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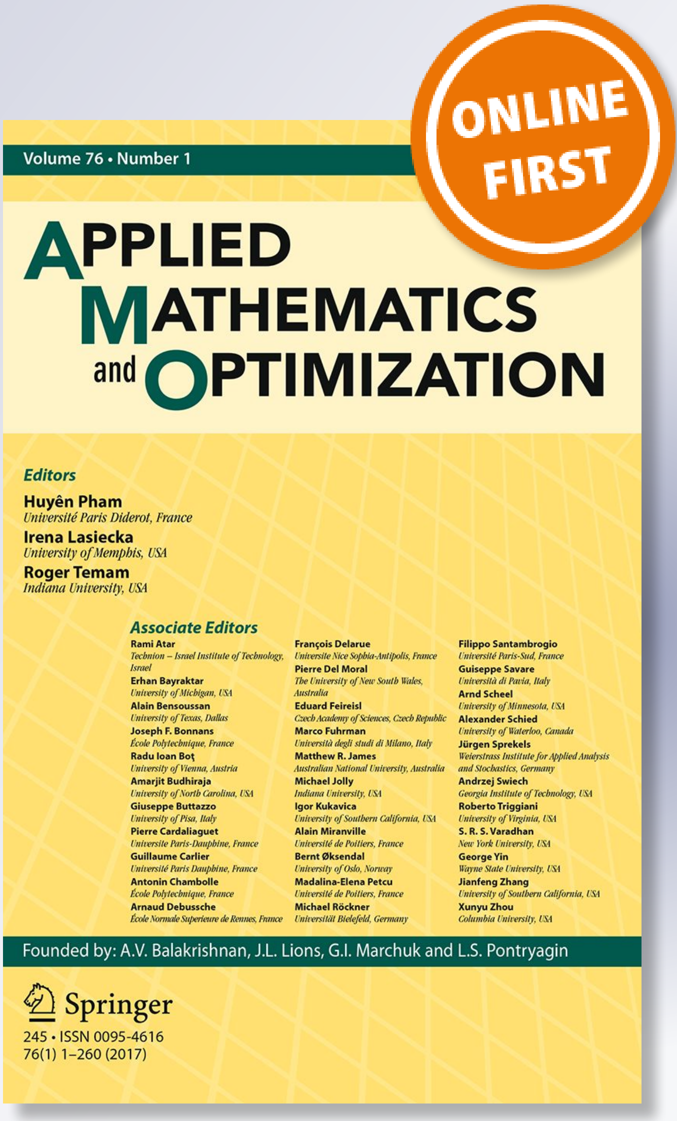

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New Stability Results for a Linear Thermoelastic Bresse System with Second Sound

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Abstract

In this paper, we consider a linear one-dimensional thermoelastic Bresse system with second sound consisting of three hyperbolic equations and two parabolic equations coupled in a certain manner under mixed homogeneous Dirichlet–Neumann boundary conditions, where the heat conduction is given by Cattaneo's law. Only the longitudinal displacement is damped via the dissipation from the two parabolic equations, and the vertical displacement and shear angle displacement are free. We prove the well-posedness of the system and some exponential, non exponential and polynomial stability results depending on the coefficients of the equations and the smoothness of initial data. Our method of proof is based on the semigroup theory and a combination of the energy method and the frequency domain approach.

Keywords Bresse system · Heat conduction · Well-posedness · Asymptotic behavior · Semigroup theory · Energy method · Frequency domain approach

Mathematics Subject Classification 35B40 · 35L45 · 74H40 · 93D20 · 93D15

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1 Introduction

In this paper, we consider the following linear Bresse system with second sound:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\, w\right)_x - lk_0\left(w_x - l\varphi\right) = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\, w\right) = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\, w\right) + \delta\theta_x = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \rho_3\theta_t + q_x + \delta w_{xt} = 0 & \text{in } (0,1)\times(0,\infty)\,, \\ \tau q_t + \beta q + \theta_x = 0 & \text{in } (0,1)\times(0,\infty) \end{cases} \tag{1.1}$$

with the initial data

$$\begin{cases} \varphi\left(x,0\right) = \varphi_0\left(x\right),\ \varphi_t\left(x,0\right) = \varphi_1\left(x\right) & \text{in } (0,1)\,, \\ \psi\left(x,0\right) = \psi_0\left(x\right),\ \psi_t\left(x,0\right) = \psi_1\left(x\right) & \text{in } (0,1)\,, \\ w\left(x,0\right) = w_0\left(x\right),\ w_t\left(x,0\right) = w_1\left(x\right) & \text{in } (0,1)\,, \\ \theta\left(x,0\right) = \theta_0\left(x\right),\ q\left(x,0\right) = q_0\left(x\right) & \text{in } (0,1) \end{cases} \tag{1.2}$$

and mixed homogeneous Dirichlet–Neumann boundary conditions

$$\begin{cases} \varphi\left(0,t\right) = \psi_x\left(0,t\right) = w_x\left(0,t\right) = \theta\left(0,t\right) = 0 & \text{in } (0,\infty)\,, \\ \varphi_x\left(1,t\right) = \psi\left(1,t\right) = w\left(1,t\right) = q\left(1,t\right) = 0 & \text{in } (0,\infty)\,, \end{cases} \tag{1.3}$$

where $\rho_1,\ \rho_2,\ \rho_3,\ b,\ k,\ k_0,\ \tau,\ \beta,\ \delta$ and l are positive constants, the initial data φ_0, φ_1, ψ_0, ψ_1, w_0, w_1, θ_0 and q_0 belong to a suitable Hilbert space, and the unknowns of (1.1)–(1.3) are the following variables:

$$(\varphi, \psi, w, \theta, q) : (0,1)\times(0,\infty) \to \mathbb{R}^5.$$

The Bresse system [3] is consisting of three coupled hyperbolic equations

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\, w\right)_x - lk_0\left(w_x - l\varphi\right) = F_1 & \text{in } (0,L)\times(0,\infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\, w\right) = F_2 & \text{in } (0,L)\times(0,\infty)\,, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\, w\right) = F_3 & \text{in } (0,L)\times(0,\infty)\,, \end{cases} \tag{1.4}$$

where $L > 0$,

$$F_i\,:\ (0,L)\times(0,\infty) \to \mathbb{R}$$

are the external forces (controllers) and $w,\ \varphi$ and ψ represent, respectively, the longitudinal, vertical and shear angle displacements. For more details, we refer to [15] and [16].

For the last few years, many researchers studied the well-posedness and the stability of Bresse systems (1.4). Under different types of controls F_i, various stability results have been obtained depending on the nature and the number of controls, the regularity of initial data and the following parameters:

$$s_1 = \frac{k}{\rho_1}, \quad s_2 = \frac{b}{\rho_2} \quad \text{and} \quad s_3 = \frac{k_0}{\rho_1}; \tag{1.5}$$

for this purpose, we refer the reader to [1,2,4,7,9,21,24–26] and [27] in case of (local or global, linear or nonlinear) frictional damping, and [5,11,12] and [10] in case of memories. In some papers, it was proved that, when each equation of (1.4) is directly damped; that is

$$F_1 F_2 F_3 \neq 0,$$

the stability of (1.4) holds regardless to s_1, s_2 and s_3. However, when at least one Eq. in (1.4) is free; that is

$$F_1 F_2 F_3 = 0 \quad \text{and} \quad (F_1, F_2, F_3) \neq (0, 0, 0),$$

system (1.4) is still stable depending on the relation between the coefficients s_1, s_2 and s_3 like:

$$s_i = s_j, \quad i, j \in \{1, 2, 3\}.$$

When

$$(F_1, F_2, F_3) = (0, 0, 0),$$

system (1.4) is conservative, which means that the energy is conserved and equal to the energy of initial data along the trajectory of solutions.

When the Bresse system is indirectly damped via the coupling (in a certain manner) with other equations, we mention here the work [18], where the authors studied the stability of a thermoelastic Bresse system consisting of the following equations:

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l w)_x - l k_0 (w_x - l\varphi) + l\delta\theta = 0 & \text{in } (0, L) \times (0, \infty), \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l w) + \delta q_x = 0 & \text{in } (0, L) \times (0, \infty), \\ \rho_1 w_{tt} - k_0 (w_x - l\varphi)_x + lk(\varphi_x + \psi + l w) + \delta\theta_x = 0 & \text{in } (0, L) \times (0, \infty), \\ \rho_3 \theta_t - \theta_{xx} + \beta (w_x - l\varphi)_t = 0 & \text{in } (0, L) \times (0, \infty), \\ \rho_3 q_t - q_{xx} + \beta \psi_{xt} = 0 & \text{in } (0, L) \times (0, \infty) \end{cases} \tag{1.6}$$

with homogeneous Dirichlet–Neumann–Neumann boundary conditions

$$\varphi(x, t) = \psi_x(x, t) = w_x(x, t) = \theta(x, t) = q(x, t) = 0, \quad x = 0, \ L, \ t \in (0, \infty) \tag{1.7}$$

or homogeneous Dirichlet–Dirichlet–Dirichlet boundary conditions

$$\varphi(x, t) = \psi(x, t) = w(x, t) = \theta(x, t) = q(x, t) = 0, \quad x = 0, \ L, \ t \in (0, \infty). \tag{1.8}$$

They proved that the norm of solutions in the energy space decays exponentially to zero at infinity if

$$s_1 = s_2 = s_3. \tag{1.9}$$

Otherwise, the norm of solutions decays polynomially to zero with rates depending on the regularity of the initial data. For the classical solutions, these rates were $t^{-\frac{1}{4}+\epsilon}$ in case (1.7), and $t^{-\frac{1}{8}+\epsilon}$ in case (1.8), where ϵ is an arbitrary positive constant.

In [8], the authors considered the following coupled Bresse system with only one heat equation:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0 & \text{in } (0, L) \times (0, \infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0 & \text{in } (0, L) \times (0, \infty)\,, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0, L) \times (0, \infty)\,, \\ \rho_3\theta_t - \theta_{xx} + (\beta\psi_t)_x = 0 & \text{in } (0, L) \times (0, \infty) \end{cases} \tag{1.10}$$

with (1.7) or (1.8). They proved that the exponential stability of (1.10) is equivalent to (1.9). On the other hand, when (1.9) is not satisfied, the obtained decay rate in [8] for classical solutions is $t^{-\frac{1}{6}+\epsilon}$ in general, and $t^{-\frac{1}{3}+\epsilon}$ when $s_1 \neq s_2$ and $s_1 = s_3$. The results of [8] were extended in [20] to the case where the thermal dissipation is locally distributed; that is δ and β are non negative functions on x such that theirs minimums on some open interval $I \subset (0, L)$ are positive. Moreover, when (1.9) is not satisfied, the authors of [20] improved the polynomial stability estimates of [8] by getting the decay rates $t^{-\frac{1}{4}}$ and $t^{-\frac{1}{2}}$ instead of $t^{-\frac{1}{6}+\epsilon}$ and $t^{-\frac{1}{3}+\epsilon}$, respectively.

In [14], the authors considered the following coupled system:

$$\begin{cases} \rho_1\varphi_{tt} - k\left(\varphi_x + \psi + l\,w\right)_x - lk_0\left(w_x - l\varphi\right) = 0 & \text{in } (0, 1) \times (0, \infty)\,, \\ \rho_2\psi_{tt} - b\psi_{xx} + k\left(\varphi_x + \psi + l\,w\right) + \delta\theta_x = 0 & \text{in } (0, 1) \times (0, \infty)\,, \\ \rho_1 w_{tt} - k_0\left(w_x - l\varphi\right)_x + lk\left(\varphi_x + \psi + l\,w\right) = 0 & \text{in } (0, 1) \times (0, \infty)\,, \\ \rho_3\theta_t + q_x + \delta\psi_{xt} = 0 & \text{in } (0, 1) \times (0, \infty)\,, \\ \tau q_t + \beta q + \theta_x = 0 & \text{in } (0, 1) \times (0, \infty)\,. \end{cases} \tag{1.11}$$

They proved that (1.11) is exponentially stable if

$$s_1 = s_3, \quad \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k \rho_3}{\rho_1}\right) = \frac{\tau\delta^2}{b} \quad \text{and} \quad l \quad \text{small,}$$

and (1.11) is not exponentially stable if

$$s_1 \neq s_3 \quad \text{or} \quad \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k \rho_3}{\rho_1}\right) \neq \frac{\tau\delta^2}{b}.$$

Moreover, when

$$s_1 = s_3, \quad \left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right)\left(1 - \frac{\tau k \rho_3}{\rho_1}\right) \neq \frac{\tau \delta^2}{b} \quad \text{and} \quad l \quad \text{small},$$

the polynomial stability for (1.11) was proved in [14] with the decay rate $t^{-\frac{1}{2}}$.

In (1.6) and (1.10), the heat equations are governed by Fourier's law of heat conduction. However, the heat conduction in (1.1) and (1.11) is given by Cattaneo's law (for more details, see [14]).

In [6], the author considered the following coupled system:

$$\begin{cases} \rho_1 \varphi_{tt} - k(\varphi_x + \psi + l\,w)_x - lk_0(w_x - l\varphi) = 0, \\ \rho_2 \psi_{tt} - b\psi_{xx} + k(\varphi_x + \psi + l\,w) + \delta\theta_x = 0, \\ \rho_1 w_{tt} - k_0(w_x - l\varphi)_x + lk(\varphi_x + \psi + l\,w) = 0, \\ \rho_3 \theta_t - k_1 \int_0^\infty g(s)\theta_{xx}(t-s)ds + \gamma\psi_{xt} = 0, \end{cases} \tag{1.12}$$

with homogeneous Dirichlet–Neumann boundary conditions

$$\varphi(x,t) = \psi_x(x,t) = w_x(x,t) = \theta(x,t) = 0, \quad x = 0,\ L,\ t \in (0,\infty) \tag{1.13}$$

He proved that (1.12) is exponentially stable if and if

$$k = k_0, \quad \left(\frac{\rho_1}{\rho_3 k} - \frac{1}{g(0)k_1}\right)\left(\frac{\rho_1}{k} - \frac{\rho_2}{b}\right). - \frac{1}{g(0)k_1}\frac{\rho_1\gamma^2}{\rho_3 kb} = 0. \tag{1.14}$$

On the other hand if (1.14) is not satisfied no decay rates was derived in [6]. We need to mention here, that the coupling (through the second equation) and the boundary conditions considered in [6] are not the same as the one considered in this paper. Notice that, when the three hyperbolic equations in Bresse system are (all or some of them) directly damped; that is

$$(F_1, F_2, F_3) \neq (0,0,0),$$

system (1.4) is dissipative. However, systems (1.1), (1.6), (1.10) and (1.11) are consisting of coupled conservative three hyperbolic equations with one or two parabolic equations, so the stability of the overall system is preserved thanks to the dissipation generated by the parabolic equations. On the other hand, we remark that in (1.6), the second and third hyperbolic equations are indirectly damped by the coupling with the heat equations, and the first hyperbolic one is only weakly damped through the coupling with the second and the third hyperbolic equations. On the other hand, in (1.10) and (1.11), only the second hyperbolic equation is effectively damped by the dissipation coming from the parabolic equations.

In our case (1.1), only the third hyperbolic equation is indirectly damped through the coupling with the heat equations. Our objective, first is to consider (1.1)–(1.3), we prove the well-posedness and we establish some decay rates for the solutions (like: exponential stability, non exponential stability and polynomial stability) depending on the relationship between the coefficients of (1.1) and the smoothness of the initial data.

Without loss of generality, we consider the domain $(0, 1)$ instead of $(0, L)$. The proof of the well-posedness is based on the semigroup theory. However, the stability results are proved using the energy method combining with the frequency domain approach.

The paper is organized as follows. In Sect. 2, we prove the well-posedness of (1.1)–(1.3). In Sects. 3 and 4, we show, respectively, our non exponential and exponential stability results for (1.1)–(1.3). The proof of our polynomial decay for (1.1)–(1.3) is proved in Sect. 5.

2 Well-posedness of (1.1)–(1.3)

In this section, we prove the existence, uniqueness and smoothness of solutions for (1.1)–(1.3) using the semigroup theory. In order to transform (1.1)–(1.3) into a first order evolution system on a suitable Hilbert space, we introduce the vector functions

$$\Phi = \left(\varphi,\ \tilde{\varphi},\ \psi,\ \tilde{\psi},\ w,\ \tilde{w},\ \theta,\ q\right)^T \text{ and } \Phi_0 = \left(\varphi_0,\ \varphi_1,\ \psi_0,\ \psi_1,\ w_0,\ w_1,\ \theta_0,\ q_0\right)^T,$$

where $\tilde{\varphi} = \varphi_t$, $\tilde{\psi} = \psi_t$ and $\tilde{w} = w_t$. System (1.1) with initial data (1.2) can be written as

$$\begin{cases} \Phi_t = \mathcal{A}\Phi \quad \text{in } (0, \infty), \\ \Phi(0) = \Phi_0, \end{cases} \tag{2.1}$$

where $\mathcal{A}$ is a linear operator defined by

$$\mathcal{A}\Phi = \begin{pmatrix} \tilde{\varphi} \\ \dfrac{k}{\rho_1}(\varphi_x + \psi + l\,w)_x + \dfrac{lk_0}{\rho_1}(w_x - l\varphi) \\ \tilde{\psi} \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x + \psi + l\,w) \\ \tilde{w} \\ \dfrac{k_0}{\rho_1}(w_x - l\varphi)_x - \dfrac{lk}{\rho_1}(\varphi_x + \psi + l\,w) - \dfrac{\delta}{\rho_1}\theta_x \\ -\dfrac{1}{\rho_3}q_x - \dfrac{\delta}{\rho_3}\tilde{w}_x \\ -\dfrac{\beta}{\tau}q - \dfrac{1}{\tau}\theta_x \end{pmatrix}. \tag{2.2}$$

Now, we introduce the following spaces:

$$\begin{cases} H_*^1(0,1) = \left\{ f \in H^1(0,1) : f(0) = 0 \right\}, \\ \overset{\sim}{H_*^1}(0,1) = \left\{ f \in H^1(0,1) : f(1) = 0 \right\}, \\ H_*^2(0,1) = H^2(0,1) \cap H_*^1(0,1), \\ \overset{\sim}{H_*^2}(0,1) = H^2(0,1) \cap \overset{\sim}{H_*^1}(0,1) \end{cases}$$

and the energy space is given by

$$\mathcal{H} = H_*^1(0,1) \times L^2(0,1) \times \overset{\sim}{H_*^1}(0,1) \times L^2(0,1) \times \overset{\sim}{H_*^1}(0,1) \times \left(L^2(0,1) \right)^3$$

equipped with the inner product, for $\Phi_j = (\varphi_j, \tilde{\varphi}_j, \psi_j, \tilde{\psi}_j, w_j, \tilde{w}_j, \theta_j, q_j)^T \in \mathcal{H},\ j = 1, 2,$

$$\begin{aligned} \langle \Phi_1, \Phi_2 \rangle_{\mathcal{H}} = {} & k \left\langle (\varphi_{1x} + \psi_1 + l\, w_1), (\varphi_{2x} + \psi_2 + l\, w_2) \right\rangle_{L^2(0,1)} + b \left\langle \psi_{1x}, \psi_{2x} \right\rangle_{L^2(0,1)} \\ & + k_0 \left\langle (w_{1x} - l\varphi_1), (w_{2x} - l\varphi_2) \right\rangle_{L^2(0,1)} + \rho_1 \left\langle \tilde{\varphi}_1, \tilde{\varphi}_2 \right\rangle_{L^2(0,1)} \\ & + \rho_2 \langle \tilde{\psi}_1, \tilde{\psi}_2 \rangle_{L^2(0,1)} + \rho_1 \left\langle \tilde{w}_1, \tilde{w}_2 \right\rangle_{L^2(0,1)} + \rho_3 \left\langle \theta_1, \theta_2 \right\rangle_{L^2(0,1)} \\ & + \tau \left\langle q_1, q_2 \right\rangle_{L^2(0,1)}, \end{aligned}$$

and the corresponding norm in the energy space will be given by

$$\begin{aligned} \|\Phi\|_{\mathcal{H}}^2 = {} & k \left\| \varphi_x + \psi + l\, w \right\|_{L^2(0,1)}^2 + b \left\| \psi_x \right\|_{L^2(0,1)}^2 + k_0 \left\| w_x - l\varphi \right\|_{L^2(0,1)}^2 \\ & + \rho_1 \left\| \tilde{\varphi} \right\|_{L^2(0,1)}^2 + \rho_2 \| \tilde{\psi} \|_{L^2(0,1)}^2 + \rho_1 \left\| \tilde{w} \right\|_{L^2(0,1)}^2 + \rho_3 \left\| \theta \right\|_{L^2(0,1)}^2 \\ & + \tau \left\| q \right\|_{L^2(0,1)}^2. \end{aligned}$$

The domain of the operator $\mathcal{A}$ will be

$$D(\mathcal{A}) = \left\{ \Phi \in \mathcal{H} \mid \mathcal{A}\Phi \in \mathcal{H},\ \varphi_x(1) = \psi_x(0) = w_x(0) = 0 \right\}.$$

Based on the definition of $\mathcal{A}$ and $\mathcal{H}$, one can see that

$$D(\mathcal{A}) = \left\{ \begin{array}{c} \Phi \in \mathcal{H} \mid \varphi \in H_*^2(0,1);\ \psi, w \in \overset{\sim}{H_*^2}(0,1);\ \tilde{\varphi}, \theta \in H_*^1(0,1); \\ \tilde{\psi}, \tilde{w}, q \in \overset{\sim}{H_*^1}(0,1);\ \varphi_x(1) = \psi_x(0) = w_x(0) = 0 \end{array} \right\}.$$

Since the homogeneous Dirichlet–Neumann boundary conditions in (1.3) are included in the definition of $H_*^1(0,1)$, $\overset{\sim}{H_*^1}(0,1)$ and $D(\mathcal{A})$, it follows that, if $\Phi \in D(\mathcal{A})$ and satisfies (2.1), then (1.1)–(1.3) holds.

It is clear from the homogeneous Dirichlet boundary conditions in $H_*^1(0,1)$ and $\widetilde{H}_*^1(0,1)$ that, if $(\varphi, \psi, w) \in H_*^1(0,1) \times \widetilde{H}_*^1(0,1) \times \widetilde{H}_*^1(0,1)$ satisfying

$$k \left\|(\varphi_x + \psi + l\, w)\right\|_{L^2(0,1)}^2 + b \left\|\psi_x\right\|_{L^2(0,1)}^2 + k_0 \left\|(w_x - l\varphi)\right\|_{L^2(0,1)}^2 = 0,$$

then

$$\psi = 0, \quad \varphi = -c \sin(lx) \quad \text{and} \quad w = c \cos(lx),$$

where c is a constant such that $c = 0$ or $l = \frac{\pi}{2} + m\pi$, for some $m \in \mathbb{N}$. Furthermore, we get $\varphi = \psi = w = 0$ if

$$l \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N}. \tag{2.3}$$

Here and after we assume that (2.3) is satisfied. Thus, $\mathcal{H}$ is a Hilbert space and $D(\mathcal{A})$ is dense in $\mathcal{H}$. If the domain $(0,1)$ is replaced by $(0, L)$, then (2.3) becomes

$$lL \neq \frac{\pi}{2} + m\pi, \quad \forall m \in \mathbb{N}.$$

Now, we prove that the operator $\mathcal{A}$ generates a C_0 semigroup of contractions on $\mathcal{H}$. For this purpose, it is sufficient to prove that $\mathcal{A}$ is maximal monotone. A direct calculation gives

$$\langle \mathcal{A}\Phi, \Phi \rangle_{\mathcal{H}} = -\beta \left\|q\right\|_{L^2(0,1)}^2 \leq 0. \tag{2.4}$$

Hence, $\mathcal{A}$ is dissipative in $\mathcal{H}$. On the other hand, it is easy to show that $0 \in \rho(\mathcal{A})$; that is, for any $F = (f_1, \ldots, f_8)^T \in \mathcal{H}$, there exists $Z = (z_1, \ldots, z_8)^T \in D(\mathcal{A})$ satisfying

$$\mathcal{A}Z = F. \tag{2.5}$$

Indeed, from the 1st, 3rd and 5th Eqs. in (2.5), we get

$$z_2 = f_1, \quad z_4 = f_3 \quad \text{and} \quad z_6 = f_5, \tag{2.6}$$

and then

$$z_2 \in H_*^1(0,1) \quad \text{and} \quad z_4, z_6 \in \widetilde{H}_*^1(0,1). \tag{2.7}$$

Substituting z_2 into the 7th Eq. in (2.5), we conclude from the last two equations in (2.5) that

$$z_{7x} = -\beta z_8 - \tau f_8 \quad \text{and} \quad z_{8x} = -\delta f_{5x} - \rho_3 f_7. \tag{2.8}$$

By a direct integration, we see that (2.8) has a unique solution satisfying

$$z_7 \in H_*^1(0,1) \quad \text{and} \quad z_8 \in \overset{\sim}{H_*^1}(0,1). \tag{2.9}$$

Finally, the second, fourth and sixth equations in (2.5) become

$$\begin{cases} k(z_{1x} + z_3 + l z_5)_x + lk_0(z_{5x} - lz_1) = \rho_1 f_2, \\ bz_{3xx} - k(z_{1x} + z_3 + l z_5) = \rho_2 f_4, \\ k_0(z_{5x} - lz_1)_x - lk(z_{1x} + z_3 + l z_5) = \delta z_{7x} + \rho_1 f_6. \end{cases} \tag{2.10}$$

To prove that (2.10) admits a solution satisfying

$$z_1 \in H_*^2(0,1), \quad z_3, z_5 \in \overset{\sim}{H_*^2}(0,1) \quad \text{and} \quad z_{1x}(1) = z_{3x}(0) = z_{5x}(0) = 0, \tag{2.11}$$

we define the following bilinear form:

$$\begin{aligned} G_1((v_1, v_2, v_3), (w_1, w_2, w_3)) = {} & k\langle v_{1x} + v_2 + lv_3, w_{1x} + w_2 + lw_3\rangle_{L^2(0,1)} \\ & + b\langle v_{2x}, w_{2x}\rangle_{L^2(0,1)} \\ & + k_0\langle v_{3x} - lv_1, w_{3x} - lw_1\rangle_{L^2(0,1)}, \\ & \forall (v_1, v_2, v_3)^T, (w_1, w_2, w_3)^T \in \mathcal{H}_0 \times \mathcal{H}_0, \end{aligned}$$

and the following linear form:

$$\begin{aligned} G_2(v_1, v_2, v_3) = {} & \langle v_1, \rho_1 f_2\rangle_{L^2(0,1)} + \langle v_2, \rho_2 f_4\rangle_{L^2(0,1)} \\ & + \langle v_3, \delta z_{7x} + \rho_1 f_6\rangle_{L^2(0,1)}, \quad \forall (v_1, v_2, v_3)^T \in \mathcal{H}_0, \end{aligned}$$

where

$$\mathcal{H}_0 = H_*^1(0,1) \times \overset{\sim}{H_*^1}(0,1) \times \overset{\sim}{H_*^1}(0,1)$$

Thus, the variational formulation of (2.10) is given by

$$G_1((z_1, z_3, z_5), (w_1, w_2, w_3)) = G_2(w_1, w_2, w_3), \ \forall (w_1, w_2, w_3)^T \in \mathcal{H}_0. \tag{2.12}$$

From Lax–Milgram theorem, it follows that (2.12) has a unique solution

$$(z_1, z_3, z_5) \in \mathcal{H}_0.$$

Therefore, using classical elliptic regularity arguments, we conclude that (z_1, z_3, z_5) solves (2.10) and satisfies the regularity and boundary conditions (2.11). This proves that (2.5) has a unique solution $Z \in D(\mathcal{A})$. By the resolvent identity, we have $\lambda I - \mathcal{A}$ is surjective, for any $\lambda > 0$ (see [19]), where I denotes the identity operator. Consequently, the Lumer-Phillips theorem implies that $\mathcal{A}$ is the infinitesimal generator of a linear C_0 semigroup of contractions on $\mathcal{H}$. Thus, the well-posedness result for (2.1) is stated in the following (see [22]):

Theorem 2.1 *Assume that (2.3) holds. For any $p \in \mathbb{N}$ and $\Phi_0 \in D(\mathcal{A}^p)$, system (2.1) admits a unique solution*

$$\Phi \in \cap_{j=0}^{p} C^{p-j}\left(\mathbb{R}_+; D\left(\mathcal{A}^j\right)\right), \tag{2.13}$$

where $D\left(\mathcal{A}^j\right)$ is endowed by the graph norm $\|\cdot\|_{D(\mathcal{A}^j)} = \sum_{r=0}^{j} \|\mathcal{A}^r \cdot\|_{\mathcal{H}}$.

In the next three sections, we will show some exponential, non exponential and polynomial stability results for (2.1). The proof of these results is based on the following frequency domain theorems:

Theorem 2.2 ([13] *and* [23]) *A C_0 semigroup of contractions on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ is exponentially stable if and only if*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad and \quad \sup_{\lambda \in \mathbb{R}} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.14}$$

Theorem 2.3 ([17]) *If a bounded C_0 semigroup $e^{t\mathcal{A}}$ on a Hilbert space $\mathcal{H}$ generated by an operator $\mathcal{A}$ satisfies, for some $j \in \mathbb{N}^*$,*

$$i\mathbb{R} \subset \rho(\mathcal{A}) \quad and \quad \sup_{|\lambda| \geq 1} \frac{1}{\lambda^j} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{L}(\mathcal{H})} < \infty. \tag{2.15}$$

Then, for any $p \in \mathbb{N}^*$, there exists a positive constant c_p such that

$$\left\| e^{t\mathcal{A}} z_0 \right\|_{\mathcal{H}} \leq c_p \|z_0\|_{D(\mathcal{A}^p)} \left(\frac{\ln t}{t}\right)^{\frac{p}{j}} \ln t, \quad \forall z_0 \in D\left(\mathcal{A}^p\right), \ \forall t > 0. \tag{2.16}$$

3 Lack of Exponential Stability of (1.1)-(1.3)

Our objective here is to show that the semigroup associated with our Bresse system with second sound (2.1) is not exponentially stable depending on the following relations:

$$(k - k_0)\left(\rho_3 - \frac{\rho_1}{\tau k}\right) - \delta^2 = b\rho_1 - k\rho_2 = 0 \tag{3.1}$$

and

$$l^2 \neq \frac{\rho_2 k_0 + \rho_1 b}{\rho_2 k_0}\left(\frac{\pi}{2} + m\pi\right)^2 + \frac{\rho_1 k}{\rho_2 (k + k_0)}, \quad \forall m \in \mathbb{Z}. \tag{3.2}$$

Theorem 3.1 *We assume that* (2.3) *holds, and* (3.1) *or* (3.2) *does not hold. Then the semigroup associated with* (2.1) *is not exponentially stable.*

Proof We use Theorem 2.2 by proving that the first or second condition in (2.14) is not satisfied. First, we prove that the first condition in (2.14) is equivalent to (3.2). Note that, according to the fact that $0 \in \rho(\mathcal{A})$ (see Sect. 2), $\mathcal{A}^{-1}$ is bounded and it is a bijection between $\mathcal{H}$ and $D(\mathcal{A})$. Since $D(\mathcal{A})$ has a compact embedding into $\mathcal{H}$, so it follows that $\mathcal{A}^{-1}$ is a compact operator, which implies that the spectrum of $\mathcal{A}$ is discrete. Let $\lambda \in \mathbb{R}^*$. We will prove that the unique

$$\Phi = \left(\varphi, \tilde{\varphi}, \psi, \tilde{\psi}, w, \tilde{w}, \theta, q\right)^T \in D(\mathcal{A})$$

satisfying

$$\mathcal{A}\Phi = i\lambda\Phi \tag{3.3}$$

is $\Phi = 0$ if and only if (3.2) holds; that is the fact that $i\lambda$ is not an eigenvalue of $\mathcal{A}$ is equivalent to (3.2). But Eq. (3.3) is equivalent to

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \quad \tilde{\psi} = i\lambda\psi, \quad \tilde{w} = i\lambda w, \\ \dfrac{k}{\rho_1}(\varphi_x + \psi + l\,w)_x + \dfrac{lk_0}{\rho_1}(w_x - l\varphi) = i\lambda\tilde{\varphi}, \\ \dfrac{b}{\rho_2}\psi_{xx} - \dfrac{k}{\rho_2}(\varphi_x + \psi + l\,w) = i\lambda\tilde{\psi}, \\ \dfrac{k_0}{\rho_1}(w_x - l\varphi)_x - \dfrac{lk}{\rho_1}(\varphi_x + \psi + l\,w) - \dfrac{\delta}{\rho_1}\theta_x = i\lambda\tilde{w}, \\ -\dfrac{1}{\rho_3}q_x - \dfrac{\delta}{\rho_3}\tilde{w}_x = i\lambda\theta, \quad -\dfrac{\beta}{\tau}q - \dfrac{1}{\tau}\theta_x = i\lambda q. \end{cases} \tag{3.4}$$

Using (2.4), we find

$$-\beta\,\|q\|^2_{L^2(0,1)} = Re\,\langle\mathcal{A}\Phi, \Phi\rangle_{\mathcal{H}} = Re\,\langle i\lambda\,\Phi, \Phi\rangle_{\mathcal{H}} = Re\,i\lambda\,\|\Phi\|^2_{\mathcal{H}} = 0.$$

Then

$$q = 0. \tag{3.5}$$

Taking into account that $\theta \in H^1_*(0,1)$, using (3.5) and the eight equation in (3.4), we deduce that

$$\theta = 0. \tag{3.6}$$

Inserting (3.5) and (3.6) into the seventh equation in (3.4), we find

$$\tilde{w}_x = 0. \tag{3.7}$$

Then, the third equation in (3.4), implies that

$$w_x = 0. \tag{3.8}$$

As $w \in \tilde{H}{}_{*}^{1}(0,1)$, we have

$$w = \tilde{w} = 0. \tag{3.9}$$

Using (3.5), (3.6) and (3.9), then the system (3.4) is reduced into:

$$\begin{cases} \tilde{\varphi} = i\lambda\varphi, \quad \tilde{\psi} = i\lambda\psi, \\ k\left(\varphi_x + \psi\right)_x - l^2 k_0 \varphi = -\rho_1 \lambda^2 \varphi, \\ b\psi_{xx} - k\left(\varphi_x + \psi\right) = -\rho_2 \lambda^2 \psi, \\ -k_0 \varphi_x - k\left(\varphi_x + \psi\right) = 0, \end{cases} \tag{3.10}$$

which is equivalent to $\tilde{\varphi} = i\lambda\varphi$, $\tilde{\psi} = i\lambda\psi$ and

$$\begin{cases} \left(l^2 k_0 - \rho_1 \lambda^2\right)\varphi - k\left(\varphi_x + \psi\right)_x = 0, \\ -\rho_2 \lambda^2 \psi - b\psi_{xx} + k\left(\varphi_x + \psi\right) = 0, \\ \varphi_x + \psi = -\dfrac{k_0}{k}\varphi_x. \end{cases} \tag{3.11}$$

By deriving $(3.11)_3$ and combining with $(3.11)_1$, we see that φ satisfy the following equation:

$$\varphi_{xx} + \alpha\varphi = 0, \tag{3.12}$$

where $\alpha = \frac{l^2 k_0 - \rho_1 \lambda^2}{k_0}$. At this stage, we distinguish three cases.

Case 1 $\lambda^2 = \frac{l^2 k_0}{\rho_1}$. Then

$$\varphi(x) = c_1 x + c_2,$$

for $c_1,\ c_2 \in \mathbb{C}$. Using the boundary conditions

$$\varphi(0) = \varphi_x(1) = 0, \tag{3.13}$$

we find

$$\varphi = 0, \tag{3.14}$$

which implies that, using the first two equations in (3.10) and the last one in (3.11),

$$\overset{\sim}{\varphi} = 0 \tag{3.15}$$

and

$$\psi = \overset{\sim}{\psi} = 0. \tag{3.16}$$

Consequently, we get

$$\Phi = 0. \tag{3.17}$$

Case 2 $\lambda^2 > \frac{l^2 k_0}{\rho_1}$. Then

$$\varphi(x) = c_1 e^{\sqrt{-\alpha}x} + c_2 e^{-\sqrt{-\alpha}x}.$$

Using again the boundary conditions (3.13), we find (3.14), and similarly as before, we arrive at (3.17).

Case 3 $\lambda^2 < \frac{l^2 k_0}{\rho_1}$. Then

$$\varphi(x) = c_1 \cos\left(\sqrt{\alpha}x\right) + c_2 \sin\left(\sqrt{\alpha}x\right).$$

Using the boundary conditions (3.13), we deduce that $c_1 = 0$, and

$$c_2 = 0 \quad \text{or} \quad \exists m \in \mathbb{Z} : \; \alpha = \left(\frac{\pi}{2} + m\pi\right)^2. \tag{3.18}$$

If $c_2 = 0$, then (3.14) holds, and as before, we find (3.17).

If $c_2 \neq 0$, then, by (3.18),

$$\exists m \in \mathbb{Z} : \; \frac{l^2 k_0 - \rho_1 \lambda^2}{k_0} = \left(\frac{\pi}{2} + m\pi\right)^2. \tag{3.19}$$

Therefore, $(3.11)_3$ is equivalent to

$$\psi(x) = -c_2\left(1 + \frac{k_0}{k}\right)\sqrt{\alpha}\cos\left(\sqrt{\alpha}x\right), \tag{3.20}$$

and then the first two equations in (3.11) are reduced to

$$\lambda^2 = \frac{k_0\left[k k_0 + b l^2 (k + k_0)\right]}{(k + k_0)(k_0 \rho_2 + b\rho_1)}. \tag{3.21}$$

We see that (3.19) and (3.21) lead to

$$\exists m \in \mathbb{Z} : \; l^2 = \frac{\rho_2 k_0 + \rho_1 b}{\rho_2 k_0}\left(\frac{\pi}{2} + m\pi\right)^2 + \frac{\rho_1 k}{\rho_2 (k + k_0)};$$

that is (3.2) does not hold. So, if (3.2) holds, we get a contradiction, and hence, $c_2 = 0$ and, as before, we find (3.17).

If (3.2) does not hold, then, for $\lambda \in \mathbb{R}$ satisfying (3.21), the function

$$\Phi(x) = c_2\Big(\sin\left(\sqrt{\alpha}x\right), i\lambda \sin\left(\sqrt{\alpha}x\right), -\left(1+\frac{k_0}{k}\right)\sqrt{\alpha}\cos\left(\sqrt{\alpha}x\right), -i\lambda\left(1+\frac{k_0}{k}\right)\sqrt{\alpha}\cos\left(\sqrt{\alpha}x\right), 0, 0, 0, 0\Big)^T$$

is a solution of (3.3), for any $c_2 \in \mathbb{C}$, and then $i\lambda \notin \rho(\mathcal{A})$. Thus, we proved that $i\mathbb{R} \subset \rho(\mathcal{A})$ is equivalent to (3.2).

Now, we show that the second condition in (2.14) does not hold if (3.1) is not satisfied, i.e. we assume that (3.1) is not satisfied and we will prove that there exists a sequence $(\lambda_n)_n \subset \mathbb{R}$ such that

$$\left\|(\lambda_n I - \mathcal{A})^{-1}\right\|_{\mathcal{L}(\mathcal{H})} \longrightarrow \infty,$$

which is equivalent to prove that there exists $(F_n)_n \subset \mathcal{H}$ with $\|F_n\|_{\mathcal{H}} \leq 1$, for which we have

$$||\underbrace{(\lambda_n I - \mathcal{A})^{-1} F_n}_{\Phi_n}||_{\mathcal{H}} \longrightarrow \infty, \tag{3.22}$$

therefore, we have

$$\lambda_n \Phi_n - \mathcal{A}\Phi_n = F_n. \tag{3.23}$$

Our objective is to show that the solution Φ_n is not bounded when F_n is bounded in $\mathcal{H}$. The equation (3.23) implies that

$$\begin{cases} i\lambda_n \varphi_n - \widetilde{\varphi}_n = f_{1n}, \\ i\lambda_n \rho_1 \widetilde{\varphi}_n - k(\varphi_{nx} + \psi_n + l\,w_n)_x - lk_0(w_{nx} - l\varphi_n) + \delta\theta_{nx} = \rho_1 f_{2n}, \\ i\lambda_n \psi_n - \widetilde{\psi}_n = f_{3n}, \\ i\lambda_n \rho_2 \widetilde{\psi}_n - b\psi_{nxx} + k(\varphi_{nx} + \psi_n + l\,w_n) = \rho_2 f_{4n}, \\ i\lambda_n w_n - \widetilde{w}_n = f_{5n}, \\ i\lambda_n \rho_1 \widetilde{w}_n - k_0(w_{nx} - l\varphi_n)_x + lk(\varphi_{nx} + \psi_n + l\,w_n) = \rho_1 f_{6n}, \\ i\lambda_n \tau q_n + \beta q_n + \theta_{nx} = \tau f_{7n}, \\ i\lambda_n \rho_3 \theta_n + q_{nx} + \delta \widetilde{w}_{nx} = \rho_3 f_{8n}. \end{cases} \tag{3.24}$$

We will show that, for all $n \in \mathbb{N}$, given $c_4 \in \mathbb{C}^*$ and

$$F_n(x) = (0, 0, 0, c_4\cos(Nx), 0, 0, 0, 0)^T,$$

where $N = \frac{(2n+1)\pi}{2}$, there exists $\lambda_n \in \mathbb{R}$ and $\Phi_n = (i\lambda_n - \mathcal{A})^{-1} F_n \in D(\mathcal{A})$ such that

$$\lim_{\lambda_n \to \infty} \|\Phi_n\|_{\mathcal{H}} = \infty.$$

The system (3.24) will be written as

$$\begin{cases} i\lambda_n \varphi_n - \widetilde{\varphi}_n = 0, \; i\lambda_n \psi_n - \widetilde{\psi}_n = 0, \; i\lambda_n w_n - \widetilde{w}_n = 0, \\ -\lambda_n^2 \rho_1 \varphi_n - k\left(\varphi_{nx} + \psi_n + l\, w_n\right)_x - lk_0\left(w_{nx} - l\varphi_n\right) = 0, \\ -\lambda_n^2 \rho_2 \psi_n - b\psi_{nxx} + k\left(\varphi_{nx} + \psi_n + l\, w_n\right) = \rho_2 c_4 \cos\left(Nx\right), \\ -\lambda_n^2 \rho_1 w_n - k_0\left(w_{nx} - l\varphi_n\right)_x + lk\left(\varphi_{nx} + \psi_n + l\, w_n\right) + \delta\theta_{nx} = 0, \\ i\lambda_n \rho_3 \theta_n + q_{nx} + \delta \widetilde{w}_{nx} = 0, \\ i\lambda_n \tau q_n + \beta q_n + \theta_{nx} = 0. \end{cases} \tag{3.25}$$

Because of the boundary conditions, one can take the following solution:

$$\begin{cases} \varphi_n(x) = \alpha_1 \sin\left(Nx\right), \; \psi_n(x) = \alpha_2 \cos\left(Nx\right), \; w_n(x) = \alpha_3 \cos\left(Nx\right), \\ \theta_n(x) = \alpha_4 \sin\left(Nx\right), \; q_n(x) = \alpha_5 \cos\left(Nx\right), \end{cases} \tag{3.26}$$

where the constants $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ and α_5 are the solution of the following system:

$$\begin{cases} (-\lambda_n^2 \rho_1 + N^2 k + l^2 k_0)\alpha_1 + kN\alpha_2 + (k + k_0) l\, N\alpha_3 = 0, \\ kN\alpha_1 + (-\lambda_n^2 \rho_2 + bN^2 + k)\alpha_2 + kl\,\alpha_3 = \rho_2 c_4, \\ (k_0 + k) l N\alpha_1 + lk\alpha_2 + (-\lambda_n^2 \rho_1 + k_0 N^2 + l^2 k + \frac{\delta(i\lambda_n \tau + \beta)\delta\lambda_n N^2}{(i\lambda_n^2 \rho_3 \tau + \lambda_n \rho_3 \beta - iN^2)})\alpha_3 = 0, \\ (i\lambda_n^2 \rho_3 \tau + \lambda_n \rho_3 \beta - iN^2)\alpha_5 + \delta\lambda_n \alpha_3 N^2 = 0, \\ (i\lambda_n \tau + \beta)\alpha_5 = -\alpha_4 N. \end{cases} \tag{3.27}$$

We distinguish two cases.

Case 1 $\dfrac{b}{\rho_2} = \dfrac{k_0}{\rho_1}$ and $[k - k_0]\left[\rho_3 - \dfrac{\rho_1}{\tau k}\right] - \delta^2 \neq 0$. Let $\lambda_n^2 = \dfrac{k}{\rho_1} N^2 + A$, where A is a constant to be chosen later. Then form (3.27) we have

$$\begin{aligned} &\left(\left(\frac{(k_0 - k)\,\rho_2}{\rho_1} N^2 + (k - A\rho_2)\right)\left(l^2 k_0 - A\rho_1\right) - k^2 N^2\right)\alpha_1 \\ &\quad = -\rho_2 k\, N c_4 - \left(\frac{l\,(k + k_0)\,(k_0 - k)\,\rho_2}{\rho_1} N^3 + + \left(kk_0 - A\rho_2\,(k + k_0) - k^2\right) Nl\right)\alpha_3, \\ &\left(\left(\frac{(k_0 - k)\,\rho_2}{\rho_1} N^2 + (k - A\rho_2)\right)\left(l^2 k_0 - A\rho_1\right) - k^2 N^2\right)\alpha_2 \\ &\quad = \rho_2\left[l^2 k_0 - A\rho_1\right] c_4 + \left(l\,(k + k_0)\, k\, N^2 - kl\,\left(l^2 k_0 - A\rho_1\right)\right)\alpha_3 \end{aligned} \tag{3.28}$$

and α_3 must satisfy

$$\frac{\left(\begin{array}{l}\left(\frac{\left[l^2k_0 - A\rho_1\right](k_0-k)\rho_2}{\rho_1} - k^2 - \frac{l^2(k+k_0)^2\rho_2}{\rho_1}\right)(k_0-k)N^4 \\ +\left(\begin{array}{c}\frac{\left(l^2k - A\rho_1\right)\left[l^2k_0 - A\rho_1\right](k_0-k)\rho_2}{\rho_1} - \left(l^2k - A\rho_1\right)k^2 \\ +(k-A\rho_2)\left[l^2k_0 - A\rho_1\right](k_0-k) - (k-A\rho_2)l^2(k+k_0)^2 + l^2k^2(k+k_0)\end{array}\right)N^2 \\ +lk^2N + \left[l^2k_0 - A\rho_1\right]\left(l^2k - A\rho_1\right)(k-A\rho_2) - l^2k^2\left[l^2k_0 - A\rho_1\right]\end{array}\right)\alpha_3}{\left(\left[\frac{(k_0-k)\rho_2}{\rho_1}N^2 + (k-A\rho_2)\right]\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)} + \frac{\delta^2N^2\left[-\frac{\tau k}{\rho_1}N^2 - A\tau + i\beta N\sqrt{\left(\frac{k}{\rho_1} + \frac{A}{N^2}\right)}\right]}{\left[\left(1 - \frac{\tau\rho_3 k}{\rho_1}\right)N^2 - A\tau\rho_3 + i\rho_3\beta N\sqrt{\left(\frac{k}{\rho_1} + \frac{A}{N^2}\right)}\right]}\alpha_3 = -\frac{\left[l^2k_0 - A\rho_1 - (k+k_0)N^2\right]\rho_2 kl}{\left(\left[\frac{(k_0-k)\rho_2}{\rho_1}N^2 + (k-A\rho_2)\right]\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)}c_4. \tag{3.29}$$

Now, we distinguish four subcases.

$k_0 - k = 0$ and $1 - \frac{\tau\rho_3 k}{\rho_1} \neq 0$, then, from (3.28) and (3.29), we have

$$\begin{cases}\left((k-A\rho_2)\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)\alpha_1 \\ = -\rho_2 kNc_4 - \left[kk_0 - k^2 - A\rho_2(k+k_0)\right]lN\alpha_3, \\ \left((k-A\rho_2)\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)\alpha_2 \\ = \rho_2\left[l^2k_0 - A\rho_1\right]c_4 + \left(l(k+k_0)kN^2 - kl\left[l^2k_0 - A\rho_1\right]\right)\alpha_3\end{cases} \tag{3.30}$$

and α_3 satisfies

$$\frac{\left(\begin{array}{c}\left[l^2k^2(k+k_0) - \left(l^2k - A\rho_1\right)k^2 - (k-A\rho_2)l^2(k+k_0)^2\right]N^2 \\ +lk^2N + \left[l^2k_0 - A\rho_1\right]\left(l^2k - A\rho_1\right)(k-A\rho_2) - l^2k^2\left[l^2k_0 - A\rho_1\right]\end{array}\right)\alpha_3}{\left((k-A\rho_2)\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)} + \frac{\delta^2N^2\left[-\frac{\tau k}{\rho_1}N^2 - A\tau + i\beta N\sqrt{\left(\frac{k}{\rho_1} + \frac{A}{N^2}\right)}\right]}{\left[\left(1 - \frac{\tau\rho_3 k}{\rho_1}\right)N^2 - A\tau\rho_3 + i\rho_3\beta N\sqrt{\left(\frac{k}{\rho_1} + \frac{A}{N^2}\right)}\right]}\alpha_3 = -\frac{\left[l^2k_0 - A\rho_1 - (k+k_0)N^2\right]}{\left([k-A\rho_2]\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)}\rho_2 klc_4. \tag{3.31}$$

We choose A so that

$$A = \frac{\left[k\rho_1 + \rho_2 l^2k_0\right] + N\sqrt{4\rho_2\rho_1 k^2 + \frac{\left[k\rho_1 + \rho_2 l^2 k_0\right]^2}{N^2}}}{2\rho_2\rho_1} \simeq \frac{Nk}{\sqrt{\rho_2\rho_1}}, \tag{3.32}$$

then with (3.32), we have

$$(k - A\rho_2)\left[l^2k_0 - A\rho_1\right] - k^2N^2 = l^2kk_0, \tag{3.33}$$

since, our concern is the asymptotic behavior of the constants, so, for N large enough, we obtain

$$\begin{aligned}&\left[\begin{array}{c}\dfrac{A\rho_1k^2 + A\rho_2l^2\,(k+k_0)^2 - \dfrac{A^3}{N^2}\rho_1^2\rho_2 + l^2k^2\,(k+k_0)}{\left((k - A\rho_2)\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)} \\ -\dfrac{\tau k\delta^2}{\left(1 - \dfrac{\tau\rho_3 k}{\rho_1}\right)\rho_1}\end{array}\right] N^2\alpha_3 \\ &\simeq -\frac{\left[l^2k_0 - A\rho_1 - (k+k_0)\,N^2\right]}{\left([k - A\rho_2]\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)}\rho_2klc_4.\end{aligned} \tag{3.34}$$

By using (3.32), we have

$$\alpha_3 \simeq \frac{\sqrt{\rho_2\rho_1}}{l\,(k+k_0)\,N}c_4$$

and

$$\alpha_2 \simeq \frac{\rho_1 k}{l^2k_0\,(k+k_0)}c_4,$$

so, we deduce with expression of α_2 that

$$\|\Phi_n\|_{\mathcal{H}} \longrightarrow \infty.$$

$k_0 - k = 0$ and $1 - \dfrac{\tau\rho_3 k}{\rho_1} = 0$, then we have from (3.28), (3.29) and (3.33)

$$\begin{cases}\left((k - A\rho_2)\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)\alpha_1 \\ = -\rho_2kNc_4 - l\left[kk_0 - A\rho_2\,(k+k_0) - k^2\right]N\alpha_3, \\ \left((k - A\rho_2)\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)\alpha_2 \\ = \rho_2\left[l^2k_0 - A\rho_1\right]c_4 + \begin{pmatrix} l\,(k+k_0)\,kN^2 \\ -kl\,\left[l^2k_0 - A\rho_1\right]\end{pmatrix}\alpha_3\end{cases} \tag{3.35}$$

and

$$\frac{\begin{pmatrix} \left(-\left(l^2k - A\rho_1\right)k^2 - (k - A\rho_2)\, l^2\,(k+k_0)^2 + l^2k^2\,(k+k_0)\right)N^2 \\ +lk^2N + \left[l^2k_0 - A\rho_1\right]\left(l^2k - A\rho_1\right)(k - A\rho_2) - l^2k^2\left[l^2k_0 - A\rho_1\right] \end{pmatrix}\alpha_3}{l^2kk_0}$$

$$+\frac{\delta^2N^2\left[-\frac{\tau k}{\rho_1}N^2 - A\tau + i\beta N\sqrt{\left(\frac{k}{\rho_1} + \frac{A}{N^2}\right)}\right]}{\left[-A\tau\rho_3 + i\rho_3\beta N\sqrt{\left(\frac{k}{\rho_1} + \frac{A}{N^2}\right)}\right]}$$

$$\alpha_3 = -\frac{\left[l^2k_0 - A\rho_1 - (k+k_0)\,N^2\right]}{l^2kk_0}\rho_2klc_4. \tag{3.36}$$

Using $1 - \dfrac{\tau\rho_3k}{\rho_1} = 0$, (3.32), (3.35) and (3.36) when N large enough, we deduce that

$$\begin{cases} \alpha_3 \simeq \dfrac{2\left[-\sqrt{\rho_2\rho_1} + i\rho_2\rho_3\beta\sqrt{\dfrac{k}{\rho_1}}\right]}{l\left[-\left(4k + \dfrac{\delta^2}{\rho_3}\right) + 4i\dfrac{\beta\sqrt{\rho_2k}}{\tau}\right]N}c_4, \\ \alpha_2 \simeq \dfrac{\dfrac{\delta^2}{\rho_3}\sqrt{\rho_2\rho_1}}{l^2k\left[-\left(4k + \dfrac{\delta^2}{\rho_3}\right) + 4i\dfrac{\beta\sqrt{\rho_2k}}{\tau}\right]}Nc_4, \end{cases}$$

so, we obtain

$$\|\Phi_n\|_{\mathcal{H}} \longrightarrow \infty.$$

$k_0 - k \neq 0$ and $1 - \dfrac{\tau\rho_3k}{\rho_1} = 0$, then we have from (3.28) and (3.29)

$$\left(\begin{bmatrix} \frac{(k_0-k)\,\rho_2}{\rho_1}N^2 \\ +(k - A\rho_2) \end{bmatrix}\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)$$

$$\alpha_1 = -\rho_2kNc_4 - \begin{bmatrix} \frac{l\,(k+k_0)\,(k_0-k)\,\rho_2}{\rho_1}N^3 \\ +\left[kk_0 - A\rho_2\,(k+k_0) - k^2\right]Nl \end{bmatrix}\alpha_3$$

and

$$\left(\begin{bmatrix} \frac{(k_0-k)\,\rho_2}{\rho_1}N^2 \\ +(k - A\rho_2) \end{bmatrix}\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)$$

$$\alpha_2 = \rho_2 \left[l^2 k_0 - A\rho_1\right] c_4 + \begin{pmatrix} l\,(k+k_0)\,kN^2 \\ -kl\,\left[l^2 k_0 - A\rho_1\right] \end{pmatrix} \alpha_3.$$

Also, we have

$$\frac{\begin{pmatrix} \left(\frac{\left[l^2k_0 - A\rho_1\right](k_0-k)\,\rho_2}{\rho_1} - k^2 - \frac{l^2\,(k+k_0)^2\,\rho_2}{\rho_1}\right)(k_0-k)\,N^4 \\ +\begin{pmatrix} \frac{\left(l^2k - A\rho_1\right)\left[l^2k_0 - A\rho_1\right](k_0-k)\,\rho_2}{\rho_1} - \left(l^2k - A\rho_1\right)k^2 \\ +(k-A\rho_2)\left[l^2k_0 - A\rho_1\right](k_0-k) - (k-A\rho_2)\,l^2\,(k+k_0)^2 + l^2k^2\,(k+k_0) \end{pmatrix} N^2 \\ +lk^2N + \left[l^2k_0 - A\rho_1\right]\left(l^2k - A\rho_1\right)(k-A\rho_2) - l^2k^2\left[l^2k_0 - A\rho_1\right] \end{pmatrix}\alpha_3}{\left(\left[\frac{(k_0-k)\,\rho_2}{\rho_1}N^2 + (k-A\rho_2)\right]\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)}$$
$$+\frac{\delta^2N^2\left[-\frac{\tau k}{\rho_1}N^2 - A\tau + i\beta N\sqrt{\left(\frac{k}{\rho_1} + \frac{A}{N^2}\right)}\right]}{\left[-A\tau\rho_3 + i\rho_3\beta N\sqrt{\left(\frac{k}{\rho_1} + \frac{A}{N^2}\right)}\right]}\alpha_3$$
$$= -\frac{\left[l^2k_0 - A\rho_1 - (k+k_0)\,N^2\right]}{\left(\left[\frac{(k_0-k)\,\rho_2}{\rho_1}N^2 + (k-A\rho_2)\right]\left[l^2k_0 - A\rho_1\right] - k^2N^2\right)}\rho_2klc_4, \tag{3.37}$$

Here we choose A as follow:

$$A = \frac{\begin{pmatrix} \left[(k_0-k)\,\rho_2N^2 - k\rho_1 - \rho_2l^2k_0\right] \\ +\sqrt{\left[(k_0-k)\,\rho_2N^2 - \left(k\rho_1 + \rho_2l^2k_0\right)\right]^2 - 4\rho_2\rho_1\left[\frac{l^2k_0\,(k_0-k)\,\rho_2}{\rho_1} - k^2\right]N^2} \end{pmatrix}}{2\rho_2\rho_1}$$
$$\simeq \frac{(k_0-k)\,N^2}{\rho_1}, \tag{3.38}$$

then we have

$$\left[\frac{(k_0-k)\,\rho_2}{\rho_1}N^2 + k - A\rho_2\right]\left[l^2k_0 - A\rho_1\right] - k^2N^2 = l^2kk_0, \tag{3.39}$$

therefore, for N large enough and using (3.37), (3.38) and (3.39), we have

$$\begin{cases} \alpha_3 \simeq -\frac{2lk_0\rho_2}{(k_0-k)^2\,N^2}c_4, \\ \alpha_2 \simeq -\frac{\rho_2\,(k_0-k)}{l^2kk_0}N^2c_4, \end{cases} \tag{3.40}$$

so, we deduce that

$$\|\Phi_n\|_{\mathcal{H}} \longrightarrow \infty.$$

$k_0 - k \neq 0$ and $1 - \dfrac{\tau\rho_3 k}{\rho_1} \neq 0$, then, using (3.28), (3.29) and (3.39), we obtain the same result as before

$$\begin{cases} \alpha_3 \simeq -\dfrac{2lk_0\rho_2}{(k_0-k)^2 N^2} c_4, \\ \alpha_2 \simeq -\dfrac{\rho_2 (k_0-k)}{l^2 k k_0} N^2 c_4, \end{cases} \tag{3.41}$$

so, we get

$$\|\Phi_n\|_{\mathcal{H}} \longrightarrow \infty.$$

Case 2 $\dfrac{b}{\rho_2} \neq \dfrac{k_0}{\rho_1}$. Let $\lambda_n^2 = \dfrac{k}{\rho_1} N^2 + A$, then from (3.25) we have

$$\left\{ \begin{array}{l} \left[-A\rho_1 + l^2 k_0\right]\alpha_1 + kN\alpha_2 + l(k+k_0)N\alpha_3 = 0, \\ kN\alpha_1 + \left[\left(b - \dfrac{\rho_2 k}{\rho_1}\right)N^2 - A\rho_2 + k\right]\alpha_2 + kl\,\alpha_3 = \rho_2 c_4, \\ \left[(k_0-k)N^2 - A\rho_1 + l^2 k\right]\alpha_3 + l(k+k_0)N\alpha_1 + lk\alpha_2 + \delta N\alpha_4 = 0, \\ \alpha_4 = \dfrac{\delta N\left[-\dfrac{\tau k}{\rho_1}N^2 - A\tau + i\beta N\sqrt{\left(\dfrac{k}{\rho_1} + \dfrac{A}{N^2}\right)}\right]}{\left[\left(1 - \dfrac{\tau\rho_3 k}{\rho_1}\right)N^2 - \tau\rho_3 A + i\rho_3\beta N\sqrt{\left(\dfrac{k}{\rho_1} + \dfrac{A}{N^2}\right)}\right]}\alpha_3, \\ \alpha_5 = -\dfrac{i\delta\lambda_n N^2}{\left[N^2 - \tau\rho_3\lambda_n^2 + i\rho_3\beta\lambda_n\right]}\alpha_3, \end{array} \right\} \tag{3.42}$$

then we obtain

$$\begin{aligned} &\left(\left[\begin{array}{c} \left(b - \dfrac{\rho_2 k}{\rho_1}\right)N^2 \\ +(k - A\rho_2) \end{array} \right] \left(l^2 k_0 - A\rho_1\right) \atop -k^2 N^2 \right)\alpha_2 \\ &= \rho_2\left(l^2 k_0 - A\rho_1\right)c_4 + \left[\begin{array}{c} (k+k_0)N^2 \\ -\left(l^2 k_0 - A\rho_1\right) \end{array} \right] kl\alpha_3, \end{aligned} \tag{3.43}$$

$$\begin{aligned} &\left(\left[\begin{array}{c} \left(b - \dfrac{\rho_2 k}{\rho_1}\right)N^2 \\ +(k - A\rho_2) \end{array} \right] \left(l^2 k_0 - A\rho_1\right) - k^2 N^2 \right)\alpha_1 \\ &= -\rho_2 kNc_4 - \left[(k+k_0)\left[\left(b - \dfrac{\rho_2 k}{\rho_1}\right)N^2 + (k - A\rho_2)\right] \atop -k^2 \right] lN\alpha_3 \end{aligned} \tag{3.44}$$

and

$$\left[(k_0-k)N^2-A\rho_1+l^2k+\frac{\left(\begin{array}{c}-\left[(k+k_0)\left[\left(b-\frac{\rho_2 k}{\rho_1}\right)N^2+k-A\rho_2\right]-k^2\right]l^2(k+k_0)N^2\\+\left[(k+k_0)N^2-\left(l^2k_0-A\rho_1\right)\right]k^2l^2\end{array}\right)}{\left[\left(b-\frac{\rho_2 k}{\rho_1}\right)N^2+k-A\rho_2\right]\left(l^2k_0-A\rho_1\right)-k^2N^2}\right]\alpha_3$$
$$+\frac{\delta^2N^2\left[-\frac{\tau k}{\rho_1}N^2-A\tau+i\beta N\sqrt{\frac{k}{\rho_1}+\frac{A}{N^2}}\right]}{\left(1-\frac{\tau\rho_3 k}{\rho_1}\right)N^2-\tau\rho_3A+i\rho_3\beta N\sqrt{\frac{k}{\rho_1}+\frac{A}{N^2}}}\alpha_3$$
$$=-\frac{lk\rho_2\left(l^2k_0-A\rho_1\right)c_4-\rho_2kNl(k+k_0)Nc_4}{\left[\left(b-\frac{\rho_2 k}{\rho_1}\right)N^2+k-A\rho_2\right]\left(l^2k_0-A\rho_1\right)-k^2N^2}. \tag{3.45}$$

Now, we choose A such that

$$A=\frac{\left(\begin{array}{c}\left[\rho_1\left(b-\frac{\rho_2 k}{\rho_1}\right)N^2+\rho_2l^2k_0+k\rho_1\right]\\+\sqrt{\left[\rho_1\left(b-\frac{\rho_2 k}{\rho_1}\right)N^2+\rho_2l^2k_0+k\rho_1\right]^2-4\rho_1\rho_2\left(\left[\left(b-\frac{\rho_2 k}{\rho_1}\right)l^2k_0-k^2\right]N^2-B\right)}\end{array}\right)}{2\rho_1\rho_2}$$
$$\simeq\frac{\left(b-\frac{\rho_2 k}{\rho_1}\right)N^2}{\rho_2}, \tag{3.46}$$

where B is another constant to be chosen later. So, by using (3.46), we have

$$\left[\left(b-\frac{\rho_2 k}{\rho_1}\right)N^2+(k-A\rho_2)\right]\left(l^2k_0-A\rho_1\right)-k^2N^2=l^2kk_0+B. \tag{3.47}$$

From (3.45) and by using (3.47), we have

$$\left[\begin{array}{c}\left[(k_0-k)N^2-A\rho_1+l^2k\right]\\+\frac{\left(\begin{array}{c}-\left[(k+k_0)\left[\left(b-\frac{\rho_2 k}{\rho_1}\right)N^2+(k-A\rho_2)\right]-k^2\right]l^2(k+k_0)N^2\\+\left[(k+k_0)N^2-\left(l^2k_0-A\rho_1\right)\right]k^2l^2\end{array}\right)}{\left(l^2kk_0+B\right)}\end{array}\right]\alpha_3$$
$$+\frac{\delta^2N^2\left[-\frac{\tau k}{\rho_1}N^2-A\tau+i\beta N\sqrt{\left(\frac{k}{\rho_1}+\frac{A}{N^2}\right)}\right]}{\left(1-\frac{\tau\rho_3 k}{\rho_1}\right)N^2-\tau\rho_3A+i\rho_3\beta N\sqrt{\left(\frac{k}{\rho_1}+\frac{A}{N^2}\right)}}\alpha_3$$
$$=-\frac{lk\rho_2\left(l^2k_0-A\rho_1\right)c_4-\rho_2kNl(k+k_0)Nc_4}{l^2kk_0+B}. \tag{3.48}$$

From (3.48) and by using (3.46), we deduce, for N large enough, the following:

$$\left[\left[k_0-\frac{b}{\rho_2}\rho_1\right]N^2+\frac{\left(k_0+\frac{b\rho_1}{\rho_2}\right)}{\left(l^2kk_0+B\right)}k^2l^2N^2\right]\alpha_3$$

$$-\frac{\frac{b\tau\delta^2}{\rho_2}}{\left[\left(1-\frac{\tau\rho_3 b}{\rho_2}\right)N^2+i\rho_3\beta N\sqrt{\frac{b}{\rho_2}}\right]}N^4\alpha_3$$

$$=-\frac{-A\rho_1-(k+k_0)\,N^2}{\left(l^2kk_0+B\right)}lk\rho_2c_4. \tag{3.49}$$

Here, we distinguish two subcases.

$1-\frac{\tau\rho_3 b}{\rho_2}=0$, then we have

$$\alpha_3=-i\left[\frac{lk\rho_2\rho_3\beta\left(\frac{b}{\rho_2}\rho_1+k_0\right)\sqrt{\frac{\rho_2}{b}}}{\tau\delta^2\left(l^2kk_0+B\right)N}\right]c_4 \tag{3.50}$$

and

$$\alpha_2=-\frac{\rho_1 b-k\rho_2}{l^2kk_0+B}N^2c_4.$$

By choosing $B=0$, we deduce that

$$\|\Phi_n\|_{\mathcal{H}}\longrightarrow\infty.$$

$1-\frac{\tau\rho_3 b}{\rho_2}\neq 0$, then, from (3.48), we have

$$\left[k_0-\frac{b}{\rho_2}\rho_1+\frac{k_0+\frac{b\rho_1}{\rho_2}}{l^2kk_0+B}k^2l^2-\frac{\frac{b\tau\delta^2}{\rho_2}}{1-\frac{\tau\rho_3 b}{\rho_2}}\right]N^2\alpha_3$$
$$=\frac{k_0+\frac{\rho_1 b}{\rho_2}}{l^2kk_0+B}lk\rho_2N^2c_4, \tag{3.51}$$

here, we choose B such that

$$B=\frac{\rho_2}{b\tau\delta^2}\left(k_0+\frac{b\rho_1}{\rho_2}\right)\left(1-\frac{\tau\rho_3 b}{\rho_2}\right)k^2l^2-l^2kk_0, \tag{3.52}$$

so, by (3.52), we obtain

$$\frac{k_0 + \dfrac{b\rho_1}{\rho_2}}{l^2 k k_0 + B} k^2 l^2 = \frac{\dfrac{b\tau\delta^2}{\rho_2}}{1 - \dfrac{\tau\rho_3 b}{\rho_2}},$$

then we deduce from (3.48) and (3.52) that

$$\alpha_3 = \frac{b\tau\delta^2\rho_2}{\rho_2 k l \left(1 - \dfrac{\tau\rho_3 b}{\rho_2}\right)\left[k_0 - \dfrac{b}{\rho_2}\rho_1\right]} c_4$$

and

$$\alpha_1 = -\frac{b\tau\delta^2}{\rho_2\left(k_0 + \dfrac{b\rho_1}{\rho_2}\right)\left(1 - \dfrac{\tau\rho_3 b}{\rho_2}\right)k^2 l^2}\left[\rho_2 k + \frac{k_0 b\tau\delta^2}{\left(1 - \dfrac{\tau\rho_3 b}{\rho_2}\right)\left[k_0 - \dfrac{b}{\rho_2}\rho_1\right]}\right] N c_4,$$

thus we have

$$w_{nx}(x) - l\varphi_n(x) = \frac{k_0\tau^2 b^2\delta^4}{\rho_2 k^2 l\left(1 - \dfrac{\tau\rho_3 b}{\rho_2}\right)^2\left[k_0 - \dfrac{b}{\rho_2}\rho_1\right]\left(k_0 + \dfrac{b\rho_1}{\rho_2}\right)} N c_4 \sin(Nx),$$

hence

$$\|\Phi_n\|_{\mathcal{H}} \longrightarrow \infty.$$

The proof of our theorem is then completed. □

4 Exponential Stability of (1.1)–(1.3)

In this section, we use again Theorem 2.2 to prove that the semigroup associated with (2.1) is exponentially stable provided that (2.3), (3.1) and (3.2) hold.

Theorem 4.1 *We assume that (2.3), (3.1) and (3.2) hold. Then the semigroup associated with (2.1) is exponentially stable.*

Proof In Sect. 3, we have proved that the first condition in (2.14) is equivalent to (3.2). Now, by contradiction, we will prove the second condition in (2.14). So, we assume that the second condition in (2.14) is false, then there exist sequences $(\Phi_n)_n \subset \mathcal{D}(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying

$$\|\Phi_n\|_{\mathcal{H}} = 1, \quad \forall n \geq 0, \tag{4.1}$$

$$\lim_{n\to\infty} |\lambda_n| = \infty \tag{4.2}$$

and

$$\lim_{n\to\infty} \|(i\,\lambda_n\, I \;-\; \mathcal{A})\;\Phi_n\|_{\mathcal{H}} \;=\; 0, \tag{4.3}$$

which implies that

$$\left\{\begin{array}{l} i\lambda_n\varphi_n - \widetilde{\varphi}_n \;\longrightarrow\; 0 \text{ in } H_*^1\,(0,1)\,, \\ i\lambda_n\rho_1\widetilde{\varphi}_n - k\,(\varphi_{nx} + \psi_n + lw_n)_x - lk_0\,(w_{nx} - l\varphi_n) \;\longrightarrow\; 0 \text{ in } L^2\,(0,1)\,, \\ i\lambda_n\psi_n - \widetilde{\psi}_n \;\longrightarrow\; 0 \text{ in } H_*^1\,(0,1)\,, \\ i\lambda_n\rho_2\widetilde{\psi}_n - b\psi_{nxx} + k\,(\varphi_{nx} + \psi_n + lw_n) \;\longrightarrow\; 0 \text{ in } L^2\,(0,1)\,, \\ i\lambda_n w_n - \widetilde{w}_n \;\longrightarrow\; 0 \text{ in } H_*^1\,(0,1)\,, \\ i\lambda_n\rho_1\widetilde{w}_n - k_0\,(w_{nx} - l\varphi_n)_x + lk\,(\varphi_{nx} + \psi_n + lw_n) + \delta\theta_{nx} \;\longrightarrow\; 0 \text{ in } L^2\,(0,1)\,, \\ i\lambda_n\rho_3\theta_n + q_{nx} + \delta\widetilde{w}_{nx} \;\longrightarrow\; 0 \text{ in } L^2\,(0,1)\,, \\ i\lambda_n\tau q_n + \beta q_n + \theta_{nx} \;\longrightarrow\; 0 \text{ in } L^2\,(0,1)\,, \end{array}\right\} \tag{4.4}$$

where the notation $\longrightarrow$ means the limit when n goes to infinity. In the following, we will check the second condition in (2.14) by finding a contradiction with (4.1). Our proof is divided into several steps.

Step 1 Taking the inner product of $(i\,\lambda_n\, I \;-\; \mathcal{A})\;\Phi_n$ with Φ_n in $\mathcal{H}$ and using (2.4), we get

$$Re\,\langle (i\,\lambda_n\, I \;-\; \mathcal{A})\,\Phi_n,\, \Phi_n\rangle_{\mathcal{H}} = \beta\,\|q_n\|^2_{L^2(0,1)}\,. \tag{4.5}$$

Using (4.1) and (4.3), we deduce that

$$q_n \longrightarrow 0 \text{ in } L^2\,(0,1)\,. \tag{4.6}$$

Step 2 Applying triangle inequality, we have

$$\left\|\frac{\theta_{nx}}{\lambda_n}\right\|_{L^2(0,1)} \le \frac{1}{|\lambda_n|}\,\|i\lambda_n\tau q_n + \beta q_n + \theta_{nx}\|_{L^2(0,1)} + \left\|i\,\tau\, q_n + \frac{\beta}{\lambda_n}\, q_n\right\|_{L^2(0,1)}.$$

By (4.2), $(4.4)_8$ and (4.6), we get

$$\frac{\theta_{nx}}{\lambda_n} \longrightarrow 0 \text{ in } L^2\,(0,1)\,. \tag{4.7}$$

Multiplying $(4.4)_1$, $(4.4)_3$ and $(4.4)_5$ by $\frac{1}{\lambda_n}$, and using (4.1) and (4.2), we deduce that

$$\left\{\begin{array}{l} \varphi_n \longrightarrow 0 \text{ in } L^2\,(0,1)\,, \\ \psi_n \longrightarrow 0 \text{ in } L^2\,(0,1)\,, \\ w_n \longrightarrow 0 \text{ in } L^2\,(0,1)\,. \end{array}\right\}. \tag{4.8}$$

Step 3 Taking the inner product of (4.4)$_7$ with $\dfrac{i\theta_n}{\lambda_n}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\rho_2 \|\theta_n\|^2_{L^2(0,1)} - \left\langle q_n, \frac{i\theta_{nx}}{\lambda_n}\right\rangle_{L^2(0,1)} - \delta \left\langle \widetilde{w}_n, \frac{i\theta_{nx}}{\lambda_n}\right\rangle_{L^2(0,1)} \longrightarrow 0,$$

then, from (4.1) and (4.7), we get

$$\theta_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.9}$$

Applying triangle inequality, we have

$$\begin{aligned}\left\|\frac{w_{nxx}}{\lambda_n}\right\|_{L^2(0,1)} \leq \frac{1}{k_0 |\lambda_n|} & \left\| i\lambda_n \rho_1 \widetilde{w}_n - k_0 (w_{nx} - l\varphi_n)_x \right.\\ & \left. + lk(\varphi_{nx} + \psi_n + lw_n) + \delta\theta_{nx} \right\|_{L^2(0,1)} \\ & + \frac{1}{k_0}\left\| i\rho_1 \widetilde{w}_n + \frac{lk_0}{\lambda_n}\varphi_{nx} + \frac{lk}{\lambda_n}(\varphi_{nx} + \psi_n + lw_n) + \delta\frac{\theta_{nx}}{\lambda_n}\right\|_{L^2(0,1)}.\end{aligned}$$

Then, by (4.1), (4.2), (4.4)$_6$ and (4.7), we deduce that

$$\left(\frac{1}{\lambda_n} w_{nxx}\right)_n \text{ is uniformly bounded in } L^2(0,1). \tag{4.10}$$

Step 4 Taking the inner product of (4.4)$_7$ with $\dfrac{iw_{nx}}{\lambda_n}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned}\rho_3 \langle \theta_n, w_{nx}\rangle_{L^2(0,1)} - \left\langle q_n, \frac{iw_{nxx}}{\lambda_n}\right\rangle_{L^2(0,1)} - \delta \left\langle \left(i\lambda_n w_{nx} - \widetilde{w}_{nx}\right), \frac{iw_{nx}}{\lambda_n}\right\rangle_{L^2(0,1)} \\ + \delta \|w_{nx}\|^2_{L^2(0,1)} \longrightarrow 0.\end{aligned}$$

Using (4.1), (4.2), (4.4)$_5$, (4.6), (4.9) and (4.10), we deduce that

$$w_{nx} \longrightarrow 0 \text{ in } L^2(0,1), \tag{4.11}$$

and from (4.4)$_5$, we have

$$\frac{\widetilde{w}_{nx}}{\lambda_n} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.12}$$

As $\widetilde{w}_n$ in $\widetilde{H}^1_*(0,1)$ and by using (4.12), we obtain

$$\frac{\widetilde{w}_n}{\lambda_n} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.13}$$

Step 5 Taking the inner product of (4.4)$_6$ with $\dfrac{i\widetilde{w}_n}{\lambda_n}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\rho_1 \left\| \widetilde{w}_n \right\|^2_{L^2(0,1)} + k_0 \left\langle (w_{nx} - l\varphi_n), \frac{i\widetilde{w}_{nx}}{\lambda_n} \right\rangle_{L^2(0,1)} + lk \left\langle (\varphi_{nx} + \psi_n + lw_n), \frac{i\widetilde{w}_n}{\lambda_n} \right\rangle_{L^2(0,1)} + \delta \left\langle \frac{\theta_{nx}}{\lambda_n}, i\widetilde{w}_n \right\rangle_{L^2(0,1)} \longrightarrow 0.$$

Using (4.1), (4.7), (4.12) and (4.13), we obtain

$$\widetilde{w}_n \longrightarrow 0 \text{ in } L^2(0,1), \tag{4.14}$$

and with (4.4)$_5$, we find

$$\lambda_n w_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.15}$$

Step 6 Taking the inner product of $k(\varphi_{nx} + \psi_n + lw_n)$ with θ_{nx} in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned} k\left\langle (\varphi_{nx} + \psi_n + lw_n), \theta_{nx} \right\rangle &= -k\left\langle (\varphi_{nx} + \psi_n + lw_n)_x, \theta_n \right\rangle_{L^2(0,1)} \\ &= \left\langle \left(i\lambda_n \rho_1 \widetilde{\varphi}_n - k(\varphi_{nx} + \psi_n + lw_n)_x - lk_0 (w_{nx} - l\varphi_n), \theta_n \right) \right\rangle_{L^2(0,1)} \\ &\quad - \lambda_n \rho_1 \left\langle i\widetilde{\varphi}_n, \theta_n \right\rangle_{L^2(0,1)} + lk_0 \left\langle (w_{nx} - l\varphi_n), \theta_n \right\rangle_{L^2(0,1)}, \end{aligned}$$

then, by using (4.1), (4.4)$_2$ and (4.9),

$$k\left\langle (\varphi_{nx} + \psi_n + lw_n), \theta_{nx} \right\rangle_{L^2(0,1)} + \lambda_n \rho_1 \left\langle i\widetilde{\varphi}_n, \theta_n \right\rangle_{L^2(0,1)} \longrightarrow 0. \tag{4.16}$$

Taking the inner product of $(\varphi_{nx} + \psi_n + lw_n)$ with $i\lambda_n \widetilde{w}_n$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned} &\left\langle (\varphi_{nx} + \psi_n + lw_n), i\lambda_n \widetilde{w}_n \right\rangle_{L^2(0,1)} \\ &= -\left\langle i\lambda_n \varphi_{nx}, \widetilde{w}_n \right\rangle_{L^2(0,1)} - \left\langle i\lambda_n \psi_n, \widetilde{w}_n \right\rangle_{L^2(0,1)} - l\left\langle i\lambda_n w_n, \widetilde{w}_n \right\rangle_{L^2(0,1)} \\ &\quad - \left\langle \widetilde{\psi}_n, \widetilde{w}_n \right\rangle_{L^2(0,1)} - l\left\langle \left(i\lambda_n w_n - \widetilde{w}_n \right), \widetilde{w}_n \right\rangle_{L^2(0,1)} - l\left\| \widetilde{w}_n \right\|^2_{L^2(0,1)} \\ &= -\left\langle \left(i\lambda_n \varphi_{nx} - \widetilde{\varphi}_{nx} \right), \widetilde{w}_n \right\rangle_{L^2(0,1)} + \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx} \right\rangle_{L^2(0,1)} - \left\langle \left(i\lambda_n \psi_n - \widetilde{\psi}_n \right), \widetilde{w}_n \right\rangle_{L^2(0,1)} \\ &\quad - \left\langle \widetilde{\psi}_n, \widetilde{w}_n \right\rangle_{L^2(0,1)} - l\left\langle \left(i\lambda_n w_n - \widetilde{w}_n \right), \widetilde{w}_n \right\rangle_{L^2(0,1)} - l\left\| \widetilde{w}_n \right\|^2_{L^2(0,1)}. \end{aligned}$$

Then, by using (4.1), $(4.4)_1$, $(4.4)_3$, $(4.4)_5$ and (4.14), we deduce that

$$\left\langle (\varphi_{nx} + \psi_n + l w_n) \, , i\lambda_n \widetilde{w}_n \right\rangle_{L^2(0,1)} - \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx} \right\rangle_{L^2(0,1)} \longrightarrow 0. \tag{4.17}$$

Taking the inner product of $\widetilde{\varphi}_n$ with $\widetilde{w}_{nx}$ in $L^2(0,1)$, we get

$$\begin{aligned} \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx} \right\rangle_{L^2(0,1)} &= \left\langle \widetilde{\varphi}_n, \left(\widetilde{w}_{nx} - \widetilde{\varphi}_n \right) \right\rangle_{L^2(0,1)} + \left\| \widetilde{\varphi}_n \right\|^2_{L^2(0,1)} \\ &= - \left\langle \widetilde{\varphi}_n, \left(i\lambda_n w_{nx} - \widetilde{w}_{nx} \right) \right\rangle_{L^2(0,1)} + \left\langle \widetilde{\varphi}_n, \left(i\lambda_n \varphi_n - \widetilde{\varphi}_n \right) \right\rangle_{L^2(0,1)} \\ &\quad + \left\langle \widetilde{\varphi}_n, i\lambda_n \left(w_{nx} - \varphi_n \right) \right\rangle_{L^2(0,1)} + \left\| \widetilde{\varphi}_n \right\|^2_{L^2(0,1)}, \end{aligned}$$

then, by (4.1), $(4.4)_1$ and $(4.4)_5$, we have

$$\lambda_n \left\langle \widetilde{\varphi}_n, i \left(w_{nx} - \varphi_n \right) \right\rangle_{L^2(0,1)} - \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx} \right\rangle_{L^2(0,1)} + \left\| \widetilde{\varphi}_n \right\|^2_{L^2(0,1)} \longrightarrow 0. \tag{4.18}$$

Taking the inner product of $(4.4)_2$ with $(w_{nx} - l\varphi_n)$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned} &\left\langle i\lambda_n \rho_1 \widetilde{\varphi}_n, \left(w_{nx} - l\varphi_n \right) \right\rangle_{L^2(0,1)} + k \left\langle (\varphi_{nx} + \psi_n + l w_n) \, , (w_{nx} - l\varphi_n)_x \right\rangle_{L^2(0,1)} \\ &- l k_0 \left\| (w_{nx} - l\varphi_n) \right\|^2_{L^2(0,1)} \longrightarrow 0, \end{aligned}$$

which implies that

$$\begin{aligned} &\lambda_n \rho_1 \left\langle i\widetilde{\varphi}_n, \left(w_{nx} - l\varphi_n \right) \right\rangle_{L^2(0,1)} \\ &\quad - \frac{k}{k_0} \langle (\varphi_{nx} + \psi_n + l w_n) \, , \\ &\quad \times \left[i\lambda_n \rho_1 \widetilde{w}_n - k_0 \left(w_{nx} - l\varphi_n \right)_x + l k \left(\varphi_{nx} + \psi_n + l w_n \right) + \delta\theta_{nx} \right] \rangle_{L^2(0,1)} \\ &\quad + \frac{k\rho_1}{k_0} \left\langle (\varphi_{nx} + \psi_n + l w_n) \, , \ i\lambda_n \widetilde{w}_n \right\rangle_{L^2(0,1)} + \frac{l k^2}{k_0} \left\| (\varphi_{nx} + \psi_n + l w_n) \right\|^2_{L^2(0,1)} \\ &\quad + \frac{\delta k}{k_0} \langle (\varphi_{nx} + \psi_n + l w_n) \, , \theta_{nx} \rangle_{L^2(0,1)} - l k_0 \left\| (w_{nx} - l\varphi_n) \right\|^2_{L^2(0,1)} \longrightarrow 0. \end{aligned}$$

Using (4.1), $(4.4)_6$, (4.8) and (4.11) , we get

$$\begin{aligned} &- \lambda_n \rho_1 \left\langle \widetilde{\varphi}_n, i \ \left(w_{nx} - l\varphi_n \right) \right\rangle_{L^2(0,1)} + \frac{k\rho_1}{k_0} \left\langle (\varphi_{nx} + \psi_n + l w_n) \, , i\lambda_n \widetilde{w}_n \right\rangle_{L^2(0,1)} \\ &+ \frac{l k^2}{k_0} \left\| (\varphi_{nx} + \psi_n + l w_n) \right\|^2_{L^2(0,1)} + \frac{\delta k}{k_0} \langle (\varphi_{nx} \end{aligned}$$

$$+\psi_n + lw_n), \theta_{nx}\rangle_{L^2(0,1)} \longrightarrow 0, \tag{4.19}$$

then, by (4.16), (4.17), (4.18) and (4.19), we obtain

$$\left(\frac{k}{k_0}-1\right)\rho_1 \left\langle \widetilde{\varphi}_n, \widetilde{w}_{nx}\right\rangle_{L^2(0,1)} - \frac{\delta}{k_0}\lambda_n\rho_1 \left\langle i\widetilde{\varphi}_n, \theta_n\right\rangle_{L^2(0,1)} + \frac{lk^2}{k_0}\|(\varphi_{nx}+\psi_n+lw_n)\|^2_{L^2(0,1)} + \rho_1\left\|\widetilde{\varphi}_n\right\|^2_{L^2(0,1)} \longrightarrow 0. \tag{4.20}$$

Step 7 Taking the inner product of (4.4)$_8$ with $(\varphi_{nx}+\psi_n+lw_n)$ in $L^2(0,1)$, we get

$$\langle i\lambda_n\tau q_n, \varphi_{nx}\rangle_{L^2(0,1)} - \tau\langle q_n, i\lambda_n\psi_n\rangle_{L^2(0,1)} - l\tau\langle q_n, i\lambda_n w_n\rangle_{L^2(0,1)} + \langle\beta q_n, (\varphi_{nx}+\psi_n+lw_n)\rangle_{L^2(0,1)} + \left\langle\theta_{nx}, (\varphi_{n,x}+\psi_n+lw_n)\right\rangle_{L^2(0,1)} \longrightarrow 0,$$

then

$$\langle i\lambda_n\tau q_n, \varphi_{nx}\rangle_{L^2(0,1)} - \tau\left\langle q_n, \left(i\lambda_n\psi_n - \widetilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \tau\left\langle q_n, \widetilde{\psi}_n\right\rangle_{L^2(0,1)} - l\tau\left\langle q_n, \left(i\lambda_n w_n - \widetilde{w}_n\right)\right\rangle_{L^2(0,1)} - l\tau\left\langle q_n, \widetilde{w}_n\right\rangle_{L^2(0,1)} + \langle\beta q_n, (\varphi_{nx}+\psi_n+lw_n)\rangle_{L^2(0,1)} + \langle\theta_{nx}, (\varphi_{nx}+\psi_n+lw_n)\rangle_{L^2(0,1)} \longrightarrow 0.$$

By using (4.1), (4.4)$_3$, (4.4)$_5$, (4.6) and (4.16), we have

$$\langle i\lambda_n\tau q_n, \varphi_{nx}\rangle_{L^2(0,1)} - \frac{\lambda_n\rho_1}{k}\left\langle\theta_n, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} \longrightarrow 0,$$

integrating by parts and using the boundary conditions, we obtain

$$-\lambda_n\tau\langle iq_{nx}, \varphi_n\rangle_{L^2(0,1)} - \frac{\lambda_n\rho_1}{k}\left\langle\theta_n, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} \longrightarrow 0,$$

therefore

$$-\lambda_n\tau\left\langle i\left(i\lambda_n\rho_3\theta_n + q_{nx} + \delta\widetilde{w}_{nx}\right), \varphi_n\right\rangle_{L^2(0,1)} - \lambda_n\tau\langle\lambda_n\rho_3\theta_n, \varphi_n\rangle_{L^2(0,1)} + \lambda_n\tau\delta\left\langle i\widetilde{w}_{nx}, \varphi_n\right\rangle_{L^2(0,1)} - \frac{\lambda_n\rho_1}{k}\left\langle\theta_n, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} \longrightarrow 0,$$

hence

$$\tau\left\langle\left(i\lambda_n\rho_3\theta_n + q_{nx} + \delta\widetilde{w}_{nx}\right), \left(i\lambda_n\varphi_n - \widetilde{\varphi}_n\right)\right\rangle_{L^2(0,1)} + \tau\left\langle\left(i\lambda_n\rho_3\theta_n + q_{nx} + \delta\widetilde{w}_{nx}\right), \widetilde{\varphi}_n\right\rangle_{L^2(0,1)}$$

$$-\lambda_n\tau\left\langle i\rho_3\theta_n,\left(i\lambda_n\varphi_n-\widetilde{\varphi_n}\right)\right\rangle_{L^2(0,1)}-\lambda_n\tau\left\langle i\rho_3\theta_n,\widetilde{\varphi_n}\right\rangle_{L^2(0,1)}$$
$$+\tau\delta\left\langle\widetilde{w}_n,\left(i\lambda_n\varphi_n-\widetilde{\varphi_n}\right)_x\right\rangle_{L^2(0,1)}-\tau\delta\left\langle\widetilde{w}_{nx},\widetilde{\varphi_n}\right\rangle_{L^2(0,1)}-\frac{\lambda_n\rho_1}{k}\left\langle\theta_n,i\widetilde{\varphi_n}\right\rangle_{L^2(0,1)}\longrightarrow 0,$$

so, using (4.1), $(4.4)_1$, $(4.4)_7$, we get

$$\left(\tau\rho_3-\frac{\rho_1}{k}\right)\lambda_n\left\langle\theta_n,i\widetilde{\varphi_n}\right\rangle_{L^2(0,1)}-\tau\delta\left\langle\widetilde{w}_{nx},\widetilde{\varphi_n}\right\rangle_{L^2(0,1)}$$
$$-\lambda_n\tau\left\langle i\rho_3\theta_n,\left(i\lambda_n\varphi_n-\widetilde{\varphi_n}\right)\right\rangle_{L^2(0,1)}\longrightarrow 0. \tag{4.21}$$

On the other hand, integrating by parts and using the boundary conditions, we find that

$$\begin{aligned}
&\lambda_n\left\langle i\rho_3\theta_n,\left(i\lambda_n\varphi_n-\widetilde{\varphi_n}\right)\right\rangle_{L^2(0,1)}\\
&=\left\langle\left(i\lambda_n\rho_3\theta_n+q_{nx}+\delta\widetilde{w}_{nx}\right),\left(i\lambda_n\varphi_n-\widetilde{\varphi_n}\right)\right\rangle_{L^2(0,1)}\\
&\quad-\left\langle q_{nx},\left(i\lambda_n\varphi_n-\widetilde{\varphi_n}\right)\right\rangle_{L^2(0,1)}-\delta\left\langle\widetilde{w}_{nx},\left(i\lambda_n\varphi_n-\widetilde{\varphi_n}\right)\right\rangle_{L^2(0,1)}\\
&=\left\langle\left(i\lambda_n\rho_3\theta_n+q_{nx}+\delta\widetilde{w}_{nx}\right),\left(i\lambda_n\varphi_n-\widetilde{\varphi_n}\right)\right\rangle_{L^2(0,1)}\\
&\quad+\left\langle q_n,\left(i\lambda_n\varphi_{nx}-\widetilde{\varphi}_{nx}\right)\right\rangle_{L^2(0,1)}+\delta\left\langle\widetilde{w}_n,\left(i\lambda_n\varphi_{nx}-\widetilde{\varphi}_{nx}\right)\right\rangle_{L^2(0,1)},
\end{aligned}$$

so, by using $(4.4)_1$, $(4.4)_7$, (4.6) and (4.14), we deduce that

$$\lambda_n\left\langle i\rho_3\theta_n,\left(i\lambda_n\varphi_n-\widetilde{\varphi_n}\right)\right\rangle_{L^2(0,1)}\longrightarrow 0, \tag{4.22}$$

therefore, (4.21) and (4.22) give

$$\left(\tau\rho_3-\frac{\rho_1}{k}\right)\lambda_n\left\langle i\widetilde{\varphi_n},\theta_n\right\rangle_{L^2(0,1)}-\tau\delta\left\langle\widetilde{\varphi_n},\widetilde{w}_{nx}\right\rangle_{L^2(0,1)}\longrightarrow 0, \tag{4.23}$$

and then, multiplying (4.23) by $\frac{\rho_1}{\tau\delta}\left(\frac{k}{k_0}-1\right)$ and adding (4.20), we obtain

$$\frac{\rho_1\lambda_n}{k_0\delta}\left[(k-k_0)\left(\rho_3-\frac{\rho_1}{\tau k}\right)-\delta^2\right]\left\langle i\widetilde{\varphi_n},\theta_n\right\rangle_{L^2(0,1)}$$
$$+\frac{lk^2}{k_0}\left\|(\varphi_{nx}+\psi_n+lw_n)\right\|^2_{L^2(0,1)}+\rho_1\left\|\widetilde{\varphi_n}\right\|^2_{L^2(0,1)}\longrightarrow 0.$$

Here we use the fact that $(k-k_0)\left(\rho_3-\dfrac{\rho_1}{\tau k}\right)-\delta^2=0$ (condition (3.1)), we deduce that

$$\frac{lk^2}{k_0}\left\|(\varphi_{nx}+\psi_n+lw_n)\right\|^2_{L^2(0,1)}+\rho_1\left\|\widetilde{\varphi_n}\right\|^2_{L^2(0,1)}\longrightarrow 0,$$

then, from (4.8), we have

$$\varphi_{nx} \longrightarrow 0 \text{ in } L^2(0,1) \tag{4.24}$$

and

$$\widetilde{\varphi}_n \longrightarrow 0 \text{ in } L^2(0,1), \tag{4.25}$$

and using (4.2), $(4.4)_1$ and (4.25), we have

$$\lambda_n \varphi_n \longrightarrow 0 \text{ in } L^2(0,1) \tag{4.26}$$

and

$$\frac{\widetilde{\varphi}_{nx}}{\lambda_n} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.27}$$

Step 8 Taking the inner product of $(4.4)_4$ with $(\varphi_{nx} + \psi_n + lw_n)$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned}&\left\langle i\lambda_n\rho_2\widetilde{\psi}_n, \varphi_{nx}\right\rangle_{L^2(0,1)} + \left\langle i\lambda_n\rho_2\widetilde{\psi}_n, \psi_n\right\rangle_{L^2(0,1)} + l\left\langle i\lambda_n\rho_2\widetilde{\psi}_n, w_n\right\rangle_{L^2(0,1)}\\ &+b\left\langle \psi_{nx}, (\varphi_{nx}+\psi_n+lw_n)_x\right\rangle_{L^2(0,1)} + k\left\|(\varphi_{nx}+\psi_n+lw_n)\right\|^2_{L^2(0,1)} \longrightarrow 0,\end{aligned}$$

then

$$\begin{aligned}&-\lambda_n\rho_2\left\langle\widetilde{\psi}_n, i\varphi_{nx}\right\rangle_{L^2(0,1)} - \rho_2\left\langle\widetilde{\psi}_n, \left(i\lambda_n\psi_n - \widetilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \rho_2\left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)}\\ &- l\rho_2\left\langle\widetilde{\psi}_n, \left(i\lambda_n w_n - \widetilde{w}_n\right)\right\rangle_{L^2(0,1)} - l\rho_2\left\langle\widetilde{\psi}_n, \widetilde{w}_n\right\rangle_{L^2(0,1)}\\ &-\frac{b}{k}\left\langle\psi_{nx}, \left[i\lambda_n\rho_1\widetilde{\varphi}_n - k(\varphi_{nx}+\psi_n+lw_n)_x - lk_0(w_{nx}-l\varphi_n)\right]\right\rangle_{L^2(0,1)}\\ &+\frac{b}{k}\left\langle\psi_{nx}, i\lambda_n\rho_1\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} - \frac{lk_0 b}{k}\left\langle\psi_{nx}, (w_{nx}-l\varphi_n)\right\rangle_{L^2(0,1)}\\ &+k\left\|\varphi_{nx}+\psi_n+lw_n\right\|^2_{L^2(0,1)} \longrightarrow 0,\end{aligned}$$

using (4.1), $(4.4)_2$, $(4.4)_3$, $(4.4)_5$, (4.8), (4.11), (4.14) and (4.24), we get

$$-\lambda_n\rho_2\left\langle\widetilde{\psi}_n, i\varphi_{nx}\right\rangle_{L^2(0,1)} - \rho_2\left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)} + \frac{b\rho_1}{k}\lambda_n\left\langle\psi_{nx}, i\widetilde{\varphi}_n\right\rangle_{L^2(0,1)} \longrightarrow 0. \tag{4.28}$$

Now, we use that

$$\lambda_n \left\langle \psi_{nx}, i\widetilde{\varphi_n} \right\rangle_{L^2(0,1)} = -\left\langle \left(i\lambda_n \psi_{nx} - \widetilde{\psi_{nx}} \right), \widetilde{\varphi_n} \right\rangle_{L^2(0,1)} - \left\langle \widetilde{\psi_{nx}}, \widetilde{\varphi_n} \right\rangle_{L^2(0,1)},$$

and by integrating by parts and using the boundary conditions, we have

$$\begin{aligned} \lambda_n \left\langle \psi_{nx}, i\widetilde{\varphi_n} \right\rangle_{L^2(0,1)} &= -\left\langle i\lambda_n \psi_{nx} - \widetilde{\psi_{nx}}, \widetilde{\varphi_n} \right\rangle_{L^2(0,1)} + \left\langle \widetilde{\psi_n}, \widetilde{\varphi_{nx}} \right\rangle_{L^2(0,1)} \\ &= -\left\langle \left(i\lambda_n \psi_{nx} - \widetilde{\psi_{nx}} \right), \widetilde{\varphi_n} \right\rangle_{L^2(0,1)} - \left\langle \widetilde{\psi_n}, \left(i\lambda_n \varphi_{nx} - \widetilde{\varphi_{nx}} \right) \right\rangle_{L^2(0,1)} \\ &\quad + \left\langle \widetilde{\psi_n}, i\lambda_n \varphi_{nx} \right\rangle_{L^2(0,1)}, \end{aligned}$$

therefore, from (4.1), $(4.4)_1$ and $(4.4)_3$, we see that

$$\lambda_n \left\langle \psi_{nx}, i\widetilde{\varphi_n} \right\rangle_{L^2(0,1)} - \lambda_n \left\langle \widetilde{\psi_n}, i\varphi_{nx} \right\rangle_{L^2(0,1)} \longrightarrow 0, \tag{4.29}$$

so, inserting (4.29) into (4.28), we obtain

$$\frac{\lambda_n}{k} (b\rho_1 - k\rho_2) \left\langle \psi_{nx}, i\widetilde{\varphi_n} \right\rangle_{L^2(0,1)} - \rho_2 \left\| \widetilde{\psi_n} \right\|^2_{L^2(0,1)} \longrightarrow 0. \tag{4.30}$$

At this stage, we use the fact that $b\rho_1 - k\rho_2 = 0$ (condition (3.1)), then we have from (4.30)

$$\widetilde{\psi_n} \longrightarrow 0 \text{ in } L^2(0,1), \tag{4.31}$$

and by $(4.4)_3$, we deduce that

$$\lambda_n \psi_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.32}$$

Step 9 Taking the inner product of $(4.4)_4$ with ψ_n in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$-\rho_2 \left\langle \widetilde{\psi_n}, i\lambda_n \psi_n \right\rangle_{L^2(0,1)} + b \left\| \psi_{nx} \right\|^2_{L^2(0,1)} + k \left\langle (\varphi_{nx} + \psi_n + l w_n), \psi_n \right\rangle_{L^2(0,1)} \longrightarrow 0,$$

and by using (4.8), (4.24), (4.31) and (4.32), we obtain

$$\psi_{nx} \longrightarrow 0 \text{ in } L^2(0,1). \tag{4.33}$$

A combination of (4.6), (4.8), (4.9), (4.11), (4.14), (4.24), (4.25), (4.31) and (4.33) leads to

$$\|\Phi_n\|_{\mathcal{H}} \longrightarrow 0,$$

which is a contradiction with (4.1). Hence, the proof of Theorem 4.1 is completed. □

5 Polynomial Stability of (1.1)–(1.3)

In this section, we prove the polynomial decay of the solutions of (2.1) using Theorem 2.3. Our main result is stated as follow:

Theorem 5.1 *We assume that* (2.3) *and* (3.2) *hold. Then, for each* $p \in \mathbb{N}^*$, *there exists a constant* $c_p > 0$ *such that*

$$\forall \Phi_0 \in D\left(\mathcal{A}^p\right), \ \forall t > 0, \ \left\| e^{t\mathcal{A}} \Phi_0 \right\|_{\mathcal{H}} \le c_p \, \|\Phi_0\|_{D(\mathcal{A}^p)} \left(\frac{\ln t}{t}\right)^{\frac{p}{8}} \ln t. \tag{5.1}$$

Proof In Sect. 3, we have proved that the first condition in (2.15) is satisfied if (3.2) holds. Now, we need to show that

$$\sup_{|\lambda| \ge 1} \frac{1}{\lambda^8} \left\| (i\lambda I - \mathcal{A})^{-1} \right\|_{\mathcal{H}} < \infty. \tag{5.2}$$

We establish (5.2) by contradiction. So, if (5.2) is false, then there exist sequences $(\Phi_n)_n \subset \mathcal{D}(\mathcal{A})$ and $(\lambda_n)_n \subset \mathbb{R}$ satisfying

$$\|\Phi_n\|_{\mathcal{H}} = 1, \ \forall n \in \mathbb{N}, \tag{5.3}$$

$$\lim_{n \to \infty} |\lambda_n| = \infty \tag{5.4}$$

and

$$\lim_{n \to \infty} \lambda_n^8 \, \|(i\lambda_n I - \mathcal{A}) \, \Phi_n\|_{\mathcal{H}} = 0, \tag{5.5}$$

which implies that

$$\left\{\begin{array}{l} \lambda_n^8\left(i\lambda_n\varphi_n-\widetilde{\varphi}_n\right)\to 0 \text{ in } H_*^1(0,1),\\ \lambda_n^8\left[i\lambda_n\rho_1\widetilde{\varphi}_n-k\left(\varphi_{nx}+\psi_n+lw_n\right)_x-lk_0\left(w_{nx}-l\varphi_n\right)\right]\to 0 \text{ in } L^2(0,1),\\ \lambda_n^8\left(i\lambda_n\psi_n-\widetilde{\psi}_n\right)\to 0 \text{ in } \widetilde{H}_*^1(0,1),\\ \lambda_n^8\left[i\lambda_n\rho_2\widetilde{\psi}_n-b\psi_{nxx}+k\left(\varphi_{nx}+\psi_n+lw_n\right)\right]\to 0 \text{ in } L^2(0,1),\\ \lambda_n^8\left(i\lambda_n w_n-\widetilde{w}_n\right)\to 0 \text{ in } \widetilde{H}_*^1(0,1),\\ \lambda_n^8\left[i\lambda_n\rho_1\widetilde{w}_n-k_0\left(w_{nx}-l\varphi_n\right)_x+lk\left(\varphi_{nx}+\psi_n+lw_n\right)+\delta\theta_{nx}\right]\to 0 \text{ in } L^2(0,1),\\ \lambda_n^8\left(i\lambda_n\rho_3\theta_n+q_{nx}+\delta\widetilde{w}_{nx}\right)\to 0 \text{ in } L^2(0,1),\\ \lambda_n^8\left(i\lambda_n\tau q_n+\beta q_n+\theta_{nx}\right)\to 0 \text{ in } L^2(0,1). \end{array}\right\}. \tag{5.6}$$

Our goal is to derive $\|\Phi_n\|_{\mathcal{H}}\to 0$ as a contradiction with (5.3). This will be established through several steps.

Step 1 Taking the inner product of $\lambda_n^8(i\lambda_n I-\mathcal{A})\ \Phi_n$ with Φ_n in $\mathcal{H}$, we get (as for (4.5))

$$Re\left(\left\langle\lambda_n^8(i\lambda_n I-\mathcal{A})\ \Phi_n,\Phi_n\right\rangle_{L^2(0,1)}\right)=\beta\left\|\lambda_n^4 q_n\right\|_{L^2(0,1)}^2,$$

so we have

$$\lambda_n^4 q_n\longrightarrow 0 \text{ in } L^2(0,1). \tag{5.7}$$

Step 2 Applying triangle inequality, we obtain

$$\left\|\lambda_n^3\theta_{nx}\right\|_{L^2(0,1)}\le\left\|\lambda_n^3\left(i\lambda_n\tau q_n+\beta q_n+\theta_{nx}\right)\right\|_{L^2(0,1)}+\left\|i\lambda_n^4\tau q_n+\beta\lambda_n^3 q_n\right\|_{L^2(0,1)},$$

then, using (5.4), $(5.6)_8$ and (5.7), we have

$$\lambda_n^3\theta_{nx}\longrightarrow 0 \text{ in } L^2(0,1). \tag{5.8}$$

Knowing that θ_n in $H_*^1(0,1)$, then we have

$$\lambda_n^3\theta_n\longrightarrow 0 \text{ in } L^2(0,1). \tag{5.9}$$

Step 3 Using (5.3), (5.4), $(5.6)_1$, $(5.6)_3$ and $(5.6)_5$, we obtain

$$\left\{\begin{array}{c} \varphi_n,\ \psi_n\ w_n\longrightarrow 0 \text{ in } L^2(0,1),\\ (\lambda_n\varphi_n)_n,\ (\lambda_n\psi_n)_n,\ (\lambda_n w_n)_n \text{ are uniformly bounded in } L^2(0,1). \end{array}\right\}. \tag{5.10}$$

Step 4 By triangle inequality, we have

$$\left\|\frac{w_{nxx}}{\lambda_n}\right\|\le\left\|\frac{1}{k_0\lambda_n}\left[i\lambda_n\rho_1\widetilde{w}_n-k_0\left(w_{nx}-l\varphi_n\right)_x+lk\left(\varphi_{nx}+\psi_n+lw_n\right)+\delta\theta_{nx}\right]\right\|_{L^2(0,1)}$$

$$+\frac{1}{k_0}\left\|i\rho_1\widetilde{w}_n+lk_0\frac{\varphi_{nx}}{\lambda_n}+\frac{lk}{\lambda_n}\left(\varphi_{nx}+\psi_n+lw_n\right)+\frac{\delta}{\lambda_n}\theta_{nx}\right\|_{L^2(0,1)},$$

then we deduce from (5.3), (5.4), $(5.6)_6$ and (5.8) that

$$\left(\frac{w_{nxx}}{\lambda_n}\right)_n \text{ is uniformly bounded in } L^2(0,1). \tag{5.11}$$

integrating by parts and using the boundary conditions, we have

$$\begin{aligned}\left\|\lambda_n^2 w_{nx}\right\|^2_{L^2(0,1)} &= \lambda_n^4\left\langle w_{nx}, w_{nx}\right\rangle_{L^2(0,1)}\\ &= \lambda_n^3\left\langle iw_{nx}, \left(i\lambda_n w_{nx}-\widetilde{w}_{nx}\right)\right\rangle_{L^2(0,1)}+\lambda_n^3\left\langle iw_{nx},\widetilde{w}_{nx}\right\rangle_{L^2(0,1)}\\ &= \left\langle iw_{nx}, \lambda_n^3\left(i\lambda_n w_{nx}-\widetilde{w}_{nx}\right)\right\rangle_{L^2(0,1)}\\ &\quad +\frac{1}{\delta}\left\langle iw_{nx},\lambda_n^3\left(i\lambda_n\rho_3\theta_n+q_{nx}+\delta\widetilde{w}_{nx}\right)\right\rangle_{L^2(0,1)}\\ &\quad +\frac{\rho_3}{\delta}\left\langle\lambda_n w_n,\lambda_n^3\theta_{nx}\right\rangle_{L^2(0,1)}+\frac{1}{\delta}\left\langle i\frac{w_{nxx}}{\lambda_n},\lambda_n^4 q_n\right\rangle_{L^2(0,1)},\end{aligned}$$

then, by using (5.3), (5.4), $(5.6)_5$, $(5.6)_7$, (5.7), (5.8), (5.10) and (5.11), we find

$$\lambda_n^2 w_{nx}\to 0 \text{ in } L^2(0,1). \tag{5.12}$$

As w_n in $\widetilde{H}^1_*(0,1)$, we deduce from (5.12) that

$$\lambda_n^2 w_n\to 0 \text{ in } L^2(0,1), \tag{5.13}$$

and using (5.4) and $(5.6)_5$, we see that

$$\lambda_n\widetilde{w}_{nx}\to 0 \text{ in } L^2(0,1) \tag{5.14}$$

and

$$\lambda_n\widetilde{w}_n\longrightarrow 0 \text{ in } L^2(0,1). \tag{5.15}$$

Also, dividing $(5.6)_6$ by λ_n^8 and using (5.3), (5.4), (5.8) and (5.15), we deduce that

$$(w_{nxx})_n \text{ is uniformly bounded in } L^2(0,1). \tag{5.16}$$

Step 5 Taking the inner product of $(5.6)_7$ with $\dfrac{iw_{nx}}{\lambda_n^4}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$-\rho_3\left\langle i\lambda_n^3\theta_{nx},\lambda_n^2 w_n\right\rangle_{L^2(0,1)}-\delta\left\langle\lambda_n^4\left(i\lambda_n w_{nx}-\widetilde{w}_{nx}\right), iw_{nx}\right\rangle_{L^2(0,1)}$$

$$- \left\langle \lambda_n^4 q_n, i w_{nxx} \right\rangle_{L^2(0,1)} + \delta \lambda_n^5 \left\| w_{nx} \right\|_{L^2(0,1)}^2 \to 0.$$

Using (5.3), (5.4), (5.6)$_5$, (5.7), (5.8), (5.13) and (5.16), we obtain

$$|\lambda_n|^{\frac{5}{2}} w_{nx} \to 0 \text{ in } L^2(0,1), \tag{5.17}$$

and from (5.6)$_5$, we get

$$|\lambda_n|^{\frac{3}{2}} \widetilde{w}_{nx} \to 0 \text{ in } L^2(0,1). \tag{5.18}$$

Step 6 Applying again triangle inequality, we have

$$\begin{aligned} \left\| \frac{\varphi_{nxx}}{\lambda_n} \right\|_{L^2(0,1)} \le \frac{1}{k} \left\| \frac{1}{\lambda_n} \left[i\lambda_n \rho_1 \widetilde{\varphi}_n - k \left(\varphi_{nx} + \psi_n + l w_n\right)_x - l k_0 \left(w_{nx} - l\varphi_n\right) \right] \right\|_{L^2(0,1)} \\ + \frac{1}{k} \left\| i\rho_1 \widetilde{\varphi}_n - \frac{k}{\lambda_n} \left(\psi_{nx} + l w_{nx}\right) - \frac{l k_0}{\lambda_n} \left(w_{nx} - l\varphi_n\right) \right\|_{L^2(0,1)}, \end{aligned}$$

and using (5.3), (5.4) and (5.6)$_2$, we deduce that

$$\left(\frac{\varphi_{nxx}}{\lambda_n}\right)_n \text{ is uniformly bounded in } L^2(0,1). \tag{5.19}$$

Taking the inner product of (5.6)$_6$ with $\frac{\varphi_{nx}}{\lambda_n^8}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we obtain

$$\begin{aligned} \rho_1 \left\langle i\lambda_n \widetilde{w}_n, \varphi_{nx} \right\rangle_{L^2(0,1)} + k_0 \left\langle \lambda_n w_{nx}, \frac{\varphi_{nxx}}{\lambda_n} \right\rangle_{L^2(0,1)} + l(k+k_0) \left\| \varphi_{nx} \right\|_{L^2(0,1)}^2 \\ + lk \left\langle (\psi_n + l w_n), \varphi_{nx} \right\rangle_{L^2(0,1)} + \delta \left\langle \theta_{nx}, \varphi_{nx} \right\rangle_{L^2(0,1)} \longrightarrow 0, \end{aligned}$$

then, from (5.3), (5.4), (5.8), (5.10), (5.12), (5.15) and (5.19), we have

$$\varphi_{nx} \longrightarrow 0 \text{ in } L^2(0,1). \tag{5.20}$$

Step 7 Taking the inner product of (5.6)$_6$ with $\frac{\varphi_{nx}}{\lambda_n^7}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned} - \rho_1 \left\langle \widetilde{w}_n, \lambda_n \left(i\lambda_n \varphi_{nx} - \widetilde{\varphi}_{nx}\right) \right\rangle_{L^2(0,1)} + \rho_1 \left\langle \lambda_n \widetilde{w}_{nx}, \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} \\ + k_0 \left\langle \lambda_n^2 w_{nx}, \frac{\varphi_{nxx}}{\lambda_n} \right\rangle_{L^2(0,1)} + l(k+k_0) \lambda_n \left\| \varphi_{nx} \right\|_{L^2(0,1)}^2 \\ + lk \left\langle \lambda_n (\psi_n + l w_n), \varphi_{nx} \right\rangle_{L^2(0,1)} + \delta \left\langle \lambda_n \theta_{nx}, \varphi_{nx} \right\rangle_{L^2(0,1)} \longrightarrow 0, \end{aligned}$$

hence, using (5.3), (5.4), $(5.6)_1$, (5.8), (5.10), (5.12), (5.19) and (5.20), we obtain

$$\lambda_n \left\|\varphi_{nx}\right\|^2_{L^2(0,1)} \longrightarrow 0. \tag{5.21}$$

Taking the inner product of $(5.6)_2$ with $\frac{\varphi_n}{\lambda_n^7}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned}
&-\rho_1\lambda_n \left\langle \widetilde{\varphi}_n, \left(i\lambda_n\varphi_n - \widetilde{\varphi}_n\right)\right\rangle_{L^2(0,1)} - \rho_1\lambda_n \left\|\widetilde{\varphi}_n\right\|^2_{L^2(0,1)} \\
&+ k\lambda_n \left\langle (\varphi_{nx} + \psi_n + lw_n), \varphi_{nx}\right\rangle_{L^2(0,1)} - lk_0\lambda_n \left\langle (w_{nx} - l\varphi_n), \varphi_n\right\rangle_{L^2(0,1)} \longrightarrow 0,
\end{aligned}$$

which implies

$$\begin{aligned}
&-\rho_1 \left\langle \widetilde{\varphi}_n, \lambda_n\left(i\lambda_n\varphi_n - \widetilde{\varphi}_n\right)\right\rangle_{L^2(0,1)} - \rho_1\lambda_n \left\|\widetilde{\varphi}_n\right\|^2_{L^2(0,1)} \\
&+ k\lambda_n \left\|\varphi_{nx}\right\|^2_{L^2(0,1)} + k\left\langle (\lambda_n\psi_n + l\lambda_n w_n), \varphi_{nx}\right\rangle_{L^2(0,1)} \\
&- lk_0 \left\langle (\lambda_n w_{nx} - l\lambda_n\varphi_n), \varphi_n\right\rangle_{L^2(0,1)} \longrightarrow 0,
\end{aligned}$$

so, using (5.3), (5.4), $(5.6)_1$, (5.10), (5.12) and (5.21), we deduce that

$$\lambda_n \left\|\widetilde{\varphi}_n\right\|^2_{L^2(0,1)} \longrightarrow 0, \tag{5.22}$$

and from $(5.6)_1$, we obtain that

$$\lambda_n^3 \left\|\varphi_n\right\|^2 \longrightarrow 0. \tag{5.23}$$

Step 8 Multiplying $(5.6)_2$ by $\frac{1}{|\lambda_n|^{\frac{1}{2}}\lambda_n^8}$, we get

$$\begin{aligned}
&i\frac{\lambda_n}{|\lambda_n|}\rho_1 |\lambda_n|^{\frac{1}{2}}\widetilde{\varphi}_n - k\frac{\varphi_{nxx}}{|\lambda_n|^{\frac{1}{2}}} - k\frac{\psi_{nx}}{|\lambda_n|^{\frac{1}{2}}} - l(k+k_0)\frac{w_{nx}}{|\lambda_n|^{\frac{1}{2}}} \\
&+l^2k_0\frac{\varphi_n}{|\lambda_n|^{\frac{1}{2}}} \longrightarrow 0 \text{ in } L^2(0,1),
\end{aligned}$$

then, using (5.3), (5.4) and (5.22), we deduce that

$$\frac{\varphi_{nxx}}{|\lambda_n|^{\frac{1}{2}}} \longrightarrow 0 \text{ in } L^2(0,1). \tag{5.24}$$

On the other hand, by integrating by parts and using the boundary conditions, we see that

$$\lambda_n \left\langle w_{nxx}, i\lambda_n\varphi_{nx}\right\rangle_{L^2(0,1)} = \lambda_n^2 \left\langle iw_{nx}, \varphi_{nxx}\right\rangle_{L^2(0,1)}$$

$$
\begin{aligned}
&= \left\langle \lambda_n \left(i\lambda_n w_{nx} - \widetilde{w}_{nx} \right), \varphi_{nxx} \right\rangle_{L^2(0,1)} + \lambda_n \left\langle \widetilde{w}_{nx}, \varphi_{nxx} \right\rangle_{L^2(0,1)} \\
&= \left\langle \lambda_n^2 \left(i\lambda_n w_{nx} - \widetilde{w}_{nx} \right), \frac{\varphi_{nxx}}{\lambda_n} \right\rangle_{L^2(0,1)} + \left\langle \lambda_n \left|\lambda_n\right|^{\frac{1}{2}} \widetilde{w}_{nx}, \frac{\varphi_{nxx}}{\left|\lambda_n\right|^{\frac{1}{2}}} \right\rangle_{L^2(0,1)},
\end{aligned}
$$

then, using (5.4), $(5.6)_5$, (5.18) and (5.24), we obtain

$$
\lambda_n \left\langle w_{nxx}, i\lambda_n \varphi_{nx} \right\rangle_{L^2(0,1)} \longrightarrow 0. \tag{5.25}
$$

Furthermore, integrating by parts and using the boundary conditions,

$$
\begin{aligned}
&\lambda_n \left\langle (\varphi_{nx} + \psi_n + l w_n)_x, \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} = -\lambda_n \left\langle (\varphi_{nx} + \psi_n + l w_n), \widetilde{\varphi}_{nx} \right\rangle_{L^2(0,1)} \\
&= -\frac{1}{lk} \left\langle \lambda_n^2 \left[i\lambda_n \rho_1 \widetilde{w}_n - k_0 (w_{nx} - l\varphi_n)_x + lk (\varphi_{nx} + \psi_n + l w_n) + \delta\theta_{nx} \right], \frac{\widetilde{\varphi}_{nx}}{\lambda_n} \right\rangle_{L^2(0,1)} \\
&\quad - \frac{1}{lk} \left\langle \left(i\lambda_n \rho_1 \widetilde{w}_n + \delta\theta_{nx} \right), \lambda_n \left(i\lambda_n \varphi_{nx} - \widetilde{\varphi}_{nx} \right) \right\rangle_{L^2(0,1)} \\
&\quad + \frac{k_0}{lk} \left\langle (w_{nx} - l\varphi_n)_x, \lambda_n \left(i\lambda_n \varphi_{nx} - \widetilde{\varphi}_{nx} \right) \right\rangle_{L^2(0,1)} - \frac{\lambda_n^3}{lk} \left\langle i\rho_1 \widetilde{w}_{nx}, i\varphi_n \right\rangle_{L^2(0,1)} \\
&\quad + \frac{\delta}{lk} \left\langle \lambda_n^2 \theta_{nx}, i\varphi_{nx} \right\rangle_{L^2(0,1)} - \frac{k_0 \lambda_n}{lk} \left\langle w_{nxx}, i\lambda_n \varphi_{nx} \right\rangle_{L^2(0,1)} - \frac{k_0 \lambda_n^2}{k} i \left\| \varphi_{nx} \right\|_{L^2(0,1)}^2,
\end{aligned}
$$

then, using $(5.6)_1$, $(5.6)_6$, (5.8), (5.15), (5.16), (5.18), (5.23) and (5.25), we find

$$
\lambda_n \left\langle (\varphi_{nx} + \psi_n + l w_n)_x, \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} + \frac{k_0}{k} i \left\| \lambda_n \varphi_{nx} \right\|_{L^2(0,1)}^2 \longrightarrow 0. \tag{5.26}
$$

Taking the inner product of $(5.6)_2$ with $\dfrac{\widetilde{\varphi}_n}{\lambda_n^7}$ in $L^2(0,1)$, we get

$$
\begin{aligned}
\rho_1 i \left\| \lambda_n \widetilde{\varphi}_n \right\|_{L^2(0,1)}^2 &- k\lambda_n \left\langle (\varphi_{nx} + \psi_n + l w_n)_x, \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} \\
&- lk_0 \left\langle (\lambda_n w_{nx} - l\lambda_n \varphi_n), \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} \longrightarrow 0,
\end{aligned}
$$

then, using (5.26), we obtain

$$
\rho_1 i \left\| \lambda_n \widetilde{\varphi}_n \right\|_{L^2(0,1)}^2 + ik_0 \left\| \lambda_n \varphi_{nx} \right\|_{L^2(0,1)}^2 - lk_0 \left\langle (\lambda_n w_{nx} - l\lambda_n \varphi_n), \widetilde{\varphi}_n \right\rangle_{L^2(0,1)} \longrightarrow 0,
$$

and from (5.3), (5.4), $(5.6)_1$, (5.12), (5.22) and (5.23), we deduce that

$$
\lambda_n \widetilde{\varphi}_n \longrightarrow 0 \text{ in } L^2(0,1) \tag{5.27}
$$

and

$$\lambda_n \varphi_{nx} \longrightarrow 0 \text{ in } L^2(0,1). \tag{5.28}$$

Step 9 Multiplying (5.6)$_4$ by $\frac{1}{\lambda_n^9}$, we obtain

$$i\rho_2 \widetilde{\psi}_n - b\frac{\psi_{nxx}}{\lambda_n} + \frac{k}{\lambda_n}(\varphi_{nx} + \psi_n + l w_n) \to 0 \text{ in } L^2(0,1).$$

By triangle inequality, we deduce from (5.3) and (5.4) that

$$\left(\frac{\psi_{nxx}}{\lambda_n}\right)_n \text{ is uniformly bounded in } L^2(0,1). \tag{5.29}$$

Taking the inner product of (5.6)$_2$ with $\frac{\psi_{nx}}{\lambda_n^8}$ in $L^2(0,1)$, we get

$$\begin{aligned}&\rho_1 \left\langle i\lambda_n \widetilde{\varphi}_n, \psi_{nx}\right\rangle_{L^2(0,1)} - k\langle \varphi_{nxx}, \psi_{nx}\rangle_{L^2(0,1)} - k\|\psi_{nx}\|^2_{L^2(0,1)}\\ &-l(k+k_0)\langle w_{nx}, \psi_{nx}\rangle_{L^2(0,1)} + l^2 k_0 \langle \varphi_n, \psi_{nx}\rangle_{L^2(0,1)} \to 0,\end{aligned}$$

then, integrating by parts and using the boundary conditions, we obtain

$$\begin{aligned}&\rho_1 \left\langle i\lambda_n \widetilde{\varphi}_n, \psi_{nx}\right\rangle_{L^2(0,1)} + k\left\langle \lambda_n \varphi_{nx}, \frac{\psi_{nxx}}{\lambda_n}\right\rangle_{L^2(0,1)} - k\|\psi_{nx}\|^2_{L^2(0,1)}\\ &- l(k+k_0)\langle w_{nx}, \psi_{nx}\rangle_{L^2(0,1)} + l^2 k_0 \langle \varphi_n, \psi_{nx}\rangle_{L^2(0,1)} \to 0,\end{aligned}$$

so, using (5.3), (5.4), (5.10), (5.12), (5.27), (5.28) and (5.29), we deduce that

$$\psi_{nx} \longrightarrow 0 \text{ in } L^2(0,1). \tag{5.30}$$

Taking the inner product of (5.6)$_4$ with $\frac{\psi_n}{\lambda_n^8}$ in $L^2(0,1)$, integrating by parts and using the boundary conditions, we get

$$\begin{aligned}&-\rho_2 \left\langle \widetilde{\psi}_n, \left(i\lambda_n \psi_n - \widetilde{\psi}_n\right)\right\rangle_{L^2(0,1)} - \rho_2 \left\|\widetilde{\psi}_n\right\|^2_{L^2(0,1)} + b\|\psi_{nx}\|^2_{L^2(0,1)}\\ &+ \langle k(\varphi_{nx} + \psi_n + l w_n), \psi_n\rangle_{L^2(0,1)} \longrightarrow 0,\end{aligned}$$

hence, using (5.3), (5.4), (5.6)$_3$, (5.10) and (5.30), we get

$$\widetilde{\psi}_n \longrightarrow 0 \text{ in } L^2(0,1). \tag{5.31}$$

A combination of (5.4) and all the above convergence leads to

$$\|\Phi_n\|_{\mathcal{H}} \longrightarrow 0,$$

which is a contradiction with (5.3). Consequently, the proof of our Theorem 5.1 is completed. □

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Compliance with Ethical Standards

Conflict of interest The authors declare that they have no conflict of interest.

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