_0^x(\kappa\theta_{xx}-\gamma\varphi_{xt})\mathrm{d}y\right)\mathrm{d}x\\={}&-\rho_3\int_0^L\left(k_1(\varphi_x+\psi)+\gamma\theta-\int_0^{+\infty}ag(s)\varphi_x(t-s)\mathrm{d}s\right)\theta\mathrm{d}x\end{aligned}$$

$$+\rho_1\int_0^L \varphi_t\left(\int_0^x \kappa\theta_x-\gamma\varphi_t)\mathrm{d}y\right)\mathrm{d}x.$$

By using Young's and Poincaré's inequalities and (3.5), (6.8) is established. □

For N, N_2, N_3, N_4, we set

$$I_6 = NE + N_2I_2 + I_3 + N_3I_4 + N_4I_5.$$

Direct calculations, using (6.4)–(6.8), yield

$$\begin{aligned}I_6'(t) \le &-(N\kappa - c_\delta(1+N_2+N_3+N_4))\int_0^L \theta_x^2\mathrm{d}x + \left(\frac{N}{2}-c_\delta\right)g'\circ\varphi_x\\ &+(1+N_2+N_3+N_4)\,c_\delta g\circ\varphi_x - \left(N_4\frac{\gamma\rho_1}{2} - N_2\rho_1 - N_3\Big(\rho_1+\frac{c}{\epsilon}\Big)\right)\int_0^L \varphi_t^2\mathrm{d}x\\ &-\left(N_2k_2 - \frac{g_0k_0||a||_\infty}{2\delta_1} - \delta - N_3\Big(\frac{g_0k_0||a||_\infty}{2\epsilon'}+\delta\Big) - \delta N_4\right)\int_0^L \psi_x^2\mathrm{d}x\\ &-(\rho_2(1-N_2)-\delta-c\epsilon' N_3)\int_0^L \psi_t^2\mathrm{d}x\\ &-\Big((N_2+N_3-1)k_1 - N_3g_0||a||_\infty\Big(1+\frac{\epsilon}{2}\Big)\Big)\int_0^L(\varphi_x+\psi)^2\mathrm{d}x\\ &+\Big(N_2+\frac{\delta_1}{2}-1\Big)g_0\int_0^L a\varphi_x^2\mathrm{d}x + \delta\,(N_2+1+N_3+N_4)\int_0^L\varphi_x^2\mathrm{d}x\\ &+\left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L \varphi_{xt}\psi_t\mathrm{d}x. \end{aligned}\tag{6.9}$$

At this point, we distinguish two cases.

Case 1 $a\equiv 0$: in this case (6.9), reduces to

$$\begin{aligned}I_6'(t) \le &-(N\kappa - c_\delta(1+N_2+N_3+N_4))\int_0^L \theta_x^2\,\mathrm{d}x\\ &-\rho_1\left(N_4\frac{\gamma}{2}-(N_2+N_3)\right)\int_0^L\varphi_t^2\,\mathrm{d}x - (N_2k_2-\delta(1+N_3+N_4))\int_0^L\psi_x^2\mathrm{d}x\\ &-(\rho_2(1-N_2)-\delta)\int_0^L\psi_t^2\mathrm{d}x - (N_2+N_3-1)\,k_1\int_0^L(\varphi_x+\psi)^2\mathrm{d}x\\ &+\delta\,(N_2+1+N_3+N_4)\int_0^L\varphi_x^2\mathrm{d}x + \left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L\varphi_{xt}\psi_t\mathrm{d}x.\end{aligned}\tag{6.10}$$

By taking $N_3 = 1, 0 < N_2 < 1$, $N_4 > \frac{2(N_2+N_3)}{\gamma}$, δ small enough, and N large enough, (6.10) becomes

$$I_6'(t) \le -cE(t) + \left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L\varphi_{xt}\psi_t\mathrm{d}x,\tag{6.11}$$

where c is a positive constant.

Case 2 $\inf\limits_{x\in[0,L]}\{a(x)\} > 0$: with the same choice of ϵ', δ_1, N_3, N_2 and ϵ as in section 3 and

$$N_4 > \frac{2\left(N_2\rho_1 + N_3(\rho_1+\frac{c_0}{\epsilon})\right)}{\gamma\rho_1},$$

δ small enough, and N large enough, (6.9) becomes

$$I_6'(t) \le -cE(t) + \left(\frac{\rho_1k_2}{k_1}-\rho_2\right)\int_0^L\varphi_{xt}\psi_t\mathrm{d}x + cg\circ\varphi_x.\tag{6.12}$$

We then proceed, as in Section 3, to complete the proof.

Remark 6.1 When $a \equiv 0$ or g satisfies (2.7), we obtain the exponential decay. That is,

$$E(t) \leq c_1'' e^{-c_2'' t},$$

for two positive constants c_1'' and c_2''.

When (1.1) does not hold, we have the following:

Theorem 6.3 Assume (2.1), (2.3), (H3), and (H4) hold and let $n \in \mathbb{N}^*$ and $U_0 \in D(A^n)$ such that $a \equiv 0$ or (2.7) or (2.20) is satisfied. Then, the energy E satisfies (2.21).

Proof The proof goes exactly like that of Theorem 2.3. □

6.2 Timoshenko-heat Type III

In this subsection, we consider a coupled Timoshenko-thermoelasticity system of type III on $]0, L[$ in the presence of an infinite memory acting on the first equation. That is,

$$\begin{cases} \rho_1 \varphi_{tt} - k_1(\varphi_x + \psi)_x + \gamma\theta_x + \displaystyle\int_0^{+\infty} (ag(s)\varphi_x(t-s))_x \, \mathrm{d}s = 0, \\ \rho_2 \psi_{tt} - k_2 \psi_{xx} + k_1(\varphi_x + \psi) = 0, \\ \rho_3 \theta_{tt} - \kappa\theta_{xx} + \gamma\varphi_{xt} - \delta\theta_{xxt} = 0, \\ \varphi(0,t) = \varphi(L,t) = \psi_x(0,t) = \psi_x(L,t) = \theta_x(0,t) = \theta_x(L,t) = 0, \\ \varphi(x,-t) = \varphi_0(x,t), \quad \varphi_t(x,0) = \varphi_1(x), \\ \psi(x,0) = \psi_0(x), \quad \psi_t(x,0) = \psi_1(x), \\ \theta(x,0) = \theta_0(x), \quad \theta_t(x,0) = \theta_1(x), \end{cases} \tag{6.13}$$

where φ, ψ, and θ are functions of (x,t) and denote the transverse displacement of the beam, the rotation angle of the filament, and the temperature displacement, respectively; ρ_i, k_i, γ, κ, δ, L are positive constants and a and g are as in Section 2. We only give brief comments and state the main results and leave the proofs for the reader since they go exactly like the ones done in Subsection 6.1.

From the third equation in (6.13) and the boundary conditions, we easily verify that

$$\partial_{tt} \int_0^L \theta(x,t)\mathrm{d}x = 0.$$

By solving this ordinary differential equation and using the initial data of θ, we get

$$\int_0^L \theta(x,t)\mathrm{d}x = t\int_0^L \theta_1(x)\,\mathrm{d}x + \int_0^L \theta_0(x)\,\mathrm{d}x.$$

So, we set

$$\tilde{\theta}(x,t) = \theta(x,t) - \frac{t}{L}\int_0^L \theta_1(x)\mathrm{d}x - \frac{1}{L}\int_0^L \theta_0(x)\mathrm{d}x$$

to conclude that $(\varphi, \tilde{\psi}, \tilde{\theta})$ satisfies (6.13), with initial data

$$\tilde{\theta}_0(x) = \theta_0(x) - \frac{1}{L}\int_0^L \theta_0(x)\mathrm{d}x$$

and

$$\tilde{\theta}_1(x) = \theta_1(x) - \frac{1}{L}\int_0^L \theta_1(x)\mathrm{d}x$$

instead of θ_0 and θ_1, respectively, and more importantly

$$\int_0^L \tilde{\theta}(x,t)\mathrm{d}x = 0;$$

which implies that Poincaré's inequality is applicable for $\tilde{\theta}$. In the sequel, we work with $\tilde{\theta}$ instead of θ, but, for simplicity of notation, we use θ instead of $\tilde{\theta}$.

6.2.1 Well-Posedness

By combining arguments from the Subsection 2.2 above and subsection 6.1 of [14], one can easily establish the well-posedness of (6.13). For this purpose, we define η as in subsection 2.2 and set

$$\mathcal{H} = \begin{cases} \tilde{\mathcal{H}} & \text{if}\, a \equiv 0, \\ \tilde{\mathcal{H}} \times L_g & \text{if } \inf\limits_{x\in[0,L]} \{a(x)\} > 0, \end{cases}$$

where L_g and its inner product are given in Subsection 2.2, and

$$\tilde{\mathcal{H}} = H_0^1(]0,L[) \times H_*^1(]0,L[) \times H_*^1(]0,L[) \times L^2(]0,L[) \times L_*^2(]0,L[) \times L_*^2(]0,L[).$$

If $a \equiv 0$, the space $\mathcal{H}$ is equipped with the inner product

$$\begin{aligned} \langle V, W\rangle = &\int_0^L (k_1(\partial_x v_1 + v_2)(\partial_x w_1 + w_2) + k_2 \partial_x v_2 \partial_x w_2)\,\mathrm{d}x \\ &+ \int_0^L (\kappa \partial_x v_3 \partial_x w_3 + \rho_1 v_4 w_4 + \rho_2 v_5 w_5 + \rho_3 v_6 w_6)\,\mathrm{d}x, \end{aligned}$$

for any $V = (v_1, v_2, v_3, v_4, v_5, v_6)^T$, $W = (w_1, w_2, w_3, w_4, w_5, w_6)^T \in \mathcal{H}$; and if $\inf\limits_{x\in[0,L]} \{a(x)\} > 0$, we equip $\mathcal{H}$ with the inner product

$$\begin{aligned} \langle V, W\rangle = &\langle v_7, w_7\rangle_{L_g} + \int_0^L (k_1(\partial_x v_1 + v_2)(\partial_x w_1 + w_2) + k_2 \partial_x v_2 \partial_x w_2)\,\mathrm{d}x \\ &+ \int_0^L (-g_0 a \partial_x v_1 \partial_x w_1 + \kappa \partial_x v_3 \partial_x w_3 + \rho_1 v_4 w_4 + \rho_2 v_5 w_5 + \rho_3 v_6 w_6)\,\mathrm{d}x, \end{aligned}$$

for any $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)^T$, $W = (w_1, w_2, w_3, w_4, w_5, w_6, w_7)^T \in \mathcal{H}$. By letting

$$U = \begin{cases} (\varphi, \psi, \theta, \varphi_t, \psi_t, \theta_t)^T & \text{if } a \equiv 0, \\ (\varphi, \psi, \theta, \varphi_t, \psi_t, \theta_t, \eta)^T & \text{if } \inf\limits_{x\in[0,L]} \{a(x)\} > 0 \end{cases}$$

and

$$U_0 = \begin{cases} (\varphi_0, \psi_0, \theta_0, \varphi_1, \psi_1, \theta_1)^T & \text{if } a \equiv 0, \\ (\varphi_0(.,0), \psi_0, \theta_0, \varphi_1, \psi_1, \theta_1, \eta_0)^T & \text{if } \inf\limits_{x\in[0,L]} \{a(x)\} > 0, \end{cases}$$

problem (6.13) can be written as

$$\begin{cases} U' + AU = 0, & \text{in } \mathbb{R}_+, \\ U(0) = U_0, \end{cases} \tag{6.14}$$

where, if $a \equiv 0$,

$$AV = \begin{cases} -v_4, \\ -v_5, \\ -v_6, \\ -\dfrac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2) + \dfrac{\gamma}{\rho_1}\partial_x v_3, \\ -\dfrac{k_2}{\rho_2}\partial_{xx} v_2 + \dfrac{k_1}{\rho_1}(\partial_x v_1 + v_2), \\ -\dfrac{\kappa}{\rho_3}\partial_{xx} v_3 + \dfrac{\gamma}{\rho_3}\partial_x v_4 - \dfrac{\delta}{\rho_3}\partial_{xx} v_6, \end{cases}$$

for any $V = (v_1, v_2, v_3, v_4, v_5, v_6)^T \in D(A)$ and, if $\inf\limits_{x\in[0,L]}\{a(x)\} > 0$,

$$AV = \begin{cases} -v_4, \\ -v_5, \\ -v_6, \\ -\dfrac{k_1}{\rho_1}\partial_x(\partial_x v_1 + v_2) + \dfrac{g_0}{\rho_1}\partial_x(a\partial_x v_1) - \dfrac{1}{\rho_1}\displaystyle\int_0^{+\infty} g(s)\partial_x(a\partial_x v_6(s))\mathrm{d}s + \dfrac{\gamma}{\rho_1}\partial_x v_3, \\ -\dfrac{k_2}{\rho_2}\partial_{xx} v_2 + \dfrac{k_1}{\rho_1}(\partial_x v_1 + v_2), \\ -\dfrac{\kappa}{\rho_3}\partial_{xx} v_3 + \dfrac{\gamma}{\rho_3}\partial_x v_4 - \dfrac{\delta}{\rho_3}\partial_{xx} v_6, \\ -v_4 + \partial_s v_7, \end{cases}$$

for any $V = (v_1, v_2, v_3, v_4, v_5, v_6, v_7)^T \in D(A)$. By noting that (6.14) is linear and exploiting the semigroup theory [19, 32], one can easily show that Theorem 6.1 also holds for (6.14). Hence, the well-posedness for (6.13) is established.

6.2.2 Stability

Similarly to (6.1), we establish a general stability result for solutions of (6.13), under the hypotheses (H3) and (H4). We define the first-order energy of (6.13) by

$$E(t) = \frac{1}{2}g \circ \varphi_x + \frac{1}{2}\int_0^L \left(\rho_1\varphi_t^2 + \rho_2\psi_t^2 + k_1(\varphi_x + \psi)^2 + k_2\psi_x^2 + \rho_3\theta_t^2 + \kappa\theta_x^2 - g_0 a\varphi_x^2\right)\mathrm{d}x. \quad (6.15)$$

Straightforward computations yield

$$E'(t) = -\kappa\int_0^1 \theta_{xt}^2\,\mathrm{d}x + \frac{1}{2}g' \circ \varphi_x \le 0. \quad (6.16)$$

Remark 6.3 By adopting the same functionals used in the subsection 6.1 and repeating the same steps, one can easily show that Theorems 6.2 and 6.3 remain valid for problem (6.13). In particular, we obtain the exponential stability if $a \equiv 0$ or g decays exponentially.

Acknowledgements This work has been funded by KFUPM under the scientific project IN141015. The authors would like to express their sincere thanks to KFUPM for its continuous support.

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